Schroeder 4.3

A power plant produces 1 GW of electricity, at an efficiency of 40% (typical of today's coal-fired plants).

a) At what rate does this plant expel waste heat into its environment?

Solution: To calculate the rate, consider what occurs during one second of operation. The efficiency is given by

$$\epsilon = 1 - \frac{Q_c}{Q_h}$$

Therefore,

$$\frac{Q_c}{Q_h} = 1 - \epsilon = 0.6$$

therefore, $Q_c = 0.6Q_h$. Further, we have

$$\frac{W}{Q_h} = \frac{1\text{GJ}}{Q_h} = 0.4 \implies Q_h = 2.5 \text{ GJ}$$

Finally, putting it all together, we have:

$$Q_c = 0.6Q_h = (0.6)(2.5 \text{ GJ}) = 1.5 \text{ GJ}$$

We took this over 1 second, so the rate of heat expulsion is 1.5 GW. □

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b) Assume first that the cold reservoir for this plant is a river whose flow rate is 100 m^3/s. By how much will the temperature of the river increase?

Solution: Again, consider a unit time of 1 second. There are 100m^3 of water, during which 1.5 GJ of energy is being dumped into it. From the back of the book, we have $C_P = 75.29$ J/K at a volume of 18.068 cm^3, so this corresponds to $C_P = 4.18$ J/cm^3K. We want this value in units of J/m^3K since our heat is in units of cubic meters, so therefore this corresponds to $C_P = 4.18 \times 10^6$ J/m^3K. Now, we divide:

$$Q_c = (C_P V)\Delta T \implies \Delta T = \frac{Q_c}{C_P V} = \frac{1.5 \times 10^9}{(4.18 \times 10^6 \text{ J/m}^3\text{K})(100 \text{ m}^3)} = 3.58 \text{ K}$$

So we see that the water warms up by 3.58 K. □

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c) To avoid this "thermal pollution" of the river, the plant could instead be cooled by evaporation of water. (This is more expensive, but in some areas it is environmentally preferable.) At what rate must the water evaporate? What fraction of the river must be evaporated?

Solution: We follow the outline in the problem statement, by considering ΔH. Specifically, let

$$\Delta H_f = H_{\text{H}_2\text{O},g} - H_{\text{H}_2\text{O},l} = 44.01 \text{ kJ/mol}$$

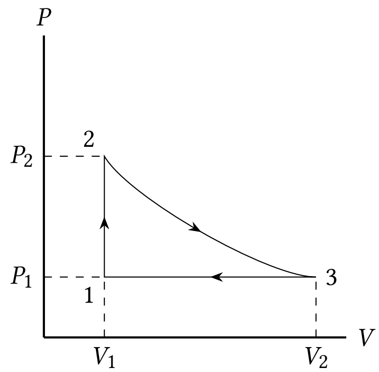
This is the energy required to evaporate one mole of water, according to the table. Therefore, the required volume is:

$$V_{\text{req}} = \frac{Q}{\Delta H_f} = \frac{1.5 \times 10^6 \text{ kJ}}{44.01\text{kJ / mol}} \cdot 18.068 \text{ cm}^3\text{/mol} = 615,815 \text{ cm}^3$$

Converting this to cubic meters, this comes out to approximately 0.6 m^3 per second, so about 0.6% of the river.
□

Problem 2

An ideal gas with f degrees of freedom per molecule undergoes the cyclic process shown below. Step $2 \longrightarrow 3$ is adiabatic.



a) For each of the steps $1 \longrightarrow 2$, $2 \longrightarrow 3$, $3 \longrightarrow 1$, compute $\Delta U, Q, W$ with $\Delta U = Q + W$. Express your answers in terms of $V_1, V_2, P_1, f, \gamma = (f+2)/f$. Summarize the final answer into a 3-row table with columns $\Delta U, Q, W$.

