Collaborators

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Problem 1

An infinitely large sheet, parallel with the xy-plane, lies at $z = +d/2$ and has a uniform current density $\mathbf{K} = kt\hat{x}$ flowing on it. Another infinitely large sheet, again parallel with the xy-plane, lies at $x = -d/2$ and has a uniform current density $\mathbf{K} = +kt\hat{x}$. In below we consider the fields in the quasi-static limit.

a) Find the magnetic field due to the electric currents everywhere in the space.

Solution: To find the **B** field everywhere, we use an amperian loop to find the contribution from one of the two sheets, then superimpose the two sheets together. From Ampere's law:

$$\int B dl = \mu_0 K l$$

We know that the magnitude $|\mathbf{B}|$ is the same both above and below the plate, so:

$$2Bl = \mu_0 kl \implies B = \frac{\mu_0 k}{2}$$

This is true for any given moment in time, so therefore the B field for the top plate in this case is:

$$\mathbf{B}_{\text{top}}(t) = \begin{cases} \frac{\mu_0}{2} kt\hat{y} & \text{above} \\ -\frac{\mu_0}{2} kt\hat{y} & \text{below} \end{cases}$$

Below the plate we have the exact opposite situation, so therefore when superimposing, we get:

$$\mathbf{B}_{\text{bottom}}(t) = \begin{cases} -\frac{\mu_0}{2} kt\hat{y} & \text{above} \\ \frac{\mu_0}{2} kt\hat{y} & \text{below} \end{cases}$$

Now, we superimpose the fields on top of each other. Above both plates, we take the **B** generated above both plates, and we get that $\mathbf{B} = 0$ above. Below both plates, the same situation happens: $\mathbf{B} = 0$. In between the plates, we take the **B** generated below the top plate, and above the bottom plate. This gives us $\mathbf{B}(t) = -\mu_0 kt\hat{y}$. Therefore, combining both:

$$B(t) = \begin{cases} 0 & |z| > \frac{d}{2} \\ -\mu_0 kt\hat{y} & |z| < \frac{d}{2} \end{cases}$$

□

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b) Find the electric field due to the time-varying magnetic field everywhere in the space.

Solution: From the symmetry of the problem, we know that $E(-z) = -E(z)$ in terms of magnitude. From this, we can infer that $B(0) = 0$. Now, consider a loop of length l that extends a height $z = a < \frac{d}{2}$. Then, the flux through this loop is:

$$\Phi_B(t) = B(t) \cdot lz = -\mu_0 kt \cdot lz$$

Taking the derivative, we get:

$$\frac{\mathrm{d}\Phi_B(t)}{\mathrm{d}t} = -\mu_0 klz$$

Then, using $\int E \cdot dl$, since $E(0) = 0$, then the only contribution will be from the segment at height z. Thus:

$$El = -\mu_0 klz \implies E = -\mu_0 kz$$

If $z > \frac{d}{2}$, then the loop extends outside the sheets, so the flux then becomes:

$$\Phi_B(t) = -\mu_0 ktl\frac{d}{2} \implies \frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\mu_0 \frac{kld}{2}$$

Solving for E, we get:

$$El = -\mu_0 \frac{kld}{2}$$
$$\therefore E = -\mu_0 \frac{kd}{2}$$

To find the direction of E, we use the anti-right hand rule: since B points in the $-\hat{y}$ direction, then we know that E above points in the $\hat{x}$ direction, and below it points in the $-\hat{x}$ direction. Therefore:

$$\mathbf{E}_{\text{above}} = \begin{cases} \mu_0 kz\hat{x} & z < \frac{d}{2} \\ \mu_0 \frac{kd}{2}\hat{x} & z > \frac{d}{2} \end{cases}$$

And below:

$$\mathbf{E}_{\text{below}} = \begin{cases} -\mu_0 kz\hat{x} & |z| < \frac{d}{2} \\ \mu_0 \frac{kd}{2}\hat{x} & |z| > \frac{d}{2} \end{cases}$$

