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1 Linear Maps

1.1 Vector Space of Linear Maps

Problem: Suppose $b, c \in \mathbf{R}$. Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$$

Show that T is linear if and only if $b = c = 0$.

Solution: We first show that if $b = c = 0$, then T is linear. Recall the facts of linearity:

$$T(u + v) = Tu + Tv \quad T(\lambda v) = \lambda(Tv)$$

for all $v \in V$. If $b = c = 0$, then we can define T as:

$$T(x, y, z) = (2x - 4y + 3z, 6x)$$

Now suppose we have two vectors $u = (x_1, y_1, z_1), v = (x_2, y_2, z_2)$. Then:

$$Tu+Tv = (2x_1-4y_1+3z_1, 6x_1)+(2x_2-4y_2+3z_2, 6x_2) = (2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2), 6(x_1+x_2)) = T(u+v)$$

Now for homogeneity:

$$T(\lambda u) = T(\lambda x_1, \lambda y_1, \lambda z_1) = (2\lambda x_1 - 4\lambda y_1 + 3\lambda z_1, 6\lambda x_1) = \lambda(Tu)$$

Therefore, both conditions are satisfied, indeed T is linear. Now we show that if T is linear, then $b = c = 0$ is necessary. Consider what we had earlier:

$$Tu + Tv = (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + 2b, 6(x_1 + x_2) + cx_1y_1z_1 + cx_2y_2z_2)$$

This is only equal to $T(u + v)$ if $2b = 0$ and $c(x_1y_1z_1 + x_2y_2z_2) = 0$, since they are the only nonlinear terms. Thus, if T is linear, then $b = c = 0$. □

Problem: Suppose that $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Solution: We return to the definition of linear independence: a set of vectors $v_1, \ldots v_m$ is linearly independent the solution to the equation:

$$a_1v_1 + \cdots + a_mv_m = 0$$

is that $a_1, \ldots, a_m = 0$. Since we know that the list $Tv_1, \ldots, Tv_m$ is linearly independent in w, then the solution to the equation:

$$a_1Tv_1 + \cdots + a_mTv_m = 0$$

is $a_1 = \cdots = a_m = 0$. Now, apply the rules of T being a linear map:

$$a_1Tv_1 + \cdots + a_mTv_m = T(a_1v_1) + \cdots + T(a_mv_m) = T(a_1v_1 + \cdots a_mv_m) = 0$$

Now, we use the fact that since linear maps take 0 to 0, this implies that $a_1v_1 + \cdots + a_mv_m = 0$. Further, since the only values of a_i that satisfy this equation is $a_1 = \cdots = a_m = 0$, then this satisfies the condition that $v_1, \ldots, v_m$ is linearly independent. □

Problem: Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution: Since V is one-dimensional, this implies that there is only one basis vector, v_1. Therefore, for all vectors $v \in V$, then $v = \alpha v_1$ for some $\alpha \in \mathbf{F}$. Then, becuase $T \in \mathcal{L}(V)$, then T must map every vector $v \in V$ to another vector in V, which must be expressed as a scalar times v. Thus, $Tv = \lambda v$ is the only option for a linear map on this space. More precisely:

$$Tv = T(\lambda v_1) = \lambda T v_1 = \alpha \lambda v_1 = \lambda(\alpha v_1) = \lambda v$$

as desired. □

Problem: Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}$ but φ is not linear.

Problem: Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T : V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that T is not a linear map on V.

Solution: A linear map must satisfy $T(\lambda v) = \lambda(Tv)$ for all $v \in V$. However, consider some λ such that $v \in U$ but $\lambda v \notin U$. Then:

$$T(\lambda v) = 0 \quad \lambda(Tv) = \lambda S v \neq 0$$

Hence, T is not linear on V. Alternatively, we could define $v \in V$ and $w \in V$ but $w \notin U$, then we have:

$$T(v + w) = 0$$

But:

$$Tv + Tw = Sv \neq 0$$

so this also violates linearity. □

Problem: Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

Solution: Since v_i is linearly dependent, then we can write $v_i = \sum_{j \neq i} a_j v_j$ for some set of a_j. Then, consider some nonzero $w \in W$, and set the w_k's as follows:

$$w_k = \begin{cases} w & k = i \\ 0 & \text{else} \end{cases}$$

Then, suppose for all $j \neq i$, that $Tv_j = w_j = 0$. Then, let's write Tv_i:

$$Tv_i = T\left(\sum_{j \neq i} a_j v_j\right) = \sum_{j \neq i} a_j T v_j \sum_{j \neq i} a_j w_j = w_i$$

