Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2.1 – 2.3

Sets

Section 2.1

Sets

A *set* is an unordered collection of objects.

- the students in this class
- the chairs in this room

The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.

The notation $a \in A$ denotes that a is an element of the set A.

If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

$S = \{a,b,c,d\}$

Order not important

$S = \{a,b,c,d\} = \{b,c,a,d\}$

Each distinct object is either a member or not; listing more than once does not change the set.

$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$S = \{a,b,c,d, \ldots\ldots,z\}$

Roster Method

Set of all vowels in the English alphabet:

$V = \{a,e,i,o,u\}$

Set of all odd positive integers less than 10:

$O = \{1,3,5,7,9\}$

Set of all positive integers less than 100:

$S = \{1,2,3,........,99\}$

Set of all integers less than 0:

$S = \{...., -3,-2,-1\}$

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

A predicate may be used:

$S = \{x \mid P(x)\}$

Positive rational numbers:

$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

Z$^+$ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

R$^+$ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Interval Notation

closed interval [a,b]

open interval (a,b)

$$[a, b] = \{\, x \mid a \le x \le b \}$$

$$[a,b) = \{\, x \mid a \le x < b \}$$

$$(a,b] = \{\, x \mid a < x \le b \}$$

$$(a, b) = \{\, x \mid a < x < b \}$$

Some things to remember

Sets can be elements of sets.

$\{\{1,2,3\}, a, \{b,c\}\}$

$\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

The empty set is different from a set containing the empty set.

$\emptyset \neq \{\emptyset\}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$
- We write *A* = *B* if *A* and *B* are equal sets.

{1,3,5} = {3, 5, 1}

{1,5,5,5,3,3,1} = {1,3,5}

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.

Showing a Set is or is not a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of negative integers.

Def 1 : A set is an unordered collection of objects.

Def 2 : The objects in a set are called the elements (元素), or members of the set.

Def 3 : A,B: sets.

$A=B$ iff $\forall x\,(x \in A \leftrightarrow x \in B)$

Def 4 : $A \subseteq B$ iff $\forall x\,(x \in A \rightarrow x \in B)$

(A是B的子集合，補充： $A \subset B$ 表示$A \subseteq B$ 但 $A \neq B$

$x \in A$ 表示A是一個集合，而x是A中的元素，例如：$A = \{x, y, z\}$

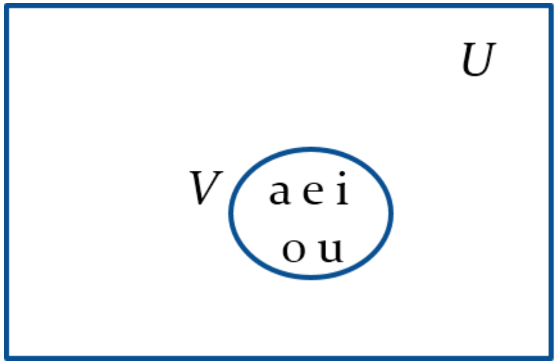
- $\varnothing = \{\ \}$ 表示空集合
- 當集合的元素很多，無法一一列舉，元素又具備某些特性時，可使用如下表示法：
  $Q = \{\ x \mid x$的特性 $\}$
  如$Q = \{\ x \mid x$ 是奇數，$3<x<100\ \}$

Universal Set and Empty Set

The *universal set U* is the set containing everything currently under consideration.

- Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

Venn Diagram



The empty set is the set with no elements. Symbolized ∅, but {} also used.

John Venn (1834-1923)
Cambridge, UK

Another look at Equality of Sets

Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x\left(x \in A \leftrightarrow x \in B\right)$$

Using logical equivalences we have that $A = B$ iff

$$\forall x\left[\left(x \in A \rightarrow x \in B\right) \wedge \left(x \in B \rightarrow x \in A\right)\right]$$

This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

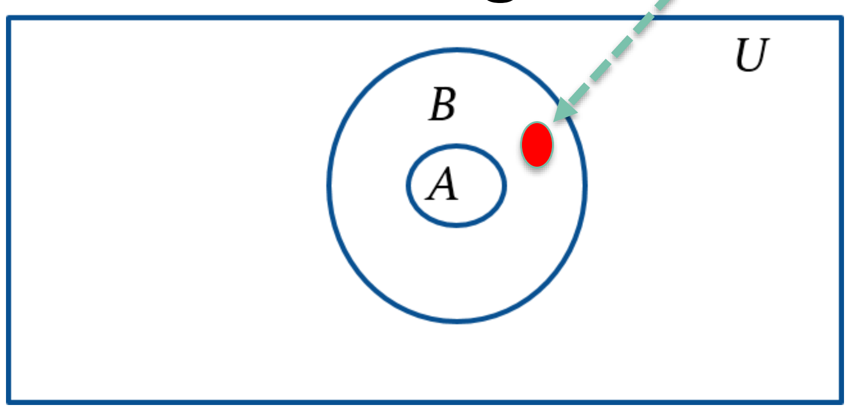
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x \wedge (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality
基數,元素個數

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A*, denoted by $|A|$, is the number of (distinct) elements of *A*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets
冪集合

Definition: The set of all subsets of a set *A*, **denoted *P*(*A*),** is called the *power set* of *A*.

Example: If *A* = {a,b} then

$$P(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

If a set has *n* elements, then the cardinality of the power set is 2^n.

Exercise

$S = \{0,1,2\}$，求 $|S|$=? P(S)=?

Tuples

The *ordered n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product(笛卡爾積)

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product 2

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n =$$

$$\left\{ \left(a_1, a_2 \cdots, a_n \right) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n \right\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** $\{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

Exercise

- $A = \{0,1\}$, $B = \{x,y\}$, $C=\{a,b,c\}$
- $A \times B \times C$ =?

- ANS：

Truth Sets of Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the truth set of P to be the set of elements x in D for which P (x) is true. The truth set of P (x) is denoted by

$$\{x \in D \mid P(x)\}$$

Example: The truth set of $P(x)$ where the domain is the integers and $P(x)$ is "$|x| = 1$" is the set {-1,1}

Exercise

What are the truth sets of the predicates P (x), Q(x), and R(x), where the domain is the set of integers and P (x) is "$|x| = 1$," Q(x) is "$x^2 = 2$," and R(x) is "$|x| = x$."

Solution:

1. The truth set of P, $\{x \in Z \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x, we see that the truth set of P is the set $\{-1, 1\}$.

Set Operations

Section 2.2

Section Summary 2

Set Operations

- Union
- Intersection
- Complementation
- Difference

More on Set Cardinality

Set Identities

Proving Identities

Membership Tables

Boolean Algebra

Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*.

