P1

Claim:

Any tournament of 2^n vertices contains an acyclic subtournament of at least $n+1$ vertices.

Proof:

Prove by induction. Let P(n) be the claim that any tournament of 2^n vertices contains an acyclic subtournament of at least $n+1$ vertices.

Base case: P(0) holds because the subtournament of $2^0 = 1$ vertices has $0+1 = 1$ vertices and is trivially acyclic.

Inductive Hypothesis: Assume that P$(n-1)$ holds for $n-1 > 0$.

Inductive Step: Goal, show P(n) holds. Let graph $G = (V, E)$ be a tournament with 2^n vertices. Let v be an arbitrary vertex in G. Partition G into two subtournaments S_1 and S_2 such that $S_1 = \{u \in V \setminus v \mid (u, v) \in E\}$ and $S_2 = \{w \in V \setminus v \mid (v, w) \in E\}$. Because the in-degree of $v \geq 2^{n-1}$ or the out-degree of $v \geq 2^{n-1}$, S_1 or S_2 contains at least 2^{n-1} vertices. Let S' be a tournament with 2^{n-1} vertices from S_1 or S_2. S' has 2^{n-1} vertices, hence by the inductive hypothesis, S' is an acyclic subtournament of G with $n+1-1$ vertices. Let tournament $S^* = S' \cup v$. S^* is acyclic since S' is a subset of S_1 or S_2 which were defined by their having only incoming or outgoing edges with v and S^* has $n+1-1+1 = n+1$ edges. Thus, G has an acyclic subtournament S^* with at least $n+1$ vertices. $\square$