1.

a. Answer: $\boxed{P(X>40) = 0.3974}$

Explanation:

Because the probabilities are the unchanging and independent, we know that X is a binary distribution. We also know then it's expected value and variance. We apply the continuity correction because X is discrete, standardize, and apply the CLT.

$X \sim Bin(n,p)$

$E[X] = np = 234(\frac{1}{6}) = 39$

$Var(X) = np(1-p) = 234(\frac{1}{6})(1-\frac{1}{6}) = \frac{65}{2} = 32.5$

standardize X to $\frac{X-39}{\sqrt{\frac{65}{2}}}$

$$P(X>40) = P\left(\frac{X-39}{\sqrt{\frac{65}{2}}} > \frac{40.5-39}{\sqrt{\frac{65}{2}}}\right)$$
$$= P\left(\frac{X-39}{\sqrt{\frac{65}{2}}} > 0.2631174058\right)$$
$$= 1-\Phi(0.2631174058)$$
$$= 1-0.6026$$
$$= 0.3974$$

b. Answer: $\boxed{P(X\geq 40) = 0.4641}$

Explanation:

Same as part a. but with $\geq$ instead of $>$.

$$P(X\geq 40) = P\left(\frac{X>39}{\sqrt{\frac{65}{2}}} > \frac{39.5-3}{\sqrt{\frac{65}{2}}}\right)$$
$$= 1-\Phi(0.2631174058)$$
$$= 1-0.6026$$
$$= 0.4641$$

c. Answer: $\boxed{P(37\leq X\leq 41) = 0.3400}$

Explanation:

Here we want X between a range, so we do that, apply continuity, standardize and apply CLT.

$$P(37\leq X\leq 41) = P(36.5\leq X\leq 41.5)$$
$$= P\left(\frac{36.5-39}{\sqrt{\frac{65}{2}}} \leq X \leq \frac{41.5-39}{\sqrt{\frac{65}{2}}}\right)$$
$$= \Phi(0.44)-(1-\Phi(0.44))$$
$$= 0.6700-(1-0.6700)$$
$$= 0.3400$$

2. Answer: $\boxed{P(X > 130) = 0.9441}$

Explanation:

Here we just need to find λ then we know the expected value and variance. Apply continuity because poisson is discrete, standardize, and apply CLT.

$\lambda = np = 50{,}000{,}000 \cdot \frac{3}{1{,}000{,}000} = 150$

$X \sim Poi(\lambda)$

$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$

$E[X] = \lambda = 150$

$Var(X) = \lambda = 150$

$$P(X > 130) = P\left(\frac{X-150}{\sqrt{150}} > \frac{130.5-150}{\sqrt{150}}\right)$$
$$= P\left(\frac{X-150}{\sqrt{150}} > \left(-\frac{13\sqrt{6}}{20}\right)\right)$$
$$= 1-(1-\phi(1.59))$$
$$= 1-(1-0.9441)$$
$$= 0.9441$$

3 Answer: $\boxed{P(|Z|>70) = 0.0872}$

Explanation:

We let Z be the difference between X and Y. Because they're uniform then Z is too so we know it's expected value and variance. We want the absolute value of Z so we have to find not just the probability that $Z > 70$ but also $Z < -70$, but because the normal distribution will be symmetric about zero we can just find two times $Z > 70$. We don't apply continuity because since X and Y are in $\mathbb{R}$ Z is continuous.

$$Z = X - Y$$

$$Z \sim \text{Uni}(a, b)$$

$$E[Z] = \frac{a+b}{2} = \frac{0.5 - 0.5}{2} = 0$$

$$\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{(0.5+0.5)^2}{12} = \frac{1}{12}$$

$$\begin{aligned}
P(|Z| > 70) &= P(Z < -70) + P(Z > 70) \\
&= 2P(Z > 70) \\
&= 2P\left(\frac{Z - 20{,}000(0)}{\sqrt{20{,}000 \cdot \frac{1}{12}}} > \frac{70 - 20{,}000(0)}{\sqrt{20{,}000 \cdot \frac{1}{12}}}\right) \\
&= 2(1 - \Phi(1.71)) \\
&= 2(1 - 0.9564) \\
&= 2(0.0436) \\
&= 0.0872
\end{aligned}$$

