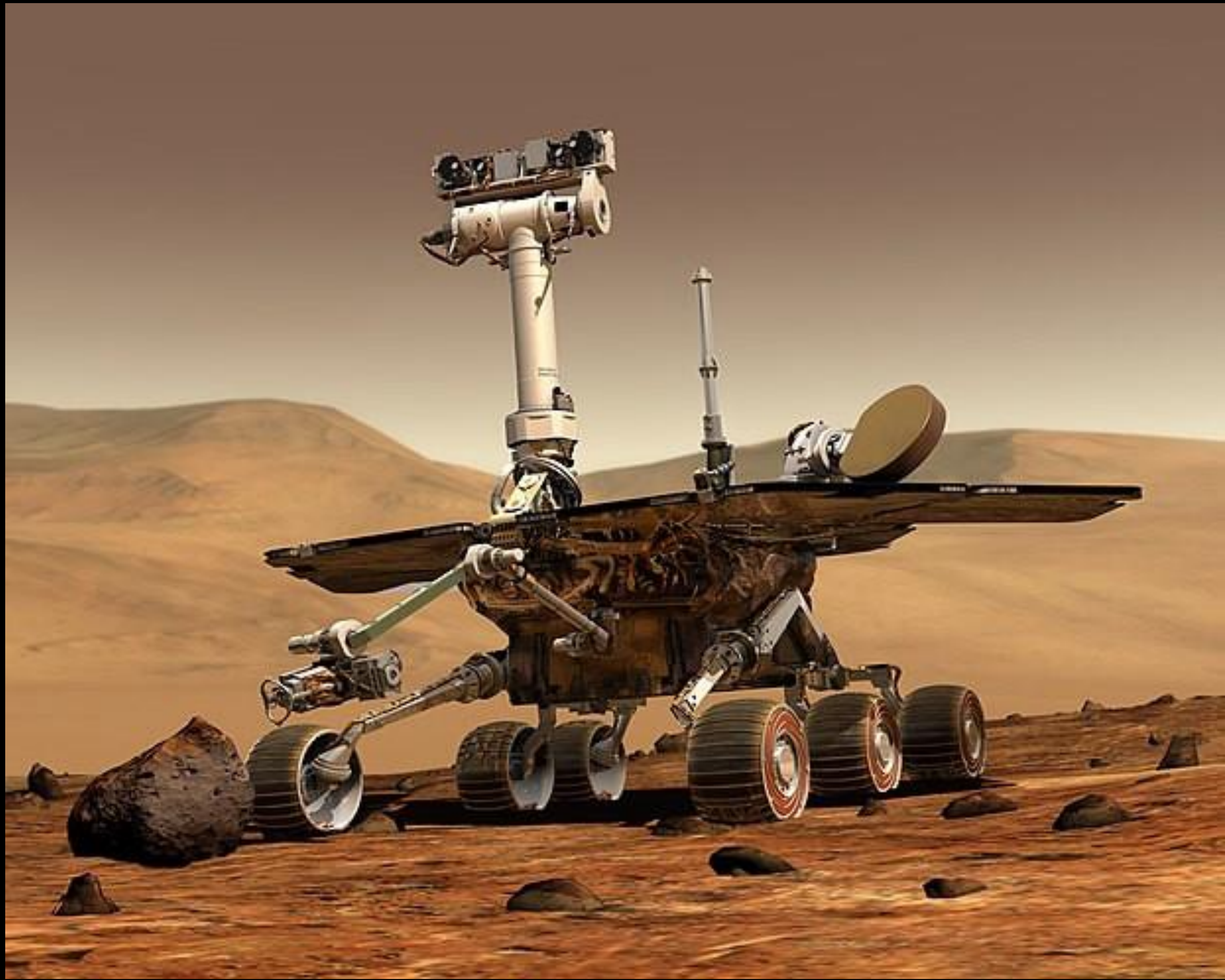


Uncertainty



NEXT 36 HOURS

HOURLY → | 10 DAYS →

TONIGHT
CLEAR



LOW
20°

0%

THU



HIGH
36°

0%

THU NIGHT



LOW
25°

0%

FRI



HIGH
46°

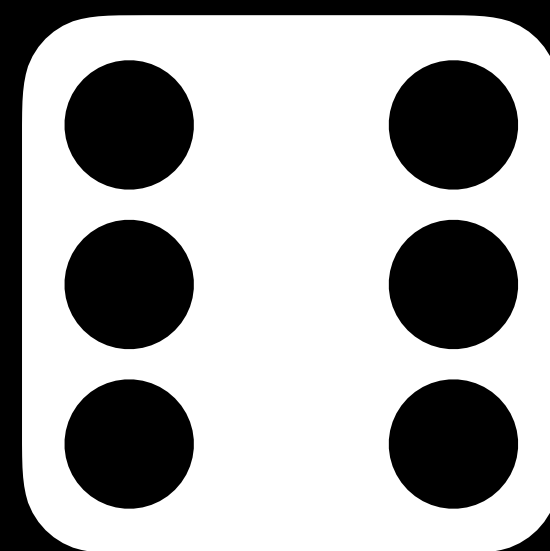
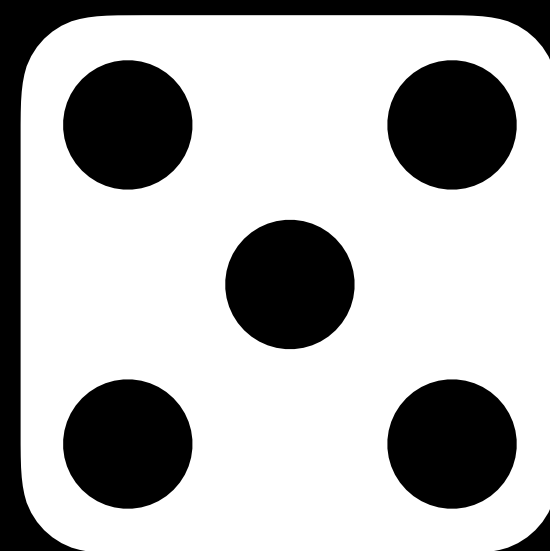
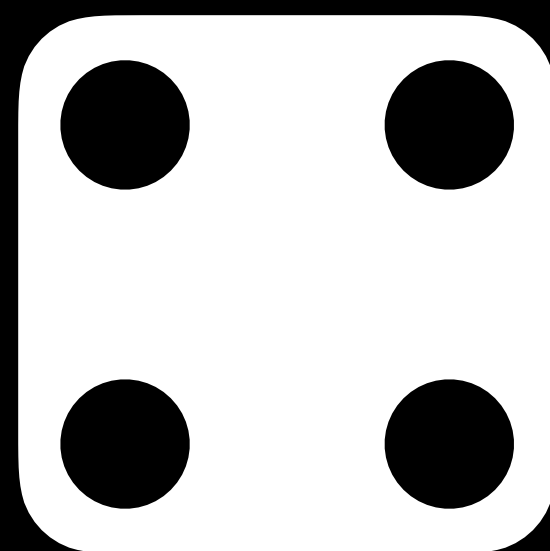
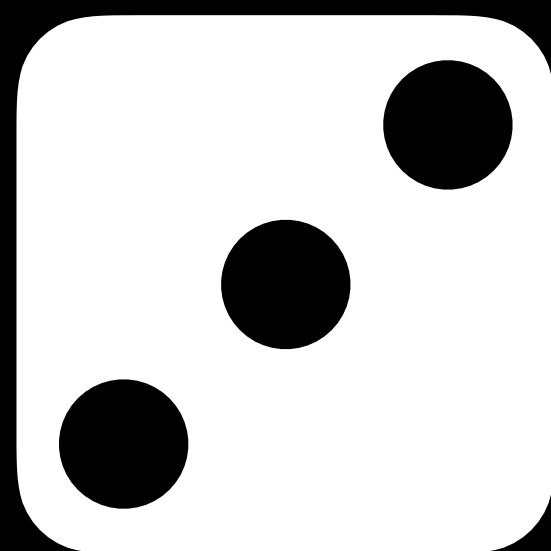
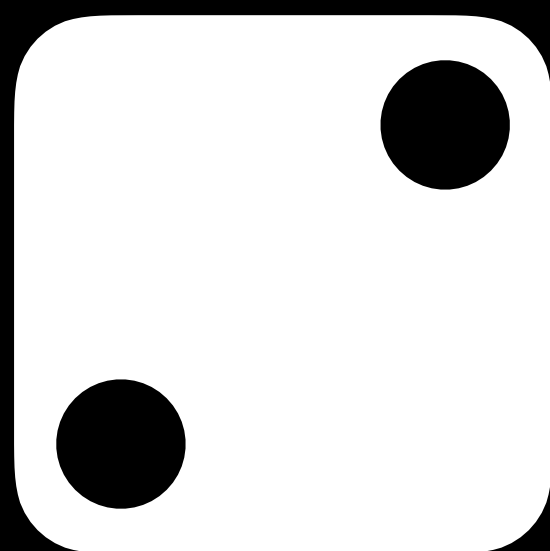
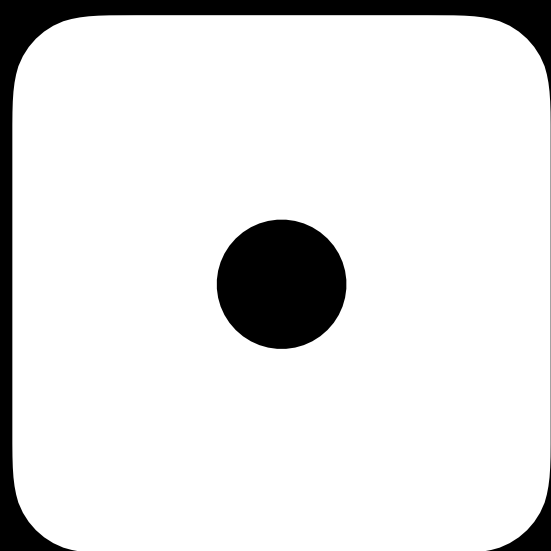
0%

