On the thinness of trees*

Flavia Bonomo-Braberman^{a,b}, Eric Brandwein^a, Carolina Lucía Gonzalez^{a,b}, Agustín Sansone^a

^a Universidad de Buenos Aires. Facultad de Ciencias Exactas y Naturales. Departamento de Computación. Buenos Aires, Argentina.

^bCONICET-Universidad de Buenos Aires. Instituto de Investigación en Ciencias de la Computación (ICC). Buenos Aires, Argentina.

Abstract

The study of structural graph width parameters like tree-width, clique-width and rank-width has been ongoing during the last five decades, and their algorithmic use has also been increasing [Cygan et al., 2015]. New width parameters continue to be defined, for example, mim-width in 2012, twin-width in 2020, and mixed-thinness, a generalization of thinness, in 2022.

The concept of thinness of a graph was introduced in 2007 by Mannino, Oriolo, Ricci and Chandran, and it can be seen as a generalization of interval graphs, which are exactly the graphs with thinness equal to one. This concept is interesting because if a representation of a graph as a k-thin graph is given for a constant value k, then several known NP-complete problems can be solved in polynomial time. Some examples are the maximum weighted independent set problem, solved in the seminal paper by Mannino et al., and the capacitated coloring with fixed number of colors [Bonomo, Mattia and Oriolo, 2011].

In this work we present a constructive $\mathcal{O}(n \log(n))$ -time algorithm to compute the thinness for any given n-vertex tree, along with a corresponding thin representation. We use intermediate results of this construction to improve known bounds of the thinness of some special families of trees.

Keywords: trees, thinness, polynomial time algorithm

^{*}Partially supported by CONICET (PIP 11220200100084CO) and UBACyT (20020170100495BA and 20020160100095BA).

Email addresses: fbonomo@dc.uba.ar (Flavia Bonomo-Braberman),

1. Introduction

The definition of k-thin graphs involves a vertex ordering and partition into k classes satisfying certain properties. In that case, the order and partition are said to be consistent.

A wide family of problems can be solved in XP parameterized by thinness, given a consistent representation. This family includes weighted variations of list matrix assignment problems with matrices of bounded size, and the possibility of adding bounds on the weight of the sets and their unions and intersections [3].

For a given order of the vertices of G, there exists an algorithm to compute an optimal consistent partition of the vertices of G with time complexity $\mathcal{O}(n^3)$, where n is the number of vertices of G [3], since the problem can be reduced in linear time to the optimal coloring of an auxiliary co-comparability graph of n vertices, and the latter can be solved in $\mathcal{O}(n^3)$ time [9]. On the other hand, computing a consistent ordering of the vertices for a given partition, or detect it does not exist, is NP-complete [3]. Very recently, by a reduction from that problem, it was proved that deciding whether the thinness of a graph is at most k, without any given order or partition, is NP-complete [17]. In this work we solve this problem in polynomial time for trees. This is the first non-trivial class in the literature for which we know how to compute the thinness (and consistent order and partition of the vertices) efficiently. Some efforts were made before to study the thinness of trees in [16].

The design of this algorithm was heavily inspired by the proof and the algorithm by Høgemo, Telle, and Vågset for another graph invariant, the linear maximum induced matching width (linear mim-width) [11]. The linear mim-width is a known lower bound for the thinness [3], and there are families with bounded linear mim-width and unbounded thinness [6]. However, we prove here that, for trees, the two parameters behave alike and the difference between thinness and linear mim-width is at most 1.

ebrandwein@dc.uba.ar (Eric Brandwein), cgonzalez@dc.uba.ar (Carolina Lucía Gonzalez), asansone@dc.uba.ar (Agustín Sansone)

¹A polynomial-time algorithm and forbidden induced subgraphs characterization are known for thinness of cographs [3], but the algorithm and proofs follow from their decomposition theorem without much more complication.

2. Definitions and preliminary results

All graphs in this work are finite, undirected and have no loops or multiple edges.

Let G be a graph, we denote by V(G) its vertex set and by E(G) its edge set. We denote by N(v) and N[v], respectively, the neighborhood and closed neighborhood of a vertex $v \in V(G)$. Let $X \subseteq V(G)$. We denote by N(X) the set of vertices not in X having at least one neighbor in X, and by N[X] the closed neighborhood $N(X) \cup X$.

We denote by G[X] the subgraph of G induced by X, and by G-W or $G \setminus W$ the graph $G[V(G) \setminus W]$. We use $G \setminus (u,v)$ to denote the graph with vertices V(G) and edges $E(G) \setminus \{(u,v)\}$. A subgraph H of G is a spanning subgraph if V(H) = V(G).

A tree is a connected graph with no cycles. A leaf of a tree T is a vertex with degree one in T. The diameter of a tree is the maximum number of edges in a simple path joining two vertices.

A rooted tree on vertex r is a tree in which vertex r is labeled as the root, and we will usually denote it by T_r . The ancestors of a vertex v in a rooted tree T_r are all vertices in the simple path between v and r which are not v. Note that r has no ancestors in T_r . The descendants of a vertex v in T_r are all vertices for which v is an ancestor in T_r . The children of a vertex v are those neighbors of v which are also descendants. Conversely, the parent of a vertex $v \neq r$ is the only neighbor of v which is also an ancestor of v, if any. In a rooted tree T_r , the vertex v has no parent. The grandchildren of a vertex v are the children's children, and the grandparent is the parent's parent, if any. A strict subtree of a tree T is a tree induced by some vertices of T which is different from T.

The *height* of a rooted tree is the maximum number of edges in a simple path from the root to a leaf. A rooted tree of height h has h+1 levels of vertices, being the root the only vertex at level 1, and saying that a vertex has level t+1 if and only if its parent has level t.

Let T be a tree containing the adjacent vertices v and u. The dangling tree from v in u, $T\langle v, u \rangle$, is the component of $T \setminus (u, v)$ containing u.

A graph G is k-thin if there exists an ordering $\sigma = v_1, \ldots, v_n$ of V(G) and a partition S of V(G) into k classes such that, for each triple (r, s, t) with r < s < t, if v_r, v_s belong to the same class and $(v_t, v_r) \in E(G)$, then $(v_t, v_s) \in E(G)$. An order and a partition satisfying those properties are said to be consistent. We call the tuple (σ, S) a consistent solution or consistent

layout. The minimum k such that G is k-thin is called the thinness of G, and denoted by thin(G).

A complete graph is a graph where all vertices are pairwise adjacent. We denote by K_n the complete graph of n vertices. A clique of a graph G is a complete induced subgraph of G. The maximum size (number of vertices) of a clique of G is denoted by $\omega(G)$.

A coloring of a graph is an assignment of colors to its vertices such that any two adjacent vertices are assigned different colors. The minimum number of colors needed to color G is called the *chromatic number* of G and denoted by $\chi(G)$.

A graph G is perfect whenever for every induced subgraph H of G, $\chi(H) = \omega(H)$.

A graph G is an interval graph if to each vertex $v \in V(G)$ can be associated a closed interval $I_v = [l_v, r_v]$ of the real line, such that two distinct vertices $u, v \in V(G)$ are adjacent if and only if $I_u \cap I_v \neq \emptyset$. The family $\{I_v\}_{v \in V(G)}$ is an interval representation of G. Graphs of thinness one are exactly the interval graphs [15].

A graph G is a comparability graph if there exists a partial order in V(G) such that two vertices of G are adjacent if and only if they are comparable by that order. A graph G is a co-comparability graph if its complement \overline{G} is a comparability graph.

Let G be a graph and < an ordering of its vertices. The graph $G_{<}$ has V(G) as a vertex set, and $E(G_{<})$ is such that for v < w in the ordering, $(v, w) \in E(G_{<})$ if and only if there is a vertex z in G such that w < z, $(z, v) \in E(G)$, and $(z, w) \notin E(G)$. An edge of $G_{<}$ represents that its endpoints cannot belong to the same class in a vertex partition that is consistent with the ordering <.

Theorem 1. [3, 4] Given a graph G and an ordering < of its vertices, the graph $G_<$ has the following properties:

- 1. the chromatic number of $G_{<}$ is equal to the minimum integer k such that there is a partition of V(G) into k sets that is consistent with the order <, and the color classes of a valid coloring of $G_{<}$ form a partition consistent with <;
- 2. G_{\leq} is a co-comparability graph;
- 3. if G is a co-comparability graph and < a comparability ordering of \overline{G} , then $G_{<}$ is a spanning subgraph of G.

Since co-comparability graphs are perfect [14], $\chi(G_{<}) = \omega(G_{<})$. We thus have the following.

Corollary 2. Let G be a graph, and k a positive integer. Then $thin(G) \ge k$ if and only if, for every ordering < of V(G), the graph $G_{<}$ has a clique of size k.

The pathwidth of a graph G, denoted by pw(G), is the minimum clique number of an interval supergraph of G minus one [12]. The linear mimwidth of a graph G, denoted by lmimw(G), is the smallest integer k such that V(G) can be arranged in a linear layout v_1, \ldots, v_n in such a way that for every $1 \leq i \leq n-1$, the size of a maximum induced matching in the bipartite graph formed by the edges of G with an endpoint in $\{v_1, \ldots, v_i\}$ and the other one in $\{v_{i+1}, \ldots, v_n\}$ is at most k [18].

The following relations are known.

Theorem 3. [3, 13] For every graph G, $\liminf_{G \to G} G \leq \min_{G \to G} G \leq \min_$

3. Structural characterization and polynomial time algorithm for thinness of trees

A considerable part of the ideas related to the construction of the algorithm to compute the thinness (and a consistent ordering and partition of the vertices) we are presenting in this section was inspired by an algorithm to compute the linear mim-width of a tree and an optimal layout [11], which was at the same time inspired by the framework behind the pathwidth algorithm presented in [8].

Even if the structure results for trees are very similar to those for linear mim-width, the arguments in the proofs are different, because the definitions of the concepts are different.

3.1. Path layout lemma

Lemma 4 (Path Layout Lemma). Let T be a tree. If there exists a path $P = (x_1, \ldots, x_p)$ in T such that every connected component of $T \setminus N[P]$ has thinness less than or equal to k then $thin(T) \leq k+1$. Moreover, given the consistent orderings and partitions for the components in at most k classes we can in linear time compute a consistent ordering and partition for T in at most k+1 classes.

Proof. Using the consistent orderings $\{\sigma_{T\langle v_{i,j},u_{i,j,m}\rangle}\}$ and partitions $\{S_{T\langle v_{i,j},u_{i,j,m}\rangle}\}$ into k classes of the connected components $T\langle v_{i,j},u_{i,j,m}\rangle$ of $T\setminus N[P]$, we give the Algorithm 1 constructing an order σ_T and partition S_T into k+1 classes of the vertices of T, showing that $\text{thin}(T) \leq k+1$. Here, $v_{i,j}$ corresponds to the neighbors of $x_i \in P$ which are not themselves in P, and $u_{i,j,m}$ the neighbors of $v_{i,j}$ different from x_i . We will use C_c to denote the c-th class of S_T , and $C_{T\langle v_{i,j},u_{i,j,m}\rangle,c}$ to the c-th class of $S_{T\langle v_{i,j},u_{i,j,m}\rangle}$. The \oplus will denote the list append operation or the list concatenation operation interchangeably.

