Parameterized algorithms for THINNESS and SIMULTANEOUS INTERVAL NUMBER via the cluster module number

Flavia Bonomo, Eric Brandwein, Ignasi Sau

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Thinness

Consistent solution

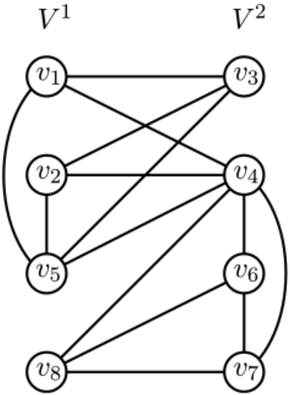
A *consistent solution using k classes* for a graph G is a pair $(\prec, S)$ where

- $\prec$ is a strict total order on $V(G)$, and
- S is a partition of $V(G)$ into k classes,

such that, for each triple of vertices $u \prec v \prec w$ of G, if

- u and v belong to the same class in S, and
- $(u, w) \in E(G)$,

then $(v, w) \in E(G)$.



Definition

A graph is *k-thin* if it admits a consistent solution using k classes. The *thinness* of a graph G, denoted by $\text{thin}(G)$, is the minimum integer k such that G admits a consistent solution using k classes.

THINNESS problem

Input: A graph G and an integer k.
Output: Is $\text{thin}(G) \leq k$?

Applications

Many NP-complete problems can be solved in polynomial time on graphs with bounded thinness, including:

- Maximum Independent Set (Mannino et al., “The stable set problem and the thinness of a graph”)
- Capacitated Coloring (Bonomo, Mattia, and Oriolo, “Bounded coloring of co-comparability graphs and the pickup and delivery tour combination problem”)
- Clique and other list matrix partition problems with cardinality constraints (Bonomo and Estrada, “On the thinness and proper thinness of a graph”)

Thinness

Theorem (Shitov. "Graph thinness, a lower bound and complexity")

THINNESS is NP-complete.

Kernelization

Kernelization

A *kernel* for a parameterized problem is a polynomial-time algorithm that, given an instance (I, k) for the problem, outputs an *equivalent* instance (I', k') such that:

- $|I'| + k' \leq g(k)$ for some computable function g, and
- (I, k) is a YES instance iff (I', k') is a YES instance.

Kernelization

Theorem (Cygan et al., *Parameterized Algorithms*)

A parameterized problem is in FPT iff it admits a kernel.

We are especially interested in kernels of **polynomial size**.

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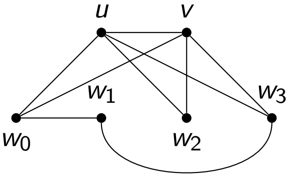
Reducing thinness instances

Reducing thinness instances

Lemma 1 (Bonomo-Braberman et al. "Thinness of product graphs")

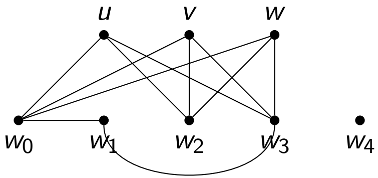
If two adjacent vertices have the same neighborhood, removing one of them does not modify the thinness of the graph.



Reducing thinness instances

Lemma 2

If three **non-adjacent** vertices have the same neighborhood, removing one of them does not modify the thinness of the graph.

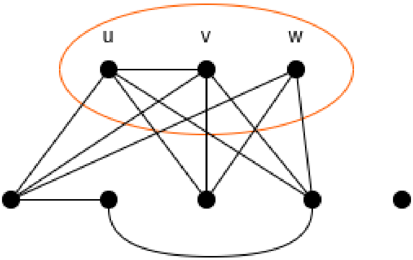


Cluster-modular cardinality

Modules

Definition (Module)

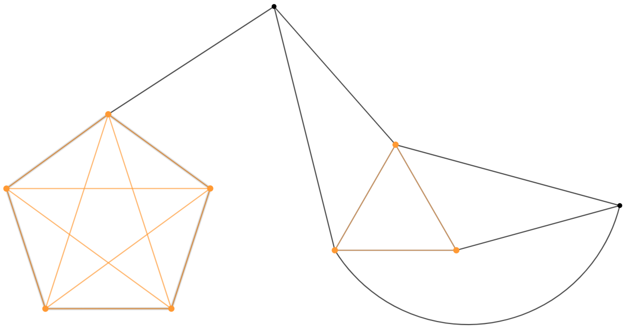
A *module* of a graph G is a subset W of $V(G)$ such that every vertex outside W is either adjacent to all vertices in W or to none of them.



