# Results and open questions on thinness

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A graph $G = (V, E)$ is *$k$-thin* if there exist an ordering $v_1, \dots, v_n$ of $V$ and a partition of $V$ into $k$ classes $(V^1, \dots, V^k)$ such that, for each triple $(r, s, t)$ with $r < s < t$, if $v_r$, $v_s$ belong to the same class and $v_t v_r \in E$, then $v_t v_s \in E$. Such an ordering and partition are said to be *consistent*. The minimum $k$ such that $G$ is $k$-thin is called the *thinness* of $G$ and denoted $\text{thin}(G)$.

A graph $G = (V, E)$ is *proper $k$-thin* if there exist an ordering $v_1, \dots, v_n$ of $V$ and a partition of $V$ into $k$ classes $(V^1, \dots, V^k)$ such that, for each triple $(r, s, t)$ with $r < s < t$, if $v_r$, $v_s$ belong to the same class and $v_t v_r \in E$, then $v_t v_s \in E$, and if $v_s$, $v_t$ belong to the same class and $v_t v_r \in E$, then $v_s v_r \in E$. Such an ordering and partition are said to be *strongly consistent*. The minimum $k$ such that $G$ is proper $k$-thin is called the *proper thinness* of $G$ and denoted by $\text{pthin}(G)$.

If there is a $k$-partition and an ordering consistent with it, then there is an ordering which is consistent with the partition and is such that the vertices that are singletons in the partition are the lowest. This does not hold for strong consistency.

## 1 Results

### 1.1 Thinness of graph families

- For every $t \geq 1$, $\text{thin}(\overline{tK_2}) = t$.
- Let $CR_n$ be the graph on $2n$ vertices obtained from a complete bipartite graph $K_{n,n}$ by removing a perfect matching. For every $n \geq 1$, $\text{thin}(CR_n) \geq \frac{n}{2}$.
- For every fixed value $m$, the thinness of the complete $m$-ary tree on $n$ vertices is $\Theta(\log n)$.
- Let $GR_r$ be the $r \times r$-grid. For every $r \geq 2$, $fracr4 \leq \text{thin}(GR_r) \geq r + 1$.
- Let $Q_n$ be the hypercube of dimension $n$. For every $n \geq 1$, $\text{thin}(Q_n) \geq n - 2$.
- Mycielski graphs have unbounded thinness.

### 1.2 Upper bounds

- $\text{thin}(G) \leq \text{pw}(G) + 1$. Moreover, given a path decomposition of width $k$, a vertex ordering and a consistent partition into at most $k + 1$ classes can be obtained.
- $\text{thin}(G) \leq |V(G)| - \frac{\log(|V(G)|)}{4}$

- $\text{thin}(G) \le |V(G)| \frac{\Delta(G)+3}{\Delta(G)+4}$

- For every graph $G$ with $|E(G)| \ge 1$, $\text{thin}(G) \le \text{pthin}(G) \le \text{bandw}(G)$. Moreover, a linear layout realizing the bandwidth leads to a (strongly) consistent partition into at most $\text{bandw}(G)$ classes.

- As a corollary, if $G$ is connected, $\text{thin}(G) \le |V(G)| - \mathit{diam}(G)$.

- For every graph $G$, $\text{thin}(G) \le \text{cutw}(G) + 1$. Moreover, a linear layout realizing the cutwidth leads to a consistent partition into at most $\text{cutw}(G) + 1$ classes.

- Let $S \subseteq V(G)$. Then $\text{thin}(G) \le |V(G)| - |S| + \text{thin}(G[S])$.

- Let $G$ be a graph. Let $H$ be an interval completion of $G$. Let $F$ be the subgraph of $H$ whose edges are $E(H) - E(G)$. Then $\text{thin}(G) \le \tau(F) + 1$.

- If $V(G)$ is not a stable set then $\text{thin}(G) \le |V(G)| - \alpha(G)$.

- If $G$ is not a complete graph then $\text{thin}(G) \le |V(G)| - \omega(G)$.

- If $G$ is a non trivial co-comparability graph, then $\text{thin}(G) \le \frac{|V(G)|}{2}$.