Algorithm 1 Consistent layout given the layouts of the dangling subtrees of a path.

```
function Consistent Layout (T: tree, P = (x_1, ..., x_p):
\{\sigma_{T\langle v_{i,j}, u_{i,j,m}\rangle}\}: orderings, \{S_{T\langle v_{i,j}, u_{i,j,m}\rangle}\}: partitions)
     for class \leftarrow 1, k+1 do
           C_{class} \leftarrow \emptyset
     end for
     for x_i \in P do
           for v_{i,j} \in N(x_i) \setminus P do
                 \sigma_T \leftarrow \sigma_T \oplus v_{i,i}
                  C_{k+1} \leftarrow C_{k+1} \cup \{v_{i,j}\}
                 for u_{i,j,m} \in N(v_{i,j}) \setminus x_i do
                       \sigma_T \leftarrow \sigma_T \oplus \sigma_{T\langle v_{i,j}, u_{i,j,m}\rangle}
                        for c \leftarrow 1, k do
                              C_c \leftarrow C_c \cup C_{T\langle v_{i,i}, u_{i,i,m} \rangle, c}
                        end for
                  end for
           end for
           \sigma_T \leftarrow \sigma_T \oplus x_i
           C_{k+1} \leftarrow C_{k+1} \cup \{x_i\}
     end for
     S_T \leftarrow \{C_1, \ldots, C_{k+1}\}
     return \sigma_T, S_T
end function
```

Firstly, from the algorithm we can see that each vertex of T is added exactly once to σ_T and to only one class of S_T , and as those are the only operations performed in the algorithm apart from the initialization of the

consistent solution, it has linear time complexity on the size of the tree. Now we must show that the ordering and partition are consistent, meaning, there are no three vertices a < b < c in σ_T such that a and b are in the same class of the partition, and $(c, a) \in E(T)$ and $(c, b) \notin E(T)$.

We will prove it by contradiction, assuming there are three vertices a < b < c of T that violate consistency. We will separate by cases and prove each one separately.

- $a \in N[P], b \notin N[P]$, or $a \notin N[P], b \in N[P]$: If one of $\{a, b\}$ belongs to N[P] and the other does not, they are both added to different classes in S_T , so this triple is consistent.
- $a \notin N[P], c \in N[P]$: This cannot happen. The only vertices of N[P] adjacent to vertices not in N[P] are the $v_{i,j}$ for some i, j, and they are all appended to the order before all their neighbors, so there cannot be a vertex a such that $a < v_{i,j}$ and $(v_{i,j}, a) \in E(T)$.
- $a \in N[P], c \notin N[P]$: As before, the only possibility for a is to be $v_{i,j}$ for some i, j so as to be adjacent to a vertex $c \notin N[P]$. Also, the only vertices not in N[P] adjacent to $v_{i,j}$ are the $u_{i,j,m}$ for some m, so $c = u_{i,j,m}$. And the only possible vertices between $v_{i,j}$ and $u_{i,j,m}$ in the order are the vertices of $T\langle v_{i,j}, u_{i,j,m}\rangle$, which means that $b \in T\langle v_{i,j}, u_{i,j,m}\rangle$. But then $a \in N[P]$ and $b \notin N[P]$, which means they are in different classes of the partition, and so this triple is consistent.

Combining the last three cases, we see that, to violate consistency, the three vertices must belong to N[P], or none of the three can.

• $\{a,b,c\} \subseteq N[P]$: We begin by noting that no vertex $v_{i,j}$ is adjacent to another vertex α in N[P] such that $\alpha < v_{i,j}$, because $v_{i,j}$ is always added to the order before their corresponding x_i , which is the only vertex in N[P] adjacent to $v_{i,j}$. This means that $c = x_i$ for some $1 \le i \le p$. The only vertices adjacent to x_i that are lower in the order are all $v_{i,j}$, and x_{i-1} if i > 1.

If $a = v_{i,j}$ for some j, then all vertices b such that $a < b < x_i$ are some $v_{i,k}$ for some k > j. As $(x_i, v_{i,k}) \in T$, this triple is consistent. If, on the other hand, $a = x_{i-1}$, then again, $b = v_{i,k}$ for some k, which means it is also adjacent to x_i , and so this triple is also consistent.

- $\{a,b,c\} \subseteq T\langle v_{i,j}, u_{i,j,m}\rangle$ for some i,j,m: Because $\sigma_{T\langle v_{i,j}, u_{i,j,m}\rangle}$ is a subsequence of σ_T , and all classes of $S_{T\langle v_{i,j}, u_{i,j,m}\rangle}$ are subsets of the corresponding classes of S_T , the consistency is preserved between three vertices of the same dangling tree.
- $a \in T\langle v_{i,j}, u_{i,j,m} \rangle$, $c \in T\langle v_{i',j'}, u_{i',j',m'} \rangle$ for some $i \neq i', j \neq j', m \neq m'$: Vertices in different dangling trees are not adjacent, so this cannot happen.
- $\{a,c\} \subseteq T\langle v_{i,j}, u_{i,j,m} \rangle$, $b \in T\langle v_{i',j'}, u_{i',j',m'} \rangle$ for some $i \neq i', j \neq j', m \neq m'$: Either all the vertices of $T\langle v_{i,j}, u_{i,j,m} \rangle$ are added before all the vertices of $T\langle v_{i',j'}, u_{i',j',m'} \rangle$, or vice versa. This means that either a < b and c < b, or b < a and b < c, so an inconsistent triple cannot be found this way.

We have proven that any possible triple in the order and partition generated by the algorithm is consistent, and then the order and partition given by the algorithm are consistent.

Lemma 5. If $D(x,k) \geq 3$ for some vertex x in T, then $thin(T) \geq k+1$.

Proof.

Let x be a vertex in T such that $D(x,k) \geq 3$. Let v_1, v_2, v_3 be three neighbors of x such that they have neighbors u_1, u_2, u_3 respectively which satisfy that thin $(T\langle v_i, u_i \rangle) \geq k$. Let $T_i = T\langle v_i, u_i \rangle$ for $i \in \{1, 2, 3\}$. Let < be an ordering of the vertices of T which is part of a consistent solution for T that minimizes the amount of classes used in the partition. Let a and b be, respectively, the lowest and greatest vertex in the order < that belong to any subtree T_i .

There must be at least one subtree T_j such that $a \notin T_j$ and $b \notin T_j$. As thin $(T_j) \geq k$, we know that $T_{j<}$ has a clique C of size at least k. Let t be the greatest vertex according to < that belongs to C.

We know that a < t < b, because a is lower in the order than all the vertices in T_j , and b is greater. Also, there is a simple path P between a and b that does not include any vertex of T_j . Let us see why. If a and b belong to the same subtree T_i , then there is a simple path P between the two that includes only vertices of T_i , because T_i is a tree. If a and b belong to different subtrees T_h and T_i , then the simple path is $a \to \cdots \to u_h \to v_h \to x \to v_i \to u_i \to \cdots \to b$, which does not include any vertex of T_j .

Because P begins with a vertex which is lower than t in <, and finishes in a vertex greater than t, there exist two adjacent vertices $a', b' \in P$ such that a' < t < b'. We will see that a' is adjacent in $T_{<}$ to all the vertices in C, which means that there is a clique of size k + 1 in $T_{<}$.

We know that a' < t < b', and $(b', a') \in E(T)$, but $(b', t) \notin E(T)$, so this means that a' is adjacent to t in $T_{<}$.

Now, given t' < t a vertex of C, let us see that $(a', t') \in E(T_{<})$. As t' is adjacent to t in $T_{<}$, there is a vertex c > t adjacent to t' but not to t in T_{j} . As a' < t < c, a' < c. And as t' < t < b', t' < b'. This leaves us with two possibilities:

- a' < t': b' is adjacent in T to a' but not to t', so a' and t' are adjacent in $T_{<}$.
- t' < a': c is adjacent in T to t' but not to a', so a' and t' are adjacent in $T_{<}$.

This shows that every vertex of C is adjacent to a vertex a' in $T_{<}$, and so $T_{<}$ has a clique of size at least k+1, which implies that thin $(T) \geq k+1$. \square

3.2. k-component index theorem

Definition 6 (k-neighbor). Let x be a vertex of a tree T, and v a neighbor of x in T. If there exists a vertex $u \neq x$ neighbor of v such that $thin(T\langle v, u \rangle) > k$, then v is a k-neighbor of x.

Definition 7 (k-component index, k-saturation). The k-component index of x is the number of k-neighbors of x, and we note it as D(x,k). If $D(x,k) \geq 3$ for some vertex x in T, we say that x is k-saturated in T.

Lemma 8. If $thin(T) \ge k + 1$, then there exists a vertex x in T such that $D(x,k) \ge 3$.

Proof. To prove this we first prove the following partial claim: if $\operatorname{thin}(T) \geq k+1$ then there exists a vertex $x \in T$ such that $D(x,k) \geq 3$; or there exists a strict subtree S of T with $\operatorname{thin}(S) \geq k+1$. We will prove the contrapositive statement, so let us assume that every vertex in T has D(x,k) < 3 and no strict subtree of T has $\operatorname{thinness} \geq k+1$ and show that then $\operatorname{thin}(T) \leq k$. For every vertex $x \in T$, it must then be true that $D(x,k) \leq 2$ and that

D(x, k+1) = 0. The strategy of this proof is to show that there is always a path P in T such that all the connected components in $T \setminus N[P]$ have $thinness \leq k-1$. When we have shown this, we proceed to use the Path Layout Lemma, to get that $thin(T) \leq k$.

We begin by defining the following two sets of vertices:

$$X = \{x \mid x \in V(T) \text{ and } D(x, k) = 2\}$$

$$Y = \{ y \mid y \in V(T) \text{ and } D(y, k) = 1 \}$$

Case 1: $X \neq \emptyset$

If x_i and x_j are in X, take the simple path $P = (x_i, \ldots, x_j)$ connecting x_i and x_j . Both x_i and x_j have at least one k-neighbor outside of P, as they each have only one neighbor in P, and $D(x_i, k) = D(x_j, k) = 2$. This means that each element of P has two dangling subtrees with thinness greater or equal to k; one in the direction of x_i , and another one in the direction of x_j . So for all $v \in P$, $D(v, k) \ge 2$. As we assumed no vertex in T has more than 2 k-neighbors, D(v, k) = 2, and so v is also in X.

The fact that every pair of vertices in X are connected by a path in X means that X must be a connected subtree of T. Furthermore, this subtree must be a path, otherwise there is a vertex $w \in X$ with three neighbors in X, all of which have D(x,k) = 2, meaning they have at least one k-neighbor different from w. Then, $D(w,k) \geq 3$, which cannot happen as w is in X.

We therefore conclude that all vertices in X must lie on some path $P=(x_1,\ldots,x_p)$. The final part of the argument lies in showing that we can apply the Path Layout Lemma. For some $x_i \in P$ with $i \in \{2,\ldots,p-1\}$, its k-neighbors are x_{i-1} and x_{i+1} . For x_1 , these neighbors are x_2 and some $x_0 \notin X$. For x_p , these neighbors are x_{p-1} and some $x_{p+1} \notin X$. Vertices x_0 and x_{p+1} only have one k-neighbor $(x_1$ and x_p respectively) or else they would be in X. If we make $P'=(x_0,\ldots,x_{p+1})$, we then see that every connected component in $T \setminus N[P']$ must have $thinness \leq k-1$. By the Path Layout Lemma, $thin(T) \leq k$.

Case 2: $X = \emptyset, Y \neq \emptyset$

We construct the path P in a simple greedy manner as follows. We start with $P = (y_1, y_2)$, where y_1 is some arbitrary vertex in Y, and y_2 its only k-neighbor. Then, if the last vertex in P has a k-neighbor $y' \notin P$, then we append y' to P, and repeat this process exhaustively. Since we look at finite

graphs, we will eventually reach some vertex y_p such that either $y_p \notin Y$ or the k-neighbor of y_p is in P. We are then done and have $P = (y_1, \ldots, y_p)$, which is a path in T by construction.

One property of P is that no vertex $y_i \in P$ can have a k-neighbor outside P. In the case of i = p, this is by construction. In the case of $i \neq p$, if y_i had a k-neighbor outside P, it would have at least two k-neighbors (the other one being y_{i+1}) which cannot happen because X is empty. This means that $T \setminus N[P]$ has no subtrees with thinness greater than k-1. By the Path Layout Lemma, thin $(T) \leq k$.