The operators in set theory are analogous to the corresponding operator in propositional calculus.

As always there must be a universal set U. All sets are assumed to be subsets of U.

Union

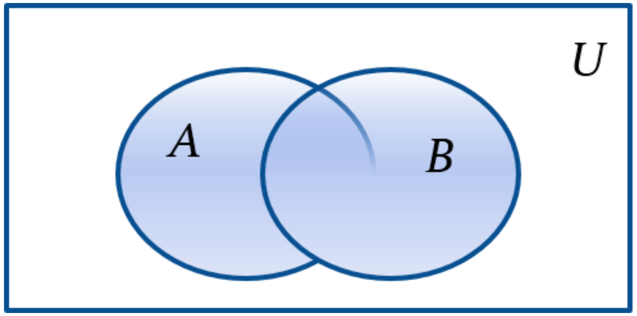
Definition: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

Example: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

Definition: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is

$$\{x \mid x \in A \wedge x \in B\}$$

Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.

Example: What is? $\{1,2,3\} \cap \{3,4,5\}$?

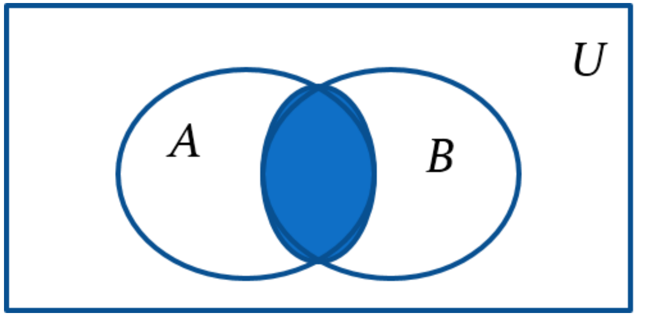
Solution: $\{3\}$

Example: What is?

$\{1,2,3\} \cap \{4,5,6\}$?

Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

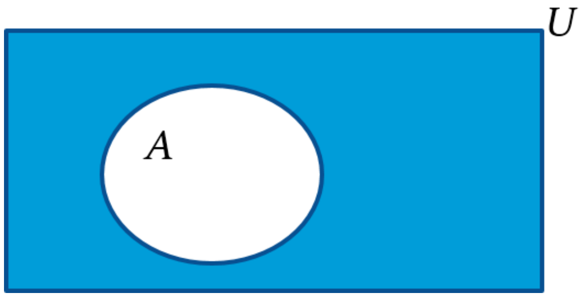
$$\bar{A} = \{x \mid x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

$$\text{Solution}: \{x \mid x \leq 70\}$$

Venn Diagram for Complement

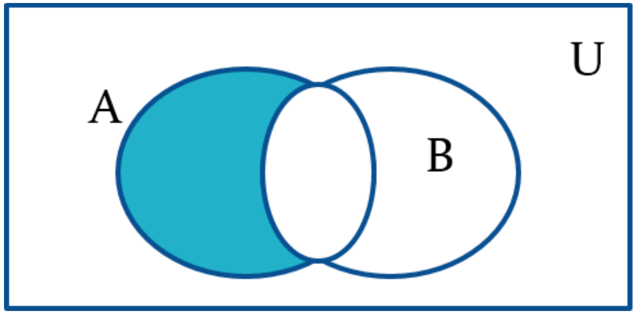


Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$



The Cardinality of the Union of Two Sets

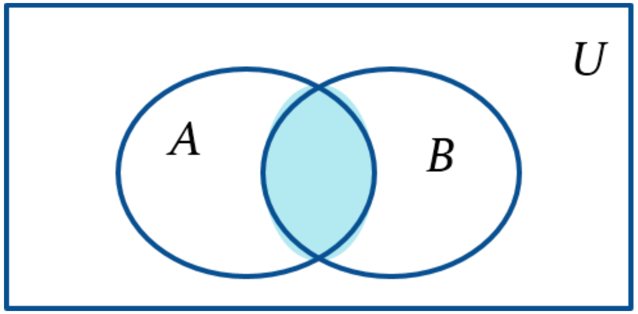
Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Venn Diagram for A, B, A ∩ B, A ∪ B



Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution:

2. $A \cap B$

   Solution:

3. $\bar{A}$

   Solution:

4. $A - B$

   Solution:

5. $B - A$

   Solution:

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

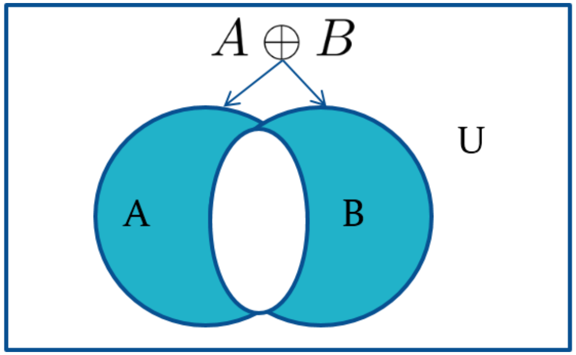
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

Solution: $\{1,2,3,6,7,8\}$



Venn Diagram

Set Identities [1]

Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities$_2$

Commutative laws

$$A \cup B = B \cup A \qquad\qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities 3

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not

Proof of Second De Morgan Law [1]

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Proof of Second De Morgan Law 2

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \land (x \in B))$	by defn. of intersection
$\neg(x \in A) \lor \neg(x \in B)$	1st De Morgan law for Prop Logic
$x \notin A \lor x \notin B$	defn. of negation
$x \in \overline{A} \lor x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	by defn. of union

Set-Builder Notation: Second De Morgan Law

$\overline{A \cap B} = x \in \overline{A \cap B}$	by defn. of complement
$= \left\{ x \mid \neg\left(x \in \left(A \cap B \right) \right) \right\}$	by defn. of does not belong symbol
$= \left\{ x \mid \neg\left(x \in A \wedge x \in B \right\}$	by defn. of intersection
$= \left\{ x \mid \neg\left(x \in A \right) \vee \neg\left(x \in B \right) \right\}$	by 1st De Morgan law for Prop Logic
$= \left\{ x \mid x \notin A \vee x \notin B \right\}$	by defn. of not belong symbol
$= \left\{ x \mid x \in \overline{A} \vee x \in \overline{B} \right\}$	by defn. of complement
$= \left\{ x \mid x \in \overline{A} \cup \overline{B} \right\}$	by defn. of union
$= \overline{A} \cup \overline{B}$	by meaning of notation

Proof of Second De Morgan Law [3]

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$\left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$	by defn. of union
$\left(x \notin A\right) \vee \left(x \in \overline{B}\right)$	defn. of complement
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	defn. of negation
$\neg\left(\left(x \in A\right) \wedge \neg\left(x \in B\right)\right)$	1st De Morgan law for Prop Logic
$\neg\left(x \in A \cap B\right)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Exercise

Let A, B, and C be sets. Prove or disprove that $A-(B\cap C)=(A-B)\cup(A-C)$.

