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HW 07 | Quantum Mechanics | November 26, 2022

Collaborators

I worked with **Andrew Binder** to complete this assignment.

Problem 1

An operator $\hat{A}$, representing observable A, has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2 respectively. Operator $\hat{B}$, representing observable B, has two normalized eigenstates ϕ_1 and ϕ_2, with eigenvalues b_1 and b_2. The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5, \qquad \psi_2 = (4\phi_1 - 3\phi_2)/5$$

(a) Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?

Solution: If the value a_1 is obtained, then the wavefunction collapses to solely that of ψ_1. □

(b) If B is now measured, what are the possible results, and what are their probabilities? *Solution:* Since we know that the state has collapsed into a state of ψ_1, then the only possible values we could get for b will be the eigenfunctions of b related to ψ_1. Therefore, we could only measure b_1 and b_2 with probabilities $\frac{9}{25}$ and $\frac{16}{25}$, respectively. More explicitly,

$$P(b_1) = \frac{9}{25}$$
$$P(b_2) = \frac{16}{25}$$

□

(c) Right after the measurement of B, A is measured again. What is the probability of getting a_1? (Note that the answer would be quite different if I had told you the outcome of the B measurement)

Solution: By measuring B we've collapsed the wavefunction into either one of ϕ_1 or ϕ_2. Now in order to find the measurement of A, we need to rearrange each ϕ and express them in terms of ψ:

$$\phi_1 = \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2$$
$$\phi_2 = \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2$$

Therefore, to measure the value a_1, we basically just need to calculate the probability of the wavefunction collapsing into ϕ_1 then measuring ψ_1, and add it to the probability of the wavefunction collapsing into ϕ_2 then measuring ψ_1. Therefore,

$$P(a_1) = \frac{9}{25} \cdot \frac{9}{25} + \frac{16}{25} \cdot \frac{16}{25} = \frac{337}{625}$$

$\square$

Problem 2

(a) Work out all of the **canonical commutation relations** for components of operators $\mathbf{r}$ and $\mathbf{p}$: $[x, y]$, $[x, p_y]$, $[x, p_x]$, $[p_y, p_z]$, and so on.

Answer:

$$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij}, \quad [r_i, r_j] = [p_i, p_j] = 0$$

where the indices stand for x, y, or z, and $r_x = x, r_y = y$, and $r_z = z$.

Solution: The idea is that since measuring the position along one axis does not affect our measurements along another orthogonal axis (becuase we've determined no information about that other axis by measuring one), then it makes sense that the commutator of $[r_x, r_y] = 0$, and in general $[r_i, r_j] = 0$. The same argument applies when we try to measure the momentum along two orthogonal axes, so $[p_i, p_j] = 0$ for any i and j.

However, when we try to measure the position then the momentum, we do have the restriction that measuring momentum and position along the *same* axis does not commute, but measuring along different axes does commute. Therefore $[r_x, p_x] = i\hbar$, and more generally $[r_i, p_j] = i\hbar\delta_{ij}$. Therefore, to summarize:

$$[r_i, p_j] = i\hbar\delta_{ij}, [r_i, r_j] = [p_i, p_j] = 0$$

□

(b) Confirm Ehrenfest's theorem for 3-dimensions:

$$\frac{d}{dt}\langle \mathbf{r} \rangle = \frac{1}{m}\langle \mathbf{p} \rangle, \text{ and } \frac{d}{dt}\langle \mathbf{p} \rangle = \langle -\nabla V \rangle$$

(Each of these, of course, stands for *three* equations - one for each component.) *Hint:* First check that Equation 3.71 is valid in three dimensions.

Solution: Using the hint, we first verify that Equation 3.71 holds in three dimensions. This is relatively obvious to see, since the only thing that changes with the Schrödinger equation is the fact that our spatial derivative becomes a 3-dimensional spatial derivative, but it does not change the fact that:

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$$

which really is the only thing that we need to derive Equation 3.71 in the first place. Therefore, since it holds in one direction then it must hold in 3 dimensions. Now, we can use this to verify Ehrenfest's theorem. First, we know that

$$\begin{aligned}\frac{d}{dt}\langle r \rangle &= \left\langle \frac{dr}{dt} \right\rangle + \frac{i}{\hbar}\left\langle [\hat{H}, r] \right\rangle \\ &= \langle v \rangle \\ &= \frac{\langle p \rangle}{m}\end{aligned}$$

This is true because position and the hamiltonian commute. Now for the time derivative of the momentum, we have

$$\frac{dp}{dt} = \frac{\partial}{\partial t}\left(i\hbar\frac{\partial}{\partial x}\right) = 0$$

and so therefore we only need to calculate $\frac{i}{\hbar}\left\langle[\hat{H}, \hat{p}]\right\rangle$. However, since the x, y and z component of momentum are independent of each other (because they are orthogonal), then we know that from the original Ehrenfest's theorem, that

$$\frac{d\langle p_x\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

And so therefore we have

$$\begin{aligned}\frac{d\langle p\rangle}{dt} &= \left\langle -\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial V}{\partial z}\right\rangle \\ &= \langle -\nabla V\rangle\end{aligned}$$

And so therefore Ehrenfest's theorem is verified. □

(c) Formulate Heisenberg's uncertainty principle in three dimensions. *Answer:*

$$\sigma_x\sigma_{p_x} \geq \hbar/2, \quad \sigma_y\sigma_{p_y} \geq \hbar/2, \quad \sigma_z\sigma_{p_z} \geq \hbar/2$$

But there is no restriction on, say, $\sigma_x\sigma_{p_y}$.

