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Introduction to Probability

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1 Introduction to Probability

1. Alex and Shruti are playing Yahtzee, a game involving rolling 5 dice.

   (a) First, define a probability space representing the possible outcome of Alex or Shruti's rolls of the 5 dice. Assume all dice are fair and labeled 1 through 6.

   Solution: Our probability space can be represented by

   $$\Omega = \{(d_1, d_2, d_3, d_4, d_5) \mid d_i \in \{1, 2, 3, 4, 5, 6\}\}$$

   or similar, since there are 5 rolls and each one takes on a value of 1 through 6.

   Alex and Shruti each roll 1 die to see who goes first. The person with the higher roll goes first, and in case of a tie, they both roll their die again.

   (b) What's the chance Shruti rolls a higher number on the first roll?

   Solution: Let p be the chance Shruti rolls a higher number. By symmetry, Alex has the same chance p of rolling the higher number. Therefore the chance of a tie (neither Alex nor Shruti wins) is $1 - 2p$. A tie happens when they both roll 1, 2, 3, 4, 5, or 6, which happens with probability $6/36 = 1/6$. Thus $1 - 2p = 1/6$. Solving for p, we get $p = 5/12$.

   (c) What's the chance Shruti goes first?

   Solution: **Solution 1:** By symmetry, it's $1/2$ since Shruti's chance of going first, even if rerolls are needed, is always the same as Alex's.
   Solution 2: By the law of total probability, we sum up Shruti's chance of going first after $n = 1, 2, 3, \ldots$ rolls:

   $$\begin{aligned} P(1) &= \frac{5}{12} \\ P(2) &= \frac{1}{6} \cdot \frac{5}{12} \\ &\vdots \\ P(\text{Shruti wins}) &= \sum_{i=1}^{\infty} \frac{5}{12} \cdot \left(\frac{1}{6}\right)^{i-1} \\ &= \frac{5}{12} \cdot \frac{1}{1 - 1/6} \\ &= \frac{5}{12} \cdot \frac{6}{5} = \frac{1}{2} \end{aligned}$$

   (d) They finally begin playing. Partway through the game, Alex is missing the "three of a kind" category while Shruti is missing the "four of a kind" category. What is the probability of rolling...

   1. exactly 3 of a kind?

Solution:

$$\frac{\binom{5}{3} \cdot 6 \cdot 5^2}{6^5}$$

2. exactly 4 of a kind?

Solution:

$$\frac{\binom{5}{4} \cdot 6 \cdot 5}{6^5}$$

3. Which one is more likely? 3 of a kind or 4 of a kind?

Solution: We can just compare the numerators of the 2 above parts, since the denominators are equal.

$$\binom{5}{3} \cdot 6 \cdot 5^2 = 10 \cdot 6 \cdot 5^2 = 1500$$
$$\binom{5}{4} \cdot 6 \cdot 6 = 5 \cdot 6^2 \cdot 5 = 900$$

Lining up with intuition, 3 of a kind is more likely.

2. Suppose two integers a and b are drawn uniformly from $[-n \,..\, n]$, that is $a, b \in \mathbb{Z}$ and $-n \le a, b \le n$.

(a) Define a probability space for (a, b). Does each sample point occur with uniform probability?

Solution: A probability space is defined by a sample space and probabilities for each sample point. The sample space is defined as

$$\Omega = \{(i, j) : -n \le i, j \le n\}$$

Clearly $\mathbb{P}[(i, j)] = \mathbb{P}[(k, l)]$ by uniform symmetry, so $\forall \omega \in \Omega$,

$$\mathbb{P}[\omega] = \frac{1}{|\Omega|} = \frac{1}{(2n+1)^2} \tag{1}$$

(b) Find the probability that $\max\{0, a\} = \min\{0, b\}$.

Solution: The problem is true if and only if $a \le 0$ and $b \ge 0$. There are $n+1$ such values for a and $n+1$ such values for b, meaning there are $(n+1)^2$ sample points. Given each sample point occurs with equal probability, we simply divide the number of sample points by the cardinality of the sample space:

$$\frac{(n+1)^2}{(2n+1)^2}$$

(c) Find the probability that $|a - b| \le k$. You may assume $k < \frac{n}{2}$.

Solution: We begin by finding a formula for arbitrary n.
If $a < -n + k$, then $a = -n + i$ where $i \in [0 .. k - 1]$ then, b can be in $[-n .. -n + k + i]$, meaning there are $k + i + 1$ satisfying sample points for b.