4. Answer: $\boxed{P(X<2.0) = 22.66\%}$

Explanation:

We recognize that these two probabilities are a system of linear equations. We solve it by setting both equations equal to μ then substituting for σ. While finding the probability of $X<2.0$, we don't apply continuity because the grades are continuous real numbers.

$$\phi\left(\frac{2.8-\mu}{\sigma}\right) = 0.5987 \qquad \phi\left(\frac{3.8-\mu}{\sigma}\right) = (1-0.0668) = 0.9332$$

$$\frac{2.8-\mu}{\sigma} = 0.25 \qquad \frac{3.8-\mu}{\sigma} = 1.50$$

$$\mu = 2.8-0.25\sigma \qquad \mu = 3.8-1.50\sigma$$

$$2.8-0.25\sigma = 3.8-1.50\sigma$$

$$1.25\sigma = 1.0$$

$$\sigma = 0.8$$

$$\mu = 2.8-0.25\sigma$$
$$= 2.8-0.25(0.8)$$
$$= 2.6$$

$$P(X<2.0) = P\left(\frac{X-2.6}{0.8} < \frac{2.0-2.6}{0.8}\right)$$
$$= 1-\phi(0.75)$$
$$= 1-0.7734$$
$$= 0.2266$$

5. Answer: $\boxed{E[X] = 328}$

Explanation:

We let Y a bernoulli distribution with that represents the probability of winning any one game. We know Y's expected value and variance. Then, we let X_i be the average of 45 Bernoulli r.v. games. We then find the expected value and variance of this average X_i. With these we can find the probability of winning less than $\frac{31}{90}$ games, a bad run. Finally we use this to find how many bad runs we expect in 8800 games. We define X to be the sum of X_i games from 1 to 8756. Note, because X_i represents 45 games already summing from 1 to 8800 would overcount. Instead we sum to $8800-45+1 = 8756$. We find the expectation of X to solve.

$p = 0.477$

$Y \sim Ber(p)$

$E[Y] = p = 0.477$

$Var(Y) = p(1-p) = 0.477(1-0.477) = 0.2495$

$\overline{X}_i = \frac{1}{n}\sum_{i=1}^{n} Y$

$E[\overline{X}_i] = E[\frac{1}{n}\sum_{i=1}^{n} Y_i] = \frac{1}{n}\sum_{i=1}^{n} E[Y_i] = E[Y_i] = 0.477$

$Var(\overline{X}_i) = Var(\frac{1}{n}\sum_{i=1}^{n} Y_i) = \frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i) = \frac{Var(Y_i)}{n} = \frac{0.2495}{45} = 5.5438 \times 10^{-3}$

$P(\overline{X}_{45} < \frac{31}{90}) = P\left(\frac{\overline{X}_{45} - 0.477}{\sqrt{5.5438\times10^{-3}}} < \frac{\frac{31}{90} - 0.477}{\sqrt{5.5438\times10^{-3}}}\right)$

$= 1 - \Phi(1.78)$

$= 1 - 0.9625$

$= 0.0375$

$X = \sum_{i=1}^{8800-45+1} \overline{X}_{45} = \sum_{i=1}^{8756} \overline{X}_{45}$

$E[X] = E[\sum_{i=1}^{8756} \overline{X}_{45}]$

$= \sum_{i=1}^{8756} i\, p(\overline{X}_{45} < \frac{31}{90})$

$= \sum_{i=1}^{8756} (0.0375)$

$= 328.35$

$= 328$

6.

a. Answer: $\boxed{P(X<900) = 0.0125}$

Explanation:

Let X be the sum of service times of A. Then X will be normally distributed. We then also know it's mean and variance. We can then find the probability of $X < 900$ milliseconds.

