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HW 01 | Signals and Systems | January 25, 2024

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Collaborators

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I did this homework in LaTeX, please let me know if there are any formatting preferences that would make grading nicer.

Problem 1

Euler's formula is

$$e^{i\theta} = \cos\theta + i\sin\theta$$

a) Derive the following identities using Euler's formula:

i) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

Solution: The first thing to figure out is what $e^{-i\theta}$ equals:

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

Since cosine is even, then $\cos(-\theta) = \cos(\theta)$, and since sine is odd, then $\sin(-\theta) = -\sin(\theta)$. Therefore:

$$e^{-i\theta} = \cos(\theta) - i\sin\theta$$

Therefore, we simplify the right hand side:

$$\begin{aligned}\frac{e^{i\theta}+e^{-i\theta}}{2} &= \frac{(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)}{2} \\ &= \frac{2\cos\theta}{2} \\ &= \cos\theta\end{aligned}$$

as desired. $\square$

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ii) $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ *Solution:* Similar to the previous problem, simplify the right hand side:

$$\begin{aligned}\frac{e^{i\theta}-e^{-i\theta}}{2i} &= \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{2i} \\ &= \frac{2i\sin\theta}{2i} \\ &= \sin\theta\end{aligned}$$

as desired. $\square$

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b) Derive **de Moivre's Theorem:** for any real integer n,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Solution: We use Euler's formula on the left hand side:

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)$$

$\square$

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c) Show that any linear combination of a set sinusoids of frequency ω, is always a sinusoid of the same frequency, *even if* each sinusoid has a distinct phase. In particular, show that

$$\sum_{k=1}^{N} A_k\cos(\omega t + \phi_k) = A\cos(\omega t + \phi)$$

holds for some A and ϕ, and derive expressions for A and ϕ in terms of A_k and ϕ_k.

Hint: Recognize that any complex number can be written as $Ae^{i\phi}$ for suitable A and ϕ. Further notice that $\sum_{k=1}^{N} A_k e^{i\phi_k}$ is a complex number.

Solution: Here, we can express this sum in a slightly different way:

$$\sum_{k=1}^{N} A_k \cos(\omega t + \phi_k) = \sum_{k=1}^{N} \mathrm{Re}\left[A_k e^{i\omega t + \phi_k}\right]$$

Now, because the real part adds linearly, we can rewrite this as:

$$\sum_{k=1}^{N} \mathrm{Re}\left[A_k e^{i\omega t + \phi_k}\right] = \mathrm{Re}\left[\sum_{k=1}^{N} A_k e^{i\omega t + \phi_k}\right] = \mathrm{Re}\left[e^{i\omega t} \sum_{k=1}^{N} A_k e^{\phi_k}\right]$$

Now, since the sum of $A_k e^{i\phi_k}$ is some general complex number, we can write it as some $Ae^{i\phi}$. Therefore:

$$\mathrm{Re}\left[e^{i\omega t} \sum_{k=1}^{N} A_k e^{i\phi_k}\right] = \mathrm{Re}\left[e^{i\omega t} A e^{i\phi}\right] = A\cos(\omega t + \phi)$$

as desired. Because the expressions for A depend on ϕ, the furthest I've gotten was the expression:

$$\sum_{k=1}^{N} A_k e^{i\phi_k} = Ae^{i\phi}$$

which relates ϕ_k and A_k to A and ϕ. $\square$

Problem 2

Evaluate the following integrals:

a) $\int_{-1}^{\infty} e^{-2t} dt$ *Solution:* The integral is simple:

$$\begin{aligned}\int_{-1}^{\infty} e^{-2t} dt &= -\frac{1}{2}\left[e^{-2t}\right]_{-1}^{\infty} \\ &= \frac{1}{2}\left[\underbrace{-e^{-2\cdot\infty}}_{=0} + e^{2}\right] \\ &= \frac{e^2}{2}\end{aligned}$$

□

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b) $X(\omega) = \int_{-\infty}^{\infty} e^{a|t|} e^{-i\omega t} dt$

