Eric Du

Electromagnetism and Optics II

Header styling inspired by the Berkeley EECS Department: https://eecs.berkeley.edu/

Collaborators

I worked with **Teja Nivarthi** on this assignment. He was particularly helpful in solving problem 3b and onwards.

Problem 1

A current $I(t)$ flows around the circular ring in Fig. 11.8. Derive the general formula for the power radiated (analogous to Eq. 11.60), expressing your answer in terms of the magnetic dipole moment $m(t)$, of the loop. [Answer $P = \mu_0 \ddot{m}^2/6\pi c^3$.]

Solution: The first steps of my solution follow what's been done in lecture. The vector potential is given by:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(t_r, \mathbf{r}')}{\mathscr{r}} \, d\tau$$

Here, we use the same approximations as we derived in lecture:

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi} \approx r\left(1 - \frac{b}{r}\sin\theta\cos\phi\right)$$

so:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi\right)$$

So now the vector potential reads:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \oint_C I\left(t - \frac{\mathscr{r}}{c}\right)\frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi\right) \, d\mathbf{l}$$

Firstly, we can integrate over ϕ, while setting $d\mathbf{l} = \cos\phi\hat{\boldsymbol{\phi}}$ (we also ignore the $\hat{\mathbf{x}}$ terms since they cancel, just like in lecture.) Since $\frac{\mathscr{r}}{c}$ is small, we can Taylor expand the current term around $t - \frac{\mathscr{r}}{c}$, which gives us:

$$I(t) = I\left(t - \frac{\mathscr{r}}{c}\right) + \dot{I}\left(t - \frac{\mathscr{r}}{c}\right)\left(\frac{b}{c}\sin\theta\cos\phi\right)$$

For notational purposes I will denote $t' = t - \frac{\mathscr{r}}{c}$. So, the integral is:

$$I(t) = \frac{\mu_0 b}{4\pi r}\hat{\boldsymbol{\phi}} \int_0^{2\pi} \left[I(t') + \dot{I}(t')\left(\frac{b}{c}\sin\theta\cos\phi\right)\right]\left(1 + \frac{b}{r}\sin\theta\cos\phi\right)\cos\phi \, d\phi$$

Now, the $\int_0^{2\pi}\cos\phi\,d\phi$ and the $\int_0^{2\pi}\cos^3\phi\,d\phi$ terms all go to zero, so there are only two terms that survive. Those are:

$$\frac{\mu_0 b\hat{\boldsymbol{\phi}}}{4\pi r}\int_0^{2\pi} I(t')\frac{b}{r}\sin\theta\cos^2\phi + \dot{I}(t')\frac{b}{c}\sin\theta\cos^2\phi \, d\phi$$

The integral evaluates to π, so the vector potential is written as (from this point onward, $\dot{I} = \dot{I}(t')$ and likewise, $\ddot{I} = \ddot{I}(t')$):

$$\mathbf{A} = \frac{\mu_0 b^2}{4r}\left(\frac{I}{r} + \frac{\dot{I}}{c}\right)\sin\theta\hat{\boldsymbol{\phi}}$$

Now, with $\mathbf{A}$ determined, we can calculate $\mathbf{E}$ and $\mathbf{B}$:

$$\begin{aligned}
\mathbf{E} &= -\frac{\partial A}{\partial t} = -\frac{\mu_0 b^2}{4r}\left(\frac{\dot{I}}{r} + \frac{\ddot{I}}{c}\right)\sin\theta\hat{\boldsymbol{\phi}} \\
\mathbf{B} &= \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\mathbf{A}_\phi \sin\theta) - \frac{\partial}{\partial r}(r\mathbf{A}_\phi) \\
&= \frac{1}{r\sin\theta}\frac{\mu_0 b^2}{4r}\left(\frac{\dot{I}}{r} + \frac{\ddot{I}}{c}\right)(2\sin\theta\cos\phi)\hat{\mathbf{r}} - \frac{1}{r}\left(\frac{\mu_0 b^2}{4}\left(\frac{\dot{I}}{r}\sin\theta\right) - \frac{\mu_0 b^2}{4}\left(\frac{\ddot{I}}{c^2}\right)\sin\theta\right)\hat{\boldsymbol{\theta}}
\end{aligned}$$