Solution: Step $1 \longrightarrow 2$ is held at constant volume, so $W = -P\Delta V = 0$. Therefore, $\Delta U = Q = \Delta(\frac{f}{2}PV)$, so we have:

$$\Delta U = \frac{f}{2}V_1(P_2 - P_1) = Q$$

Step $2 \longrightarrow 3$ is adiabatic, so $Q = 0$, and $\Delta U = W$. In order to calculate the work done, we use the integral form of the work:

$$W = -\int_{V_1}^{V_2} P(V)dV$$

Since this is an adiabat, then we know that $PV^\gamma = \text{const.}$, so we have $P(V) = \frac{c}{V^\gamma}$. Now we can integrate:

$$\begin{aligned}
W &= -\int_{V_1}^{V_2} \frac{c}{V^\gamma} dV \\
&= -\left[\frac{c}{1-\gamma}V^{1-\gamma}\right]_{V_1}^{V_2} \\
&= \frac{c}{1-\gamma}\left[V_1^{1-\gamma} - V_2^{1-\gamma}\right] \\
&= \frac{c}{1-\gamma}V_1^{1-\gamma}\left[1 - \left(\frac{V_2}{V_1}\right)^{1-\gamma}\right] \\
&= \frac{P_1V_1}{1-\gamma}\left(\frac{V_2}{V_1}\right)^{1-\gamma}\left[1 - \left(\frac{V_2}{V_1}\right)^{1-\gamma}\right] = \Delta U
\end{aligned}$$

Finally, for step $3 \longrightarrow 1$, we're at constant pressure, so $W = -P_1\Delta V = -P_1(V_1 - V_2)$. As for Q:

$$\begin{aligned}
Q &= \Delta U - W \\
&= \frac{f}{2}P_1(V_1 - V_2) + P_1(V_1 - V_2) \\
&= P_1(V_1 - V_2)\left(\frac{f}{2} + 1\right)
\end{aligned}$$

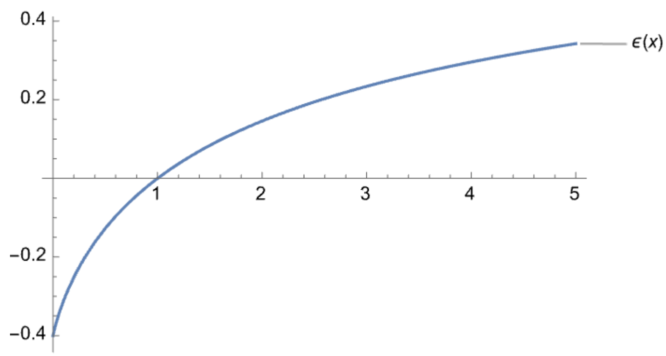
□

b) Viewed as an engine, what is the thermodynamic efficiency ϵ of the process? (I apologize, the expression isn't very beautiful). Your final answer should depend only on f, $\gamma\, V_2/V_1$. Using $f = 5$, produce a plot of $\epsilon(V_2/V_1)$.

Solution: The efficiency is calculated using $\epsilon = 1 - \frac{Q_c}{Q_h}$, so we need to identify which process does heat enter and leave the system. Clearly, since the heat from $1 \longrightarrow 2$ is positive and $3 \longrightarrow 1$ is negative, then Q_h is the heat from $1 \longrightarrow 2$ and Q_c is the heat from $3 \longrightarrow 1$. In order for the efficiency to be less than 1, we need to flip Q_c so that it's positive. In the end, the efficiency is:

$$\begin{aligned}
\epsilon &= 1 - \frac{P_1(V_2 - V_1)(\frac{f}{2} + 1)}{\frac{f}{2} P_1 V_1 \left[\left(\frac{V_2}{V_1}\right)^\gamma - 1\right]} \\
&= 1 - \frac{\left(\frac{V_2}{V_1} - 1\right)(f + 2)}{f\left[\left(\frac{V_2}{V_1}\right)^\gamma - 1\right]} \\
&= 1 - \frac{\gamma\left(\frac{V_2}{V_1} - 1\right)}{\left(\frac{V_2}{V_1}\right)^\gamma - 1}
\end{aligned}$$

A plot of this is shown below, for $f = 5$:



Note that the efficiency is negative when $\frac{V_2}{V_1} < 1$, which makes sense since that means that $V_2 < V_1$, so this regime doesn't model our system at all. Therefore in some sense, we only care about the regime where $\frac{V_2}{V_1} > 1$, where our efficiency is positive. □

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c) In order to compare with Carnot efficiency, find an expression for the hottest T_h and coldest T_c temperatures of the gas during the cycle. Compute the resulting Carnot efficiency ϵ_C for reservoirs at these temperatures. In order to compare with part b, plot $\epsilon_C(V_2/V_1)$ for $f = 5$ on the same plot.

