

P1

Prompt:

Given a rectangular slab of size $W \times H$ and a set of rectangular plates of sizes $W_1 \times H_1, W_2 \times H_2, \dots, W_n \times H_n$, design an algorithm that runs in time polynomial in n, W , and H and that outputs the minimum possible waste area.

Algorithm:

Input: Slab dimensions W and H and a set of plates P .

Output: The minimum possible waste area.

Let t be an $(W + 1) \times (H + 1)$ table

Let every entry in $t = -1$

$\text{maxCut}(W, H, P)$:

 If $t[W][H] \neq -1$ then

 Return $t[W][H]$

 EndIf

 Let C be a copy of P

 For each plate p in P **do**

 If $p.w = W$ and $p.h = H$ then

 Set $t[W][H] = W * H$

 Return $t[W][H]$

 Else **if** $W < p.w$ or $H < p.h$ then

 Remove p from C

 EndIf

 EndFor

 Let $\text{vertCutMax} = 0$

 Let $\text{horizCutMax} = 0$

 For each plate p in C **do**

 If $p.w \neq W$ then

 Let $\text{vertCut} = \text{maxCut}(W - p.w, H, C) + \text{maxCut}(p.w, H, C)$

 Set $\text{vertCutMax} = \max\{\text{vertCut}, \text{vertCutMax}\}$

 EndIf

 If $p.h \neq H$ then

 Let $\text{horizCut} = \text{maxCut}(W, H - p.h, C) + \text{maxCut}(W, p.h, C)$

 Set $\text{horizCutMax} = \max\{\text{horizCut}, \text{horizCutMax}\}$

 EndIf

 EndFor

 Let $\text{maxCut} = \max\{\text{vertCutMax}, \text{horizCutMax}\}$

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t [W] [H] = maxCut
return t [W] [H]

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Let $\text{maxCut} = \text{maxCut}(W, H, P)$
Output $(W * H) - \text{maxCut}$

Claim:

The algorithm terminates in time polynomial in n , W , and H .

Proof:

The algorithm iterates over each plate p in P at most $2n$ times. In the first of these loops, the work performed is constant in time, making the dominant runtime factor the n loop iterations. In the second loop, the work is divided into 4 subproblems: 2 subproblems of size $W-p.w$ and $p.w$ and 2 sub problems of size $H-p.h$ and $p.h$, respectively. In each case, when a solution to a subproblem is found it is stored in the $(W+1) \times (H+1)$ table making future checks of that same sized subproblem accessible in constant time. Therefore, the runtime of the problem is bounded by $W * H * 2n$ since there are $W * H$ entries that must be checked and for each entry as many as $2n$ checks against the n plates in P . Since 2 is a constant factor, we ignore that and can say that the runtime of the problem is bounded by $W * H * n$. Since each subproblem is monotonically smaller in size than the calling problem, since each loop iterates at most n times, and since the problems are bounded on the low end (a plate p is removed from P if it's larger than $W \times H$), the algorithm terminates. Thus, that algorithm terminates in time polynomial in n , W , and H .

Claim:

The algorithm outputs the minimum possible waste area.

Proof:

Let $\text{OPT}(W, H)$ denote the maximum usable area cut from a slab with dimensions $W \times H$ into plates from a set $P = \{W_1 \times H_1, W_2 \times H_2, \dots, W_n \times H_n\}$.

Let $v(W, H)$ be a function that takes the slab dimensions $W \times H$ and returns the area of that slab if there is a plate p of size $W \times H$ in P and returns 0 otherwise.

$$v(W, H) = \begin{cases} W * H & \text{if } p.w = W \text{ and } p.h = H \text{ for any } p \text{ in } P \\ 0 & \text{otherwise} \end{cases}$$

Case 1: If a slab with dimensions $W \times H$ is not cut, then the maximum usable area of that slab is $W * H$ if there is a plate p in P such that $p.w = W$ and $p.h = H$. Otherwise, if no such plate p exists, then the maximum usable area of a slab with dimensions $W \times H$ is 0. This is given by the defined $v(W, H)$ function.

$$OPT(W, H) = v(W, H)$$

Case 2: If a slab with dimensions $W \times H$ is cut vertically, then the cut that yields the maximum usable area for the slab with dimensions $W \times H$ will be the cut yielding two slabs with dimensions $(W - p.w, H)$ and $(p.w, H)$ for some plate p in P such that the sum of the maximum usable area of the two cut slabs is the maximum over all plates p in P .

$$OPT(W, H) = \max\{OPT(W - p.w, H) + OPT(p.w, H)\} \text{ for all } p \text{ in } P$$

Case 3: If a slab with dimensions $W \times H$ is cut horizontally, then the cut that yields the maximum usable area for the slab with dimensions $W \times H$ will be the cut yielding two slabs with dimensions $(W, H - p.h)$ and $(W, p.h)$ for some plate p in P such that the sum of the maximum usable area of the two cut slabs is the maximum over all plates p in P .

$$OPT(W, H) = \max\{OPT(W, H - p.h) + OPT(W, p.h)\} \text{ for all } p \text{ in } P$$

Thus, the recurrence relation that maximizes the usable area of the slab:

$$OPT(W, H) = \max \begin{cases} v(W, H) & \text{If not cut} \\ \max\{OPT(W - p.w, H) + OPT(p.w, H)\} \text{ for all } p \text{ in } P & \text{If cut vertically} \\ \max\{OPT(W, H - p.h) + OPT(W, p.h)\} \text{ for all } p \text{ in } P & \text{If cut horizontally} \end{cases}$$

To find the minimal wasted area, simply subtract the maximum usable area given by $OPT(W, H)$ from the total area of the slab, $W * H$. \square