FRI NIGHT



LOW
32°

20%



Probability

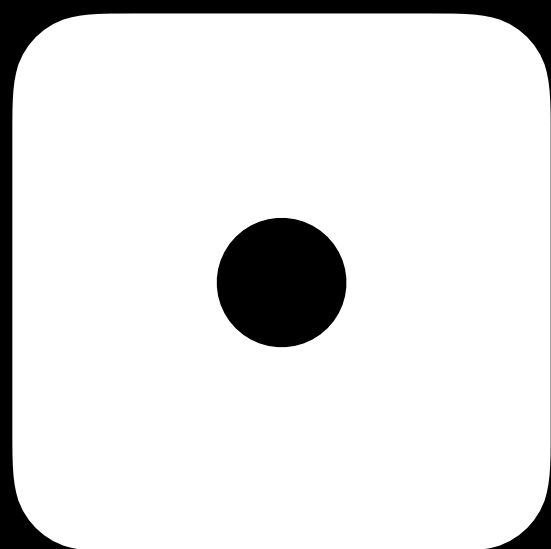
Possible Worlds

ω

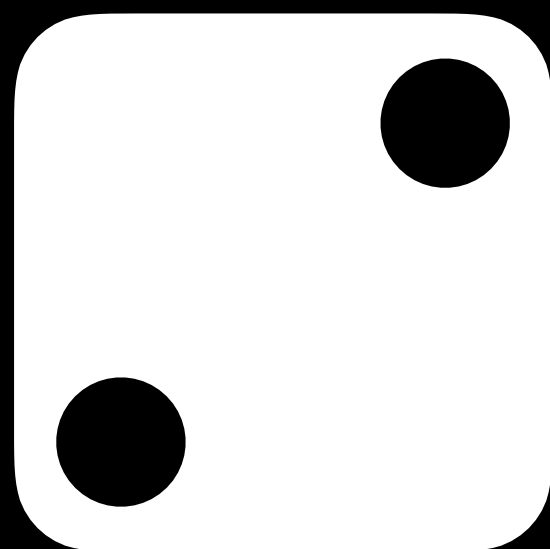
$$P(\omega)$$

$$0 \leq P(\omega) \leq 1$$

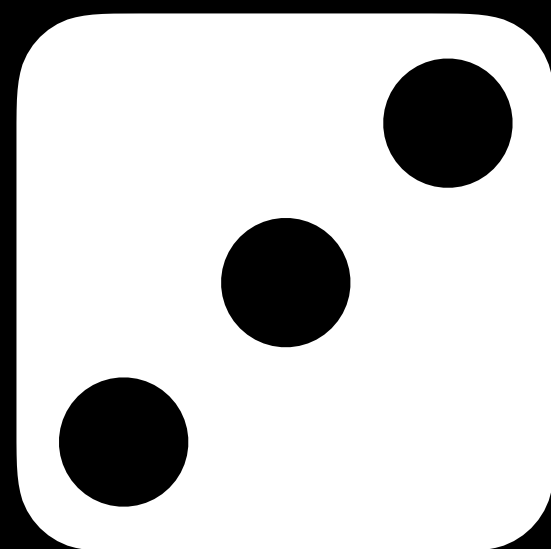
$$\sum_{\omega \in \Omega} P(\omega) = 1$$



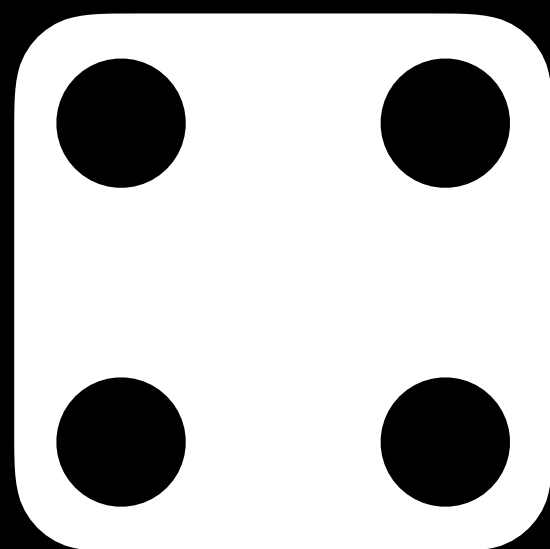
$$\frac{1}{6}$$



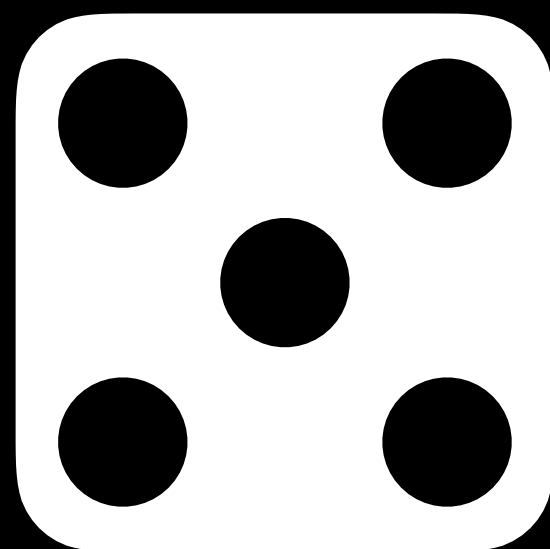
$$\frac{1}{6}$$



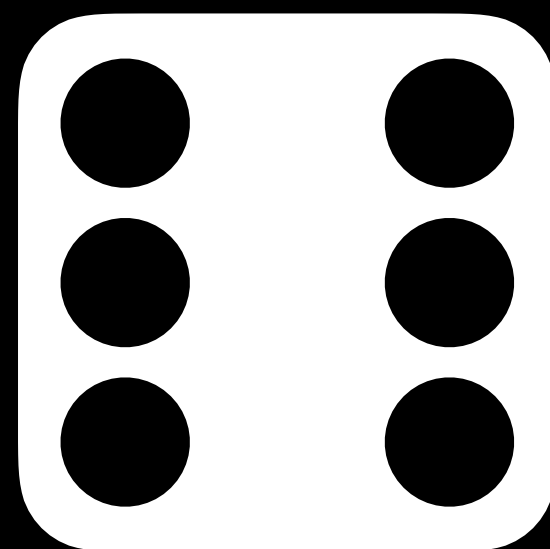
$$\frac{1}{6}$$



$$\frac{1}{6}$$

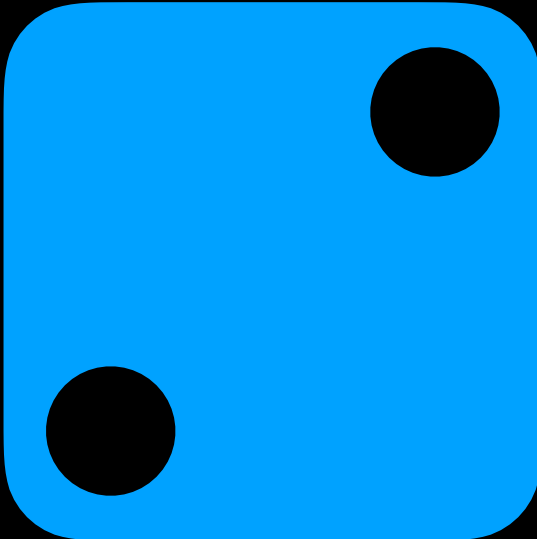


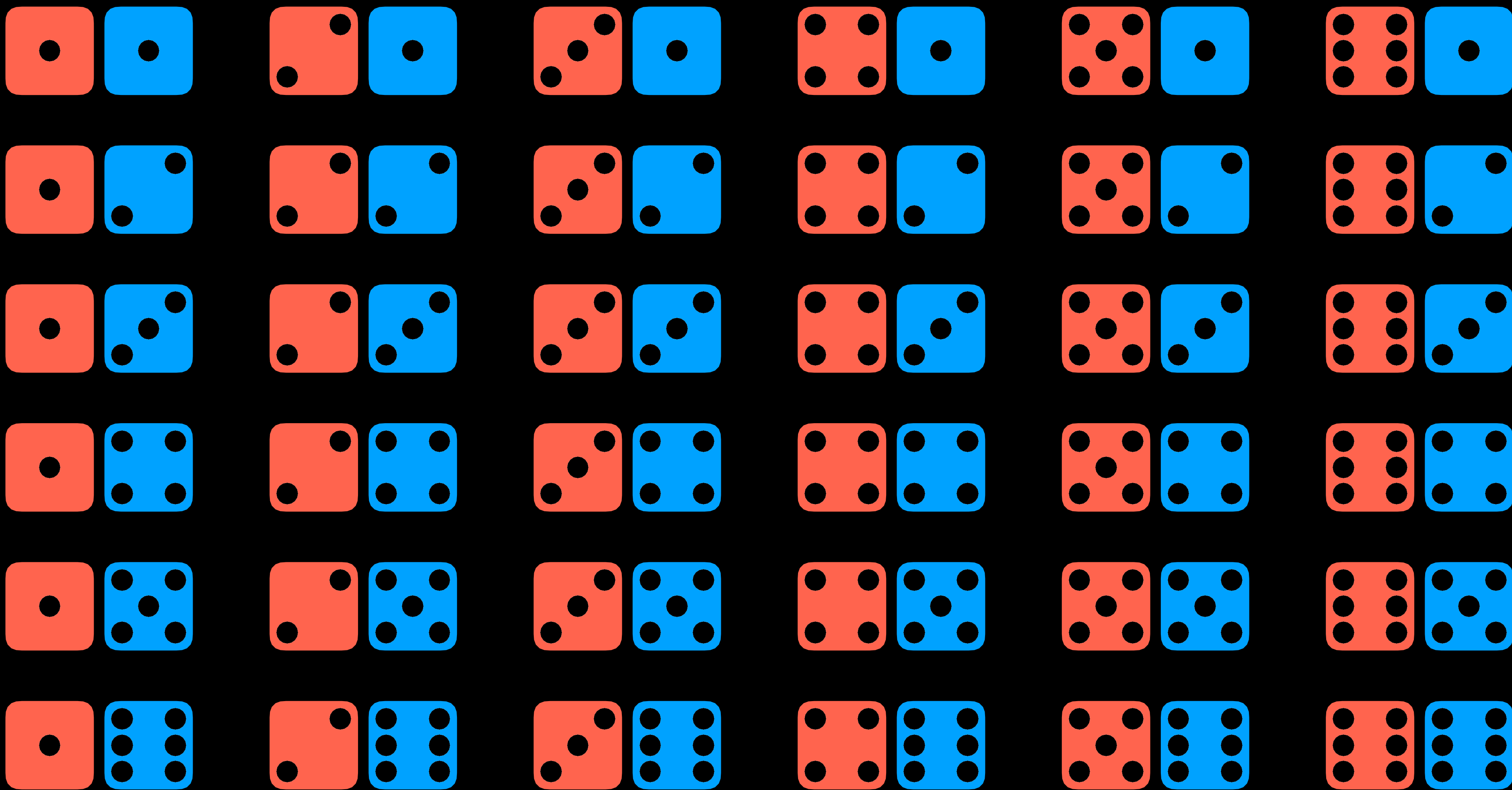
$$\frac{1}{6}$$



$$\frac{1}{6}$$

$$P(\text{🎲}) = \frac{1}{6}$$





2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

$$P(\textit{sum to } 12) = \frac{1}{36}$$

$$P(\textit{sum to } 7) = \frac{6}{36} = \frac{1}{6}$$

unconditional probability

degree of belief in a proposition
in the absence of any other evidence

conditional probability

degree of belief in a proposition
given some evidence that has already
been revealed

