CS325 Winter 2017: practice question set 2

This set of practice questions help you review the following concepts:

- Designing divide and conquer algorithms and characterize its run time using recurrence relations.
- More proof by induction, particularly using it to prove correctness of recursive algorithms

1. The well-known mathematician George Polya posed the following false "proof" showing through mathematical induction that actually, all horses are of the same color.

   Base case: If there's only one horse, there's only one color, so of course its the same color as itself.
   Inductive case: Suppose within any set of n horses, there is only one color. Now look at any set of $n+1$ horses. Number them: $1, 2, 3, \ldots, n, n+1$. Consider the sets $\{1, 2, 3, ..., n\}$ and $\{2, 3, 4, ..., n+1\}$. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all $n+1$ horses.

   Identify what is wrong with this proof.

2. DPV 2.17. You need to 1) describe the algorithm in clear pseudo-code; 2) prove its correctness (via induction) and 3) show that its run time is $O(\log n)$.

3. Given two sorted arrays $a[1, ..., n]$ and $b[1, ..., n]$, given an $O(\log n)$ algorithm to find the median of their combined $2n$ elements. (Hint: use divide and conquer).

4. Given an array A of n distinct numbers whose values $A[1], A[2], ..., A[n]$ is unimodel: that is, for some index $p \in [1, n]$, the values in the array first increases up to position p, then decrease the remainder of the way. For example $[1, 2, 5, 9, 7, 3]$ is one such array with $p = 4$. Please design an $O(\log n)$ algorithm to find the peak p given such an array A.

5. Divide and conquer for the Tromino Puzzle.
   Use divide and conquer to design an algorithm/strategy to fill up any $2^n \times 2^n$ board with one square removed using tromino tiles. A tromino tile is a piece formed by three adjacent squares in the shape of an L and it can be rotated to fit the board. See the figure below for an example solution for a 8×8 board with cell $(4, 5)$ removed. The following page contains an interactive version of this puzzle (`https://www3.amherst.edu/~nstarr/trom/puzzle-8by8/`). You can play with it to get some idea. (Hint: assuming that you can solve the problem of $2^{n-1} \times 2^{n-1}$ with an arbitrary cell removed, how can you use that to solve the problem of $2^n \times 2^n$ size?)