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Lecture Notes

Introduction to Quantum Computing

Physics C191
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1 Introduction: Axioms of Quantum Mechanics

- Also called postulates.
- Typically four axioms:
  1. Quantum states & superposition
  2. Unitary Evolution: deterministic
  3. Measurements: introduces statistical nature to quantum behavior
  4. Observables: quantities we can measure in the real world

1.1 Quantum States

- A quantum state is denoted by $|\psi\rangle$. It's a vector in a complex-valued vector space, with a particular inner product structure. This combination of the vector space with the inner product structure is called a Hilbert space, denoted by $\mathcal{H}$.
- Vectors in $\mathcal{H}$ are denoted by *kets* $|v\rangle$, and because it's a vector space (hence it's linear), we can make other vectors by adding together two vectors: $|w\rangle = |u\rangle + |v\rangle$.

  We also have a null vector $|0\rangle = |u\rangle - |u\rangle$.
- Linearly independent vectors:

  $$a_1 |u_1\rangle + a_2 |u_2\rangle + \cdots + a_n |u_n\rangle = 0$$

  if the only solution to this is to set $a_1, a_2, \dots, a_n$ to 0, then the set of vectors $|u_1\rangle, |u_2\rangle, \dots, |u_n\rangle$ is linearly independent.

  We will only work with finite dimensional vector spaces, for the sake of quantum information
- If the set of vectors $\{|u_i\rangle\}$ spans the space, then they are referred to as a basis. This means that any vector $|w\rangle$ can be written as a linear combination of some $|u_i\rangle$:

  $$|w\rangle = \sum_i a_i |u_i\rangle$$

  It can also be represented as a column vector of n values:

  $$|w\rangle = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$
- An example where $n = 2$, is *spin projection*, which has two possible values: $\pm\hbar/2$. In this case, the general state $|\psi\rangle$ can be written as $|\psi\rangle = a_1 |+\hbar/2\rangle + a_2 |-\hbar/2\rangle$.

  We'll be dealing with mostly two-state systems in this class, and any other two-state system that we choose is sometimes called "pseudo-spin" since the math is nearly identical.

- In all cases, we should have $\sum_i |a_i|^2 = 1$; we call the states that follow this behavior (and they should) to be **normalized to 1**.

1.2 Inner Product

- Given $|w\rangle = \sum_i a_i |u_i\rangle$ and $v = \sum_i b_i |u_i\rangle$, then the complex-valued inner product $\langle v|w\rangle = \sum_i b_i^* a_i$. It can be real-valued, but in general it's considered complex.

  This gives a way for us to talk about how far apart two vectors are from one another, similar to a dot product.

- If the inner product is 0 and our vectors are not the zero vector themselves, then we call these two vectors **orthogonal**.
- An **orthonormal basis** is one where all the vectors are orthogonal, and also normalized to 1. In other words, we have $\langle u_i|u_j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta.
- So what is $\langle u|$? $\langle u|$ lives in the *dual space*, and is defined as follows: if $|w\rangle = \sum_i a_i |u_i\rangle$, then $\langle w| = \sum_i a_i^* \langle u_i|$.

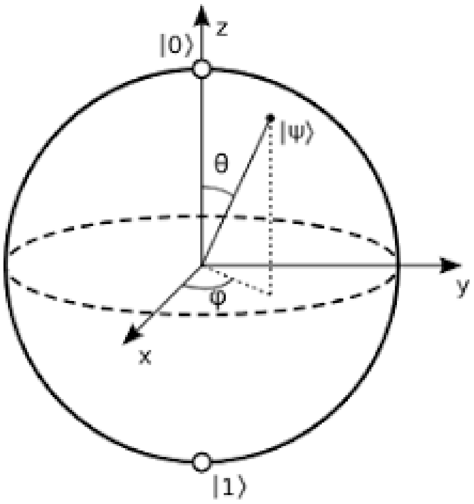
  So if $|w\rangle$ is represented as a column vector (earlier), then $\langle w|$ is represented as a row vector:

$$\langle w| = \begin{bmatrix} a_1^* & \dots & a_n^* \end{bmatrix} = w^{\top *} = w^\dagger$$

- The properties of the inner product:
  - $\langle u|v\rangle = \langle v|u\rangle^*$
  - Antilinearity: $\langle u|av\rangle = a\langle u|v\rangle$, but $\langle au|v\rangle = a^*\langle u|v\rangle$.
  - Norm of $|v\rangle$: $\langle v|v\rangle = \|v\|^2$. Hence, $\|v\| = \sqrt{\langle v|v\rangle}$.
- Conventionally, although we denote $|w\rangle = \sum_i^{n-1} a_i |u_i\rangle$, we generally deal with $n = 2$, so we have $|0\rangle$ and $|1\rangle$ as our states. This is called the **computational basis**.

