Kernelization dichotomies for hitting minors under structural parameterizations

Eric Brandwein

Departamento de Computación, FCEyN, Universidad de Buenos Aires, Argentina

Joint work with Marin Bougeret and Ignasi Sau

Let $\mathcal{G}$ be a class of graphs.

Vertex Deletion

- *Input:* A graph G and an integer k.
- *Question:* Does G have a set S of at most k vertices such that $G \setminus S \in \mathcal{G}$?

Let $\mathcal{G}$ be a class of graphs.

VERTEX DELETION(k)

- *Input:* A graph G and an integer k.
- ***Parameter:* k.**
- *Question:* Does G have a set S of at most k vertices such that $G \setminus S \in \mathcal{G}$?

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- Admits FPT algorithms.
- Also admits a **polynomial kernel** (Fomin, Lokshtanov, Saurabh, Zehavi 2019).

Kernel

A *kernel* for a parameterized problem Π is a polynomial-time algorithm that, given an instance (I, k) of Π, outputs an instance (I', k') of Π such that:

- $|I'| + k' \leq f(k)$ for some computable function f, and
- (I, k) is a yes-instance if and only if (I', k') is a yes-instance.

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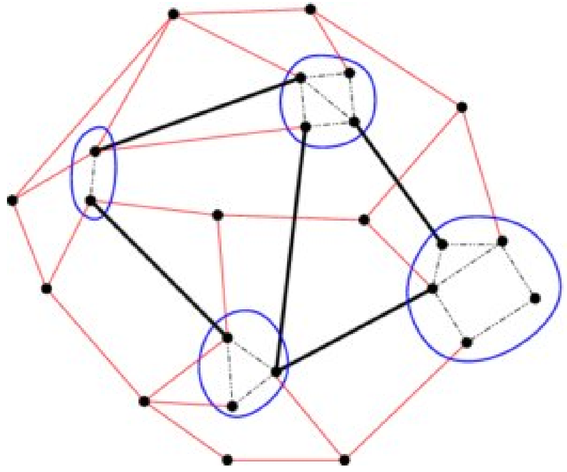
We say a kernel is *polynomial* if f is a polynomial function.

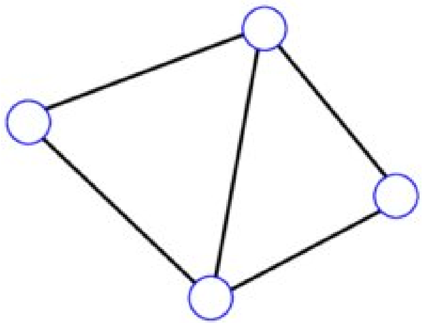
Vertex Cover(k) admits a polynomial kernel.

Can we generalize to other problems?

Minor

A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions.





Minor

A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions.

A class of graphs $\mathcal{G}$ is $\mathcal{F}$*-minor-free* if no graph in $\mathcal{G}$ contains any graph in $\mathcal{F}$ as a minor.

A class of graphs $\mathcal{G}$ is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is also in $\mathcal{G}$.

Let $\mathcal{F}$ be a fixed finite set of graphs.

$\mathcal{F}$-MINOR DELETION(k)

- *Input:* A graph G and an integer k.
- *Parameter:* k.
- *Question:* Does G have a set S of at most k vertices such that $G \setminus S$ is **$\mathcal{F}$-minor-free**?

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VERTEX COVER = $\{K_2\}$-MINOR DELETION.

Other problems

- Feedback Vertex Set = $\{K_3\}$-Minor Deletion.
- Planar Vertex Deletion = $\{K_5, K_{3,3}\}$-Minor Deletion.
- Treewidth-t Vertex Deletion.
- etc.

- $\mathcal{F}$-Minor Deletion(k) admits a polynomial kernel whenever $\mathcal{F}$ contains a planar graph (Fomin, Lokshtanov, Misra, Saurabh 2012; Gupta, Lee, Li, Manurangsi, Włodarczyk 2019).

- $\mathcal{F}$-Minor Deletion(k) admits a polynomial kernel whenever $\mathcal{F}$ contains a planar graph (Fomin, Lokshtanov, Misra, Saurabh 2012; Gupta, Lee, Li, Manurangsi, Włodarczyk 2019).
- Planar Vertex Deletion(k) admits an *approximate* polynomial kernel (Jansen, Włodarczyk 2025).

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- Other $\mathcal{F}$? 🤷

Smaller parameters?

Vertex Cover(fvs) admits a polynomial kernel (Jansen, Bodlaender 2013).

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Many other poly kernels for vertex-deletion distance to a **minor-closed** graph class.

Let $\mathcal{F}$ be a fixed finite set of graphs, and $\mathcal{G}$ be a fixed minor-closed graph class.

$\mathcal{F}$-MINOR DELETION$(|X_{\mathcal{G}}|)$

- *Input:* A graph G, an integer k, **and a set $X_{\mathcal{G}}$ such that $G \setminus X_{\mathcal{G}} \in \mathcal{G}$.**
- *Parameter:* $|X_{\mathcal{G}}|$.
- *Question:* Does G have a set S of at most k vertices such that $G \setminus S$ is $\mathcal{F}$-minor-free?

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Dichotomies: $\mathcal{F}$-MINOR DELETION$(|X_{\mathcal{G}}|)$ admits a polynomial kernel *if and only if* $\mathcal{G}$ meets some conditions.

