P3

Prompt:

Let IndSetExists(G, k) be a polynomial time algorithm that, given a graph G and an integer k, outputs **true** if G has an independent set of size k and false otherwise. Now, given a graph G with n vertices and an integer $1 \leq k \leq n$ design a polynomial time algorithm that outputs an independent set of size k in G if it exists and outputs "Impossible" otherwise.

Algorithm:

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Input: A graph G with n vertices and an integer k
Output: An independent set of size k in G, if one exists
IndSet(G, k):
    If IndSetExists(G, k) = false then
        Return "Impossible"
    EndIf
    Let S be an empty set
    For each vertex v in G do
        If k = 1 then
            S = S ∪ v
            Return S
        Else if IndSetExists((G − v), k) then
            G = G − {v}
        Else then
            G = G − {v} − N(v)
            S = S ∪ v
            k = k − 1
        EndIf
    EndFor
    Return S
```

Claim:

The algorithm terminates in polynomial time.

Proof:

The algorithm loop iterates at most v times. On each loop iteration, IndSetExists, which runs in polynomial time, is called at most once. Thus, the polynomial runtime of IndSetExists dominates and the algorithm has a polynomial runtime.

Claim:

The algorithm outputs a set of vertices of size k, if one exists, or "Impossible" otherwise.

Proof:

Prove by induction. Let P(n, k) be the claim that, given a graph G with n vertices and an integer k such that $1 \leq k \leq n$, the algorithm outputs an independent set of k vertices in G if such a set exists, otherwise it outputs "Impossible". We concern ourselves here only with the non-trivial case where an independent set of size k exists in G.

Base Case: It's obvious that P(n, 1) holds for any $n \geq 1$.

Inductive Hypothesis: Assume P(i, j) holds for $1 \leq i \leq n-1$ and $1 \leq j \leq k-1$.

Inductive Step: Goal, show that P(n, k) holds. Let G be a graph with n vertices and let k be an integer such that $1 \leq k \leq n$. Since we know that G contains an independent set of size k, our goal is only to show we can find such a set. Note the trivially obvious property, that if there is a subset $S \subseteq G$ and $|S| = k$ then $|G| \geq k$, where $|S|$ and $|G|$ denotes the size of the independent sets of S and G, respectively. Let S be such an independent subset $S \subseteq G$ where $|S| = k$. Let v be an arbitrary vertex in G. Then, there are two cases, where $v \notin S$ and where $v \in S$. We consider each in turn.

Case 1: $v \notin S$. If $v \notin S$, let $G' = G - \{v\}$. Since G' has $n-1$ vertices, $|G'| \leq k$. But, since $v \notin S$, $S \subseteq G' \subseteq G$ and hence $|G'| \geq k$. Therefore $|G'| = k$. Thus since $|G'| = |S| = |G| = k$, P(n, k) holds when $v \notin S$.

Case 2: $v \in S$. If $v \in S$, let $G' = G - \{v\} - N(v)$, where N(v) denotes the vertex set of the neighbors incident to v. The inductive hypothesis holds for G' so $|G'| = |S \backslash v| = k-1$. To show that $|S \cup v| = k$, consider adding v back to G. Since v and $N(v)$ were removed from G, v was connected to G by N(v). By definition, no vertex $n \in N(v) \in S$. Therefore in G, v was separated by at least two edges from any other vertex that appears in S. This implies that $|G' + v + N(v)| = |S \cup v| = |G| = k$. Thus, P($n$, k) holds when $v \in S$.

Because P(n, k) holds in both cases, P(n, k) holds generally. $\square$