1.3 Geometric Interpretation

- For $n = 2$, there is a nice geometric interpretation called the **Bloch sphere**:



  The sphere has radius 1, and all points on the sphere represent quantum states. A general state $|\psi\rangle$ is written as

$$|\psi\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right] = \alpha|0\rangle + \beta|1\rangle$$

If we want our state to be normalized, then we want $\|\alpha\|^2 + \|\beta\|^2 = 1$.

- There are also other orthonormal bases we can choose:
  - x-basis: $|+x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle$, $|-x\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
  - y-basis: $|+y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle$, and $|-y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$.

1.4 Unitary Evolution

- All equations we'll deal with are relations in $\mathcal{H}$, and these operations form a group called SU(2). This is called the *Special unitary group*.
- This unitary transformation takes our vectors $|0\rangle$ and $|1\rangle$ and does the following:

$$|0\rangle \xrightarrow{U} a\,|0\rangle + b\,|1\rangle$$
$$|1\rangle \xrightarrow{U} c\,|0\rangle + d\,|1\rangle$$

  In this case, we can write U as a 2x2 matrix:

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \; U^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

  Recall that $U^\dagger$ is the conjugate transpose. If U is a unitary operator, then $U \dagger U = I = UU^\dagger$. This implies that $U\dagger = U^{-1}$

- On a qubit, we will apply many gates throughout this semester. Some of these are listed below:
  - X-gate: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  - Z-gate: $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  - Hadamard gate: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

  All of these operations can be interpreted as a series of rotations on the Bloch sphere.

1.5 Observables

- An operator A, and its Hermitian conjugate is denoted by $A^\dagger = (A^\top)^*$.
- In QM, Hermitian operators are related to real observables we can measure in the lab, and because they are measurable, they must have real eigenvalues.
- They will also have mutually orthogonal eigenvectors.
- As an example, the X gate is Hermitian, with eigenvectors of $|+x\rangle$ and $|-x\rangle$. This is also sometimes called the Hadamard basis, because acting the Hadamard gate on $|0\rangle$ gives us $|+x\rangle$, and acting it on $|1\rangle$ gives $|-x\rangle$.

2 Entanglement & Bell Inequalities

2.1 Projection Operators

- The basic form of an operator is that it takes one vector and spits out another: $|c\rangle = |c\rangle \langle a|a\rangle$. So, the outer product $|c\rangle \langle a|$ is the operator.

- Consider a state $|w\rangle = \sum_{i=1}^{n} a_i |u_i\rangle$, where $\{|u_i\rangle\}$ form an orthonormal basis. If we want to find any one of the a_j, then we compute $\langle u_j|w\rangle$:

  $$\langle u_j|w\rangle = \sum_{i=1}^{n} a_i \underbrace{\langle u_j|u_i\rangle}_{\delta_{ij}} = a_j$$

  Alternatively, this allows us to write $|w\rangle$ in terms of:

  $$|w\rangle = \sum_{i}^{n} \langle u_i|w\rangle |u_i\rangle = \sum_{i=1}^{n} |u_i\rangle \langle u_i|w\rangle$$

  Now, the term $|u_i\rangle\langle u_i|$ is an operator, and is called the **projection operator**. If we act the operator on one of the basis vectors:

  $$|u_i\rangle \langle u_i|u_i\rangle = |u_i\rangle$$

  whereas if we do it on an arbitrary vector $|w\rangle$:

  $$|u_i\rangle \underbrace{\langle u_i|w\rangle}_{a_i} = a_i |u_i\rangle$$

- The projection operator is written as $P_i = |u_i\rangle\langle u_i|$, which has the property that $P_i^2 = |u_i\rangle \langle u_i|u_i\rangle \langle u_i| = P_i$. It also has the property that

  $$\sum_{i}^{n} P_i = \sum_{i}^{n} |u_i\rangle\langle u_i| = I$$

2.2 General Operators

- A general operator is defined as $A = IAI$. Now, we're going to express the identity matrices in terms of the projection operators:

  $$\begin{aligned} A &= \sum_i \sum_j |u_i\rangle \overbrace{\left\langle u_i|A|u_j\right\rangle}^{A_{ij}} \left\langle u_j\right| \\ &= \sum_{i,j} A_{ij} |u_i\rangle \left\langle u_j\right| \end{aligned}$$

  The term A_{ij} represents a *matrix element*, represented in the $\left|u_j\right\rangle$ basis.

  What does the $|u_i\rangle \left\langle u_j\right|$ operator represent?

