Physics W89 - Introduction to Mathematical Physics - Summer 2022
Midterm Exam

Released Thursday, July 14, 6:00 PM PDT
Submission Deadline: Friday, July 15, 6:00 PM PDT

- Submit your completed exam to ***Gradescope***.
- Please try to start each problem on a new sheet of paper so you can more easily parse it in the Gradescope system.
- Take care to write legibly and dark enough for your scanner/camera to read.
- Due to the nature of take-home exams, the exam will be open book/open note/open internet (with reasonable restrictions). That means you can refer to our notes. You can refer to the book. You can check your calculations online. If you use any sources, you must cite them (e.g.,"I used xxx to check this determinant").
- Even if you are using a source ***you must fully explain the steps you are taking and the motivation behind them as you understand them***.
- The following are expressly forbidden:
  - You may not plagiarize
  - You may not discuss any aspect of the exam with anyone other than the instructor (Austin Hedeman) or the GSI (Daniel Gardeazabal) during the exam period.
  - You may not collaborate on any aspect of the exam.
  - You may not copy, transcribe, upload, or post any portion of the exam or your solutions to any service other than Gradescope. (This also includes typing the question into a search engine.)
- If you have questions at any time during the exam period you may e-mail me directly or post on Piazza (posts during the exam period will be restricted to private student-to- instructor posts). If we need to make an announcement about any aspect of the exam, we will do so via bCourses and on Piazza.

All of your work must be **thoroughly explained**. If we ask for a determinant, it is not sufficient to just give the answer. You must show how you calculated it or explain clearly any simplifying steps/tricks you took (e.g. "since these two columns are equal, I can conclude that...")

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1 Nuts and Bolts [52 points]

If you need to use a row reduction for any of these parts, please explicitly show the steps of the row reduction in your work.

Taylor Series: It's the middle of summer and cookout season so let's bake some pi! We know[1] that $\arctan(1) = \pi/4$. Recall that $\frac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}$. We can approach this problem with Taylor series!

(a) [*6 points*] Use the known Taylor series for $(1+\epsilon)^p$ to expand $1/(1+x^2)$ out to three non-zero terms in the variable x^2. Then integrate your expression from $x=0$ to $x=1$ to find a Taylor series approximation for $\pi/4$ to three non-zero terms! Evaluate and compare to the exact value of $\pi/4$.

Complex Numbers: In Problem 1.3(g) of the homework we explored the three cube roots of i.

$$w_1 = e^{i\pi/6}; \qquad w_2 = e^{5i\pi/6}; \qquad w_3 = e^{3i\pi/2}.$$

(b) [*6 points*] Explicitly show that the sum of the three cube roots of i is 0. Then determine the magnitude $|z|$ of $z \equiv w_1 + 2w_2$.

Vector Identities in 3D: Consider four real vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ in 3D space. The following equation is a relationship between cross products and dot products:

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}).$$

(c) [*6 points*] Prove this relationship using index notation using the Kronecker delta, Levi-Civita symbol and manipulations.
[*Note: To get full credit for this part you must use index notation and manipulations rather than other methods you may have learned in other classes.*]

Linear Systems of Equations: Consider the following linear system of equations:

$$2x + 6z = \alpha + y; \qquad 3x - 3y = \beta - 9z; \qquad \gamma = -4x + 3y - 12z.$$

Let $\vec{x} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and let D and $\vec{b}$ be the matrix and vector such that this system can be expressed as $\mathsf{D}\vec{x} = \vec{b}$.

(d) [*7 points*] Find the matrix D and determine its rank and nullity. In terms of the constants α and γ, under what condition on β does a solution to this system exist? Find all solutions to this equation when $\alpha = 3$, $\beta = -6$, and $\gamma = 1$.

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[1] You all remember your trigonometry this clearly, for sure!

