Thinness and its variations on some graph families and coloring graphs of bounded thinness

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Abstract

Interval graphs and proper interval graphs are well known graph classes, for which there have been proposed several generalizations in the literature. In this work, we study the (proper) k-thin graphs and its variations for the classes of cographs, crown graphs and grid graphs.

We provide the exact values for several variants of thinness (proper, independent, complete, precedence, and combinations of them) for the crown graphs CR_n. For cographs, we prove that the precedence thinness can be determined in polynomial time. We also improve known bounds for the thinness of $n \times n$ grids GR_n and $m \times n$ grids $GR_{m,n}$, proving that $\left\lceil \frac{n-1}{3} \right\rceil \leq \text{thin}(GR_n) \leq \left\lceil \frac{n+1}{2} \right\rceil$. Regarding the precedence thinness, we prove that prec-thin$(GR_{n,2}) = \left\lceil \frac{n+1}{2} \right\rceil$ and that $\left\lceil \frac{n-1}{3} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq$ prec-thin$(GR_n) \leq \left\lceil \frac{n-1}{2} \right\rceil^2 + 1$. As applications, we show that the k-coloring problem is NP-complete for precedence 2-thin graphs and for proper 2-thin graphs, when k is part of the input. On the positive side, it is polynomially solvable for precedence proper 2-thin graphs, given the order and partition.

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1. Introduction

This work studies the classes of k-thin and proper k-thin graphs, which generalize interval and proper interval graphs, respectively. A graph G is an *interval graph* if there is a mapping of $V(G)$ on intervals of the real line, each $v \in V(G)$ corresponding to the interval I_v, such that $(u, v) \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$ for all distinct $u, v \in V(G)$. Moreover, G is called a *proper interval graph* if there is such a mapping of vertices to intervals such that $I_u \nsubseteq I_v$ for all distinct $u, v \in V(G)$. There are several characterizations and recognition algorithms for interval and proper interval graphs [16, 17, 8]. A well known characterization for interval graphs is the following:

Theorem 1 ([16]). *A graph G is an interval graph if and only if there is an ordering σ of $V(G)$ such that, for any triple (p, q, r) of vertices of $V(G)$ ordered according to σ, if $(p, r) \in E(G)$, then $(q, r) \in E(G)$.*

The ordering σ of Theorem 1 is said to be a *canonical* ordering or interval ordering. An analogous characterization for proper interval graphs is described next.

Theorem 2 ([17]). *A graph G is a proper interval graph if and only if there is an ordering σ of $V(G)$ for which σ and its reversal are both canonical.*

The ordering σ of Theorem 2 is said to be a *proper canonical* ordering or proper interval ordering.

A *k-thin graph* G is a graph for which there is a k-partition $\mathcal{V} = (V_1, V_2 \dots, V_k)$ of $V(G)$, and an ordering σ of $V(G)$ such that, for any triple (p, q, r) of vertices of $V(G)$ ordered according to σ, if $p, q \in V_i$ for some $1 \leq i \leq k$ and $(p, r) \in E(G)$, then $(q, r) \in E(G)$. Such an ordering σ is said to be *consistent* with $\mathcal{V}$. A graph G is called a *proper k-thin graph* if $V(G)$ admits a k-partition $\mathcal{V}$ of $V(G)$ and an ordering σ such that both σ and its reversal are consistent with $\mathcal{V}$. An ordering of this type is said to be *strongly consistent* with $\mathcal{V}$.

Although not all graphs G are (proper) interval graphs, all of them are (proper) k-thin graphs for some $k \geq 1$. This is so because if $k = |V(G)|$, any ordering of $V(G)$ is (strongly) consistent with the unique possible partition in which each part consists of a single vertex. Moreover, since (proper) interval graphs are precisely the (proper) 1-thin graphs, the parameter k of (proper) k-thin graphs measures "how far" a graph is from being a (proper) interval graph.

Under such classes, some NP-complete problems can be solved in polynomial time, as proved in [15, 1, 3]. For instance, generalized versions of k-coloring are polynomial-time solvable for graphs of bounded thinness, when the number of colors k is also a constant [1, 3]. Despite that, the complexity of k-coloring on graphs with bounded thinness at least two is open when $k \geq 3$ is part of the input (it is polynomial-time solvable for 1-thin graphs, i.e., interval graphs [13]).

Variations of the concept of (proper) thinness have been studied in the literature, by constraining either the vertices that can share a part in the partition, or how vertices can be arranged in the ordering. The classes of *precedence k-thin* and *precedence proper k-thin graphs* were defined in [7], as (proper) k-thin graphs such that the vertices belonging to a same part must be consecutive in the (strongly) consistent ordering. It still holds that precedence (proper) 1-thin graphs are equivalent to (proper) interval graphs. In [7], among other results, it was presented a characterization of such classes based on threshold graphs. In [6], there have been defined the classes of (proper) k-independent-thin and (proper) k-complete-thin graphs, for which each part of the partition must be an independent and a complete set, respectively. The authors proved some bounds on the thinness for such variants for some graphs operations. In [5], it is proved that 2-independent-thin graphs are equivalent to interval bigraphs, and that proper independent 2-thin graphs are equivalent to bipartite permutation graphs, so (proper) k-independent-thin graphs can be seen as generalizations of those classes.

The *crown graph* CR_n is the graph obtained from a complete bipartite graph $K_{n,n}$ by removing a perfect matching. It was proven in [6] that $\text{thin}(CR_n) \geq \frac{n}{2}$. The *grid graph* $GR_{n,m}$ is defined as the Cartesian product of two path graphs P_n and P_m. We denote $GR_{n,n}$ simply by GR_n. It was proved in [15] that $\frac{n}{4} \leq \text{thin}(GR_n) \leq n+1$. In [9], the upper bound for $\text{thin}(GR_n)$ was improved to $\left\lceil \frac{2n}{3} \right\rceil$.

In this work, we provide a characterization of consistent solutions for the crown graphs CR_n, and solve all known variants of the thinness parameter for this class. With respect to the grid graphs, we prove that the thinness of GR_n is at least $\left\lceil \frac{n-1}{3} \right\rceil$ and at most $\left\lceil \frac{n+1}{2} \right\rceil$, which are tighter lower and upper bounds than the ones found in the literature. Also, we show that the precedence thinness of $GR_{2,n}$ is exactly $\left\lceil \frac{n+1}{2} \right\rceil$, whereas for GR_n is between $\left\lceil \frac{n-1}{3} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 1$ and $\left\lceil \frac{n-1}{2} \right\rceil^2 + 1$. Finally, we prove that the k-coloring problem is NP-complete for both precedence 2-thin and proper 2-thin graphs when k

is part of the input, but polynomially solvable for precedence proper 2-thin graphs.

This paper is structured as follows. Section 2 presents the remaining definitions used throughout the paper. In Section 3, we present a characterization of consistent solutions for crown graphs and solve all known variants of the thinness parameter for this class, summarized in Table 1. Section 4 provides formulas for precedence thinness of the disjoint union and join of two graphs in terms of their precedence thinness, thus proving that it is possible to efficiently compute the precedence thinness of cographs. In Section 5, we prove lower and upper bounds for the thinness and precedence thinness of grid graphs, improving the previously known ones. In Section 6, we prove that the k-coloring problem is NP-complete for both precedence 2-thin and proper 2-thin graphs when k is part of the input, but polynomially solvable for precedence proper 2-thin graphs. Also, we show that precedence proper 2-thin graphs are perfectly orderable. Finally, some concluding remarks are presented in Section 7.

2. Preliminaries

Let G be a graph. We denote by $V(G)$ its vertex set and by $E(G)$ its edge set. The *size* (*cardinality*) of a set S is denoted by $|S|$. Let $u, v \in V(G)$, u and v are *adjacent* if $(u, v) \in E(G)$.

Two graphs G and H are said to be *isomorphic*, denoted by $G \cong H$, if there is a bijection $\theta : V(G) \to V(H)$ such that $(u, v) \in E(G)$ if, and only if, $(\theta(u), \theta(v)) \in E(H)$.

The *subgraph* of G *induced* by the subset of vertices $V' \subseteq V(G)$, denoted by $G[V']$, is the graph $G[V'] = (V', E')$, where $E' = \{(u, v) \in E(G) \mid u, v \in V'\}$. An *induced subgraph* of G is a subgraph of G induced by some subset of $V(G)$.

Let $u \in V(G)$. The *(open) neighborhood* of u, denoted by $N_G(u)$, is defined as $N_G(u) = \{v \in V(G) \mid (u, v) \in E(G)\}$. The *closed neighborhood* of u is denoted by $N_G[u] = \{u\} \cup N_G(u)$. Let $U \subseteq V(G)$. Define $N_G(U) = \bigcup_{u \in U} N_G(u)$ and $N_G[U] = \bigcup_{u \in U} N_G[u]$. When G is clear in the context, it may be omitted from the notation.

A *clique* or *complete set* (resp. *stable set* or *independent set*) is a set of pairwise adjacent (resp. nonadjacent) vertices. The clique graph K_n is the graph such that $V(K_n)$ is a clique.

Let $G = (V, E)$, the *complement* of G is defined as $\overline{G} = (V, E')$ such that $E' = \{(u, v) \mid u, v \in V \text{ and } (u, v) \notin E\}$.

The *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and the *join* of G_1 and G_2 is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{(v, v') : v \in V_1, v' \in V_2\})$ (i.e., $\overline{G_1 \vee G_2} = \overline{G_1} \cup \overline{G_2}$). (The join is sometimes also noted by $G_1 \otimes G_2$, but we follow the notation in [1]).

The class of *cographs* can be defined as the graphs that can be obtained from trivial graphs by the union and join operations [12].

The *thinness* (resp. *proper thinness*) of G, or $\text{thin}(G)$ (resp. $\text{pthin}(G)$), is defined as the minimum value k for which G is a k-thin graph (resp. proper k-thin graph). Similarly, the other types of thinness have their related parameters, listed next. With respect to each restriction, we have

- the *precedence thinness* of G, or $\text{prec-thin}(G)$
- the *independent thinness* of G, or $\text{thin}_{\text{ind}}(G)$
- the *complete thinness* of G, or $\text{thin}_{\text{cmp}}(G)$
- the *precedence proper thinness* of G, or $\text{prec-pthin}(G)$
- the *independent proper thinness* of G, or $\text{pthin}_{\text{ind}}(G)$
- the *complete proper thinness* of G, or $\text{pthin}_{\text{cmp}}(G)$

Also, as some of those restrictions can be combined, this leads to the following parameters:

- the *precedence independent thinness* of G, or $\text{prec-thin}_{\text{ind}}(G)$
- the *precedence complete thinness* of G, or $\text{prec-thin}_{\text{cmp}}(G)$
- the *precedence independent proper thinness* of G, or $\text{prec-pthin}_{\text{ind}}(G)$
- the *precedence complete proper thinness* of G, or $\text{prec-pthin}_{\text{cmp}}(G)$

A graph G is *bipartite* if its vertex set can be partitioned into two independent sets, and this is denoted by $G = (V_1 \cup V_2, E)$, where V_1 and V_2 are independent. The *complete bipartite* graph $K_{p,q}$ is a bipartite graph such that $|V_1| = p$, $|V_2| = q$, and for all $v_1 \in V_1$ and $v_2 \in V_2$, $(v_1, v_2) \in E$.

For a graph G and $A, B \subseteq V(G)$ such that $A \cap B = \emptyset$, the *bipartite graph induced* by the two subsets, denoted by $G[A, B]$, is the bipartite graph $G[A, B] = (A \cup B, E')$, where $E' \subseteq E(G)$ are the edges with one endpoint in A and one endpoint in B.

A *matching* $\mathcal{M}$ of a graph G is defined as a set of pairwise non-adjacent edges of G. If all vertices in $V(G)$ are endpoints of the edges in $\mathcal{M}$, then $\mathcal{M}$ is said to be a *perfect matching*.

Let $\mathcal{M}$ be a perfect matching of $K_{n,n}$. The *crown graph* CR_n (also known as *Hiraguchi graph*) is the graph on $2n$ vertices such that $V(CR_n) = V(K_{n,n})$ and $E(CR_n) = E(K_{n,n})) \setminus \mathcal{M}$.

For positive integers n, m, the $(n \times m)$-*grid*, noted $GR_{n,m}$, is the graph whose vertex set is $\{(i,j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and whose edge set is $\{((i,j),(k,l)) : |i-k| + |j-l| = 1, \text{ where } 1 \leq i,k \leq n, 1 \leq j,l \leq m\}$. We denote simply by GR_n the $(n \times n)$-grid $GR_{n,n}$.