conditional probability

$$P(a \mid b)$$

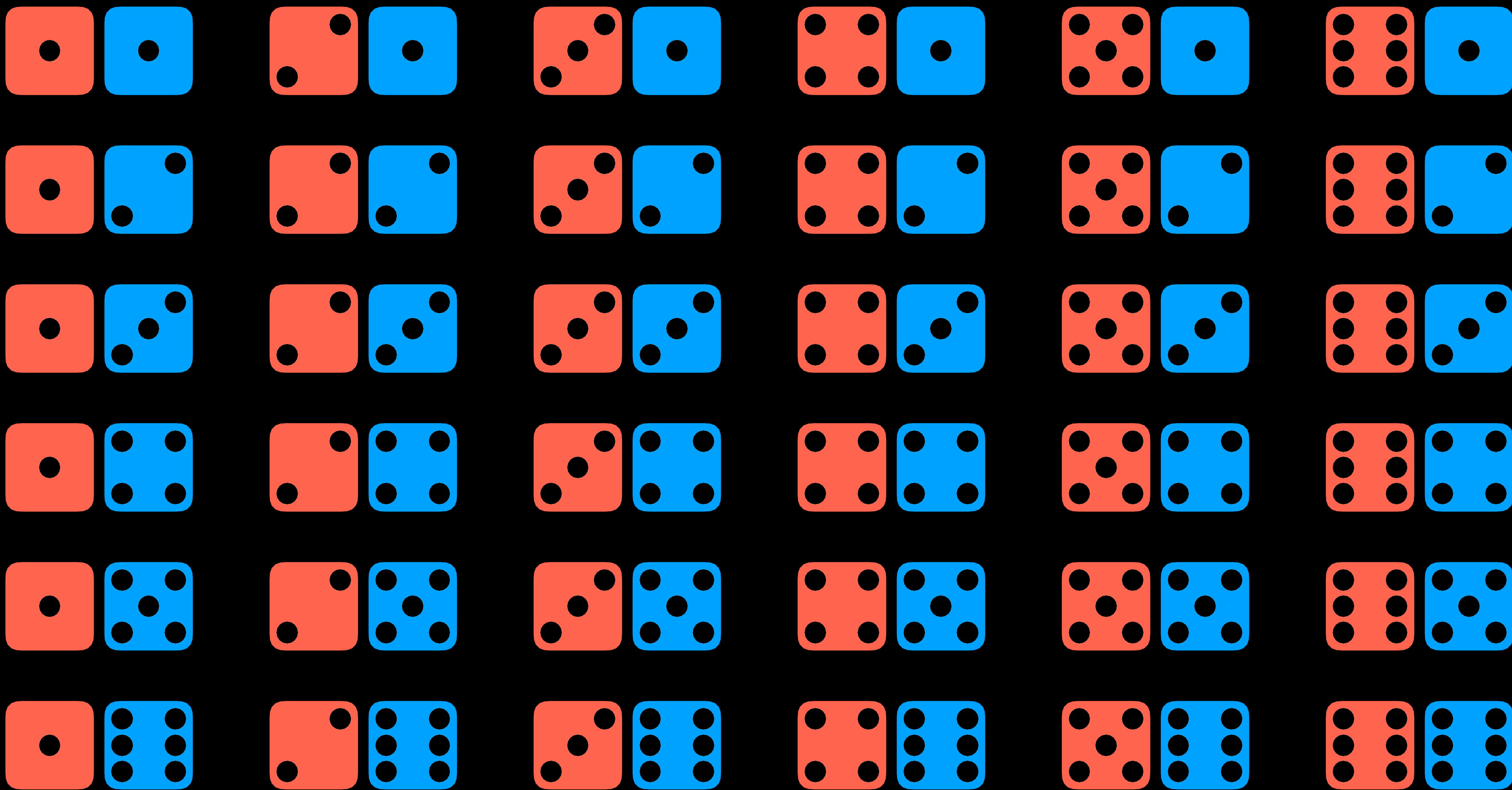
$P(\text{rain today} \mid \text{rain yesterday})$

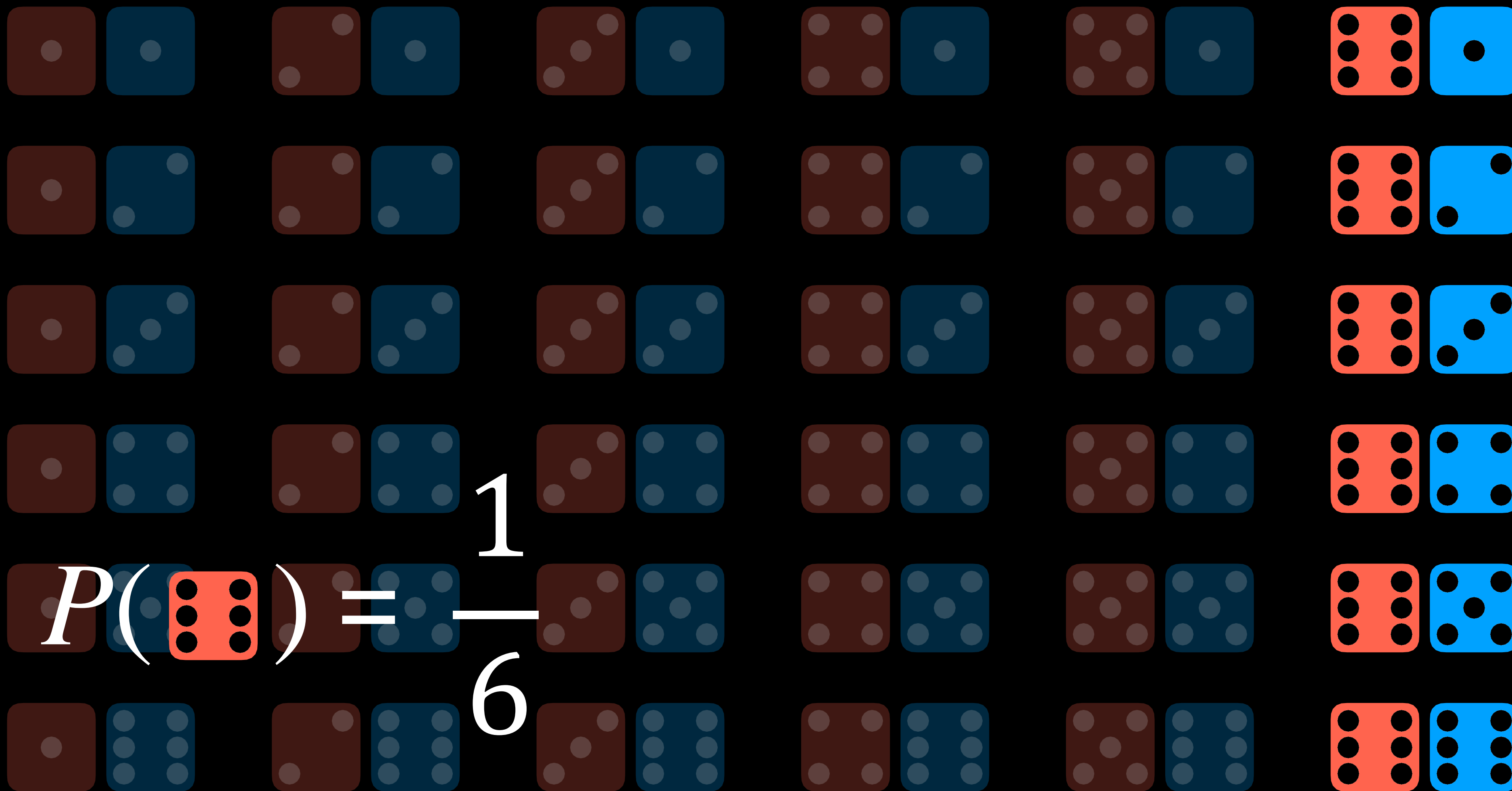
P(route change | traffic conditions)

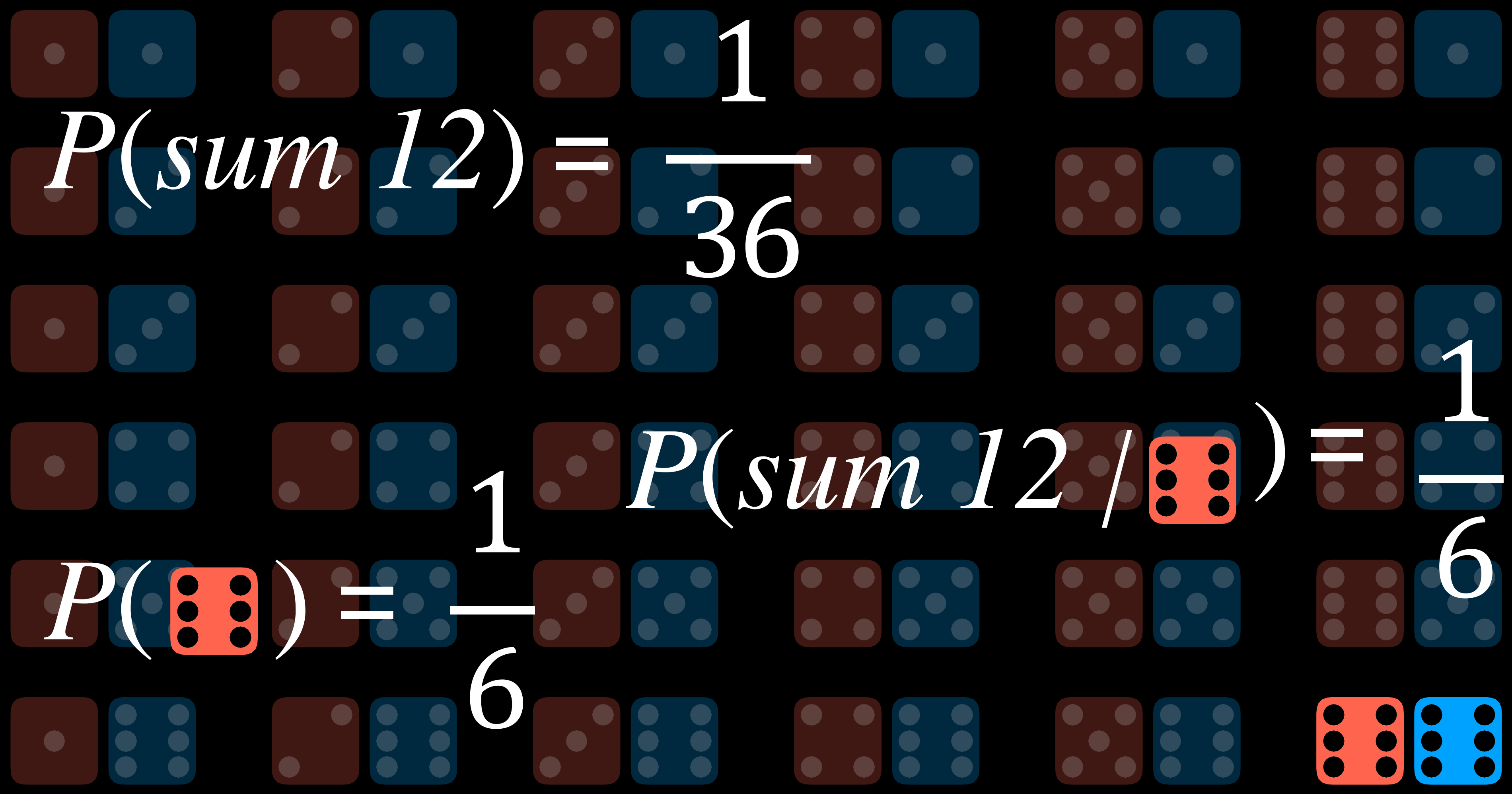
P(disease | test results)