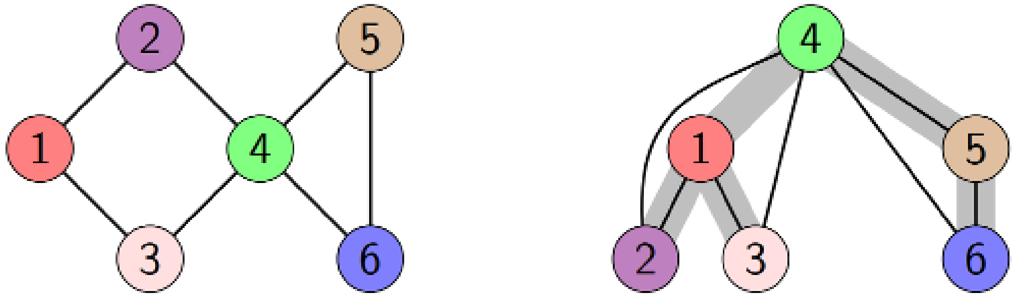
Theorem (Bougeret, Jansen, Sau 2022)

Assuming $\mathsf{NP} \not\subseteq \mathsf{coNP}/\mathsf{poly}$, Vertex Cover($|X_{\mathcal{G}}|$) admits a polynomial kernel if and only if every graph in $\mathcal{G}$ has bounded *bridge-depth*.

Assuming $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$, Feedback Vertex Set$(|X_{\mathcal{G}}|)$ admits a polynomial kernel **if and only if** $\mathcal{G}$ has bounded $\mathsf{ed}_{\mathcal{F}}$ (Dekker, Jansen 2024).

Treedepth decomposition

A *treedepth decomposition* of a connected graph G is a rooted tree T on $V(G)$ such that for every edge $\{u, v\} \in E(G)$ either u is an ancestor of v or v is an ancestor of u.



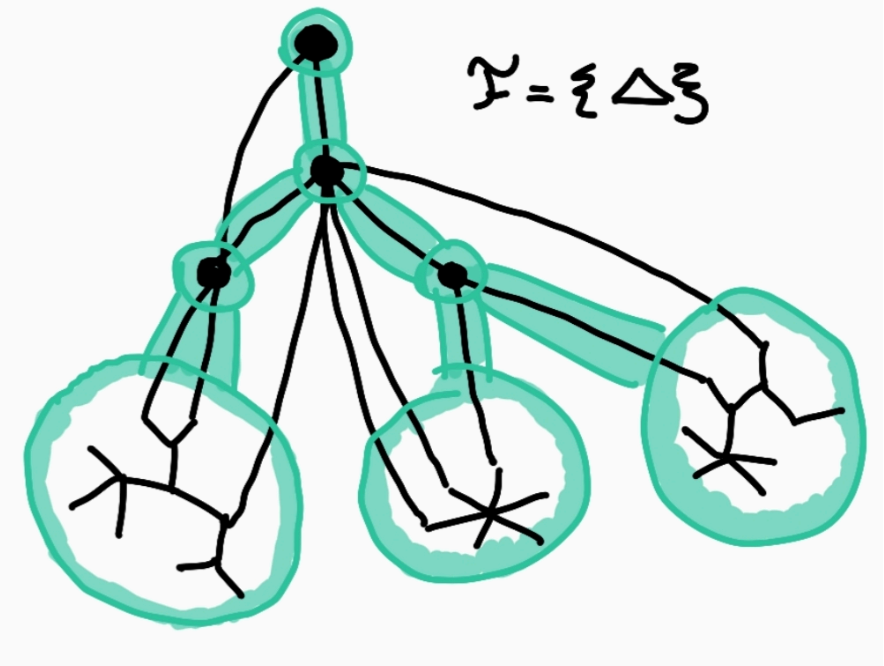
Treedepth decomposition

A *treedepth decomposition* of a connected graph G is a rooted tree T on $V(G)$ such that for every edge $\{u, v\} \in E(G)$ either u is an ancestor of v or v is an ancestor of u.

The *treedepth* of G is the minimum depth (in number of vertices) of a treedepth decomposition of G.

$\mathcal{F}$-elimination forest

An $\mathcal{F}$*-elimination forest* is a treedepth decomposition where the leaves are instead $\mathcal{F}$-minor-free induced subgraphs of G. The edges still cannot cross between different branches.



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The *elimination distance to* $\mathcal{F}$*-minor-free* $\mathsf{ed}_{\mathcal{F}}(G)$ is the minimum depth of an $\mathcal{F}$-elimination forest of G.

Assuming $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$, Feedback Vertex Set($|X_{\mathcal{G}}|$) admits a polynomial kernel **if and only if** $\mathcal{G}$ has bounded $\mathsf{ed}_{\mathcal{F}}$ (Dekker, Jansen 2024).

What about other $\mathcal{F}$-Minor Deletion problems?

What we know: poly kernels

- $\mathcal{F}$-Minor Deletion(k) admits a polynomial kernel whenever $\mathcal{F}$ contains a planar graph.
- Planar Vertex Deletion(k) admits an approximate polynomial kernel.

What we know: poly kernels

- $\mathcal{F}$-Minor Deletion(k) admits a polynomial kernel whenever $\mathcal{F}$ contains a planar graph.
- Planar Vertex Deletion(k) admits an approximate polynomial kernel.
- For all connected $\mathcal{F}$, $\mathcal{F}$-Minor Deletion$(|X_{\mathcal{G}}|)$ admits a polynomial kernel whenever $\mathcal{G}$ has bounded treedepth (Jansen, Pieterse 2020).

What we know: no poly kernels

Assuming $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$, for biconnected $\mathcal{F}$ on at least three vertices with at least one planar graph, $\mathcal{F}$-Minor Deletion$(|X_{\mathcal{G}}|)$ **does not** admit a polynomial kernel whenever $\mathcal{G}$ has unbounded $\mathsf{ed}_{\mathcal{F}}$ (Dekker, Jansen 2024).