Case 3: $X = \emptyset, Y = \emptyset$

As both X and Y are empty, all vertices $t \in T$ have no k-neighbors. Taking P = (t) then gives us a path such that no subtree of $T \setminus N[P]$ has thinness greater than k - 1. By the Path Layout Lemma, thin $(T) \leq k$.

We have proven the partial claim that if $\operatorname{thin}(T) \geq k+1$ then there exists a vertex $x \in T$ such that $D(x,k) \geq 3$; or there exists a strict subtree S of T with $\operatorname{thin}(S) \geq k+1$. To finish proving the theorem we need to show that if $\operatorname{thin}(T) \geq k+1$ then there exists a vertex $x \in T$ with $D(x,k) \geq 3$. Assume that there is no vertex with k-component index at least 3 in T. By the partial claim, there must then exist a strict subtree S with $\operatorname{thin}(S) \geq k+1$. But since we look at finite trees, we know that in S there must exist a minimal subtree S_0 with thinness k+1 with no strict subtree with thinness > k. By the partial claim, S_0 must contain a vertex x_0 with $D_{S_0}(x_0,k) \geq 3$. But every dangling tree $S_0\langle v,u\rangle$ is a subtree of $T\langle v,u\rangle$, and so if $D_{S_0}(x_0,k) \geq 3$, then $D_T(x_0,k) \geq 3$ contradicting our assumption.

The thinness of a tree can be characterized in terms of the component index as follows.

Theorem 9 (Classification of Thinness of Trees). Let T be a tree, then $thin(T) \ge k + 1$ if and only if $D(x, k) \ge 3$ for some vertex x in T.

Proof. It follows straightforward from Lemmas 5 and 8. \square

Changing thin by lmimw in the definition of k-saturation, the following result follows.

Theorem 10 (Classification of Linear MIM-width of Trees). [11] Let

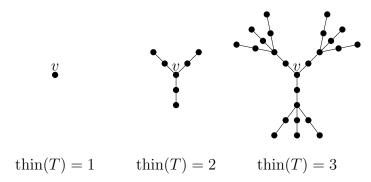
T be a tree and $k \ge 1$, then $\operatorname{lmimw}(T) \ge k+1$ if and only if $D(x,k) \ge 3$ for some vertex x in T.

We can then prove the following.

Corollary 11. For any given tree T, $thin(T) - lmimw(T) \le 1$.

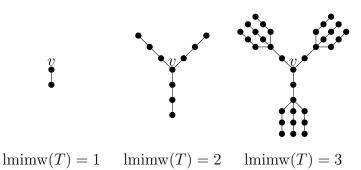
Proof. The proof follows easily by induction using Theorems 9 and 10 and considering that, for the base cases, the thinness and linear mim-width of a single edge is one, but the thinness of the trivial graph is one while the linear mim-width of the trivial graph is zero. \Box

The difference arises from the fact that every graph has thinness at least one, while edgeless graphs have linear mim-width zero. Indeed, Theorem 9 also suggests how to build the minimum trees for each thinness value:



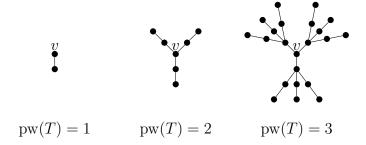
For each thinness value k, v is such that D(v, k-1) = 3. The minimum tree with thinness k can be constructed by replacing each leaf in the minimum tree with thinness 2 into the minimum tree with thinness k-1, thus achieving D(v, k) = 3 with the minimum amount of vertices.

Compare this with the minimum trees with linear mim-width 1, 2 and 3 [10]:



These are pretty similar, except that the leaves in the trees with thinness k are replaced by two vertices. This is because a theorem very similar to Theorem 9 is also true for linear mim-width [11], and the smallest tree with linear mim-width 1 is the path of two vertices, while for thinness 1 it is a single vertex. This produces slightly bigger trees than for the thinness, which corresponds with the fact that the linear mim-width is a lower bound for the thinness.

Regarding the pathwidth, instead, the minimum trees are smaller, which also corresponds with the fact that the pathwidth plus one is an upper bound for the thinness. Again, a theorem similar to Theorem 9 holds for pathwidth [8], but with subtrees instead of dangling trees.



3.3. Consequences of the main theorem

Corollary 12 (Bound on the number of vertices). The thinness of an n-vertex tree T is $\mathcal{O}(\log(n))$. In fact thin $(T) \leq \log_3(n+2)$.

Proof. Let us do an induction in the thinness of the tree.

• Base case:

For any given tree T such that thin(T) = 1, since $n \ge 1$ we know that $1 \le log_3(1+2) \le log_3(n+2)$.

• Inductive step:

Suppose the property holds for all trees such that their thinness is strictly less than a given k > 1. Then for a given tree T such that thin(T) = k, we want to prove that $thin(T) = k \le log_3(n+2)$, where n is the number of vertices in T.

Since thin(T) = k, because of Theorem 9, there must exist a vertex x in T such that $D(x, k - 1) \ge 3$. We name T_1 , T_2 and T_3 the tree disjoint subtrees of thinness equal to k - 1 such that they have their root adjacent to one of the children of x. Without loss of generality,

 T_1 has the lowest number of vertices (we call it n_1) among $\{T_1, T_2, T_3\}$. Notice this implies that $n \geq 3n_1 + 4$ (implying $n_1 \leq \frac{n-4}{3}$), because of the three subtrees, x and their tree children. From the Inductive Hypothesis we can say that $thin(T_1) \leq log_3(n_1 + 2)$. Then

$$thin(T) - 1 = thin(T_1) \le \log_3(n_1 + 2) \le \log_3(\frac{n - 4}{3} + 2)$$

Which implies that

$$thin(T) \le \log_3(\frac{n-4}{3}+2) + 1 = \log_3(3\left[\frac{n-4}{3}+2\right]) = \log_3(n+2).$$

Corollary 13 (Bound on the number of leaves). A nontrivial tree of thinness k has at least $\frac{3^{k-1}+3}{2}$ leaves. In particular, the thinness of a tree with ℓ leaves is at most $\log_3(6\ell-9)$.

Proof. Let us do an induction in the thinness of the tree.

• Base case:

A nontrivial tree of thinness 1 has at least 2 leaves. Since every path has thinness 1, every tree of thinness 2 has at least 3 leaves.

• Inductive step:

Suppose the property holds for all trees with thinness at most k > 1, and let T be a tree with $\mathrm{thin}(T) = k + 1$. By Theorem 9, there must exist a vertex x in T such that $D(x,k) \geq 3$. We name T_1, T_2 and T_3 the tree disjoint subtrees of thinness equal to k such that they have their root adjacent to one of the neighbors of x. From the Inductive Hypothesis, each of T_1, T_2, T_3 has at least $\frac{3^{k-1}+3}{2}$ leaves. Since some of them can be rooted at a leaf, we can ensure that T has at least $3\frac{3^{k-1}+3}{2}-3=\frac{3^k+3}{2}$ leaves.

Corollary 13 establishes an upper bound for the thinness of a graph in terms of the number of leaves. Let us call an *almost-leaf* a vertex which is

not a leaf that has at most one neighbor that is not a leaf. Using some ideas from [5], we can prove this other bound, useful for trees with a big number of leaves but a small number of internal vertices of degree greater than two after trimming the leaves.

Theorem 14. Let T be a tree with t almost-leaves, $t \geq 2$. Then thin $(T) \leq t - 1$.

Proof. We start by trimming all the leaves of T, obtaining a tree T'. The leaves of T' are the almost-leaves of T. We then root T' at a leaf and start a new class of the partition, containing the root. If a vertex has one child, then it is assigned to the same class as its parent. If it has more than one child, then one child is assigned to the same class as its parent, and the other children are assigned to a new class each. So, we have at most t-1 classes.

Now we order the vertices of T' by postorder (meaning, setting the children of a vertex to be all smaller than the vertex), and we add the trimmed leaves adjacent to a vertex v right before v and in its same class.

We will now show that the order and the partition are consistent. Suppose x < y < z, with x, y in the same class of the partition and $(z, x) \in E(T)$. Notice that z cannot be a leaf of T, since, by the way of defining the order, if w is a leaf then its only neighbor is greater than it. So z is a vertex of T'. If x is a leaf of T, then y is also a leaf of T adjacent to z, since we added the trimmed leaves adjacent to a vertex v right before v. If x is not a leaf in T, then z is the parent of x in T', since we have ordered T' by postorder. By the way of defining the ordering and the classes, either x and z are in the same class or x is the greatest vertex in his class, contradicting the existence of y; moreover, no vertex between x and z in the order of V(T') belongs to the same class as x, so y must be a leaf of T adjacent to z, otherwise the vertex adjacent to y in T has to be in the same class and between y and z. This completes the proof of consistency.

It was proven in [3] that for a fixed value m, the thinness of a complete m-ary tree on n vertices is $\Theta(\log(n))$, and it was also proven in [16] that the thinness of a non-trivial tree is less than or equal to its height; but, until now, it was an open problem to compute the exact thinness of a complete m-ary tree. As a consequence of Theorem 9, we have the following results.

Theorem 15 (Thinness of complete m-ary trees). Let be $m \geq 3$ and T a complete m-ary tree with height h, then $thin(T) = \lceil \frac{h+1}{2} \rceil$.

Proof. For a given $m \geq 3$ we proceed by induction on h.

• Base case:

For h = 0 and h = 1, since they are interval graphs, thin(T) = 1. Then the condition thin $(T) = \left\lceil \frac{h+1}{2} \right\rceil$ holds.

• Inductive step:

Assume the property holds for all m-ary trees of height less or equal than a given $h \ge 1$. We want to see the property also holds for h + 1.

Let T be a complete m-ary tree with height h+1. Let x be the only vertex at level 1; v_1, v_2, \ldots, v_m the level 2 vertices; and $t_{i,1}, t_{i,2}, \ldots, t_{i,m}$ the vertices adjacent to v_i at level 3. Then by the inductive hypothesis we can say that thin $(T\langle v_i, t_{i,j}\rangle) = \lceil \frac{h}{2} \rceil$.

Since there are at least three vertices v_i , each one with at least one dangling tree $T\langle v_i, t_{i,j} \rangle$, $D(x, \lceil \frac{h}{2} \rceil) \geq 3$. Due to Theorem 9, we have that $thin(T) \geq \lceil \frac{h}{2} \rceil + 1$. On the other hand, applying the Path Layout Lemma with P = (x) shows us that $thin(T) \leq \lceil \frac{h}{2} \rceil + 1$. As $\lceil \frac{h}{2} \rceil + 1 = \lceil \frac{h+2}{2} \rceil$, we have that $\lceil \frac{h+2}{2} \rceil \leq thin(T) \leq \lceil \frac{h+2}{2} \rceil$, and so the property holds for h+1.

Theorem 16 (Thinness of complete binary trees). Let T be a complete binary tree with height h, then thin $(T) = \lceil \frac{h+1}{3} \rceil$.

Proof. First we prove that thin $(T) \ge \left\lceil \frac{h+1}{3} \right\rceil$. We proceed by induction on h.

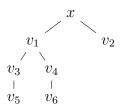
• Base case:

For $h \in \{0, 1, 2\}$, since T is an interval graph, thin $(T) = 1 \ge \lceil \frac{h+1}{3} \rceil$.

• Inductive step:

Suppose the property holds for all trees of height less of equal than h (with $h \geq 2$), we want to see that the property is true for any given tree T of height h + 1.