- One basis that we'll use very frequently is to express A in terms of the eigenbasis. That is, the set $|a_i\rangle$ of vectors such that

  $$A |a_i\rangle = a_i |a_i\rangle$$

  In this basis, then A is written as:

  $$\begin{aligned} A &= IAI \\ &= \sum_{ij} |a_i\rangle \langle a_i|A|a_j\rangle \langle a_j| \\ &= \sum_{i,j} a_j |a_i\rangle \langle a_i|a_j\rangle \langle a_j| \end{aligned}$$

  Here we've used the property that $A|a_j\rangle = a_j |a_j\rangle$. Then, if we choose the eigenvectors to be orthogonal (which is okay for a Hermitian A), then $\langle a_i|a_j\rangle = \delta_{ij}$, so:

  $$A = \sum_i a_i |a_i\rangle \langle a_i|$$

  Why can we choose the $\{|a_i\rangle\}$ to be orthogonal?

- We choose A to be Hermitian (which is the only way we were able to make this simplification). Since they have real eigenvalues, they have mutually orthogonal eigenvectors.

2.3 Measurement Postulate

- An observable A can be measured by a set of operators $\{M_m\}$ with outcomes (observable values) m.
- For example, a qubit (so any 2-level system) with states $|0\rangle$ and $|1\rangle$, we can make a general state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ with normalization constraint $\|\alpha\|^2 + \|\beta\|^2 = 1$.

  By measuring, we "learn" the value of α and β. Our measurement operators consist of

  $$M_0 = |0\rangle \langle 0| , \quad M_1 = |1\rangle \langle 1|$$

  You'll notice that these are projections onto a given state – this is intentional.
- Upon measuring $|\psi\rangle$, we will get one outcome (either 0 or 1), with probability $p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$.
- After measurement, the state "collapses" into the state $\frac{M_m|\psi\rangle}{\sqrt{p(m)}}$. This is a fancy way to say that it will only give us $|0\rangle$ if the outcome was 0. This probabilistic determination of the final state is intrinsic to quantum mechanics.

  As an example, if we have one state $|\psi\rangle$, we either get 0 or 1 but have no information about α or β. However, if we have many identical $|\psi\rangle$, then we get 0 with probability $\|\alpha\|^2$, and we get 1 with probability $\|\beta\|^2$. This is because:

  $$p(m = 0) = \langle \psi | 0 \rangle \langle 0 | 0 \rangle \langle 0 | \psi \rangle = \|\alpha\|^2$$
  $$p(m = 1) = \langle \psi | 1 \rangle \langle 1 | 1 \rangle \langle 1 | \psi \rangle = \|\beta\|^2$$

  Note that $M_m^\dagger = M$, based on the way we've defined them. If we get 0, then the final state is written as:

  $$\frac{|0\rangle \langle 0 | \psi \rangle}{\sqrt{\|\alpha\|^2}} = |0\rangle = e^{i\theta} |0\rangle$$

  the $e^{i\theta}$ is just some overall phase factor.
- We introduce an average over many measurements to be the quantity $\langle A \rangle$, which is calculated as:

  $$\langle A \rangle = \sum_m p(m) a_m$$

  This is also sometimes called the *average value* of an operator. The measurement basis we choose for a Hermitian A is given by the eigenvectors of A, so we have:

  $$M_m = |a_m\rangle \langle a_m|$$

  where $|a_i\rangle$ is the i-th eigenvector of A. Then, this means that $\langle A \rangle = \sum_m p(m) a_m$. Remember that A is represented as:

  $$A = \sum_m a_m |a_m\rangle \langle a_m|$$
- Some cool expansion:

  $$\begin{aligned} \langle A \rangle &= \sum_m p(m) a_m \\ &= \sum_m a_m \langle \psi | M_m^\dagger M_m | \psi \rangle \\ &= \sum_m a_m \langle \psi | a_m \rangle \langle a_m | a_m \rangle \langle a_m | \psi \rangle \\ &= \sum_m a_m \langle \psi | a_m \rangle \langle a_m | \psi \rangle \end{aligned}$$

But now let's throw a $\langle\psi|$ to the left:

$$\langle\psi| \sum_m a_m |a_m\rangle \langle a_m|\psi\rangle = \langle\psi|A|\psi\rangle$$

This is the matrix element we've come across earlier.

2.3.1 Specific Examples

- Suppose we want to measure Z for a qubit. Recall that $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This has eigenvalues ± 1, with eigenvectors $|0\rangle, |1\rangle$.
- Now, we compute $\langle Z\rangle$ for a general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

$$\begin{aligned} \langle Z\rangle &= p(+1)(+1) + p(-1)(-1) \\ &= \|\alpha\|^2 - \|\beta\|^2 \end{aligned}$$

Remember that the equation is (probability of obtaining state) × (eigenvalue of that state).

