and the spectral center of gravity f is equal to (3.12), i.e., the Gabor functions follow the Heisenberg's uncertainty principle.

3.3.1 Filter normalization

The Gabor filter can be more appropriately defined by taking the following justifications. First, we must remember that we use the Gabor function as a linear filter to analyze a signal. Under this condition, the temporal analysis of the signal is carried out using the convolution operator. Considering that the Gabor function is concentrated near the time instant t_0 and that a convolution centered at the origin is preferable, then $t_0 = 0$. Another parameter that we can omit is the phase shift ϕ. There is no evidence that any specific phase would be more beneficial than any other. Moreover, for the functions to be similar at all locations, the phase shift should depend on the location t_0, and thus, the phase shift can be removed from the origin centered filter (ϕ = 0). The Gabor filter function in its compact form is defined as

$$\begin{aligned} g(t) &= e^{-\alpha^2 t^2} e^{j2\pi f t} \\ G(\upsilon) &= \sqrt{\frac{\pi}{\alpha^2}} e^{-\left(\frac{\pi}{\alpha}\right)^2 (\upsilon - f)^2} \end{aligned} \tag{3.23}$$

The normalization of the Gabor filter can be performed taking into account the application in which it will be used. However, in this thesis we use the general normalization which is based on the multi domain representation property of the function and following the next conditions [4]:

1. Maximum condition:

$$\max |G(\upsilon)| = 1 \tag{3.24}$$

2. Constant spectra condition:

$$\int_{-\infty}^{\infty} |g(t)| dt = 1 \tag{3.25}$$

From the equation (3.22), it is evident that the maximum response of the Gabor filter in the frequency domain is a function of $\sqrt{\pi/\alpha^2}$, therefore, its inverse

$$\sqrt{\frac{\alpha^2}{\pi}} \tag{3.26}$$

can be used as the Gabor filter normalization factor in the time domain and fulfill the two conditions mentioned above. In summary, the function of the normalized Gabor filter can be defined as