Solution: First, we note that the integral is symmetric around $x = 0$, so we can instead compute the integral from 0 to ∞ and double it. Namely:

$$\int_{-\infty}^{\infty} e^{a|t|} e^{-i\omega t} dt = 2\int_{0}^{\infty} e^{at} e^{-i\omega t} = 2\int_{0}^{\infty} e^{(a-i\omega)t} dt$$

Since t is positive on the interval $[0, \infty)$, we can get rid of the absolute value. Now, this integral becomes much more tractable:

$$2\int_{0}^{\infty} e^{(a-i\omega)t} dt = \frac{2}{a - i\omega}\left[e^{(a-i\omega)t}\right]_{0}^{\infty}$$

What matters now is the sign of a on the e^{at} term, since $e^{i\omega t}$ is always bounded. If $a > 0$, then the exponential is growing, which leads us to an unbounded integral. However, if $a < 0$, then as $t \to \infty$, then the exponential term goes to 0:

$$\frac{2}{a - i\omega}\left[e^{(a-i\omega)t}\right]_{0}^{\infty} = \frac{2}{a - i\omega}(-1) = \frac{2}{i\omega - a}$$

Therefore, we have:

$$X(\omega) = \begin{cases} \frac{2}{i\omega - a} & a < 0 \\ 2\pi\delta(\omega) & a = 0 \\ \infty & a > 0 \end{cases}$$

the middle case of $a = 0$ yields a Dirac delta because then the integral simplifies to:

$$X(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} dt = 2\pi\delta(\omega)$$

□

Problem 3

Consider a pair of complex numbers v and z. Prove each of the following assertions

a) $z + z^* = 2\mathrm{Re}(z)$, where $\mathrm{Re}(z)$ denotes the real part of z.

Solution: Consider $z = a + bi$. Then, $z^* = a - bi$, so

$$z + z^* = (a + bi) + (a - bi) = 2a = 2\mathrm{Re}(z)$$

as desired. □

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b) Let $z = a + ib$, where $a, b \in \mathbb{R}$. Then $zz^* \geq 0$, with equality if, and only if, $z = 0$.

Solution: Let's write out the multiplication:

$$zz^* = (a + ib)(a - ib)$$

Since this is of the form $(x + y)(x - y)$, this is a difference of squares:

$$zz^* = a^2 - (ib)^2 = a^2 + b^2$$

Since $a, b \in \mathbb{R}$, then this implies that $a^2 + b^2 \geq 0$. This quantity is equal to zero if and only if $a = b = 0$, which is the case where $z = 0$. □

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c) z is real if, and only if, $z = z^*$

Solution: We prove the forward case: if $z = z^*$, then z is real.

If this is the case, then it must hold that the real and imaginary parts of z and z^* must equal. Writing $z = a+ib$, then this implies that $a = a$, and $b = -b$. The only solution for b is $b = 0$, so z must be real since it has no imaginary part.

Now, we prove that if z is real, then $z = z^*$. This is fairly trivial – if z is real, then we can write it as $z = a+0i = a$, and its conjugate is $z^* = a - 0i = a$. Clearly, they're the same, so $z = z^*$. □

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d) $(zv)^* = z^*v^*$

Solution: Let $z = a + bi$ and $v = c + di$. We show equality by expanding out both sides, starting with the left:

$$(zv)^* = ((a + bi)(c + di))^* = (ac + (bc + ad)i - bd)^* = ac - (bc + ad)i - bd$$

Now, the right:

$$z^*v^* = (a - bi)(c - di) = ac - (bc + ad)i - bd$$

as desired. □

Problem 4

Two complex numbers z_1 and z_2 are described below:

$$z_1 = 1 + i\sqrt{3} \quad z_2 = \exp\left(i\frac{2\pi}{3}\right)$$

Throughout this problem, express each of your answers in Cartesian form ($a + ib$), in polar form ($re^{i\theta}$, where $r > 0$), as a real number, as an imaginary number, or graphically in a well-labeled complex-plane diagram, whichever form is less cluttered or otherwise more appropriate.

a) Identify each of the following complex numbers as points (or vectors) on the complex plane, using a well-labeled sketch: $z_1, z_2, z_1^*, z_2^*, 1/z_1, 1/z_2$.