$$P(a|b) = \frac{P(a \wedge b)}{P(b)}$$

$$P(\textit{sum } 12 \text{ / } \text{🎲})$$







$$P(\text{sum } 12) = \frac{1}{36}$$

$$P(\text{sum } 12 / \text{red die}) = \frac{1}{6}$$

$$P(\text{red die}) = \frac{1}{6}$$

$$P(a|b) = \frac{P(a \wedge b)}{P(b)}$$

$$P(a \wedge b) = P(b)P(a|b)$$

$$P(a \wedge b) = P(a)P(b|a)$$

random variable

a variable in probability theory with a domain of possible values it can take on

random variable

Roll

$\{1, 2, 3, 4, 5, 6\}$

random variable

Weather

$\{sun, cloud, rain, wind, snow\}$

random variable

Traffic

$\{none, light, heavy\}$

random variable

Flight

{on time, delayed, cancelled}

probability distribution

$$P(\textit{Flight} = \textit{on time}) = 0.6$$

$$P(\textit{Flight} = \textit{delayed}) = 0.3$$

$$P(\textit{Flight} = \textit{cancelled}) = 0.1$$

probability distribution

$$\mathbf{P}(\textit{Flight}) = \langle 0.6, 0.3, 0.1 \rangle$$

independence

the knowledge that one event occurs does
not affect the probability of the other event

independence

$$P(a \wedge b) = P(a)P(b | a)$$

independence

$$P(a \wedge b) = P(a)P(b)$$

independence

$$P(\text{red die} \text{ blue die}) = P(\text{red die})P(\text{blue die})$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

independence

$$P(\text{🎲🎲}) \neq P(\text{🎲})P(\text{🎲})$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

independence

$$P(\text{🎲} \text{🎲}) \neq P(\text{🎲})P(\text{🎲} \mid \text{🎲})$$

$$= \frac{1}{6} \cdot 0 = 0$$

Bayes' Rule

$$P(a \wedge b) = P(b) P(a | b)$$

$$P(a \wedge b) = P(a) P(b | a)$$

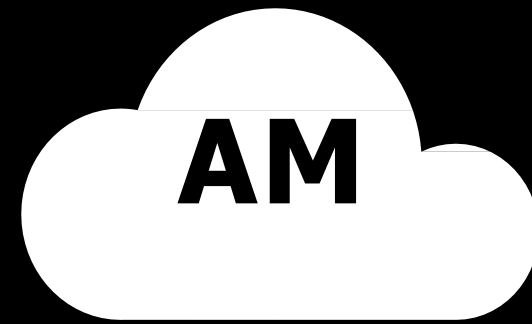
$$P(a) P(b | a) = P(b) P(a | b)$$

Bayes' Rule

$$P(b | a) = \frac{P(b) P(a | b)}{P(a)}$$

Bayes' Rule

$$P(b | a) = \frac{P(a | b) P(b)}{P(a)}$$



Given clouds in the morning,
what's the probability of rain in the afternoon?

- 80% of rainy afternoons start with cloudy mornings.
- 40% of days have cloudy mornings.
- 10% of days have rainy afternoons.

$$P(\textit{rain} \mid \textit{clouds}) = \frac{P(\textit{clouds} \mid \textit{rain})P(\textit{rain})}{P(\textit{clouds})}$$

$$= \frac{(.8)(.1)}{.4}$$

$$= 0.2$$

Knowing

P(cloudy morning | rainy afternoon)

we can calculate

P(rainy afternoon | cloudy morning)

Knowing

P(visible effect | unknown cause)

we can calculate

P(unknown cause | visible effect)

Knowing

$$P(\text{medical test result} \mid \text{disease})$$

we can calculate

$$P(\text{disease} \mid \text{medical test result})$$

Knowing

$$P(\textit{blurry text} \mid \textit{counterfeit bill})$$

we can calculate

$$P(\textit{counterfeit bill} \mid \textit{blurry text})$$

Joint Probability



$C = \textit{cloud}$	$C = \neg \textit{cloud}$
0.4	0.6

$R = \textit{rain}$	$R = \neg \textit{rain}$
0.1	0.9



	$R = \textit{rain}$	$R = \neg \textit{rain}$
$C = \textit{cloud}$	0.08	0.32
$C = \neg \textit{cloud}$	0.02	0.58

$\mathbf{P(C \mid rain)}$

$$\mathbf{P(C \mid rain)} = \frac{\mathbf{P(C, rain)}}{\mathbf{P(rain)}} = \alpha \mathbf{P(C, rain)}$$

$$= \alpha \langle 0.08, 0.02 \rangle = \langle 0.8, 0.2 \rangle$$

	$R = rain$	$R = \neg rain$
$C = cloud$	0.08	0.32
$C = \neg cloud$	0.02	0.58

Probability Rules

Negation

$$P(\neg a) = 1 - P(a)$$

Inclusion-Exclusion

$$P(a \vee b) = P(a) + P(b) - P(a \wedge b)$$

Marginalization

$$P(a) = P(a, b) + P(a, \neg b)$$

Marginalization

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j)$$

Marginalization

	$R = \textit{rain}$	$R = \neg \textit{rain}$
$C = \textit{cloud}$	0.08	0.32
$C = \neg \textit{cloud}$	0.02	0.58

$$P(C = \textit{cloud})$$

$$= P(C = \textit{cloud}, R = \textit{rain}) + P(C = \textit{cloud}, R = \neg \textit{rain})$$

$$= 0.08 + 0.32$$

$$= 0.40$$

Conditioning

$$P(a) = P(a | b)P(b) + P(a | \neg b)P(\neg b)$$

Conditioning

$$P(X = x_i) = \sum_j P(X = x_i | Y = y_j) P(Y = y_j)$$

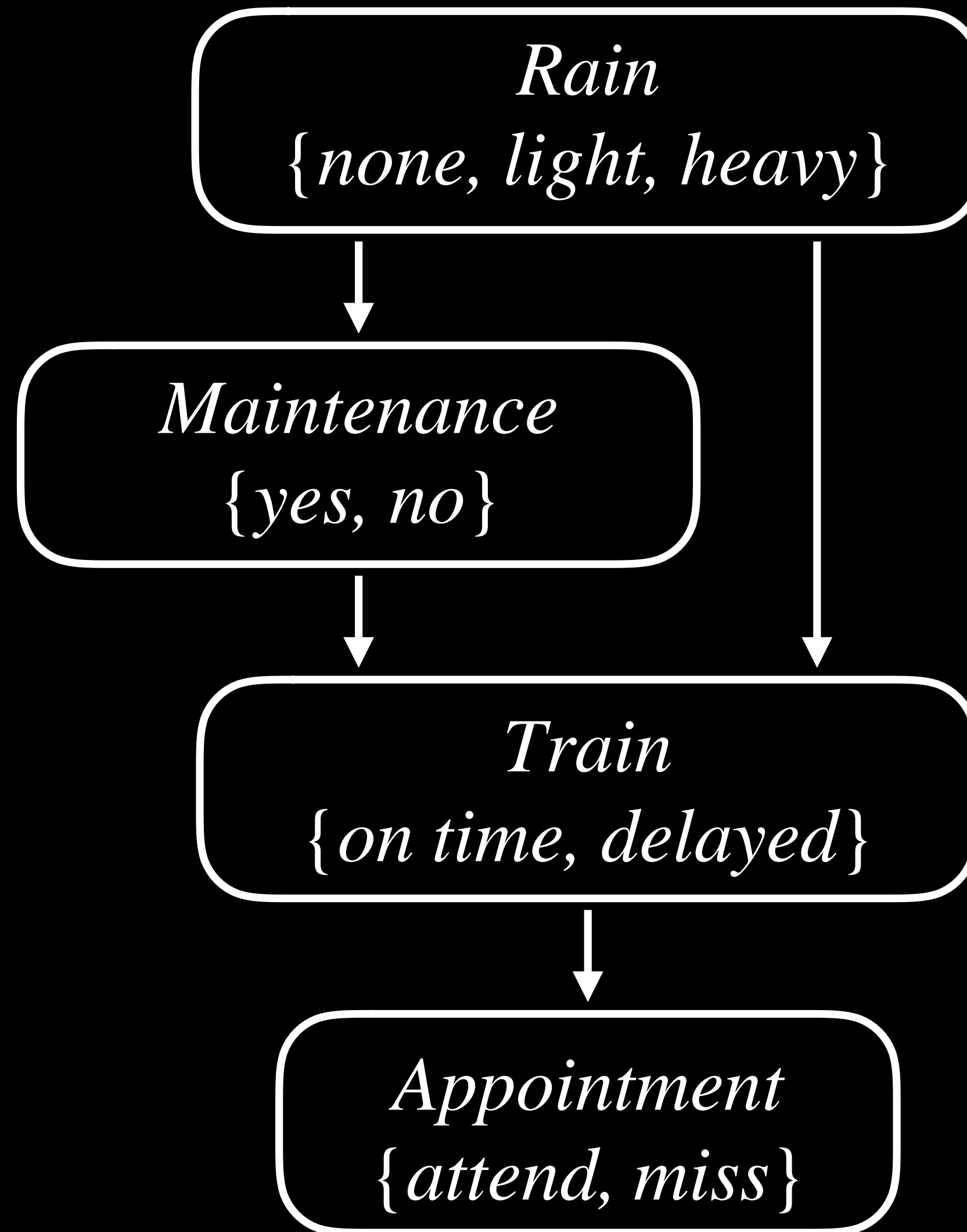
Bayesian Networks

Bayesian network

data structure that represents the dependencies among random variables

Bayesian network

- directed graph
- each node represents a random variable
- arrow from X to Y means X is a parent of Y
- each node X has probability distribution $\mathbf{P}(X / \textit{Parents}(X))$



Rain
{none, light, heavy}

<i>none</i>	<i>light</i>	<i>heavy</i>
0.7	0.2	0.1

Rain
{none, light, heavy}



Maintenance
{yes, no}



<i>R</i>	<i>yes</i>	<i>no</i>
<i>none</i>	0.4	0.6
<i>light</i>	0.2	0.8
<i>heavy</i>	0.1	0.9