- Now let's measure $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the same state $|\psi\rangle$. It's eigenvalues are ± 1, with eigenvectors $|+\rangle, |-\rangle$. Recall that

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

This means that we can solve for $|0\rangle$ and $|1\rangle$:

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |0\rangle) \\ |1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \end{aligned}$$

Therefore, the average $\langle X\rangle$:

$$\langle X\rangle = p_m(+1)(+1) + p_m(-1)(-1)$$

Then, we expand the probabilities:

$$\begin{aligned} p_m(+1) &= \langle\psi|+\rangle \langle +|+\rangle \langle +|\psi\rangle = \langle\psi|+\rangle \langle +|\psi\rangle \\ p_m(-1) &= \langle\psi|-\rangle \langle -|-\rangle \langle -|\psi\rangle = \langle\psi|-\rangle \langle -|\psi\rangle \end{aligned}$$

To complete the computation, we have to express $|\psi\rangle$ in the $|\pm\rangle$ basis:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \frac{\alpha}{\sqrt{2}}(|+\rangle + |-\rangle)$$

3 Multiple Qubits, Entanglement

3.1 Multiple Qubits

- Suppose we have two qubits $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $|\psi_2\rangle = x|0\rangle + y|1\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$

- Then the combined state, if the two qubits live on their own, is given by $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. The $\otimes$ symbol denotes a tensor product.

$$|\psi_1\rangle \otimes |\psi_2\rangle = (\alpha |0\rangle + \beta |1\rangle) \otimes (x |0\rangle + y |1\rangle)$$

  In matrix form, this is represented as:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \beta x \\ \beta y \end{bmatrix}$$

  the resulting vector lives in $\mathbb{C}^4$, with the basis states $|00\rangle = |0\rangle_1 |0\rangle_2$, $|10\rangle = |1\rangle_1 |0\rangle_2$, $|01\rangle = |0\rangle_1 |1\rangle_2$, $|11\rangle = |1\rangle_1 |1\rangle_2$

- In general given n qubits, there are 2^n basis states, and hence we will be working with superpositions over these 2^n basis states. This fact underscores the power of quantum computers, since they scale much more efficiently than classical computers. This is also sometimes referred to as "quantum parallelism".

- If we measure all qubits, then the outcome is just some sort of bitstring, so we have to be clever about how we are measuring to get the information we want.

- With multiple qubits, operators are also tensor products. Given the two operators:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

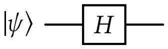
  Then $A \otimes B$, the operator that acts on the multi-qubit state, is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}b \\ a_{21}B & a_{22}B \end{pmatrix}$$

  Note that $A \otimes B$ is not the same as $B \otimes A$.

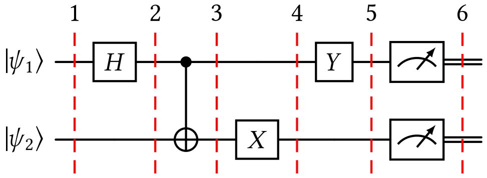
3.2 Quantum Circuits

- A generic quantum circuit is written as:



  the box with an H denotes a gate (in this case, a Hadamard gate), which corresponds to a rotation on the Bloch sphere.

- Let's analyze the following quantum circuit:



  Let's analyze this in stpes:

  - Initially, we have $|\psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$ and $|\psi_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$, whose combination can be written as:

$$|\psi_{12}^{(1)}\rangle = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

- At step 2, we run the first qubit through a Hadamard gate, and leave the second qubit untouched. This means we act the operator $H \otimes I$ on the state:

$$|\psi_{12}^{(2)}\rangle \, H \otimes I \, |\psi_{12}\rangle = \alpha_1\alpha_2 \left(\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |10\rangle \right) + \alpha_1\beta_2 \left(\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |11\rangle \right) \\ + \beta_1\alpha_2 \left(\frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |10\rangle \right) + \beta_1\beta_2 \left(\frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |11\rangle \right)$$

- At step 3, we apply a CNOT gate, which flips the state of the second bit if the value of the first bit is 1. As a truth table:

Input	Output
00	00
01	01
10	11
11	10

  As a matrix, it's written as;

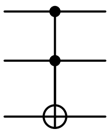
$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

  We then apply this CNOT gate to each component of $|\psi_{12}^{(2)}\rangle$ to get $|\psi_{12}^{(3)}\rangle$.

- Apply $I \otimes X$ to $|\psi_{12}^{(3)}\rangle \to |\psi_{12}^{(4)}\rangle$
- Apply $Y \otimes I$ tp $|\psi_{12}\rangle^{(4)} \to |\psi_{12}^{(5)}\rangle$
- Measurement in the Z basis, by applying projection operators to the final resulting state.