Let σ be a sequence $(e_1, e_2, \dots, e_k)$. We denote by $\overline{\sigma} = (e_k, e_{k-1}, \dots, e_1)$ the *reversal* of σ, and by $\sigma_1 \oplus \sigma_2$ (or $\sigma_1\sigma_2$) the concatenation of the sequences σ_1 and σ_2. If a precedes b in σ, then we will denote $a <_\sigma b$. Given an ordering $<$ (that is, a linear order) of a set of elements X, $\sigma_<$ denotes the sequence of vertices ordered according to $<$.

For $S \subseteq X$, let $\text{first}(S)$ and $\text{last}(S)$ be the smallest and the greatest elements of S according to $<$, respectively. We say S is *consecutive in X according to* $<$ if there is no z in $X \setminus S$ such that $\text{first}(S) < z < \text{last}(S)$. Notice that for each $u \in S$, there are at most two elements $x \in X$ such that $\{u, x\}$ is consecutive in S according to $<$. Namely, if the elements of X are ordered $s_1 < \cdots < s_r$ and $u = s_i$, such two elements are s_{i-1} when $i > 1$ and s_{i+1} when $i < r$.

Lemma 3 (Alternative characterization for strong consistency [4])**.** *Given a graph G, an ordering $<$ of $V(G)$ and a partition $\mathcal{V}$ of $V(G)$ are strongly consistent if and only if for every $v \in V(G)$ and every $V \in \mathcal{V}$, the set $N[v] \cap (V \cup \{v\})$ is consecutive in $V \cup \{v\}$ according to $<$. In particular, $N[v] \cap V$ is consecutive in V according to $<$.*

Given a graph G, a partition $\mathcal{V}$ of $V(G)$ and an ordering σ of $V(G)$, we say that the pair $(\mathcal{V}, \sigma)$ is a *solution* which is *(strongly) consistent* if σ is a (strongly) consistent ordering with respect to $\mathcal{V}$ and G.

Given an order $<$ and a partition of the vertices, we say that a triple (r, s, t) *breaks the consistency* meaning that $r < s < t$, r and s belong to the same class, and there is an edge between r and t but there is no edge between s and t.

Let G be a graph, H be an induced subgraph of G, and let f be one of the parameters under study in this paper (i.e., those appearing in Table 1). An observation that we will use often in the proofs of lower bounds is that $f(H) \leq f(G)$. This holds because the order and partition of $V(G)$ involved in the definition of f, when restricted to $V(H)$, preserve all the desired properties in H, i.e., consistency, strong consistency, precedence, and independence or completeness of the classes.

A *coloring* of a graph G is a labeling of its vertices such that adjacent vertices have distinct labels, regarded as colors. A *k-coloring* is a coloring

	CR_1	CR_2	CR_3	CR_n, $n \geq 4$ even	CR_n, $n \geq 4$ odd
thin	1	1	2	$n-1$	$n-1$
pthin	1	1	2	$n-1$	n
thin_{ind}	1	2	3	n	n
$\text{pthin}_{\text{ind}}$	1	2	3	n	n
thin_{cmp}	2	2	3	$2n-4$	$2n-4$
$\text{pthin}_{\text{cmp}}$	2	2	3	$2n-4$	$2n-4$
prec-thin	1	1	2	$n-1$	$n-1$
prec-pthin	1	1	3	$n+1$	$n+1$
$\text{prec-thin}_{\text{ind}}$	1	3	4	$n+1$	$n+1$
$\text{prec-pthin}_{\text{ind}}$	1	3	4	$n+1$	$n+1$
$\text{prec-thin}_{\text{cmp}}$	2	2	4	$2n-2$	$2n-2$
$\text{prec-pthin}_{\text{cmp}}$	2	2	4	$2n-2$	$2n-2$

Table 1: The value of different thinness variations on crown graphs.

that maps the vertex set into a set of size k. Along this paper, we will regard a k-coloring of a graph G as a function $f : V(G) \to \mathbb{N}$ such that $f(v) \leq k$ for every $v \in V(G)$, and $f(u) \neq f(v)$ for adjacent vertices u and v. A graph is *k-colorable* if it admits a k-coloring. The *k-coloring problem* takes as input a graph G and a natural number k, and consists of deciding whether G is k-colorable.

It is known that generalized versions of k-coloring are polynomial-time solvable for graphs of bounded thinness, when k is a constant [1, 3]. But the complexity of k-coloring on graphs with bounded thinness at least two remained open when k is part of the input (it is polynomial-time solvable for 1-thin graphs, i.e., interval graphs [13]).

3. The thinness of crown graphs

In this section, we prove the exact value of the thinness and its variations for CR_n, along with consistent orderings and partitions in each case. The main results are summarized in Table 1.

We will introduce next some additional definitions and notations that are necessary for this section. Let $G = CR_n$. Define $\{A_n, A'_n\}$ as the bipartition of $V(G)$ such that $A_n = \{v_1, v_2, \ldots, v_n\}$, $A'_n = \{v'_1, v'_2, \ldots, v'_n\}$ and v'_i is the only vertex of A'_n that is not adjacent to v_i, for all $1 \leq i \leq n$. We will also define $\text{mirror}(v_i) = v'_i$, $\text{mirror}(v'_i) = v_i$, $\text{side}(v_i) = A_n$, and $\text{side}(v'_i) = A'_n$. Given an ordering $<$ of $V(G)$, we say that v is a *little* vertex if $v < \text{mirror}(v)$, and v is a *big* vertex otherwise. Similarly, for a set of vertices $S \subseteq V(G)$, we define $\text{Little}(S)$ as the subset of S that are little vertices.

3.1. Characterization of consistent solutions for CR_n

Let $<$ and $\mathcal{V}$ be an ordering and a partition of $V(G)$, respectively. Next, we define two conditions that will be used to characterize consistency of $<$ and $\mathcal{V}$ in G.

Condition 1. *For any $V \in \mathcal{V}$ and $v \in \mathit{Little}(V)$, $v = \text{first}(V \cap \text{side}(v))$.*

Condition 2. *For any $V \in \mathcal{V}$ and $1 \leq i, j \leq n$ such that $v_i, v'_j \in V$, there is no $v'_z \in A'_n$ with $z \neq i$ such that $v_i < v'_j < v'_z$ (resp. $v_z \in A_n$ with $z \neq j$ such that $v'_j < v_i < v_z$).*

These two conditions are necessary and sufficient to characterize consistency in CR_n, as shown in the following theorem.

Theorem 4. *Let $G = CR_n$, and let $<$ and $\mathcal{V}$ be an ordering and a partition of $V(G)$, respectively. Then $<$ and $\mathcal{V}$ are consistent for G if, and only if, both Conditions 1 and 2 hold.*

Proof. Suppose that $<$ and $\mathcal{V}$ are consistent for G. First suppose that Condition 1 does not hold. That is, there are $V \in \mathcal{V}$ and $v \in V$ such that $v < \text{mirror}(v) = v'$ and $w = \text{first}(V \cap \text{side}(v)) < v$. There is a contradiction with the fact that $<$ and $\mathcal{V}$ are consistent, as $(w, v') \in E(G)$ and $(v, v') \notin E(G)$. Now suppose that Condition 2 does not hold. That is, there are $V \in \mathcal{V}$, $v_i, v'_j \in V$ and $z \notin \{i, j\}$, such that either $v_i < v'_j < v'_z$, or $v'_j < v_i < v_z$. In either case, this contradicts the fact that $<$ and $\mathcal{V}$ are consistent, as $(v_i, v'_z) \in E(G)$ and $(v'_j, v'_z) \notin E(G)$ (resp. $(v'_j, v_z) \in E(G)$ and $(v_i, v_z) \notin E(G)$).

Consider now that both conditions hold for $<$ and $\mathcal{V}$. Suppose that $<$ is not consistent with $\mathcal{V}$. That is, there are $V \in \mathcal{V}$ and $p < q < r$ such that $p, q \in V$, $(p, r) \in E(G)$ and $(q, r) \notin E(G)$. Consider, without loss of generality that $\text{side}(p) = A_n$. If $\text{side}(q) = A_n = \text{side}(p)$, then $r = q'$, as q' is the only vertex of A'_n that is not adjacent to q. This fact contradicts Condition 1, as $q < q'$, and therefore q is little, but $q \neq \text{first}(V \cap \text{side}(q))$, since $p < q$. If $\text{side}(q) = A'_n \neq \text{side}(p)$, then $\text{side}(r) = A'_n$, as r is adjacent to p. Then there is $v'_z \in A'_n$ with $z \neq i$ such that $v_i < v'_j < v'_z$, for $v_i = p$, $v'_j = q$, and $v'_z = r$, contradicting Condition 2. $\square$

3.2. Consequences for consistent solutions for CR_n

Corollary 5. *If v and w are two distinct little vertices of $G = CR_n$ and* $\mathrm{side}(v) = \mathrm{side}(w)$*, then they cannot belong to the same class in a consistent solution.*

Proof. Otherwise, it would contradict Condition 1. □

Corollary 6. *Let $(\mathcal{V}, <)$ be a consistent solution for $G = CR_n$. If v and w are two vertices of G such that* $\mathrm{side}(v) \neq \mathrm{side}(w)$*, v and w belong to the same class of $\mathcal{V}$, and $v < w$, then w must be either the last vertex or the penultimate vertex of* $\mathrm{side}(w)$*. Moreover, if w is the penultimate vertex of* $\mathrm{side}(w)$*, then the last vertex of* $\mathrm{side}(w)$ *must be* $\mathrm{mirror}(v)$*.*

Proof. Otherwise, it would contradict Condition 2. □

Claim 7. *Given four distinct vertices of $G = CR_n$, v, w in one side of the bipartition and x, y in the other side, and such that, in a consistent solution:*

- *v and x belong to the same class;*
- *w and y belong to the same class;*
- *$v < x$;*
- *$w < y$;*

then at least one of $\{x, y\}$ is not a little vertex.

Proof. Without loss of generality let us assume that $x < y$. By Corollary 6 we know that x must be either the last vertex or the penultimate vertex of $\mathrm{side}(x)$; but since $x < y$, we can say that x and y are respectively the penultimate and last vertex of $\mathrm{side}(x)$, and that $y = \mathrm{mirror}(v)$. And since $v < x < y$, this implies that y is not a little vertex. □

Claim 8. *In a consistent solution for $G = CR_n$, the number of classes containing a little vertex is at least $n - 1$. In other words, at most one class can contain two little vertices, one of each side, since by Corollary 5 no class can contain more than one little vertex of the same side.*

Proof. Suppose, on the contrary, that there are 4 distinct little vertices v, w, x and y such as $v, w \in A_n$ and $x, y \in A'_n$, v and x belong to the same class, w and y belong to the same class (by Corollary 5, no class can contain more than one little vertex of the same side). By Claim 7, it is neither possible that $v < x$ and $w < y$, nor that $v > x$ and $w > y$.

Suppose then, without loss of generality, that $v < x$ and $w > y$. By Corollary 6, w and x can only be the last or penultimate vertices in A_n and A'_n, respectively. Moreover, if w is the penultimate vertex of A_n then $\mathrm{last}(A_n) = \mathrm{mirror}(y)$, and if x is the penultimate vertex of A'_n then $\mathrm{last}(A'_n) = \mathrm{mirror}(v)$. Without loss of generality, let us assume $w > x$. Notice as $\mathrm{mirror}(w) > w > x$ the vertex x cannot be the last vertex in A'_n, implying that x is the penultimate vertex in A'_n and $\mathrm{mirror}(w) = \mathrm{last}(A'_n)$. But then $\mathrm{mirror}(w) = \mathrm{last}(A'_n) = \mathrm{mirror}(v)$; thus $w = v$, a contradiction.