Rain
{none, light, heavy}



Maintenance
{yes, no}



Train
{on time, delayed}

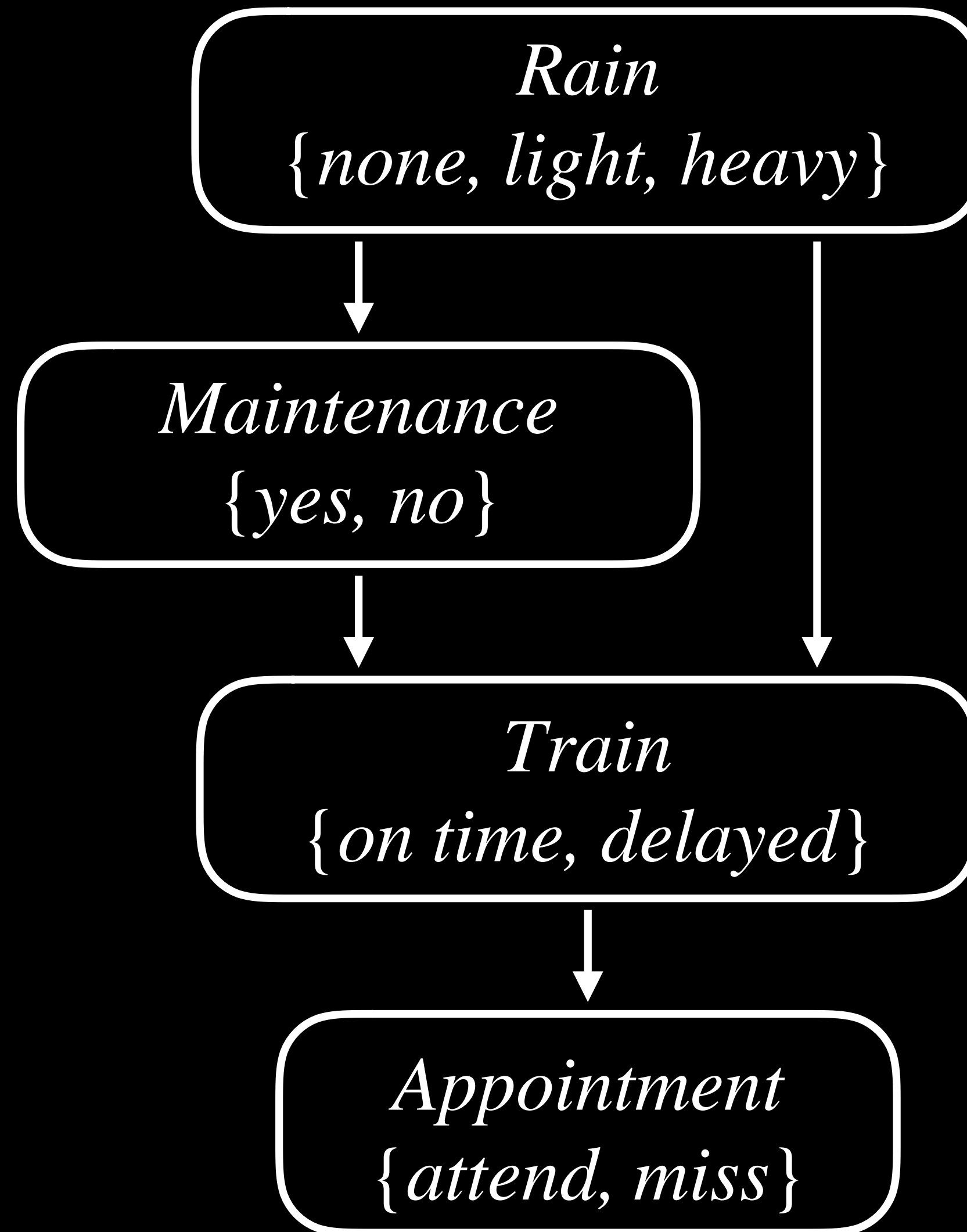
<i>R</i>	<i>M</i>	<i>on time</i>	<i>delayed</i>
<i>none</i>	<i>yes</i>	0.8	0.2
<i>none</i>	<i>no</i>	0.9	0.1
<i>light</i>	<i>yes</i>	0.6	0.4
<i>light</i>	<i>no</i>	0.7	0.3
<i>heavy</i>	<i>yes</i>	0.4	0.6
<i>heavy</i>	<i>no</i>	0.5	0.5