But since all $w_j = 0$, this means that $w_i = 0$, but we set $w_i \neq 0$ purposefully, therefore there is no T that would stasify this. □

1.2 Null Spaces and Ranges

Problem: Suppose $S, T \in \mathcal{L}(V)$ are such that range $S \subseteq$ null T. Prove that $(ST)^2 = 0$.

Solution: Consider a vector $v \in V$. Then:

$$\begin{aligned}(ST)^2 v &= (STS)(Tv) \\ &= ST(Sv')\end{aligned}$$

Now, Sv' will exist within range S, and since we know that range $S \subseteq$ null T, then this impleis that $T(Sv') = 0$. Finally, $S(0) = 0$, so hence $(ST)^2 v = 0$, so $(ST)^2 = 0$. □

Problem: Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m$$

a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?

Solution: If T is surjective, then $v_1, \ldots, v_m$ spans V. This is because if the range of T is V, then the set of vectors T applies a linear combination to must span V. □

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b) What property of T corresponds to the list $v_1, \ldots, v_m$ being linearly independent?

Solution: The set $v_1, \ldots, v_m$ is linearly independent if and only if T is injective, since linear independence means that there is only one way to express every vector (i.e. the solution to $T(z_1, \ldots, z_m) = 0$ is $z_1 = \cdots = z_m = 0$). □

Problem: Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Solution: The proof of this is very similar to the one we did earlier. If $v_1, \ldots, v_n$ is linearly dependent, then this means that

$$a_1 v_1 + \cdots + a_n v_n = 0$$

is solved by setting $a_i = 0$. Then, now let's consider the list $Tv_1, \ldots, Tv_n$:

$$a_1 Tv_1 + \cdots + a_n Tv_n = T(a_1 v_1 + \cdots + a_n v_n) = 0$$

where the last equality we obtain from the fact that linear maps map 0 to 0. This implies that the only solution to this equation is $a_i = 0$, hence the list $Tv_1, \ldots, Tv_n$ is linearly independent. □

Problem: Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu : u \in U\}.$$

Solution: We can define $U = \{0\} \cup (V \setminus \operatorname{null} T)$. This way, null $T \cap U = \{0\}$ by definition, and range $T = \{Tv : v \in V\}$, but since U defines the same set of vectors (since we only get rid of the null space), then range $T = \{Tu : u \in U\}$, as desired. □

Problem: Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that T is surjective.

Solution: We know that $T \in \mathcal{L}(\mathbf{F}^4, F^2)$. Because the null space can be determined by two variables (x_1, x_3), so dim null $T = 2$. This implies that since dim $V = 4$, then dim range $T = 2$, and since this equals the dimension that T maps to $\mathbf{F}^2$, then this implies that T is indeed surjective by (3.19). □

Problem: Suppose V and W are both finite-dimensional. Prove that there exists an injective map from V to W if and only if $\dim V \leq \dim W$.

Solution: Recall the definition of injectivity: a linear map $T \in \mathcal{L}(V, W)$ is injective if and only if $Tu = Tv$ implies that $u = v$, or equivalently that $\operatorname{null} T = \{0\}$.

We prove the forward case: if $\dim V \leq \dim W$, we show that there exists an injective map from V to W. Let $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_m$ be the basis vectors of V and W respectively, and $n \leq m$. Then, define a map $Tv_i = w_i$ for all $i = 1, \ldots, n$. Then, $\dim \operatorname{range} T = \dim V$, implying that $\dim \operatorname{null} T = 0$ from FTLM, as desired.