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	*B*	*C*	*B∩C*	*A∪ (B∩C)*	*A∪B*	*A∪C*	*(A∪B) ∩ (A∪C)*
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

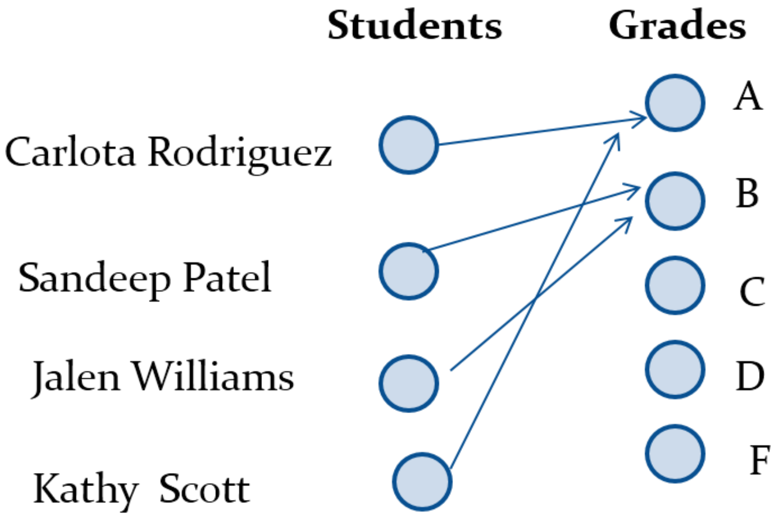
Functions
(函數)

Section 2.3

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

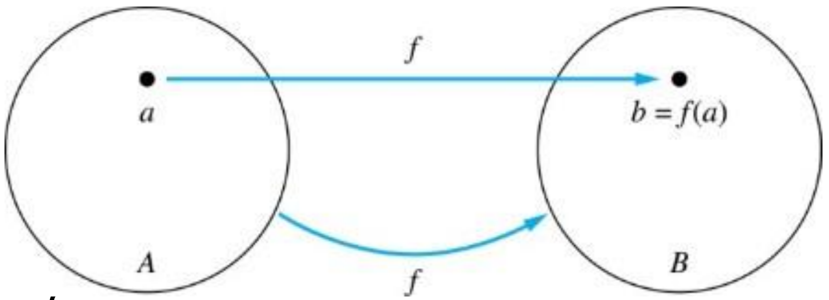
- Functions are sometimes called *mappings* or *transformations*.



Functions

Given a function $f: A \to B$**:**

- We say f *maps* A to B *or* f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Jump to long description

Functions

A function f: $A \to B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.

Specifically, a function *f* from *A* to *B* contains one, and only one ordered pair (*a, b*) for every element $a \in A$.

$$\forall x\left[x \in A \to \exists y\left[y \in B \wedge (x, y) \in f\right]\right]$$

所有X都要參與

and

$$\forall x, y_1, y_2\left[\left[(x, y_1) \in f \wedge (x, y_2) \in f\right] \to y_1 = y_2\right]$$

同一個X(domain)對應到唯一的一個y (codomain)

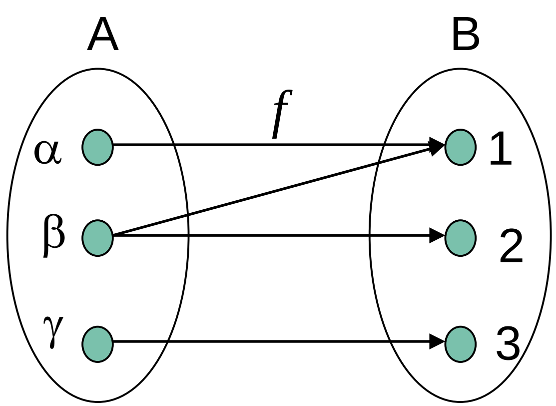
Functions (函數)

- Def 1 : A,B : sets

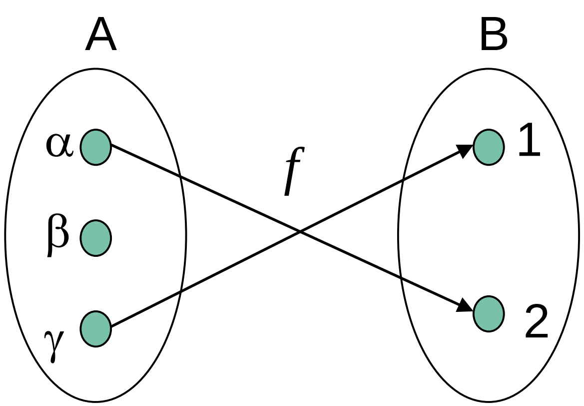
  A function $f : A \to B$ is an assignment of exactly one element of B to each element of A.
  We write $f(a) = b$ if b is the unique element of B assigned by f to a ∈ A.