Our results

Theorem 1

For connected $\mathcal{F}$, and $\mathcal{G}$ with bounded $\mathsf{ed}_{\mathcal{F}}$, if $\mathcal{F}$-Minor Deletion(k) admits an (approximate) polynomial kernel, then $\mathcal{F}$-Minor Deletion$(|X_{\mathcal{G}}|)$ admits an (approximate) polynomial kernel.

Our results

Corollary 1

Assuming $\mathsf{NP} \not\subseteq \mathsf{coNP}/\mathsf{poly}$, for connected $\mathcal{F}$ with at least one planar graph, $\mathcal{F}$-MINOR DELETION$(|X_{\mathcal{G}}|)$ admits a polynomial kernel whenever $\mathcal{G}$ has bounded $\mathsf{ed}_{\mathcal{F}}$.

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Corollary 1

Assuming $\mathsf{NP} \not\subseteq \mathsf{coNP}/\mathsf{poly}$, for connected $\mathcal{F}$ with at least one planar graph, $\mathcal{F}$-Minor Deletion$(|X_{\mathcal{G}}|)$ admits a polynomial kernel whenever $\mathcal{G}$ has bounded $\mathsf{ed}_{\mathcal{F}}$.

Only known before for Vertex Cover and Feedback Vertex Set.

Our results

Includes:

- Cactus Vertex Deletion($|X_{\mathcal{G}}|$) ($\mathcal{F} = \{K_3\}$).
- Outerplanar Vertex Deletion($|X_{\mathcal{G}}|$) ($\mathcal{F} = \{K_4, K_{2,3}\}$).
- d-Pseudoforest Deletion($|X_{\mathcal{G}}|$).
- Treewidth-t Vertex Deletion($|X_{\mathcal{G}}|$) for every fixed t.
- Pathwidth-t Vertex Deletion($|X_{\mathcal{G}}|$) for every fixed t.
- Treedepth-t Vertex Deletion($|X_{\mathcal{G}}|$) for every fixed t.
- Branchwidth-t Vertex Deletion($|X_{\mathcal{G}}|$) for every fixed t.
- etc.

Our results

Corollary 2

Assuming $\mathsf{NP} \not\subseteq \mathsf{coNP}/\mathsf{poly}$, for biconnected $\mathcal{F}$ on at least three vertices with at least one planar graph, $\mathcal{F}$-Minor Deletion$(|X_{\mathcal{G}}|)$ admits a polynomial kernel **if and only if** $\mathcal{G}$ has bounded $\mathsf{ed}_{\mathcal{F}}$.

Our results

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Assuming $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$, for biconnected $\mathcal{F}$ on at least three vertices with at least one planar graph, $\mathcal{F}$-Minor Deletion$(|X_{\mathcal{G}}|)$ admits a polynomial kernel **if and only if** $\mathcal{G}$ has bounded $\mathsf{ed}_{\mathcal{F}}$.

Generalizes the dichotomy for Feedback Vertex Set.

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Corollary 3

Planar Vertex Deletion($|X_{\mathcal{G}}|$) admits an approximate polynomial kernel whenever $\mathcal{G}$ has bounded elimination distance to planar graphs.

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Generalizes the approximate kernel for Planar Vertex Deletion(k).

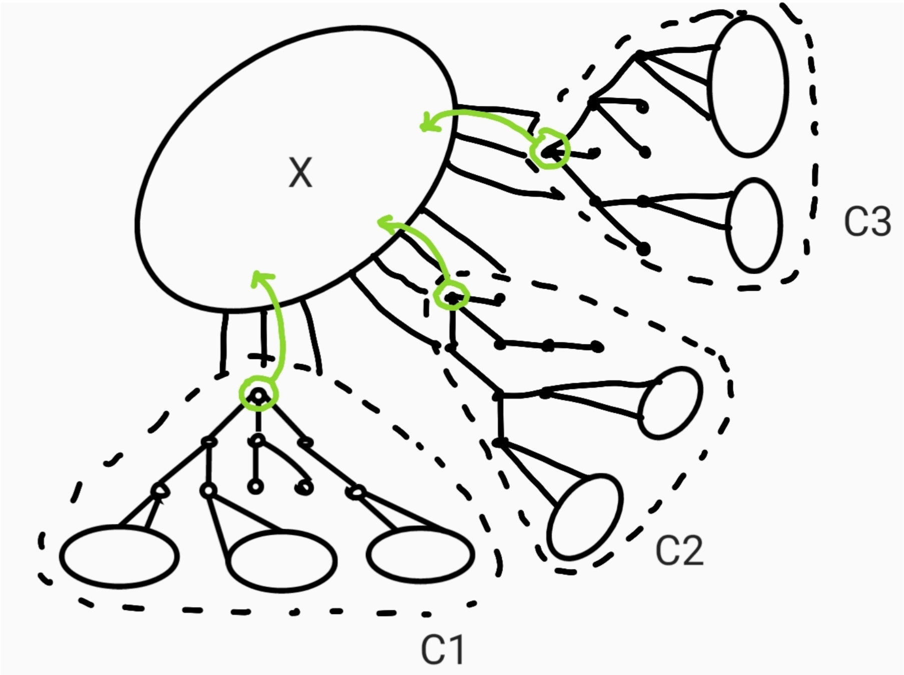
Proof of Theorem 1

Follow the approach of the kernel for $\mathcal{F}$-Minor Deletion$(|X_{\mathcal{G}}|)$ when $\mathcal{G}$ has bounded treedepth (Jansen, Pieterse 2020).