3.2.1 Other Common Gates

- There are many quantum gates that we'll study, here's a list of them that will be useful:
- CPHASE, or controlled Z gate
- Swap gate: swaps the
- S-phase: rotation by 90 degrees, $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
- P-phase: a general phase gate $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$
- Toffoli gate: controlled-controlled NOT gate:



- T-gate: $\begin{pmatrix} 1 & 0 \\ 9 & e^{i\pi/4j} \end{pmatrix}$

3.2.2 Universal Gate Sets

- A set G of quantum gates is considered universal if for $\epsilon > 0$ and for any unitary matrix U on n qubits, there is a sequence of gates from G such that

  $$\|U - U_{g_\ell} \cdots U_{g_2} U_{g_1}\| < \epsilon$$

  In this definition, we define $U_g = V \otimes I$, where V is an operator acting on k qubits, and I acts on the remaining $n - k$ qubits. The double bar represnets an operator norm, defined as:

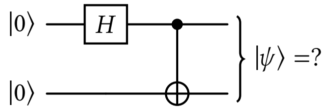
  $$\|U - U'\| = \max_{|v\rangle \text{ unit vectors}} \|(U - U')\,|v\rangle\,\|$$

  where $\|w\| = \sqrt{\langle w|w\rangle}$.

- Examples of universal gate sets:
  - Barenco et al. (1995): CNOT and all single qubit (continuous) gates.
  - CNOT, H, S, T gates
  - Rotation operators $R_x(\theta), R_y(\theta), R_z(\theta)$, the phase operator P_ϕ and CNOT.

3.3 Entanglement

- Consider 2 qubits:

  Well, we first start with the state $|00\rangle$, and after passing the first bit through a Hadamard gate, we get the state

  $$\frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

  Then, running it through the CNOT, then we have:

  $$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \Phi^+$$

  This is one of four states called the "Bell states", because there is no way to express this state as a product state of two individual qubits.

4 More on Multiple Qubits

- Last time, we looked at multiple-qubit states, and talked about how the combination is the tensor product, written like this:

  $$|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$$

- We also talked about how an entangled state is defined as a state where we cannot express as a (tensor) product state. In other words, the state is not separable.

- There are an infinite number of entangled states, called the Bell states:

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

- To quantify entanglement, we use a Schmidt decomposition for qubits: ($d = 2$ for qubits)

$$|\psi_{AB}\rangle = \sum_{i=0}^{d-1} c_i \, |i\rangle_A \, |i\rangle_B$$

  This state ψ_{AB} is separable if only one $c_i \neq 0$. The number of nonzero c_i is called the schmidt rank, and it's what we use to quantify how entangled a state is. If all c_i are equal, then the state is maximally entangled.

  The bell states $\Phi^\pm$ are easily seen to be maximally entangled, since $|00\rangle$ and $|11\rangle$ are the basis states, and they each have a coefficient of $1/\sqrt{2}$.

4.1 Measurement

- Given a state $|00\rangle$ and we measure the first qubit in the Z basis, what happens?
- Recall our measurement operator is a projection operator:

$$M_1 = |1\rangle \langle 1|$$
$$M_2 = |0\rangle \langle 0|$$

- Then, applying the measurement operators, we get an outcome of measuring 0 with probability 1. The state after measurement is given by $|00\rangle$. Note that the second qubit is not affected by this measurement.

  Are these two states identical?

- Now suppose we had a state of the form

$$|\psi\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

  This is the state that results when the second qubit is passed through a Hadamard gate. Now, if we measure the first state, we again certainly get a result of 0, so the measurement is given by: =

$$|\psi\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

  If we measure the second qubit (in the Z basis), then we get the state $|0\rangle \otimes |0\rangle$ with probability $\frac{1}{2}$, and $|0\rangle \otimes |1\rangle$ also with probability $\frac{1}{2}$.

- Another example, given the state:

$$|\psi\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

  And now we measure the first qubit, we get 0 and 1 with probability $\frac{1}{2}$, and we get the resulting states:

$$|\psi'\rangle = |0 \text{ or } 1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

4.2 Measurement with Entangled States

- Suppose we have a qubit in the state $|\Psi^-\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Now, we send the first qubit to Alice, and the second one to Bob.
- Alice will measure the first qubit in the Z basis, which will give her 0 or 1 with probability $\frac{1}{2}$.

  The thing is, if alice measures 0, then it means that the state now collapses to the first term in the superposition: $|\psi'\rangle = |01\rangle$, so Bob must get a result of 1 upon measurement. The flip is also true.
- This is an example where the outcomes of the measurements are now correlated!
- Now suppose we change our measurement basis: if we measure in the X basis, where measurements are given by $M_1 = |+\rangle \langle +|$ and $M_2 = |-\rangle \langle -|$.