- For a tree $T$, $\text{thin}(T) \le \mathit{pthin}(T) \le \mathit{height}(T)$ (a partition of the vertices according to its height in the tree, and a postorder of the vertices of the tree, are strongly consistent).

- For a tree $T$, $\text{thin}(T) \le |\mathit{leaves}(T)| - 1$ (vertex order as in convex.tex).

- For a tree $T$, let $T' = T \setminus \mathit{leaves}(T)$. Then $\text{thin}(T) \le |\mathit{leaves}(T')| - 1$ (a modification of the vertex order and partition as in convex.tex, adding pendant vertices to each vertex).

- A *$k$-nested interval graph* is an interval graph admitting an interval representation in which there are no chains of $k + 1$ intervals nested in each other. It is easy to see that $k$-nested interval graphs are a superclass of *$k$-length interval graphs*. Every $k$-nested interval graph is proper $k$-thin. They are the same class for $k = 1$, but for every $k \ge 2$, there are 2-nested interval graphs that are not $k$-length interval, and for every $k \ge 3$, there are proper 3-thin graphs that are interval but not $k$-nested interval.

### 1.3 Lower bounds

- For every graph $G$, $\text{thin}(G) \ge \text{box}(G)$. Moreover, given a linear ordering realizing the thinness, a box intersection model can be constructed.

- For every graph $G$, $\text{lmimw}(G) \le \text{thin}(G)$. Moreover, a linear ordering $v_1, \dots, v_n$ realizing the thinness, satisfies that the size of a maximum induced matching in the bipartite graph formed by the edges of $G$ with an endpoint in $\{v_1, \dots, v_i\}$ and the other one in $\{v_{i+1}, \dots, v_n\}$ is at most $\text{thin}(G)$.

- The *vertex isoperimetric peak* of a graph $G$, denoted as $b_v(G)$, is defined as $\max_s \min_{X \subset V, |X|=s} |N(X)|$. For every graph $G$ with at least one edge, $\text{thin}(G) \ge \frac{b_v(G)}{\Delta(G)}$.

- Let $G$ be a graph. If $|N(u) \setminus N[v]| \geq k$ for all $u, v \in V(G)$ then $\text{thin}(G) \geq k+1$. Moreover, for every order $<$ of $V(G)$, the first $k+1$ vertices induce a complete graph in $G_<$.
- As a corollary, if $G$ is a graph with $\delta(G) \geq d$ and such that for all $u, v \in V(G)$, $|N(u) \cap N(v)| \leq c < d$, then $\text{thin}(G) \geq d - c$.
- Let $G$ be a graph. Let $S \subseteq V(G)$ and $p = |S|$. If $|N(u) \setminus N[v]| \geq k$ for all $u, v \in S$ and $|V(G)| - p \leq k$ then $\text{thin}(G) \geq 1 + k + p - |V(G)|$.
- $\text{thin}(G) \geq \frac{\chi(G)}{\omega(G)}$ (because a $k$-thin graph can be partitioned into $k$ perfect graphs, so $\chi(G) \leq \text{thin}(G)\omega(G)$)

## 1.4 Incomparable parameters

- Interval graphs have thinness 1 and unbounded clique-width, while cographs have clique-width 2 and unbounded thinness, because $\overline{tK_2}$ is a cograph for every $t$, so the parameters are not comparable.
- Complete graphs have high treewidth and thinness 1, and trees instead have treewidth 1 but the thinness of the complete $m$-ary tree on $n$ vertices is $\Theta(\log n)$, so the parameters are not comparable.

## 1.5 Operators

- If $G$ is not complete, then $\text{thin}(G \vee 2K_1) = \text{thin}(G) + 1$.

  This implies that if there is some constant value $k$ such that recognizing $k$-thin graphs is NP-complete, then for every $k' > k$, recognizing $k'$-thin graphs is NP-complete. The existence of such $k$ is still not known, and in general the complexity of recognition of $k$-thin and proper $k$-thin graphs, both with $k$ as a parameter and with constant $k$, is open.

- For every graph $G$, $\text{pthin}(3G \vee K_1) = \text{pthin}(G) + 1$.

  If there is some constant value $k$ such that recognizing proper $k$-thin graphs is NP-complete, then for every $k' > k$, recognizing proper $k'$-thin graphs is NP-complete.