Let T be a complete binary tree with height h+1. We name some of the vertices of the first 4 levels as described in the following figure:



From the inductive hypothesis we know that $\operatorname{thin}(T\langle v_3,v_5\rangle) \geq \left\lceil \frac{h-2+1}{3} \right\rceil = \left\lceil \frac{h-1}{3} \right\rceil$ and $\operatorname{thin}(T\langle v_4,v_6\rangle) \geq \left\lceil \frac{h-1}{3} \right\rceil$. In addition to that, since that $T\langle v_3,v_5\rangle$ is an induced subgraph of $T\langle x,v_2\rangle$, we can also say that $\operatorname{thin}(T\langle x,v_2\rangle) \geq \operatorname{thin}(T\langle v_3,v_5\rangle) \geq \left\lceil \frac{h-1}{3} \right\rceil$.

These conditions imply that $D(v_1, \lceil \frac{h-1}{3} \rceil) \geq 3$, so due to Theorem 9 we can say that $thin(T) \geq \lceil \frac{h-1}{3} \rceil + 1 = \lceil \frac{h-1+3}{3} \rceil = \lceil \frac{h+2}{3} \rceil$. Then the property holds for h+1.

Now we show that thin $(T) \leq \left\lceil \frac{h+1}{3} \right\rceil$. By proceed by an induction on h.

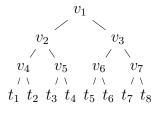
• Base case:

For $h \in \{0, 1, 2\}$, since T is an interval graph, thin(T) = 1; then the property is true since $1 \leq \left\lceil \frac{h+1}{3} \right\rceil$.

• Inductive step:

Suppose the property holds for some height $h \geq 2$ and for all heights lesser than h; we want to see that the property is also true for h + 1.

Let T be a complete binary graph with height h+1. Let us call the vertices of the first 4 levels of T as it is shown in the next figure:



Due to the inductive hypothesis we know that $\operatorname{thin}(T\langle v_i,t_j\rangle) \leq \left\lceil\frac{h-1}{3}\right\rceil$. This means that for all trees $T\langle v_i,t_j\rangle$ there exists a consistent ordering and partition of its vertices in no more than $\left\lceil\frac{h-1}{3}\right\rceil$ classes. We can then apply the Path Layout Lemma with $P=(v_2,v_1,v_3)$ to deduce that $\operatorname{thin}(T) \leq \left\lceil\frac{h-1}{3}\right\rceil + 1 = \left\lceil\frac{h-1+3}{3}\right\rceil = \left\lceil\frac{h+2}{3}\right\rceil$.

We proved
$$\left\lceil \frac{h+1}{3} \right\rceil \leq thin(T) \leq \left\lceil \frac{h+1}{3} \right\rceil$$
.

From [16] it can be shown by construction that it is always possible to have a consistent solution for a given tree with approximately $\frac{diameter}{2}$ classes. Using the theorems above, we can adjust this bound.

Theorem 17 (Bound on the diameter). Let T be a tree and d its diameter, then thin $T \leq \lceil \frac{d+1}{4} \rceil$. Moreover, if the maximum degree of a vertex in T is at most 3, then thin $T \leq \lceil \frac{d+3}{6} \rceil$.

Proof. Let m be the maximum degree among all vertices of T. If d is even, consider the complete m-ary tree T' with height $h' = \frac{d}{2}$. Since the diameter of T' is equal to 2h' = d and it is an complete m-ary tree, and every vertex of T has degree at most m, T is an induced subgraph of T'. This implies that thin(T) < thin(T'). Now from the result proven in Theorem 15 we can say that

$$thin(T') = \left\lceil \frac{h'+1}{2} \right\rceil = \left\lceil \frac{\frac{d}{2}+1}{2} \right\rceil = \left\lceil \frac{d+2}{4} \right\rceil$$

By transitivity, if d is even, $\mathrm{thin}(T) \leq \left\lceil \frac{d+2}{4} \right\rceil$. If d is odd, consider the tree T'' obtained by joining the roots of two complete (m-1)-ary trees of height $h'' = \frac{d-1}{2}$ by an edge (notice that every vertex that is not a leaf has degree m in T''). The diameter of T'' is equal to 2h'' + 1 = d. Let P be a maximum length path in T, and uv the middle edge of that path. Every leaf w of T such that u is not in the path between w and v is at distance at most h'' from v, otherwise there will be a path of length greater than d in T. Symmetrically, every leaf w of T such that v is not in the path between w and u is at distance at most h'' from u. This and the way of building T'' shows that T is an induced subgraph of T'' in this case. This implies that thin(T) < thin(T''). Now from the Path Layout Lemma applied to the path uv, and the result proven in Theorem 15, we can say that

$$thin(T'') \le \left\lceil \frac{h'' - 1}{2} \right\rceil + 1 = \left\lceil \frac{\frac{d - 1}{2} - 1}{2} \right\rceil + 1 = \left\lceil \frac{d - 3}{4} \right\rceil + 1 = \left\lceil \frac{d + 1}{4} \right\rceil$$

By transitivity, if d is odd, thin $(T) \leq \lceil \frac{d+1}{4} \rceil$. It is easy to see that we can combine the even and odd case and say that for every tree T with diameter d, thin $(T) \leq \left\lceil \frac{d+1}{4} \right\rceil$.

For $m \leq 3$ and d odd, we can use Theorem 16 instead of Theorem 15, obtaining $\operatorname{thin}(T) \leq \left\lceil \frac{d-1}{3} \right\rceil + 1 = \left\lceil \frac{d+3}{6} \right\rceil$. For d even, we make a similar construction by joining the roots of three complete binary trees to a new vertex x, and applying the Path Layout Lemma to the path x, $\operatorname{thin}(T) \leq \left\lceil \frac{d+4}{6} \right\rceil$. Again, we can combine the even and odd case and say that for every tree T with diameter d and maximum degree at most 3, $\operatorname{thin}(T) \leq \left\lceil \frac{d+3}{6} \right\rceil$. \square

3.4. The algorithm: rooted trees, k-critical vertices and labels

Our algorithm computing thinness will work on a rooted tree, processing it bottom-up. We will choose an arbitrary vertex r of the tree T and denote by T_r the tree rooted at r. During the bottom-up processing of T_r we will compute a label for various subtrees. The notion of a k-critical vertex is crucial for the definition of labels.

Definition 18 (rooted complete subtree). We define the rooted complete subtree $T_r[x]$ of T_r as the subtree of T_r rooted at x induced by x and the descendants of x.

Definition 19 (k-critical vertex). Let T_r be a rooted tree. We call a vertex x in T_r k-critical if it has exactly two children v_1 and v_2 that have at least one child each, u_1 and u_2 respectively, such that $thin(T_r[u_1]) = thin(T_r[u_2]) = k$.

Thus x is k-critical if and only if $D_{T_r[x]}(x,k) = 2$. Note that this is in the subtree rooted at x. It could be the case that x has a k-neighbor index greater than 2 in T_r , even if in $T_r[x]$ it is equal to 2.

Lemma 20. If T_r has thinness k, then it has at most one k-critical vertex.

Proof. For a contradiction, let x and x' be two k-critical vertices in T_r . There are then four vertices, v_1 , v_2 , v'_1 , v'_2 , the two k-neighbors of x and the two neighbors of x' respectively, such that there exist dangling trees $T\langle v_1, u_1 \rangle$, $T\langle v_2, u_2 \rangle$, $T\langle v'_1, u'_1 \rangle$, $T\langle v'_2, u'_2 \rangle$ that all have thinness k. If x and x' have a descendant/ancestor relationship in T_r , then assume without loss of generality that x' is v_1 or a descendant of v_1 . Let p be the direct parent of x'. Then, as $T\langle v_2, u_2 \rangle$ is a subtree of $T\langle x', p \rangle$, and $T\langle v'_1, u'_1 \rangle$ and $T\langle v'_2, u'_2 \rangle$ are disjoint trees in different neighbors of x', then $D_{T_r}(x', k) = 3$, and, by Theorem 9, T_r should have thinness greater or equal than k+1. Otherwise, all the dangling

trees are disjoint, thus $D_T(x,k) = D_T(x',k) = 3$ and we arrive to the same conclusion.

Definition 21 (label and last type). Let T_r be a rooted tree with $thin(T_r) = k$. Then $label(T_r)$ consists of a list of decreasing numbers, (a_1, \ldots, a_p) , where $a_1 = k$, and $lastType(T_r)$ is an integer between 0 and 3 which will have the information of where in the tree an a_p -critical vertex lies, if it exists at all, according to the following list. If p = 1 then we define the label as being simple, otherwise it is complex. The $label(T_r)$ and $lastType(T_r)$ are defined recursively, with type 0 being a base case for singletons and for stars, and with type 4 being the only one defining a complex label.

- **Type 0:** In this type of trees, r is a leaf, i.e. T_r is a singleton, or all children of r are leaves. $label(T_r) = (1)$ and $lastType(T_r) = 0$.
- **Type 1:** Trees of this type are not Type 0 trees, and have no k-critical vertex. $label(T_r) = (k)$ and $lastType(T_r) = 1$.
- **Type 2:** r is the k-critical vertex of trees of this type. $label(T_r) = (k)$ and $lastType(T_r) = 2$.
- **Type 3:** In these trees a child of r is k-critical. $label(T_r) = (k)$ and $lastType(T_r) = 3$.
- **Type 4:** There is a k-critical vertex u_k in T_r that is neither r nor a child of r. Let w be the parent of u_k . Then $label(T_r) = k \oplus label(T_r \setminus T_r[w])$, and $lastType(T_r) = lastType(T_r \setminus T_r[w])$.

In type 4 we note that $\operatorname{thin}(T_r \setminus T_r[w]) < k$ since otherwise u_k would have three k-neighbors (two children in the tree and also its parent) and by Theorem 9 we would then have $\operatorname{thin}(T_r) = k + 1$. Therefore, all numbers in $label(T_r \setminus T_r[w])$ are smaller than k and a complex label is a list of decreasing numbers. We also note that each element of a complex label corresponds to the thinness of some subtree of T_r , with the first element being the thinness of T_r . We now give a proposition that for any vertex x in T_r will be used to compute $label(T_r[x])$ and $lastType(T_r[x])$ based on the labels of the subtrees rooted at the children and grandchildren of x. The subroutine underlying this proposition will be used when reaching vertex x in the bottom-up processing of T_r .

Proposition 22. Let x be a vertex of T_r with children Child(x), and assume we are given $label(T_r[v])$ and $lastType(T_r[v])$ for all $v \in Child(x)$. Define $k = \max_{v \in Child(x)} \{ thin(T_r[v]) \}$, meaning, the maximum thinness of a subtree rooted on a child of x. Also define $N_k = \{v \in Child(x) \mid thin(T_r[v]) = k\}$, meaning, the set of children for whom the subtrees rooted at them have thinness k. Denote $N_k = \{v_1, \ldots, v_q\}$, $l_i = label(T_r[v_i])$, and $t_i = lastType(T_r[v_i])$. Define $d_k = D_{T_r[x]}(x, k)$ by noting that $d_k = |\{v_i \in N_k \mid v_i \text{ has child } u_j \text{ with } thin(T_r[u_j]) = k\}|$. Given this information, we can find $label(T_r[x])$ and $lastType(T_r[x])$ as follows.