  The same correlation follows: if Alice measures $|+\rangle$, then Bob will certainly get $|-\rangle$, and if Alice gets $|-\rangle$, Bob will certainly get $|+\rangle$.

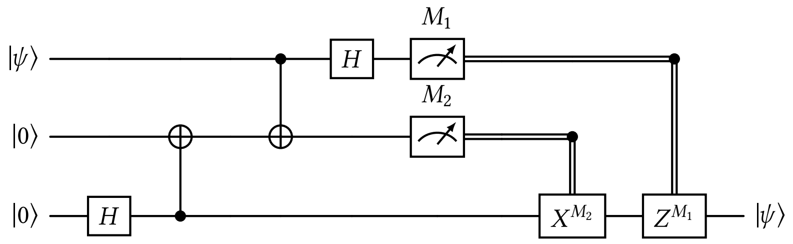
  How is the maesurement carried out? Do we express the state $|\Psi^-\rangle$ in terms of the $|\pm\rangle$ basis, and then carry out the probabilities?
- This idea that you can glean information about a quantum state without making a full measurement was problematic, and led Einstein, Podolsky and Rosen to speculate the presence of "hidden variables".

  John Bell proposed a set of inequalities (now called Bell inequalities) that would tell us for sure whether these hidden variables actually exist. He proposed a set of measurements that can be made called g, and if the systems were truly classical, then we would be able to determine that $\langle g\rangle \leq 2$. Otherwise, $\langle g\rangle > 2$ was possible.

  What we found through experiment was that $\langle g\rangle > 2$ was indeed possible, which leads us to the conclusion that there are no hidden variables are present.

4.3 Quantum Teleportation

- Consider the following circuit:



  Initially, the state is in $|\psi\rangle\,|0\rangle\,|0\rangle$. After the third qubit passes through the Hadamard gate, the state is

$$|\psi_2\rangle = |\psi\rangle\,|0\rangle\,\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

  After the first CNOT, we have:

$$|\psi_3\rangle = |\psi\rangle\,\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle)$$

  Note that this is the resulting state because the third qubit is the control bit, which flips the state of the second qubit.

  After the second CNOT, then:

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle)$$

Finally, we send the first qubit thorugh a Hadamard gate:

$$|\psi_4\rangle = \frac{1}{2}(\alpha\,|000\rangle + \alpha\,|100\rangle + \alpha\,|011\rangle + \alpha\,|111\rangle + \beta\,|010\rangle - \beta\,|110\rangle + \beta\,|001\rangle - \beta\,|101\rangle)$$

The way to compute this is to look at the first qubit, and recall that:

$$H\,|\psi\rangle = H\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha+\beta\\ \alpha-\beta\end{pmatrix}$$

so on the states $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, we have:

$$H\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(\,|0\rangle + |1\rangle)$$
$$H\begin{pmatrix}0\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(\,|0\rangle - |1\rangle)$$

so basically, each term gets split up into two terms: the last two qubits aren't changed, and the first qubit is split into a superposition between 0 and 1, with the sign determined by the coefficient.

- We then measure this state, and depending on the state of measurement, we either apply (or don't) the X gate or Z gate onto the third qubit.

  What we will find is that upon performing this procedure, the state $|\psi\rangle$ is now encoded in qubit 3 rather than qubit 1.

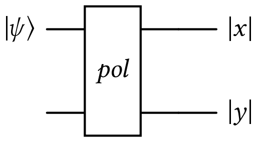
- Note however, that even though this feels like the state has "teleported", no information is being sent here that violates causality. Because we will only know what gate to apply based upon the measurement results, it means that we need to transfer this information over thorugh some *classical* channel, meaning that the information could not possibly travel faster than light.

5 Quantum Key Distribution

- Qubits used for communication are usually photons, which have a momentum $\vec{k}$, and an electric field E_x and E_y that propagates in the plane perpendicular to $\vec{k}$. The E_x vector is denoted as $|v\rangle$, and E_y as $|H\rangle$, and this means that the general electric field $\overline{E} = \alpha\,|v\rangle + \beta\,|H\rangle$.

- We can pass these photons thorugh polarizers, which only transmit light with specific oscillations.

- So as a quantum circuit, it's written as:



  Only one of these channels can be measured,

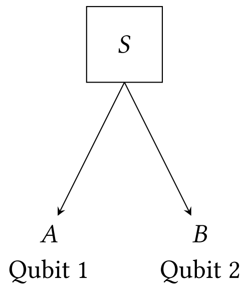
5.1 Distributed Entanglement

- One of the ways to do quantum communication and computation
- It includes:
  - teleportation

  - Secure QKD: communication
  - Distributed quantum computation – quantumgatsby teleportation
  - "Blind quantum teleportation"

5.1.1 QKD Secureness

- The way QKD works is a server $|\psi\rangle = |HV\rangle + |VH\rangle$, and the first qubit is sent to Bob, and the second is sent to Alice:



- Classically, if an observer were to say, measure the second qubit, then send an identical copy through that channel, then Alice and Bob won't be able to tell at all that the state has been measured.

