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1 Introduction

In this project paper, we will first introduce the classical framing of convex optimization, then look at the current quantum algorithms that provably achieve a speedup over their classical counterparts.

1.1 Contribution Statement

1.2 Linear Programs

To begin, let's look at a simpler problem: linear programming. In linear programming (LP), the goal is to find the optimal (either a maximum or minimum) value of a *linear objective equation*, which is an equation of the form:

$$c_1x_1 + c_2x_2 + c_3x_3$$

For convenience, we also employ a matrix vector representation – the above equation can also be written as $\mathbf{c}^\top\mathbf{x}$, where the vector $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$. Along with this equation, we also place any number of linear constraints on the variables x_1, x_2, and x_3. For instance, if we had 2 constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 &\leq b_2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

this can also be written using matrix-vector notation: $\mathbf{Ax} \leq \mathbf{b}$. Here, $\mathbf{A}$ is the matrix that gives us the coefficients of the constraint equations, $\mathbf{x}$ is the same vector as before, and $\mathbf{b}$ is the constraint value. Finally, we have a nonnegativity constraint on the variables, which can be written as $\mathbf{x} \geq 0$. Putting this all together, the following is the statement of linear programs:

$$\begin{aligned} \min_x \quad & \mathbf{c}^\top\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b}, \quad \mathbf{x} \geq 0 \end{aligned}$$

Note that here we've written the problem in terms of a minization, but it is always possible to convert a maximization problem into this form. Another thing to note is that in this formulation, the set of constraints $\mathbf{Ax} \leq \mathbf{b}$ defines a *convex* region, also called the feasible region, in which the optimal value is to be found – this will be important later.

1.3 Semidefinite Programs

Now we move to the main focus of this project: semidefinite programs (SDPs). Here, instead of dealing with vectors as our variables, we now deal with positive semidefinite matrices. These are matrices that are symmetric and also have real eigenvalues, or more formally speaking, a matrix $\mathbf{M}$ is positive semidefinite if and only if:

1. $\mathbf{M}$ is symmetric
2. For any vector $\mathbf{v}$, we have $\mathbf{v}^\top \mathbf{M}\mathbf{v} \geq 0$.

With that in mind, the spirit of the problem is the exact same as an LP: we have an objective function to maximize, subject to a set of d constraints:

$$\begin{aligned} \min \quad & \text{Tr}\left(\mathbf{C}^\top \mathbf{X}\right) \\ \text{s.t.} \quad & \text{Tr}\left(\mathbf{A}_i^\top \mathbf{X}\right) \leq b_i \quad \text{for } i = 1, 2, \ldots, d \\ & \mathbf{X} \succeq 0 \end{aligned}$$

Here, $\mathbf{C}, \mathbf{X}$ and $\mathbf{A}_i$ are all positive semidefinite matrices instead of simple vectors. Note that this is a more general class of problem than an LP; if $\mathbf{C}$ and $\mathbf{X}$ are diagonal, then this problem reduces immediately to an LP.

While LPs and SDPs share thematic similarities, the generalization of moving from vectors to semidefinite matrices introduces some key differences [**?**]:

1. Properties concerning duality are lost. For LPs the concept of duality states that for every LP, also called the *primal* LP, we can construct another "equivalent" LP called the *dual* LP. Moreover, the strong duality theorem states that if the primal LP has an optimal value, then the dual LP also has an optimal value, and the optimal value for the dual is equal to that of the primal.

   With SDPs, strong duality fails. The duality gap, defined to be the difference between the optimal primal and dual problems, can be finite or even infinite. However, if both problems contain solutions in the semidefinite cone, then the primal and dual share the same optimal value.

2. There are no known finite algorithms for solving SDPs. As we'll see, all of the solution methods always approximate the solution, to some tolerance of ϵ.

Despite these differences, however, SDPs are not that much harder to solve than LPs, and in fact many LP algorithms have been generalized to also work for SDPs as well.

2 Classical Algorithms

Before diving into the quantum algorithms for this problem, it is useful to first take a look at the leading philosophies developed to solve this problem classically. In particular, these classical methods give us insight into the philosophy behind their corresponding quantum algorithms.

