

Collaborators

I worked with **Andrew Binder**, **Adarsh Iyer**, **Nathan Song** and **Teja Nivarthi** on this homework assignment.

Problem 1 (25 pts)

In this class (and in pretty much all of physics) it is key to be proficient at computing Gaussian integrals. Consider

$$I = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2}$$

a) Show that

$$I^2 = \int_{\mathbb{R}} dx dy e^{-\frac{1}{2}(x^2+y^2)}$$

Solution: The definition of I^2 is that we multiply two integrals together, with separate integration variables. Therefore, we can write,

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \right)$$

Then notice that since these integrals are independent of each other, this is the same as writing a double integral:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx dy = \int_{\mathbb{R}^2} e^{-\frac{1}{2}(x^2+y^2)} dx dy$$

as desired. □

b) Compute I^2 by expressing this integral in polar coordinates. Conclude that

$$I = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$$

Solution: Following the hint, we compute this integral in polar coordinates, where our integration bounds now go from $r \in [0, \infty)$ and $\theta \in [0, 2\pi]$. Our integral becomes:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{1}{2}r^2} r dr d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{1}{2}r^2} r dr$$

The second integral can be solved via a u -substitution of $u = \frac{r^2}{2}$ so $du = r dr \implies dr = \frac{du}{r}$, hence:

$$I^2 = 2\pi \int_0^{\infty} e^{-u} du = 2\pi [-e^{-u}]_{u=0}^{\infty} = 2\pi$$

Since this is I^2 , then we conclude that $I = \sqrt{2\pi}$. □

c) Show that

$$I = \int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}$$

Solution: The same steps done to part (b) can solve this problem as well. We split into polar coordinates, and

instead of letting $u = \frac{r^2}{2}$, we can let $u = \frac{ar^2}{2}$, meaning $dr = \frac{du}{ar}$. Therefore, the integral becomes:

$$I^2 = 2\pi \int_0^\infty e^{-u} \frac{du}{a} = \frac{1}{a} \underbrace{\left(2\pi \int_0^\infty e^{-u} du \right)}_{\text{same integral as part (b)}} = \frac{2\pi}{a}$$

Hence, $I = \sqrt{\frac{2\pi}{a}}$, as desired. □

d) Show that

$$I = \int_{-\infty}^\infty dx e^{-\frac{a}{2}x^2 + bx} = e^{\frac{1}{2a}b^2} \sqrt{\frac{2\pi}{a}}$$

Solution: Here, we complete the square on the exponent. First, factor out $-\frac{a}{2}$ from the exponent:

$$I = \int_{-\infty}^\infty e^{-\frac{a}{2}(x^2 + \frac{2b}{a}x)} dx$$

And now we complete the square:

$$I = \int_{-\infty}^\infty e^{-\frac{a}{2}\left(\left(x - \frac{b}{a}\right)^2 - \frac{b^2}{a^2}\right)} dx = \int_{-\infty}^\infty e^{-\frac{a}{2}\left(x - \frac{b}{a}\right)^2 + \frac{b^2}{2a}} dx = e^{\frac{b^2}{2a}} \int_{-\infty}^\infty e^{-\frac{a}{2}\left(x - \frac{b}{a}\right)^2} dx$$

This integral gives the same result as the previous integral, since we can perform a u -substitution $u = x - \frac{b}{a}$ so $du = dx$. Therefore:

$$I = e^{\frac{b^2}{2a}} \underbrace{\int_{-\infty}^\infty e^{-\frac{a}{2}u^2} du}_{\text{same as part (c)}} = e^{\frac{b^2}{2a}} \sqrt{\frac{2\pi}{a}}$$

as desired. □

e) Show that

$$I = \int_{-\infty}^\infty dx e^{-\frac{a}{2}x^2} x^2 = \frac{1}{a} \sqrt{\frac{2\pi}{a}}$$

Hint: Compute

$$\left[\frac{\partial^2}{\partial b^2} \int_{-\infty}^\infty dx e^{-\frac{a}{2}x^2 + bx} \right]_{b=0}$$

by both differentiating under the integral sign and explicitly computing its derivatives using the result in d).