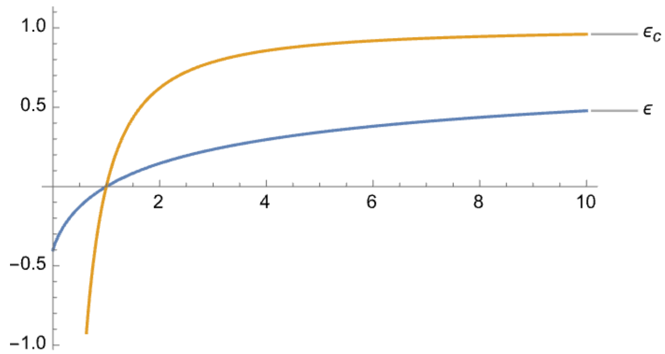
Solution: Since $PV = NkT$ then $T = \frac{PV}{Nk}$, hence $T \propto PV$. Therefore, we are looking for the values that maximize and minimize PV. For the minimum, this is fairly obvious: point 1 should be the one we choose. For the maximum, we need to compare the temperature at points 2 and 3. Looking at point 2 more closely, we can use PV^γ to compare with point 3:

$$\begin{aligned}
P_2 V_1 &= P_1 \left(\frac{V_2}{V_1}\right)^\gamma V_1 \\
&= P_1 V_2 \cdot V_2^{\gamma-1} V_1^{1-\gamma} \\
&= P_1 V_2 \left(\frac{V_2}{V_1}\right)^{\gamma-1} > P_1 V_2
\end{aligned}$$

so we conclude that point 2 is T_h. Thus, we have:

$$\begin{aligned}\epsilon_C &= 1 - \frac{T_c}{T_h} \\ &= 1 - \frac{P_1}{P_1\left(\frac{V_2}{V_1}\right)^\gamma} \\ &= 1 - \left(\frac{V_1}{V_2}\right)^\gamma \\ &= 1 - \left(\frac{V_2}{V_1}\right)^{-\gamma}\end{aligned}$$

Plotting both of these below:



Again like the previous plot, we only care about the regime where $\frac{V_2}{V_1} > 1$. One thing to note is that when $V_2 \gg V_1$, that both efficiencies approach 1, However, ϵ approaches this limit much more slowly than ϵ_C. ☐

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d) Viewed as a heat pump, what is the thermodynamic efficiency of the process?

Solution: A heat pump is the same thing as a refrigerator, except we have a different interpretation of what is considered a "benefit" in our system. In this case, the "benefit" is Q_h, and the cost is still W, so we have:

$$\text{COP} = \frac{Q_h}{W}$$

Note that our expression for ϵ in part (b) is the inverse of this, so we conclude:

$$\begin{aligned}\text{COP} &= \frac{1}{\epsilon} \\ &= \left[1 - \frac{\gamma\left(\frac{V_2}{V_1} - 1\right)}{\left(\frac{V_2}{V_1}\right)^\gamma - 1}\right]^{-1} \\ &= \left[\frac{\left(\frac{V_2}{V_1}\right)^\gamma - \gamma\left(\frac{V_2}{V_1} - 1\right) - 1}{\left(\frac{V_2}{V_1}\right)^\gamma - 1}\right]^{-1} \\ &= \frac{\left(\frac{V_2}{V_1}\right)^\gamma - 1}{\left(\frac{V_2}{V_1}\right)^\gamma - \gamma\left(\frac{V_2}{V_1} - 1\right) - 1}\end{aligned}$$

Unfortunately this expression is also not very clean, but there's nothing much I can do about it. ☐