$$X \sim N(n\mu, n\sigma^2) = X \sim (20(50), 20(10^2)) = X \sim N(1000, 2000)$$

$$P(X<900) = P\left(\frac{X-1000}{\sqrt{2000}} < \frac{900-1000}{\sqrt{2000}}\right)$$

$$= 1 - \phi(2.2361)$$

$$= 1 - 0.9875$$

$$= 0.0125$$

b. Answer: $\boxed{P(Y<900) = 0.0183}$

Explanation:

We repeat the same process from part a. Let Y be the sum of service times of B. Then find E[B] and Var(B). Plug these in for the probability that $Y < 900$.

$$Y \sim N(n\mu, n\sigma^2) = Y \sim (20(52), 20(15^2)) = Y \sim N(1040, 4500)$$

$$P(Y<900) = P\left(\frac{Y-1040}{\sqrt{4500}} < \frac{900-1040}{\sqrt{4500}}\right)$$

$$= 1 - \phi(2.0870)$$

$$= 1 - 0.9817$$

$$= 0.0183$$

c. Answer: $\boxed{E[X_1 - X_2] = E[X_1] - E[X_2],\ Var(X_1 - X_2) = Var(X_1) + Var(X_2)}$

Explanation:

We know X_1 and X_2 are i.i.d. Then we can just apply linearity and solve.

$$E[X_1 - X_2] = E[X_1 + (-X_2)]$$

$$= E[X_1] - E[X_2]$$

$$Var(X_1 - X_2) = Var(X_1 + (-X_2))$$

$$= Var(X_1) + Var(-X_2)$$

$$= Var(X_1) + (-1)^2 Var(X_2)$$

$$= Var(X_1) + Var(X_2)$$

d. Answer: $\boxed{P(Y-X<0) = 0.3085}$

Explanation:

We want to know the probability that B finishes before A or, put differently, $Y-X<0$. First we find the $E[Y-X]$ and $Var(Y-X)$ so we use our formulae from part c. Then we simply plug those in and find the probability.

d. $E[Y-X] = 1040 - 1000 = 40$

$Var(Y-X) = 4500 + 2000 = 6500$

$$P(Y-X<0) = P\left(\frac{(Y-X)-40}{\sqrt{6500}} < \frac{0-40}{\sqrt{6500}}\right)$$

$$= 1 - \Phi(0.50)$$

$$= 1 - 0.6915$$

$$= 0.3085$$

e. Answer: $\boxed{\text{B has a higher variance.}}$

Explanation:

Even though B has a higher mean, it also has a significantly larger variance. Because of this B will often, ~30% of the of the time, finish before A.

7.

a. Answer: $\boxed{P(X \geq 135) \leq 0.667}$

Explanation:

We find the expected value of X and plug in to the formula.

$$E[X] = np = 900(0.1) = 90$$

$$\begin{aligned} P(X \geq 135) &\leq \frac{E[X]}{a} \\ &= \frac{900 \cdot 0.1}{135} \\ &= \frac{2}{3} \\ &= 0.667 \end{aligned}$$

b. Answer: $\boxed{P(X \geq 135) \leq 0.0385}$

Explanation:

We find the variance of X and plug into the formula.

$$Var(X) = np(1-p) = 900(0.1)(1-0.1) = 81$$

$$\begin{aligned} P(X \geq 135) &= P(X - 90 \geq 45) \\ &= P(X - \mu \geq \alpha) \\ &\leq \frac{Var(X)}{Var(X) + \alpha^2} \\ &= \frac{81}{81 + 45^2} \\ &= \frac{1}{26} \\ &= 0.0385 \end{aligned}$$

c. Answer: $\boxed{P(X \geq 135) \leq 0.000553}$

Explanation:

We first solve for δ and find that it equals $\frac{1}{2}$. Then, we plug that in to the formula and solve.

$$(1+\delta)\mu = 135$$

$$(1+\delta)90 = 135$$

$$1+\delta = \frac{3}{2}$$

$$\delta = \frac{1}{2}$$

$$\begin{aligned} P(X \geq 135) &= P(X \geq (1+\tfrac{1}{2})90) \\ &\leq e^{-\frac{1}{3}(\frac{1}{2})^2 90} \\ &= 0.000553 \end{aligned}$$