Solution: Following the hint, we know the result of the integral from the previous problem, so we're basically just left with

$$I = \frac{\partial^2}{\partial b^2} \left[e^{\frac{b^2}{2a}} \sqrt{\frac{2\pi}{a}} \right]_{b=0} = \sqrt{\frac{2\pi}{a}} \frac{\partial}{\partial b} \left[\frac{b}{a} e^{\frac{b^2}{2a}} \right]_{b=0} = \sqrt{\frac{2\pi}{a}} \left[\frac{1}{a} e^{\frac{b^2}{2a}} + \frac{b^2}{a^2} e^{\frac{b^2}{2a}} \right]_{b=0} = \frac{1}{a} \sqrt{\frac{2\pi}{a}}$$

as desired. □

Problem 2 (20 pts)

Consider a system of N particles with spin. Label the particles using an index $i = 1, \dots, N$ so that the i -th particle has spin s_i . Unlike the systems we have encountered so far, in this system the possible values for each spin are $s_i = -1, 0, 1$. The energy of this system is given by

$$E = D \sum_{i=1}^N s_i^2$$

In other words, when the spin is ± 1 , this costs the system an energy D , while whenever the spin is 0, this spin does not contribute to the energy of the system.

- a) Explain why the number of accessible states that the system when it has an energy E is

$$\Omega(N, E) = \binom{N}{E/D} 2^{E/D}$$

Solution: Since each particle contributes an energy D to the system, if we have a total system with energy E then we can divide this up into $\frac{E}{D}$ quanta of energy, and we need to count the number of ways to allocate $\frac{E}{D}$ quanta of energy among N particles. This is a problem that we've done countless times before, where $q = \frac{E}{D}$, where we have $\binom{N}{q}$ possible ways.

Further, since a spin of ± 1 contributes the same toward the total energy, then we have 2 choices for the spin of each particle among those that we've selected to carry energy. This explains the $2^{E/D}$ term. Putting these two together, we get:

$$\Omega(N, E) = \binom{N}{E/D} 2^{E/D}$$

□

- b) Compute the entropy of the system $S(N, E)$ in the limit of many particles and high energies $E \gg D$.

Solution: We use the definition of $S = k \ln \Omega$, so we have:

$$S = k \ln \left[\binom{N}{E/D} \right] + k \frac{E}{D} \ln 2$$

For this, we expand the binomial coefficient into an expression involving factorials, and use Stirling's formula of the form $\ln x! = x \ln x - x$, which gives us:

$$\begin{aligned} S &= k \left[N \ln N - N - \frac{E}{D} \ln \frac{E}{D} + \frac{E}{D} - \left(N - \frac{E}{D} \right) \ln \left(N - \frac{E}{D} \right) + \left(N - \frac{E}{D} \right) + \frac{E}{D} \ln 2 \right] \\ &= k \left[N \ln N - \frac{E}{D} \ln \frac{E}{D} - \left(N - \frac{E}{D} \right) \ln \left(N - \frac{E}{D} \right) + \frac{E}{D} \ln 2 \right] \end{aligned}$$

We can't simplify this further, so that's where we'll stop. In the limit where $E \gg D$, we can replace all $\frac{E}{D} = E$, so this gives us:

$$S = k [N \ln N - E \ln E - (N - E) \ln(N - E) + E \ln 2]$$

□

- c) Compute the temperature of this system as a function of the energy and the number of particles. Can the temperature of this system be negative?

Solution: To calculate temperature, we use the definition that $\frac{1}{T} \equiv \frac{dS}{dU} = \frac{dS}{dE}$. Therefore, computing this derivative:

$$\begin{aligned}\frac{1}{T} &= k \left[-\frac{1}{D} \ln \frac{E}{D} - \frac{E}{D} \cdot \frac{1}{E} \frac{1}{D} - \left(-\frac{1}{D} \right) \ln \left(N - \frac{E}{D} \right) - \left(N - \frac{E}{D} \right) \frac{1}{\left(N - \frac{E}{D} \right)} \left(-\frac{1}{D} \right) + \frac{1}{D} \ln 2 \right] \\ &= \frac{k}{D} \left[\ln \left(N - \frac{E}{D} \right) + \ln 2 - \ln \frac{E}{D} \right]\end{aligned}$$

We can combine this into a single logarithm, giving us:

$$\frac{1}{T} = \frac{k}{D} \ln \left[\frac{2 \left(N - \frac{E}{D} \right)}{\frac{E}{D}} \right]$$

From here, we can see that it is indeed possible for T to be negative, and precisely it is negative when the logarithm returns a negative number, which occurs when:

$$\frac{2 \left(N - \frac{E}{D} \right)}{\frac{E}{D}} < e$$

Specifically in terms of E , this occurs when $E > \frac{2}{3}ND$. □

- d) Obtain the energy for the system as a function of temperature. Discuss the low and high temperature limits. What happens to the entropy in these limits? What is the physical meaning of this?