The extra factor of $-\frac{1}{c}$ comes from the chain rule, since t' has r dependence. Now, we get rid of all the terms with greater than $\frac{1}{r^2}$ dependence, since we are interested in the radiation zone, so therefore $\mathbf{B}$ simplifies to only the last term:

$$\mathbf{B} = \frac{\mu_0 b^2}{4rc^2}\ddot{I}\sin\theta\hat{\boldsymbol{\theta}}$$

Similarly, we drop the first term in $\mathbf{E}$:

$$\mathbf{E} = -\frac{\mu_0 b^2}{4r}\frac{\ddot{I}}{c}\sin\theta\hat{\boldsymbol{\phi}}$$

The Poynting vector is then calculated as:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0^2 b^4}{16r^2c^3}\ddot{I}^2\sin^2\theta\hat{\mathbf{r}}$$

The power is then:

$$P = \oint \mathbf{S}\, d\mathbf{a} = \frac{\mu_0^2 b^4}{16c^3}\ddot{I}^2\int_0^\pi\int_0^{2\pi}\frac{\sin^2\theta}{r^2}r^2\, d\theta\, d\phi$$

The θ integral integrates to $4/3$ and the ϕ integral integrates to 2π. So, we have:

$$P = \frac{\mu_0 b^2\pi}{6c^3}\ddot{I}^2$$

Now, $m = \pi b^2 I$, so $\ddot{m} = \pi b^2 \ddot{I}$, and hence:

$$P = \frac{\mu_0 \ddot{m}^2}{6\pi c^3}$$

□

Problem 2

An ideal electric dipole is situated at the origin; its dipole moment points in the $\hat{\mathbf{z}}$ direction, and is quadratic in time:

$$\mathbf{p}(t) = \frac{1}{2}\ddot{p}_0 t^2 \hat{\mathbf{z}} \quad (-\infty < t < \infty)$$

where $\ddot{p}_0$ is a constant.

(a) Use the method of Section 11.1.2 to determine the (exact) electric and magnetic fields, for all $r > 0$ (there's also a delta-function term at the origin, but we're not concerned with that). [*Partial answer:* $V = (\mu_0 \ddot{p}_0 / 8\pi) \cos\theta[(ct/r)^2 - 1]$, $\mathbf{A} = (\mu_0 \ddot{p}/4\pi c)[(ct/r) - 1]\hat{\mathbf{z}}$.]

Solution: In section 11.1.2, we have $\mathbf{p}(t) = p_0 \cos(\omega t)\hat{\mathbf{z}}$. The retarded potential is then:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 d}\left[\frac{\ddot{p}_0 \left(t - \mathscr{r}_+/c\right)^2}{2\mathscr{r}_+} - \frac{\ddot{p}_0 \left(t - \mathscr{r}_-/c\right)^2}{2\mathscr{r}_-}\right]$$

Applying the same approximations as we did in lecture (I don't want to write them all out), the vector potential is given by:

$$V = \frac{1}{4\pi\epsilon_0 d}\left[\frac{\ddot{p}_0}{2}\left(t - \frac{\mathscr{r}_+}{c}\right)^2 \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right) - \frac{\ddot{p}_0}{2}\left(t - \frac{\mathscr{r}_-}{c}\right)^2 \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)\right]$$

Now, we can expand the $\mathscr{r}$ term:

$$t - \frac{\mathscr{r}_\pm}{c} = t - \frac{r}{c} \pm \frac{d}{2c}\cos\theta$$

Since we've already approximated $\frac{d}{r} \ll 1$, it is also reasonable to assume that $\frac{d}{c} \ll 1$, and so we drop these $\frac{d}{c}$ terms. Therefore, the potential is:

$$\frac{1}{4\pi\epsilon_0 d}\frac{\ddot{p}_0}{2}\left(t - \frac{r}{c}\right)^2 \frac{1}{r}\left(\frac{d}{r}\cos\theta\right) = \frac{\ddot{p}_0}{8\pi\epsilon_0}\left(\frac{r}{c}\right)^2\left(\frac{ct}{r} - 1\right)^2 \frac{\cos\theta}{r^2}$$

Finally, we make the following simplification:

$$\left(\frac{ct}{r} - 1\right)^2 \approx \left(\frac{ct}{r}\right)^2 - 1$$

And after this we arrive at the partial answer provided:

$$V = \frac{\ddot{p}_0 \mu_0}{8\pi}\left[\left(\frac{ct}{r}\right)^2 - 1\right]\cos\theta$$