Solution: Going through these one by one:

- z_1 : the number is already in Cartesian form, with $a = 1$ and $b = \sqrt{3}$.
- z_2 : the number is already in polar form, with $r = 1$, $\theta = 2\pi/3$.
- z_1^*: We just flip the imaginary part, so $z_1^* = 1 - i\sqrt{3}$, so $a = 1$, $b = -\sqrt{3}$.
- z_2^*: Taking the complex conjugate here is the same as flipping the phase, so we plot $z_2^* = \exp(-i2\pi/3)$. Hence, $r = 1$, $\theta = -2\pi/3$.
- $1/z_1$: we take the reciprocal, then we simplify the denominator:

$$\frac{1}{1 + i\sqrt{3}} = \frac{1 - i\sqrt{3}}{4}$$

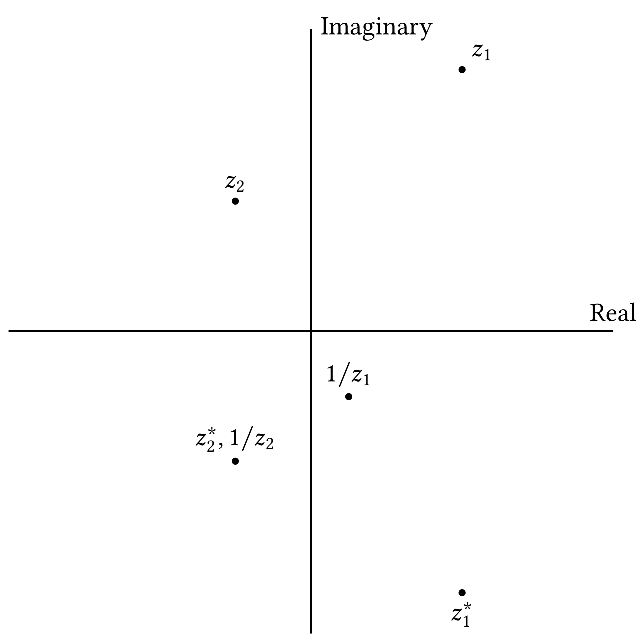
  So here, $a = 1/4$ and $b = -3/4$.

- $1/z_2$: taking the reciprocal:

$$\frac{1}{z_2} = \frac{1}{\exp(2\pi i/3)} = \exp(-i2\pi/3)$$

  Therefore, $r = 1$ and $\theta = -2\pi/3$.

Graphically, this is how they're plotted:



$\square$

b) Simplify the following expressions:

i) $|z_1 z_2|$

Solution: Here it's nice to have both z_1 and z_2 in the same form. I'm going to change z_1 to its polar form:

$$z_1 = 1 + i\sqrt{3} \implies |z| = 1^2 + (\sqrt{3})^2 = 4, \theta = \tan^{-1}(\sqrt{3}) = \pi/3$$

Therefore, $z_1 = 4e^{i\pi/3}$. Thus:

$$z_1 z_2 = 4e^{i\pi/3}e^{2\pi i/3} \implies |z_1 z_2| = \sqrt{4} = 2$$

$\square$

ii) $|z_1 z_2^*|$

Solution: We can use the same calculation as the first part for z_1 :

$$z_1 z_2^* = 4e^{i\pi/3}e^{-2\pi i/3} \implies |z_1 z_2^*| = \sqrt{4} = 2$$

$\square$

iii) z_1^3

Solution: We've transformed z_1 into its polar form already in the previous parts, so it's not hard to cube it:

$$z_1 = 4e^{i\pi/3} \implies z_1^3 = 64e^{i\pi} = 64(-1) = -64$$

$\square$

iv) z_2^4

Solution: Likewise, we can take the fourth power fairly easily:

$$z_2 = e^{2\pi i/3} \implies z_2^4 = \left(e^{2\pi i/3}\right)^4 = e^{8\pi i/3}$$

$\square$

c) Determine $z_2^{1/4}$. Be mindful of how many fourth roots of z_2 has and identify each of them graphically on a well-labeled sketch of the complex plane.