Maintenance
{*yes*, *no*}

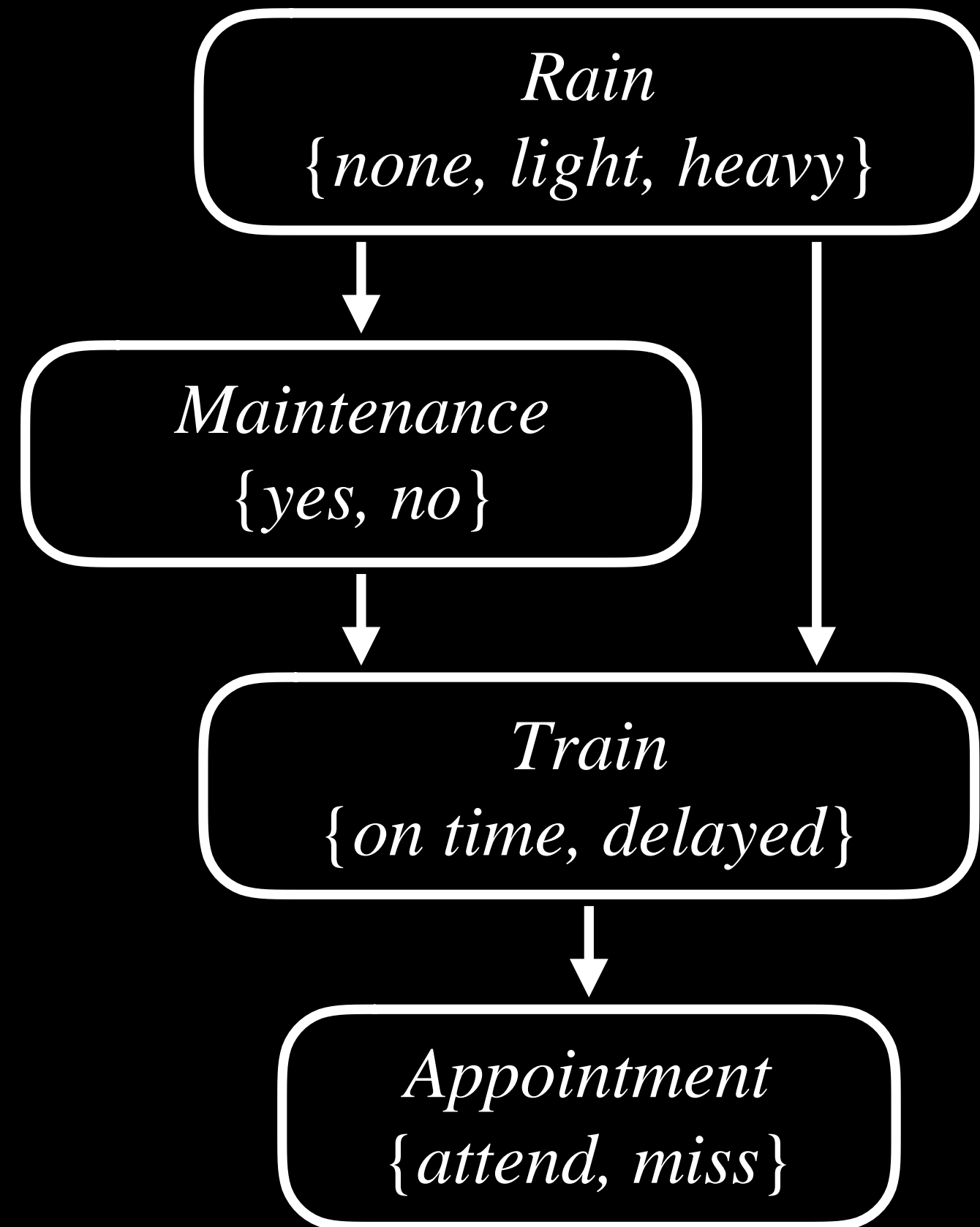
Train
{*on time*, *delayed*}

Appointment
{*attend*, *miss*}

<i>T</i>	<i>attend</i>	<i>miss</i>
<i>on time</i>	0.9	0.1
<i>delayed</i>	0.6	0.4



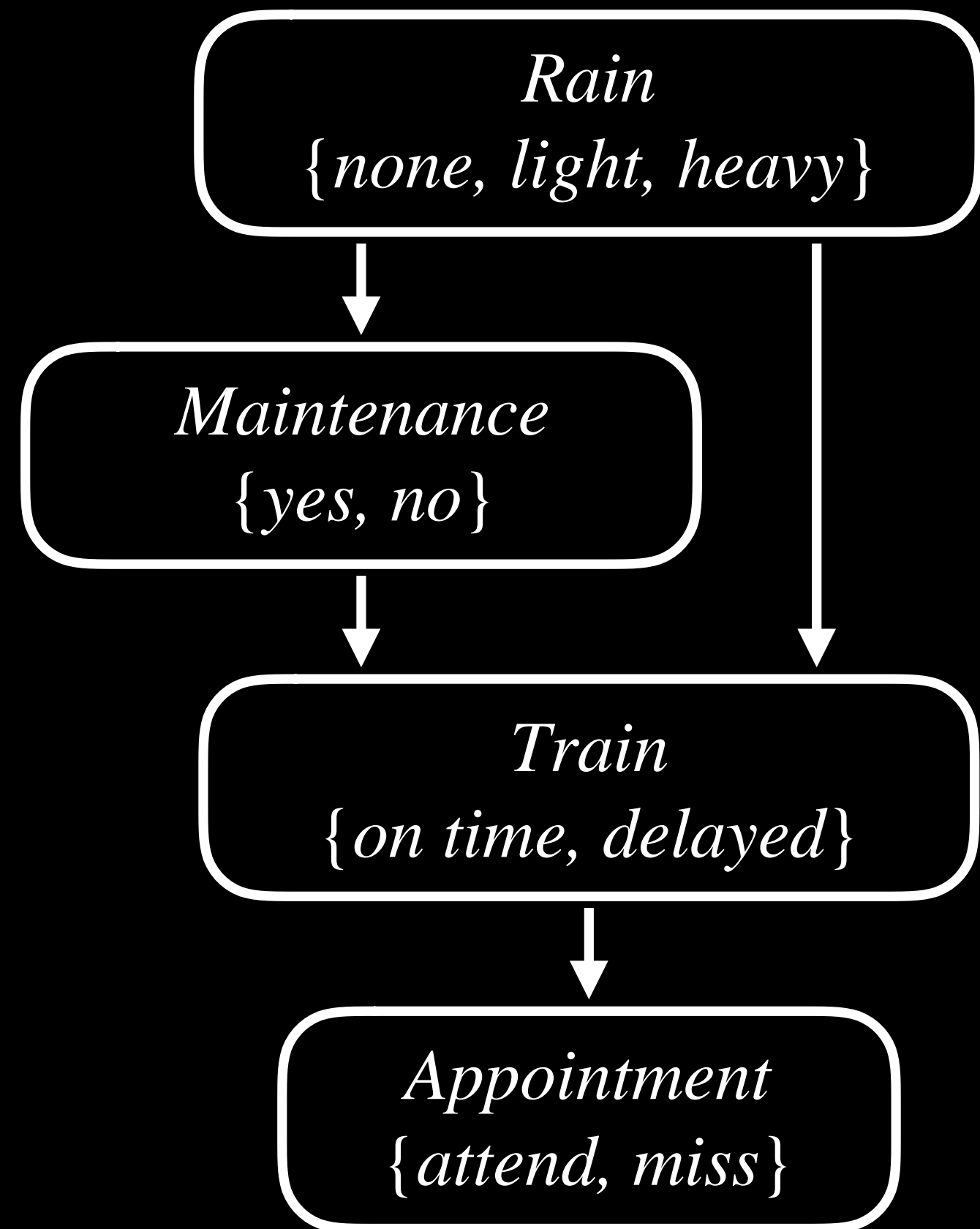
Computing Joint Probabilities



$P(\textit{light})$

$P(\textit{light})$

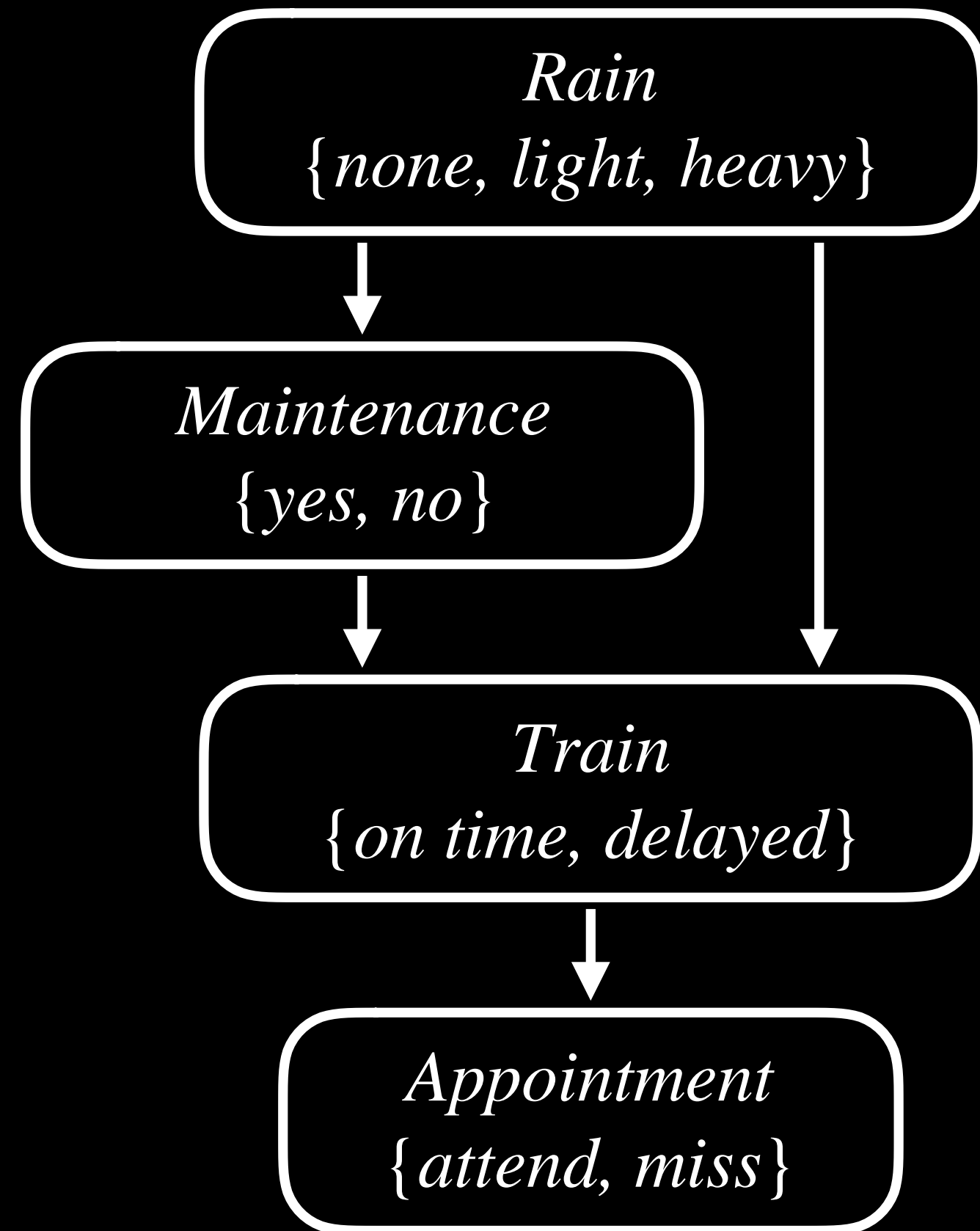
Computing Joint Probabilities



$P(\textit{light}, \textit{no})$

$P(\textit{light}) P(\textit{no} \mid \textit{light})$

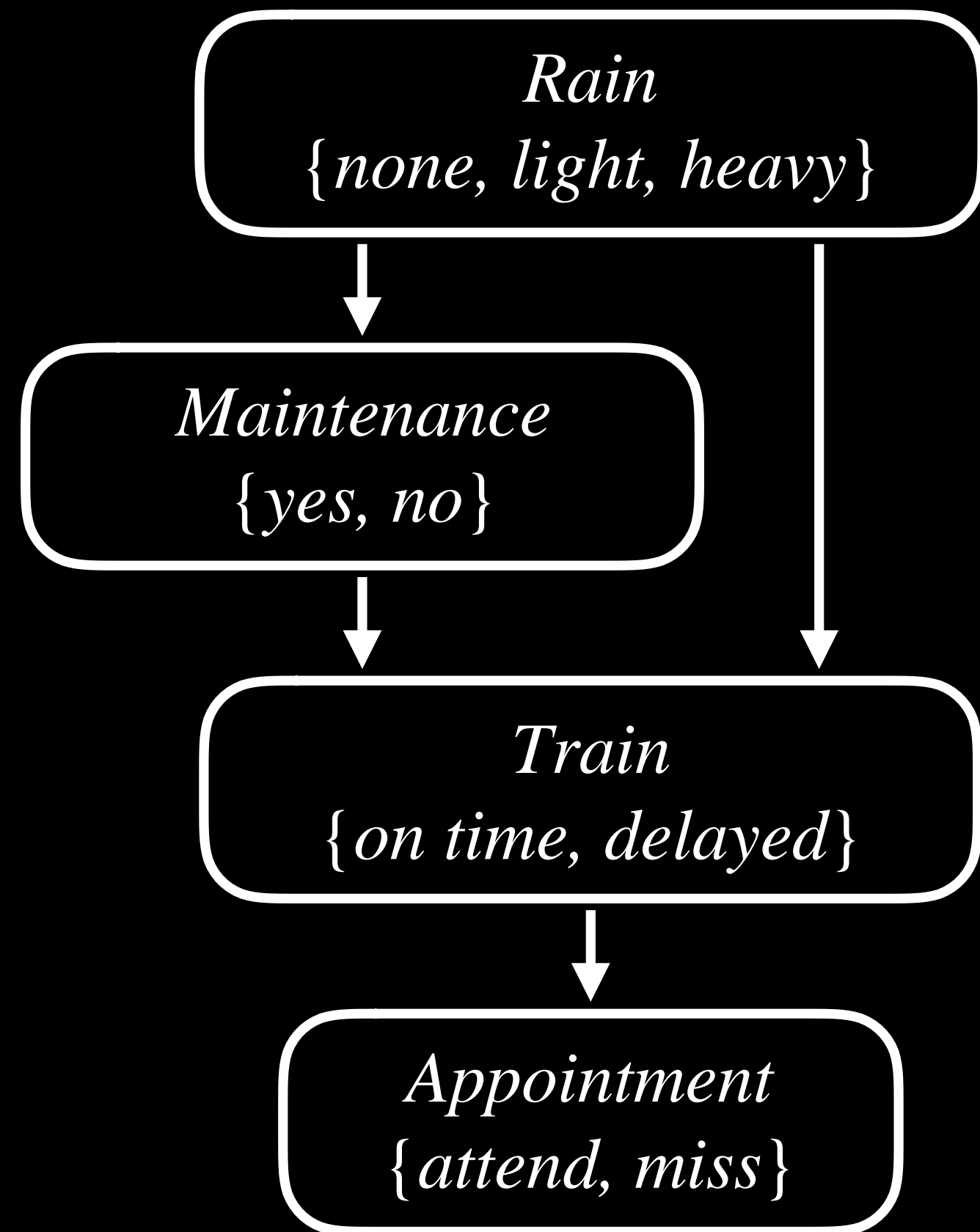
Computing Joint Probabilities



$P(\text{light}, \text{no}, \text{delayed})$

$P(\text{light}) P(\text{no} \mid \text{light}) P(\text{delayed} \mid \text{light}, \text{no})$

Computing Joint Probabilities



$$P(\text{light}, \text{no}, \text{delayed}, \text{miss})$$

$$P(\text{light}) P(\text{no} \mid \text{light}) P(\text{delayed} \mid \text{light}, \text{no}) P(\text{miss} \mid \text{delayed})$$

Inference

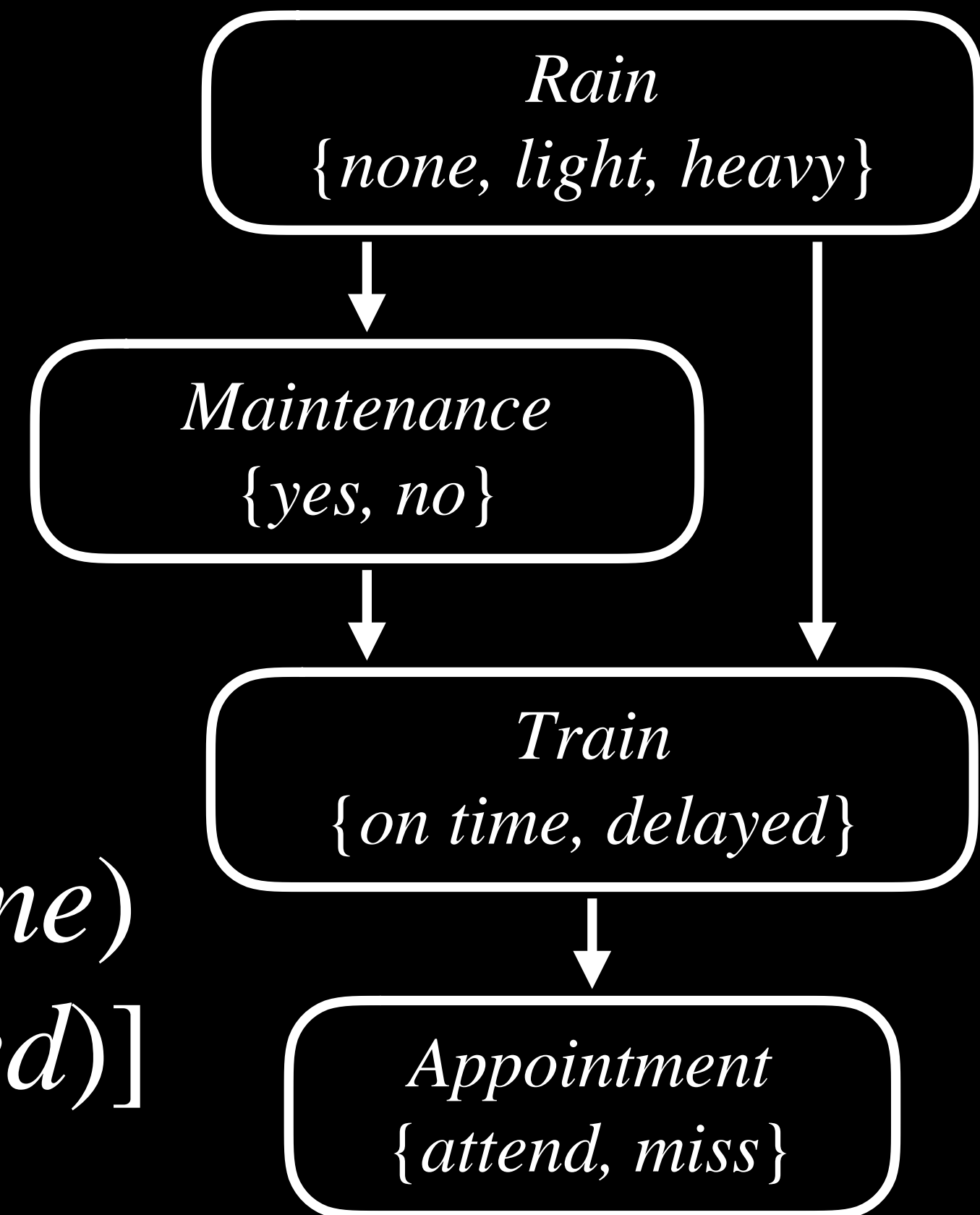
Inference

- Query X : variable for which to compute distribution
- Evidence variables E : observed variables for event e
- Hidden variables Y : non-evidence, non-query variable.
- Goal: Calculate $P(X \mid e)$

$P(\text{Appointment} \mid \text{light}, \text{no})$

$= \alpha P(\text{Appointment}, \text{light}, \text{no})$

$= \alpha [P(\text{Appointment}, \text{light}, \text{no}, \text{on time})$
 $+ P(\text{Appointment}, \text{light}, \text{no}, \text{delayed})]$



Inference by Enumeration

$$\mathbf{P}(X \mid \mathbf{e}) = \alpha \mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$

X is the query variable.