Schroeder 4.6

To get more than an infinitesimal amount of power out of a Carnot engine, we would have to keep the temperature of its working substance below that of the hot reservoir and above that of the cold reservoir by non-infinitesimal amounts. Consider, then, a Carnot cycle in which the working substance is at temperature T_{hw} as it absorbs heat from the hot reservoir, and at temperature T_{cw} as it expels heat to the cold reservoir. Under most circumstances the rates of heat transfer will be directly proportional to the temperature differences:

$$\frac{Q_h}{\Delta t} = K(T_h - T_{hw}) \qquad \text{and} \qquad \frac{Q_c}{\Delta t} = K(T_{cw} - T_c)$$

I've assumed here for simplicity that the constants of proportionality (K) are the same for both of these processes. Let us also assume that both processes take the same amount of time, so the Δt's are the same in both of these equations.

a) Assuming that no new entropy is created during the cycle except during the two heat transfer processes, derive an equation that relates the four temperatures T_h, T_c, T_{hw} and T_{cw}.

   Solution: Here we use the relations given in the problem statement, and since K and Δt are the same in both expressions, we have:

$$K\Delta t = \frac{Q_h}{T_h - T_{hw}}$$
$$K\Delta t = \frac{Q_c}{T_{cw} - T_c}$$

   We can now set these two equal to each other, giving us the relation:

$$\frac{Q_h}{T_h - T_{hw}} = \frac{Q_c}{T_{cw} - T_c} \tag{1}$$

   Individually, since this process is a Carnot cycle (and thus quasistatic), we have $Q_h = T_h\Delta S_h$ and $Q_c = T_c\Delta S_c$, From which we can solve for ΔS_h and ΔS_c:

$$\Delta S_h = \frac{Q_h}{T_{hw}}$$
$$\Delta S_c = \frac{Q_c}{T_{cw}}$$

   Since there is no entropy created in this system (by the problem statement), then we know that $\Delta S_h + \Delta S_c = 0$ for the system. Therefore, this implies that $\Delta S_h = -\Delta S_c$, but since $\Delta S_c < 0$ due to cooling we can reverse the sign and end up with $\Delta S_h = \Delta S_c$. Now let $\Delta S_h = S$. Then, we have $Q_h = T_{hw}\Delta S$ and $Q_c = T_{cw}\Delta S$. Now let's plug it into equation 1:

$$\frac{T_{hw}\Delta S}{T_h - T_{hw}} = \frac{T_{cw}\Delta S}{T_{cw} - T_c}$$
$$\therefore \frac{T_{hw}}{T_{cw}} = \frac{T_h - T_{hw}}{T_{cw} - T_c}$$

$\square$

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b) Assuming that the time required for the two adiabatic steps is negligible, write down an expression for the power (work per unit time) output of this engine. Use the first and second laws to write the power entirely in terms of the four temperatures (and the constant K), then eliminate T_{cw} using the result from part (a).

Solution: Since the adiabatic steps is negligible, then the power is given by:

$$P = \frac{W}{T} = \frac{W}{2\Delta t}$$

we have $2\Delta t$ in the denominator to account for the absorption and expulsion steps. The rest of this is just algebra:

$$\begin{aligned}
P = \frac{W}{2\Delta t} &= \frac{Q_h - Q_c}{2\Delta t} \\
&= \frac{Q_h}{2\Delta t}\left(1 - \frac{Q_c}{Q_h}\right) \\
&= \frac{Q_h}{2\Delta t}\left(1 - \frac{T_{cw} - T_c}{T_h - T_{hw}}\right) \\
&= \frac{K}{2}(T_h - T_{hw})\left(1 - \frac{T_{cw} - T_c}{T_h - T_{hw}}\right) \\
&= \frac{K}{2}(T_h - T_{hw} - T_{cw} - T_c) \\
&= \frac{K}{2}\left(T_h - T_{hw} - T_c\left(1 - \frac{T_{hw}}{T_h - 2T_{hw}}\right)\right)
\end{aligned}$$

In the second to third step, I used equation 1 to express the ratio in terms of temperatures, and in the final step I cancelled out T_{cw} as instructed in the problem statement, using the following expression for T_{cw} (which can be derived from the final equation in part (a)):

$$T_{cw} = -\frac{T_{hw}T_c}{T_h - 2T_{hw}}$$

□

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c) When the cost of building an engine is much greater than the cost of fuel (as is often the case), is desirable to optimize the engine for maximum power output, not maximum efficiency. Show that, for fixed T_h and T_c, the expression you found in pat (b) has a maximum value at $T_{hw} = \frac{1}{2}(T_h + \sqrt{T_h T_c})$. (Hint: You'll have to solve a quadratic equation.) Find the corresponding expression for T_{cw}.

