4. Claim: Given a graph $G = (V, E)$, with n vertices such that the degree of every vertex of G is at most k and in every connected component of G there is at least one vertex such that the degree of that vertex is less than k, the vertices of G can be colored with k distinctive colors such that the endpoints of every edge of G have distinct colors.

Proof: Prove by induction on n. Let $P(n)$ be "every graph with n vertices such that the degree of every vertex is at most k and in every connected component of G there is at least one vertex such that the degree of that vertex is less than k, can be colored using k distinct colors such that the endpoints of every edge of G have distinct colors." Let G be an arbitrary graph fitting the above definition.

Base Case: $P(1)$ holds because every connected component is made up of at most 1 vertex. We color these vertices with a single color. We can do so because $k \geq 0$.

Inductive Hypothesis: Assume $P(n-1)$ holds.

Inductive Step: Goal, prove $P(n)$ holds. Let V be a set of vertices chosen from G such that every vertex v that has degree k is $v \in V$ but not including vertices with degree k such that all of that vertex's neighbors are also in V. Let $G' = G - V$ and all edges incident to every $v \in V$. Note that removing V (and all edges incident to every $v \in V$ from G can only reduce the degree of every remaining vertex in G'. Specifically, every vertex of G' has at most degree $k-1$ and every $v \in V$ also has at most degree $k-1$. Since every connected component of G' has $n-1$ vertices, by the Inductive Hypothesis we color the vertices of G' with $k-1$ colors such that every edge has endpoints with distinct colors. Color every vertex of G, except the vertices of V, with the same vertex colors as those in G'. Now, to color $v \in V$ of G, note that every $v \in V$ of G has at most $k-1$ neighbors. Color every uncolored $v \in V$ of G with the one color not used by any of the neighbors of that v. □