1.

a. Answer: $\boxed{F_X(2) \approx 4.770 \times 10^{-1}}$

Explain:

The distribution is Binomial because we want to know the probability of some number of successes in some number of trials.

$$X \sim Bin(n,p) = X \sim Bin(80, 0.02)$$

$$F_X(2) = P(X=2)$$

$$= 1 - \left(\binom{80}{0} 0.02^0 (1-0.02)^{80} + \binom{80}{1} 0.02^1 (1-0.02)^{80-1}\right)$$

$$\approx 0.4770264967$$

b. Answer: $\boxed{F_X(2) \approx 2.584 \times 10^{-1}}$

Explanation:

We take λ to be $n \cdot p$ and solve for k.

$$X \sim Poi(\lambda) = X \sim Poi(80 \cdot 0.02) = X \sim Poi(1.6)$$

$$F_X(2) = P(X=2)$$

$$= 1 - \left(e^{-(1.6)} \frac{(1.6)^0}{0!} + e^{-(1.6)} \frac{(1.6)^1}{1!}\right)$$

$$\approx 0.4750690532$$

c. Answer: $\boxed{\text{Percent error} \approx 0.41\%}$

Explanation:

We plug in the answers from part a and part b to the percent error formula and solve.

$$\text{Percent error} = \frac{|\text{Approximate Value} - \text{Actual Value}|}{|\text{Actual Value}|} \cdot 100$$

$$= \frac{|0.4770264967 - 0.4750690532|}{|0.4770264967|} \cdot 100$$

$$\approx 0.4120334661\%$$

2.

a. Answer: $\boxed{7}$

Explanation:

We know that we want to use Poisson distribution. Because we want to find less than we use complent. We find the solution of 7 by guess and checking values for n.

$$0.2 > 1 - P(X=k)$$
$$> 1 - \sum_{i=0}^{n} P_X(i)$$
$$> 1 - \sum_{i=0}^{n} e^{-\lambda} \frac{\lambda^i}{i!}$$
$$> 1 - \sum_{i=0}^{n} e^{-5} \frac{5^i}{i!}$$
$$> 1 - \sum_{i=0}^{7} e^{-5} \frac{5^i}{i!}$$

b. Answer: $\boxed{P(x=7) \approx 1.04 \times 10^{-1}}$

Explanation:

We plug in the value we found in part a to the formula for the poisson distribution.

$$P(X=7) = e^{-5} \frac{5^7}{7!}$$
$$= 0.104444863$$

3.

a. Answer: $\boxed{c = \frac{3}{4},\ E[X] = 0,\ Var(X) = \frac{1}{5}}$

Explanation:

We solve for c by setting the integral of $f_X(x) = 1$ because we that density must sum to one and integrating x will leave c as the only unknown. Then we plug in $f_X(x)$ to the $E[X], E[X^2]$ and, $Var(X)$ formulas and solve.

$$\begin{aligned}
1 &= \int_{-1}^{1} c(1-x^2)\,dx \\
&= c\int_{-1}^{1}(1-x^2)\,dx \\
&= c\left[\int_{-1}^{1} 1\,dx - \int_{-1}^{1} x^2\,dx\right] \\
&= c\left[x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1}\right] \\
&= c\left[(1+1) - \left(\tfrac{1}{3} + \tfrac{1}{3}\right)\right] \\
&= c\left[\tfrac{4}{3}\right] \\
c &= \tfrac{3}{4}
\end{aligned}$$

$$\begin{aligned}
E[X] &= \int_{-1}^{1} x f(x)\,dx \\
&= \int_{-1}^{1} x\left[c(1-x^2)\right]dx \\
&= c\int_{-1}^{1}(x - x^3)\,dx \\
&= c\left(\int_{-1}^{1} x\,dx - \int_{-1}^{1} x^3\,dx\right) \\
&= c\left(\frac{x^2}{2}\Big|_{-1}^{1} - \frac{x^4}{4}\Big|_{-1}^{1}\right) \\
&= c\left(\left[\tfrac{1}{2} - \tfrac{1}{2}\right] - \left[\tfrac{1}{4} - \tfrac{1}{4}\right]\right) \\
&= c(0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \int_{-1}^{1} x^2 c(1-x^2)\,dx \\
&= c\int_{-1}^{1}(x^2 - x^4)\,dx \\
&= c\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{-1}^{1} \\
&= c\left[\left(\tfrac{1}{3} - \tfrac{1}{5}\right) - \left(-\tfrac{1}{3} + \tfrac{1}{5}\right)\right] \\
&= c\left[\tfrac{4}{15}\right] \\
&= \left(\tfrac{3}{4}\right)\tfrac{4}{15} \\
&= \tfrac{1}{5}
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
&= \left(\tfrac{1}{5}\right) - (0)^2 \\
&= \tfrac{1}{5}
\end{aligned}$$