- Example

$f(\alpha)=1$
$f(\beta)=1$
$f(\beta)=2$
$f(\gamma)=3$

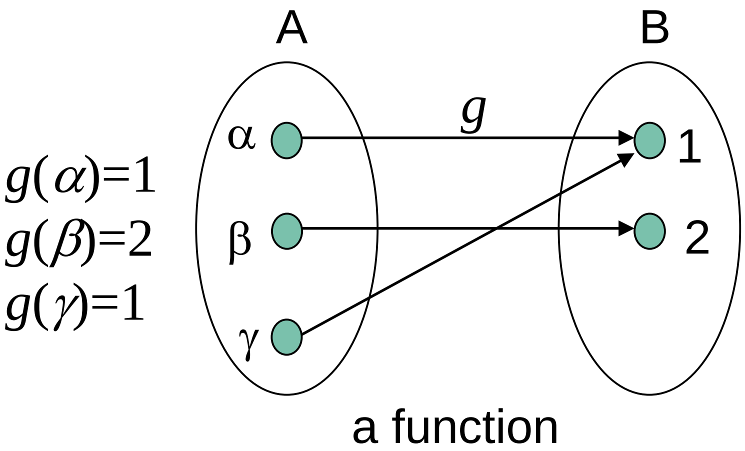


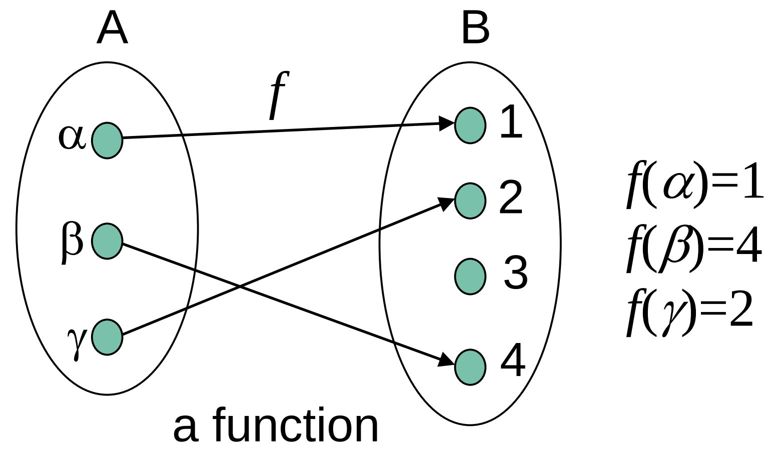
Not a function
(X)同一個X對應到2個y



$f(\alpha)=1$
$f(\beta)=?$
$f(\gamma)=2$

Not a function
(X)有一個X沒有參與動應





- Def 2 : (以 f : A→B 為例，右上圖)

  A: domain (定義域) of f, B: codomain (對應域) of f

  $f(\alpha) = 1$, $f(\beta) = 4$, $f(\gamma) = 2$

■ 1稱為α的image (映像, 必定唯一),

■ α稱為1的pre-image(前像, 可能不唯一)

■ range(值域) of $f = \{ f(a) \mid a \in A\} = f(A) = \{1,2,4\}$ (未必等於B集合)

Questions

$f(a) = ?$ z

The image of d is ? z

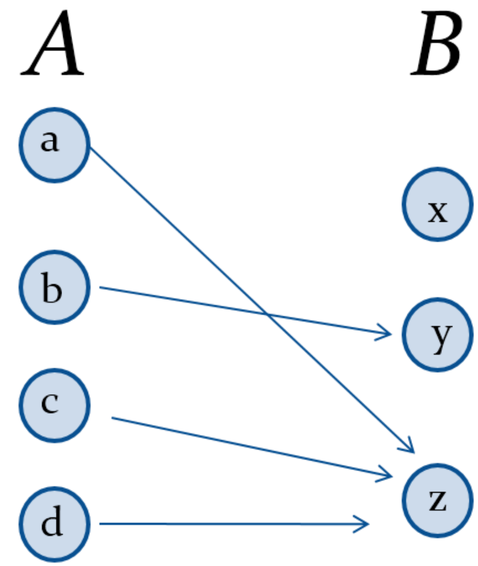
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A) = ?$ {y,z}

The preimage(s) of z is (are) ? {a,c,d}



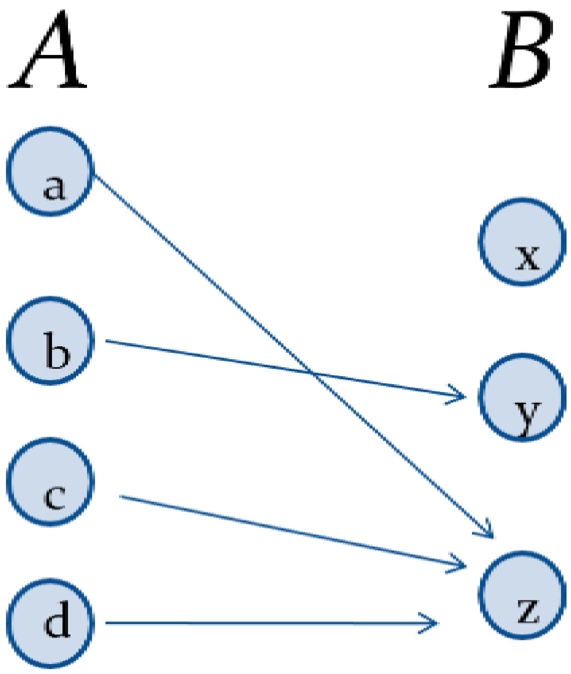
Question on Functions and Sets

If $f: A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{ f(s) \mid s \in S \}$$

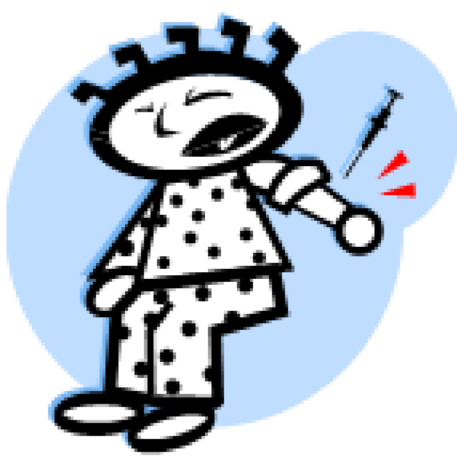
f {a,b,c,} is ? {y,z}

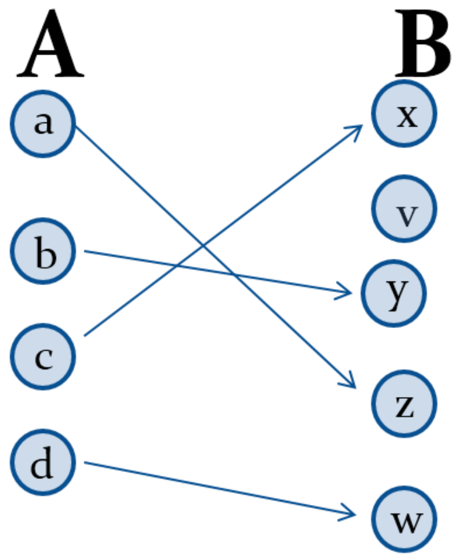
f {c,d} is ? {z}



Injections (嵌射)

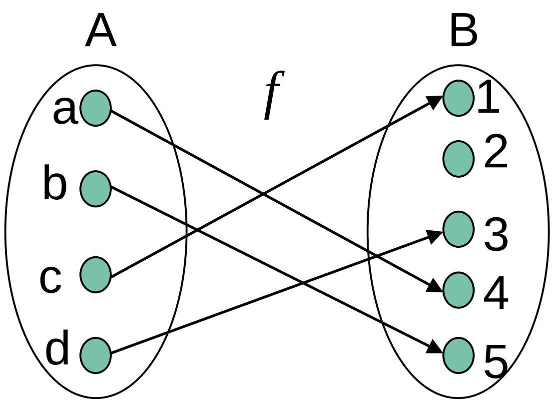
Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.