## 1.6 Characterization for graph classes

- Interval graphs = 1-thin graphs
- Proper interval graphs = proper 1-thin graphs
- Interval graphs have unbounded proper thinness
- Let $G$ be a cograph and $t \geq 1$. Then $G$ has thinness at most $t$ if and only if $G$ contains no $\overline{(t+1)K_2}$ as induced subgraph.

| **Upper bounds** | **High thinness** |
|---|---|
| $\text{(p)thin}(G_1 \cup G_2) = \max\{\text{(p)thin}(G_1), \text{(p)thin}(G_2)\}$<br>$\text{(p)thin}_{\text{ind}}(G_1 \cup G_2) = \max\{\text{(p)thin}_{\text{ind}}(G_1), \text{(p)thin}_{\text{ind}}(G_2)\}$<br>$\text{(p)thin}_{\text{cmp}}(G_1 \cup G_2) = \text{(p)thin}_{\text{cmp}}(G_1) + \text{(p)thin}_{\text{cmp}}(G_2)$ | |
| $\text{thin}(G_1^* \vee G_2^*) = \text{thin}(G_1^*) + \text{thin}(G_2^*)$<br>$\text{thin}(G_1 \vee K_n) = \text{thin}(G_1)$<br>$\text{pthin}(G_1 \vee G_2) \leq \text{pthin}(G_1) + \text{pthin}(G_2)$<br>$\text{(p)thin}_{\text{ind}}(G_1 \vee G_2) = \text{(p)thin}_{\text{ind}}(G_1) + \text{(p)thin}_{\text{ind}}(G_2)$<br>$\text{(p)thin}_{\text{cmp}}(G_1 \vee G_2) \leq \text{(p)thin}_{\text{cmp}}(G_1) + \text{(p)thin}_{\text{cmp}}(G_2)$<br>$\text{thin}_{\text{cmp}}(G_1 \vee K_n) = \text{thin}_{\text{cmp}}(G_1)$ | |
| $\text{(p)thin}(G_1 \bullet G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \text{(p)thin}(G_2)$<br>$\text{(p)thin}(G_1 \bullet K_n) = \text{(p)thin}(G_1)$<br>$\text{(p)thin}_{\text{ind}}(G_1 \bullet G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \text{(p)thin}_{\text{ind}}(G_2)$<br>$\text{(p)thin}_{\text{cmp}}(G_1 \bullet G_2) \leq \lvert V(G_1)\rvert \cdot \text{(p)thin}_{\text{cmp}}(G_2)$<br>$\text{(p)thin}_{\text{cmp}}(G_1 \bullet K_n) = \text{(p)thin}_{\text{cmp}}(G_1)$ | $K_n \bullet S_2$ |
| $\text{(p)thin}(G_1 \square G_2) \leq \text{(p)thin}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}_{\text{ind}}(G_1 \square G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}_{\text{cmp}}(G_1 \square G_2) \leq \text{(p)thin}_{\text{cmp}}(G_1) \cdot \lvert V(G_2)\rvert$ | $P_n \square P_n$<br>$K_n \square K_n$<br>$K_n \square K_{n,n}$ |
| $\text{(p)thin}(G_1 \times G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}_{\text{ind}}(G_1 \times G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \lvert V(G_2)\rvert$ | $K_n \times K_2$<br>$P_n \times P_n$ |
| $\text{(p)thin}(G_1 \boxtimes G_2) \leq \text{(p)thin}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}(G_1 \boxtimes K_n) = \text{(p)thin}(G_1)$<br>$\text{(p)thin}_{\text{ind}}(G_1 \boxtimes G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}_{\text{cmp}}(G_1 \boxtimes G_2) \leq \text{(p)thin}_{\text{cmp}}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}_{\text{cmp}}(G_1 \boxtimes G_2) = \text{(p)thin}_{\text{cmp}}(G_1)$ | $P_n \boxtimes P_n$<br>$(K_n \square K_2) \boxtimes (K_n \square K_2)$ |
| $\text{(p)thin}(G_1 * G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}_{\text{ind}}(G_1 * G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \lvert V(G_2)\rvert$ | $K_n * S_2$<br>$nK_2 * nK_2$ |
| | $K_n \diamond K_2$<br>$nK_2 \diamond K_1$ |
| $\text{(p)thin}(G_1 \ltimes G_2) \leq \text{(p)thin}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}_{\text{ind}}(G_1 \ltimes G_2) \leq \text{(p)thin}_{\text{ind}}(G_1) \cdot \lvert V(G_2)\rvert$<br>$\text{(p)thin}_{\text{cmp}}(G_1 \ltimes G_2) \leq \text{(p)thin}_{\text{cmp}}(G_1) \cdot \lvert V(G_2)\rvert$ | $K_2 \ltimes nK_2$<br>$K_2 \ltimes (K_n \square K_2)$ |
| $\text{(p)thin}(G_1 \circ G_2) \leq \lvert V(G_1)\rvert \cdot \text{(p)thin}_{\text{ind}}(G_2)$<br>$\text{(p)thin}_{\text{ind}}(G_1 \circ G_2) \leq \lvert V(G_1)\rvert \cdot \text{(p)thin}_{\text{ind}}(G_2)$ | $K_2 \circ K_n$<br>$nK_2 \circ K_1$<br>$(K_n \square K_2) \circ K_1$ |