By the arguments above, there is at most one pair of little vertices (v, x) such that they belong to the same class. Then there must be at least

$$|Little(A_n)| + |Little(A'_n)| - 1 = n - 1$$

different classes containing the little vertices. ☐

3.3. Consequences for strongly consistent solutions for CR_n

Corollary 9. *If v and w are two distinct big vertices of $G = CR_n$ and* $\mathrm{side}(v) = \mathrm{side}(w)$*, then they cannot belong to the same class in a strongly consistent solution.*

Proof. Otherwise, the reverse order would contradict Corollary 5, since big vertices for an ordering are little vertices for its reverse. ☐

Corollary 10. *No class contains three vertices of the same side in a strongly consistent solution of $G = CR_n$.*

Proof. Since every vertex is either little or big, if a class contains three vertices of the same side then either there are at least two little vertices of the same side, contradicting Corollary 5, or there are at least two big vertices of the same side, contradicting Corollary 9. ☐

Corollary 11. *Let $(\mathcal{V}, <)$ be a strongly consistent solution for $G = CR_n$. If v and w are two vertices of G such that* $\mathrm{side}(v) \neq \mathrm{side}(w)$*, v and w belong to the same class of $\mathcal{V}$, and $v < w$, then v must be either the first or second vertex of* $\mathrm{side}(v)$*. Moreover, if v is the second vertex of* $\mathrm{side}(v)$*, then the first vertex of* $\mathrm{side}(v)$ *must be* $\mathrm{mirror}(w)$*.*

Proof. It follows from applying Corollary 6 to the reverse order of $<$. ☐

Claim 12. *Let $(\mathcal{V}, <)$ be a strongly consistent solution for $G = CR_n$. If $n \geq 4$, for every class $V \in \mathcal{V}$ the vertices of each side are consecutive in the order restricted to V.*

Proof. Suppose, on the contrary, that in some class V there are three vertices $a < b < c$ such that $\text{side}(a) = \text{side}(c) \neq \text{side}(b)$. Since $n \geq 4$, there is $d \in V(G)$ such that $\text{side}(d) = \text{side}(b)$, and $d \notin \{b, \text{mirror}(a), \text{mirror}(c)\}$. Then $\{a, c\} \subseteq N[d]$ and $b \notin N[d]$, which contradicts Lemma 3. $\square$

Corollary 13. *No class contains four vertices in a strongly consistent solution of $G = CR_n$, $n \geq 4$.*

Proof. Let $(\mathcal{V}, <)$ be a strongly consistent solution for $G = CR_n$, where $n \geq 4$. Suppose a class $V \in \mathcal{V}$ contains at least four vertices. By Corollary 10, it contains exactly four vertices, two of each side. Moreover, by Claim 12, the class must be composed by four distinct vertices $a < b < c < d$ such that $\text{side}(a) = \text{side}(b)$, $\text{side}(c) = \text{side}(d)$, and $\text{side}(a) \neq \text{side}(c)$.

Applying Corollary 6 to $a < c$ and $b < c$, it follows that c must be the penultimate vertex of $\text{side}(c)$, thus $d = \text{mirror}(a)$ and $d = \text{mirror}(b)$, a contradiction. $\square$

Claim 14. *In a strongly consistent solution for $G = CR_n$, the number of classes containing a big vertex is at least $n - 1$. In other words, at most one class can contain two big vertices, one of each side, since by Corollary 9 no class can contain more than one big vertex of the same side.*

Proof. It follows from Claim 8 applied to the reverse order of the solution. $\square$

Claim 15. *There are at most two classes of three vertices in a strongly consistent solution of $G = CR_n$, $n \geq 4$. Moreover, if there are two such classes, then one of them has vertices (a, b, c), where a and b are the first two vertices of their side and c belongs to the opposite side, and the other one has vertices (x, y, z), where y and z are the last two vertices of the side of a and x belongs to the side of c. In particular, they contain four vertices of one side and two of the other side, and the four vertices of the same side are the first two and the last two of the side in the order.*

Proof. Let $(\mathcal{V}, <)$ be a strongly consistent solution for $G = CR_n$, where $n \geq 4$. Suppose a class $V \in \mathcal{V}$ contains three vertices. Then it contains either two little vertices or two big vertices. By Claims 8 and 14, this may happen only once for little vertices and once for big vertices, so there are at most two classes of three vertices. Suppose there are indeed two classes V, W of three vertices each. By Corollary 10 and Claim 12, we may assume that $V = \{a, b, c\}$ where $a < b < c$, $\text{side}(a) = \text{side}(b)$ and $\text{side}(b) \neq \text{side}(c)$.

By Corollary 11 applied to $b < c$, it follows that a and b are the first and second vertices of side(a), and $a = \text{mirror}(c)$. In particular, a is little and c is big. By Corollary 5, b has to be big too. Suppose $W = \{x, y, z\}$, $x < y < z$, has two vertices of side(c) and one of side(a). By Corollary 10 and Claim 12, there are two cases: either side(x) = side(y) = side(c), and side(z) = side(a), or side(y) = side(z) = side(c), and side(x) = side(a). In the first case, reasoning as above, x and y are the first and second vertices of side(c), $x = \text{mirror}(z)$, x is little and y, z are big. This is a contradiction to Claim 14. In the second case, by Corollary 11 applied to $x < y$, it follows that x is either the first or the second vertex of side(a), but $x \notin \{a, b\}$, a contradiction. Thus, suppose $W = \{x, y, z\}$, $x < y < z$, has two vertices of side(a) and one of side(c). By Corollary 10 and Claim 12, there are two cases: either side(x) = side(y) = side(a), and side(z) = side(c), or side(y) = side(z) = side(a), and side(x) = side(c). In the first case, reasoning as above, x and y are the first and second vertices of side(a), a contradiction because $x \neq a$. So the second case holds, and y and z are the penultimate and last vertices of side(a). □

3.4. Proofs of the results in Table 1

Theorem 16. *For $n \geq 1$, $\text{thin}(CR_n) = \max(1, n-1)$. For n odd, $n \neq 3$, $\text{pthin}(CR_n) = n$; for $n = 3$ or $n \geq 1$ even, $\text{pthin}(CR_n) = n - 1$.*

Proof. The thinness of a non-empty graph is always at least 1, and by Claim 8, $\text{thin}(CR_n) \geq n - 1$.

Suppose $n \geq 5$ is odd, and suppose $(\mathcal{V}, <)$ is a strongly consistent solution for $G = CR_n$, with $n-1$ classes. By Corollary 13 and Claim 15, the solution has to have two classes of three vertices each, and $n-3$ classes of two vertices each. Moreover, since for one of the sides, the first two and the last two vertices belong to these classes of size three, by Corollary 6 and Corollary 11, no other class can contain elements of both sides. But since n is odd, both $n-4$ and $n-2$ are odd, so there is no way to partition the remaining $2n-6$ vertices into classes of size 2 where no class can contain elements of both sides. Hence, for $n \geq 5$ odd, $\text{pthin}(CR_n) \geq n$.

We will conclude the proof by construction. We will first show strongly partitions and orders for $n \leq 4$.

For $n = 1$, let $\sigma = (v_1, v'_1)$ and $\mathcal{V} = \{\{v_1, v'_1\}\}$.

For $n = 2$, let $\sigma = (v_1, v'_2, v_2, v'_1)$ and $\mathcal{V} = \{\{v_1, v'_2, v_2, v'_1\}\}$.

For $n = 3$, let $\sigma = (v_1, v'_2, v'_3, v_3, v_2, v'_1)$ and $\mathcal{V} = \{\{v_1, v'_3, v_2\}, \{v'_2, v_3, v'_1\}\}$.

For $n = 4$, let $\sigma = (v_1, v'_3, v_4, v'_4, v'_2, v_2, v_3, v'_1)$ and $\mathcal{V} = \{\{v_1, v'_2, v'_1\}, \{v'_3, v'_4, v_3\}, \{v_4, v_2\}\}$.

In all the cases, it is not hard to check that Condition 1 and Condition 2 are satisfied for both $(\mathcal{V}, \sigma)$ and $(\mathcal{V}, \overline{\sigma})$.

For $n \geq 6$ even, we build a strongly consistent solution $(\mathcal{V}, \sigma)$ with $n-1$ classes for CR_n, from the solution for $n = 4$ by inserting the remaining vertices into the order and distributing them into $n-4$ classes of two vertices each. The construction is shown in Algorithm 1.

For $n \geq 5$ odd, we can build a strongly consistent solution $(\mathcal{V}, \sigma)$ with n classes for CR_n following Algorithm 3, and a consistent solution $(\mathcal{V}, \sigma)$ with $n-1$ classes for CR_n following Algorithm 2.

In each case, it is not hard to check that Condition 1 and Condition 2 hold either for $(\mathcal{V}, \sigma)$ or for both $(\mathcal{V}, \sigma)$ and $(\mathcal{V}, \overline{\sigma})$, as required. □

Algorithm 1 Strongly consistent layout of CR_n for even $n \geq 6$.

function StConsistentLayoutCR($v_1, v_2, \ldots v_n$: vertices in A_n, $v'_1, v'_2, \ldots v'_n$: vertices in A'_n)
  $C_1 \leftarrow \{v_1, v'_2, v'_1\}$
  $C_2 \leftarrow \{v'_3, v'_4, v_3\}$
  $C_3 \leftarrow \{v_4, v_2\}$
  $C_4 \leftarrow \{v_5, v_n\}$
  $C_5 \leftarrow \{v'_6, v'_5\}$
  $\sigma \leftarrow (v'_6, v'_5)$
  for odd $i \in [7..n-1]$ **do**
    ▷ in each iteration we set the order and classes of 4 vertices
    $\sigma \leftarrow \sigma \oplus (v_i, v_{i-1}, v'_{i+1}, v'_i)$
    $C_{i-1} \leftarrow \{v_i, v_{i-1}\}$
    $C_i \leftarrow \{v'_{i+1}, v'_i\}$
  end for
  $\sigma \leftarrow (v_1, v'_3, v_4, v'_4, v_5) \oplus \sigma \oplus (v_n, v'_2, v_2, v_3, v'_1)$
    ▷ we set the order of the remaining vertices
  $\mathcal{V} \leftarrow \{C_1, \ldots, C_{n-1}\}$
return $\sigma, \mathcal{V}$
end function

Next, we deal with the independent and complete versions of thinness in crowns.

Algorithm 2 Consistent layout of CR_n for odd $n > 1$.

```
function ConsistentLayoutCR(v_1, v_2, ... v_n: vertices in A_n, v'_1, v'_2, ... v'_n: vertices in A'_n )
    σ ← ∅                                   ▷ the ordering starts out empty
    for odd i ∈ [1..n − 2) do
                ▷ in each iteration we set the order and classes of 4 vertices
        σ ← σ ⊕ (v_i, v'_{i+1}, v_{i+1}, v'_i)
        C_i ← {v_i, v_{i+1}}
        C_{i+1} ← {v'_{i+1}, v'_i}
    end for
    σ ← (v_n) ⊕ σ ⊕ (v'_n)        ▷ we set the order and class of the remaining vertices
    C_{n−1} ← C_{n−1} ∪ {v_n}
    C_{n−2} ← C_{n−2} ∪ {v'_n}
    𝒱 ← {C_1, ..., C_{n−1}}
return σ, 𝒱
end function
```

Algorithm 3 Strongly consistent layout of CR_n for odd $n > 4$.

```
function StConsistentLayoutCR(v_1, v_2, ... v_n: vertices in A_n, v'_1, v'_2, ... v'_n: vertices in A'_n )
    σ ← ∅                                   ▷ the ordering starts out empty
    for even i ∈ [2..n) do
                ▷ in each iteration we set the order and classes of 4 vertices
        σ ← σ ⊕ (v'_i, v'_{i−1}, v_{i+1}, v_i)
        C_i ← {v'_i, v'_{i−1}}
        C_{i+1} ← {v_{i+1}, v_i}
    end for
    σ ← (v_1) ⊕ σ ⊕ (v'_n)        ▷ we set the order and class of the remaining vertices
    C_1 ← {v_1, v'_n}
    𝒱 ← {C_1, ..., C_n}
return σ, 𝒱
end function
```

Theorem 17. *For every $n \geq 1$, $\text{thin}_{\text{ind}}(CR_n) = \text{pthin}_{\text{ind}}(CR_n) = n$.*

Proof. Let $G = CR_n$. We prove that $\text{thin}_{\text{ind}}(G) \geq n$ and $\text{pthin}_{\text{ind}}(G) \leq n$.

Let $<$ and $\mathcal{V}$ be an ordering and a partition of $V(G)$ into independent sets, respectively, that are consistent. Let $V \in \mathcal{V}$, and $v \in V$. Since V is an independent set of G, either $V = \{v, \text{mirror}(v)\}$ or $V \subseteq \text{side}(v)$. Thus, by Condition 1, if v is a little vertex then $v = \text{first}(V)$. This implies that for each pair $\{v_i, v'_i\}$ there is a class $V_i \in \mathcal{V}$ in which either v_i or v'_i is the first vertex of V_i. In particular, there is at least one class in $\mathcal{V}$ for each pair of vertices $\{v_i, v'_i\}$ in G, in other words, $|\mathcal{V}| \geq n$.