- Case 0: x is a leaf or all children of x are leaves, and then label($T_r[x]$) = (1) and $lastType(T_r[x]) = 0$.
- Case 1: x is not a leaf and not all children of x are leaves, and for every $v_i \in N_k$, l_i is simple and t_i is equal to 1 or 0, and $d_k \leq 1$. Then, $label(T_r[x]) = (k)$, and $lastType(T_r[x]) = 1$.
- Case 2: For every $v_i \in N_k$, l_i is simple and t_i is equal to 1 or 0, but $d_k = 2$. Then, $label(T_r[x]) = (k)$ and $lastType(T_r[x]) = 2$.
- Case 3: For every $v_i \in N_k$, l_i is simple and t_i is equal to 1 or 0, but $d_k \geq 3$. Then, $label(T_r[x]) = (k+1)$, and $lastType(T_r[x]) = 1$.
- Case 4: $|N_k| \ge 2$ and for some $v_i \in N_k$, either l_i is a complex label, or t_i is equal to either 2 or 3. Then, $label(T_r[x]) = (k+1)$, and $label(T_r[x]) = 1$.
- Case 5: $|N_k| = 1$, l_1 is a simple label and t_1 is equal to 2. Then, $label(T_r[x]) = (k)$, and $lastType(T_r[x]) = 3$.
- Case 6: $|N_k| = 1$, l_1 is either complex or l_1 is equal to 3, and $k \notin label(T_r[x] \setminus T_r[w])$, where w is the parent of the k-critical vertex in $T_r[v_1]$. Then, $label(T_r[x]) = k \oplus label(T_r[x] \setminus T_r[w])$, and $lastType(T_r[x]) = lastType(T_r[x] \setminus T_r[w])$.
- Case 7: $|N_k| = 1$, l_1 is either complex or t_1 is equal to 3, and $k \in label(T_r[x] \setminus T_r[w])$, where w is the parent of the k-critical vertex in $T_r[v_1]$. Then, $label(T_r[x]) = k + 1$, and $lastType(T_r[x]) = 1$.

Proof. We will assume for the time being that this Proposition covers all possible trees, as we will prove later. Now, we will prove that each case assigns values to label and lastType according to Definition 21. We will use k

as it was used in the Proposition, meaning to signify the maximum thinness of a child subtree of x.

We will start with the following lemma, which will make proving each case easier.

Lemma 23. Let T_r be a rooted tree on r, and x be a vertex of T_r . Let k the maximum thinness of a child subtree of x as in Proposition 22. If no descendant of x is k-critical, then no descendant v of x has $D_{T_r[x]}(v,k) \geq 3$.

Proof. For there to be some v such that $D(v,k) \geq 3$, v should have at least 3 k-neighbors. As v has no more than one parent, at least 2 of those k-neighbors must be children of v. And v cannot have more than 2 children k-neighbors, as this would mean that $D_{T_r[v]}(v,k) \geq 3$. This would then mean that thin $(T_r[v]) \geq k+1$. But then some child subtree of x has thinness greater or equal than k+1, which cannot happen by the definition of k. So v has exactly 2 k-children, and so is a k-critical vertex. But this also cannot happen by the statement, which means that v cannot exist. $\diamondsuit \square$

Now, we go case by case, proving that the assignments are correct.

- Case 0: As x is a leaf or all children of x are leaves, $T_r[x]$ is a type 0 tree. As the assignments in Case 0 are the same as the assignments for type 0 trees, they are correct.
- Case 1: We must prove two things: that thin $(T_r[x]) = k$, to show that $label(T_r[x])$ is being assigned the correct value; and that $T_r[x]$ is a type 1 tree, to show that $lastType(T_r[x])$ is being assigned the correct value and that $T_r[x]$ has a simple label.
 - thin $(T_r[x]) = k$: All children subtrees of x which have thinness k are type 0 or 1 trees, meaning that they do not have any k-critical vertex. As seen in Lemma 23, this means that no descendant v of x has $D_{T_r[x]}(v,k) \geq 3$. On the other hand, as $d_k \leq 1$, $D_{T_r[x]}(x,k) = 1$, and so there is no vertex $u \in T_r[x]$ such that $D_{T_r[x]}(u,k) \geq 3$. By Theorem 9, thin $(T_r[x]) \leq k$. Given that there is at least one child v of x such that thin $(T_r[v]) = k$, thin $(T_r[x]) \geq k$, and so thin $(T_r[x]) = k$.
 - $-T_r[x]$ is a type 1 tree: First of all, $T_r[x]$ is not a type 0 tree, as x is not a leaf and not all children of x are leaves. We must now

show that it has no k-critical vertex. We know that there are no k-critical vertices in any child subtree of x, so we must only show that x is not a k-critical vertex, and this is given by the fact that $d_k \leq 1$.

By Definition 21, this tree should have $label(T_r[x]) = (k)$ and $lastType(T_r[x]) = 1$, which are the values that the Proposition assigns. In the following cases we will do the same as in this one: prove that the thinness and the type of $T_r[x]$ are correctly determined to show that label and lastType are assigned the correct values.

• Case 2:

- thin $(T_r[x]) = k$: Again by Lemma 23, x has no descendant k-saturated in $T_r[x]$. And, as $d_k = 2$, $D_{T_r[x]}(x,k) = 2$, and so x is not k-saturated in $T_r[x]$. As there is no vertex k-saturated in $T_r[x]$, and as in Case 1, thin $(T_r[x]) \geq k$, thin $(T_r[x]) = k$.
- $T_r[x]$ is a type 2 tree: As $d_k = 2$, x is a k-critical vertex. There are no other k-critical vertices in $T_r[x]$, and so this is a type 2 tree.

• Case 3:

- thin $(T_r[x]) = k + 1$: $d_k \geq 3$, which means that x is a k-saturated vertex in $T_r[x]$. So we have thin $(T_r[x]) \geq k + 1$. By Lemma 23, no vertex v other that x is k-saturated in $T_r[x]$, which in turn means that v is not k + 1-saturated. As there is no child subtree of x with thinness greater than k, x is also not k + 1-saturated, and so thin $(T_r[x]) \leq k + 1$ by Theorem 9. This means that thin $(T_r[x]) = k + 1$.
- $T_r[x]$ is a type 1 tree: As $d_k \geq 3$, x has at least three grandchildren, which means that it is not a type 0 tree. We must then show that there is no k+1-critical vertex in $T_r[x]$. As no strict subtree of $T_r[x]$ has thinness equal to k+1, there cannot be a vertex with two k+1-neighbors in $T_r[x]$, and so by definition there is no k+1-critical vertex in $T_r[x]$.

• Case 4:

- thin $(T_r[x]) = k + 1$: There is some $v_i \in N_k$ for which either l_i is a complex label, meaning that $T_r[v_i]$ is a type 4 tree; or t_i is either

2 or 3, meaning that $T_r[v_i]$ is a type 2 or 3 tree. In both cases we know that $T_r[v_i]$ has a k-critical vertex. Let u be the k-critical vertex in $T_r[v_i]$, and let p be the parent of u in $T_r[x]$. As $|N_k| \geq 2$, there exists a vertex $v_j \in N_k$, $v_j \neq v_i$. As thin $(T_r[v_j]) = k$ and $T_r[v_j]$ is a subtree of $T_r[x]\langle u, p \rangle$, thin $(T_r[x]\langle u, p \rangle) \geq k$. This, compounded with the fact that u is k-critical, means that u is k-saturated in $T_r[x]$, and so thin $(T_r[x]) \geq k + 1$.

As no strict subtree of $T_r[x]$ has thinness greater than k, we know that there is no descendant k+1-critical vertex, and so by Lemma 23, no descendant of x is k+1-saturated. For the same reason, x is not k+1-saturated, as it has no k+1-neighbors in $T_r[x]$. This means that thin $(T_r[x]) \leq k+1$. With both results we arrive at the conclusion that thin $(T_r[x]) = k+1$.

 $-T_r[x]$ is a type 1 tree: As seen earlier, $T_r[x]$ has no k+1-critical vertex. Also, it is not a type 0 tree, as it contains some k-critical vertex in some of its subtrees. We see then that the definition of type 1 trees matches.

• Case 5:

- thin $(T_r[x]) = k$: x has only one child v such that thin $(T_r[v]) = k$. Also, $T_r[v]$ is a type 2 tree, and so v is its only k-critical vertex. All other vertices in $T_r[v]$ have less than two children k-neighbors. Also, there is no other k-critical vertex in $T_r[x]$, as all other children subtrees of x have thinness lower than k, and x has only one child subtree with thinness k. As shown in the proof of Lemma 23, if a vertex u is k-saturated, it must have at least two child k-neighbors. This means that the only vertex that could be k-saturated in $T_r[x]$ is v. But for this to be true, x should be a k-neighbor of v, which means that there should be another neighbor u of x for which thin $(T_r[u]) = k$. As v is the only vertex in N_k , this cannot happen, and so there is no k-saturated vertex in $T_r[x]$. This gives us thin $(T_r[x]) \le k$. As there is some subtree of $T_r[x]$ with thinness equal to k, thin $(T_r[x]) \ge k$, and so thin $(T_r[x]) = k$.
- $-T_r[x]$ is a type 3 tree: In this case, v, which is a child of x, is k-critical, and so this tree matches the definition of a type 3 tree.

• Case 6:

- thin $(T_r[x]) = k$: We start by noting that thin $(T_r[x]) \geq k$, as it contains at least one subtree with thinness k, namely $T_r[v_1]$.

As $T_r[v_1]$ is a type 3 or 4 tree, it has a k-critical vertex $u \neq v_1$. With w being the parent of u, we know that $k \notin label(T_r[x] \setminus T_r[w])$, and so thin $(T_r[x] \setminus T_r[w]) \neq k$. Also, we know that there is no vertex in $T_r[x]$ other than u with two or more child k-neighbors. This is because, by Lemma 20, u is the only k-critical vertex in $T_r[v_1]$, and there are no other child subtrees of x with thinness greater or equal to k, which means that x has no more than one k-neighbor, and that there are no vertices in other subtrees which have child k-neighbors. This means that there is no vertex in $T_r[x] \setminus T_r[w]$ which has more than one child k-neighbor, which in turn means that there is no k-saturated vertex in $T_r[x] \setminus T_r[w]$. So, thin $(T_r[x] \setminus T_r[w]) \leq k$. This leaves us with thin $(T_r[x] \setminus T_r[w]) < k$, as it must be different from k as seen earlier.

As seen in the proof of Lemma 23, the only possible k-saturated vertex in $T_r[x]$ is the one with at least two children k-neighbors, meaning u. But as thin $(T_r[x] \setminus T_r[w]) < k$, w is not a k-neighbor of u, which tells us that u has only two k-neighbors and so is not k-saturated. We arrive at the conclusion that thin $(T_r[x]) \le k$, and so thin $(T_r[x]) = k$.

 $-T_r[x]$ is a type 4 tree: The k-critical vertex in $T_r[x]$ is u, which is a descendant of v_1 , and so is neither x nor a child of x. This is the definition of a type 4 tree.

To finish this case, we note that the label and the lastType of $T_r[x]$ are assigned exactly as written in the definition of a type 4 tree.

• Case 7:

- thin $(T_r[x]) = k + 1$: As $T_r[v_1]$ is a type 3 or 4 tree, it has a k-critical vertex $u \neq v_1$. Also, as $k \in label(T_r[x] \setminus T_r[w])$, thin $(T_r[x] \setminus T_r[w]) \geq k$. This means that u is k-saturated, and so thin $(T_r[x]) \geq k + 1$.

Let us see that there is no k + 1-saturated vertex in $T_r[x]$ to see that thin $(T_r[x]) = k + 1$. For there to be a k + 1-saturated vertex y, it must have at least two child k + 1-neighbors, which means that there must be two vertices z_1 and z_2 such that thin $(T_r[z_1]) =$

- thin $(T_r[z_2]) = k + 1$, with $T_r[z_1]$ and $T_r[z_2]$ disjoint subtrees. As k is the maximum thinness of a subtree of $T_r[x]$, this cannot happen.
- $-T_r[x]$ is a type 1 tree: First we note that $T_r[x]$ is not a type 0 tree, as it has at least one k-neighbor. We also note that, as seen earlier, $T_r[x]$ has no k+1-critical vertices, because no vertex has two child k+1-neighbors. This then follows the definition of a type 1 tree.