The current is given by:

$$\mathbf{I}(t) = \frac{\mathrm{d}q}{\mathrm{d}t}\hat{\mathbf{z}} = \ddot{p}_0 t \hat{\mathbf{z}}$$

So the vector potential is:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{\ddot{p}_0\left(t - \frac{\mathscr{r}}{c}\right)}{\mathscr{r}}\hat{\mathbf{z}}\, dz$$

Using the same method in the textbook, we just replace this by the value at the center:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\ddot{p}_0(t - r/c)}{r}\hat{\mathbf{z}} = \frac{\mu_0 \ddot{p}_0}{4\pi c}\left(\frac{ct}{r} - 1\right)\hat{\mathbf{z}}$$

Now, to find the fields we use $\mathbf{E} = -\nabla V - \partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. So computing the gradient for spherical coordinates, we have:

$$\mathbf{E} = -\frac{\ddot{p}_0 \mu_0}{8\pi}\cos\theta\left[-\frac{2(ct)^2}{r^3}\right]\hat{\mathbf{r}} + \frac{1}{r}\frac{\mu_0 \ddot{p}_0}{8\pi}\sin\theta\left[\left(\frac{ct}{r}\right)^2 - 1\right]\hat{\boldsymbol{\theta}} - \frac{\mu_0 \ddot{p}_0}{4\pi c}\left(\frac{c}{r}\right)\hat{\mathbf{z}}$$

We can convert $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$, and doing some simplification we get:

$$\mathbf{E} = \frac{\ddot{p}_0}{4\pi}\cos\theta\left[\frac{(ct)^2}{r^3} - \frac{1}{r}\right]\hat{\mathbf{r}} + \frac{\mu_0\ddot{p}_0}{8\pi r}\left[\left(\frac{ct}{r}\right)^2 + 1\right]\hat{\boldsymbol{\theta}}$$

As for the magnetic field, we get:

$$\begin{aligned}\nabla\times\mathbf{A} &= \nabla\times\left[\frac{\mu_0\ddot{p}_0}{4\pi c}\left(\frac{ct}{r} - 1\right)(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}})\right] \\ &= \frac{1}{r}\left[\frac{\mu_0\ddot{p}_0}{4\pi c}\sin\theta\hat{\boldsymbol{\theta}} - \frac{\mu_0\ddot{p}_0}{4\pi c}\left(\frac{ct}{r} - 1\right)(-\sin\theta)\right]\hat{\boldsymbol{\phi}} \\ &= \frac{\mu_0\ddot{p}_0}{4\pi}\frac{t}{r^2}\sin\theta\hat{\boldsymbol{\phi}}\end{aligned}$$

□

(b) Calculate the power $P(r,t)$, passing through a sphere of radius r. [*Answer:* $(\ddot{p}_0^2/12\pi\epsilon_0 r^3)t[t^2 + (r/c)^2]$.]

Solution: The power is calculated as

$$P = \int \mathbf{S}\cdot d\mathbf{a}$$

so our first goal is to find $\mathbf{S}$:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \left(\frac{\mu_0^2\ddot{p}_0^2}{16\pi^2}\right)^2\frac{t}{r^2}\sin\theta\cos\theta\left[\frac{(ct)^2}{r^3} - \frac{1}{r}\right](-\hat{\boldsymbol{\theta}}) + \left(\frac{\mu_0^2\ddot{p}_0^2 t}{16\pi^2 r^3}\sin\theta\cdot\frac{1}{2}\left[\left(\frac{ct}{r}\right)^2 + 1\right]\right)\hat{\mathbf{r}}$$

The unit vector $\mathbf{a}$ through a sphere of radius r is in the $\hat{\mathbf{r}}$ direction, so only the second term survives. Therefore, we have:

$$P = \frac{\mu_0^2\ddot{p}_0^2 t}{16\pi^2 r^3}\frac{1}{2}\left[\left(\frac{ct}{r}\right)^2 - 1\right]r^2\int_0^\pi\int_0^{2\pi}\sin^3\theta\,d\theta\,d\phi = \frac{\mu_0^2\ddot{p}_0^2 t}{16\pi^2 r^3}\cdot\frac{1}{2}\left[\left(\frac{ct}{r}\right)^2 + 1\right]r^2\int\sin^3\,d\theta\,d\phi = \frac{\mu_0^2\ddot{p}_0^2 t}{12\pi r}\left[\left(\frac{ct}{r}\right)^2 + 1\right]$$

This is equivalent to the answer in the prompt. □

(c) Find the total power radiated (Eq. 11.2), and check that your answer is consistent with Eq. 11.60.