□

Problem 2

A long cylinder of radius R with charge density $\rho(s) = \frac{a}{s}$ rotates around its axis, the z-axis, with angular velocity ω. (a is a constant). The permeability of the cylinder is μ. A circular wire of radius L on the xy-plane, surrounding the cylinder ($L > R$), has a constant line charge density λ. The circle is initially at rest, and then the angular velocity of the cylinder is dropped from ω to 0. Find the angular momentum of the circle after the cylinder stops rotating, assuming that the circle rotates without friction.

Solution: First, we find the enclosed current so we can find the magnetic field of the system. To do that, we first need to find the current density $\mathbf{J}$. We know that $\mathbf{J} = \rho \cdot v = \rho \cdot \omega s$ so therefore:

$$\mathbf{J}(s) = \frac{a}{s} \cdot \omega s = \omega a$$

Now we can use Ampere's law:

$$\int B \cdot dl = \mu_0 I_{\text{enc}}$$

To choose our Amperian loop properly, we note that we can think of this situation as a bunch of solenoids, which means that we can draw our Amperian loop of height h starting from the outside of the cylinder, then have one of its sides extend some distance s into the cylinder. The contribution from the horizontal sides of the loop will cancel, so our only contribution the integral will be the portion of the loop at $r = s$.

Calculating the enclosed current:

$$I_{\text{enc}} = h \cdot \int_s^R J(s)ds = \omega a h(R - s)$$

Since B is constant along z, then the left hand side evaluates to $B \cdot h$, so therefore:

$$B(s) \cdot h = \mu_0 \omega a h(R - s) \implies B(s) = \mu_0 \omega a(R - s)$$

Then, to calculate the flux, we calculate the B field through the cross section of the cylinder:

$$\begin{aligned}
\Phi &= \int_0^R B(s) \cdot 2\pi s ds \\
&= 2\pi \mu_0 \omega a \int_0^R s(R - s)ds \\
&= \frac{1}{3}\pi \mu_0 \omega a R^3
\end{aligned}$$

Then, the change in flux over the total time is equal to the total flux (since we go from having some current to no current at all), so therefore we have:

$$\begin{aligned}
\int E \cdot dl &= \frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
E \cdot 2\pi L &= \frac{1}{3}\pi \mu_0 \omega a R^3 \\
\therefore E &= \frac{1}{6}\frac{\mu_0 \omega a R^3}{L}
\end{aligned}$$

Then, the force is given by $F = E \int \lambda L d\phi$, so:

$$F = \frac{1}{6}\frac{\mu_0 \omega a R^3}{L} \cdot \lambda L 2\pi = \frac{1}{3}\pi \mu_0 \omega a R^3$$

Then, we know that $\Delta L = \tau = \mathbf{r} \times \mathbf{F}$, and then using the fact that the ring starts with 0 angular momentum, then $\Delta L = L_f$:

$$L_f = \mathbf{r} \times \mathbf{F} = \frac{1}{3}\pi\mu_0\omega a R^3 L$$

$\square$

Problem 3

Prove Alfven's theorem: In a perfectly conducting fluid (say, a gas of free electrons), the magnetic flux through any closed loop moving with the fluid is constant in time. (The magnetic field lines are, as it were, "frozen" into the fluid.)

a) Use Ohm's law, in the form of Eq. 7.2, together with Faraday's law to prove that if $\sigma = \infty$ and $\mathbf{J}$ is finite, then

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B)$$

Solution: Ohm's law states $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, and Faraday's law states $\nabla \times E = -\frac{\partial B}{\partial t}$. Taking the curl of the first equation, then, we have:

$$\nabla \times J = \sigma \nabla \times E + \sigma \nabla \times (v \times B)$$

Now, we know that $\mathbf{J}$ is curlless, so therefore:

$$\begin{aligned} 0 &= \sigma \nabla \times E + \sigma \nabla \times (v \times B) \\ -\nabla \times E &= \nabla \times (v \times B) \\ \therefore \frac{\partial B}{\partial t} &= \nabla \times (v \times B) \end{aligned}$$