b. Answer: $\boxed{c=4,\ E[X] = \frac{4}{3},\ Var(X) = \frac{2}{9}}$

Explanation:

We solve for c by setting the integral of $f_X(x) = 1$ because we that density must sum to one and integrating x will leave c as the only unknown. Then we plug in $f_X(x)$ to the $E[X], E[X^2]$ and, $Var(X)$ formulas and solve.

$$\begin{aligned}
1 &= \int_{1}^{\infty} cx^{-5}\,dx \\
&= c\int_{1}^{\infty} x^{-5}\,dx \\
&= c\left(\frac{x^{-4}}{-4}\right)\Big|_{1}^{\infty} \\
&= c\left[-\frac{1}{4(\infty)^4} + \frac{1}{4}\right] \\
&= c\,\tfrac{1}{4} \\
c &= 4
\end{aligned}$$

$$\begin{aligned}
E[X] &= \int_{1}^{\infty} x\,cx^{-5}\,dx \\
&= c\int_{1}^{\infty} x^{-4}\,dx \\
&= c\left(\frac{x^{-3}}{3}\right)\Big|_{1}^{\infty} \\
&= c\left(\tfrac{1}{3}\right) \\
&= (4)\tfrac{1}{3} \\
&= \tfrac{4}{3}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \int_{1}^{\infty} x^2 cx^{-5}\,dx \\
&= c\int_{1}^{\infty} x^{-3}\,dx \\
&= c\left(\frac{x^{-2}}{2}\right)\Big|_{1}^{\infty} \\
&= c\left(\tfrac{1}{2}\right) \\
&= (4)\tfrac{1}{2} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
&= 2 - \left(\tfrac{4}{3}\right)^2 \\
&= 2 - \tfrac{16}{9} \\
&= \tfrac{2}{9}
\end{aligned}$$

4.

a. Answer: $\boxed{P(X<\frac{1}{2}) \approx 7.67\times10^{-1}}$

Explanation:

Because we want the probability that X is less than $\frac{1}{2}$, we use that formula for the exponential distribution, $1-P(\frac{L}{3}<X)$. But we also want the probability that X is less than $\frac{2}{3}$ because that sets the other bound where the ratio is less than $\frac{1}{2}$. Then use the formula from lecture that $P(a<X<b) = P(a)-P(b)$ and solve.

$$\begin{aligned}
P(X<\tfrac{1}{2}) &= 1 - P\left(\tfrac{L}{3}\le X\le\tfrac{2L}{3}\right)\\
&= 1-\left(F_X\left(\tfrac{2L}{3}\right)-F_X\left(\tfrac{L}{3}\right)\right)\\
&= 1-\left(\left[1-e^{-\lambda\frac{2L}{3}}\right]-\left[1-e^{-\lambda\frac{L}{3}}\right]\right)\\
&= 1-\left(\left[1-e^{-\left(\frac{3}{L}\right)\frac{2L}{3}}\right]-\left[1-e^{-\left(\frac{3}{L}\right)\frac{L}{3}}\right]\right)\\
&= 1-\left(\left[1-e^{-2}\right]-\left[1-e^{-1}\right]\right)\\
&\approx 0.7674558421
\end{aligned}$$

b. Answer: $\boxed{P(X<0, X>L) \approx 4.99\times10^{-2}}$

Explanation:

Because we want to know the probability that X is less than 0 or greater than L, we do something similar to part a and the complement of the probability that X is between 0 and L.