2.1 Interior Point Methods

One way to approximate solutions to SDPs is via an Interior Point Method (IPM). In this method, the SDP is rephrased as a functional problem: the objective function is denoted $f_\eta(x)$, defined as:

$$\min f_\eta(x) \quad \text{where} \quad f_\eta(x) = \eta c^\top x + f(x)$$

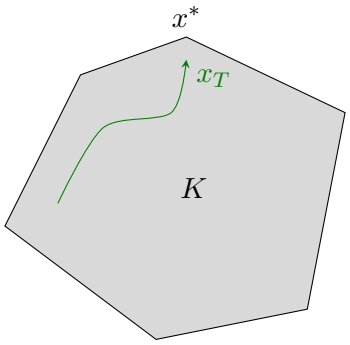
In this formulation, the original objective function $\mathbf{c}^\top \mathbf{x}$ corresponds to the first term, and $f(x)$ is a function known as a *barrier function*. Without defining them explicitly, one way to conceptually interpret $f(x)$ is

that it's a function that tends toward infinity as we approach the barrier of the feasible region. In other words, $f(x)$ is the funciton that encodes the information about the constraints $\mathbf{A}_i^\top \mathbf{X}$. There are multiple ways this barrier can be implemented, and as it turns out, different ways of writing $f(x)$ gives rise to drastically different runtime bounds.

The parameter η controls the contribution of the $c^\top x$ term in $f_\eta(x)$. When η is small, the constraints dominate, and the objective function dominates when η is large. One approach approach IPMs take to find the optimal point is via a *Newton's Method*, which starts with $\eta = 0$ (so the constraints dominate) and a point inside the feasible region x_0, then iteratively compute an update step $x_{n+1} = x_n + n(x)$, where $n(x)$ is defined as:

$$n(x) = -H(x)^{-1} g(x)$$

where $H(x)$ and $g(x)$ are the Hessian matrix and gradient vectors respectively. At each step, we also increase the value of η, thereby increasing the weight of the objective function, allowing us to converge on the optimal point x^*. Below we've included a diagram showing the path the IPM method takes from x_0 to x_T in T steps (this diagram is inspired by [**?**]).



In theory, our point x_n equals x^* when $\eta \to \infty$ and $T \to \infty$. As this is not possible in a finite process, the best we can hope to do is to get approximately close to x^*, which echoes what we mentioned earlier about the lack of a finite algorithm for solving SDPs.

As this is an iterative method, the runtime of this algorithm is heavily dependent on the number of iterations T we choose to run, and here we come to the main issue for IPM algorithms: each Newton step is an incredibly costly computation, since it requires calculating a matrix inverse and a matrix multiplication. Classically, the fastest known matrix multiplication algorithm runs in $O(n^{2.37})$ [1], and the fastest matix inverse algorithms run close to $O(n^3)$, both of which are incredibly slow [**?**].

With this in mind, there are several approaches we can take to easing the computational burden here. Perhaps the most obvious way is to find a faster way to perform matrix multiplication and inversion, which is a valid approach, however it's hard to imagine that this will generate appreciable speedup as it does nothing to affect the number of iterations we have to compute. On the other hand, a more promising approach, and one that we will focus on in subsequent sections, lies in finding ways to shrink the matrices H and g to make them more tractable, by way of approximations. This lessens the computational burden in two ways: first, a smaller H and g maeans calculating H^{-1} and multiplying it with g is faster, and second, approximations reduce the number of times we need to compute Newton's step.

[1] This algorithm is also considered a "galactic algorithm", or in other words, extremely impractical, and in practice the slower (but much easier) Strassen's algorithm is used instead, which has a runtime of roughly $O(n^{2.8})$.

In sum, there are three main steps that incur a high computational cost when using an IPM which are the following:

1. Calculating and storing the Hessian matrix $H(x)$ and the gradient $g(x)$
2. Calculating the inverse of the Hessian matrix $H(x)^{-1}$.
3. Calculating the matrix-vector product $H(x)^{-1}g(x)$.

Putting these elements together, the best classical runtime achieved is somewhere on the order of $\widetilde{O}(nd + d^3)$. As we'll see, the quantum algorithm we focus on has an optimal runtime of $\widetilde{O}(\sqrt{n}\,\mathrm{poly}(d)$, which is much faster.