In the last step, I've used Faraday's law to convert $\nabla \times E$ into $\frac{\partial B}{\partial t}$. $\square$

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b) Let $\mathcal{S}$ be the surface bounded by the loop $\mathcal{P}$ at time t, and $\mathcal{S}'$ a surface bounded by the loop in its new position $\mathcal{P}'$ at time $t + dt$ (see Fig. 7.58). The change in flux is

$$d\Phi = \int_{\mathcal{S}'} B(t + dt) \cdot da - \int_{\mathcal{S}} B(t) \cdot da$$

Use $\nabla \cdot B = 0$ to show that

$$\int_{\mathcal{S}'} B(t + dt) \cdot da + \int_{\mathcal{R}} B(t + dt) \cdot da = \int_{\mathcal{S}} B(t + dt) \cdot da$$

(where $\mathcal{R}$ is the "ribbon" joining $\mathcal{P}$ and $\mathcal{P}'$), and hence that

$$d\Phi = dt \int_{\mathcal{S}} \frac{\partial B}{\partial t} \cdot da - \int_{\mathcal{R}} B(t + dt) \cdot da$$

(for infinitesimal dt). Use the method of Sect. 7.1.3 to rewrite the second integral as

$$dt \oint (B \times V) \cdot dl$$

and invoke Stokes' theorem to conclude that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \int_{\mathcal{S}} \left(\frac{\partial B}{\partial t} - \nabla \times (v \times B) \right) \cdot da$$

Together with the result in (a), this proves the theorem.

Solution: First, since $\nabla \cdot B = 0$, then we know that $\oint B \cdot da = 0$. Therefore, choosing the surface to be both volumes plus the "ribbon":

$$0 = \oint B \cdot da = \int_{\mathcal{S}'} B(t + dt) \cdot da + \int_{\mathcal{R}} B(t + dt) \cdot da - \int_{\mathcal{S}} B(t + dt) \cdot da$$

Moving the last term to the right:

$$\int_{\mathcal{S}'} B(t + dt) \cdot da + \int_{\mathcal{R}} B(t + dt) \cdot da = \int_{\mathcal{S}} B(t + dt) \cdot da$$

as desired. The change in flux is given by the first equation, so we can now plug what we just derived into the first term:

$$\begin{aligned}
\therefore d\Phi &= \int_{\mathcal{S}} B(t + dt) \cdot da - \int_{\mathcal{S}} B(t) \cdot da - \int_{\mathcal{R}} B(t + dt) \cdot da \\
&= \int_{\mathcal{S}} \frac{\partial B}{\partial t} dt \cdot da - \int_{\mathcal{R}} B(t + dt) \cdot da \\
&= dt \int_{\mathcal{S}} \frac{\partial B}{\partial t} da - \int_{\mathcal{R}} B(t + dt) \cdot da
\end{aligned}$$

as desired. Now we consider an infinitesimal area element as $da = (v \times dl)dt$, so therefore, substituting this into the second term, we get:

$$\int_{\mathcal{R}} B(t + dt) \cdot da = \int_{\mathcal{R}} B(t + dt) \cdot (dl \times v) dt = dt \int_{\mathcal{P}} (B \times v) \cdot dl$$

note that I have $(dl \times v)$ instead of $v \times dl$, which takes out the negative sign. Finally, invoking Stokes' theorem:

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \frac{\partial B}{\partial t} da - \int_{\mathcal{P}} (B \times v) \cdot dl = \int_{\mathcal{S}} \left(\frac{\partial B}{\partial t} - \nabla \times (v \times B) \right) \cdot da$$

which proves the final statement in the problem. □