Now we prove by induction that there is an ordering σ_n, and an n-partition $\mathcal{V}$, of $V(G)$ such that both σ_n and $\overline{\sigma_n}$ respect Condition 1 and Condition 2. Moreover, if n is even, then each class of $\mathcal{V}$ is composed by either vertices of A_n or A'_n. The case where $n = 1$ is trivial, $\mathcal{V}_1 = \{\{v_1, v'_1\}\}$ and $\sigma_1 = (v_1, v'_1)$. For $n \geq 2$, define $\mathcal{V}^2_n = \{\{v_{n-1}, v_n\}, \{v'_{n-1}, v'_n\}\}$ and $\sigma^2_n = (v_{n-1}, v'_n, v_n, v'_{n-1})$. One can easily verify that both σ^2_2 and $\overline{\sigma^2_2}$ are in accordance with Condition 1 and Condition 2 with respect to $\mathcal{V}^2_2$. Suppose that the claim holds for all $n' < n$. There are two cases to consider.

If n is even, then define $\mathcal{V} = \mathcal{V}_{n-2} \cup \mathcal{V}^2_n$ and $\sigma_n = \sigma_{n-2} \oplus \sigma^2_n$. By the induction hypothesis, Condition 2 does not apply to $\mathcal{V}$ and Condition 1 holds for $\mathcal{V}_{n-2}$ with the orderings σ_{n-2} and $\overline{\sigma_{n-2}}$, and also for $\mathcal{V}^2_n$ with the orderings σ^2_n and $\overline{\sigma^2_n}$. Therefore, $\mathcal{V}_n$ and σ_n are strongly consistent.

If n is odd, then define $\mathcal{V} = \mathcal{V}_{n-1} \cup \{\{v_n, v'_n\}\}$ and $\sigma_n = (v_n) \oplus \sigma_{n-1} \oplus (v'_n)$. Regarding the induction hypothesis, it is possible to conclude two things. First, Condition 1 holds for both σ_{n-1} and $\overline{\sigma_{n-1}}$. Second, Condition 2 only applies to the class $\{v_n, v'_n\}$. Since all the neighbors of v_n and v'_n are in between them in σ_n, then both σ_n and $\overline{\sigma_n}$ are in accordance with Condition 1 and Condition 2. Hence, $\mathcal{V}_n$ and σ_n are strongly consistent. $\square$

Theorem 18. *For $n \geq 4$, $\text{thin}_{\text{cmp}}(CR_n) = 2n - 4$. For $n < 4$, $\text{thin}_{\text{cmp}}(CR_3) = 3$, $\text{thin}_{\text{cmp}}(CR_1) = \text{thin}_{\text{cmp}}(CR_2) = 2$. The same holds for the proper version.*

Proof. Let $n \geq 4$. In a partition of CR_n into completes, the classes have either size 1 or size 2, and classes of size 2 are $\{v_i, v'_j\}$ with $i \neq j$. Suppose there are r classes of size 2, and suppose that the greatest vertex of the order is v'_n. Then for every class $\{v_i, v'_j\}$ of size 2, $j \neq n$, except perhaps the class where $i = n$, we have $v'_j < v_i$ (by Condition 2). So, at least $r - 2$ classes $\{v_i, v'_j\}$ satisfy $v'_j < v_i$. Let v_k be the greatest vertex among the ones in those $r - 2$ classes. If $r - 2 > 2$, there is a class $\{v_i, v'_j\}$ satisfying

$v'_j < v_i$ and such that $i \neq k$ and $j \neq k$. So $v'_j < v_i < v_k$ and the indices are all different, contradicting Condition 2. Then $r - 2 \leq 2$, so $r \leq 4$, and $\text{thin}_{\text{cmp}}(CR_n) \geq 2n - 4$.

A complete proper thin representation satisfying the bound is the following: $\sigma = (v'_1, v'_2, v'_5, \ldots, v'_n, v_4, v_3, v_5, \ldots, v_n, v_2, v_1, v'_3, v'_4)$ where the classes of size 2 are $\{v'_1, v_2\}$, $\{v'_2, v_1\}$, $\{v_4, v'_3\}$, $\{v_3, v'_4\}$. For $n = 1$, we need two classes, $\mathcal{V} = \{\{v_1\}, \{v'_1\}\}$. For $n = 2$, $\mathcal{V} = \{\{v_1, v'_2\}, \{v_2, v'_1\}\}$. In both cases, any order works. For $n = 3$, the order $\sigma = (v_1, v'_2, v_3, v'_1, v_2, v'_3)$ and the classes $\mathcal{V} = \{\{v_1, v'_3\}, \{v'_2, v_3\}, \{v'_1, v_2\}\}$ are a complete proper thin representation. □

Now we study the precedence thinness and its variations in crowns.

Theorem 19. *For $n \geq 1$,* $\text{prec-thin}(CR_n) = \max(1, n - 1)$.

Proof. By Theorem 16, $\text{prec-thin}(CR_n) \geq \max(1, n-1)$. We prove constructively that $\text{prec-thin}(CR_n) \leq \max(1, n - 1)$.

For $n = 1$, $V_1 = (v_1, v'_1)$. Since we are dealing with precedence thinness, the order of the vertices is implied by the internal order of each of the classes which are, for $n \geq 2$, in order, $\mathcal{V} = (V_1, V_2, \ldots, V_{n-1})$.

For $n = 2$, let $V_1 = (v_1, v'_2)$, $V_2 = (v_2, v'_1)$.

For $n = 3$, let $V_1 = (v'_1)$, $V_2 = (v_2, v'_3, v_1, v'_2, v_3)$.

The consistency for these cases is easy to check.

For $n \geq 4$ let $V_1 = (v'_2)$, $V_2 = (v_1, v_2)$, $V_3 = (v'_3, v'_1)$; for $3 < i < n - 1$, $V_i = (v_i, v_{i-1})$ if i is even and $V_i = (v'_i, v'_{i-1})$ if i is odd; $V_{n-1} = (v_{n-1}, v'_n, v_{n-2}, v'_{n-1}, v_n)$.

Note that, for all $V \in \mathcal{V}$, $G[V]$ is an interval graph and the order is a canonical order of $G[V]$. It is easy to see that both Conditions 1 and 2 are being satisfied for V_1 and V_{n-1}. Each one of the remaining parts consists of vertices of either A_n or A'_n, where the first vertex is a little vertex and the second one is a big vertex. Thus, for these remaining parts, both conditions are also satisfied. Therefore, the orders and partitions defined are consistent. □

Theorem 20. *For $n \geq 4$,* $\text{prec-pthin}(CR_n) = n+1$. *For $n < 4$,* $\text{prec-pthin}(CR_1) = \text{prec-pthin}(CR_2) = 1$, $\text{prec-pthin}(CR_3) = 3$.

Proof. The cases of CR_1 and CR_2 are trivial, since they are proper interval graphs. Let $n \geq 3$, and suppose $(\mathcal{V}, <)$ is a strongly consistent solution for $G = CR_n$.

We claim that if V is the first or the last part in the solution, then $|V| = 1$. Let V be the first part, and suppose by contradiction that $|V| > 1$. Let $a < b \in V$ be the first two vertices of V. If $\text{side}(a) = \text{side}(b)$, then they are two little vertices of the same side in the same class, contradicting Corollary 5. If $\text{side}(a) \neq \text{side}(b)$, then, by Corollary 6, b is either the penultimate of the last vertex of $\text{side}(b)$, a contradiction because it is the first vertex of $\text{side}(b)$ and there are at least three vertices on its side. The argument for the last class is analogous, by using Corollary 5 and Corollary 11.

In particular, this implies that $\text{prec-pthin}(CR_3) \geq 3$. A strongly consistent solution with three classes is $(v_1)(v'_2, v_3, v'_1, v_2)(v'_3)$.

Suppose now $n \geq 4$. By Corollary 13, every class has size at most three. Moreover, given that the first class and the last class have size one, in order to obtain a solution with less than $n+1$ classes, at least two classes of size three are needed. In that case, by Claim 15, one of them, say V, has vertices (a, b, c), where a and b are the first two vertices of their side, say A_n, and c belongs to A'_n, and the other one, say W, has vertices (x, y, z), where y and z are the last two vertices A_n and x belongs to A'_n. In particular, V precedes W, but these are not the first or last classes, because the first and the last class contain only one vertex. Indeed, this situation implies that the only vertex v in the first class and the only vertex w in the last class belong to A'_n. But then we have $c < x < w \in A'_n$, which contradicts Corollary 6 applied to $b < c$. Hence, for $n \geq 4$, $\text{prec-pthin}(CR_n) \geq n + 1$.

We will complete the proof by construction. Define the following ordered sets:

- $V_1 = (v_1)$
- for all $2 \leq i \leq n$, if i is even, then $V_i = (v'_i, v'_{i-1})$, otherwise, $V_i = (v_i, v_{i-1})$
- if n is even, then $V_{n+1} = (v_n)$, otherwise, $V_{n+1} = (v'_n)$

Let $\mathcal{V} = (V_1, V_2, \ldots, V_n, V_{n+1})$ be an ordered $(n+1)$-partition of $V(CR_n)$. Note that, for all parts $V_i = \{v_i, v_{i-1}\}$ (resp. $V_i = \{v'_i, v'_{i-1}\}$), $v_i < v'_i$ and $v_{i-1} > v'_{i-1}$ (resp. $v'_i < v_i$ and $v'_{i-1} > v_{i-1}$). Thus, Condition 1 is satisfied for both the order and its reversal. Moreover, note that all parts consist of vertices of either A_n or A'_n, so Condition 2 is also satisfied for both the order and its reversal. Therefore, the order defined is strongly consistent with $\mathcal{V}$, and for $n \geq 4$, $\text{prec-pthin}(CR_n) \leq n + 1$. □

Theorem 21. *For $n \geq 2$, $\text{prec-thin}_{\text{ind}}(CR_n) = n+1$, and $\text{prec-thin}_{\text{ind}}(CR_1) = 1$. The same holds for the proper version.*

Proof. Let $G = CR_n$, $n \geq 2$. We prove that $\text{prec-thin}_{\text{ind}}(G) \geq n+1$ and $\text{prec-pthin}_{\text{ind}}(G) \leq n+1$.

Let $<$ and $\mathcal{V}$ be an ordering and a precedence partition of $V(G)$ into independent sets, respectively, that are consistent. Let $V \in \mathcal{V}$, and $v \in V$. Since V is an independent set of G, either $V = \{v, \text{mirror}(v)\}$ or $V \subseteq \text{side}(v)$. Thus, by Condition 1, if v is a little vertex then $v = \text{first}(V)$. This implies that for each pair (v_i, v'_i) there is a class of the partition in which either v_i or v'_i (the little one) is the first vertex of the class.

By the precedence constraint and Condition 2, at most one class is composed by a vertex and its mirror. Since $n \geq 2$, there is a little vertex that is not in the same class as its mirror. Let v be the greatest such vertex according to $<$, and let $V \in \mathcal{V}$ such that $\text{mirror}(v) \in V$. Then $v < \text{mirror}(v)$, $v \notin V$, and by the precedence constraint, $v < \text{first}(V)$. Since $\text{mirror}(\text{first}(V)) \neq \text{mirror}(v)$, $\text{mirror}(\text{first}(V))$ is not in the same class as its mirror. By definition of v, $\text{first}(V)$ is not a little vertex. Therefore, there is a class of the partition for each little vertex w, where w is the first of the class, plus the class V whose first vertex is not little. Hence $\text{prec-thin}_{\text{ind}}(G) \geq n+1$.

An order an precedence partition into $n+1$ independent sets that are strongly consistent can be defined as follows.

For $n = 2$, $(v_1)(v'_2, v'_1)(v_2)$.

For $n \geq 3$ odd, $(v_1)[(v'_{i+1}, v'_i)(v_{i+2}, v_{i+1})]_{1 \leq i \leq n-2}(v'_n)$.