Table 1: This table summarizes the upper bounds for products or operators (when needed, graphs with an asterisk are not complete). The table also summarizes the families of graphs with bounded parameters whose product have high thinness, used to show the nonexistence of bounds in terms of certain parameters. Recall that the lexicographic, homomorphic, and hom-product are not necessarily commutative.

## 1.7 Problems polynomially solvable

- If a partition is given, it is NP-complete to decide the existence of a (strongly) consistent ordering.

- In contrast, we can compute the minimum size partition (strongly) consistent with a vertex ordering $<$ of $V(G)$. We can build the compatibility graph $G_<$ ($\tilde{G}_<$) in such a way that any partition (strongly) compatible with the ordering is a coloring of $G_<$ ($\tilde{G}_<$), and conversely. It can be proved that $G_<$ and $\tilde{G}_<$ are co-comparability, thus an optimum coloring of them can be found in polynomial time. Moreover, if $G$ is co-comparability, then $G_<$ and $\tilde{G}_<$ are spanning subgraphs of $G$, thus for co-comparability graphs, $\text{thin}(G) \leq \text{pthin}(G) \leq \chi(G) = \omega(G)$.

- Let $G$ be a graph, and $k$ a positive integer. Then $\text{thin}(G) \geq k$ (resp. $\text{pthin}(G) \geq k$) if and only if, for every ordering $<$ of $V(G)$, the graph $G_<$ (resp. $\tilde{G}_<$) has a clique of size $k$.

- $k$-thin graphs have $O(n^k)$ maximal cliques, so any problem that can be solved by enumerating the maximal cliques (in particular, maximum weighted clique, which is also included in the other framework).

- We describe now a framework of problems that can be solved for graphs with bounded thinness, given the representation.

  *Instance:*

  - A $k$-thin representation of $G = (V, E)$, with ordering $<$ of $V$, namely $v_1 < \cdots < v_n$, and partition of $V$ into $k$ classes $V^1, \ldots, V^k$.
  - A family of arbitrary nonnegative weights $w_1, \ldots, w_t$ on $V$.
  - A family of nonnegative weights $b_1, \ldots, b_p$ on $V$ bounded by a fixed polynomial in $n$ ($p$ fixed, $q(n)$ the bound for the weights).

  *Question:* find sets $S_1, \ldots, S_r$ ($r$ fixed, not necessarily disjoint), $S_j \subseteq V$ for $1 \leq j \leq r$, such that:

  - the objective is to minimize or maximize a linear function $\sum_{1 \leq i \leq t; 1 \leq j \leq r} c_{ij} w_i(S_j)$.
  - each vertex $v$ has a list $L(v)$ of combinations of the sets $S_1, \ldots, S_r$ to which it can belong (that may include the empty combination).
  - there is an $r \times r$ symmetric matrix $M$ over $0, 1, *$, stating the adjacency conditions on the sets $S_j$, such that for $1 \leq i < j \leq r$, $M_{ii} = 1$ means $S_i$ is a clique, $M_{ii} = 0$ means $S_i$ is a stable set, $M_{ij} = 1$ means all the edges joining $S_i$ and $S_j$ have to be present, $M_{ij} = 0$ means there are no edges from $S_i$ to $S_j$.
  - there is a family of restrictions on the weight of the intersection and of the union of some families of sets. Such restrictions can be expressed as
    - $0 \leq l_{iJ\cap} \leq b_i(\bigcap_{j \in J} S_j) \leq u_{iJ\cap}$, such that $1 \leq i \leq p$, $J \subseteq \{1, \ldots, r\}$.

* $0 \le l_{iJ\cup} \le b_i(\bigcup_{j\in J} S_j) \le u_{iJ\cup}$, such that $1 \le i \le p$, $J \subseteq \{1,\ldots,r\}$.

Notice that some of these restrictions can be of cardinality, if the corresponding weight function $b_i$ is constant.

The family of problems that can be modeled within this framework includes weighted variations of list matrix partition problems with matrices of bounded size, which in turn generalize coloring, list coloring, list homomorphism, equitable coloring with different objective functions, all for fixed number of colors (or graph size in the case of homomorphism), clique cover with fixed number of cliques, weighted stable sets, and other graph partition problems. It models also sum-coloring and its more general version optimum cost chromatic partition problem for fixed number of colors, but it does not include dominating-like problems (which are, however, covered by the bonded mim-width approach).

## 2 Open questions

1. What is the computational complexity of computing the thinness/proper thinness of a graph? Or deciding if it is at most $k$ for some fixed values $k$?

2. Study the thinness of certain graph classes: interval graphs have thinness 1 (thinness of circular-arc graphs can be arbitrarily large), permutation graphs have thinness at most $\chi(G)$ (can it be bounded better? can it be calculated? treewidth and pathwidth can be calculated in polytime), trees? they can have arbitrarily large thinness but we don't know how to compute it. For permutation graphs the order of the permutation can be arbitrarily bad: example of a threshold graph such that has thinness $k$ for an order of representation. (1a2b3c4d5 vs 54321abcd)

3. Characterize (proper) $k$-thin graphs by minimal forbidden induced subgraphs (or at least within some graph class, we did it for thinness in cographs).

4. Find sufficient conditions, for instance a family subgraphs to forbid as induced subgraphs, for a graph to be (proper) $k$-thin, even if these graphs are not necessarily forbidden induced subgraphs for (proper) $k$-thin graphs. These kind of results have been obtained for mim-width.

5. Are interval graphs which are proper 2-thin graphs 2-nested interval?

6. From Daniel's mails: Are there examples of problems that are hard for bounded mim-width but easy for bounded thinness? Colouring is a candidate for a problem that might make a difference between mim-width and thinness.

7. And if there are no examples known of such problems, do we know of any examples that are easy for bounded thinness but whose complexity is unknown for bounded mim-width? (Maybe some of the weighted list matrix partition problems.)

8. But, as Colouring is polynomial-time solvable for tree convex graphs, it would even be better to have a problem that is hard for tree convex graphs and for bounded mim-width but easy for bounded thinness?

9. As we know have thinness and mim-width we could have go into more detail and explain that

    (a) tree convex graphs contain chordal bipartite so have unbounded sim-width even (if I remember correctly, I will check)

    (b) circular convex graphs have bounded linear mim-width (if I remember correctly) but unbounded thinness and unbounded clique-width.

    (c) $(t, \Delta)$-tree convex graphs have bounded thinness, but already convex graphs have unbounded clique-width.

    (d) star convex and comb convex have unbounded mim-width.

10. A few questions:

    (a) Are treewidth and thinness incomparable? **YES**

    (b) Are clique-width and thinness incomparable? **YES**

    (c) Do we know if star convex and comb convex graphs have even unbounded sim-width?

        (Nick) I think the answer is yes using essentially the same argument as for mim-width. That is, grids are also known to have unbounded sim-width, and you can add a dominating vertex to get a star convex graph, without decreasing the sim-width (and similar for comb convex, but you add $|A|$ dominating vertices).