We have gone through each case of the Proposition showing that the assignments of labels and lastTypes correspond to the tree types in Definition 21. Now we will continue by showing that every possible tree is covered by exactly one of these eight cases. Observe the decision tree in Figure 1. We will show that cases of Proposition 22 correspond to cases in the decision tree, and with that prove that exactly one case applies to every rooted tree. In the following case analysis, k represents the maximum thinness of a child subtree of $T_r[x]$, as in the Proposition.

- Case 0: This case is reached if and only if x is a leaf or all its children are leaves, and so corresponds to Case 0 in Proposition 22.
- Case 1: Firstly, this case is not reached if x is a leaf or all its children are leaves, which matches the first condition of Case 1. Secondly, it is reached if and only if $T_r[x]$ has no child subtree with a k-critical vertex, and $D_{T_r[x]}(x,k) \leq 1$. We know $T_r[v_i]$ does not have a k-critical vertex if and only if it has thinness lower than k or is a type 0 or type 1 tree by Definition 21, which is the second condition of Case 1. And the third condition, $d_k \leq 1$, is given by the fact that $D_{T_r[x]}(x,k) \leq 1$.
- Case 2 and 3: This proofs are very similar as the one for Case 1, and so they are omitted.
- Case 4: In this case of the Proposition, the condition dictates that some child subtree $T_r[v_i]$ with thinness k must be of type 2, 3, or 4. This happens if and only if $T_r[v_i]$ has a k-critical vertex, which is one of the conditions checked in the decision tree. The following condition is true if and only if $|N_k| \geq 2$, which is the other condition in the Proposition. Finally, given that in a Case 4 tree there is at least one k-critical vertex, x has at least one grandchild, and so the statement "Is x a leaf or all children of x are leaves?" is not true. This last remark

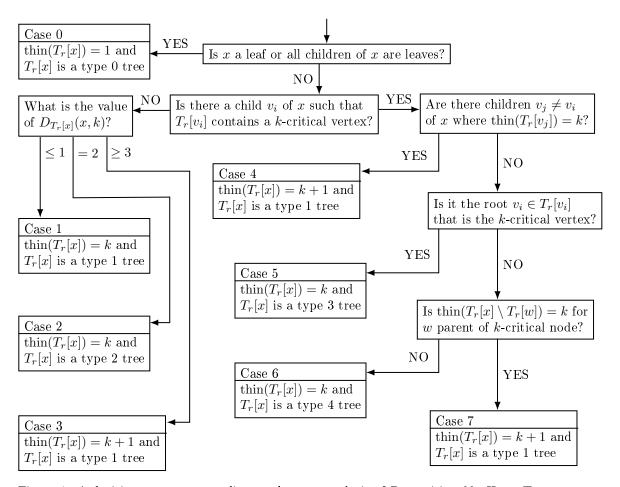


Figure 1: A decision tree corresponding to the case analysis of Proposition 22. Here, T_r and k are defined as in the Proposition.

will be applied to all following cases, because all of their corresponding trees will contain at least one k-critical vertex.

- Case 5: $|N_k| = 1$ if and only if there are no two vertices $v_i \neq v_j$ such that $\text{thin}(T_r[v_i]) = \text{thin}(T_r[v_j]) = k$, and so this two conditions correspond to each other. And $t_1 = 2$ if and only if $T_r[v_1]$ is a type 2 tree, which means that v_1 is the k-critical vertex in $T_r[v_1]$, which in turn is the last condition in the decision tree before reaching Case 5. Of course, Case 5 trees do have a k-critical vertex, because v_1 is a k-critical vertex, so that condition in the decision tree is correct as well.
- Case 6 and 7: The only difference between these cases is if k is either included or not in $label(T_r[x] \setminus T_r[w])$, which is the last condition in the decision tree. In both cases $|N_k| = 1$, which as we have seen is already checked in the decision tree, and also $T_r[v_1]$ is a type 3 or 4 tree, which happens if and only if $T_r[v_1]$ has a k-critical vertex different from v_1 . This is the contrary to the condition to enter Case 5 in the decision tree. Lastly, $|N_k| = 1$, which as seen is also checked in the decision tree.

This proves that every possible tree is covered by the cases in Proposition 22. Combined with the fact that each case assigns the label and lastType following Definition 21, we have shown that the thinness of every possible tree can be calculated as the first element of its label following this case analysis.

3.5. Computing thinness of trees and finding a consistent solution

The subroutine underlying Proposition 22 will be used in a bottom-up algorithm that starts out at the leaves and works its way up to the root, computing labels and lastTypes of subtrees $T_r[x]$. However, in cases 6 and 7 we need the label and lastType of $T_r[x] \setminus T_r[w]$, which is not a complete subtree rooted at any vertex of T_r . Note that the label and lastType of $T_r[x] \setminus T_r[w]$ are again given by a recursive call to Proposition 22, and then the label is stored as a suffix of the complex label of $T_r[x]$, and the lastType is the same. We will compute these labels and lastTypes by iteratively calling Proposition 22, substituting the recursion by iteration. We first need to carefully define the subtrees involved when dealing with complex labels.

From the definition of labels it is clear that only type 4 trees lead to a complex label. In that case we have a tree $T_r[x]$ of thinness k and a k-critical vertex u_k that is neither x nor a child of x, and the recursive definition gives $label(T_r[x]) = k \oplus label(T_r[x] \setminus T_r[w])$ for w the parent of u_k . Unravelling this recursive definition, we have the following:

Definition 24. Let x be a vertex in T_r , and let $h = |label(T_r[x])|$. Denote $label(T_r[x]) = (a_1, \ldots, a_h)$. Then $\omega(T_r[x])$ is a list $(\omega_1, \ldots, \omega_h)$ of vertices in $T_r[x]$ in which $\omega_h = x$, and every other ω_i with $1 \leq i < h$ is the parent of the a_i -critical vertex in $T_r[x] \setminus (T_r[\omega_1] \cup \cdots \cup T_r[\omega_{i-1}])$. We will use $\omega(T_r[x])_i$ to denote the element number i of the list, or simply use ω_i when it is clear which tree we are referring to.

Now, in the first level of a recursive call to Proposition 22 the role of $T_r[x]$ is taken by $T_r[x] \setminus T_r[\omega_1]$, and in the next level it is taken by $(T_r[x] \setminus T_r[\omega_1]) \setminus T_r[\omega_2]$ etc. The following definition gives a shorthand for denoting these trees.

Definition 25. Let x be a vertex in T_r , and label $(T_r[x]) = (a_1, a_2, \ldots, a_p)$. For any non-negative integer s, the tree $T_r[x, s]$ is the subtree of $T_r[x]$ obtained by removing all trees $T_r[\omega_i]$ from $T_r[x]$, where $a_i \geq s$. In other words, if q is such that $a_q \geq s > a_{q+1}$, then $T_r[x, s] = T_r[x] \setminus (T_r[\omega_1] \cup T_r[\omega_2] \cup \cdots \cup T_r[\omega_q])$.

To ease the following proofs, we will define the following list of vertices.

Definition 26. Let x be a vertex in T_r . Let $h = |label(T_r[x])|$. Denote $label(T_r[x]) = (a_1, \ldots, a_h)$. Then $\varphi(T_r[x])$ is the list of vertices $(\varphi_1, \ldots, \varphi_{h-1})$, where $\varphi_i \in T_r[x]$ is the a_i -critical vertex in $T_r[x]$. We will use $\varphi(T_r[x])_i$ to denote the element number i of the list, or simply use φ_i when it is clear which tree we are referring to.

Note that each ω_i is the parent of the corresponding φ_i , except when $i = |label(T_r[x])|$, in which case there is no φ_i .

Lemma 27. Some important properties of $T_r[x, s]$ are the following. Let $T_r[x, s]$, label $(T_r[x, s])$, and q be as in the definition. Then

- 1. if $s > a_1$, then $T_r[x, s] = T_r[x]$
- 2. $label(T_r[x, s]) = (a_{q+1}, \dots, a_p)$

- 3. $thin(T_r[x, s]) = a_{q+1} < s$
- 4. $thin(T_r[x, s+1]) = s$ if and only if $s \in label(T_r[x])$
- 5. $T_r[x, s+1] \neq T_r[x, s]$ if and only if $s \in label(T_r[x])$

 ${\it Proof.}$ The first ones follow from the definitions. We will show a proof for the last one:

Backward direction: Let $s = a_q$ for some $1 \le q \le p$. Then $T_r[x, s+1] = T_r[x] \setminus (T_r[\omega_1] \cup \cdots \cup T_r[\omega_{q-1}])$ and $T_r[x, s] = T_r[x] \setminus (T_r[\omega_1] \cup \cdots \cup T_r[\omega_q])$. These two trees are clearly different.

Forward direction: Let $T_r[x, s] = T_r[x] \setminus (T_r[\omega_1] \setminus \cdots \setminus T_r[\omega_q])$ and $T_r[x, s+1] = T_r[x] \setminus (T_r[\omega_1] \cup \cdots \cup T_r[\omega_{q_0}])$ with $q_0 < q$ and $q_0 > a_q$ (because numbers in a label are strictly descending). $a_q < s+1$ and $a_q \ge s$, ergo $a_q = s$.

Lemma 28. Let $x \in V(T_r)$, and let u be a child of x in T_r . Let $s \in \mathbb{N}$ such that $T_r[x,s]$ and $T_r[u,s]$ are not empty, that is to say, s is greater both than the last element of label($T_r[x]$) and than the last element of label($T_r[u]$). Let $T_s^* = T_r[x,s] \cap T_r[u]$. Then $T_s^* = T_r[u,s]$, meaning, the child subtree of $T_r[x,s]$ rooted at u is equal to $T_r[u,s]$.

Proof. We will do induction on s, and prove that $T_s^* = T_r[u, s]$. Let $a_1 = label(T_r[x])_1$.

• Base case, $s > a_1$: In this case, $T_r[x, s] = T_r[x]$. Note that $label(T_r[x])_1 \ge label(T_r[u])_1$, because $T_r[u]$ is a subtree of $T_r[x]$ and so its thinness is smaller or equal to thin $(T_r[x])$. So $T_r[u, s] = T_r[u]$. Then

$$T_s^* = T_r[x, s] \cap T_r[u] = T_r[x] \cap T_r[u] = T_r[u] = T_r[u, s]$$

- Inductive step, $s \leq a_1$: We will assume the statement holds for s+1, meaning that $T_{s+1}^* = T_r[u, s+1]$. Note that this implies that $T_r[u, s+1]$ is a child subtree of $T_r[x, s+1]$. We have four possible cases here:
 - 1. $s \notin label(T_r[x])$, $s \notin label(T_r[u])$: This means by Lemma 27 that $T_r[x,s] = T_r[x,s+1]$, and that $T_r[u,s] = T_r[u,s+1]$. This in turn shows that

$$T_s^* = T_r[x, s] \cap T_r[u] = T_r[x, s+1] \cap T_r[u] = T_{s+1}^* = T_r[u, s+1] = T_r[u, s]$$