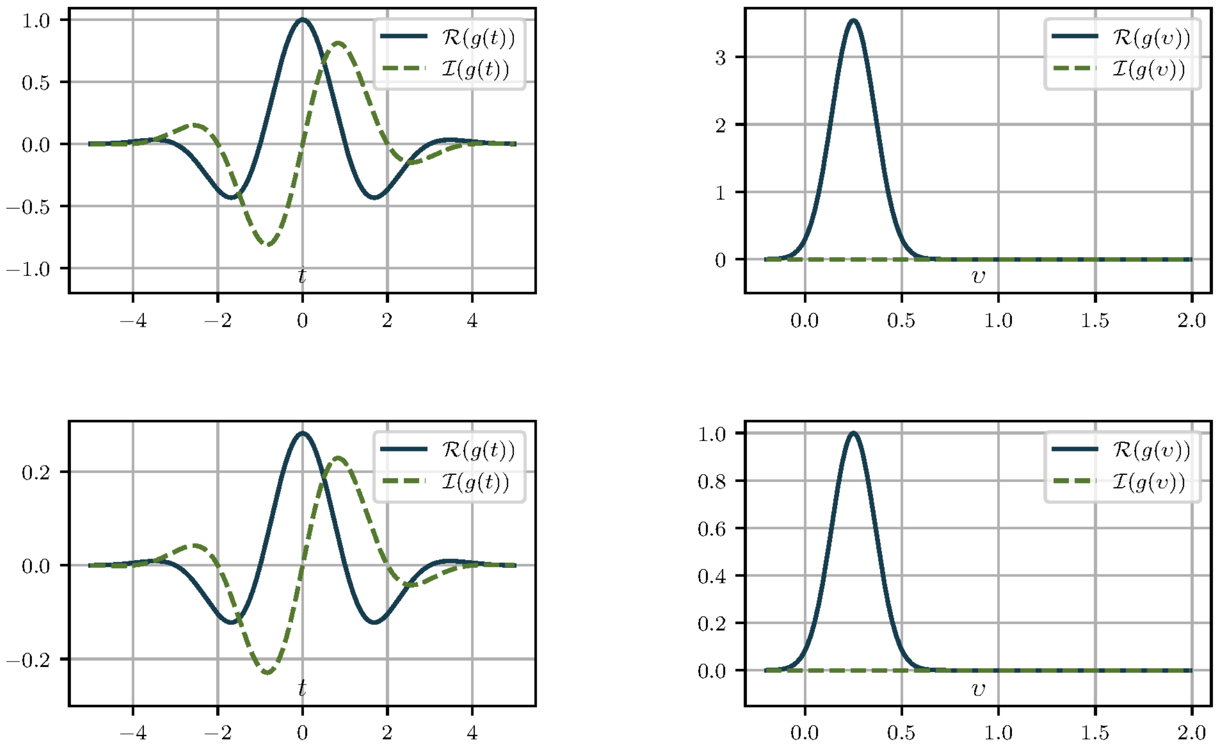

Figure 3.1: 1-d Gabor filter in the time domain (first column) and in the frequency domain (second column). From top to bottom: no normalized filter and normalized Gabor filter with $[f = 1/4, \alpha = 0.5, t_0 = 0]$.

$$
\begin{aligned}
g(t) &= \sqrt{\frac{\alpha^2}{\pi}} e^{-\alpha^2 t^2} e^{j2\pi f t} \\
G(v) &= e^{-\left(\frac{\pi}{\alpha}\right)^2 (v-f)^2}
\end{aligned}
\tag{3.27}
$$

The figure 3.1 shows a Gabor filter in the time and frequency domain before and after normalization following the two conditions described above. At this point, it is important to note that normalization is an essential step in the multi-spectral analysis and the feuture extraction of a signal.

3.3.2 Frequency filter spacing

Our main interest in Gabor filters is multispectral analysis of a function. To accomplish this, we can generate a bank containing different Gabor functions that work at different frequencies f. We can define the separation of the filters in octaves by means of the half-response spatial frequency bandwidth B_f measured between two central frequencies $f_2 < f_1$ [15]

$$B_f = \log_2\left(\frac{f_1}{f_2}\right) \tag{3.28}$$

The octave spacing between two different two adjacent filters is an interesting property of the Gabor filters, however, the filters denoted by the equations (3.27) have a spread that only depends on the parameter α, regardless of its central frequency f. This means that when implementing the Gabor function in a filter bank at different frequencies to obtain a multi-spectral decomposition of a signal, all of the filters will have the same duration. We can see this effect in Figure 3.2a, where we show a filter bank with an adjacent filter's spacing of one octave, that is, $B_f = 1$.

Frequency crossing point

The fact that the bank filters have the same width at all frequencies is not a problem nor is it a requirement to analyze a signal with the Gabor function, however, making the filter width dependent on its frequency implies a multi-resolution analysis, since the filters behave like a scaled version of each other. One way to accomplish this, is to have the same relative window size in relation to the central frequency f. We must remember that window size of a Gabor function is denoted by the effective width of a Gaussian function, which in the time domain has a form of

$$w(t) = e^{-\frac{(t-t_0)^2}{2\sigma^2}} \tag{3.29}$$

A Gaussian window is infinite in extent, so it is characterized by its locality t_0 and its standard deviation σ, which in this context is implicit in the parameter α of th Gabor fuction, therefore, $\alpha^2 = 1/2\sigma^2$. A peculiarity of the Gabor filter is that its analytical form in the frequency domain is completely defined by the fourier transform of the normalized Gaussian function.

$$G(\upsilon) = w(\upsilon) = e^{-\left(\frac{\pi}{\alpha}\right)^2(\upsilon-f)^2} \tag{3.30}$$

Since the center frequencies of the bank filters are chosen to have a constant separation between them and the effective width of the window is a function of this central frequency, there is a point on the frequency axis where two adjacent functions intersect. In a filter bank with two functions with central frequencies f_1 and f_2, the low cut-off frequency of the function at f_1 coincides with the high cut-off frequency of the function at f_2. Normally in the literature this crossing point c_1, corresponds to the points where the Gabor function has decreased half of its maximum value, i.e., $c_1 = 1/2 = 0.5$ [15]. However, we can obtain this crossing point c_1 by defining a frequency interval Δf that represents the distance between the points where the function $G(\upsilon)$ begins to decrease, therefore, evaluating Eq. (3.30) at $\upsilon = f + \frac{\Delta f}{2}$

$$G\left(f + \frac{\Delta f}{2}\right) = e^{-\left(\frac{\pi}{\alpha}\right)^2\left(\frac{\Delta f}{2}\right)^2} = c_1 G(f) \tag{3.31}$$

we obtain the expression of the half-frequency interval

$$\frac{\Delta f}{2} = \frac{\alpha}{\pi}\sqrt{\ln\left(\frac{1}{c_1}\right)} \tag{3.32}$$

from which we obtain the crossing point defined as

$$c_1 = e^{-\left(\frac{\alpha}{\pi}\right)^2\left(\frac{\Delta f}{2}\right)^2} \tag{3.33}$$