1. Reduce the number of connected components of $G \setminus X$ to $\mathsf{poly}(|X|)$.
2. Reduce the size of each connected component to $\mathsf{poly}(|X|)$.

Step 2: Reducing component size

1. Add the root of the $\mathsf{ed}_{\mathcal{F}}$-decomposition of each component to X.



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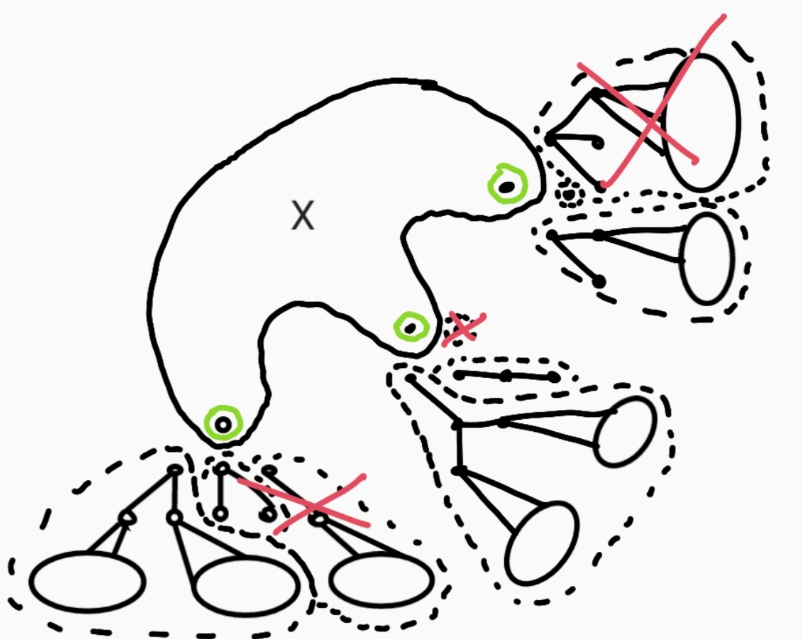
2. Reduce the number of components using Step 1 again.

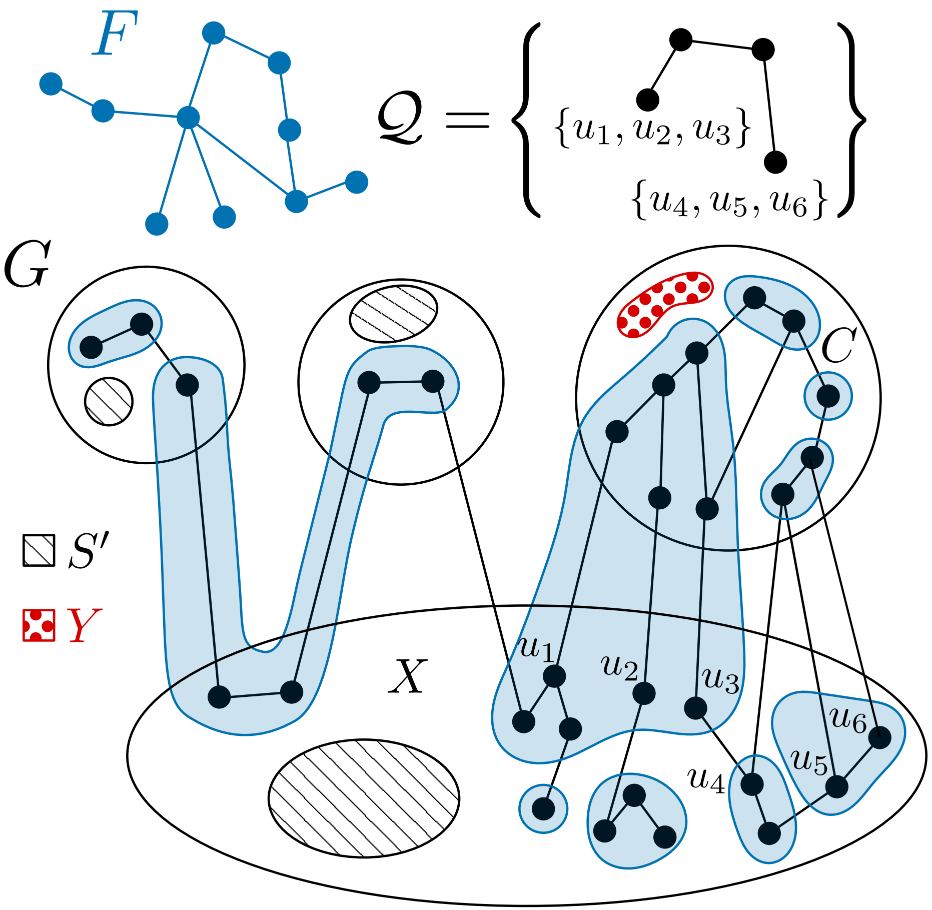


Step 2: Reducing component size

1. Add the root of the $\mathsf{ed}_{\mathcal{F}}$-decomposition of each component to X.
2. Reduce the number of components using Step 1 again.
3. Repeat until $\mathsf{ed}_{\mathcal{F}}(G \setminus X) = 0$.
4. Apply poly (approximate) kernel for $\mathcal{F}$-Minor Deletion(k).

Step 1: Reducing number of components

Idea: remove components for which an optimal solution can be added freely.


F
Q = { {u1, u2, u3} {u4, u5, u6} }
G
C
S′
Y
X
u1
u2
u3
u4
u5
u6

Labeled minors

Like normal minors, but vertices are marked with a set of labels from a finite set X.