Solution: Here, we want to solve the equation $z^4 = e^{i(2\pi/3)}$. Since z is some complex number, we'll write it as $z = e^{i\theta}$. Then, this means that we have to solve the equation:

$$z^{4(i\theta)} = e^{i(2\pi/3)} \implies 4\theta = \frac{2\pi}{3} + 2\pi n,\ n \in \mathbb{Z}$$

This implies:

$$\theta = \frac{\pi}{6} + \frac{\pi}{2}n,\ n \in \mathbb{Z}$$

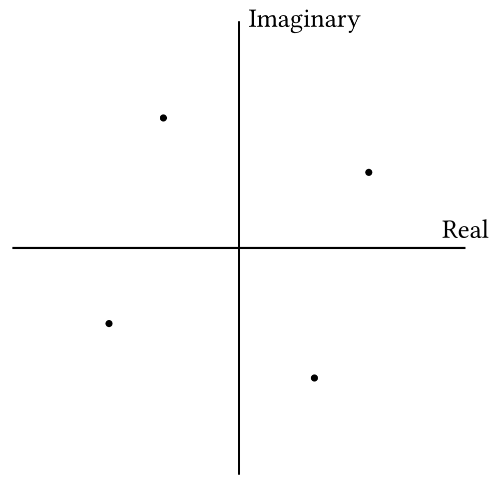
We only need to do this for $n = 0, 1, 2, 3$ though, since after $n = 4$, the values just repeat. Thus, we have:

$$\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$$

Then, since the magnitude of z can be ± 1, then the full set of solutions is:

$$z = \pm e^{i\frac{\pi}{3}}, \pm e^{i\frac{2\pi}{3}}, \pm e^{i\frac{7\pi}{6}}, \pm e^{i\frac{5\pi}{3}}$$

Graphically, this is what they look like:


Imaginary
Real

□

Problem 5

For each set defined below, provide a well-labeled diagram identifying all the points on the *complex plane* that belong to it. The symbol $\mathbb{C}$ refers to the set of complex numbers, $\mathbb{R}$ to the set of real numbers, and $\mathbb{Z}$ to the set of integers.

a) $\{z \in \mathbb{C} \mid |z - i| = |z + i|\}$

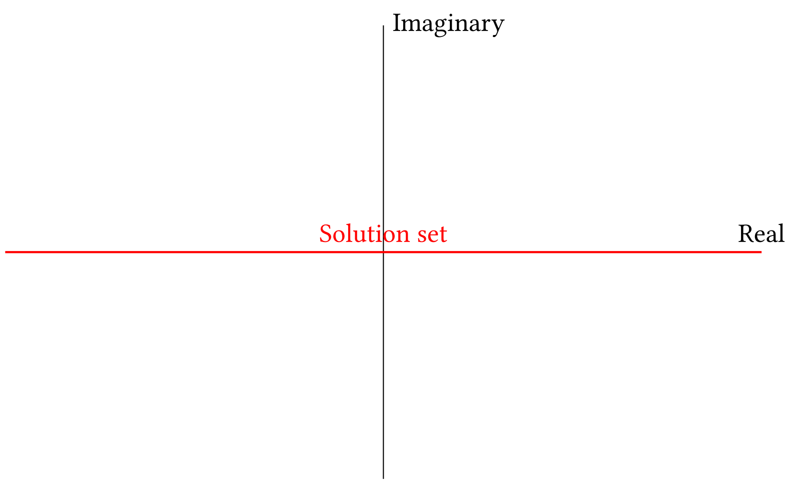
Solution: Consider $z = a + bi$. Then, on the left we have $z - i = a + (b-1)i$ and on the right we have $z + i = a + (b+1)i$. The magnitude of $z - i$ is:

$$|z - i| = \sqrt{(a + (b-1)i)(a - (b-1)i)} = \sqrt{a^2 + (b-1)^2}$$

Likewise,

$$|z + i| = \sqrt{(a + (b+1)i)(a - (b+1)i)} = \sqrt{a^2 + (b+1)^2}$$

We want to find when these two are equal, which occurs when the argument inside the square root is equal. This occurs when $(b+1)^2 = (b-1)^2$, which means that $b = 0$, with no restriction on a. Therefore, any $z \in \mathbb{R}$ satisfies this equation. As a diagram:



$\square$

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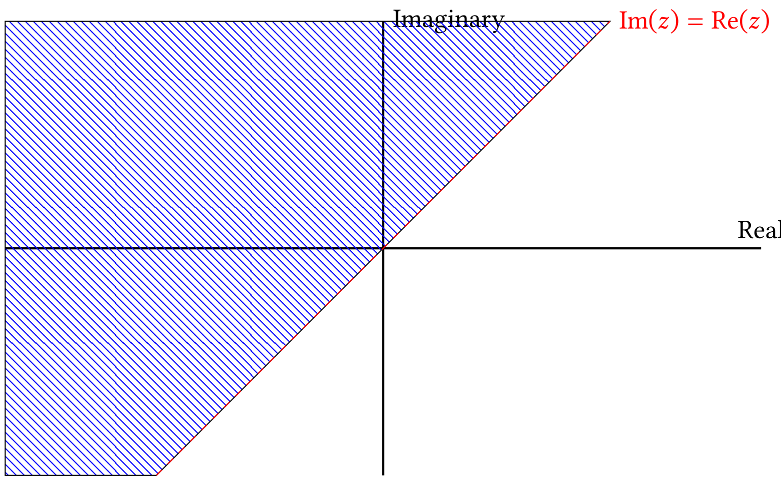
b) $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > \mathrm{Re}(z)\}$

Solution: Consider z in the magnitude-phase representation $z = me^{i\theta}$. then, $\mathrm{Im}(z) = m \sin\theta$, $\mathrm{Re}(z) = m\cos\theta$. So the condition that $\mathrm{Im}(z) > \mathrm{Re}(z)$ translates to the condition that $\sin\theta > \cos\theta$, which occurs when $\theta \in (\pi/4, 5\pi/4)$ over the interval $[0, 2\pi]$. There is no constraint on the magnitude, so the solution set is the upper half of the plane when the line $\mathrm{Im}(z) = \mathrm{Re}(z)$ passes:



$\square$

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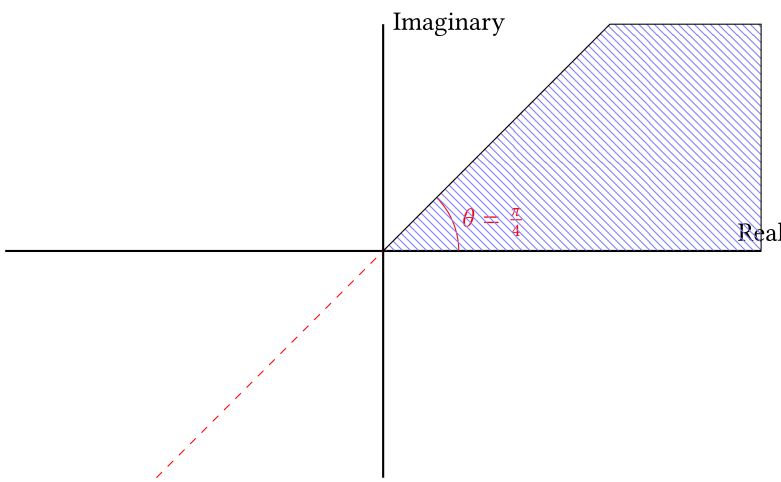
c) $\{z \in \mathbb{C} \mid 0 < \angle z < \pi/4\}$