3 Quantum IPM Algorithms

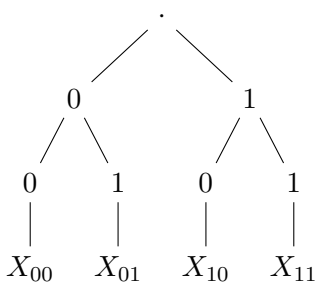
Now that we are familiar with the problem and challenges in implementing an efficient algorithm classically, we move to investigating how quantum algorithms can be used to generate speedup for these operations. In particular, we will focus on the paper by Apers and Gribling, who propose a new quantum IPM algorithm that bypasses many of the challenges encountered by the algorithms that precede it.

Before we begin discussing the algorithm, it is useful to first take a look at some elements of quantum computing that will be instrumental in showing a speedup over classical algorithms.

3.1 Quantum Random Access Memory

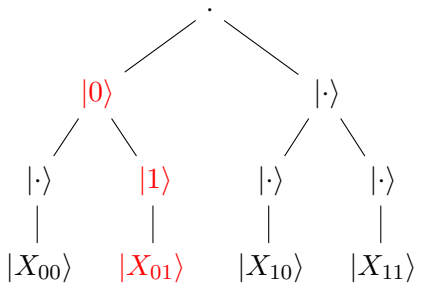
One subroutine this paper, along with many other papers employ is the quantum implementation of classical random access memory, also called QRAM. In essence, this subroutine provides an $O(n) \to O(\log n)$ speedup in query length, and is one of the major subroutines used to achieve a better runtime.

The philosophy behind QRAM is relatively simple: instead of accessing a single memory location, QRAM uses superposition to access multiple memory locations at once. This can be implemented in different ways, one of which is the "bucket-brigade" implementation, detailed in [**?**]. To explain how this works, consider a classical RAM with n bits, which encodes for 2^n memory locations. Now, since there are 2^n locations, we can imagine each memory location being a leaf in a full binary tree with n layers:



Then, the data at location X_{ab} can be specified by the binary string ab, where the value of a and b trace the path from the root node to the leaf in question. Note that the path is traced from most to least significant bit – for instance, X_{01} is encoded by the bits $a = 0, b = 1$.

Quantumly, the idea is more or less the same, except now, each node is replaced by a three-state system, sometimes called a qutrit. The states are $|\cdot\rangle$, $|0\rangle$ and $|1\rangle$. The $|\cdot\rangle$ state is referred to as a "wait" has the special property that whenever a state, either $|0\rangle$ or $|1\rangle$ is received, the state transforms into the received state. That is, if it receives $|0\rangle$, then $|\cdot\rangle \to |0\rangle$, and the same goes for $|1\rangle$. Then, to access a particular cell, we specify the path to that cell by feeding in the path one qubit at a time, just like we did classically; the diagram below shows an example where we try to access $|X_{01}\rangle$:



the red markers indicates the path taken to $|X_{01}\rangle$. Now, how is this faster than the classical approach? Well, the key comes when we consider superpositions of states. What if, instead of feeding in just $|0\rangle$ or $|1\rangle$, we instead fed $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ twice? Well, then both paths would be activated due to superposition, and as a result instead of getting a single qubit as earlier, we would get a superposition of all four memory states:

$$\frac{1}{2}(|X_{00}\rangle + |X_{01}\rangle + |X_{10}\rangle + |X_{11}\rangle)$$

In general, for a QRAM of n qubits, if we gave $|+\rangle^{\otimes n}$ as an input (in succession), then the output would just be a superposition of all the memory states. This is the power of QRAM – with only n queries, we managed to access 2^n different memory locations, or equivalently, for n memory locations we only ened $\log n$ queries. This is why QRAM introduces an $O(n) \to O(\log n)$ speedup.

As we'll see, QRAM is one of the central subroutines that are assumed to be implemented in the algorithm we will focus on. This is also true for many other quantum algorithms out there – they assume access to QRAM, and as a result they are able to claim a $O(n) \to O(\log n)$ query time basically for free.

3.2 Quantum Gradient Estimation

Gradient estimation has been an area of major research for quantum algorithms. One of the first such algorithms is Jordan's algorithm [?] which supposes black-box access to a function and can compute the gradient of said function in constant time. This algorithm is of little practical use for this problem, however, as we have analytical access to the functions in question. Another approach is a parameter shift technique for

3.3 Main Result

Previously, one of the main challenges in devising an efficient quantum SDP solver is that regardless of the method chosen, there was always dependence on what is called a *condition number*. In essence, the dependence on a condition number implies that the runtime of the algorithm was highly dependent on the values to the input matrices; this is problematic because it means that the runtime of such algorithms varies

wildly depending on the values to the input matrix. However, this algorithm we will focus on removes that dependence entirely, and as such is seen as a monumental achievement in the search for a faster quantum IPM algorithm.

