3. Claim: An instance of the stable matching problem has exactly one stable matching if and only if the man optimal matching is equal to the woman optimal matching.

Proof: Suppose for a contradiction that there is exactly one stable matching but that the man optimal matching is not equal to the woman optimal matching. We know that a man optimal matching ⇒ a woman pessimal matching. So, because we have a man optimal matching and a woman optimal matching and they are not the same, there are at least two stable optimal matchings, i.e. one stable man optimal / woman pessimal and one stable woman optimal / man pessimal. (Stable since we know that GS only produces stable matchings.) But this is a contradiction since 2 stable matchings ≠ 1 stable matching.

Now suppose for a contradiction that there is not exactly one stable matching but that the man optimal matching equals the woman optimal matching. We know that a man optimal matching = a woman pessimal matching. And we know that a woman optimal matching ⇒ a man pessimal matching, which (since man optimal matching = woman optimal matching) = a man optimal matching. Thus, the man optimal matching = woman pessimal matching = man pessimal matching = woman optimal matching, i.e. they are all the same which is a contradiction.

Thus because both implications are proved, the claim is proved. □