Now we prove the reverse: we want to show that if there exists an injective map from V to W, then $\dim V \leq \dim W$. This is trivial by contradiction: if $\dim V > \dim W$, then by (3.22) this is impossible, so we're done. □

Problem: Suppose V and W are finite-dimensional and U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution: We prove the forward case: there exists a T such that $\operatorname{null} T = U$ if $\dim U \geq \dim V - \dim W$. Let $\{u_i\}$ be a basis of U, and $\{w_i\}$ be a basis of W. Then, define a linear map T as follows:

$$Tv_i = \begin{cases} \vec{0} & v_i \in \{u_i\} \\ w_i & v_i \notin \{u_i\} \end{cases}$$

One can check very easily that this is linear, with $\operatorname{null} T = U$. Here, $\dim \operatorname{null} T = \dim U$ and $\dim \operatorname{range} T = \dim W - \dim U$, since the basis vectors that map to a nonzero vector in W are those that do not form a basis of U. Therefore, $\dim V = \dim U + (\dim W - \dim U) = \dim W \leq \dim U + \dim W$, so the inequality is satisfied.

Now we prove the reverse case: if such a T exists, then $\dim U \geq \dim V - \dim W$. From the fundamental theorem of linear maps, we know that $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. Now suppose we have a T such that $\operatorname{null} T = U$. Then, we have $\dim V = \dim U + \dim \operatorname{range} T$, and since $\dim \operatorname{range} T \leq \dim W$, then $\dim V \leq \dim U + \dim W$, which is the inequality we wanted to satisfy. □

Problem: Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V.

Solution: We show the forward case: T is injective if there exists an $S \in \mathcal{L}(W, V)$ such that ST is the identity. Suppose T is not injective, so there exists two vectors $u \neq v$ such that $Tu = Tv$. Then, acting S on the left of both sides gives: $STu = STv$ and since ST is the identity, then we're left with $u = v$, which is a contradiction.

Now we show the reverse case: if T is injective, there exists an $S \in \mathcal{L}(W, V)$ such that ST is the identity. Since T is injective, then we know that for any two vectors $v_1, v_2 \in V$, if $Tv_1 = Tv_2$ then $v_1 = v_2$. Now, let $\{w_1, w_2, \ldots, w_m\}$ be a basis for W and $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. Then, let $S \in \mathcal{L}(W, V)$ be defined as:

$$Sw_i = \begin{cases} v_i & i \in \{1, 2, \ldots, n\} \\ 0 & \text{else} \end{cases}$$

S is clearly linear, and take any vector $v = \sum_i \alpha_i v_i$. Then:

$$STv = S\left(\sum_i \alpha_i w_i\right) = \sum_i \alpha_i v_i = v$$

so therefore ST is indeed the identity.

As an aside, we also should prove that T transforms in a way such that $Tv_i = w_i$, or in other words T transforms each basis vector in V to a basis vector in W. This needs to be true since FTLM says that $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$, and since $\dim \operatorname{null} T = 0$, then $\dim \operatorname{range} T = \dim V$. $\square$

Problem: Suppose $\phi \in \mathcal{L}(V, \mathbf{F})$ and $\phi \neq 0$. Suppose $u \in V$ is not in $\operatorname{null} \phi$. Prove that

$$V = \operatorname{null} \phi \oplus \{au : a \in \mathbf{F}\}$$

Solution: What we need to show is that the vector space V can be constructed as a direct sum of $\operatorname{null} \phi$ and the set of scalar multiples of u. Suppose by contradiction that this is not the case. In other words, there is an element $w \in V$ such that there are two eleemnts $v_1, v_2 \in \operatorname{null} \phi$ and two distinct values $a_1, a_2 \in \mathbf{F}$ such that $w = v_1 + a_1$ and $w = v_2 + a_2$. Acting ϕ on these, we get that $\phi w = a_1 = a_2$, which is a contradiction to our original statement. $\square$

1.3 Matrices

Problem: Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Solution: For some v_k, if $Tv_k \neq \mathbf{0}$, then this implies that $v_k \notin \operatorname{null} T$. Then, this means that Tv_k must map to some linear combination of $\{w_i\}$, where one of $A_{i,k} \neq 0$. This must also be true for all the basis vectors $\{v_i\}$ that are not part of $\operatorname{null} T$, meaning there are at least $\dim \operatorname{range} T$ nonzero columns, leading to $\dim \operatorname{range} T$ nonzero entries. $\square$

Problem: Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W.

a) Show that if $S, T \in \mathcal{L}(V, W)$ then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$

   Solution: One way to show this is to show that the map $(S + T)v = Sv + Tv$, but this is already given to us based on definition 3.5. Then, since $\mathcal{M}(S + T)$ is just the matrix representation of $S + T$ and $\mathcal{M}(S) + \mathcal{M}(T)$ is the matrix representation of $S + T$, then the proof holds. $\square$

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b) Show that if $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$, then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

   Solution: This works in the same way as the previous problem: Definition 3.5 guarantees the things we want. $\square$

Problem: Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k equal 1 if $1 \leq k \leq \dim \operatorname{range} T$.

