Eric Du | Physics 221B
Final Exam | Quantum Mechanics II | March 2, 2025

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Problem 1

We are given the Lagrangian density of the Dirac field

$$\mathcal{L}(\psi, \psi_{,\mu}) = \frac{i}{2}\overline{\psi}\gamma^{\mu}\psi_{,\mu} - \frac{i}{2}\text{-}\psi_{,\mu}\psi - m\overline{\psi}\psi$$

a) Show that $\mathcal{L}$ is not invariant under local $U(1)$ gauge transformations $\psi \to \psi' = e^{i\lambda(x)}\psi$.

b) In $\mathcal{L}$, replace the ordinary derivatives $\psi_{,\mu} = \partial_\mu \psi$ by *covariant derivatives* $\psi_{;\mu} = D_\mu \psi = (\partial_\mu + igA_\mu)\psi$, with a new vector field $A_\mu(x)$ and a constant g. Find a condition on A_μ so that the new Lagrangian density $\mathcal{L}'$ is invariant under local gauge transformations.

c) Derive the Dirac equation from the new Lagrangian $\mathcal{L}'$. Waht is the interpretation of A_μ?

Problem 2

Calculate the propagator $\Delta(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \sin(kx)$ for both spacelike and timelike x explicitly in terms of Bessel functions.

Problem 3

A particle of mass m and energy E scatters at a scattering center. At scattering resonance, one of the partial amplitudes has a maximum. State the scattering cross section σ at that E, assuming that the other partial amplitudes are neglegible.

Problem 4

The exchange integral ($e = \hbar = 1$)

$$K_{10}^{nl} = \int d^3r_1 d^3r_2 \frac{\psi_{100}^*(\mathbf{r}_1)\psi_{nl0}^*(\mathbf{r}_2)\psi_{100}(\mathbf{r}_2)\psi_{nl0}(\mathbf{r}_1)}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

where ψ_{nlm} is the usual hydrogenlike wave function, is responsible for the energy difference between the ortho- and para-helium states.

a) Expand $1/|\mathbf{r}_1 - \mathbf{r}_2|$ in terms of spherical harmonics and express K_{10}^{nl} as a purely radial integral $K_{10}^{nl} \int dr_1 r_1^2 \int dr_2 r_2^2 \ldots$. Denote R_{nl} are the radial functions corresponding to ψ_{nlm}, $r_> = \max(r_1, r_2)$, and $r_< = \min(r_1, r_2)$.

b) For $l = n - 1$, argue why K can't be negative. Hint: How many zeros do the Radial functions have?

Problem 5

A charged, spinless particle with mass m (such as an ion) is trapped in a 3-D harmonic potential $V = \frac{1}{2}m\omega^2 r^2$, where r is the distance from the center. Denote $|l, m, n\rangle$ the energy eigenstates. The electromagnetic field is initially in the vacuum state. Give an expression for the decay time constant of the state $|1, 0, 0\rangle$ into the state $|0, 0, 0\rangle$. Please don't evaluate any matrix elements explicitly. Instead, state only whether they are zero, linear in the electromagnetic field operator $\mathbf{A}$, or quadratic.