Cluster modules

Definition (Cluster)

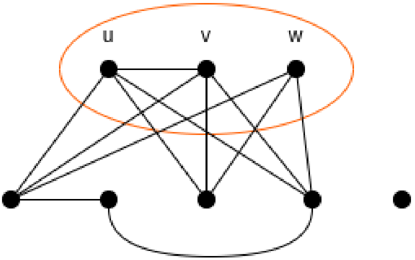
A *cluster* is a subset C of vertices that induces a union of cliques.



Cluster modules

Definition (Cluster module)

A *cluster module* is a subset C of vertices that is a cluster and a module of G.



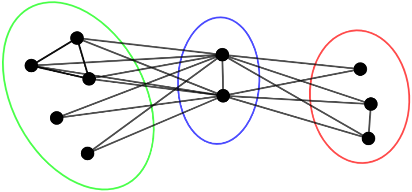
Cluster-modular cardinality

Definition

A *cluster module partition* of a graph G is a partition of the vertices of G into cluster modules.

Definition

The *cluster-modular cardinality* cluster-mc(G) of a graph G is the minimum size of a cluster module partition of G.



Cluster-modular cardinality

Theorem

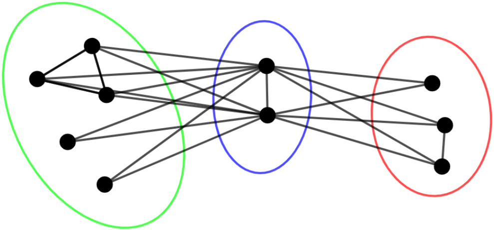
The cluster-modular cardinality of a graph can be calculated in linear time.

Results

Kernels for THINNESS

Theorem

THINNESS admits a kernel of size $2 \cdot \text{cluster-mc}(G)$ when parameterized by the cluster-modular cardinality of G.



Kernels for Thinness

Lemma

For every graph G, the following inequalities hold:

1. $\text{cluster-mc}(G) \leq \text{nd}(G)$
2. $\text{cluster-mc}(G) \leq 2^{\text{tc}(G)} + \text{tc}(G)$
3. $\text{cluster-mc}(G) \leq 2^{\text{vc}(G)} + \text{vc}(G)$

This gives a polynomial kernel for Thinness when parameterized by neighborhood diversity, and exponential kernels when parameterized by twin-cover and vertex cover.

Kernels for THINNESS

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Simultaneous interval number

The *simultaneous interval number* of a graph G, si(G), is the smallest number d of labels such that G admits a *d-simultaneous interval representation*, that is, an assignment of intervals and label sets from $\{1, \ldots, d\}$ to the vertices such that two vertices are adjacent if and only if the corresponding intervals, as well as their label sets, intersect.

Parameterizations for Simultaneous Interval Number

- Clique is FPT when parameterized by si, and Independent Set and Dominating Set are FPT parameterized by si + solution size.
- Simultaneous Interval Number is NP-complete. (Beisegel et al., "The Simultaneous Interval Number: A New Width Parameter that Measures the Similarity to Interval Graphs")
- We show FPT algorithms for Simultaneous Interval Number parameterized by cluster-mc + d, nd + d, tc, and vc.

Parameterizations for SIMULTANEOUS INTERVAL NUMBER

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Kernel size

The kernels when parameterized by cluster-modular cardinality and neighborhood diversity have **linear size** in the parameter, while the kernels when parameterized by twin-cover and vertex cover number have **exponential size** in the parameter.

Size lower bound

Theorem

Let p be a graph parameter such that

1. for every graph G, we have $\mathrm{p}(G) \leq \max\{|V(G)|, |E(G)|\}$, and
2. if H is the disjoint union of two graphs G_1 and G_2, we have $\mathrm{p}(H) \leq \max\{\mathrm{p}(G_1), \mathrm{p}(G_2)\}$.

Then THINNESS and SIMULTANEOUS INTERVAL NUMBER parameterized by p have no polynomial kernels assuming $\mathrm{NP} \not\subseteq \mathrm{coNP/poly}$.

Size lower bound

This does not apply when parameterizing Thinness or Simultaneous Interval Number by the twin-cover or the vertex cover, but it does apply when parameterizing by

- treewidth,
- bandwidth,
- **thinness,**
- **simultaneous interval number,**
- . . . and many more.

Thank you!

Any questions?