- When contracting edges, labelsets are combined.
- Can remove labels from vertices.

Checking if a component is “independent”

Lemma 5

For connected $\mathcal{F}$ and fixed $\eta \in \mathbb{N}$, given an X-labeled graph C with $\mathsf{ed}_{\mathcal{F}}(C) \leq \eta$ and connected X-labeled graphs $\mathcal{Q}$, one can:

1. compute the size of an optimal $\mathcal{F}$-Minor Deletion set for C in $O(|V(C)|)$; and
2. find if there exists an optimal $\mathcal{F}$-Minor Deletion set that hits all labeled $\mathcal{Q}$-minors in time $f(\mathcal{Q}) \cdot \mathsf{poly}(|V(C)|)$.

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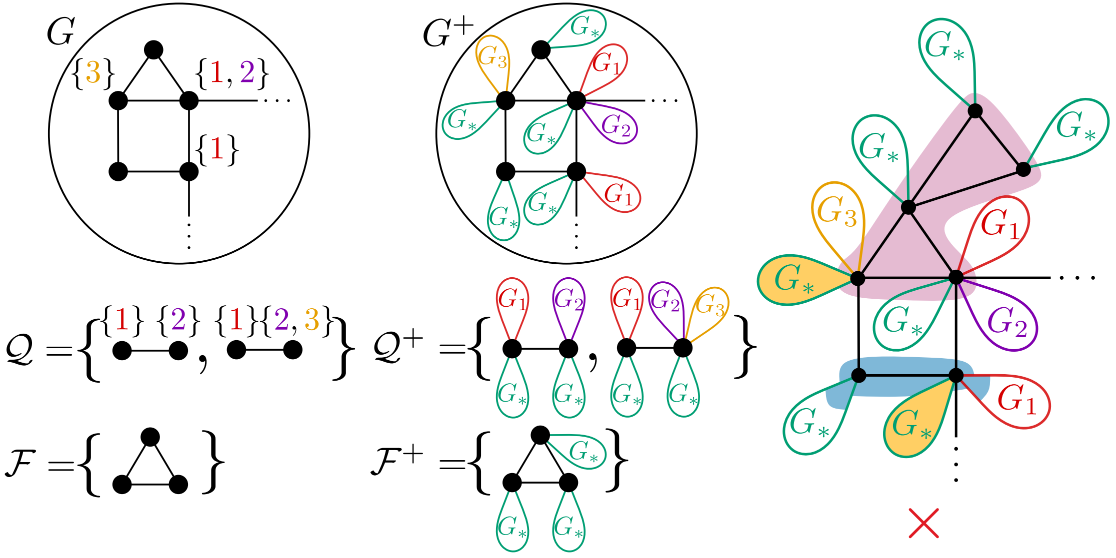
"Easy" when C has bounded treedepth or when $\mathcal{F}$ contains a planar graph by using Courcelle's theorem.

Proof of Lemma 5

1. Reduce the problem to the *unlabeled* setting.
2. Solve the unlabeled problem using dynamic programming on the $\mathcal{F}$-elimination forest adapting (Jansen, Kroon, Włodarczyk 2021).

Proof of Lemma 5: Step 1

$G_1, \ldots, G_\ell$ biconnected gadgets that are pairwise minor-incomparable and none is a minor of G.



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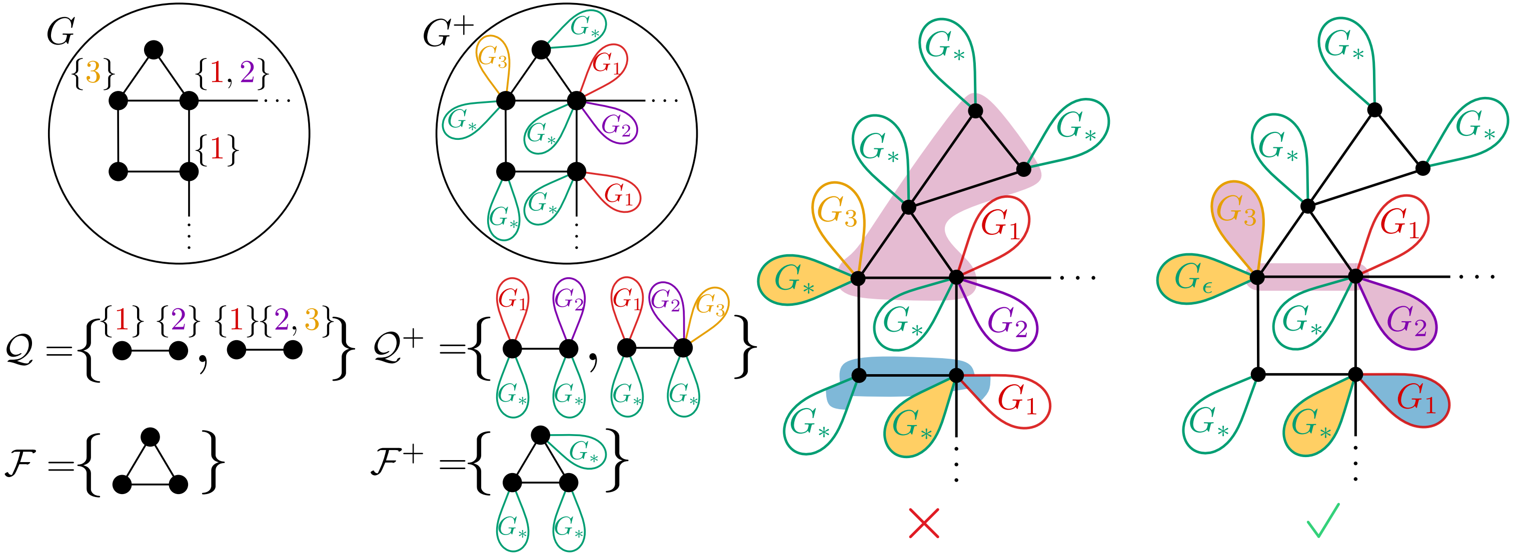

Important: $\mathsf{ed}_{\mathcal{F}}(G) \geq \mathsf{ed}_{\mathcal{F}^+}(G^+)$.

