CSE 312: Foundations of Computing II
Assignment #2
January 13, 2020
due: Tuesday, January 21, 2019, 10:00 p.m.

Instructions:

- **Answers:** When asked for a short answer (such as a single number), also *show and explain your work* briefly. Simplify your final formula algebraically as much as possible, without using your calculator, and only then use your calculator, paying attention to the exercise's instructions for the form of your final answer. If it asks for a simplified fraction, this means an integer or a fraction a/b, where $\gcd(a, b) = 1$. If it asks for an answer to (say) 3 significant digits, this means that, when written in scientific notation, your answer would look like $a.bc \times 10^d$, where a, b, c are digits and d is an integer. (Don't actually give your answer in scientific notation unless it is unwieldy when written in the usual way, say absolute value greater than 10^{13} or less than 10^{-8}.) Highlight or box your final answer to each exercise part.

- **Turn-in:** Do not write your name on your pages (your Gradescope account will identify you to us) and do not include a copy of the exercise's question in what you turn in. You must use Gradescope to upload your homework solutions. You will submit a single PDF file containing your solutions to all the exercises in the homework. Each numbered homework question must be answered on its own page (or pages). You must follow the Gradescope prompts that have you link exercise numbers to your pages. You may typeset your solutions on a computer or you can handwrite them, take a picture of (or scan) each handwritten page, and convert the pictures into a single PDF file. You are responsible for making sure that your solution is easily readable. Popular choices for typesetting mathematics are Microsoft Word (be sure to use its Equation Editor and save as PDF) and LaTeX.

1. Give your answers for this exercise to 3 significant digits.

   (a) The Schnapsen deck is well shuffled and you are dealt a 5-card hand. What is the probability that you have at least one spade and at least one heart?

   (b) All 20 cards are well shuffled again and you are dealt a 5-card hand. What is the probability that you have at least one spade and at least one heart and at least one club?

2. Final exams are given over a period of 5 days, with 6 time slots per day, so that there are 30 final exam slots in total. You are taking CSE 331, CSE 341, and CSE 351, each with a final exam. Assume that no two of your exams are given on the same day and time, but otherwise each exam is assigned to a randomly chosen day and time slot independently. Give exact answers expressed as simplified fractions.

   (a) What is the probability that your 3 final exams fall on 3 different days? Use the sample space consisting of ordered triples $(x_{331}, x_{341}, x_{351})$, where x_i is the day and time slot to which the exam for i is assigned.

   (b) What is the probability that your 3 final exams fall on 3 different days? Use the sample space consisting of sets $\{x_{331}, x_{341}, x_{351}\}$ of size 3, where x_i is the day and time slot to which the exam for i is assigned.

(c) What is the probability that your 3 final exams all fall on the same day? Use the sample space consisting of ordered triples $(x_{331}, x_{341}, x_{351})$, where x_i is the day and time slot to which the exam for i is assigned.

(d) What is the probability that your 3 final exams all fall on the same day? Use the sample space consisting of sets $\{x_{331}, x_{341}, x_{351}\}$ of size 3, where x_i is the day and time slot to which the exam for i is assigned.

3. An urn contains 6 red, 7 green, and 8 blue balls. The following is repeated 3 times: a ball is selected from the urn at random and removed (called "sampling *without* replacement"). Give your answers to 3 significant digits.

   (a) What is the probability that all 3 selected balls are the same color?

   (b) What is the probability that all 3 selected balls are different colors?

   (c) Repeat part (a) assuming "sampling *with* replacement". That is, the following is repeated 3 times: a single ball is selected from the urn at random, its color is noted, then it is returned to the urn before the next ball is drawn.

   (d) Repeat part (b) assuming sampling *with* replacement.

4. Suppose you choose 26 random, distinct, odd integers from the set $[1, 97]$ (integers from 1 to 97, inclusive), with each odd integer equally likely to be chosen. ("Distinct" means that no two of the chosen integers are equal.) What is the probability that there is a pair of chosen integers that sum to 100? Give an exact answer as a simplified fraction and justify your answer.