In particular, this algorithm develops two main subroutines [**?**]:

a) An efficient algorithm to spectrally approximate the Hessian.

b) An efficient algorithm to approximate a matrix-vector product.

The implementation of these algorithms allow us to calculate an approximate Hessian $Q(x)$ and gradient $\tilde{g}(x)$, which is then used to approximate the update step via $\tilde{n}(x) = -Q(x)^{-1}\tilde{g}(x)$. To be more specific, the form of $g(x)$ allows us to write it in terms of a matrix-vector product, which we then use the latter subroutine to efficiently approximate it.

3.4 Quantum Spectral Approximation

As this is one of the main subroutines the paper leverages, it is one that is important to look at if one is to understand the entire algorithm. As outlined in Theorem 3.1 of [**?**], the following is the formal statement of the spectral approximation:

Theorem 1 (Quantum Spectral Approximation)**.** *Consider query access to a matrix $B \in \mathbb{R}^{n\times d}$ with row sparsity r. For any $0 < \epsilon \leq 1$, there is a quantum algorithm that, with high probability, returns a matrix $\tilde{B}^{\widetilde{O}(d/\epsilon^2)\times d}$ satisfying*

$$(1-\epsilon)\tilde{B}^\top\tilde{B} \preceq B^\top B \preceq (1+\epsilon)\tilde{B}^\top\tilde{B}$$

while making $\widetilde{O}(\sqrt{nd}/\epsilon)$ row queries to B, and taking time $\widetilde{O}(r\sqrt{nd}/\epsilon + d^\omega)$.

Intuitively, this theorem is best undestood as the following: given a matrix B in $\mathbb{R}^{n\times d}$, we can use this algorithm to find a matrix $\tilde{B}$ such that $\tilde{B}^\top\tilde{B}$ is within ϵ of the true value of $B^\top B$.[2]

The reason this spectral approximation is phrased in this way is in part due to the form of the Hesssian matrix for $f(x)$. In summary, becuase all the Hessian matrices considered in the paper are of the form $B^\top B$, then the above algorithm perfectly allows us to approximate H.

The method in which this is done is actually qutie complicated – the authors make use of an algorithm called the *repeated halving algorithm* developed in [**?**]. In essence, the algorithm works in two parts: first, a set of matrices $A_1, \ldots, A_L$ is recursively constructed by subsampling A; in other words, to generate A_{i+1}, we keep each row in A_i with probability $\frac{1}{2}$. This is then combined another subsampling procedure that allows us to construct a matrix B_i from each A_i. In the paper, this is denoted as the step $B \overset{w,\epsilon}{\leftarrow} A$. Here, instead of each row being kept with uniform probability, there is a distribution p_i instead, which is dependent on the dimension of A along with other factors. This process is recursively done, and once all the computation is done we are left with a matrix B which approximates A in the sense that $B^\top B \approx A^\top A$.

Quantumly, this latter process of $B \overset{w,\epsilon}{\leftarrow} A$ is interesting, because its quantum implementation is actually rather simple. In particular, this step is actually implemented via Grover search [**?**], which is used to select the rows in A that we want to use to construct B. It should also be noted that this step in particular is

[2] More accurately, we should say that if $A \preceq B$ then this is the same as $0 \preceq B - A$, or in other words the matrix $B - A$ is positive semidefinite. However, the analogy of "values" suffices for a conceptual understanding.

one that benefits greatly from QRAM access, since accessing multiple rows of A, which is necessary to construct B would be made far more efficient with QRAM as we can access all these rows at once.

This result is then used in conjunction with calculations of leverage scores and Lewis weights in order to fully compute an approximation to $H(x)$. However, the process by which the Lewis weights are calculated are relatively complicated, and we will not analyze them in detail in this report. However, we will point to [**?**] for further reading.