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2. find if there exists an optimal $\mathcal{F}$-Minor Deletion set that hits all labeled $\mathcal{Q}$-minors in time $f(\mathcal{Q}) \cdot \mathsf{poly}(|V(C)|)$.

$\mathcal{Q}$ could have size $\mathsf{poly}(|X|)$!

Considering a small $\mathcal{Q}$

Lemma 6

For connected $\mathcal{F}$, and $\mathcal{Q}$ meeting some ✨*magical*✨conditions, all optimal solutions to $\mathcal{F}$-Minor Deletion in a graph C leave a $\mathcal{Q}$-minor if and only if they all leave a $\mathcal{Q}^\star$-minor for some $\mathcal{Q}^\star \subseteq \mathcal{Q}$ of size $g(\mathsf{ed}_{\mathcal{F}}(C), \mathcal{F})$.

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$\mathcal{Q}$ corresponds to *blocking sets* in the literature.

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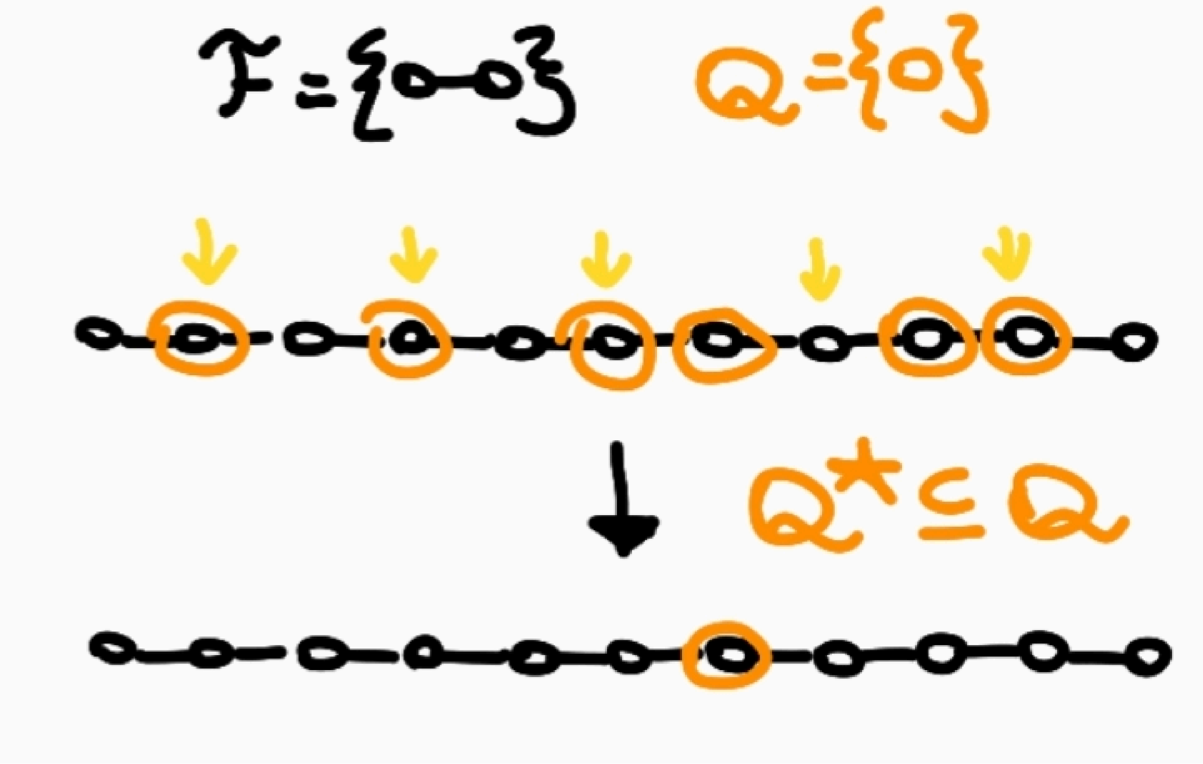
Generalizes result for treedepth $\Rightarrow$ we had to add the base case for the $\mathcal{F}$-elimination forest.

Considering a small Q



Step 1: Reducing number of components

Check if a component C of $G \setminus X$ has an optimal solution that hits all $\mathcal{Q}^\star$-minors in C for every $\mathcal{Q}^\star \subseteq \mathcal{Q}$ of size at most $g(\eta, \mathcal{F})$, and if so remove it.