Solution: From the textbook, we also have the relation, given that both x and p don't depend on t:

$$\sigma_x\sigma_p \geq \frac{1}{2i}\langle[x, p]\rangle$$

Now note that from part (a), we know that $[r_i, p_j] = i\hbar\delta_{ij}$, so therefore we know that the only commutators that will give us nonzero values will be when we compute $[x, p_x], [y, p_y], [z, p_z]$ and so therefore we get the relations

$$\sigma_x\sigma_{p_x} \geq \hbar/2, \quad \sigma_y\sigma_{p_y} \geq \hbar/2, \quad \sigma_z\sigma_{p_z} \geq \hbar/2$$

□

Problem 3

The raising and lowering operators change the value of m by one unit:

$$L_\pm f_l^m = (A_l^m) f_l^{m\pm 1}$$

where A_l^m is some constant. *Question:* What is A_l^m, if the eigenfunctions are to be *normalized*? *Hint:* First show that $L_\mp$ is the Hermitian conjugate of $L_\pm$ (since L_x and L_y are *observables*, you may assume they are Hermitian ... but *prove* it if you like); then use Equation 4.112. *Answer:*

$$A_l^m = \hbar\sqrt{l(l+1) - m(m\pm 1)} = \hbar\sqrt{(l\mp m)(l\pm m+1)}$$

Note what happens at the top and bottom of the ladder (i.e., when you apply L_+ to f_l^l or L_- to f_l^{-l})

Solution: The hint wants us to show that $L_\pm$ is the Hermitian conjugate of $L_\mp$. In lecture, we derived the following relation:

$$\begin{aligned}\hat{L}_+ &= \hat{L}_x + i\hat{L}_y\\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y\end{aligned}$$

And since L_x and L_y are observables, then they are Hermitian, so therefore L_+ is the Hermitian conjugate of L_- and vice versa. Now, to normalize, we require that $\langle f_l^m | L_\pm L_\mp f_l^m\rangle = 1$, so therefore

$$\begin{aligned}1 &= \left\langle f_l^m \middle| L^2 - L_z^2 \pm \hbar L_z f_l^m\right\rangle\\ &= \left\langle f_l^m \middle| (l(l+1)\hbar^2 + m^2 \mp \hbar^2 m) f_l^m\right\rangle\end{aligned}$$

And since we know that $\langle f_l^m | f_l^m\rangle = 1$, the our normalization constant is just the constants that we just factored out. Therefore,

$$\begin{aligned}(A_l^m)^2 &= l(l+1)\hbar^2 + m(m\mp 1)\hbar^2\\ A_l^m &= \hbar\sqrt{l(l+1) - m(m\pm 1)}\end{aligned}$$

as desired. $\square$

Problem 4

(a) Starting with the canonical commutation relations for position and momentum (Equation 4.10), work out the following commutators:

$$[L_z, x] = i\hbar y, \qquad [L_z, y] = -i\hbar x, \qquad [L_z, z] = 0$$
$$[L_z, p_x] = i\hbar p_y, \qquad [L_z, p_y] = -i\hbar p_x, \qquad [L_z, p_z] = 0$$

Solution: Using a bunch of relations we've derived from lecture, we obtain the following results:

$$\begin{aligned}
[L_z, x] &= [xp_y - yp_x, x] = [xp_y, x] - [yp_x, x] = -y[p_x, x]i\hbar y \\
[L_z, y] &= [xp_y - yp_x, y] = [xp_y, y] - [yp_x, y] = x[p_y, y] = -i\hbar x \\
[L_z, z] &= [xp_y - yp_x, z] = [xp_y, z] - [yp_x, z] = 0 \\
[L_z, p_x] &= [xp_y - yp_x, p_x] = [xp_y, p_x] - [yp_x, p_x] = p_y[x, p_x] = i\hbar p_y \\
[L_z, p_y] &= [xp_y - yp_x, p_y] = [xp_y, p_y] - [yp_x, p_y] = -p_x[x, p_y] = -i\hbar p_x \\
[L_z, p_z] &= [xp_y - yp_x, p_z] = [xp_y, p_z] - [yp_x, p_z] = 0
\end{aligned}$$

And so we're done. □

(b) Use these results to obtain $[L_z, L_x] = i\hbar L_y$ directly from Equation 4.96.