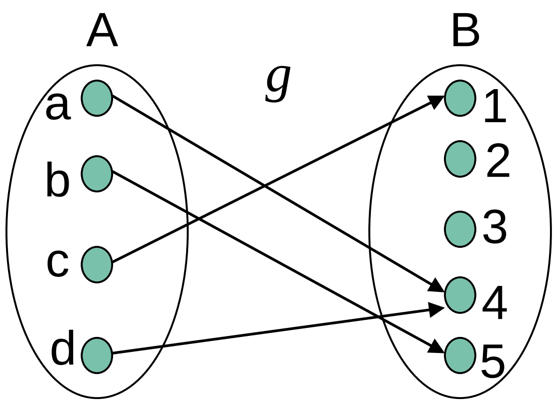


※ 一對一函數與映成函數

- A function *f* is said to be one-to-one (一對一), or injective (嵌射), iff *f* (x) ≠ *f* (y) whenever x ≠ y.
- **Example 8 :**



是 1-1

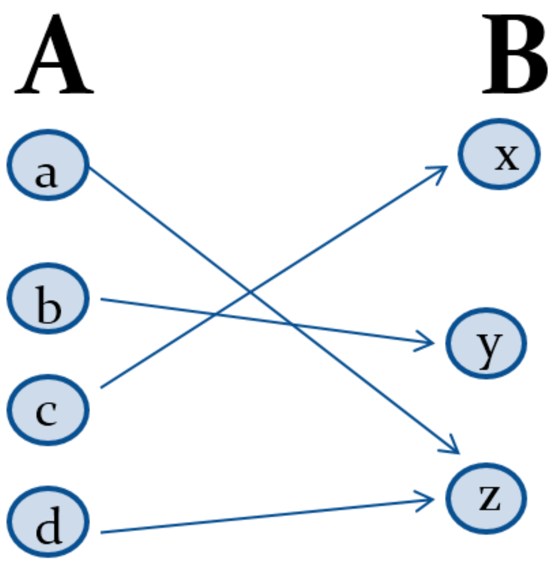


不是 1-1 , 因 g(a) = g(d) = 4

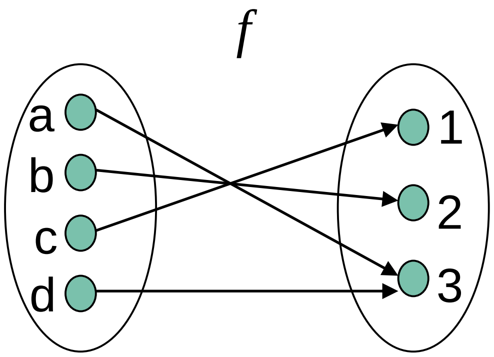
Surjections(蓋射)

Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

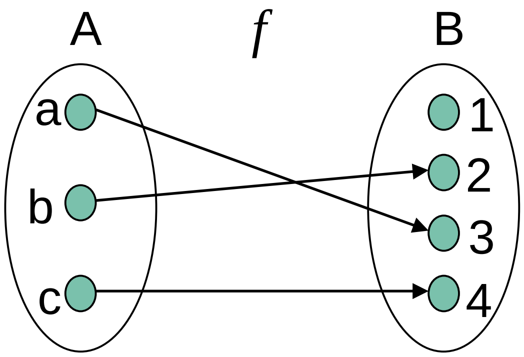
A function *f* is called a *surjection* if it is *onto*.



- A function f : A → B is called onto (映成), or surjective (蓋射), iff for every element b ∈ B, ∃a ∈ A with
  f (a) = b. (即 B 的所有元素都被 f 對應到)



onto

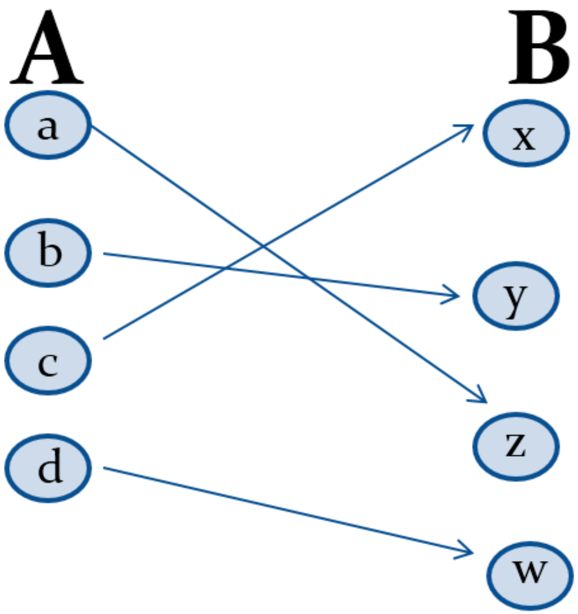


not onto

Note :
當|A| < |B| 時，
必定不會onto.

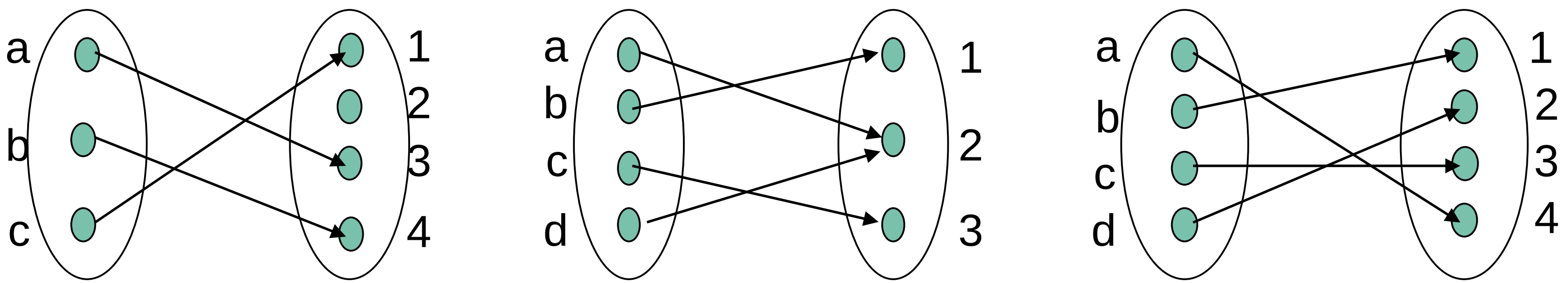
Bijections(對射)

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



- The function f is a one-to-one correspondence (一對一對應關係), or a bijection (雙射), if it is both 1-1 and onto.

- 圖5



1-1 , not onto　　not 1-1, onto　　1-1 and onto

※補充 : f : A →B

(1) If f is 1-1 , then $|A| \leq |B|$

(2) If f is onto , then $|A| \geq |B|$

(3) if f is 1-1 and onto , then $|A| = |B|$.