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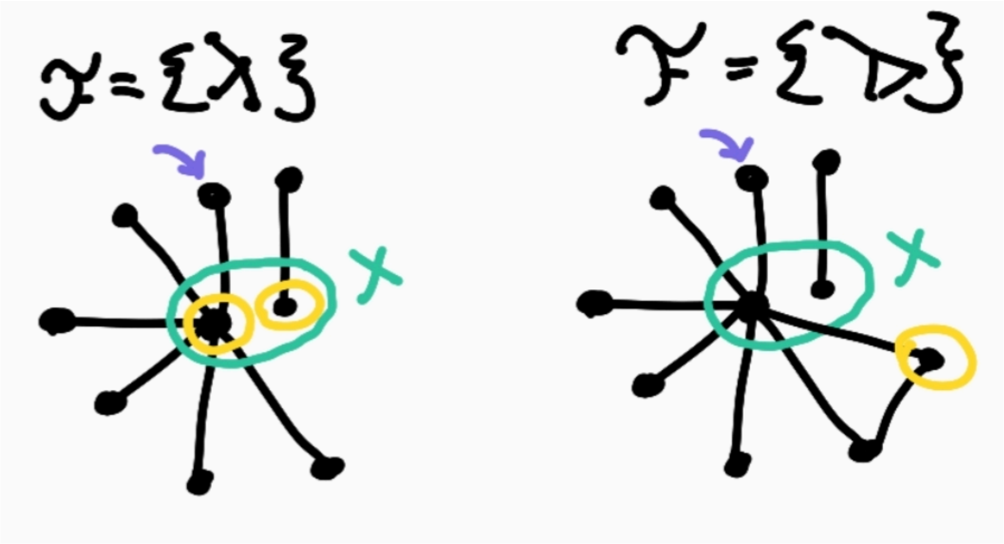
Problem: there could still be waaaay too many components remaining.

What about the other components?

Key observation 1: For every optimal solution Y to $\mathcal{F}$-Minor Deletion in G, there are at most $|X|$ components C in $G \setminus X$ such that the set $Y \cap C$ is not an optimal solution to $\mathcal{F}$-Minor Deletion in C.

What about the other components?

Key observation 2: If many ($O(|X|)$) components leave the same fragments with an optimal solution, then adding another such component with its optimal solution doesn't change anything.



Step 1: Reducing number of components

For each subset $\mathcal{Q}^\star \subseteq \mathcal{Q}$ of size at most $g(\eta, \mathcal{F})$ mark at most $h(\eta, \mathcal{F}) \cdot |X|$ components that leave some fragment of $\mathcal{Q}^\star$ with all optimal solutions. Delete the rest, and decrease k by $\mathrm{OPT}(C)$ for each deleted C.

Lemma 6

For connected $\mathcal{F}$, and $\mathcal{Q}$ meeting some ✨*magical* ✨conditions, all optimal solutions to $\mathcal{F}$-Minor Deletion in a graph C leave a $\mathcal{Q}$-minor if and only if they all leave a $\mathcal{Q}^\star$-minor for some $\mathcal{Q}^\star \subseteq \mathcal{Q}$ of size $g(\mathsf{ed}_{\mathcal{F}}(C), \mathcal{F})$.

▶ **Lemma 60** (Inductive Version of the Main Lemma – Adaptation of [37, Lemma 3]). *Let:*

- X *be a finite set;*
- $t \in \mathbb{N}$*;*
- $\mathcal{F}$ *be a set of connected graphs;*
- $\mathcal{Q}$ *be a set of connected* X*-labeled graphs such that each graph in* $\mathcal{Q}$ *has at most* $\max_{H\in\mathcal{F}}|E(H)|+1$ *vertices and* $\mathcal{Q}$ *is* $n_{\mathcal{F}}$*-saturated, with* $n_{\mathcal{F}} := \min_{H\in\mathcal{F}}|V(H)|$*;*
- $\Pi_A, \Pi_B, \Pi_C \subseteq \textsc{mpcs}_{+t}(\mathcal{F})$ *such that* $\Pi_A \odot \Pi_B \odot \Pi_C \supseteq \textsc{ext}_{+t}(\mathcal{F})$*;*
- G_A*,* G_B *and* G_C *be three* X*-labeled* t*-boundaried graphs;*
- $G := G_A \oplus G_B \oplus G_C$*;*
- $S := \mathsf{Boundary}(G)$ *such that* $\mathsf{ed}_{\mathcal{F}}(G) \geq \mathsf{ed}_{\mathcal{F}}(G_A \setminus S) + |S|$*;*
- $R_B \subseteq \textsc{mpcs}_{+t}(\mathcal{Q})$ *be a set of isomorphism classes of* X*-labeled* t*-boundaried graphs;*
- $\mathcal{Y} := \textsc{optsol}_{\mathcal{F}}\textsc{st}_{\mathcal{Q}}(G_A, G_B, G_C, \Pi_A, \Pi_B, \Pi_C, R_B)$*;*
- $\mathcal{R}_{\mathcal{Q}}$ *be the set of remainders of* $G_A \oplus G_B$ *with respect to* $\mathcal{Y}$ *that leave a* $\mathcal{Q}$*-minor;*
- $\mathcal{R}_N$ *be the set of remainders of* $G_A \oplus G_B$ *with respect to* $\mathcal{Y}$ *that do not leave a* $\mathcal{Q}$*-minor;*
- $\nu(\Pi_A) := |\textsc{mpcs}_{+t}(\mathcal{F}) \setminus \Pi_A|$*;*
- $\xi(R_B) := \mathsf{numberOf}(t \cdot \min_{H\in\mathcal{F}}|V(H)|, t, t + \max_{H\in\mathcal{Q}}|V(H)|, \min_{H\in\mathcal{F}}|V(H)|) - |R_B|$*; and*
- $\mu(G_A, \Pi_A, S) := \textsc{opt}_{\mathcal{F}}(G_A, \Pi_A, S) - \sum_{C\in cc(G_A\setminus S)} \textsc{opt}_{\mathcal{F}}(C)$*.*