Then, we know that for a filter whose center frequency is f and whose cut-off frequency interval is Δf, the full bandwidth expressed in octaves, B_f, is defined as [8]

$$B_f = \log_2\left(\frac{f + \frac{\Delta f}{2}}{f - \frac{\Delta f}{2}}\right) \tag{3.34}$$

It is clear that using expression (3.32) in equation (3.34), we find the expression that relates the frequency bandwidth to the central frequency and the effective width of the Gaussian window.

$$B_f = \log_2\left(\frac{\frac{f}{\alpha}\pi + \sqrt{\ln\left(\frac{1}{c_1}\right)}}{\frac{f}{\alpha}\pi - \sqrt{\ln\left(\frac{1}{c_1}\right)}}\right) \tag{3.35}$$

The above analysis allows us to rewrite the expression of a Gabor filter in 1-d that belongs to a bank of filters spaced from each other by a bandwidth B_f and a center frequency f as follows.

$$\begin{aligned} g(t) &= \frac{f}{\gamma\sqrt{\pi}} e^{-\left(\frac{f}{\gamma}\right)^2 t^2} e^{j2\pi f t} \\ G(\upsilon) &= e^{-\left(\frac{\gamma\pi}{f}\right)^2(\upsilon - f)^2} \end{aligned} \tag{3.36}$$

where now the effective bandwidth α of each filter in the bank will be determined based on the ratio $\gamma = \frac{f}{\alpha}$ and crossing point between adjacent filters c_1.

The figure 3.2 shows the octave spacing for a bank of filters in the frequency domain. Particulary, the figure 3.2a shows a bank without the relationship between the effective width and the central frequency of the filter, whereas the figures 3.2b and 3.2c show the interdependence between α, B_f, f and c_1 and the behavior of the bank with a different crossing point.

3.4 2-d Gabor filters

The generalization of the previously defined Gabor's function theory in 1-d to two dimensions is straightforward. First, we replace the time variable t with the pair of spatial coordinates (x, y) and the frequency variable f with the pair of frequency variables (u, v). Then, as for the 1-d case, the 2-d Gabor functions follows the Heinsenberg principle where the uncertainty measures for the space and spatial-frequency domains are expressed in terms of Δx, Δy, Δu, and Δv, for which it holds that

$$\begin{gathered}\Delta x \Delta u \geq \frac{1}{4\pi},\ \Delta y \Delta v \geq \frac{1}{4\pi} \\ \Delta x \Delta y \Delta u \Delta v \geq \frac{1}{16\pi^2}\end{gathered} \tag{3.37}$$

In case 2d, the Gabor function is represented by the modulated product of a harmonic oscillation on any spatial frequency and any orientation, represented by a complex exponential, with a pulse in the form of a probability function, represented by an elliptical Gaussian ellipse on any orientation. For simplicity, it can be assumed that the orientation of the Gaussian and the harmonic modulation are the same, therefore, applying the given simplifications, a compact form of the 2-d GEF in the space domain can be defined as

$$\begin{gathered}g(x, r) = e^{-\left(\alpha^2 x_r^2 + \beta^2 y_r^2\right)} e^{j2\pi f x_r} \\ x_r = x\cos\theta + y\sin\theta \\ y_r = -x\sin\theta + y\cos\theta\end{gathered} \tag{3.38}$$

whereas the analytical expression for the 2-d GEF in the spatial-frequency domain is obtained from the Fourier transform of (3.38), $G(u, v) = \mathcal{F}\{g(x, y)\}$, and is given by

$$\begin{gathered}G(u, v) = \frac{\pi}{\alpha\beta} e^{-\pi^2\left(\frac{(u_r - f)^2}{\alpha^2} + \frac{v_r^2}{\beta^2}\right)} \\ u_r = u\cos\theta + v\sin\theta \\ v_r = -u\sin\theta + v\cos\theta\end{gathered} \tag{3.39}$$

The above expressions can be normalized by following the same reasoning as in the 1-d case. We