For $n \geq 4$ even, $(v_1)[(v'_{i+1}, v'_i)(v_{i+2}, v_{i+1})]_{1 \leq i \leq n-3}(v'_n, v'_{n-1})(v_n)$. ∎

Theorem 22. *For* $n \geq 2$, $\text{prec-thin}_{\text{cmp}}(CR_n) = 2n-2$, *and* $\text{prec-thin}_{\text{cmp}}(CR_1) = 2$. *The same holds for the proper version.*

Proof. Let $n \geq 2$. In a partition of CR_n into completes, the classes have either size 1 or size 2, and classes of size 2 are $\{v_i, v'_j\}$ with $i \neq j$. Suppose that $\{v_i, v'_j\}$ is a class in a precedence partition consistent with a vertex order, so v_i and v'_j are consecutive in the order, and without loss of generality, $v_i < v'_j$. Suppose that there is another class $\{v_a, v'_b\}$ of size two in the partition. Then either both are greater than v'_j, or both are smaller than v_i. In the first case, by consistency, it must be $b = i$. In the second case, by consistency, either $b = i$ or $v_a < v'_b$, and either $a = j$ or $v'_b < v_a$. In particular, in any case, either $b = i$ or $a = j$. So, there are at most two more classes of size two.

Suppose that there are two such classes. By the observations above, one of them contains v'_i and the other one contains v_j. Namely, the classes are $\{v'_i, v_k\}$ and $\{v'_\ell, v_j\}$, and both v'_ℓ and v_j are smaller than v_i, since $i \neq \ell$. By the same observations, since $i \neq \ell$, then $v_j < v'_\ell$. Doing the same analysis as before but for the class $\{v'_i, v_k\}$ with respect to the class $\{v_j, v'_\ell\}$, since $i \neq j$, v'_i and v_k are smaller than v_j, $v_k < v'_i$, and $k = \ell$. But since v'_i and v_k are then smaller than v_i, it follows that either $k = j$ or $v'_i < v_k$, a contradiction in both cases.

Therefore, there are at most two classes of size two in a precedence partition consistent with a vertex order, and $\text{prec-thin}_{\text{cmp}}(CR_n) \geq 2n - 2$.

A complete proper precedence thin representation satisfying the bound is the following: $\sigma = (v'_3, \dots, v'_n, v_1, v'_2, v_2, v'_1, v_3, \dots, v_n)$ where the classes of size 2 are $\{v_1, v'_2\}$ and $\{v_2, v'_1\}$. For $n = 1$, two classes are needed because the graph is edgeless. □

4. Precedence thinness of cographs

In this section, we describe the behavior of the precedence thinness under the union and join operations, which allows to compute the precedence thinness of cographs efficiently.

With respect to the behavior of the thinness under the union and join operations, the following results were proved in the literature. These results allowed to compute the thinness of cographs in polynomial time, as well as to characterize k-thin graphs by forbidden induced subgraphs within the class of cographs.

Theorem 23 ([1]). *Let G_1 and G_2 be graphs. Then,* $\text{thin}(G_1 \cup G_2) = \max\{\text{thin}(G_1), \text{thin}(G_2)\}$.

Theorem 24 ([6]). *Let G_1 and G_2 be graphs. If G_2 is complete, then* $\text{thin}(G_1 \vee G_2) = \text{thin}(G_1)$. *If neither G_1 nor G_2 are complete, then* $\text{thin}(G_1 \vee G_2) = \text{thin}(G_1) + \text{thin}(G_2)$.

Also in [6], it was observed that the proper thinness of the join $G_1 \vee G_2$ cannot be expressed as a function whose only parameters are the proper thinness of G_1 and G_2 (even excluding simple particular cases, like trivial or complete graphs).

Next, we present a theorem that describes the precedence thinness over the union of two graphs.

Theorem 25 (Precedence thinness of union). *Let G_1 and G_2 be graphs. Then,* $\text{prec-thin}(G_1 \cup G_2) = \text{prec-thin}(G_1) + \text{prec-thin}(G_2) - 1$.

Proof. Let σ_1 be an order of $V(G_1)$, and let $V_1, \ldots, V_{k_1}$ be a precedence partition of $V(G_1)$, consistent with σ_1, such that $k_1 = \text{prec-thin}(G_1)$. Define σ_2 and $W_1, \ldots, W_{k_2}$ in the same way for G_2. It is not difficult to see that $V_1, \ldots, V_{k_1} \cup W_1, \ldots, W_{k_2}$ is a precedence partition of $V(G_1 \cup G_2)$ into $k_1 + k_2 - 1$ sets, consistent with the order $\sigma_1 \oplus \sigma_2$. So $\text{prec-thin}(G_1 \cup G_2) \leq \text{prec-thin}(G_1) + \text{prec-thin}(G_2) - 1$.

Now let σ be an order of $V(G_1 \cup G_2)$, and let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be a precedence partition of $V(G_1 \cup G_2)$, consistent with σ, such that $k = \text{prec-thin}(G_1 \cup G_2)$. We call a class *mixed* if it contains vertices of both G_1 and G_2.We will prove that $G_1 \cup G_2$ admits a vertex ordering and a consistent precedence partition into the same number k of classes, such that at most one class is mixed. Moreover, the order restricted to each of $V(G_1)$ and $V(G_2)$ coincides with σ.

Suppose $\mathcal{V}$ contains more than one mixed class, and let i be the smallest index such that V_i is mixed. Notice that $i < k$. Let $V_i^1 = V_i \cap V(G_1)$ and $V_i^2 = V_i \cap V(G_2)$. Suppose that $\text{last}(V_i)$ belongs to $V(G_1)$. Then, for every $j > i$, the vertices of V_i^2 have no neighbors in V_j: they have no neighbors in $V(G_1) \cap V_j$ by definition of disjoint union, and they cannot have a neighbor in $V(G_2) \cap V_j$, since such a vertex should be adjacent to $\text{last}(V_i)$ by consistency, and that contradicts the definition of disjoint union. We will define a partition $\mathcal{V}' = \mathcal{V} \setminus \{V_i, V_{i+1}\} \cup \{V_i^1, V_i^2 \cup V_{i+1}\}$, and the order σ' such that $v <_{\sigma'} w$ if, and only if, either $v \in V_i^1$ and $w \in V_i^2$, or $v <_\sigma w$ and it is not the case that $v \in V_i^2$ and $w \in V_i^1$.

Let $x <_{\sigma'} y <_{\sigma'} z$ such that x and y are in the same class of $\mathcal{V}'$, x and z are adjacent. Since σ and $\mathcal{V}$ are consistent, if $x <_\sigma y <_\sigma z$ and x and y are in the same class of $\mathcal{V}$, then z is adjacent to y. So, suppose first that x and y are not in the same class of $\mathcal{V}$. Then $x \in V_i^2$ and $y \in V_{i+1}$. This implies that $z \in V_j$ with $j \geq i + 1$, but x and z are adjacent, and vertices of V_i^2 have no neighbors in V_j with $j > i$, a contradiction. Suppose then that some of the two inequalities do not hold for σ. By definition of σ', this can only happen if one of the vertices belongs to V_i^1 and the other one to V_i^2. Since x and y are in the same class of $\mathcal{V}'$, it must be $y \in V_i^1$ and $z \in V_i^2$, implying that $x \in V_i^1$. But this contradicts the fact that x and z are adjacent. We conclude that z is adjacent to y, and that σ' and $\mathcal{V}'$ are consistent.

The case in which $\text{last}(V_i)$ belongs to $V(G_2)$ is symmetric, and we can repeat this procedure until obtaining an ordering and a consistent partition into k classes, such that at most one class is mixed (the repetition stops because the index of the smallest mixed class is strictly increasing after each step). If there are k_1 classes containing at least one vertex of G_1, k_2 classes containing at least one vertex of G_2, and k_3 mixed classes, then $k = k_1 + k_2 - k_3$, thus $k_1 + k_2 - k = k_3 \leq 1$. Since that ordering and partition restricted to each of $V(G_1)$ and $V(G_2)$ are consistent, $\text{prec-thin}(G_1) + \text{prec-thin}(G_2) \leq k_1 + k_2 \leq k + 1 = \text{prec-thin}(G_1 \cup G_2) + 1$. $\square$

Theorem 26 (Precedence thinness of join)**.** *Let G_1 and G_2 be graphs. If G_2 is complete, then* $\text{prec-thin}(G_1 \vee G_2) = \text{prec-thin}(G_1)$*. If neither G_1 nor G_2 are complete, then* $\text{prec-thin}(G_1 \vee G_2) = \text{prec-thin}(G_1) + \text{prec-thin}(G_2)$.

Proof. The proof is similar to the one of Theorem 25. Let σ_1 be an order of $V(G_1)$, and let $V_1, \ldots, V_{k_1}$ be a precedence partition of $V(G_1)$, consistent with σ_1, such that $k_1 = \text{prec-thin}(G_1)$. Define σ_2 and $W_1, \ldots, W_{k_2}$ in the same way for G_2. It is not difficult to see that $V_1, \ldots, V_{k_1}, W_1, \ldots, W_{k_2}$ is a precedence partition of $V(G_1 \vee G_2)$ into $k_1 + k_2$ sets, consistent with the order $\sigma_1 \oplus \sigma_2$. So $\text{prec-thin}(G_1 \vee G_2) \leq \text{prec-thin}(G_1) + \text{prec-thin}(G_2)$. Moreover, if G_2 is complete, $V_1, \ldots, V_{k_1} \cup V(G_2)$ is a precedence partition of $V(G_1 \vee G_2)$ into k_1 sets, consistent with the order $\sigma_1 \oplus \sigma_2$.

Now let σ be an order of $V(G_1 \vee G_2)$, and let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be a precedence partition of $V(G_1 \vee G_2)$, consistent with σ, such that $k = \text{prec-thin}(G_1 \vee G_2)$. We call a class *mixed* if it contains vertices of both G_1 and G_2, and *uniform* otherwise. We will prove that $G_1 \cup G_2$ admits a vertex ordering and a consistent precedence partition into the same number k of classes, such that at most one class is mixed. Moreover, if none of G_1 and G_2 is complete, then no class is mixed.

Suppose $\mathcal{V}$ contains a mixed class, and let i be the smallest index such that V_i is mixed. Let $V_i^1 = V_i \cap V(G_1)$ and $V_i^2 = V_i \cap V(G_2)$. Suppose that $\text{first}(V_i)$ belongs to $V(G_1)$. Then, V_i^2 induces a complete graph and for every $j > i$, the vertices of V_i^2 are adjacent to all the vertices of V_j: they are adjacent to every vertex in $V(G_1) \cap V_j$ by definition of join, and they cannot have a non-neighbor in $V(G_2) \cap V_j$, since such a vertex should be adjacent to $\text{first}(V_i)$ by definition of join, and that would break consistency.

If G_2 is not a complete graph, then there is another class containing vertices of G_2. Let j be the greatest index different from i such that $V_j \cap V(G_2) \neq \emptyset$. If $j > i$, V_j may be either uniform or mixed, while if $j < i$, V_j is uniform. Moreover, if $j < i$, the vertices of V_i^2 are also adjacent to all the vertices of classes between V_{j+1} and V_i (both included), since those classes contain only vertices of G_1.

We will define a partition $\mathcal{V}' = \mathcal{V} \setminus \{V_i, V_j\} \cup \{V_i^1, V_j \cup V_i^2\}$, and the order σ' obtained by σ by deleting the vertices of V_i^2 and reinserting them right after the vertices of V_j.

Let $x <_{\sigma'} y <_{\sigma'} z$ such that x and y are in the same class of $\mathcal{V}'$, x and z are adjacent. Since σ and $\mathcal{V}$ are consistent, if $x <_\sigma y <_\sigma z$ and x and y are in the same class of $\mathcal{V}$, then z is adjacent to y. So, suppose first that x and y are not in the same class of $\mathcal{V}$. Then $x \in V_j$ and $y \in V_i^2$. This implies that either $z \in V_i^2$ or $z \in V_\ell$ with $\ell > j$. In either case, y is adjacent to z. Suppose now that some of the two inequalities do not hold for σ. By definition of σ', this implies that at least one of the vertices belongs to V_i^2. If $y \in V_i^2$, then y is adjacent to z, since every vertex of V_i^2 is adjacent to every vertex greater than itself, both according to σ and according to σ'. If $y \in V_i^2$, since x and y are in the same class of $\mathcal{V}'$, and by the definition of the order and partition, it must be $y \in V_i^2$, thus adjacent to z. By last, if $z \in V_i^2$, there are two cases. If $z <_\sigma y$, then z is adjacent to y. If $y <_\sigma z$, since x and y are in the same class of $\mathcal{V}'$, then either both of them belong to V_i^2, or both of them belong to V_i^1, or both of them belong to V_ℓ with $\ell < i$. Since $y <_\sigma z$, in all those cases $x <_\sigma y <_\sigma z$, contradicting what we were supposing.

The case in which $\text{first}(V_i)$ belongs to $V(G_2)$ is symmetric, and we can repeat this procedure until obtaining an ordering and a consistent partition into k classes, such that at most one class is mixed, and this happens only in the case in which one of the graphs is complete (the repetition of the procedure stops because the index of the smallest mixed class is strictly increasing after each step).