  However, if the system was quantum, this measurement is now impossible.

- The proof of this is called the No cloning theorem, whose proof is below:

  Proof. Suppose we have an unknown state $|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Now suppose there is a U_{cl} (a "cloning matrix") which can clone $|\phi\rangle$. That is:

$$|\phi\rangle\,|0\rangle \mapsto |\phi\rangle\,|\phi\rangle = \alpha^2\,|00\rangle + \beta\alpha\,|10\rangle + \alpha\beta\,|01\rangle + \beta^2\,|11\rangle$$

  But if we do this on the initial state $|\phi\rangle$:

$$(\alpha\,|0\rangle + \beta\,|1\rangle)\,|0\rangle \mapsto \alpha\,|00\rangle + \beta\,|11\rangle$$

  We are cloning this exactly based on what we want: we clone the information of the second qubit onto the first qubit, but we see that even if we could "copy", we don't get the desired product state.

  But this is not equal to the copied state that we should expect. Therefore, no such U_{cl} can exist. □

5.2 Quantum Algorithms

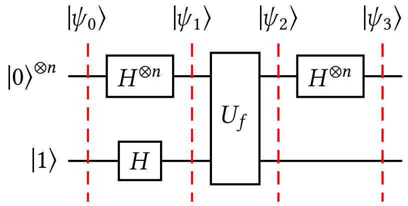
- The Deutsch-Josza is a *promise problem*: we are given a function $f(x)$, and it's one of two types:
  - $f(x)$ is either constant for all x : it is either always 0 or always 1.
  - $f(x)$ is balanced: it is 0 half the time, $f(x)$ is 1 half the time.

  More generally, we can write $f : \{0,1\}^n \mapsto \{0,1\}$, and we ask whether f is constant or balanced.

- For a function on n bits, this implies that the total domain space is of size 2^n. We need to measure a little more than half, or $2^n/2 + 1 = 2^{n-1} + 1$ measurements in order to determine the identity of f.

  Quantumly, we only need a single measurement!

- The quantum circuit is as follows:



Initially, the state is in $|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$. Then, after passing through both Hadamard gates, we have:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\sum_x |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

To explain what's happening here, here's a convenient way to denote H :

$$H = \frac{1}{\sqrt{2}}\sum_{x,y\in\{0,1\}} (-1)^{xy}|y\rangle\langle x|$$

(check for yourself that this does indeed generate the correct Hadamard matrix). Therefore, the general n-qubit Hadamard gate $H^{\otimes n}$:

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x,y\in\{0,1\}^n} (-1)^{x\cdot y}|y\rangle\langle x|$$

Therefore, we can write:

$$|\psi_1\rangle = H^{\otimes n}|0^{\otimes n}\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2^n}}\sum_x |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- Now we send $|\psi_1\rangle$ through U_f. What it does is it sends $|y\rangle$ to $|y \oplus f(x)\rangle$. If $y = 0$, then we just output $f(x)$, and if $y = 1$, thne we output the *complement* of $f(x)$, since if $f(x) = 1$ then the addition modulo 2 would return us 0, and vice versa. Therefore, we can write $|\psi_2\rangle$ as:

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}}\sum_x |x\rangle \otimes \frac{1}{\sqrt{2}}(|f(x)\rangle - |\overline{f(x)}\rangle)$$

If $f(x) = 0$, then the ancilla (the last qubit) is $|0\rangle - |1\rangle$, and if $f(x) = 1$, then the ancilla is $|1\rangle - |0\rangle = -(|0\rangle - |1\rangle)$. So in general, the ancilla is $(-1)^{f(x)}(|0\rangle - |1\rangle)$. Therefore, we can write:

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}}\sum_x (-1)^{f(x)}|x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

We can write $(-1)^{f(x)}$ because when $f(x) = 0$ then $(-1)^{f(x)} = 1$, which doesn't change the product at all, but it changes when $f(x) = 1$, which is what we want.

- Finally, we act $H^{\otimes n}$ on the data register. Note that $H^{\otimes n}|x\rangle = \sum_y (-1)^{xy}|y\rangle$, so this gives us:

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}}\sum_y \sum_x (-1)^{f(x)+(x\cdot y)}|y\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- Now we measure all n qubits. If $f(x)$ is constant,

6 Complexity Classes

- Last lecture we saw the Deutsch-Josza algorithm, which required only a single query to determine the identity of $f(X)$. However, in order to show that quantum computation is truly beneficial, then what we really need to show is that no classical algorithm could *possibly* do as well as a quantum one.