Solution: We can use the magnitude-phase representation, and this just means the region where $\theta \in (0, \pi/4)$. Therefore:



$\square$

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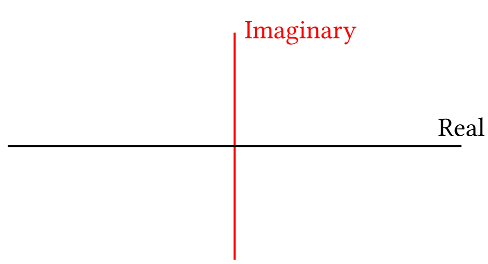
d) $\{z \in \mathbb{C} \mid z + z^* = 0\}$

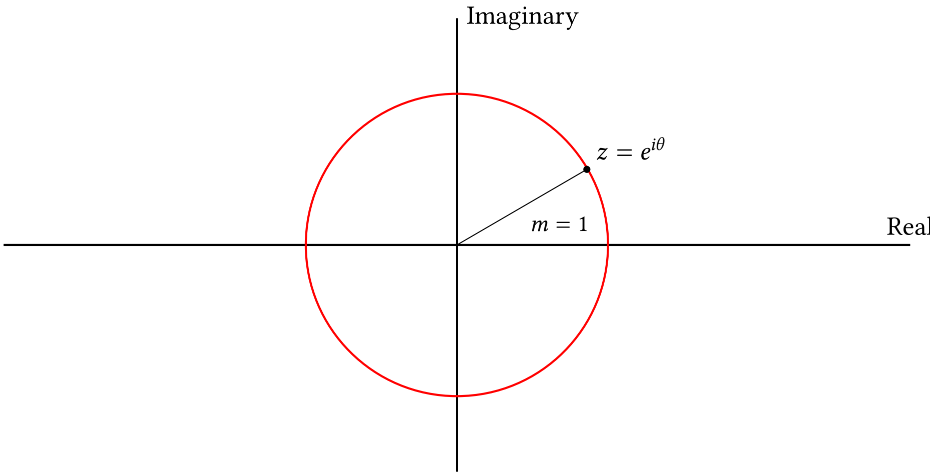
Solution: We proved earlier that $z + z^* = 2\text{Re}(z)$, so this means that $2\text{Re}(z) = 0$, implying that $\text{Re}(z) = 0$. Therefore, the solution set is the set of numbers that have no real part: $z = bi$ for any $b \in \mathbb{R}$



□

e) $\{z \in \mathbb{C} \mid z = e^{i(2\pi/3)t}, t \in \mathbb{R}\}$

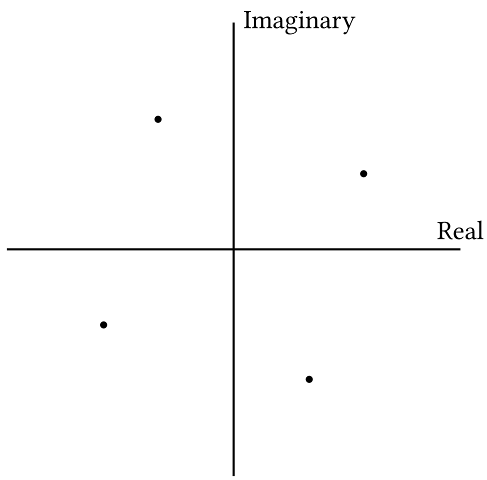
Solution: Since $t \in \mathbb{R}$, this means that any $z = e^{i\theta}$ belongs to this set. Hence, all complex numbers with magnitude 1 belong to this set:



Here, the red circle denotes the set of numbers that satisfy this relation. □

f) $\{z \in \mathbb{C} \mid z = e^{i(2\pi/3)n},\ n \in \mathbb{Z}\}$

Solution: The set actually only contains the values $\{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\}$. This is when $n = 3$, then we have $e^{i(2\pi)} = 1$, and the cycle repeats again. Therefore, plotting this on the complex plane:



□

Problem 6

Consider a complex number $z = e^{i\theta}$.