- 2. $s \in label(T_r[x])$, $s \notin label(T_r[u])$: This means that $T_r[u, s+1]$ has thinness smaller than s, while $T_r[x, s+1]$ has thinness s. This also means that $T_r[u, s] = T_r[u, s+1]$, which gives us $T_{s+1}^* = T_r[u, s]$. Let i be the position of s in $label(T_r[x])$. As s is not the last element of $label(T_r[x])$, $v = \varphi(T_r[x])_i$ exists as a vertex in $T_r[x]$. The fact that $T_r[u, s+1]$ has thinness smaller than s tells us in particular that it has no s-critical vertices. Also, by inductive hypothesis, $T_r[u, s+1]$ is a subtree of $T_r[x, s+1]$. These two facts together show that v, which is an s-critical vertex in $T_r[x, s+1]$, cannot be in $T_r[u, s+1]$. As v is outside $T_r[u, s+1] = T_r[u, s]$, also $w = \omega(T_r[x])_i$ is. As $T_r[x, s]$ is not empty, w is not x but instead is a descendant of x in $T_r[x, s+1]$. The only vertices removed from $T_r[x, s+1]$ to get $T_r[x, s]$ are w and its descendants, which are not descendants of u and so are not in T_{s+1}^* . We conclude then that $T_s^* = T_{s+1}^* = T_r[u, s]$.
- 3. $s \notin label(T_r[x])$, $s \in label(T_r[u])$: This means that $T_r[u, s+1]$ has thinness s, while $T_r[x, s+1]$ has thinness smaller than s. But this cannot happen, as by inductive hypothesis $T_r[u, s+1]$ is a subtree of $T_r[x, s+1]$, and so must have thinness smaller or equal to that of $T_r[x, s+1]$. This case then is nonexistent.
- 4. $s \in label(T_r[x])$, $s \in label(T_r[u])$: As s is not the last element of $label(T_r[x])$, $label(T_r[x,s+1])$ has at least two elements, and so $T_r[x,s+1]$ is a type 4 tree. This means that there is an s-critical vertex $v \in \varphi(T_r[x])$ in $T_r[x,s+1]$. By a similar argument, there is also an s-critical vertex $v' \in \varphi(T_r[u])$ in $T_r[u,s+1]$. By Lemma 20 there can only be one s-critical vertex in $T_r[x,s+1]$ and in $T_r[u,s+1]$, because their thinness is s. By the inductive hypothesis, $T_r[u,s+1]$ is a subtree of $T_r[x,s+1]$, and so their s-critical vertices are the same, in other words, v' = v. As both s-critical vertices are the same, the parents are also the same, and so the subtree U removed when going from $T_r[x,s+1]$ to $T_r[x,s]$ is the same as the one removed when going from $T_r[u,s+1]$ to

 $T_r[u,s]$. So we have

$$T_r[u, s] = T_r[u, s + 1] \setminus U$$

$$= T_{s+1}^* \setminus U$$

$$= T_r[x, s + 1] \cap T_r[u] \setminus U$$

$$= T_r[x, s] \cap T_r[u]$$

$$= T_s^*$$

Corollary 29. If $s \in label(T_r[x])$ and $s \in label(T_r[u])$, $T_{s+1}^* = T_r[u, s+1]$.

Proof. That s be both in $label(T_r[x])$ and in $label(T_r[u])$ means that s+1 is bigger than the last element of both labels, and so the conditions for the Lemma are satisfied for s+1.

Note that, for any s, the tree $T_r[x,s]$ is defined only after we know $label(T_r[x])$. In the algorithm, we compute $label(T_r[x])$ by iterating over increasing values of s until $s > thin(T_r[x])$ since by Lemma 27.1 we then have $T_r[x,s] = T_r[x]$. This poses a problem: we cannot know which subtrees to calculate the labels for until we have finished with all subtrees. To solve this, each iteration of the loop will correctly compute the label of another subtree called $T_{union}[x,s]$, which is not always equal to $T_r[x]$. Nonetheless, for $s > thin(T_r[x])$, the equality $T_{union}[x,s] = T_r[x,s] = T_r[x]$ will hold, and so calculating labels for these subtrees will aid in calculating labels for bigger subtrees.

Definition 30. Let x be a vertex in T_r with children v_1, \ldots, v_d . $T_{union}[x, s]$ is then equal to the tree induced by x and the union of all $T_r[v_i, s]$ for $1 \le i \le d$. More technically, $T_{union}[x, s] = T_r[V']$ where $V' = x \cup V(T_r[v_1, s]) \cup \cdots \cup V(T_r[v_d, s])$.

Given a tree T, we find its thinness by rooting it at an arbitrary vertex r, and computing labels by processing T_r bottom-up. The answer is given by the first element of $label(T_r[r])$, which by definition is equal to thin(T). At a vertex x of T_r which is a leaf or all their children are leaves we initialize by $label(T_r[x]) \leftarrow (1)$, according to Definition 21. When reaching a higher vertex x we compute the label of $T_r[x]$ by calling function MAKELABEL (T_r, x) .

```
Algorithm 2 Compute cur\_label = label(T_r[x]) and cur\_type =
lastType(T_r[x])
   function MakeLabel(T_r: tree, x: vertex)
       cur\ label \leftarrow (1)
       cur\_type \leftarrow 1
        \{v_1, \dots, v_d\} \leftarrow \text{children of } x \text{ in } T_r
       for s \leftarrow 1, \max_{i=1}^d \{ \text{ first element of } label(T_r[v_i]) \} do
            \{l'_1, \dots, l'_d\} \leftarrow \{label(T_r[v_i, s+1]) \mid 1 \le i \le d\}
            \{t'_1, \dots, t'_d\} \leftarrow \{lastType(T_r[v_i, s+1]) \mid 1 \le i \le d\}
            N_s \leftarrow \{v_i \mid 1 \le i \le d, s \in l_i'\}
            d_s \leftarrow |\{v_i \mid v_i \in N_s, v_i \text{ has a child } u_j \text{ s.t. } s \in label(T_r[u_j, s+1])\}|
            if N_s \neq \emptyset then
                              \leftarrow
                                                        from Prop.
                 case
                                   _{
m the}
                                             case
                                                                               22
                                                                                      applying
   s, \{l'_1, \ldots, l'_d\}, \{t'_1, \ldots, t'_d\}, N_s \text{ and } d_s
                 cur\_label \leftarrow as given by case in Prop. 22 (s \oplus cur\_label) if Case
   6)
                 cur type \leftarrow as given by case in Prop. 22
            end if
       end for
   end function
```

Lemma 31. Given labels at descendants of vertex x in T_r , MAKELABEL (T_r, x) computes label $(T_r[x])$ as the value of cur_label and $lastType(T_r[x])$ as the value of cur_type .

Proof. Assume that x has children v_1, \ldots, v_d , and denote their set of labels as $L = \{l_1, \ldots, l_d\}$. MAKELABEL keeps variables cur_label and cur_type that are updated maximally k times in a for loop, where k is the biggest number in any label of children of x. The following claim will suffice to prove the lemma, since for $s > thin(T_r[x])$, we have $T_{union}[x, s] = T_r[x]$.

Claim: At the end of the iteration number s of the for loop the value of cur_label is equal to $label(T_{union}[x, s + 1])$, and cur_type is equal to $lastType(T_{union}[x, s + 1])$.

- Base case: We have to show that before the first iteration of the loop we have $cur_label = label(T_{union}[x, 1])$ and $cur_type = lastType(T_{union}[x, 1])$. As no $l_i \in L$ has 0 as an element by Definition 21, then $T_{union}[x, 1]$ is by definition the singleton vertex x and by Proposition 22 the label of this tree is (1), which is what cur_label is initialized to, and its last type is 1, which is what cur_type is initialized to.
- Inductive step: We assume $cur_label = label(T_{union}[x, s])$ at the start of iteration number s of the for loop and show that at the end of the iteration $cur_label = label(T_{union}[x, s + 1])$.

The first thing done in the for loop is the computation of $\{l'_i \mid 1 \leq i \leq d, l'_i = label(T_r[v_i, s+1])\}$. By Lemma 27.2, $label(T_r[v_i, s+1])$ is contained in $label(T_r[v_i])$ for all i, therefore l'_1, \ldots, l'_d are trivial to compute. Next, the $\{t'_1, \ldots, t'_d\}$ are calculated, following the same strategy as for the labels. Afterwards N_s is set as the set of all children of x whose labels contain s, and d_s as the number of vertices in N_s that themselves have children whose labels contain s. Let us first look at what happens when $N_s = \emptyset$:

By Lemma 27.5, for every child v_i of x, $T_r[v_i, s+1] = T_r[v_i, s]$ if $s \notin label(T_r[v_i])$. Therefore, if N_s is empty, then $T_{union}[x, s+1] = T_{union}[x, s]$, and from the inductive hypothesis, $label(T_{union}[x, s+1]) = cur_label$, and indeed when $N_s = \emptyset$ then iteration s of the loop does not alter cur_label .

Otherwise, we have $|N_s| > 0$ and make a call to the subroutine given by Proposition 22 to compute $label(T_{union}[x, s + 1])$ and argue first

Proposition 1	for loop iteration s	Explanation
$T_r[x]$	$T_{union}[x,s+1]$	Tree needing label
k	s	Max thinness of children
$T_r[v_1],\ldots,T_r[v_d]$	$T_r[v_i, s], \ldots, T_r[v_d, s]$	Subtrees of children
l_1,\ldots,l_d	l'_1,\ldots,l'_d	Child labels
t_1,\ldots,t_d	t_1',\ldots,t_d'	Child lastTypes
N_k	N_s	Children with max thinness
d_k	d_s	Root k -component index
$label(T_r[x] \setminus T_r[w])$	cur_label	This is also $label(T_{union}[x, s+1] \setminus T_r[w])$

Table 1: Correspondence between values computed in Proposition 22 and variables in MAKELABEL.

that the variables used in that call correspond to the variables used in Proposition 22 to compute $label(T_r[x])$. The correspondence is given in Table 1. Most of these are just observations: $T_{union}[x, s+1]$ corresponds to $T_r[x]$ in Proposition 22, and $T_r[v_1, s+1], \ldots, T_r[v_d, s+1]$ corresponds to $T_r[v_1], \ldots, T_r[v_d]$. $\{l'_i \mid 1 \leq i \leq d, l'_i = label(T_r[v_i, s+1])\}$ correspond to $\{label(T_r[v]) | v \in Child\}$ in Proposition 22, and similarly for the t'_i with the t_i . N_s is defined in the algorithm so that it corresponds to N_k in Proposition 22. Since $|N_s| > 0$, some v_i has s in its label l'_i . By Lemma 27.3 and 27.4, we can infer that s is the maximum thinness of all $T_r[v_i, s+1]$, therefore s corresponds to k in Proposition 22.

It takes a bit more effort to show that d_s computed in iteration s of the for loop corresponds to $d_k = D_{T_r[x]}(x,k)$ in Proposition 22 – meaning we need to show that $d_s = D_{T_{union}[x,s+1]}(x,s)$. Consider v_i , a child of x. In accordance with MAKELABEL we say that v_i contributes to d_s if $v_i \in N_s$ and v_i has a child u_j with s in its label. We thus need to show that v_i contributes to d_s if and only if v_i is an s-neighbor of x in $T_{union}[x,s+1]$. Observe that by Lemma 27.4, thin $(T_r[v_i,s+1]) =$ thin $(T_r[u_j,s+1]) = s$ if and only if s is in the labels of both $T_r[v_i]$ and $T_r[u_j]$. If $s \notin label(T_r[u_j,s+1])$, then thin $(T_r[u_j,s+1]) < s$, and if this is true for all children of v_i , then v_i is not an s-neighbor of x in $T_{union}[x,s+1]$. If $s \notin label(T_r[v_i,s+1])$, then thin $(T_r[v_i,s+1]) < s$ and no subtree of $T_r[v_i,s+1]$ can have thinness s. However, if $s \in label(T_r[u_j,s+1])$ and $s \in label(T_r[v_i,s+1])$ (this is when v_i contributes

to d_s), then, by Corollary 29, $T_r[u_j, s+1]$ must be a child subtree of $T_r[v_i, s+1]$. This means that $T_r[u_j, s+1] \subseteq T_{union}[x, s+1]$, and we conclude that v_i is an s-neighbor of x in $T_{union}[x, s+1]$ if and only if v_i contributes to d_s , so $d_s = D_{T_{union}[x, s+1]}(x, s)$.