Solution: To do this, we take the derivative of P with respect to T_{hw} and set it equal to zero to find the maximum. Again, this is basically just a lot of algebra:

$$\begin{aligned}
\frac{dP}{dT_{hw}} = 0 &= \frac{K}{2}\left[-1 - T_c\left(-\frac{(T_h - 2T_{hw}) - T_{hw}(-2)}{(T_h - 2T_{hw})^2}\right)\right] \\
&= -1 + T_c\left(\frac{(T_h - 2T_{hw}) + 2T_{hw}}{(T_h - 2T_{hw})^2}\right)
\end{aligned}$$

Simplifying this further gets us the equation

$$\frac{T_c T_h}{(T_h - 2T_{hw})^2} = 1$$

from which we get:

$$\pm\sqrt{T_c T_h} = T_h - 2T_{hw} \tag{2}$$

Now we need to identify which root to take. To do this, recall the equation I mentioned at the end of part (b):

$$T_{cw} = -\frac{T_{hw}T_c}{T_h - 2T_{hw}} = \frac{T_{hw}T_c}{2T_{hw} - T_h}$$

Since T_{cw} is a positive quantity (by definition of temperature), then it must be the case that $2T_{hw} - T_h > 0$, so $T_{hw} > T_h/2$. This implies that in equation 2, the right hand side is negative, and hence in order for the signs to be consistent then we are forced to choose the negative root. Therefore, we have:

$$-\sqrt{T_c T_h} = T_h - 2T_{hw} \implies T_{hw} = \frac{1}{2}(T_h + \sqrt{T_c T_h})$$

Plugging this expression back for the derived expression at the end of part (b) will get us:

$$T_{cw} = \frac{1}{2}(T_c + \sqrt{T_c T_h})$$

To show that this is really a maximum, we need to do a second derivative test. We can take our expression for the derivative:

$$\frac{T_c T_h}{(T_h - 2T_{hw})^2} = 1$$

and differentiate on both sides with respect to T_{hw}:

$$\frac{\mathrm{d}^2 P}{\mathrm{d}T_{hw}^2} = -\frac{2(-2)}{(T_h - 2T_{hw})^3} = \frac{4}{(T_h - 2T_{hw})^3}$$

And again since $T_{hw} > T_h/2$, then the second derivative is negative, so we are indeed at a maximum around this point. □

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d) Show that the efficiency of this engine is $1-\sqrt{T_c/T_h}$. Evaluate this efficiency numerically for a typical coal-fired steam turbine with $T_h = 600^\circ$C and $T_c = 25^\circ$C, and compare to the ideal Carnot efficiency for this temperature range. Which value is closer to the actual efficiency, about 40%, of a real coal-burning power plant?

Solution: The efficiency can be calculated using

$$\epsilon = 1 - \frac{T_{cw}}{T_{hw}}$$

since these are the temperatures our engine is working between. Therefore, we just have to do the algebra:

$$\begin{aligned}
\epsilon &= 1 - \frac{T_c + \sqrt{T_c T_h}}{T_h + \sqrt{T_c T_h}} \\
&= 1 - \frac{(T_c + \sqrt{T_c T_h})(T_h - \sqrt{T_c T_h})}{T_h^2 - T_c T_h} \\
&= 1 - \frac{\sqrt{T_c T_h}(T_h - T_c)}{T_h(T_h - T_c)} \\
&= 1 - \sqrt{\frac{T_c}{T_h}}
\end{aligned}$$

as desired. Computing this with $T_h = 600^\circ\text{C} = 873$ K and $T_c = 298$ K, we get $\epsilon = 0.41$. The Carnot efficiency is the same equation without the square root, so that is $\epsilon = 0.65$. The former is clearly closer to an actual power plant, which makes sense since this is likely what real power plants do. □