Suppose that at the end of the process there are k_1 classes containing at least one vertex of G_1 and k_2 classes containing at least one vertex of G_2. In the case in which we have a mixed class, $k = k_1 + k_2 - 1$, thus $k_1 + k_2 = k + 1$. Since that ordering and partition restricted to each of $V(G_1)$ and $V(G_2)$ are consistent, $\text{prec-thin}(G_1) + \text{prec-thin}(G_2) \leq k_1 + k_2 = k + 1 = \text{prec-thin}(G_1 \cup G_2) + 1$. If none of the graphs is complete, we have no mixed class at the end of the process, so $k = k_1 + k_2$. Since that ordering and partition restricted to each of $V(G_1)$ and $V(G_2)$ are consistent, $\text{prec-thin}(G_1) + \text{prec-thin}(G_2) \leq k_1 + k_2 = k = \text{prec-thin}(G_1 \cup G_2)$. $\square$

Corollary 27. *The precedence thinness of cographs can be computed in polynomial time.*

The thinness of the complement of an interval graph can be arbitrarily large, being $\overline{tK_2}$ an example of it, but it can be also equal to one, as for example in the case of complete graphs or edgeless graphs. So, we cannot express the thinness of a graph in terms of the thinness of its complement. However, one can aim to relate the complete thinness of a graph in terms of the independent thinness of its complement, and viceversa. As we will show next, this is not possible for the regular complete/independent thinness or proper thinness (Proposition 28), but a relation exists for the precedence versions of complete and independent thinness.

Proposition 28. *For a matching nK_2, $n \geq 2$, it holds that* $\text{pthin}_{\text{ind}}(nK_2) = 2$*, while* $\text{thin}_{\text{cmp}}(\overline{nK_2}) = n$*. For the crown CR_n, $n \geq 2$, it holds that* $\text{pthin}_{\text{cmp}}(\overline{CR_n}) = 2$*, while* $\text{thin}_{\text{ind}}(CR_n) = n$*.*

Proof. Let $G = nK_2$, $n \geq 2$, and $V(G) = V \cup W$, where $V = \{v_1, \dots, v_n\}$, $W = \{w_1, \dots, w_n\}$, such that the only edges of G are $v_i w_i$ for $1 \leq i \leq n$. Then it is easy to verify that the partition $\{V, W\}$ and the ordering $v_1, w_1, v_2, w_2, \dots, v_n, w_n$ form a strongly consistent solution, and both V and W are independent sets. On the other hand, $\text{pthin}_{\text{ind}}(nK_2) \geq 2$, since $V(G)$ is not an independent set. As for $\overline{G}$, it is known that $\text{thin}(\overline{nK_2}) = n$ (e.g. [10, 6]), and it is not hard to verify that a consistent solution where the classes are complete sets is given by the vertex order $v_1 v_2 w_1 [v_i w_{i-1} v_{i+1} w_i]_{2<i\leq n-1} w_n$ and the partition $\{\{v_1, w_n\}\} \cup \bigcup_{2\leq i\leq n} \{\{v_i, w_{i-1}\}\}$. Indeed, the solution is also strongly consistent.

The equality $\text{thin}_{\text{ind}}(CR_n) = n$ is proved in Theorem 17. Now let $G = \overline{CR_n}$, $n \geq 2$, let $V(G) = V \cup W$, where $V = \{v_1, \dots, v_n\}$, $W = \{w_1, \dots, w_n\}$, such that V and W are complete sets and the only edges of G joining vertices of V and W are $v_i w_i$ for $1 \leq i \leq n$. Then it is easy to verify that the partition $\{V, W\}$ and the ordering $v_1, w_1, v_2, w_2, \dots, v_n, w_n$ form a strongly consistent solution, and both V and W are complete sets. On the other hand, $\text{pthin}_{\text{cmp}}(\overline{CR_n}) \geq 2$, since $V(G)$ is not a complete graph. □

Theorem 29. *Let G be a graph. Then* $\text{prec-thin}_{\text{ind}}(G) = \text{prec-thin}_{\text{cmp}}(\overline{G})$*, and* $\text{prec-pthin}_{\text{ind}}(G) = \text{prec-pthin}_{\text{cmp}}(\overline{G})$*.*

Proof. Let G be a graph and σ be an order of $V(G)$. Let $V_1, \dots, V_k$ be a precedence partition of $V(G)$ into independent sets which is consistent (respectively strongly consistent) with σ (i.e., $v <_\sigma w$ for $v \in V_i$ and $w \in V_j$ with $1 \leq i < j \leq k$).

Consider now the same ordered partition $V_1, \dots, V_k$ in $\overline{G}$ (now into complete sets), but where we order internally the vertices of each part according to $\overline{\sigma}$. We will prove that the partition and the order are consistent (respectively strongly consistent) for $\overline{G}$, implying that prec-thin$_{\text{ind}}(G) \geq$ prec-thin$_{\text{cmp}}(\overline{G})$, and prec-pthin$_{\text{ind}}(G) \geq$ prec-pthin$_{\text{cmp}}(\overline{G})$.

The (strong) consistency cannot be broken by a triple of vertices of the same class, since each class is a complete set. So let $v < w < z$ according to the order defined for $\overline{G}$, such that v and w are in the same class V_i of the partition, z belongs to a different class, and $vz \in E(\overline{G})$. By the definition of the order, $z \in V_j$ with $j > i$, and $w <_\sigma v <_\sigma z$. If $wz \notin E(\overline{G})$, then $wz \in E(G)$, and by consistency of σ and $V_1, \ldots, V_k$ in G, $vz \in E(G)$, a contradiction. Thus, $wz \in E(\overline{G})$. Similarly, if σ and $V_1, \ldots, V_k$ were strongly consistent, let $v < w < z$ according to the order defined for $\overline{G}$, such that w and z are in the same class V_j of the partition, v belongs to a different class, and $vz \in E(\overline{G})$. By the definition of the order, $v \in V_i$ with $i < j$, and $v <_\sigma z <_\sigma w$. If $vw \notin E(\overline{G})$, then $vw \in E(G)$, and by strong consistency of σ and $V_1, \ldots, V_k$ in G, $vz \in E(G)$, a contradiction. Thus, $vw \in E(\overline{G})$.

To prove the converse inequalities, it is enough to observe that the same argument, applied to $\overline{G}$ and exchanging independent sets by complete sets, proves that prec-thin$_{\text{ind}}(G) \leq$ prec-thin$_{\text{cmp}}(\overline{G})$, and prec-pthin$_{\text{ind}}(G) \leq$ prec-pthin$_{\text{cmp}}(\overline{G})$. □

5. Bounds for the thinness of grids

In this section, we show that the linear decomposition in [18] for mim-width (a width parameter with several algorithmic applications), leads to a consistent ordering and partition of $V(GR_n)$ into $\lceil \frac{n+1}{2} \rceil$ classes, thus improving the previously known upper bound for the thinness of the grid.

Theorem 30 (Bound for the thinness of $GR_{n,m}$)**.** *For* $1 \leq n \leq m$, $\lceil \frac{n-1}{3} \rceil \leq \text{thin}(GR_{n,m}) \leq \lceil \frac{n+1}{2} \rceil$.

Proof. To prove the upper bound, we will define a vertex partition and consistent ordering as follows. The first class V_1 consists of the union of vertices $(1, j)$, $1 \leq j \leq m$ and the vertices $(2, j)$, $1 \leq j \leq m$, j odd. For $2 \leq i \leq \frac{n}{2}$, the i-th class V_i consists of the union of vertices $(2i - 1, j)$, $1 \leq j \leq m$ and the vertices $(2i - 2, j)$ and $(2i, j)$, $1 \leq j \leq m$, $j \equiv i \mod 2$. The last class V_i, $i = \lceil \frac{n+1}{2} \rceil$, is composed by the union of the vertices $(2i-2, j)$, $1 \leq j \leq m$, $j \equiv i \mod 2$, and the vertices (n, j), $1 \leq j \leq m$, in the case in which n is odd. Examples for $n = 7$ and $n = 6$ can be seen in Figure 1, where the bold edges are the ones that are internal to a class.

Each class induces an interval graph. We will define first an internal ordering for each class, that satisfies consistency (i.e., a canonical ordering). For the first class, $(x, j) < (x', j')$ if $j < j'$, and $(2, j) < (1, j)$ for $1 \leq j \leq m$, j odd. For the last class, $(x, j) < (x', j')$ if $j < j'$, and $(n-1, j) < (n, j)$ for the corresponding values of j, when n is odd. For the i-th class, $2 \leq i \leq \frac{n}{2}$, $(x, j) < (x', j')$ if $j < j'$, and $(2i, j) < (2i-2, j) < (2i-1, j)$ for $2 \leq i \leq \frac{n}{2}$, $1 \leq j \leq m$, $j \equiv i \mod 2$.

There are edges joining V_i and V_j if and only if $|i - j| = 1$. So, it is enough to prove that we can combine the defined orderings for two consecutive classes into a single one, consistent with the graph induced by the union of the two classes. In this way, we can insert class by class the vertices, obtaining a total ordering of the vertices of the grid which is consistent with the defined partition.

Assuming that vertex $(2i, j)$ belongs to V_{i+1}, the joint ordering for $V_i \cup V_{i+1}$ follows the pattern $(2i+2, j)$, $(2i, j)$, $(2i-2, j-1)$, $(2i-1, j-1)$, $(2i-1, j)$, $(2i, j+1)$, $(2i+1, j)$, $(2i+1, j+1)$, and repeating from $(2i+2, j+2)$ (restricted to the vertices that actually do exist). This ordering is illustrated bottom-up in the right part of Figure 1, where the bold edges are the ones that are internal to a class, the horizontal dotted lines are drawn to help in visualizing the vertex ordering, and the vertices in the described pattern are shaded.

Let us argue consistency (that can be verified also in the figure). Vertices in column $2i-2$ do not have neighbors in V_{i+1}, and vertices in column $2i+2$ do not have neighbors in V_i. Recall that we are assuming that vertex $(2i, j)$ belongs to V_{i+1}. So the vertex $(2i-1, j-1)$ has no neighbors in V_{i+1}, and the vertex $(2i+1, j)$ has no neighbors in V_i.

The only neighbor of $(2i-1, j)$ in V_{i+1} is $(2i, j)$, and the only vertices between those two in the ordering are $(2i-2, j-1)$ and $(2i-1, j-1)$, both belonging to V_i. The only neighbor of $(2i+1, j+1)$ in V_i is $(2i, j+1)$, and the only vertex in between in the ordering is $(2i+1, j)$, belonging to V_{i+1}.

The only neighbor of $(2i, j+1)$ in V_{i+1} smaller than itself is $(2i, j)$, and the only vertices between those two in the ordering are $(2i-2, j-1)$, $(2i-1, j-1)$, and $(2i-1, j)$, all of them belonging to V_i. Finally, the only neighbor of $(2i, j)$ in V_i smaller than itself is $(2i, j-1)$, and the only vertices between those two in the ordering are $(2i+1, j-2)$, $(2i+1, j-1)$, and $(2i+2, j)$, all of them belonging to V_{i+1}.