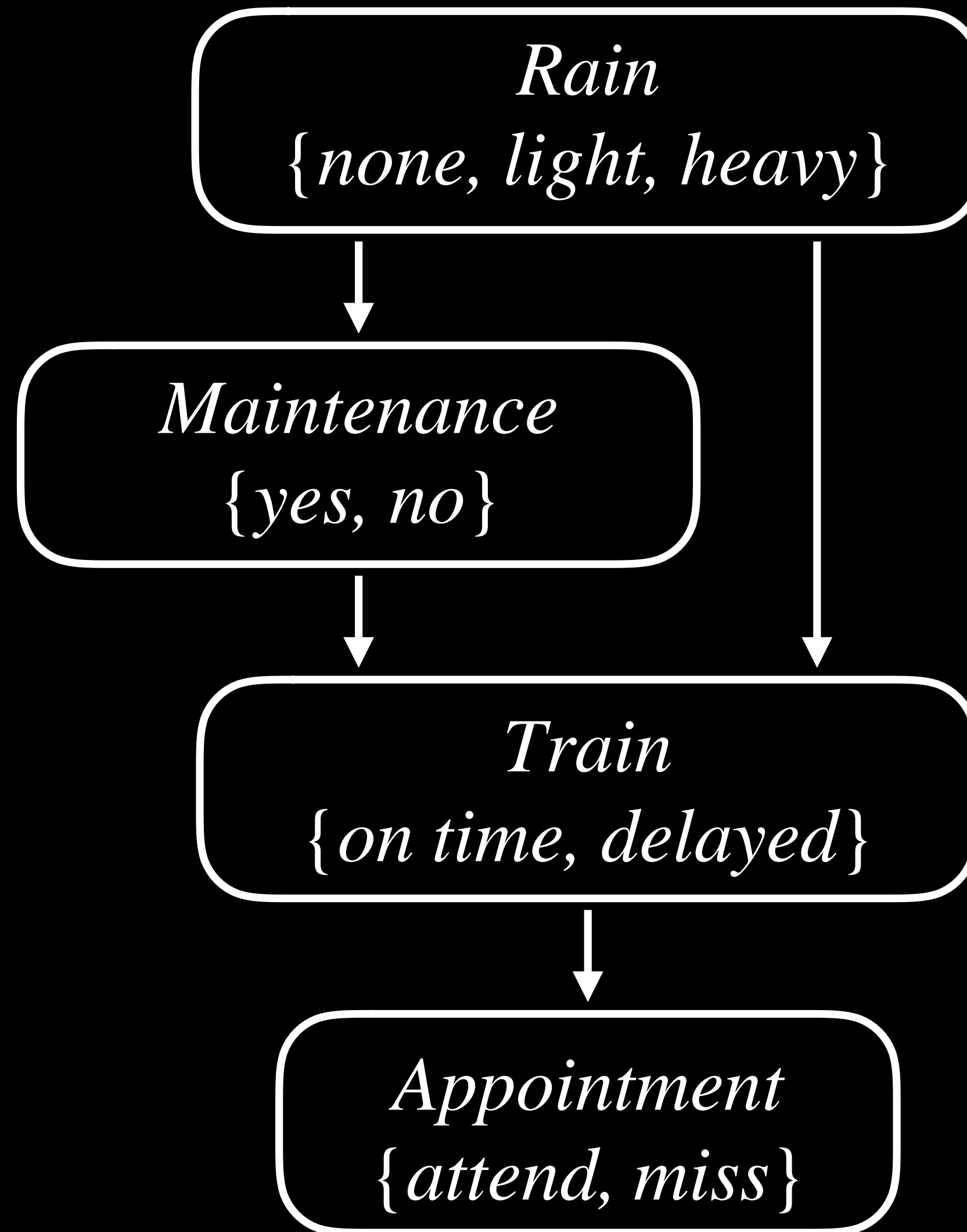
\mathbf{e} is the evidence.

\mathbf{y} ranges over values of hidden variables.

α normalizes the result.

Approximate Inference

Sampling



Rain
{none, light, heavy}

$R = \textit{none}$

<i>none</i>	<i>light</i>	<i>heavy</i>
0.7	0.2	0.1

Rain
{*none, light, heavy*}



Maintenance
{*yes, no*}



$R = none$
$M = yes$

<i>R</i>	<i>yes</i>	<i>no</i>
<i>none</i>	0.4	0.6
<i>light</i>	0.2	0.8
<i>heavy</i>	0.1	0.9

Rain
{none, light, heavy}



Maintenance
{yes, no}



Train
{on time, delayed}



$R = none$
$M = yes$
$T = on\ time$

<i>R</i>	<i>M</i>	<i>on time</i>	<i>delayed</i>
<i>none</i>	<i>yes</i>	0.8	0.2
<i>none</i>	<i>no</i>	0.9	0.1
<i>light</i>	<i>yes</i>	0.6	0.4
<i>light</i>	<i>no</i>	0.7	0.3
<i>heavy</i>	<i>yes</i>	0.4	0.6
<i>heavy</i>	<i>no</i>	0.5	0.5

Maintenance
{yes, no}

Train
{on time, delayed}

Appointment
{attend, miss}

R = none

M = yes

T = on time

A = attend

<i>T</i>	<i>attend</i>	<i>miss</i>
<i>on time</i>	0.9	0.1
<i>delayed</i>	0.6	0.4

$R = \textit{none}$

$M = \textit{yes}$

$T = \textit{on time}$

$A = \textit{attend}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{yes}$
$T = \textit{delayed}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{heavy}$
$M = \textit{no}$
$T = \textit{delayed}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$P(\textit{Train} = \textit{on time}) ?$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{yes}$
$T = \textit{delayed}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{heavy}$
$M = \textit{no}$
$T = \textit{delayed}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{yes}$
$T = \textit{delayed}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{heavy}$
$M = \textit{no}$
$T = \textit{delayed}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$P(\text{Rain} = \textit{light} \mid \text{Train} = \textit{on time}) ?$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{yes}$
$T = \textit{delayed}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{heavy}$
$M = \textit{no}$
$T = \textit{delayed}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{yes}$
$T = \textit{delayed}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{heavy}$
$M = \textit{no}$
$T = \textit{delayed}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{yes}$
$T = \textit{delayed}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{none}$
$M = \textit{yes}$
$T = \textit{on time}$
$A = \textit{attend}$

$R = \textit{heavy}$
$M = \textit{no}$
$T = \textit{delayed}$
$A = \textit{miss}$

$R = \textit{light}$
$M = \textit{no}$
$T = \textit{on time}$
$A = \textit{attend}$

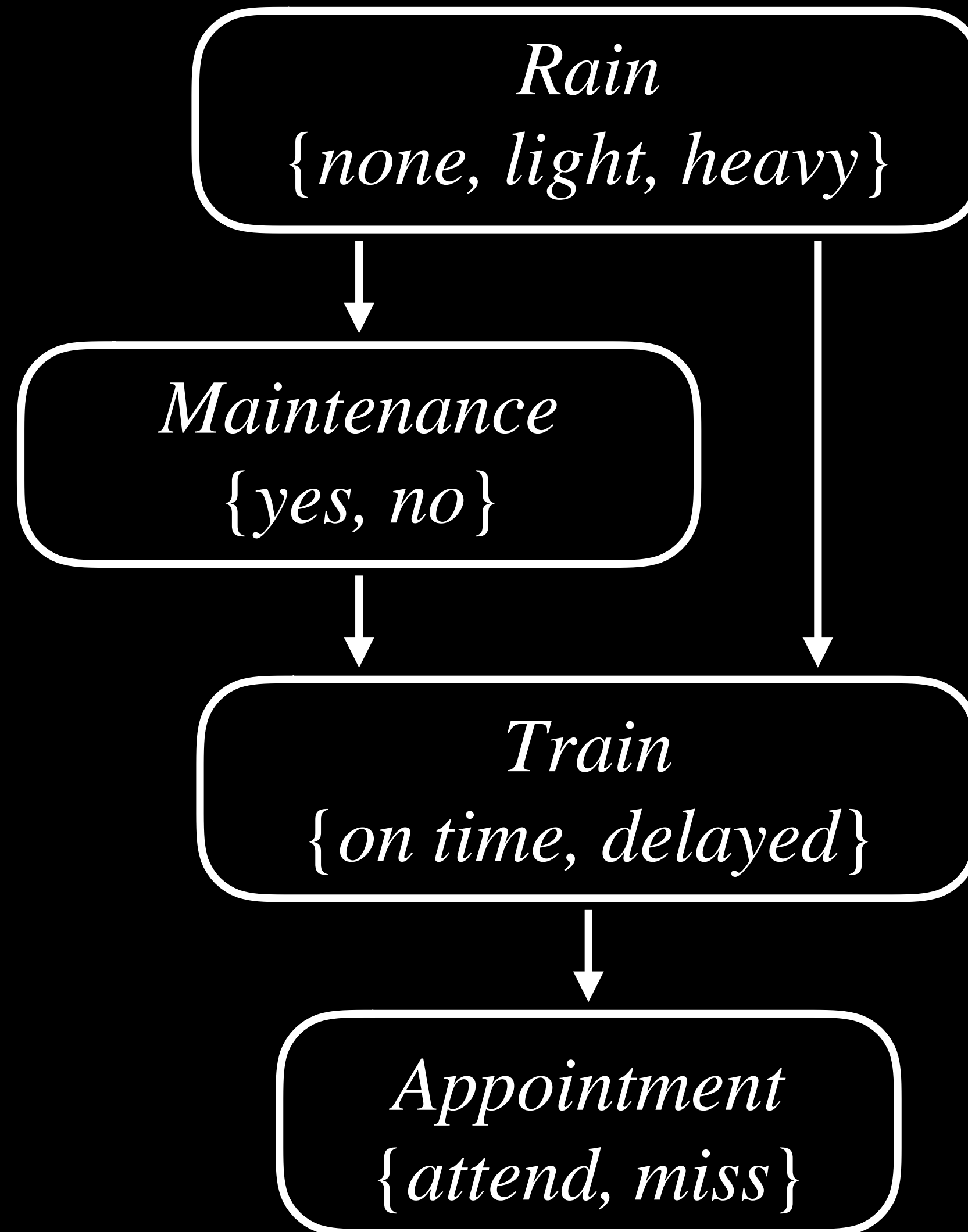
Rejection Sampling

Likelihood Weighting

Likelihood Weighting

- Start by fixing the values for evidence variables.
- Sample the non-evidence variables using conditional probabilities in the Bayesian Network.
- Weight each sample by its likelihood: the probability of all of the evidence.