Solution: We know that T maps in the form of $Tv_k = \lambda_k w_k$ for some λ_k if $v_k \notin \operatorname{null} T$. Therefore, we can construct $\mathcal{M}(T)$ as follows: place every $v_k \notin \operatorname{null} T$ in colums $1, \dots, \dim \operatorname{range} T$, and populate the corresponding row with the basis vector that v_k maps to: $\lambda_k w_k$. This will guarantee that the matrix is diagonal (i.e. row k and column k equals 1 for $1 \leq k \leq \dim \operatorname{range} T$), and zero otherwise.

To be a bit more specific, the basis for V is just the standard basis, and the basis for W is the basis formed by the images of v_i when acted on by T. $\square$

Problem: Suppose $v_1, \dots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \dots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$] are 0 except possibly a 1 in the first row, first column.

Note: This exercise, unlike the previous one, we are given a basis of V instead of being able to choose a basis of V.

Solution: Essentially, we want to prove that there exists a basis $w_1, \ldots, w_n$ such that either $Tv_1 = w_1$ or $Tv_1 = \mathbf{0}$. The latter case is easy: all entries in the first column are zero becuase $v_1 \in \operatorname{null} T$, so any basis $w_1, \ldots, w_n$ will work.

Now suppose that $v_1 \notin \operatorname{null} T$. Then, we know that $Tv_1 = \sum_k A_{k,1} w_k = \mathbf{w}$. Now, we can just choose $\mathbf{w}$ as the first basis vector – nothing prevents us from doing this, and the other bases of w can just be the basis vectors $\{w_i\}$ that don't contribute (i.e. have a prefactor of zero) to $\mathbf{w}$. This ensures that the first column will be all zeros except for possibly the top left corner, which is allowed. □

Problem: Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.

Solution: Consider the matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then:

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad BA = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

They're not equal, so we're done. □

Problem: Prove that matrix multiplication is associative. In other words, suppose that A, B and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that

$$(AB)C = A(BC)$$

Try to find a clean proof that illustrates the following quote from Emil Artin: "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out"

Solution: Let A, B, C be linear maps instead, so we show that $(AB)Cv = A(BC)v$ for some vector v. Linear maps are then associative, so this completes the proof immediately.

This is becuase the product of two matrices is defined in the context of the product of linear maps, so if linear maps are associative, then the matrix multiplication must be as well. □

Problem: Suppose m and n are positive integers. Prove that the function $A \mapsto A^t$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

Solution: We know that $A \in \mathbf{F}^{m,n}$. Then, $A^t \in \mathbf{F}^{n,m}$, by the definition 3.54 (transpose). Therefore, this is indeed a valid map from $F^{m,n} \to \mathbf{F}^{n,m}$. Now we need to prove it is linear, which is done for us since the transpose is a linear operation:

$$(A + B)^t = A^t + B^t \quad (\lambda A)^t = \lambda A^t$$

these are the additivity and homogeneity properties respectively, so indeed the transpose operation is linear. □

1.4 Invertibility and Isomorphisms

Problem: Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T$$

Solution: Since T is invertible, then T^{-1} is the unique element in $\mathcal{L}(W, V)$ such that $T^{-1}T = I$. Then, this also means that T is the unique element in $\mathcal{L}(V, W)$ such that $TT^{-1} = I$, hence T^{-1} is invertible. Further, we then call T the inverse of T^{-1}, hence $(T^{-1})^{-1} = T$. □