To prove the lower bound, we will use a result by Jelínek [14], namely, for every order $v_1, \ldots, v_{nm}$ of the vertices of $GR_{n,m}$, there exists an index t

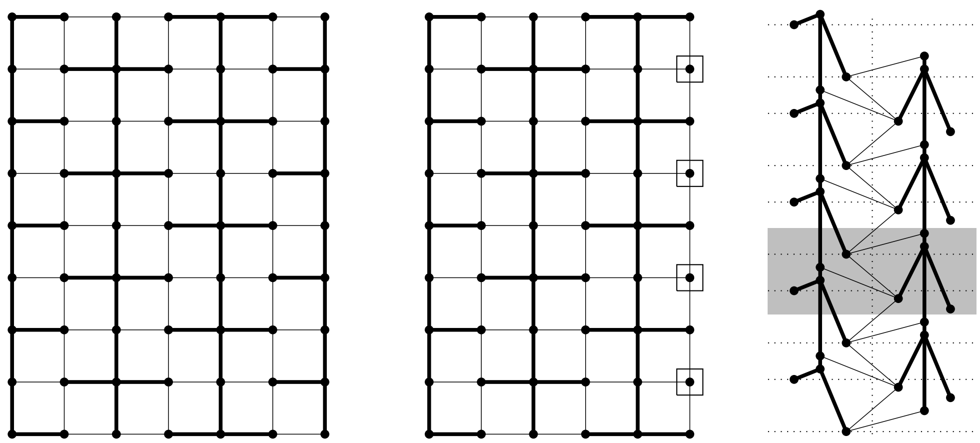

Figure 1: Scheme for the consistent partition and ordering of a grid in Theorem 30. The third drawing shows the combined ordering of two consecutive classes (bottom up).

such that the induced bipartite graph $G[A, A']$ where $A = \{v_1, \ldots, v_t\}$ and $A' = \{v_{t+1}, \ldots, v_{nm}\}$, contains a matching of size $n-1$. Let $B \subseteq A$ and $B' \subseteq A'$ the matched vertices, $|B| = |B'| = n-1$. Suppose that there is a partition of $V(GR_{n,m})$ into k classes, consistent with that ordering, suppose that there is a class W such that $|W \cap B| \geq 4$, and let $w < x < y < z$ be four vertices of $W \cap B$. Let w', x', y', z' be their respective matches in B'. Notice that all of them are greater than z. Then, by consistency, w' is adjacent also to x, y, and z; x' is adjacent also to y, and z. But then there are two cycles C_4 consisting of x', y, w', z and x', y, w', x, having two common edges ((y, x') and (y, w')), which is not possible in $GR_{n,m}$. Therefore, $|W \cap B| \leq 3$ for each class W, hence $k \geq \frac{n-1}{3}$. $\square$

Corollary 31 (Bound for the thinness of GR_n)**.** *For $n \geq 1$, $\lceil \frac{n-1}{3} \rceil \leq \operatorname{thin}(GR_n) \leq \lceil \frac{n+1}{2} \rceil$.*

The grid $GR_{2,n}$ can be also viewed as the Cartesian product of the path P_n and the complete graph K_2. So, as a corollary of the results in [1] for thinness of Cartesian products of graphs, $\operatorname{thin}(GR_{2,n}) = 2$, for $n \geq 2$.

The precedence thinness instead, grows linearly in n for $GR_{n,2}$ and quadratically in n for GR_n, as it can be seen in the following theorems, whose proofs are based on Theorem 25.

Theorem 32. *For $n \geq 1$,* prec-thin$(GR_{2,n}) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. Let $GR_{2,n} = (V, E)$ with $V = \{(i,j) : 1 \leq i \leq 2, 1 \leq j \leq n\}$. Let $(\mathcal{V}, \sigma)$ be any solution for the precedence thinness of $GR_{2,n}$, with $\mathcal{V} = (V_1, \ldots, V_k)$.

Let us show first that $|V_1| \leq 2$. Suppose that $|V_1| \geq 3$ and let v_1, v_2, v_3 be the three first elements of V_1. (Ordering relations are assumed to be according to σ.) Note that $d(v_1) \geq \delta(GR_{2,n}) = 2$. Let $u_1 < u_2$ be two first neighbors of v_1. If $u_1 \in V_1$, then $(u_1, u_2) \in E$ or otherwise (v_1, u_1, u_2) would break the consistency. But then v_1, u_1, u_2 induce a C_3, which is not possible. Therefore, $u_1 \notin V_1$ and, consequently, all neighbors of v_1 are not in V_1. Thus, v_2, v_3 are also neighbors of u_1, u_2, to avoid (v_1, v_i, u_j) break the consistency, for all $i \in \{2, 3\}, j \in \{1, 2\}$. As $(u_1, u_2) \notin E$ (avoiding the cycle v_1, u_1, u_2), there are two cycles isomorphic to C_4 consisting of v_1, u_1, v_2, u_2 and v_1, u_1, v_3, u_2, having two common edges ((v_1, u_1) and (v_1, u_2)), which is not possible in $GR_{2,n}$. Therefore, $|V_1| \leq 2$ and, by previous arguments, V_1 must be an independent set.

We prove that prec-thin$(GR_{2,n}) = \left\lceil \frac{n+1}{2} \right\rceil$ holds by induction on n. Since prec-thin$(GR_{2,1}) = 1$ and prec-thin$(GR_{2,2}) = 2$, the result holds for $n \leq 2$. Suppose $n \geq 3$ and that the result holds for all $GR_{2,n'}$ with $n' < n$. Let $(\mathcal{V}, \sigma)$ be a solution for the precedence thinness of $GR_{2,n}$, with $\mathcal{V} = (V_1, \ldots, V_k)$. We proceed the proof by analyzing the cases in which $|V_1| = 1$ or $|V_1| = 2$.

If $|V_1| = 2$, then one of the elements of V_1 is a "corner" of the grid (a degree-two vertex), and the other is that one that dominates its neighborhood; without loss of generality, $V_1 = \{(1,1), (2,2)\}$. To see that, let $u < v$ be the two vertices of V_1 and let $H = GR_{2,n} \setminus N[v]$. We claim that u must be disconnected from the other vertices of H. Indeed, suppose there is $(u, w) \in E(H)$. Then, $w \notin V_1$. To avoid (u, v, w) break the consistency, we have that $(v, w) \in E$. But then w could not be in H, since $w \in N[v]$. Thus, the claim holds. Therefore, u must be a corner of the grid (say, vertex $(1,1)$) and v must the vertex that dominates its neighborhood (say, vertex $(2,2)$).

Let $R = \{(i,j) : 1 \leq i \leq 2, 1 \leq j \leq 2\}$. Let $G = GR_{2,n}$ and $G' = G[V \setminus R]$; therefore, G' is isomorphic to $GR_{2,n-2}$. By induction hypothesis, we have that prec-thin$(G') = \left\lceil \frac{n-2+1}{2} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil$. We have that, without loss of generality, $V_1 \subseteq R$. Consequently, $(V_2 \setminus R, \ldots, V_k \setminus R)$ is a partition of $V(G')$ and, considering the ordering σ restricted to the elements of $V(G')$, such a partition consists of a solution for the precedence thinness of G'. Therefore, $k - 1 \geq$ prec-thin$(G') = \left\lceil \frac{n-1}{2} \right\rceil$, and therefore $k \geq \left\lceil \frac{n-1}{2} \right\rceil + 1 = \left\lceil \frac{n-1}{2} + 1 \right\rceil =$

$\lceil \frac{n+1}{2} \rceil$. Thus, prec-thin$(GR_{2,n}) \geq \lceil \frac{n+1}{2} \rceil$.

If $|V_1| = 1$, let $v = (a, b)$ be the unique vertex of V_1. If $b = 1$ or $b = n$ (that is, v is a corner of the grid), without loss of generality, we may assume $b = 1$. Letting $R = \{(i,j) : 1 \leq i \leq 2, 1 \leq j \leq 2\}$, $G = GR_{2,n}$ and $G' = G[V \setminus R]$, we may proceed the same reasoning as in the previous paragraph, to conclude that prec-thin$(GR_{2,n}) \geq \lceil \frac{n+1}{2} \rceil$. On the other hand, if $1 < b < n$, the proof proceeds as follows. Let $V' = \{(i,j) : 1 \leq i \leq 2,\ 1 \leq j < b\}$, $V'' = \{(i,j) : 1 \leq i \leq 2,\ b < j \leq n\}$, $G' = G[V']$, and $G'' = G[V'']$. Therefore, G' is isomorphic to $GR_{2,b-1}$, whereas G'' is isomorphic to $GR_{2,n-b}$. By Theorem 25, we have that prec-thin$(G' \cup G'') =$ prec-thin$(G') +$ prec-thin$(G'') - 1 = \lceil \frac{b-1+1}{2} \rceil + \lceil \frac{n-b+1}{2} \rceil - 1$, the latter holding by the induction hypothesis. Therefore, prec-thin$(G' \cup G'') \geq \lceil \frac{b}{2} + \frac{n-b+1}{2} - 1 \rceil = \lceil \frac{n-1}{2} \rceil$. Let $R = \{(3-a, b)\}$. As $G' \cup G'' \subset GR_{2,n}$, we have that $(V_2 \setminus R, \ldots, V_k \setminus R)$ is a partition of $V(G' \cup G'')$ and, again considering the respective restriction in σ, such a partition consists of a solution for the precedence thinness of $G' \cup G''$. Therefore, $k - 1 \geq$ prec-thin$(G' \cup G'') \geq \lceil \frac{n-1}{2} \rceil$, and we have prec-thin$(GR_{2,n}) \geq \lceil \frac{n+1}{2} \rceil$.

To show an upper bound on prec-thin$(GR_{2,n})$, we present the partition of Figure 2, for odd and even values of n. The respective consistent orderings consists of all the elements that are alone in one part (the elements depicted in a square) coming first, in any order, followed by the vertices of the bold path. Those vertices of the path must be ordered according to any canonical ordering of the induced path. It is straightforward to check that this partition and ordering indeed is a solution for the precedence thinness of $GR_{2,n}$. Regarding the size of the partition, note that for each two units in n, there is one part having only one element. Therefore, there are $\lfloor \frac{n}{2} \rfloor$ such parts. Considering the part having all the remaining vertices, we have that the size of the partition is $\lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{n+2}{2} \rfloor = \lceil \frac{n+1}{2} \rceil$. Therefore, prec-thin$(GR_{2,n}) \leq \lceil \frac{n+1}{2} \rceil$.

□

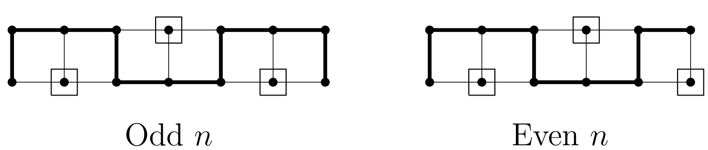

Figure 2: Scheme for the partition of a grid in Theorem 32.

Theorem 33. *For $n \geq 1$, $\left\lceil \frac{n-1}{3} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq \text{prec-thin}(GR_n) \leq \left\lceil \frac{n-1}{2} \right\rceil^2 + 1$.*

Proof. Notice that GR_n has the disjoint union of $r = \left\lceil \frac{n-1}{3} \right\rceil$ copies $G_1, \dots, G_r$ of $GR_{2,n}$ as induced subgraph. By successive applications of Theorem 25, we have that $\text{prec-thin}(G_1 \cup \ldots \cup G_r) = \sum_{i=1}^{r} \text{prec-thin}(G_i) - (r-1) = r \left\lceil \frac{n+1}{2} \right\rceil - r + 1 = r \left\lceil \frac{n-1}{2} \right\rceil + 1 = \left\lceil \frac{n-1}{3} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 1$. Consequently, $\text{prec-thin}(GR_n) \geq \text{prec-thin}(G_1 \cup \ldots \cup G_r) = \left\lceil \frac{n-1}{3} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 1$.

For the upper bound, we present the partition depicted in Figure 3, consisting of $\left\lceil \frac{n-1}{2} \right\rceil^2 + 1$ parts: each set of vertices reached by a same drawing of a "claw" is a part, and the caterpillar in the top and right sector of the grid is also a part. (Note that the diagonal edge of the claw is not part of the grid; each claw only represents the vertices belonging to a same part.) Depending on the position in the grid, the portions of the claws that would cross the left and the bottom sides of the grid are absent, but nevertheless they will be called a claw anyway. Also, depending on the parity of n, the last part can be either a caterpillar or an independent set, but we will refer to it as a caterpillar.

We build an ordering σ of the vertices of the grid as follows. Enumerate all parts $V_1, \dots, V_r$ with $r = \left\lceil \frac{n-1}{2} \right\rceil^2$ according to the relative positions among the corresponding claws in the scheme, from bottom to top, and for claws at the same level, from left to right. Therefore, V_1 is the bottommost leftmost claw (indeed, the corner and the vertex that dominates its neighborhood), V_2 is the first (partial) claw at the right of V_1, and so on. For each $1 \leq i \leq r$, if the vertices of $V_i = \{a, b, c, d\}$ are layout as depicted in the right portion of the figure, then order them in σ as $a < b < c < d$. A vertex of a claw in the relative position of a will be called of type a, and analogously for types b, c, d. Moreover, for all $u \in V_i$, $v \in V_j$, we let $u < v \iff i < j$. Finally, define a new part V_{r+1} having all remaining vertices. The vertices in V_{r+1} are ordered according to a canonical order of the caterpillar. Besides, all vertices in V_{r+1} come in σ after all other vertices of the grid.