a) Show that

$$\sum_{n=0}^{N} z^n = \begin{cases} N+1 & \text{if } \theta = 0 \\ \dfrac{\sin\left[\frac{(N+1)\theta}{2}\right]}{\sin\frac{\theta}{2}} \exp\left(i\frac{N\theta}{2}\right) & \text{if } \theta \neq 0 \end{cases}$$

Solution: For $\theta = 0$, this is obviously true, since $z^n = 1$, and there are $N+1$ numbers between $n = 0$ and $n = N$, hence the sum becomes $N+1$. For $\theta \neq 0$, we write the series out:

$$\sum_{n=0}^{N} z^n = \sum_{n=0}^{N} e^{in\theta} = 1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{iN\theta}$$

This is a geometric series, with $N+1$ terms, and a common ratio of $e^{i\theta}$. Therefore, we can write it as follows:

$$\sum_{n=0}^{N} e^{in\theta} = \frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}}$$

Now, we'll factor out this in a clever way:

$$\begin{aligned} \frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}} &= \frac{e^{i(N+1)\theta/2}\left(e^{-i(N+1)\theta/2} - e^{i(N+1)\theta/2}\right)}{e^{i\theta/2}\left(e^{-i\theta/2} - e^{i\theta/2}\right)} \\ &= \exp\left(i\frac{N\theta}{2}\right)\frac{2i\sin\left(\frac{(N+1)\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)} \\ &= \frac{\sin\left(\frac{(N+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\exp\left(i\frac{N\theta}{2}\right) \end{aligned}$$

as desired. □

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b) With a little algebraic manipulation, determine each of the following sums:

i) $\sum_{n=0}^{N} \cos(n\theta)$

Solution: We note that the sum in part (a) can also be decomposed as follows:

$$\sum_{n=0}^{N} e^{in\theta} = \sum_{n=0}^{N} \cos(n\theta) + i\sin(n\theta) = \sum_{n=0}^{N} \cos(n\theta) + i\sum_{n=0}^{N} \sin(n\theta)$$

Hence, $\text{Re}(\sum z^n) = \sum_n \cos(n\theta)$, and $\text{Im}(\sum z^n) = \sum_n \sin(n\theta)$. Now, with the result above, we can write:

$$e^{i\frac{N\theta}{2}}\frac{\sin\left(\frac{(N+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \left(\cos\left(\frac{N\theta}{2}\right) + i\sin\left(\frac{N\theta}{2}\right)\right)\frac{\sin\left(\frac{(N+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

Taking the real part, this implies that:

$$\sum_{n=0}^{N} \cos(n\theta) = \cos\left(\frac{N\theta}{2}\right)\frac{\sin\left(\frac{(N+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

□

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ii) $\sum_{n=1}^{N} \sin(n\theta)$

Solution: Taking the complex part of the previous expansion:

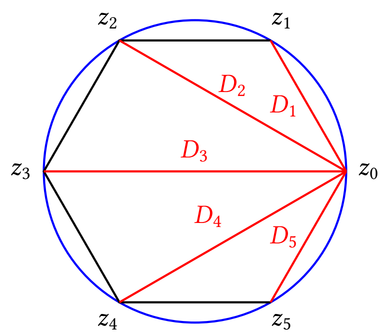
$$\sum_{n=0}^{N} \sin(n\theta) = \sin\left(\frac{N\theta}{2}\right) \frac{\sin\left(\frac{(N+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

□

Problem 7 (Optional)

A regular convex polygon whose vertices are the points $z_0, z_1, \dots, z_{N-1}$ is inscribed within the unit circle. Denote the distance from the k^{th} vertex z_k to the reference vertex z_0 by $D_k = |z_k - z_0|$, where $k = 1, \dots, N-1$.

The figure below illustrates a particular example of this – a hexagon inscribed within the unit circle (i.e. the particular case $N = 6$):



For a general convex regular polygon described above, show that

$$\prod_{k=1}^{N-1} D_k = N$$