Problem: Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Solution: We're given that S and T are invertible, so $T^{-1} \in \mathcal{L}(V, U)$ and $S^{-1} \in \mathcal{L}(W, V)$ both exist and are unique. Then, $T^{-1}S^{-1} \in \mathcal{L}(W, U)$ is a unique linear map determined by S and T. We now prove that $T^{-1}S^{-1}$ is the inverse of ST by showing that we get the identity:

$$T^{-1}S^{-1}(ST) = T^{-1}(S^{-1}S)T = T^{-1}IT = T^{-1}T = I$$

Therefore, $(ST)^{-1} = T^{-1}S^{-1}$. □

Problem: Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

Solution: Recall that $\mathcal{L}(V) = \mathcal{L}(V, V)$. The set of noninvertible linear maps from V to itself cannot be a subspace of $\mathcal{L}(V)$ becuase the zero element does not exist. The zero mapping maps any vector in V to the zero vector, so it is not a member of $\mathcal{L}(V)$, and hence this set is not a subspace of $\mathcal{L}(V)$.

Don't know if this is right, I haven't really used the noninvertible property? The set of linear maps from V to itself is just not a linear map? □

Problem: Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Solution: We prove the forward direction first: that there exists an invertible linear map if S is injective. Consider the linear map:

$$Tv_i = \begin{cases} Sv_i & v_i \in \{u_i\} \\ v_i & v_i \notin \{u_i\} \end{cases}$$

Then, T is basically the map S except it maps all basis vectors that are not part of the subspace U onto themselves. Firstly, it's obvious that $T \in \mathcal{L}(V)$, and $Tu = Su$ for every $u \in U$ by the first case. T is also invertible, becuase it is both injective and surjective: it's injective since $\operatorname{null} T = \{0\}$ (and S is injective), and it is surjective since $\operatorname{range} T = V$. Therefore, such a T exists.

Now we prove the reverse case: that if there exists a linear map satisfying these properties, then S must be injective. Since T is invertible and $T \in \mathcal{L}(V)$, then this implies that T is injective (by Theorem 3.65), hence S must also be injective since $Tu = Su$. □

Problem: Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S = \operatorname{range} T$ if and only if there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.

Solution: We prove the forward direction first. Suppose an invertible $E \in \mathcal{L}(V)$ such that $S = TE$ exists. Since E is invertible, this implies that E is a bijective map, or in other words, $\dim \operatorname{null} E = 0$. Then, since $S = TE$, then $\dim \operatorname{null} S = \dim \operatorname{null} TE$, and since $\dim \operatorname{null} E = 0$, then $\dim \operatorname{null} S = \dim \operatorname{null} T$. Finally, since $S, T \in \mathcal{L}(V, W)$ and their null spaces have the same dimension, then $\operatorname{range} S = \operatorname{range} T$.[1]

Now the reverse direction. Suppose that $\operatorname{range} S = \operatorname{range} T$, we aim to find an invertible $E \in \mathcal{L}(V)$ such that $S = TE$. This is relatively simple, as we can just let E be the identity map, which is invertible, and we also get $S = TE$ automatically. □

Problem: Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) \mid U \subseteq \operatorname{null} T\}$$

[1]More precisely, we actually should have $\dim \operatorname{range} S = \dim \operatorname{range} T$, from which we conclude that $\operatorname{range} S = \operatorname{range} T$.

a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.

Solution: We just have to show the properties of a subspace: closedness under addition and scalar multiplication.

- **Addition:** Let $T_1, T_2 \in \mathcal{E}$ two linear maps. Then, consider $v \in U$. Then, $(T_1 + T_2)v = 0$, so $T_1 + T_2 \in \mathcal{E}$, as desired.
- **Scalar Multiplication:** Let $T \in \mathcal{E}$ be a linear map and $v \in U$. Then, $(\lambda T)v = \lambda(Tv) = 0$, so $\lambda T \in \mathcal{E}$, as desired.

with both conditions satisfied, we've proven that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$. $\square$

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b) Find a formula for $\dim E$ in terms of $\dim V, \dim W$ and $\dim U$.

Hint: Define $\Phi = \mathcal{L}(V, W) \to (U, W)$ by $\Phi(T) = T|_U$. What is null Φ? What is range Φ?

Solution: what is a formula here? $\square$

Problem: Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

Solution: We prove first that if S and T are invertible, then ST is invertible. This is realtively easy, since S, T being inverible implies that they are both bijective maps, and hence their composition must also be a bijective map. This implies that ST is invertible.