It is easy to verify that all vertices belonging to a same part are consecutive in σ. Let us show that σ is consistent to $(V_1, \dots, V_{r+1})$. Suppose there is (u, v, w) breaking the consistency. It is trivial to check that u, v, w cannot all be in any V_i, $1 \leq i \leq r$, and also cannot all be in V_{r+1} since we have used a canonical order. Therefore, $u, v \in V_i$ and $w \in V_j$ with $i \neq j$. Since $v < w$, then $i < j$.

Notice that, by the chosen ordering among classes, vertices of type a and b of V_i have no neighbors in V_j, and all possible neighbors in V_j of the vertex of type c of V_i are also neighbors of the vertex of type d. Since $a < b < c < d$ in V_i, it is not possible to break consistency with vertices $u, v \in V_i$ and $w \in V_j$, where $i < j$. Thus, σ is consistent with the partition. Therefore, prec-thin$(GR_n) \leq \left\lceil \frac{n-1}{2} \right\rceil^2 + 1$. □

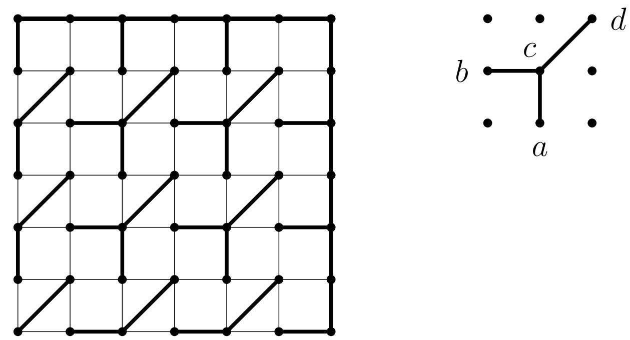

Figure 3: Scheme for the partition of a grid in Theorem 33.

6. Coloring of thin graphs

In this section, we prove that k-coloring (k part of the input) is NP-complete for precedence 2-thin graphs and for proper 2-thin graphs, while it is polynomial-time solvable for precedence proper 2-thin graphs, given the order and partition.

In order to prove the hardness results, we will reduce from μ-coloring, which is NP-complete on proper interval graphs [2].

Given a graph G and a function $\mu : V(G) \to \mathbb{N}$, the *$\mu$-coloring* problem consists of deciding whether G is *μ-colorable*, i.e. whether there exists a coloring $f : V(G) \to \mathbb{N}$ such that $f(v) \leq \mu(v)$ for every $v \in V(G)$.

Theorem 34. *The k-coloring problem is NP-complete for precedence 2-thin graphs and for proper 2-thin graphs, when k part of the input.*

Proof. Let (G, μ') be a μ-coloring instance, where G is a proper interval graph and $v_1, \ldots, v_n$ is a proper interval order of $V(G)$. Notice that the instance is equivalent, in terms of feasibility, to the instance (G, μ) with $\mu(v) = \min\{\mu'(v), n\}$ for every $v \in V(G)$. We will firstly show a polynomial time reduction from (G, μ) to an n-coloring instance of a precedence 2-thin graph G', and secondly show a polynomial time reduction from (G, μ) to an n-coloring instance of a proper 2-thin graph G''.

Let G' be a graph such that $V(G') = V(G) \cup A$, where $V(G)$ induces G, $A = \{w_1, \dots, w_n\}$ induces a complete graph, and $v \in V(G)$ is adjacent to w_i if and only if $\mu(v) < i$. Let us see that G' is precedence 2-thin with partition $\{A, V(G)\}$, and the vertex order $w_1, \dots, w_n, v_1, \dots, v_n$. Let $x < y < z$ in $V(G')$ such that x, y belong to the same class and $(x, z) \in E(G')$. If $x, y \in V(G)$, then $(y, z) \in E(G) \subseteq E(G')$ because the order restricted to $V(G)$ is an interval order. If $x, y \in A$, then $x = w_i$ and $y = w_j$ with $i < j$. If $z \in A$, then $(y, z) \in E(G')$ because A induces a complete graph in G'. If z in $V(G)$, since $(x, z) \in E(G')$, $\mu(z) < i < j$, so $(y, z) \in E(G')$ as well.

Now suppose there is a μ-coloring of G. We can extend it to an n-coloring of G' by giving color i to w_i, for $i = 1, \dots, n$. Since $v \in V(G)$ is adjacent to w_i if and only if $\mu(v) < i$, the coloring is valid. Conversely, suppose there is an n-coloring of G'. Since A is a complete subgraph of G', we can rename the colors and obtain a coloring f such that $f(w_i) = i$ for $i = 1, \dots, n$. Since $v \in V(G)$ is adjacent to w_i if and only if $\mu(v) < i$, for every vertex v of $V(G)$, $f(v) \leq \mu(v)$, so G admits a μ-coloring.

Let G'' be a graph such that $V(G'') = V(G) \cup B$, where $V(G)$ induces G, $B = w_1^1, \dots, w_n^1, w_1^2, \dots, w_n^2, w_1^n, \dots, w_n^n$ is a proper interval graph whose maximal cliques are $w_1^i, \dots, w_n^i$, for $i = 1, \dots, n$, and $w_j^i, \dots, w_{j-1}^{i+1}$ for $i = 1, \dots, n-1$, $j = 2, \dots, n$, and $v_k \in V(G)$ is adjacent to w_j^i if and only if $i = k$ and $\mu(v_k) < j$. Let us see that G' is proper 2-thin with partition $V(G), B$, and the vertex order $w_1^1, \dots, w_n^1, v_1, w_1^2, \dots, w_n^2, v_2, \dots, w_1^n, \dots, w_n^n, v_n$. Let $x < y < z$ in $V(G'')$ such that x, y belong to the same class and $(x, z) \in E(G'')$. If the three vertices belong to the same class, $(y, z) \in E(G'')$ because the order restricted to each class is a proper interval order. Otherwise, z in $V(G)$ and $x, y \in B$, since no vertex of B has a neighbor smaller than itself in $V(G)$. Let $x = w_j^i$ and $y = w_{j'}^{i'}$ and $z = v_k$. Since $(x, z) \in E(G')$, $i = k$ and $\mu(z) < j$, so, by the order definition, $i' = k$ and $j < j'$, thus $\mu(z) < j$ and $(y, z) \in E(G'')$ as well. It is easy to see that the case in which $x < y < z$ in $V(G'')$, y, z belong to the same class, and $(x, z) \in E(G'')$, may only arise when the three vertices belong to the same class, so $(x, y) \in E(G'')$ because the order restricted to each class is a proper interval order.

Now suppose there is a μ-coloring of G. We can extend it to an n-coloring of G'' by giving color j to w_j^i, for $i, j \in \{1, \dots, n\}$. Since $v_k \in V(G)$ is adjacent to w_j^i if and only if $i = k$ and $\mu(v) < j$, the coloring is valid. Conversely, suppose there is an n-coloring of G'. By the definition of the maximal cliques

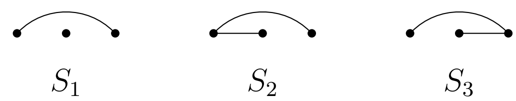

Figure 4: The forbidden ordered induced subgraphs for an interval order ($\{S_1, S_2\}$) and a proper interval order ($\{S_1, S_2, S_3\}$).

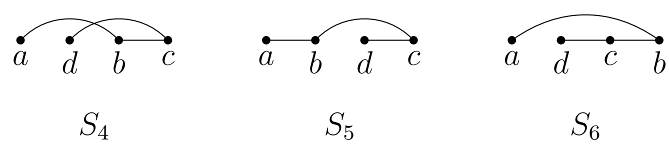

Figure 5: The forbidden ordered induced subgraphs for a perfect order.

of $G''[B]$, the n color classes are $\{w_j^1, w_j^2, \dots, w_j^n\}$ for $j = 1, \dots, n$. So, we can rename the colors and obtain a coloring f such that $f(w_j^i) = j$ for $i, j \in \{1, \dots, n\}$. Since $v_k \in V(G)$ is adjacent to w_j^i if and only if $i = k$ and $\mu(v) < j$, for every vertex v_k of $V(G)$, $f(v_k) < \mu(v_k)$, so G admits a μ-coloring. $\square$

A vertex order $<$ of G is *perfect* if G contains no P_4 $abcd$ with $a < b$ and $d < c$. A graph is *perfectly orderable* if it admits a perfect order. Perfectly ordered graphs can be optimally colored by applying the greedy coloring algorithm in the perfect order [11].

The polynomiality of the k-coloring problem on precedence proper 2-thin graphs, given the order and partition, is then a consequence of the following result.

Theorem 35. *Precedence proper 2-thin graphs are perfectly orderable. Moreover, given a partition of $V(G)$ into two sets V^1, V^2, strongly consistent with an order $<$ such that every vertex of V^1 is smaller than every vertex of V^2, a perfect order $\prec$ of G can be obtained by taking the vertices on V^1 ordered according to the reverse of $<$, followed by the vertices of V^2 ordered according to $<$.*

Proof. Suppose $(G, \prec)$ contains one of the ordered induced subgraphs S_4, S_5 or S_6 (Figure 5). Since $\prec$ restricted to both V^1 and V^2 is a proper interval order, thus it does not contain any of the ordered induced subgraphs S_1, S_2

or S_3 (Figure 4), it follows that in any of the cases, the first vertex of the ordered subgraph of Figure 5 belongs to V^1 and the last vertex belongs to V^2. Moreover, in the case of S_4 and S_5 the first two vertices belong to V^1, and in the case of S_4 the last two vertices belong to V^2.

Suppose first that $(G, \prec)$ contains S_4. Then $a, d \in V^1$ and $b, c \in V^2$, so $d < a < c$, $(d, c) \in E(G)$ and $ac \notin E(G)$, a contradiction because $<$ is strongly consistent with the partition V^1, V^2 in G.

Suppose now that $(G, \prec)$ contains S_5. Then $a, b \in V^1$ and $c \in V^2$, so $b < a < c$, $(b, c) \in E(G)$ and $(a, c) \notin E(G)$, a contradiction because $<$ is strongly consistent with the partition V^1, V^2 in G.

Finally, suppose that $(G, \prec)$ contains S_6. If $c \in V^1$, then $d \in V^1$, $c < d < b$, $(c, b) \in E(G)$ and $(d, b) \notin E(G)$, a contradiction to the strong consistency of $<$ and V^1, V^2 in G. If $c \in V^2$, then $a < c < b$, $(a, b) \in E(G)$ and $(a, c) \notin E(G)$, a contradiction to the strong consistency of $<$ and V^1, V^2 in G.

Therefore, $\prec$ is a perfect order for G. □

The complexity of the k-coloring problem (k part of the input) remains open for precedence proper t-thin graphs, $t \geq 3$.

7. Conclusion

Interval graphs and proper interval graphs are well known graph classes, to which hundreds of papers have been dedicated. Due its importance, and since not all graphs are (proper) interval graphs, several authors have defined larger classes of graphs by relaxing the definition of (proper) interval graphs or some characterization for this class. This paper aims to one of such generalizations, namely, the (proper) k-thin graphs and its variations.

We provide several properties of such generalizations, some of them restricted to the special classes of crown and grid graphs, in order to present the exact values for (proper) thinness, (proper) independent thinness, (proper) complete thinness, precedence (proper) thinness, precedence (proper) independent thinness, and precedence (proper) complete thinness for the crown graphs CR_n. The exact values are provided in Table 1. For cographs, we prove that the precedence thinness can be determined in polynomial time. In particular, we compute the precedence thinness of the union and join of two graphs in terms of their own precedence thinness. We also provide bounds for the thinness of $n \times n$ grids GR_n and $n \times m$ grids $GR_{n,m}$, proving that for $1 \leq n \leq m$, $\left\lceil \frac{n-1}{3} \right\rceil \leq \text{thin}(GR_{n,m}) \leq \left\lceil \frac{n+1}{2} \right\rceil$. For precedence thinness, it is proved that $\text{prec-thin}(GR_{2,n}) = \left\lceil \frac{n+1}{2} \right\rceil$ and that $\left\lceil \frac{n-1}{3} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq \text{prec-thin}(GR_n) \leq \left\lceil \frac{n-1}{2} \right\rceil^2 + 1$.

Regarding applications, we show that the k-coloring problem (k being part of the input) is NP-complete for both precedence 2-thin graphs and proper 2-thin graphs. On the positive side, it is polynomially solvable for precedence proper 2-thin graphs, given the order and partition. This last result is obtained by showing that precedence proper 2-thin graphs are perfectly orderable graphs, which are optimally colored by the greedy algorithm on any of its corresponding perfect orderings, and that the from the order and partition in the precedence proper 2-thin representation a perfect order can be computed.

Acknowledgments

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