$P(\text{Rain} = \textit{light} \mid \text{Train} = \textit{on time}) ?$



$R = \textit{light}$

$T = \textit{on time}$

Rain
 $\{\textit{none}, \textit{light}, \textit{heavy}\}$

none

light

heavy

0.7

0.2

0.1

Rain
{none, light, heavy}



Maintenance
{yes, no}



$R = \textit{light}$
$M = \textit{yes}$
$T = \textit{on time}$

R	yes	no
$none$	0.4	0.6
$light$	0.2	0.8
$heavy$	0.1	0.9

Rain
{none, light, heavy}



Maintenance
{yes, no}



Train
{on time, delayed}



$R = \textit{light}$
$M = \textit{yes}$
$T = \textit{on time}$

<i>R</i>	<i>M</i>	<i>on time</i>	<i>delayed</i>
<i>none</i>	<i>yes</i>	0.8	0.2
<i>none</i>	<i>no</i>	0.9	0.1
<i>light</i>	<i>yes</i>	0.6	0.4
<i>light</i>	<i>no</i>	0.7	0.3
<i>heavy</i>	<i>yes</i>	0.4	0.6
<i>heavy</i>	<i>no</i>	0.5	0.5

Maintenance
{yes, no}

Train
{on time, delayed}

Appointment
{attend, miss}

$R = \textit{light}$

$M = \textit{yes}$

$T = \textit{on time}$

$A = \textit{attend}$

<i>T</i>	<i>attend</i>	<i>miss</i>
<i>on time</i>	0.9	0.1
<i>delayed</i>	0.6	0.4

Rain
{none, light, heavy}



Maintenance
{yes, no}



Train
{on time, delayed}



$R = \textit{light}$

$M = \textit{yes}$

$T = \textit{on time}$

$A = \textit{attend}$

<i>R</i>	<i>M</i>	<i>on time</i>	<i>delayed</i>
<i>none</i>	<i>yes</i>	0.8	0.2
<i>none</i>	<i>no</i>	0.9	0.1
<i>light</i>	<i>yes</i>	0.6	0.4
<i>light</i>	<i>no</i>	0.7	0.3
<i>heavy</i>	<i>yes</i>	0.4	0.6
<i>heavy</i>	<i>no</i>	0.5	0.5

Rain
{none, light, heavy}



Maintenance
{yes, no}



Train
{on time, delayed}



$R = \textit{light}$

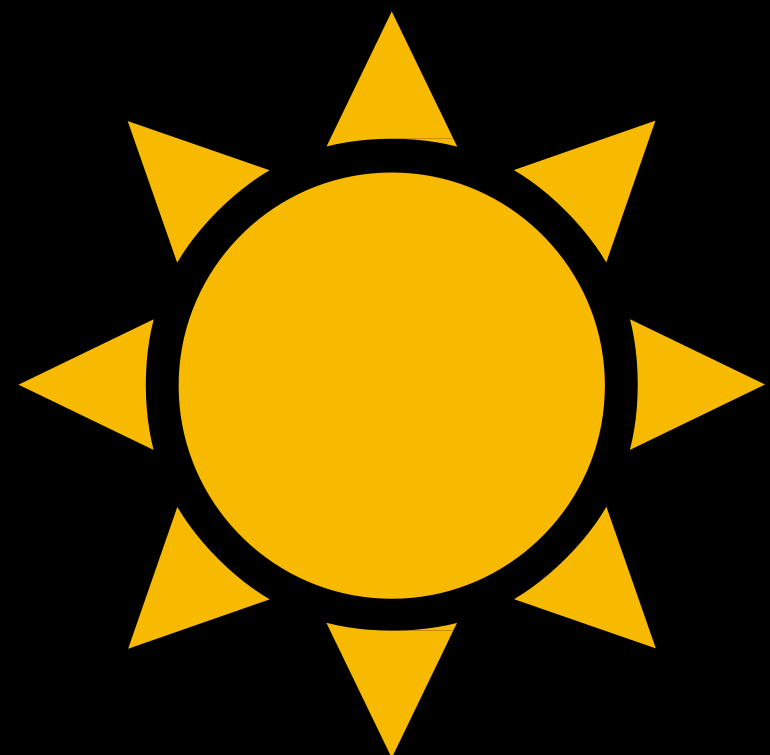
$M = \textit{yes}$

$T = \textit{on time}$

$A = \textit{attend}$

<i>R</i>	<i>M</i>	<i>on time</i>	<i>delayed</i>
<i>none</i>	<i>yes</i>	0.8	0.2
<i>none</i>	<i>no</i>	0.9	0.1
<i>light</i>	<i>yes</i>	0.6	0.4
<i>light</i>	<i>no</i>	0.7	0.3
<i>heavy</i>	<i>yes</i>	0.4	0.6
<i>heavy</i>	<i>no</i>	0.5	0.5

Uncertainty over Time



X_t : Weather at time t

Markov assumption



the assumption that the current state depends on only a finite fixed number of previous states

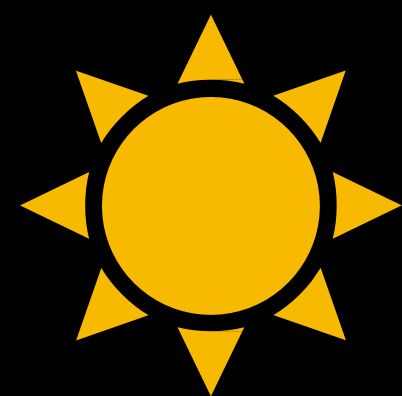
Markov Chain

Markov chain

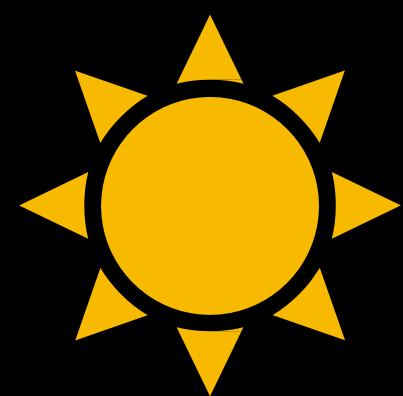
a sequence of random variables where the distribution of each variable follows the Markov assumption

Transition Model

		Tomorrow (X_{t+1})	
Today (X_t)		0.8	0.2
		0.3	0.7



X_0



X_1



X_2



X_3



X_4



Sensor Models

Hidden State	Observation
robot's position	robot's sensor data
words spoken	audio waveforms
user engagement	website or app analytics
weather	umbrella

Hidden Markov Models





Hidden Markov Model

a Markov model for a system with hidden states that generate some observed event

Sensor Model

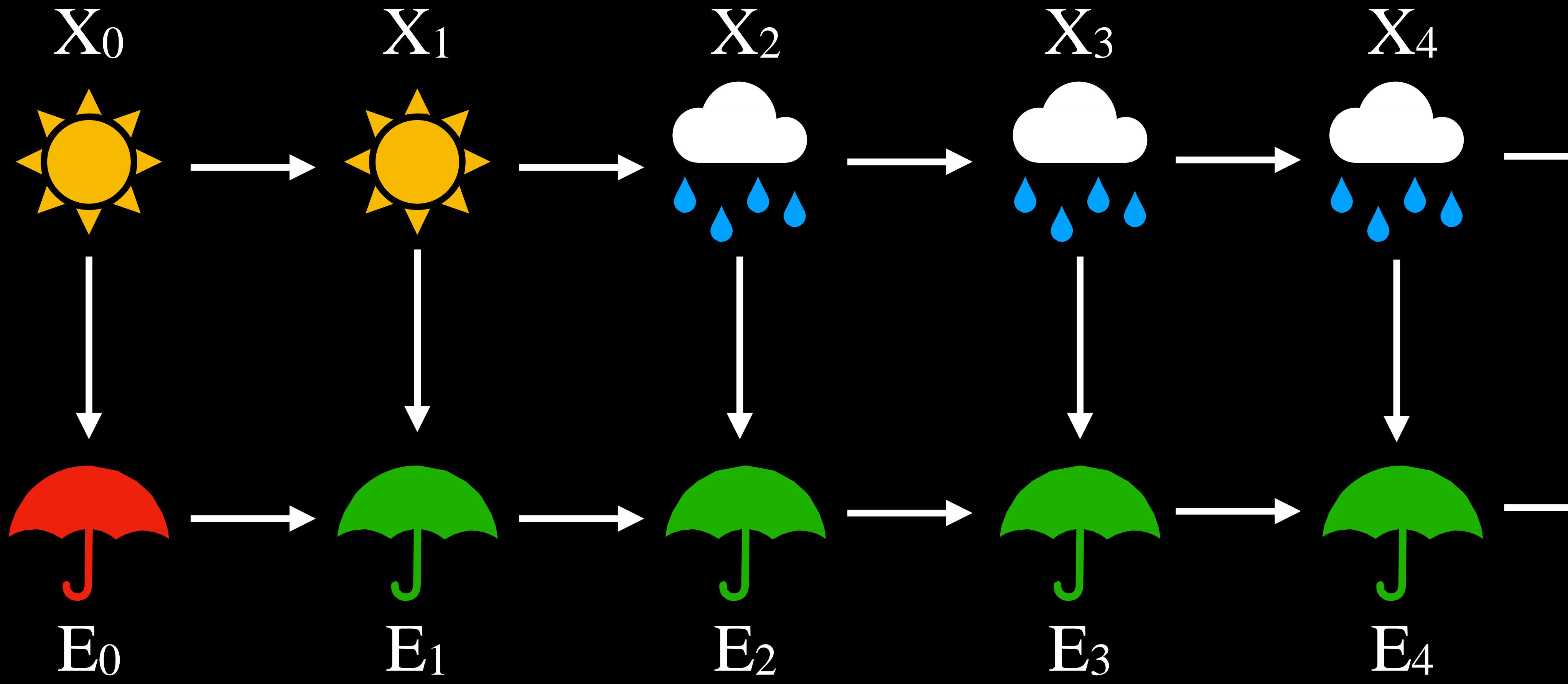
Observation (E_t)

State (X_t)

		
	0.2	0.8
	0.9	0.1

sensor Markov assumption

the assumption that the evidence variable depends only the corresponding state



Task	Definition
filtering	given observations from start until now, calculate distribution for current state
prediction	given observations from start until now, calculate distribution for a future state
smoothing	given observations from start until now, calculate distribution for past state
most likely explanation	given observations from start until now, calculate most likely sequence of states