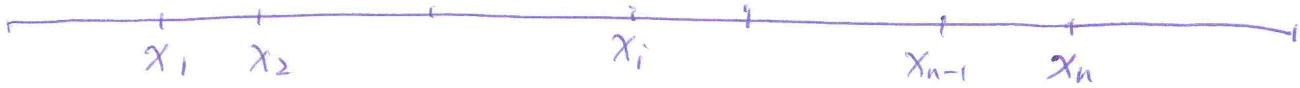
5. $x_1 \le x_2 \le \cdots \le x_n$.



Greedy Algo: ① start from x_1

② remove points in $[x_1, x_1+1]$

③ repeat on the remaining points.

output: $[x_1, x_1+1], [\quad], \cdots [x_i, x_i+1] \cdots [x_p, x_p+1]$

$\quad I_1 \qquad I_2 \quad \cdots \quad I_i \quad \cdots \quad I_p$.

Another optimal Algo: J_q.

output: $[x_1', x_1'+1], [\quad], \cdots [x_j', x_j'+1] \cdots [x_q', x_q'+1]$.

$\quad J_1 \qquad J_2 \quad \cdots \quad J_j \quad \cdots \quad J_q$.

① prove Greedy Algo stays ahead.

given the same number of intervals K, $1 \le K \le \min(p, q)$.

Greedy Algo covers more points than the other algo.

② prove Greedy Algo optimal. (base on the stay-ahead lemma).

---

① proof: by Inductive proof technique.

Base case: $K = 1$ (only 1 interval).

I_1 : $[x_1, x_1+1]$

J_1 : $[x_1', x_1'+1]$

if $x_1 < x_1'$, J_1 would not cover x_1, that's contradictio[n]
with J is another solution which is optimal

$\Rightarrow x_1' \le x_1 \Rightarrow x_1'+1 \le x_1+1$. (Greedy stays ahead).

Hypothesis: Greedy stays ahead for given $k=1, \dots, n$ $(n \geq 1)$.

Inductive step: we need to prove Greedy stays ahead for $k=n+1$.

$$\begin{array}{l|l}
\text{Greedy}: \underbrace{[x_1, x_1+1]}_{I_1} \dots \underbrace{[x_i, x_i+1]}_{I_i}, \dots \underbrace{[x_n, x_n+1]}_{I_n} & \underbrace{[x_{n+1}, x_{n+1}+1]}_{I_{n+1}} \\
\text{Optimal}: \underbrace{[x_1', x_1'+1]}_{J_1}, \dots \underbrace{[x_n', x_n'+1]}_{J_n} & \underbrace{[x_{n+1}', x_{n+1}'+1]}_{J_{n+1}}
\end{array}$$

By hypothesis, we know that Greedy stays ahead for given $k=1, \dots, n$. $(n \geq 1)$, that means $\{I_1, \dots, I_n\}$ covers more points than $\{J_1, \dots, J_n\}$, therefore, x_{n+1} is not covered in interval $\underbrace{[x_n', x_n'+1]}_{J_n}$, thus,

$$x_{n+1}' \leq x_{n+1} \Rightarrow \underbrace{x_{n+1}'+1}_{\text{ending point of } J_{n+1}} \leq \underbrace{x_{n+1}+1}_{\text{ending point of } I_{n+1}}.$$

We now proved that Greedy stays ahead given $k=n+1$.

② proof. Greedy Algo is optimal. $(p \le q)$.

suppose $p > q$.

$I_1, I_2 \text{ ------ } I_q \mid \text{ ---- } I_p$.

$J_1, J_2 \text{ ---- } J_q \mid$

sine base on the knowledge of stay-ahead lemma, $\{I_1, I_2 \ldots I_q\}$ covers more (or equal) points than $\{J_1, J_2 \text{ -- } J_q\}$, and all the points are coverd in $\{J_1, J_2 \text{ --- } J_q\}$, that means $\{I_{q+1}, \text{ --- } I_p\}$ covers some points that $\{J_1, J_2, \text{ --- } J_q\}$ not cover, which is impossible, because we assume J is an optimal solution that has the smallest intervals set cover all the points, contrradictany!

we now proved that $p \le q$.

Based on the proof ① + ②, we proved that Greedy is an optimal solution for this problem.