Solution: We simply take the expression we got in the previous part and solve for E , which gives us the following expression:

$$E = \frac{ND}{1 + \frac{1}{2}e^{D/kT}}$$

Now to analyze the bounds. At high temperature, then $e^{D/kT} \rightarrow e^0 = 1$, so we have $E \rightarrow \frac{2}{3}ND$ in this limit. At low temperature, $e^{D/kT} \rightarrow \infty$ so $E \rightarrow 0$ in this limit.

Physically, the low temperature limit makes intuitive sense, since $E \rightarrow 0$ at low temperature, matching our intuitive relationship between energy and temperature. At high temperature, we expect every single bit to be randomly distributed, so $\frac{2}{3}$ of them will be in the states ± 1 , which contributes to the total energy of the system. So, with this analysis, the high temperature limit works as well.

As for the entropy, we can substitute $E = \frac{2}{3}ND$ into the expression we derived in part (b) for the high temperature limit, which gives us:

$$S = kN \ln 3 = k \ln 3^N$$

At the low temperature limit, $E \rightarrow 0$, so all the terms with E immediately disappear (note that $\lim_{x \rightarrow 0} x \ln x = 0$). This gives us that at low temperature,

$$S = k[N \ln N - N \ln N] = 0$$

The physical interpretation of the high temperature limit is that at high temperatures, each particle has three possibilities for its spin, hence there are 3^N total microstates, which matches our expression of $S = k \ln \Omega = k \ln 3^N$. At low temperatures, this limit also makes sense (kind of), where we can think of the low temperature limit as a configuration where we have very little information, hence S approaches zero. □

Problem 3 (20 pts)

Consider an ideal gas undergoing a process described by the fact that pV^2 is a constant. What is the molar heat capacity of this process?

Solution: The heat capacity is defined as $C = \frac{Q}{\Delta T}$, and using the first law of Thermodynamics gets us

$$C = \frac{Q}{\Delta T} = \frac{\Delta U - W}{\Delta T} = \frac{\Delta U}{\Delta T} - \frac{W}{\Delta T}$$

We assume that we're working with a monoatomic ideal gas so $f = 3$ (from Ed), so we have:

$$U = \frac{3}{2}NkT = \frac{3}{2}nRT \implies \frac{\Delta U}{\Delta T} = \frac{dU}{dT} = \frac{3}{2}nR$$

Meanwhile, we can use $W = -P\Delta V$ to simplify the second term:

$$\frac{W}{\Delta T} = -\frac{P\Delta V}{\Delta T} = -P\frac{dV}{dT}$$

To do this, we use the fact that PV^2 is constant, so $\frac{d(PV^2)}{dT} = 0$:

$$\begin{aligned} \frac{d(PV^2)}{dT} &= P\frac{dV^2}{dT} + V^2\frac{dP}{dT} \\ 0 &= 2PV\frac{dV}{dT} + V^2\frac{dP}{dT} \\ V^2\frac{dP}{dT} &= -2PV\frac{dV}{dT} \end{aligned}$$

Rearranging this to match the expression we have for $\frac{W}{\Delta T}$:

$$-P\frac{dV}{dT} = \frac{V}{2}\frac{dP}{dT}$$

To calculate $\frac{dP}{dT}$, we use the ideal gas law:

$$\frac{dP}{dT} = \frac{d}{dT} \left(\frac{nRT}{V} \right) = \frac{nR}{V} - \frac{nRT}{V^2} \frac{dV}{dT}$$

Therefore, we have:

$$\begin{aligned} -P\frac{dV}{dT} &= \frac{V}{2} \left(\frac{nR}{V} - \frac{nRT}{V^2} \frac{dV}{dT} \right) \\ &= \frac{nR}{2} - \frac{P}{2} \frac{dV}{dT} \end{aligned}$$

Noticing that we have the same term on the left hand side, we get:

$$-\frac{P}{2} \frac{dV}{dT} = \frac{nR}{2} \implies -P\frac{dV}{dT} = nR = \frac{W}{\Delta T}$$

Therefore, putting them both together:

$$C = \frac{Q}{\Delta T} = \frac{\Delta U - W}{\Delta T} = \frac{3}{2}nR - nR = \frac{1}{2}nR$$

Finally, computing the molar heat capacity:

$$C_n = \frac{C}{n} = \frac{R}{2}$$

□