Solution: The total power radiated is given by:

$$P_{\text{rad}}(t_0) = \lim_{r\to\infty} P\left(r, t_0 + \frac{r}{c}\right)$$

We just take the limit as $r\to\infty$ of the previous answer, I just plugged this into mathematica and I got:

$$P_{\text{rad}}(t_0) = \frac{\mu_0\ddot{p}_0^2}{4c\pi}$$

which is exactly consistent with Eq. 11.60. □

Problem 3

Suppose the (electrically neutral) yz plane carries a time-dependent but uniform surface current $K(t)\hat{\mathbf{z}}$.

(a) Find the electric and magnetic fields at a height x above the plane if

(i) a constant current is turned on at $t = 0$:

$$K(t) = \begin{cases} 0 & t \leq 0 \\ K_0 & t > 0 \end{cases}$$

Solution: The plane is electrically neutral, so the scalar potential is zero, and we only have the vector potential.

Because we have a surface current, the integral for the vector potential becomes:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{K(t_r)}{\mathscr{r}} \, da$$

To compute this integral, we use polar coordinates. Let r define the radial distance from the origin on the plane, then we can calculate this as :

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{K(t_r)}{\mathscr{r}} r \, dr \, d\theta$$

Due to K being a constant, we only need to integrate over a finite r, just like we did in the previous homework:

$$r_c = \sqrt{(ct)^2 - x^2}$$

Therefore, we have:

$$\mathbf{A} = \frac{\mu_0 K_0}{4\pi} \int_0^{2\pi} \int_0^{r_c} \frac{r}{\sqrt{r^2 + x^2}} \, dr \, d\theta$$

Using Mathematica:

$$\mathbf{A} = \frac{\mu_0 K_0}{4\pi} (2\pi)(-x + \sqrt{x^2 + r_c^2}) = \frac{\mu_0 K_0}{2}(ct - x)\hat{\mathbf{z}}$$

Then, $\mathbf{E}$ and $\mathbf{B}$ are easy:

$$\mathbf{E} = -\partial_t \mathbf{A} = \frac{\mu_0 K_0 c}{2} \hat{\mathbf{z}} \quad \mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 K_0}{2} \hat{\mathbf{y}}$$

□

(ii) a linearly increasing current is turned on at $t = 0$:

$$K(t) = \begin{cases} 0 & t \leq 0 \\ \alpha t & t > 0 \end{cases}$$

Solution: Again, we follow the same thing as the previous problem, except now we have t dependence (or implicitly, we get r dependence). Now, the integral becomes:

$$\mathbf{A} = \frac{\mu_0 \alpha \hat{\mathbf{z}}}{4\pi} \int_0^{2\pi} \int_0^{r_c} \frac{t - \mathscr{r}/c}{\mathscr{r}} \, da = \frac{\mu_0 \alpha \hat{\mathbf{z}}}{2} \left[\int_0^{r_c} \frac{rt}{\mathscr{r}} \, dr - \int_0^{r_c} \frac{r}{c} \, dr \right]$$

so this is just equal to:

$$\mathbf{A} = \frac{\mu_0 \alpha \hat{\mathbf{z}}}{2} \left[t(ct - x) - \frac{r_c^2}{2c} \right]$$

I don't want to expand r_c out so I will leave it in this form. From here, $\mathbf{E}$ follows pretty quickly (I am skipping the algebra in the interest of clarity):

$$\mathbf{E} = -\partial_t \mathbf{A} = -\frac{\mu_0 \alpha}{2} \left[(ct - x) + ct - \frac{1}{2c}(2c^2 t) \right] = -\frac{\mu_0 \alpha}{2} [ct - x]$$

The magnetic field also follows:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial x}\hat{\mathbf{y}} = -\frac{\mu_0 \alpha}{2}\left[-t - \frac{1}{2c}(-2x)\right] = -\frac{\mu_0 \alpha}{2}\left[\frac{x}{c} - t\right]$$

□

(b) Show that the retarded vector potential can be written in the form:

$$\mathbf{A}(x,t) = \frac{\mu_0 c}{2}\hat{\mathbf{z}} \int_0^\infty K\left(t - \frac{x}{c} - u\right)\, du$$

and from this determine $\mathbf{E}$ and $\mathbf{B}$.

