1.

a. $\boxed{\text{Answer: } P_X(d) = \begin{cases} P(X=3) = \binom{21}{3}\Big/\binom{28}{3} = \frac{1330}{3276} = \frac{95}{234} \\ P(X=2) = \binom{7}{1}\binom{21}{2}\Big/\binom{28}{3} = \frac{1470}{3276} = \frac{35}{78} \\ P(X=1) = \binom{7}{2}\binom{21}{1}\Big/\binom{28}{3} = \frac{441}{3276} = \frac{7}{52} \\ P(X=0) = \binom{7}{3}\Big/\binom{28}{3} = \frac{35}{3276} = \frac{5}{468} \end{cases}}$

Explanation:

We find the probabilities for each of the outcomes of X such that the probability of choosing no defective units is P(X=3), one defective unit is P(X=2), two is P(X=1), and only choosing defective units is P(X=0).

b. $\boxed{\text{Answer: } E[X] = \frac{3}{4}}$

Explanation:

We use the formula for Expected value and substitute in our pmf from part a.

$$E[X] = \sum_{i=0}^{3} i\, P_X(i) = 3\left(\frac{5}{468}\right) + 2\left(\frac{7}{52}\right) + 1\left(\frac{35}{78}\right) + 0\left(\frac{95}{234}\right)$$
$$= \frac{5}{156} + \frac{7}{26} + \frac{35}{78} + 0$$
$$= \frac{3}{4}$$

c. $\boxed{\text{Answer: } E[X] = \frac{3}{4}}$

Explanation:

We define indicator variable X_i to be:

$$X_i = \begin{cases} 1 \text{ if defective is chosen} \\ 0 \text{ if not} \end{cases}$$

We find that the probability of choosing any one defective unit of the 28 to be:

$$P(X_i) = \frac{7}{28}$$

Then substituting that in to the formula for expected value and summing over the range of choices yields:

$$E[X_i] = E\left[\sum_{i=1}^{3} X_i\right] = \sum_{i=1}^{3} E[X_i] = (3)\left(\frac{7}{28}\right) = \frac{3}{4} = 0.75$$

2.

$\boxed{\text{Answer : } E[X] \approx 8.79 \times 10^{1}}$

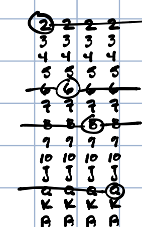
Explanation:

- Define an indicator variable X_i

$$X_i \begin{cases} 1 & : \text{A given stack has no cards of equal rank} \\ 0 & : \text{else a given stack has at least 1 pair of cards of equal rank} \end{cases}$$

- Even though the probabilities of a given stack having no cards of the same rank are dependent, the probabilities will still be the same that a given stack has no cards of the same rank. So, the probability of X_i is:

$$P(X_i = 1) = \frac{52}{52} \cdot \frac{48}{51} \cdot \frac{44}{50} \cdot \frac{40}{49} = \frac{2816}{4165} = 0.6761104442$$



That's because any of the 52 cards can be chosen first, but then there are 52-4 cards that don't have the same rank as that first card for the second card and 52-8 for the third and 52-12 for the fourth.

- The expected value then of X_i is:

$$\begin{aligned} E[X_i] &= 1 \cdot P(\text{No equal rank cards in a stack}) + 0 \cdot P(\text{At least one pair of equal rank cards in a stack}) \\ &= 1 \cdot \left(\frac{2816}{4165}\right) + 0 \cdot \left(1 - \frac{2816}{4165}\right) \\ &= \frac{2816}{4165} \end{aligned}$$

- Plugging this into the formula for expected value yields the number of stacks expected to have no cards of the same rank.

$$E[X] = E\left[\sum_{i=1}^{13} X_i\right] = \sum_{i=1}^{13} E[X_i] = \sum_{i=1}^{13} \left(\frac{2816}{4165}\right) = (13)\frac{2816}{4165} = 8.789435774 = 8.79 \times 10^{1}$$

3.

a. $\boxed{\text{Answer: } P(\text{No trumps in initial hand}) = \frac{\binom{15}{5}}{\binom{19}{5}} = \frac{1001}{3876} \approx 0.258255934 \approx 0.2583}$

Explanation: With the trump is chosen there are 19 cards to choose 5 from. There are then 15 non-trump cards to choose 5 of. So the probability of getting no trumps in an initial hand is given by $\frac{\binom{15}{5}}{\binom{19}{5}}$.

b. $\boxed{\text{Answer: } E[X] \approx 2.025 \times 10^2}$

Explanation:

Let indicator variable X_i be:

$$X_i = \begin{cases} 1 & \text{if four deals in a row have no trump} \\ 0 & \text{otherwise} \end{cases}$$

We find then, that the probability of one run is:

$$P(X_i = 1) = \left(\frac{1001}{3876}\right)^4 = 4.448373413 \times 10^{-3}$$

Then, the expected value of a run is:

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot (1 - P(X_i = 1)) = \left(\frac{1001}{3876}\right)^4$$

Finally we plus that in to the formula for expected value and solve:

$$E[X] = E\left[\sum_{i=1}^{45529} X_i\right] = \sum_{i=4}^{45529} E[X_i] = \sum_{i=4}^{45529} \left(\frac{1001}{3876}\right)^4 \approx 202.5166480 \approx 2.025 \times 10^2$$

a. $\boxed{\text{Answer: } p = \frac{n-i}{n}}$

Explanation:

We know from class that the geometric distribution $E[X_i] = \frac{1}{p}$. Further, the probability of getting a unique card is $\frac{n}{n-i}$. Plugging into the expected value formula yields:

$$E[X] = E\left[\sum_{i=0}^{n-1} X_i\right] = \sum_{i=0}^{n-1} E[X_i] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{i=0}^{n-1} \frac{1}{n-i} = nH_n$$

b. $\boxed{\text{Answer: } nH_n = 100H_{100} = 519}$

Explanation:

Just plug in 100 to the Harmonic number formula.