P2

Prompt:

Given a sequence $\{d_1, ..., d_n\}$ of positive integers, construct a tree such that the degree of vertex i is d_i or output "impossible" if no such tree exists.

Algorithm:

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// Construct a tree from given positive integer sequence K
DegreeTree(K):
    If Sum(K) does not equal 2(Length(K) - 1)
        Return "Impossible"
    Set S to SortDescending(K)
    Initialize tree T
    For each value s in S
        If tree T is empty then
            Add root r to T
            Set r to degree s
        Else
            For node n in T starting from r
                If n has degree > 0
                    Add leaf k to n
                    Set k degree to s - 1
                    Set n degree to n - 1
                EndIf
            EndFor
        EndIf
    EndFor
    Return T
```

Claim:

The algorithm terminates in polynomial time.

Proof:

The algorithm has two loops. The outer loop iterates over each value in the given sequence exactly once for a runtime of O(n). The inner loop iterates over each node in the tree as it is being constructed at most once, for a runtime of O(n). Therefore in every case, the algorithm as a whole is bounded by O(n^2). And, because each for loop is bounded by n, the algorithm is guaranteed to terminate. $\square$

Claim:

For any sequence of positive integers $\{d_1, ..., d_n\}$ with $\sum_i d_i < 2n$ and with $d_i \geq 1$ for all i, there must be at least one i with $d_i \leq 1$.

Lemma:

Assume for a contradiction that there is a sequence of positive integers $\{d_1, ..., d_n\}$ with $\sum_i d_i < 2n$ and with $d_i > 1$ for all i. Consider a sequence of positive integers of this form such that $d_i > 1$ for all i, let this sequence be $\{2_1, 2_2, ..., 2_n\}$. Then $\sum_i d_i = 2_1 + 2_2 + ... + 2_n = 2n < 2n$ which is a contradiction. $\square$

Claim:

Given a sequence of positive integers $\{d_1, ..., d_n\}$ the algorithm generates a tree with this degree sequence if and only if $\sum_i d_i = 2(n-1)$, and for all i we have $d_i \geq 1$ otherwise it outputs "Impossible".

Proof:

For a proof in the forward direction, if tree T was constructed from a sequence of positive integers $\{d_1, ..., d_n\}$ then T has $\sum_i d_i = 2(n-1)$. For a contradiction assume that tree T was constructed from a sequence of positive integers $\{d_1, ..., d_n\}$ and that $\sum_i d_i \neq 2(n-1)$. We know that T has n - 1 edges and know from lecture that the $\sum d(T) = 2m$. Therefore, T has n - 1 = m edges and degree $\sum d(T) = 2m = 2(n-1)$ which is a contradiction. $\square$

Prove the backwards direction by induction. Let P(n) be the claim that if a sequence of positive integers $\{d_1, ..., d_n\}$ has $\sum_i d_i = 2(n-1)$ and $d_i \geq 1$ for all i then a tree can be constructed from this sequence.

Base case: We use n = 2 for a base case so that the condition $d_i \geq 1$ for all i is met. P(2) holds because $\sum_i d_i = 1 = 2((2)-1) = 2(n-1)$ and $d_i \geq 1$ for all i.

Inductive Hypothesis: Assume P(n - 1) holds for all $n - 1 > 1$.

Inductive Step: Goal show P(n). By the above lemma, for any sequence of positive integers $D = \{d_1, ..., d_n\}$ with $\sum_i d_i < 2n$ and with $d_i \geq 1$ for all i there must be at least one i with $d_i \leq 1$. Let this vertex with $d_i \leq 1$ be vertex v. Let $D' = D - v$. Note, that $d_i \geq 1$ for all i in D' because removing v does not disconnect any vertices other than v. Then $\sum d(D') = 2(n-1) - 2 = 2((n-1)-1)$. By the inductive hypothesis, we can create a tree T' from D'. Because T' is a tree with degree $2((n-1)-1)$, T = T' + v is also a tree with $\sum_i d_i = 2((n-1)-1) + 2 = 2(n-1)$ and $d_i \geq 1$ for all i. $\square$

Thus, because we've shown the biconditional holds, we've shown that the algorithm, which only executes if given a sequence of positive integers $\{d_1, ..., d_n\}$ such that $\sum_i d_i = 2(n-1)$ and for all i we have $d_i \geq 1$, generates a tree. $\square$