Solution: To begin, the integral is:

$$\mathbf{A}(x,t) = \frac{\mu_0}{4\pi}\int \frac{K(t_r)}{\scriptr}\, da$$

We've seen the radial symmetry here, so the 4π in the denominator will become just a 2. Then, what we are integrating is:

$$\mathbf{A}(x,t) = \frac{\mu_0}{2}\hat{\mathbf{z}}\int \frac{K(t - \scriptr/c)}{\scriptr}\, da$$

Writing this out in terms of $\scriptr = \sqrt{r^2 + x^2}$:

$$\mathbf{A}(x,t) = \frac{\mu_0}{2}\hat{\mathbf{z}}\int_0^\infty \frac{K\left(t - \sqrt{r^2+x^2}/c\right)}{\sqrt{r^2+x^2}} r\, dr$$

Now, we make the u-substitution such that:

$$t - \frac{\sqrt{r^2+x^2}}{c} = t - \frac{x}{c} - u$$

so from here, we calculate:

$$u = \frac{1}{c}\left[\sqrt{r^2+x^2} - x\right] \implies du = \frac{1}{c}\left[\frac{1}{2}\frac{2r}{\sqrt{r^2+x^2}}\, dr\right]$$

So now, the integral becomes:

$$\mathbf{A}(x,t) = \frac{\mu_0}{2}\hat{\mathbf{z}}\int_0^\infty K\left(t - \frac{x}{c} - u\right)\cdot \frac{r}{\sqrt{r^2+x^2}}\cdot \frac{c\sqrt{r^2+x^2}}{r}\, du = \frac{\mu_0 c}{2}\hat{\mathbf{z}}\int_0^\infty K\left(t - \frac{x}{c} - u\right)\, du$$

From here, we can determine $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{E} = -\partial_t \mathbf{A} = -\frac{\mu_0 c}{2}\hat{\mathbf{z}}\int_0^\infty \frac{\partial}{\partial t}K\left(t - \frac{x}{c} - u\right)\, du$$

And we can make a change of variables $u \to t$ which costs us a negative sign, so:

$$\mathbf{E} = -\frac{\mu_0 c}{2}\hat{\mathbf{z}}\int_0^\infty -\frac{\partial}{\partial u}K\left(t - \frac{x}{c} - u\right)\, du = \frac{\mu_0 c}{2}\left[K(-\infty) - K\left(t - \frac{x}{c}\right)\right]\hat{\mathbf{z}}$$

Likewise we can do the same with $\mathbf{B}$:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial x}\hat{\mathbf{y}} = -\frac{\mu_0 c}{2}\hat{\mathbf{y}}\int_0^\infty \frac{\partial}{\partial x}K\left(t - \frac{x}{c} - u\right)\, du$$

We do the same trick as with $\mathbf{E}$, except now it costs us a factor of $\frac{1}{c}$, so we have:

$$\mathbf{B} = -\frac{\mu_0 c}{2}\int_0^\infty \frac{1}{c}\frac{\partial}{\partial u}K\left(t - \frac{x}{c} - u\right)\, du = -\frac{\mu_0}{2}\left[K(-\infty) - K\left(t - \frac{x}{c}\right)\right]\hat{\mathbf{y}}$$

□

(c) Show that the total power radiated per unit area of surface is

$$\frac{\mu_0 c}{2}[K(t)]^2$$

Explain what you mean by "radiation", in this case, given that the source is not localized.

Solution: The Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = -\frac{1}{\mu_0}\left(\frac{\mu_0}{2}\right)^2 c\left[K(-\infty) - K\left(t - \frac{x}{c}\right)\right]^2 (-\hat{\mathbf{x}})$$

If we allow $K(-\infty) \to 0$, then this equation simplifies to:

$$\mathbf{S} = \frac{\mu_0 c}{2}\left[K\left(t - \frac{x}{c}\right)\right]^2 \hat{\mathbf{x}}$$

Therefore, at any moment in time, the total power radiated should not be calculated relative to point x but instead calculated absolutely, in which case we replace the argument to K by $K(t)$. Therefore, we have the total power per unit area:

$$\mathbf{S} = \frac{\mu_0 c}{2}\left[K(t)\right]^2 \hat{\mathbf{x}}$$

Here, radiation still means the same thing – basically the portion of the EM field that has a $\frac{1}{r}$ dependence, which decays slowly enough that it is nonzero even at infinity. We can see this in the fact that the vector potential we integrated always has an $\frac{1}{z}$ dependence. $\square$