3.5 Quantum Matrix-Vector Approximation

Perhaps the more interesting of the two subroutines introduced is this efficient method of calculating the matrix-vector product. This is stated as Theorem 5.1 in [**?**]:

Theorem 2 (Approximate matrix-vector product)**.** *Assume query access to a vector $v \in \mathbb{R}^n$ and a matrix $B \in \mathbb{R}^{n\times d}$ with row sparsity r. There is a quantum algorithm that returns a vector $\tilde{y}$ satisfying*

$$\|\tilde{y} - B^\top v\|_{(B^\top B)^{-1}} \leq \delta$$

while making $\widetilde{O}(\sqrt{n}d\|v\|_\infty/\delta)$ row queries to B and v, and taking time $\widetilde{O}(r\sqrt{n}d^2\|v\|_\infty + d^\omega)$.

In other words, this algorithm takes in an input $B \in \mathbb{R}^{n\times d}$ and $v \in \mathbb{R}^n$, and finds a vector $\tilde{y}$ that gets arbitrarily close (becuase we choose the value of δ) to the value of $B^\top v$, which is the matrix vector product. In the context of our IPM algorithm, this is the algorithm that finds an approximation to the quantity equivalent to the quantity $Q(x)^{-1}\tilde{g}(x)$.

Similar to the spectral approximation, the implementation of this algorithm is also rather complex, and uses other algorithms such as the quantum multivariate mean estimation from [**?**] to arrive at a result. Despite this, the main points are relatively easy to understand. First, the input matrix B is spectrally approximated as $\tilde{W}$ using the spectral approximation we talked about earlier as a subroutine. This outputs $\tilde{W}$, which we then convert to $\tilde{W}^{-1/2}$ classically. Now, this quantity is used in conjunction with v to generate a random variable X from which the quantum multivaraite mean subroutine is used, ultimately returning a vector $\tilde{\mu}$. Then, we return $W^{1/2}\tilde{\mu} = \tilde{y}$ as the approximation.

While the details as to *why* the algorithm works is rather technical, the important point is the following: the quantum multivaraite mean algorithm is incredibly efficient, and the authors get around having to explicitly approximation $B^\top v$ by using this algorithm to output an estimate $\tilde{\mu}$ which we use to output $\tilde{y}$.

3.6 Putting it all Together

Now, we are ready to put everything together. Recall in section 2.1, we discussed that depending on the way we encode our barrier function $f(x)$, the resulting time (and space) complexity of the quantum algorithm is different. In **??**, the authors detail the implementation of their quantum algorithm for three different barriers, of which we will focus on the fastest one: the lewis weight barrier.

The idea of a Lewis weight barrier is inspired by [**?**], which defines $f(x)$ as follows:

$$f(x) = \ln\det\left(A^\top S_x^{-1} W_x^{1-\frac{2}{p}} S_x^{-1} A\right) \quad \text{where} \quad W_x = \mathrm{Diag}(w^{(p)}(S_x^{-1}A))$$

Where A is the input matrix, and S_x is a matrix calculated based on the Lewis weights, which as mentioned, is a process that we do not fully understand, and will not discuss in detail here. The reason this approach is used is because the gradient and Hessian matrices take on a relatively nice form:

$$g(x) = -A^\top S_x^{-1} W_x \mathbf{1} \quad H(x) = A^\top S_x^{-1} W_x S_x^{-1} A$$

Then, leveraging the forms of these matrices, we can apply the spectral and the matrix-vector approximation algorithms here to approximate these quantities. This is also done in a very clever way: for H, instead of spectrally approximating H, we spectrally approximate the quantity $B = W_x^{1/2} S_x^{-1} A$ outputting a matrix $\tilde{B}$, with the guarantee that $\tilde{B}^\top \tilde{B} \approx H(x)$. A similar approach is done to approximate $g(x)$, where we allow $B = W_x^{1/2} S_x^{-1} A$ and $v = W_x^{1/2} \mathbf{1}$, whose product is equal to $g(x)$.

This last step highlights the major contribution of this paper: the authors were able to find a way to separate $H(x)$ and $g(x)$ in an incredibly clever way, such that we can apply these algorithms like the spectral and matrix-vector approximation in order to approximate H and g with an extremely fast runtime. Then, combining all these steps together, we finally arrive at a runtime of

$$\widetilde{O}(\sqrt{n}\text{poly}(d, \log(n), 1/\epsilon))$$

which is incredibly fast. If we only focus on n (and ignore polylog factors), then we can bascally say that this algorithm runs in $O(\sqrt{n})$ time, which is significantly faster than the $O(nd + d^3)$ classical runtime.