If $-n + k \le a \le n - k$, then b can be in $[a - k .. a + k]$, meaning there are $2k + 1$ satisfying sample points for b.

If $a > n - k$, the case is symmetric to the first case and there are again $k + i + 1$ satisfying sample points for b.

Now, we find the number of sample points for (a, b) by summing over all possible values for a:

$$\begin{aligned}
|A| &= 2\Big(\sum_{i=0}^{k-1} k + i + 1\Big) + \Big(\sum_{i=1}^{(n-k)-(-n+k)+1} 2k + 1\Big) && (2)\\
&= 2\big(k^2 + \frac{k(k-1)}{2} + k\big) + (2n - 2k + 1)(2k + 1) && (3)\\
&= 2k(k+1) + k(k-1) + (2n - 2k + 1)(2k + 1) && (4)\\
&= -k^2 + k + 4nk + 2n + 1 && (5)
\end{aligned}$$

From (1) to (2), we use the arithmetic sum formula. Since each sample point occurs with equal probability,

$$\mathbb{P}(|A|) = \frac{-k^2 + k + 4nk + 2n + 1}{(2n+1)^2} \tag{6}$$

(d) Suppose we choose two closed intervals $u = [a .. b], v = [c .. d]$ uniformly at random from $[-n .. n]$. What is the probability that u is enveloped by v, meaning that $u \subset v$ and $c < a \le b < d$. What happens to this probability as n approaches ∞?

Solution: First, we define our sample space as the set of all pairs of intervals, so that the probability of any sample point is equal.

Notice that any interval is uniquely defined by its starting and ending index. Hence, we place bars between elements, before the first element, and after the last element (e.g. $| -n | -n+1 | \dots | n-1 | n |$) and the number of intervals is the number of ways to choose 2 bars, which is the same as choosing 2 distinct elements from $2n + 2$.

$$\binom{2n+2}{2} \tag{7}$$

Hence, the size of the sample space is $\binom{2n+2}{2}^2$.

Now, there is always exactly one arrangement of (a, b, c, d), (e, f, g, h) such that $e < f \le g < h$. Once again, this problem can be modeled as selecting 4 bars from the bars between elements, giving us

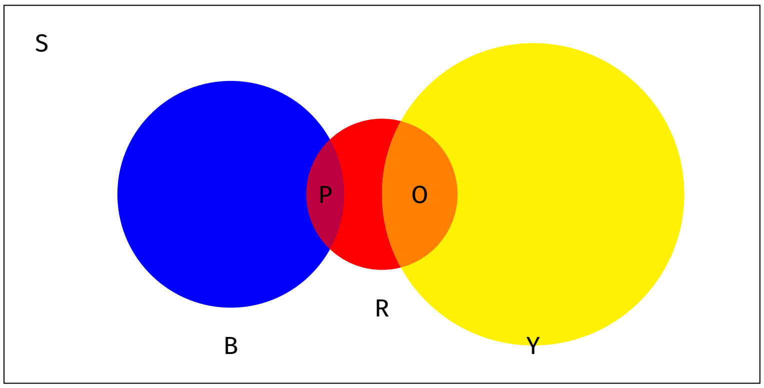
$$|A| = \binom{2n+2}{4} \tag{8}$$

Dividing $|A|$ by $|\Omega|$ gives

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\binom{2n+2}{4}}{\binom{2n+2}{2}^2} = \frac{1}{6}\frac{(2n!)^2}{(2n-2)!(2n+2)!} = \frac{1}{6}\frac{2n(2n-1)}{(2n+2)(2n+1)} \tag{9}$$

As $n \longrightarrow \infty$, the probability converges to $\frac{1}{6}$

3. Consider the drawing of the probability space S below. Here, the blue/purple region is the set of events B, the red/purple/orange region is the set of events R, and the yellow/orange region is the set of events Y. The set of events P is the set of events in both B and R, and is represented by the purple region. The set of events O is the set of events in both R and Y, and is represented by the orange region.



Assume that we are sampling from S uniformly at random. Please answer the following multiple choice questions about this space, selecting all that apply.

(a) What is $P[R]$, the probability that an element from S is in R?

Hint: Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes.

i. $\frac{|R|-|P|-|O|}{|S|}$

ii. $\frac{|R|}{|S|}$

iii. $|R|$

Solution: Option ii. Since we are sampling uniformly at random, the probabilities are computed by dividing the number of elements in the event of interest (R) by the total number of elements (S).