Lastly, we show that if $T_{union}[x, s+1]$ is a Case 6 or Case 7 tree – that is, $|N_s| = 1$, and $T_r[v_1, s+1]$ is a type 3 or type 4 tree, with w being the parent of an s-critical vertex – then cur_label has the value corresponding to $label(T_r[x] \setminus T_r[w])$ in Proposition 22, which would in this case be $label(T_{union}[x, s+1] \setminus T_r[w])$. We know, by definition of label and Lemma 27.5, that $T_r[v_i, s+1] \setminus T_r[v_i, s] = T_r[w] \cap T_r[v_i, s+1]$. As $T_r[w]$ is contained entirely in $T_r[v_i]$, we have that

$$T_r[w] \cap T_r[v_i, s+1] = T_r[w] \cap T_{union}[x, s+1]$$

But since $|N_s|=1$, for every $j\neq i,$ $T_r[v_j,s+1]\setminus T_r[v_j,s]=\emptyset$. Therefore

$$T_{union}[x, s+1] \setminus T_{union}[x, s] = T_r[w] \cap T_{union}[x, s+1]$$

which in turn gives us

$$T_{union}[x, s+1] \setminus (T_r[w] \cap T_{union}[x, s+1]) = T_{union}[x, s+1] \setminus T_r[w] = T_{union}[x, s]$$

But by the induction assumption,

$$cur_label = label(T_{union}[x, s]) = label(T_{union}[x, s] \setminus T_r[w])$$

Thus cur_label corresponds to $label(T_r[x] \setminus T_r[w])$ in Proposition 22.

We have now argued for all the correspondences in Table 1. By that, we conclude from Proposition 22 and the inductive assumption that $cur_label = label(T_{union}[x, s+1])$ at the end of the s-th iteration of the for loop in MAKELABEL. It runs for k iterations, where k is equal to the biggest number in any label of the children of x, and cur_label is then equal to $label(T_{union}[x, k+1])$. Since $k \ge thin(T_r[v_i])$ for all i, by definition $T_r[v_i, k+1] = T_r[v_i]$ for all i, and thus $T_{union}[x, k+1] = T_r[x]$. Therefore, when MAKELABEL finishes, $cur_label = label(T_r[x])$.

Theorem 32. Given any tree T, thin(T) can be computed in $\mathcal{O}(n \log(n))$ -time.

Proof. We find thin(T) by bottom-up processing of T_r and returning the first element of $label(T_r)$. After correctly initializing at leaves and vertices whose children are all leaves, we make a call to MAKELABEL for each of the remaining vertices. Correctness follows by Lemma 31 and induction on the structure of the rooted tree. We will now show that each call runs in $\mathcal{O}(\log(n))$ time to prove the time complexity.

Let m be the biggest number in any label of children of x, which is $\mathcal{O}(\log(n))$ by Corollary 12. For every integer s from 1 to m, the algorithm checks how many labels of children of x contain s to compute N_s , and how many labels of grandchildren of x contain s to compute t_s . The labels are sorted in descending order; therefore the whole loop goes only once through each of these labels, each of length $\mathcal{O}(\log(n))$. Other than this, MAKELABEL only does a constant amount of work. Therefore, MAKELABEL (T_r, x) , if x has a children and b grandchildren, takes time proportional to $\mathcal{O}(\log(n)(a+b))$. As the sum of the number of children and grandchildren over all vertices of T_r is $\mathcal{O}(n)$ we conclude that the total runtime to compute thin(T) is $\mathcal{O}(n\log(n))$. \square

Theorem 33. An optimal consistent solution can be found in $\mathcal{O}(n \log(n))$ -time.

Proof. Given T we first run the algorithm computing thin(T) finding the label and lastType of every full rooted subtree in T_r . We give a recursive layout algorithm that uses these labels in tandem with Consistent Layout presented in the Path Layout Lemma. We call it on a rooted tree where labels of all subtrees are known. For simplicity we call this rooted tree T_r even though in recursive calls this is not the original root r and tree T. The layout algorithm goes as follows:

- 1. Let thin $(T_r) = k$ and find a path P in T_r such that all trees in $T_r \setminus N[P]$ have thinness lower than k. The path depends on the type of T_r as explained in detail below.
- 2. Call this layout algorithm recursively on every rooted tree in $T_r \setminus N[P]$ to obtain linear layouts; to this end, we need the correct label for every vertex in these trees.

3. Call Consistent Layout on T_r , P and the layouts provided in step 2.

Every tree in the forest $T \setminus N[P]$ is equal to a dangling tree $T\langle v, u \rangle$ where v is a neighbor of some $x \in P$.

We observe that if thin(T) = k, then by definition $thin(T\langle v, u \rangle) = k$ if and only if v is a k-neighbor of x. It follows that every tree in $T \setminus N[P]$ has thinness at most k-1 if and only if no vertex in P has a k-neighbor that is not in P. We use this fact to show that for every type of tree we can find a satisfying path in the following way:

- Type 0 trees: Choose P = (r). Since $|T \setminus N[r]| = 0$ in these trees, this must be a satisfying path.
- Type 1 trees: These trees contain no k-critical vertices, which by definition means that for any vertex x in T_r , at most one of its children is a k-neighbor of x. Choose P to start at the root r, and as long as the last vertex in P has a k-neighbor v, v is appended to P. This set of vertices is obviously a path in T_r . No vertex in P can possibly have a k-neighbor outside of P, therefore all connected components of $T \setminus N[P]$ have thinness lower or equal to k-1. Furthermore, all components of $T \setminus N[P]$ are full rooted subtrees of T_r and so the labels are already known.
- Type 2 trees: In these trees the root r is k-critical. We look at the trees rooted in the two k-neighbors of r, $T_r[v_1]$ and $T_r[v_2]$. By Remark 20 these must both be Type 1 trees, and so we find paths P_1 , P_2 in $T_r[v_1]$ and $T_r[v_2]$ respectively, as described above. Gluing these paths together at r we get a satisfying path for T_r , and we still have correct labels for the components $T \setminus N[P]$.
- Type 3 trees: In these trees, r has exactly one child v such that $T_r[v]$ is of type 2 and none of its other children have thinness k. We choose P as we did above for $T_r[v]$. Vertex r is clearly not a k-neighbor of v, or else $D_T(v,k)=3$. Every other vertex in P has all their neighbors in $T_r[v]$. Again, every tree in $T \setminus N[P]$ is a full rooted subtree, and every label is known.
- Type 4 trees: In these trees, T_r contains precisely one vertex $w \neq r$ such that w is the parent of a k-critical vertex, x. This w is easy to find using the labels and lastTypes by annotating each subtree $T_r[v]$ with one new

piece of information: its only k'-critical vertex, with $k' = \text{thin}(T_r[v])$, if it has one. To calculate it for some subtree $T_r[v]$, given that we have already calculated it for all its child subtrees, a simple check of every child subtree will give the answer for each type:

- Type 0 and 1 trees: These do not contain any k'-critical vertex, so there is nothing to annotate.
- Type 2 trees: v is the k'-critical vertex, so we annotate this tree with v.
- Type 3 trees: We know that some child u of v is the root of a type 2 tree. We check every child of v to find the only child subtree with thinness k' that is a type 2 tree, and we copy its annotation into $T_r[v]$.
- Type 4 trees: Some child subtree will have thinness k' and will also have a k'-critical vertex, by definition of a type 4 tree. We can then do the same as with the type 3 trees, checking the lastTypes of the child subtrees with thinness k' to see which one has a k'-critical vertex.

This procedure can be done in $\mathcal{O}(n)$ by traversing the whole tree in a bottom-up fashion, so the time complexity is not affected by it. After finding the k-critical vertex of T_r , w is simply its only parent, also easy to find

Clearly, the tree $T_r[w]$ is a type 3 tree with thinness k. We find a path P that is satisfying in $T_r[w]$ as described above. w is still not a k-neighbor of x, therefore P is a satisfying path. In this case, we have one connected component of $T \setminus N[P]$ that is not a full rooted subtree of T_r , that is $T_r \setminus T_r[w]$. Thus, for every ancestor y of w, $T_r[y] \setminus T_r[w]$ is not a full rooted subtree either, and we need to update the labels of these trees.

As each $T_r[y]$ contains the k-critical vertex x, it has thinness greater or equal to k. Also, as $T_r[y]$ is a subtree of T_r , it has thinness lower or equal to k. These two facts tell us that $t \in T_r[y] = k$, which means that k is the first element of its label. Also, they are all type 4 trees, because they each have the parent w of x as a descendant. This means that $w = \omega(T_r[y])_1$ for each y. With this, we see that $T_r[y] \setminus T_r[w]$ is by definition equal to $T_r[y, k]$, whose label is equal to $t \in T_r[y]$

without its first number. Thus we quickly find the correct labels to do the recursive call.

References

- [1] Balabán, J., Hlinený, P., Jedelský, J.: Twin-width and transductions of proper k-mixed-thin graphs, arXiv: 2202.12536 [math.CO]
- [2] Bonnet, É., Kim, E.J., Thomassé, S., Watrigant, R.: Twin-width I: tractable FO model checking. In: Proceedings of the 61st IEEE Annual Symposium on Foundations of Computer Science FOCS 2020. pp. 601–612 (2020)
- [3] Bonomo, F., De Estrada, D.: On the thinness and proper thinness of a graph. Discrete Applied Mathematics **261**, 78–92 (2019)
- [4] Bonomo, F., Mattia, S., Oriolo, G.: Bounded coloring of cocomparability graphs and the pickup and delivery tour combination problem. Theoretical Computer Science **412**(45), 6261–6268 (2011)
- [5] Bonomo-Braberman, F., Brettell, N., Munaro, A., Paulusma, D.: Solving problems on generalized convex graphs via mim-width, arXiv: 2008.09004
- [6] Bonomo-Braberman, F., Brettell, N., Munaro, A., Paulusma, D.: Solving problems on generalized convex graphs via mim-width. In: Proceedings of the 17th Algorithm and Data Structures Symposium WADS 2021. Lecture Notes in Computer Science, vol. 12808, pp. 200-214 (2021)
- [7] Cygan, M., Fomin, F., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., Saurabh, S.: Parameterized Algorithms. Springer (2015)
- [8] Ellis, J., Sudborough, I., Turner, J.: The vertex separation and search number of a graph. Information and Computation 113(1), 50–79 (1994)
- [9] Golumbic, M.: The complexity of comparability graph recognition and coloring. Computing 18, 199–208 (1977)

- [10] Høgemo, S.: On the Linear MIM-Width of Trees. Master's thesis, The University of Bergen, Bergen, Norway (2019)
- [11] Høgemo, S., Telle, J.A., Vågset, E.R.: Linear MIM-width of trees. In: Sau, I., Thilikos, D.M. (eds.) Proceedings of the International Workshop on Graph-Theoretic Concepts in Computer Science WG 2019. Lecture Notes in Computer Science, vol. 11789, pp. 218–231. Springer (2019)
- [12] Kaplan, H., Shamir, R.: Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. SIAM Journal on Computing 25(3), 540–561 (1996)
- [13] Mannino, C., Oriolo, G., Ricci, F., Chandran, S.: The stable set problem and the thinness of a graph. Operations Research Letters **35**, 1–9 (2007)
- [14] Meyniel, H.: A new property of critical imperfect graphs and some consequences. European Journal of Combinatorics 8, 313–316 (1987)
- [15] Olariu, S.: An optimal greedy heuristic to color interval graphs. Information Processing Letters **37**, 21–25 (1991)
- [16] Rabinowicz, L.: Sobre la thinness de árboles. Master's thesis, Departamento de Computación, FCEyN, Universidad de Buenos Aires, Buenos Aires (2019)
- [17] Shitov, Y.: Graph thinness is NP-complete (2021), manuscript
- [18] Vatshelle, M.: New Width Parameters of Graphs. Ph.D. thesis, Department of Informatics, University of Bergen (2012)