$$\begin{aligned}
P(X\le 0, X\ge L) &= 1-P(0\le X\le L)\\
&= 1-\left(F_X(L)-F_X(0)\right)\\
&= 1-\left(\left[1-e^{-\frac{3}{L}(L)}\right]-\left[1-e^{-\frac{3}{L}(0)}\right]\right)\\
&= 1-\left(\left[1-e^{-3}\right]-\left[1-e^{0}\right]\right)\\
&\approx 0.04978706837
\end{aligned}$$

5.

a. Answer: $\boxed{F_X(x) = \begin{cases} \frac{x^3}{r^3} & \text{if } x \leq r \\ 0 & \text{otherwise} \end{cases}}$

Explanation:

We want to know the probability of the flea being at any given point in the circle. That's the probability of it being at any particular radius divided by it's probability of being in the sphere.

$$F_X(x) = \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi r^3} = \frac{x^3}{r^3}$$

b. Answer: $\boxed{f_X(x) = \begin{cases} \frac{3x^2}{r^3} & \text{if } x \leq r \\ 0 & \text{otherwise} \end{cases}}$

Explanation:

We know that the PDF is simply the derivative of the CDF with respect to x.

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\left(\frac{x^3}{r^3}\right) = \frac{3x^2}{r^3}$$

c. Answer: $\boxed{E[X] = \frac{3r}{4}}$

Explanation:

We use integration to find the expected value.

$$E[X] = \int_a^b x f_X(x)\,dx = \int_0^r x\left(\frac{3x^2}{r^3}\right)dx = \frac{3}{r^3}\int_0^r x^3\,dx = \frac{3}{r^3}\,\frac{x^4}{4}\Big|_0^r = \frac{3}{r^3}\left[\frac{r^4}{4} - 0\right] = \frac{3r}{4}$$

d. Answer: $\boxed{Var(X) = \frac{3r^2}{80}}$

We first find the $E[X^2]$ to use in the variance formula. This is simply squaring x before integrating. Then we plug this value and that from part a. into the variance formula.

$$E[X^2] = \int_0^r x^2\left(\frac{3x^2}{r^3}\right)dx = \frac{3}{r^3}\int_0^r x^4\,dx = \frac{3}{r^3}\left[\frac{r^5}{5}\right] = \frac{3r^2}{5}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{3r^2}{5} - \left(\frac{3r}{4}\right)^2 = \frac{3r^2}{5} - \frac{9r^2}{16} = \frac{48r^2}{80} - \frac{45r^2}{80} = \frac{3r^2}{80}$$

6.

a. Answer: $\boxed{P(H_1) = \frac{1}{4}}$

Explanation:

We know that $P(A|X=x)$ is just p and since p is uniformly distributed $f(x) = \frac{1}{\frac{1}{2}}$. We plug these in and integrate with respect to p to find the probability of the first flip being heads.

$$\begin{aligned} P(H_1) &= \int_{-\infty}^{\infty} P(A|X=x) f(x)\,dx \\ &= \int_0^{\frac{1}{2}} (p)\left(\frac{1}{\frac{1}{2}}\right) dp \\ &= 2\left(\frac{p^2}{2}\Big|_0^{\frac{1}{2}}\right) \\ &= 2\left(\frac{(\frac{1}{2})^2}{2} - 0\right) \\ &= 2\left(\frac{1}{8}\right) \\ &= \frac{1}{4} \end{aligned}$$

b. Answer: $\boxed{P(H_2) = \frac{1}{12}}$

Explanation:

Because p is conditionally independent for a given value, we can integrate the product of p times p for the first and second heads to find $P(H_2)$.

$$\begin{aligned} P(H_2) &= \int_{-\infty}^{\infty} P(\text{First two heads} | X = x) f(x)\,dx \\ &= \int_0^{\frac{1}{2}} (p^2)(2)\,dp \\ &= 2\int_0^{\frac{1}{2}} p^2\,dp \\ &= 2\left(\frac{p^3}{3}\Big|_0^{\frac{1}{2}}\right) \\ &= 2\left(\frac{(\frac{1}{2})^3}{3} - 0\right) \\ &= \frac{1}{12} \end{aligned}$$

c. Answer: $\boxed{P(H_2 | H_1) = \frac{1}{3}}$

Explanation:

We use the formula for conditional probability. We know $P(H_1 \cap H_2)$ from part b. We plug in and solve.

$$\begin{aligned} P(H_2 | H_1) &= \frac{P(H_2 \cap H_1)}{P(H_1)} \\ &= \frac{(\frac{1}{12})}{(\frac{1}{4})} \\ &= \frac{1}{3} \end{aligned}$$

d. Answer: $\boxed{\text{Not independent}}$

Explanation:

H_1 and H_2 are independent if and only if $P(H_2|H_1) = P(H_2)$. It doesn't, so they're not. We know $P(H_2) = P(H_1)$ because once a probability is chosen, its consistent for the following flips.

$$P(H_2|H_1) = P(H_2) = P(H_1)$$
$$\frac{1}{3} \neq \frac{1}{4}$$

e. Answer: $\boxed{P(H_k) = \frac{\left(\frac{1}{2}\right)^k}{k+1}}$

Explanation:

We follow the same steps from part b. Each flip is conditionally independent for a given probability so the probability for k flips heads is p^k. We integrate and solve.

$$
\begin{aligned}
P(H_k) &= \int_{-\infty}^{\infty} P(A \mid X = x) f(x)\, dx \\
&= \int_0^{\frac{1}{2}} (p^k)(2)\, dp \\
&= 2 \int_0^{\frac{1}{2}} p^k\, dp \\
&= 2 \left(\frac{p^{k+1}}{k+1} \Big|_0^{\frac{1}{2}} \right) \\
&= 2 \left(\frac{\left(\frac{1}{2}\right)^{k+1}}{k+1} - 0 \right) \\
&= 2 \frac{\left(\frac{1}{2}\right)^{k+1}}{k+1} \\
&= 2 \frac{\left(\frac{1}{2}\right)^{k} \cdot \frac{1}{2}}{k+1} \\
&= \frac{\left(\frac{1}{2}\right)^{k}}{k+1}
\end{aligned}
$$

7.

a. Answer: $\boxed{P(\text{at least 3 trump}) = \frac{145}{3876}}$

Explanation:

We use counting to find the probability.

$$P(\text{At least 3 trump}) = \frac{\binom{4}{4}\binom{15}{1} + \binom{4}{3}\binom{15}{2}}{\binom{19}{3}} = \frac{145}{3876}$$

b. Answer: $\boxed{X \sim Geo(p),\ E[X] = \frac{3876}{145}}$

Explanation:

We know X is geometric w/ parameter p because we want to know the number of deals up to and including which indicates geometric.

Parameter p is just the result from part a.

$$E[X] = \frac{1}{p} = \frac{1}{\frac{145}{3876}} = \frac{3876}{145}$$

c. Answer: $\boxed{P(X>t) \approx 3.19 \times 10^{-1}}$

Explanation:

Here we can use the binomial distribution with parameter k=0 to get the probability of no trumps in 30 deals.

$$P(X>t) = \binom{30}{0}\left(\frac{145}{3876}\right)^{0}\left(1-\frac{145}{3876}\right)^{30-0}$$

$$\approx 0.3185990085$$

d. Answer: $\boxed{E[Y] = \frac{3876}{145}}$

Explanation:

We plug p in to the equation and solve.

$$E[Y] = \frac{1}{\lambda} = \frac{1}{\frac{145}{3876}} = \frac{3876}{145}$$

e. Answer: $\boxed{P(Y>t) \approx 3.26 \times 10^{-1}}$

Explanation:

We take the complement of $P(Y>t)$, plug in the formula for the exponential distribution, and solve.

$$P(Y>t) = 1 - P(Y \le t) = 1 - \left(1 - e^{-\lambda t}\right) = 1 - \left(1 - e^{-\left(\frac{145}{3876}\right)(30)}\right) \approx 0.3255331362$$

f. Answer: $\boxed{P(Y>t) \approx 3.20 \times 10^{-1}}$

Explanation:

We repeat the same steps from part e using $t = 30.5$ and solve.

$$P(Y>t) = 1 - P(Y \le t) = 1 - \left(1 - e^{-\lambda t}\right) = 1 - \left(1 - e^{-\left(\frac{145}{3876}\right)(30.5)}\right) \approx 0.3195006815$$