Showing that f is one-to-one or onto [1]

Suppose that $f : A \to B$.

- *To show that f is injective* Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.
- To show that f is not injective Find particular elements $x, y \in A$ *such that* $x \neq y$ *and* $f(x) = f(y)$.
- *To show that f is surjective Consider an arbitrary element* $y \in B$ *and find an element* $x \in A$ *such that* $f(x) = y$.
- *To show that f is not surjective Find a particular* $y \in B$ *such that* $f(x) \neq y$ *for all* $x \in A$.

Showing that f is one-to-one or onto[2]

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

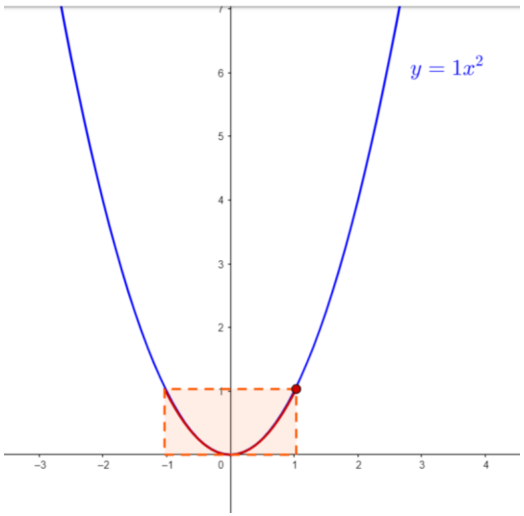
Showing that f is one-to-one or onto [2]

Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

$f(x) = x^2$, 若考慮範圍

1. $R \rightarrow R^+ \cup \{0\}$: Onto
2. $R^+ \rightarrow R^+$ ： one to one, onto, bijection



Exercise

Determine whether the function $f\,(x) = x + 1$ is one-to-one?

Sol: $x \neq y \Rightarrow x + 1 \neq y + 1$

$$\Rightarrow f\,(x) \neq f\,(y)$$

$\therefore f$ is injection, oen-to-one

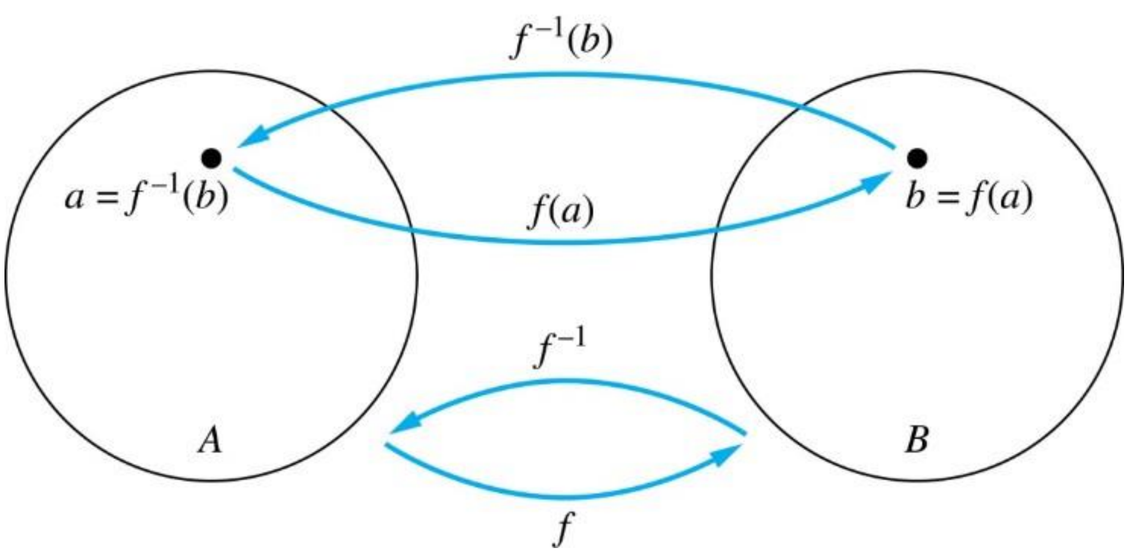
Q1 : Let f(n) = 2n + 1. Is ***f*** a one-to-one or Onto function from the set of integers to the set of integers?

If $f(n) = f(m)$, then $2n+1 = 2m+1$. It follows that $n = m$. Hence f is one-to-one. Since $f(n) = 2n+1$ is odd for every integer n, it follows that $f(n)$ is not onto; for example, 2 is not in its range.

Inverse Functions

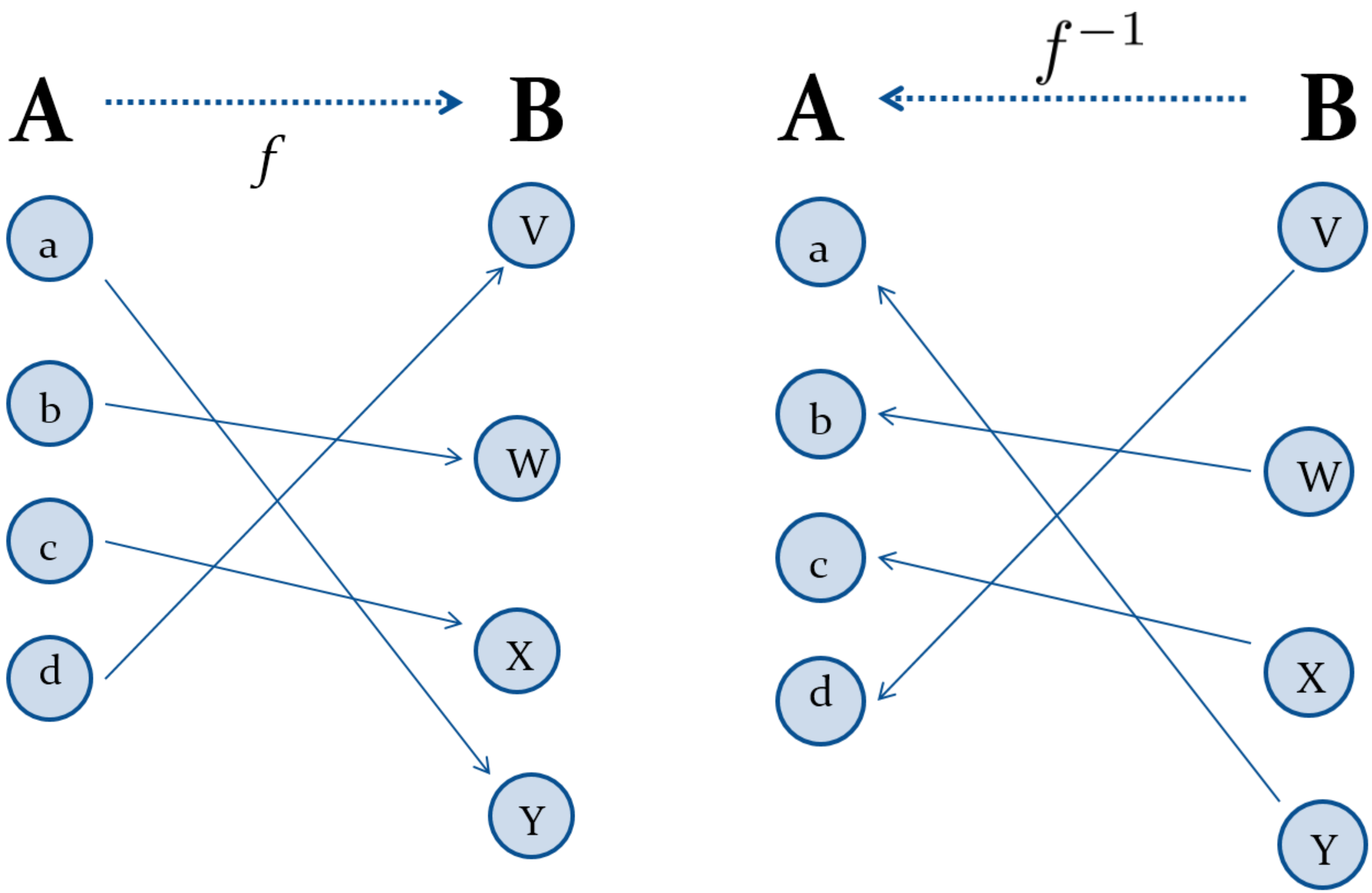
Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless *f* is a bijection. Why?