Solution: Now that we confirmed each of those commutator values, we can use Equation 4.96:

$$\begin{aligned}
[L_z, L_x] &= [L_z, yp_z - zp_y] = [L_z, y]p_z - [L_z, p_y]z \\
&= (-i\hbar x)p_z - (-i\hbar p_x)z \\
&= i\hbar(zp_x - xp_z) \\
&= i\hbar L_y
\end{aligned}$$

As desired. □

(c) Evaluate the commutators $[L_z, r^2]$ and $[L_z, p^2]$ (where, of course, $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$).

Solution: Using the fact that $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$, we can just throw them into our commutators:

$$\begin{aligned}
[L_z, r^2] &= [L_z, x^2 + y^2 + z^2] \\
&= [L_z, x^2] + [L_z, y^2] + [L_z, z^2] \\
&= [L_z, x]x + x[L_z, x] + [L_z, y]y + y[L_z, y] + [L_z, z]z + z[L_z, z] \\
&= i\hbar yx + xi\hbar y + (-i\hbar x)y + y(-i\hbar x) = 0 \\
[L_z, p^2] &= [L_z, p_x^2 + p_y^2 + p_z^2] \\
&= [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2] \\
&= [L_z, p_x]p_x + p_x[L_z, p_x] + [L_z, p_y]p_y + p_y[L_z, p_y] + [L_z, p_z]p_z + p_z[L_z, p_z] \\
&= i\hbar p_y p_z + p_x i\hbar p_y + (-i\hbar p_x)p_y + p_y(-i\hbar p_x) = 0
\end{aligned}$$

And so we find that L_z commutes with both r^2 and p^2. □

(d) Show that the Hamiltonian $H = (p^2/2m) + V$ commutes with all three components of $\mathbf{L}$, provided that V depends only on r. (Thus, H, L^2, and L_z are mutually compatible observables.)

Solution: First notice that because in the previous part we've shown that L_z commutes with r^2 and p^2, then so does L_x and L_y (this is analogous to tilting your or just rotating your coordinate axes). Therefore, all components of L commute with p^2. This takes care of the first term in the Hamiltonian (i.e. $\frac{p^2}{2m}$)

Now for the potential. Notice that we can write the potential in the form: $V(r) = v(\sqrt{r^2})$, which explicitly makes V a function of r^2, which we know all components of L commutes with, by the same argument we used to show that L commutes with p. Therefore, we know that all the components of L also commute with V. Therefore, since L commutes with both terms that make up H, then L commutes with H, as desired.

$\square$

Problem 5

Let $\hat{\mathbf{n}}$ be a unit vector in a direction specified by the polar angles (θ, ϕ). Show that the component of the angular momentum in the direction $\hat{\mathbf{n}}$ is

$$\begin{aligned} L_n &= \sin\theta\cos\phi L_x + \sin\theta\sin\phi L_y + \cos\theta L_z \\ &= \frac{1}{2}\sin\theta(e^{-i\phi}L_+ + e^{i\phi}L_-) + \cos\theta L_z \end{aligned}$$

Solution: Suppose we have a vector L which is expressed in cartesian coordinates. Then, the component along the $\hat{n}$ direction will be $L \cdot \hat{n}$. In order to do this dot product, we need to first convert $\hat{n}$ to cartesian coordinates. Since n is a unit vector, then we know that $|\hat{n}| = 1$, so therefore

$$\hat{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$

And so therefore taking the dot product:

$$\begin{aligned} L_n = L \cdot \hat{n} &= (L_x, L_y, L_z) \cdot (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ &= L_x\sin\theta\cos\phi + L_y\sin\theta\sin\phi + L_z\cos\theta \end{aligned}$$

which is exactly the equation we were asked to show.

$\square$

If the system is in simultaneous eigenstates of $\mathbf{L}^2$ and L_z belonging to the eigenvalues $l(l+1)\hbar^2$ and $m\hbar$,

(a) what are the possible results of a measurement of L_n?

Solution: Since L_n can be expressed as a linear combination of L_x, L_y, and L_z, then we expect that the possible results of a measurement would be the possible results of a measurement given these operators. Because we can just tilt our axis to have L_z be the represented by L_x or L_y, then the only possible result for measurement is $m\hbar$.

$\square$

(b) what are the expectation values of L_n and L_n^2?

Solution: The expectation values for L_n and L_n^2 are calculated using:

$$\begin{aligned} \langle L_n \rangle &= \langle \psi | L_n \psi \rangle \\ &= \sin\theta\cos\phi \langle \psi | L_x \psi \rangle + \sin\theta\sin\phi \langle \psi | L_y \psi \rangle + \cos\theta \langle \psi | L_z \psi \rangle \\ &= 0 \end{aligned}$$

I didn't have time to compute $\left\langle L_n^2 \right\rangle$ but the process is very similar. First, we square L_n in terms of L_x, L_y and L_z, then derive the expectation value for each of them. Once this is done, we can derive the expectation value. $\square$