(b) What is $P[R|Y]$, the probability that an element from S is in R given that it is also in Y?

Hint: Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. When we condition on an event Z, we are restricting the total outcomes to be outcomes for which Z occurs.

i. $\frac{|O|}{|S|}$

ii. $\frac{|O|}{|Y|}$

iii. $\frac{|R \cap Y|}{|Y|}$

Solution: Options ii and iii. Conditioning on Y is like considering everything outside of Y as not even being there. So we need to restrict R to Y which gives us $Y \cap R = O$. Then we need to divide the number of elements in that region by $|Y|$ which is our new sample space.

(c) What is $P[R|O]$, the probability that an element from S is in R given that it is in O?

Hint: Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. When we condition on an event Z, we are restricting the total outcomes to be outcomes for which Z occurs.

i. 1

ii. $\frac{|R \cap O|}{|O|}$

iii. $\frac{|R \cap O|}{|S|}$

Solution: Options i and ii. Like before, this probability is equal to $\frac{|R \cap O|}{|O|}$. But note that $O \cap R = O$, so this fraction is just 1.

(d) What is $P[B \cup R \cup Y]$, the probability that an element of S is in B or R or Y?

Hint: Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. Be careful not to double count!

i. $\frac{|B|+|R|+|Y|-|P|-|O|}{|S|}$

ii. $\frac{|B \setminus P|+|R|+|Y \setminus O|}{|S|}$ $\quad (X \setminus Z = \{x \ s.t. \ x \in X \text{ and } x \notin Z\})$

iii. $\frac{|B|+|R|+|O|}{|S|}$

Solution:

Options i and ii. We just need to count and make sure that each region inside the union is counted exactly once. This happens in the first two choices. In the last choice, O is counted twice (once from R and once from O) as is P. So the last choice is not right.

(e) What is $P[O|R \cup Y]$, the probability that an element of S is in O given that it is also in R or Y?

Hint: Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. When we condition on an event Z, we are restricting the total outcomes to be outcomes for which Z occurs. Be careful not to double count!

i. $\frac{|O|}{|S|-|B|}$

ii. $\frac{|O|}{|Y|+|R|-|O|}$

iii. $\frac{|O|}{|Y|+|R|}$

Solution: Option ii. O is a subset of $R \cup Y$. So in the numerator we just need $|O|$. In the denominator we need $P[R \cup Y]$. Only the second choice gives us that in the denominator.

2 Fill in the Blank

1. (Fall 2021 Final Q9): Given a probability space (Ω, P) and events A, B and C, fill in $<, \leq, =, \geq, >$ or "Incomparable" for the following questions such that the statement always holds. If no inequality always holds, use "Incomparable".

   (a) $P[A \cup B \cup C]$ _____ $P[A] + P[B] + P[C]$.

   > **Solution:** $\leq$. Notice that the left hand side is a union, so the maximum for $P[A \cup B \cup C]$ is when $A \cap B = \emptyset$, $A \cap C = \emptyset$ and $B \cap C = \emptyset$, in which case $P[A \cup B \cup C] = P[A] + P[B] + P[C]$.

   (b) $P[A \cap B \cap C]$ _____ $P[A]P[B]P[C]$.

   > **Solution:** Incomparable, there is no relationship relating the size of the sets. Consider the case where $A = B = C$. Then $P[A \cap B \cap C] = P[A]$, but $P[A]P[B]P[C] = P[A]^3$, so $P[A \cap B \cap C] \geq P[A]P[B]P[C]$ (remember that $0 \leq P[A] \leq 1$).
   >
   > Alternatively, there is the case that $P[A \cap B \cap C] = 0$, if A, B, C are disjoint. But, $P[A], P[B], P[C]$ could each be nonzero, so this is a case where $P[A \cap B \cap C] < P[A]P[B]P[C]$.

   (c) $P[A \cap B]$ _____ $P[A]$.

   > **Solution:** $\leq$, since $A \cap B$ is a subset of A, so $|A \cap B|$ is always smaller than $|A|$, hence $P[A \cap B] \leq P[A]$. (Equality reached when $A = B$)

   (d) $P[A|B]$ _____ $P[A]$.

   > **Solution:** Incomparable. If A and B are disjoint, then $P[A|B] = 0$ so $P[A|B] < P[A]$, and if $A = B$, then $P[A|B] = 1$, so $P[A|B] > P[A]$.

   (e) $P[A \cup B \cup C]$ _____ $P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$.

   > **Solution:** $=$. This is principle of inclusion exclusion.