[Jump to long description]

Inverse Functions



Questions [1]

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

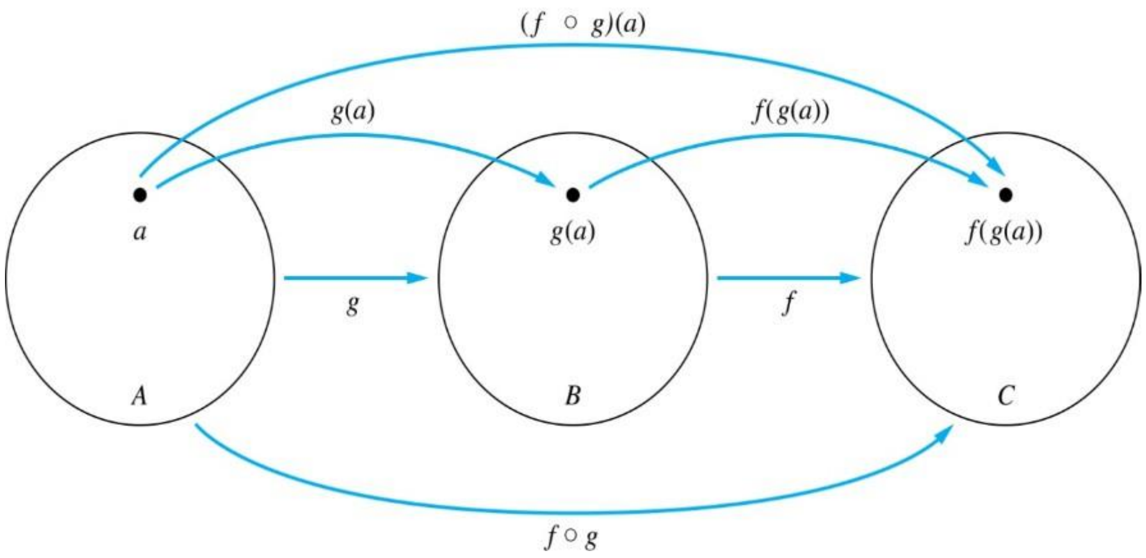
Example 2: Let f: **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Composition

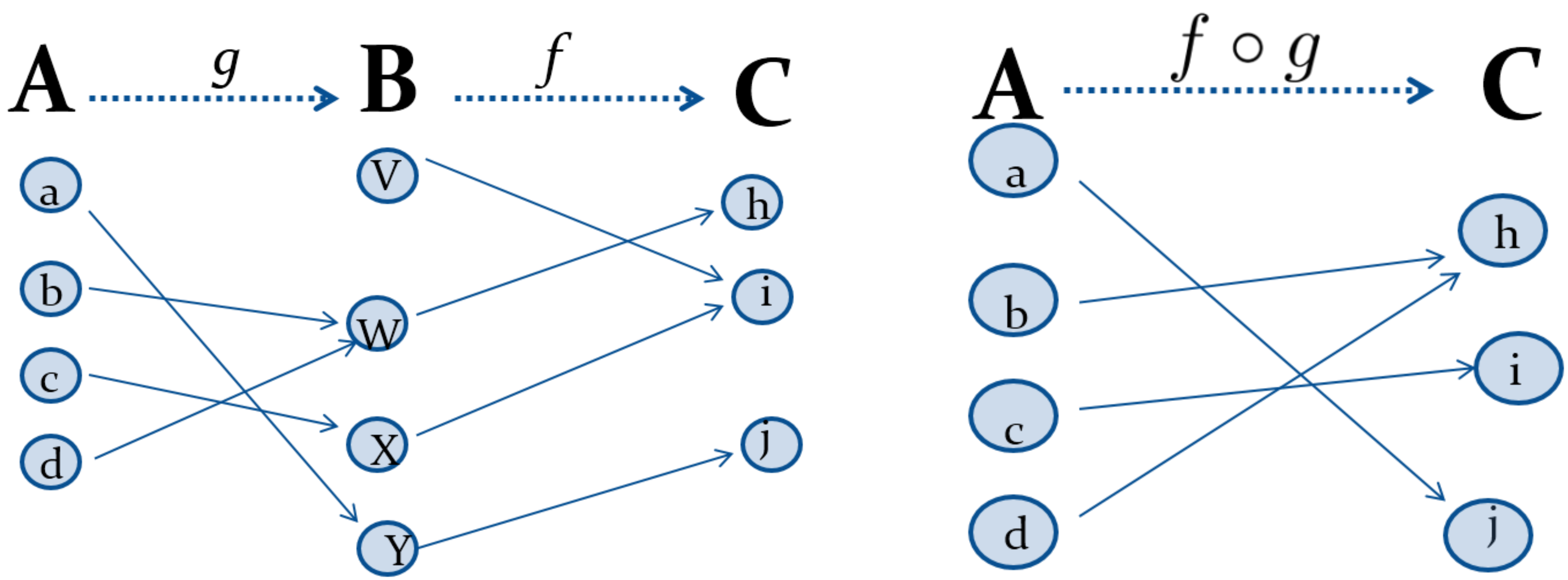
Definition: Let *f*: *B*→*C*, *g*: *A*→*B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by