5. Rosencrantz and Guildenstern are flipping coins. Guildenstern has a bag with 50 coins in it. All of them are fair coins, except for 4 that each have heads on both sides and 1 that has tails on both sides. Guildenstern reaches into the bag without looking, removes a randomly chosen coin, with each of the 50 coins equally likely, and flips it. Give exact answers expressed as simplified fractions.

   (a) What is the probability that it is one of the 2-headed coins, given that the flip came up heads?

   (b) What is the probability that it is one of the fair coins, given that the flip came up heads?

   (c) What is the probability that it is one of the fair coins, given that the flip came up tails?

6. Alice, Bob, and Chris, in that order and starting with Alice, take turns drawing a random card from a Schnapsen deck and then returning it to the deck and reshuffling the cards, continuing until either ♡K or ♡Q is drawn. The first one to draw either of these two cards wins.

   (a) Explain why $\Omega = \{0^i 1 \mid i \geq 0\} \cup \{\text{the infinite string } 000 \cdots\}$ is a reasonable definition of the sample space of this experiment.

   (b) What is the probability of each outcome in the sample space? Show that these probabilities add up to 1, as they must in any proper sample space.

   (c) Let A be the event that Alice wins, B be the event that Bob wins, and C be the event that Chris wins. Let N be the event that no one wins, ever. State in which of these four sets each element of Ω belongs and calculate the probability of each of these four events. Give each probability exactly as a simplified fraction.

7. You are playing a game of Schnapsen with the Maestro and find yourself in this situation:

Your cards	**Concealed cards**	**Face-up trump:** ♠J
♠ KQ	♠ AT	**Stock:** 5 face-down cards
♡ —	♡ AKJ	**Game points to win:** You 2, Maestro 2
♣ A	♣ TJ	**Trick points:** You 13, Maestro 27
♢ AQ	♢ TKJ	**On lead:** You

The Maestro declared the heart marriage earlier and led ♡Q, so you know he is still holding ♡K. Things are looking very good for you. You already have 13 points in your tricks and you are on lead holding the trump marriage (worth 40 trick points) and two aces. You decide that it's a good time to close the stock. Since the Maestro has fewer than 33 trick points at the time you close the stock, whoever win this deal will receive 2 game points and will win the whole game.

Your plan after closing the stock is to declare the trump marriage, bringing your trick point total to 53, and lead ♠Q. If the Maestro holds only one of ♠AT you will lose this trick, but you are guaranteed to win the next trick and can then lead one of your aces. That will take you well over 66 trick points and you win.

The only danger, in fact, is if the Maestro hold both of ♠AT and also has at least one heart in addition to the ♡K you know he holds. What will happen in this case? He will win your lead of ♠Q with ♠A and will then lead ♠T, forcing you to play your last trump ♠K. Those two tricks will bring his trick point total to 27 + 11 + 3 + 10 + 4 = 55. He will then lead his two winning hearts, one at a time. You will be forced to play one of your aces on his second heart, and the Maestro will have accumulated much more than 66 trick points himself and will win. (If the Maestro has no other heart than ♡K, he can only get to 55 + 4 + 3 = 62 and then has to lead a club or diamond, and you will be the one to reach 66 and win.)

In summary, you will lose the game if and only if the Maestro holds both of ♠AT and at least one heart in addition to ♡K.

Let S be the event that the Maestro holds both ♠AT and let H be the event that the Maestro holds at least one heart in addition to ♡K. The end result of this exercise will be to calculate your probability of losing, which is P($S \cap H$). For all parts, give exact answers as simplified fractions.

  (a) Calculate P(S). Don't forget that you know the Maestro holds ♡K.
  (b) Calculate P($H \mid S$). (Hint: the phrase "at least one heart" should suggest a good technique to use.)
  (c) Calculate your probability of losing the game.

For your interest, this deal actually arose a year ago at our Friday Schnapsen Club, with the Maestro playing against one of last year's 312 students. An interesting feature of the deal is that you have another possible way to play it, which is to lead ♢A first. If you win that trick, you will have at least 13 + 11 + 2 = 26 trick points, and then declaring the trump marriage brings you to at least 66. You could calculate the probability of losing with this alternative line of play by following a very similar outline to this exercise, and then use those two probabilities to decide which of these two plays is better. (You don't have to do what it says in the previous sentence for this homework, it is just there for your interest.)