Then there exist functions f *and* g *such that*

1. $|\mathcal{R}_N| \leq f(\mathsf{ed}_{\mathcal{F}}(G_A \setminus S), \textsc{isCon}(G_A \setminus S), \mu(G_A, \Pi_A, S), \nu(\Pi_A), \xi(R_B), \|\mathcal{F}\|, |S|)$*, and*
2. *there exists* $\mathcal{Q}^\star \subseteq \mathcal{Q}$ *such that* $|\mathcal{Q}^\star| \leq g(\mathsf{ed}_{\mathcal{F}}(G_A\setminus S), \textsc{isCon}(G_A\setminus S), \mu(G_A, \Pi_A, S), \nu(\Pi_A), \xi(R_B), \|\mathcal{F}\|, |S|)$*, and for each* $R \in \mathcal{R}_{\mathcal{Q}}$ *there exist* $q \in \mathcal{Q}^\star$ *and* $r \in R$ *with* $q \preceq_m \textsc{forget}(r)$*.*

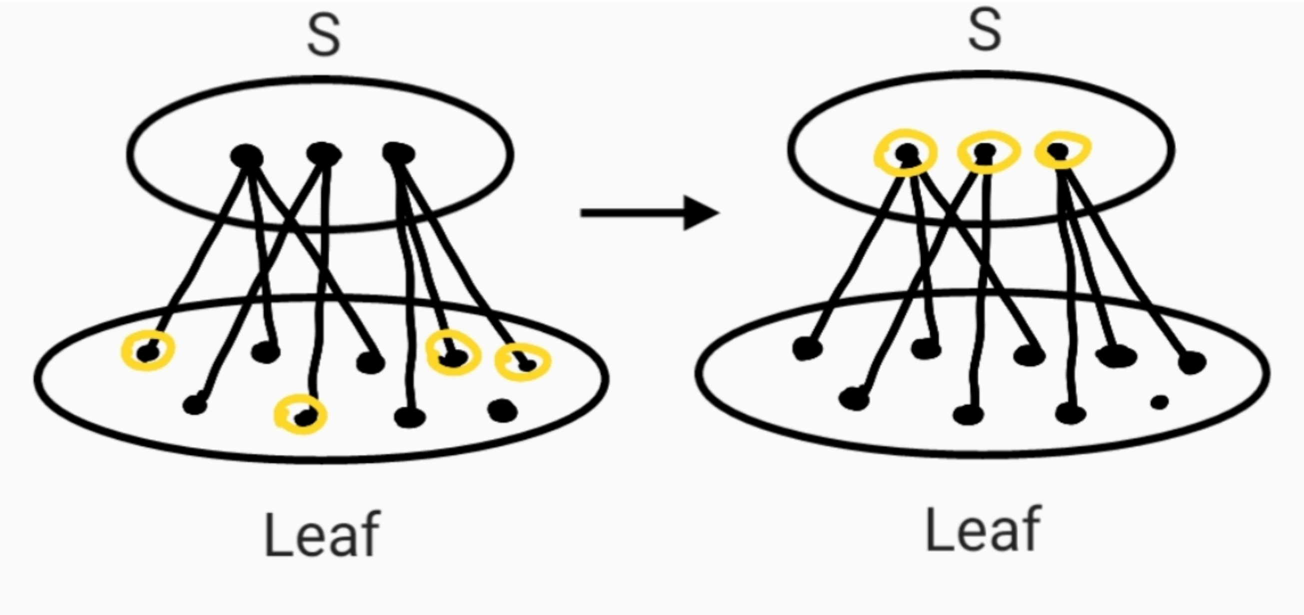
Considering a small $\mathcal{Q}$: the base case

$\mathcal{F}$-minor-free base case (simplified)

Let $\mathcal{F}$ be connected and let $\mathcal{Q}$ meet the ✨*magical* ✨conditions. For a graph G with separation $(\mathsf{Leaf}, S, \mathsf{Rest})$, where $G[\mathsf{Leaf}]$ is $\mathcal{F}$-minor-free, all optimal solutions to $\mathcal{F}$-Minor Deletion in G that leave a $\mathcal{Q}$-minor in $G[\mathsf{Leaf} \cup S]$ also leave a $\mathcal{Q}^\star$-minor for some $\mathcal{Q}^\star \subseteq \mathcal{Q}$ of size at most $g'(|S|, \mathcal{F})$.

Considering a small $\mathcal{Q}$: the base case

Key observation: Every optimal solution has at most $|S|$ vertices in Leaf.



Considering a small $\mathcal{Q}$: the base case

1. Find a small set $\mathsf{Breaker}$ that hits all $\mathcal{Q}$-minors in $G[\mathsf{Leaf} \cup S]$.
2. Mark a limited number of labels in vertices in $G[\mathsf{Leaf} \cup S]$ for each vertex $v \in \mathsf{Breaker}$, such that if a solution avoids v, it must leave a $\mathcal{Q}$-minor using only marked labels.

Summarizing

1. Reduce the number of components to $\mathsf{poly}(|X|)$.
   (i) Detect which components in $G \setminus X$ leave a $\mathcal{Q}^\star$-minor for some small $\mathcal{Q}^\star \subseteq \mathcal{Q}$.
   (ii) Mark $h(\eta, \mathcal{F}) \cdot |X|$ of them for each $\mathcal{Q}^\star \subseteq \mathcal{Q}$.
   (iii) Throw away the rest of the C, decreasing k by $\mathrm{OPT}(C)$ each time.
2. Reduce the size of each component to $\mathsf{poly}(|X|)$ by repeatedly adding the root of the $\mathcal{F}$-elimination forest to X.
3. Use the (approximate) polynomial kernel for $\mathcal{F}$-Minor Deletion(k) on each component.

The future 🔮

- Dichotomies for non-biconnected $\mathcal{F}$?
- Lower bounds for non-planar $\mathcal{F}$?

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Merci !