$$f \circ g(x) = f\left(g(x)\right)$$



Jump to long description

Composition



Composition

Example 1: If

$$f(x) = x^2 \text{ and } g(x) = 2x + 1,$$

then

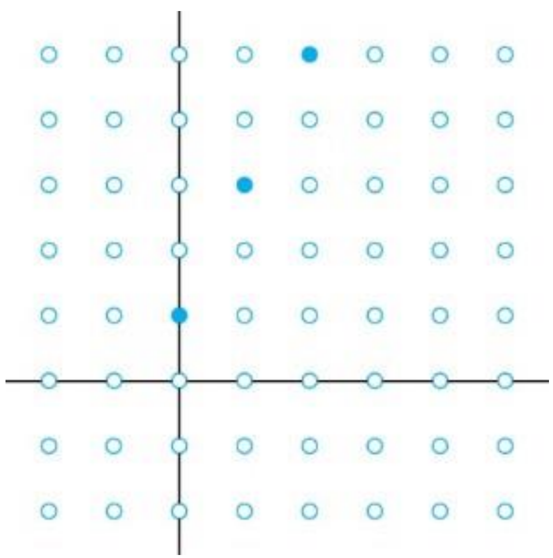
$$f(g(x)) = (2x+1)^2$$

and

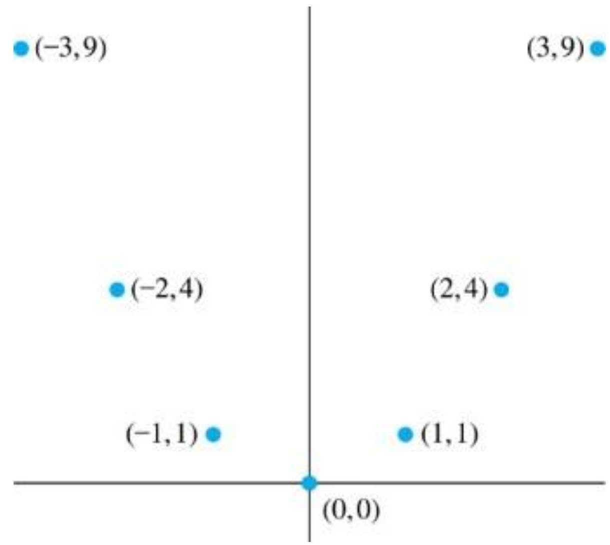
$$g(f(x)) = 2x^2 + 1$$

Graphs of Functions

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Jump to long description

Some Important Functions

The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

The *ceiling* function, denoted

$$f(x) = \lceil x \rceil$$

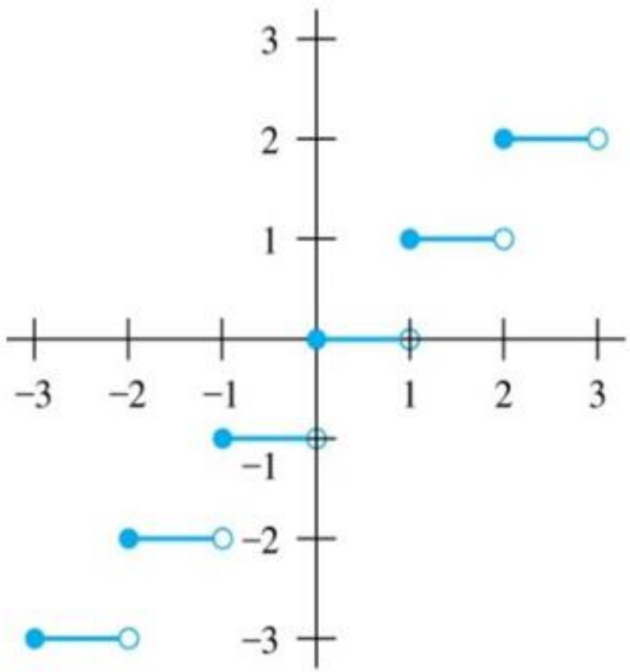
is the smallest integer greater than or equal to *x*

Example:

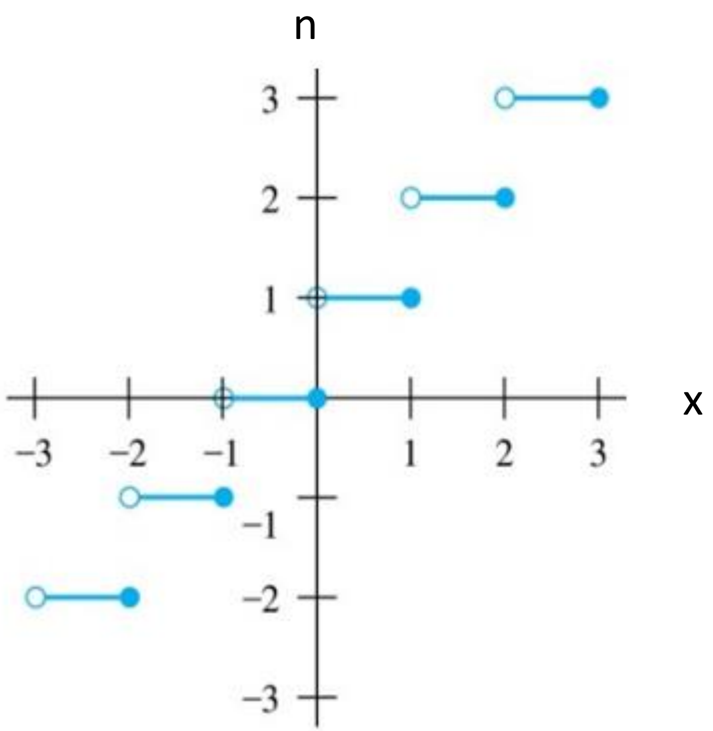
$$\lceil 3.5 \rceil = 4 \qquad \lfloor 3.5 \rfloor = 3$$

$$\lceil -1.5 \rceil = -1 \qquad \lfloor -1.5 \rfloor = -2$$

Floor and Ceiling Functions



$f(x) = \lfloor x \rfloor$



$f(x) = \lceil x \rceil$

Graph of (a) Floor and (b) Ceiling Functions

Jump to long description

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions. (*n* is an integer, *x* is a real number)	
(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

1c : -x-1 < n ≤ -x
1d : -x≥n>-x-1

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.