- In the case of the Deutscsh-Josza problem, a simple randomized algorithm that samples $f(x)$ at random also achieves a similar accuracy with comparable queries. For instance, 2 queries already give us a probability of error of less than 1/3.
- In general, for a probability of error less than $\frac{1}{2^n}$, we only need $n + 1$ queries.
- What we want is a problem in which the quantum algorithm *exponentially* speeds up the computation process.
- Some algorithms which show an exponential gap:
  - Bernstein-Vazirani
  - Simons algorithm

  We won't really go over these too in depth, but they are described in great detail in many textbooks.

6.1 Quantum Complexity

- This is the study of computation using information encoded in quantujm bits, or quantum states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.
- The quantum circuit is the description of the algorithm we apply to get a result.
- We need to determine the number of gates needed, and precisely the number of 2-qubit gates.
- For an arbitrary circuit with n qubits, we need no more than 2^n gates. This is because 2^n denotes a situation where every bit is connected with every other gate. However, for an efficient circuit, we only need a polynomial n^c number of gates, where c is some real, finite constant.
- So really, what we're studying is the class of algorithms which are efficiently implementable on n qubits, which is described by a sequence of $O(n)$ qubit gates drawn from a universl set. Mathematically, we write:

  $$U \approx U_{g_k} \cdots U_{g_2} U_{g_1}, \quad k = O(\text{poly}(n))$$

- There's an approximation symbol because of the **Solovay-Kitaev** theorem, which states that the accuracy of a combination of 1-qubit gates is approximated by $O(\log^c(1/\epsilon)$ with $c \approx 2$.

  By extension, this means that 1 qubit and m 2-qubit gates requires at most $O(m \log^c(1/\epsilon)$.
- This means that we have one universal gate set and we wanted to switch to another, we only have to pay a constant factor of change.

  The Wikipedia Article on this does a better job explaining this.

6.2 Quantum Complexity Clasification

- Generally, we will describe algorithms with respect to the size of the problem (generally the number of qubits n).

  Here, the number of qubits refers to the number of data qubits.
- We also might want to consider the resource scaling, so for instance the number of ancillas, the number of measurements, and also the number of gates.
- **Classical Church-Turing thesis:** Any computable function (classical algorithm) can be modeled by the running of some Turing machine.

  A Turing machine is a machine with a finite set of states, and an infinitely long input tape to which it can read and write from. The Turing machine also defines a transition matrix to transition between states. Note that this thesis does not talk about efficiency at all; the program could run infinitely and still be valid.

- **Strong Church-Turing Thesis (ECT):** A probabilitstic Turing machine, which has the same input and output tape but instead jumps randomly at each step, can efficiently (in polytime) simulate any realistic model of computation.
- Quantum computation challenges this fact – Shor's algorithm factors numbers in $O(\text{poly}(n))$, but there is no known polynomial time for factoring large numbers.
- **Strong Quantum Church-Turing thesis:** A quantum Turing machine can efficiently simulate any realistic (includes quantum) model of computation.

  Anything that's polynomial in L (the input size in bits) is considered easy, and anything that's superpolynomial is considered hard.
- A review on different algorithms and their classical runtimes:
  - Matrix multiplication: $O(n^3)$, with optimizations $O(n^{2.37})$
  - Sorting: $O(n \log n)$, also theorized to be a lower bound.
  - Factoring: number field sieve, has $O(e^{n^{1/3}(\log(n))^{2/3}}$
- Classical complexity classes:
  - P: problems which are solvable in polynomial time.
  - NP: problems which are verifiable in polynomial time.

    Here, the verifier takes in the proposed input from the solver, and we ask whether the verifier can check the solution in polynomial time.
  - It's believed that P $\subset$ NP.
  - NP-Hard: any problem in NP can be converted to an NP-Hard problem within polynomial time.
  - NP-Complete: A problem which is NP-Hard and is also in NP.
  - PSPACE: problems which are solvable in polynomial number of bits, with no constraints on time. It's trivial to show that P $\subset$ PSPACE, but we don't know how close the two sets are to each other.
  - BPP: problems which are efficiently solvable by a randomized algorithm up to some error.

    Does this refer to polynomial in the error?
  - BQP: problems which are efficiently solvable on a quantum computer with an allowed error.
  - Quantum Merlin Arthur: a decision of the following form:

    If the answer is YES, then Merlin (the solver) has a quantum state to which the verifier can verify in polynomial time on a quantum computer.
- What is known:
  - P $\subseteq$ BPP $\subseteq$ BQP $\subseteq$ PSPACE