Now, assume that ST is invertible. Then, by problem 2 we know that $(ST)^{-1} = T^{-1}S^{-1}$, and since these two have inverses, then S and T must both be invertible. $\square$

Problem: Show that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.

Solution: Not sure if you can do this, but theorem 3.70 gives us a nice way of showing that two finite-dimensional vector spaces are isomorphic: if they have the same dimension. Combining this with Theorem 3.72, we find that $\dim \mathcal{L}(\mathbf{F}, V) = (\dim \mathbf{F})(\dim V) = \dim V$, which has the same dimension as $\dim V$, so they are indeed isomorphic.

what happens if V is infinite-dimensional? $\square$

Problem: Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that

$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$

for all $x \in \mathbf{R}$.

Solution: Let $D \in \mathcal{L}(\mathcal{P}(\mathbf{R})$ be the differentiation linear map (it's shown to be linear on page 53). Then, using this language, we can rewrite the equation as:

$$q(x) = (x^2 + x)D^2p(x) + 2xDp(x) + p(3)$$

since $D \in \mathcal{L}(\mathcal{P}(\mathbf{R})$, then we know that $Dp(x) \in \mathcal{P}(R)$, and $D^2p(x) \in \mathcal{P}(\mathbf{R})$ as well. The set of polynomials is also a vector space, so the prefactors of $x^2 + x$ and the addition of $p(3)$ do not alter the fact that $q(x) \in \mathcal{P}(\mathbf{R})$, as desired. $\square$

Problem: Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \ldots, n$. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)).$$

Solution: To prove this, we will consider a column k for both matrices. For the left hand side, we know that the transformation $Tv_k = u_k$, so for column k, the entries $A_{j,k}$ are such that:

$$u_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n$$

Now, for the right hand side. The identity map maps $Iu_k = u_k$, and column k is filled such that:

$$u_k = B_{1,k}v_1 + \cdots + B_{n,k}v_n$$

Since $v_1, \ldots, v_n$ is the same basis used in both equations, then we require that $A_{j,k} = B_{j,k}$, and hence the matrices are the same. □

1.5 Products and Quotients of Vector Spaces

Problem: Suppose T is a function from V to W. The *graph* of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W \mid v \in V\}$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

> *Formally, a function T from V to W is a subset T of $V \times W$ such that for each $v \in V$, there exists exactly one element $(v, w) \in T$. In other words, formally a function is what is called above its graph. We do not usually think of functions in this formal manner. However, if we do become formal, then this exercise can be rephrased as follows: Prove that a function T from V to W is a linear map if and only if T is a subspace of $V \times W$.*

Solution: We first prove the forward case: the graph of T being a subspace implies that T is a linear map. To do this, consider the elements $(v_1, Tv_1), (v_2, Tv_2) \in V$. Then, T being a subspace implies the following two conditions:

$$(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2) \in V$$
$$\lambda(v_1, Tv_1) = (\lambda v_1, \lambda Tv_1) \in V$$

Then, in order for this to be true, then T must obey $Tv_1 + Tv_2 = T(v_1 + v_2)$ and $\lambda Tv_1 = T(\lambda v_1)$. This is precisely the conditions for a linear map.

Now we prove that T being a linear map implies that the graph of T is a subspace. Becuase T is a linear map, then we know that $Tv_1 + Tv_2 = T(v_1 + v_2)$, and $T(\lambda v_1) = \lambda Tv_1$. Now, we consider the addition and scalar multiplication:

$$(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, T(v_1 + v_2))$$
$$\lambda(v_1, Tv_1) = (\lambda v_1, T(\lambda v_1))$$

Since both of these are of the form (v, Tv), then it follows that both of these are also contianed within V. Therefore, the graph of T is indeed a subspace. □

Problem: Suppose that $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_k is finite-dimensional for each $k = 1, \ldots, m$.

Solution: Theorem 3.92 says that $\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$. Therefore, if one of $\dim V_k = \infty$, then $\dim(V_1 \times \cdots \times V_m) = \infty$. This is a contradiction, so therefore each V_k is finite dimensional. □

Problem: Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Solution: Consider $T \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$. We can define a linear map $T_i \in \mathcal{L}(V_i, W)$ such that for all $v_i \in V_i$, we have $T_i v_i = T(0, \ldots, v_i, \ldots)$. In other words, $T_i v_i$ produces the same vector as if we just fed v_i into T. Then, we can define a linear map: $\Lambda : T \to (T_1, \ldots, T_m)$ based on T_i.

Now we prove the injectivity and surjectivity of Λ. For injectivity, we aim to prove that $\text{null}\,\Lambda = \{0\}$. Consider $T \in \text{null}\,\Lambda$, then, this implies that $\Lambda T = (0_1, \dots, 0_m)$. Then, since each 0_i is defined such that $T_i v_i = T(0, \dots, v_i, \dots, 0)$, we can write $0_i v_i = 0 = T(0, \dots, v_i, \dots, 0)$. This holds for every i, so therefore we can write:

$$Tv = T(v_1, \dots, 0) + \cdots + T(0, \dots, v_m) = 0 + \cdots + 0 = 0$$

Therefore, T is the zero map, hence $\text{null}\,T = \{0\}$.

To prove the surjectivity of Λ, we want to show that given a set of $(T_1, \dots, T_m)$, we can find a T such that $\Lambda T = (T_1, \dots, T_m)$. To do so, let $T \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$ such that $T(v_1, \dots, v_m) = T_1 v_1 + \cdots + T_m v_m$. Then, $T(0, \dots, v_i, \dots, 0) = T_i v_i$, so based on the definition of Λ, we have $\Lambda T = (T_1, \dots, T_m)$. This is the desired result. $\square$

Problem: Suppose that v, x are vectors in V and that U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Solution: Since U, W are subspaces, then the zero element exists in both. This means that $v + 0 \in v + U$, and there exists some w such that $v = x + w$, or in other words $v - x = w \implies v - x \in W$. Then, by Theorem 3.101, $v - x \in W$ implies that $v + W = x + W$. Combining this with $v + U = x + W$, we get $v + U = v + W$, so $U = W$, as desired. $\square$

Problem: Prove that a nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.

Solution: We prove first that if A is a translate of V, then the equation holds. Since A is a translate of V, then this means that A can be represented as $A = \{x + u \mid u \in U\}$ for some subset U and $x \in V$. Now, consider $v = x + u_1$ and $w = x + u_2$ for $u_1, u_2 \in U$. Then, we have:

$$\lambda v + (1 - \lambda)w = \lambda(x + u_1) + (1 - \lambda)(x + u_2) = \lambda x + \lambda u_1 + x + u_2 - \lambda x - \lambda u_2 = x + u_2 + \lambda(u_1 - u_2)$$

Since U is a subspace, then $u_2 + \lambda(u_1 - u_2) \in U$, so we can conclude that $\lambda v + (1 - \lambda)w \in A$.

Now we prove the reverse direction: if $\lambda v + (1 - \lambda)w \in A$, then A is some translate of V. Rearranging this equation, it becomes $w + \lambda(v - w) \in A$ for all $v, w \in A$. Now, since $v, w \in V$ and V is a vector space, then $0 \in V$, so we can set $w = 0$, and then this basically gives the equation $v \in A$. In other words, we can make a "trivial" translate by letting $A = V$ and $w = 0$. $\square$

Problem: Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or the empty set.

Solution: There are two cases: either $A_1 = A_2$ or $A_1 \neq A_2$. $\square$

Problem: Suppose $U = \{(x_1, x_2, \dots,) \in \mathbf{F}^\infty \mid x_k \neq 0 \text{ for finitely many } k\}$.

a) Show that U is a subspace of $\mathbf{F}^\infty$.

b) Prove that $\mathbf{F}^\infty / U$ is infinite-dimensional

Problem: Suppose $v_1, \dots, v_m \in V$. Let

$$A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m \mid \lambda_1, \dots, \lambda_m \in \mathbf{F} \text{ and } \lambda_1 + \cdots + \lambda_m = 1\}$$

a) Prove that A is a translate of some subspace of V.

b) Prove that if B is a translate of some subspace of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.

c) Prove that A is a translate of some subspace of V of dimension less than m.