Square Matrix Classifications:

(e) [*6 points*]

- Create a (3×3) lower-triangular matrix where each entry that is not required to be zero is a distinct non-zero number. At least one of your entries should be purely imaginary.
- Determine whether or not the lower-triangular matrix you created is a normal matrix.
- Show that the (2×2) matrix $\begin{pmatrix} -2 & i \\ 3i & 2 \end{pmatrix}$ is involutory.
- Find the anti-Hermitian part of the 2×2 involutory matrix above.

Determinants and Inverses: Consider the matrix

$$\mathsf{F} = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 2 & 0 \\ 2 & 4 & 1 \end{pmatrix}.$$

(f) [*8 points*] Explicitly compute $\det F$ using the Laplace expansion. Be sure to fully explain your steps. Then, using whatever (by-hand) calculation method you would like, solve for the inverse matrix F^{-1}. Be sure to explain your steps and explicitly verify that you have found the inverse by computing either $\mathsf{F}\mathsf{F}^{-1}$ or $\mathsf{F}^{-1}\mathsf{F}$.

Linear Independence: Consider the vector space $\mathbb{R}^4$ and the following three vectors:

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 7 \\ 14 \\ 6 \\ 00 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 8 \\ 9 \\ 8 \\ 9 \end{pmatrix}$$

(g) [*6 points*] Show (however you'd like) that these three vectors are linearly independent and argue or show that they do not span the space $\mathbb{R}^4$.

The Trace as an Inner Product: Consider the space of complex (2×2)-matrices $\mathbb{C}^{2\times 2}$. We can introduce an inner product on this space,

$$\mathsf{A} \cdot \mathsf{B} \equiv \frac{1}{2}\operatorname{tr}(\mathsf{A}^\dagger \mathsf{B}). \tag{1}$$

(h) [*7 points*] Show or argue that Eq. 1 satisfies the three defining properties of an inner product. Then find the magnitude of the matrix $\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}$ with respect to the inner product and show that it is orthogonal to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

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2 An 89-Proof Problem [12 points]

Next up, some proofs! Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Let's look at how the ranks and nullities of A, B, and AB compare!

(a) [*7 points*] Show or argue the following:

- $\operatorname{Im} \mathsf{AB} \subset \operatorname{Im} \mathsf{A}$,
- $\ker \mathsf{B} \subset \ker \mathsf{AB}$,
- $\operatorname{rank}(\mathsf{AB}) \leq \operatorname{rank} \mathsf{B}$.

[*Note: For the first two bullet points we are just asking to prove that they are subsets - you do* not *have to show that they are full subspaces.*]
[*Hint: One way to think of the statement $W \subset V$ is to say "if a vector is an element of W then it must also be an element of V." This is a good way to start thinking of the proofs. You are allowed to use the rank-nullity theorem for the last bullet point.*]

You only have to do ONE of the following two parts - b.1 or b.2. Choose whichever you are more comfortable with!

(b.1) [*5 points*] Let $\{\hat{f}_i\}$ $(i = 1, \cdots N)$ be some set of N orthonormal vectors on an N-dimensional vector space. Show that the matrix U whose column vectors are the N vectors $\hat{f}_i$ is a unitary matrix.

If two matrices A and B commute then we can use the familiar rule for exponents $e^{\mathsf{A}}e^{\mathsf{B}} = e^{\mathsf{A}+\mathsf{B}}$. This is *not true in general* if A and B do not commute - an idea we *may*[2] explore on the final exam.

(b.2) [*5 points*] Show using the Taylor expansion that $(e^{\mathsf{A}})^\dagger = e^{(\mathsf{A}^\dagger)}$. Then use this to show that $\mathsf{U} \equiv e^{i\mathsf{H}}$ is unitary if H is Hermitian.

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3 Two Quantum Systems [36 points]

When dealing with a particle that can move in one dimension (like a mass at the end of a spring or a particle trapped in a "box" from which it can't escape) the quantum state of a system is represented by a complex function of one real variable, called the ***wave function***, $\psi(x)$. For a particle confined within an "infinite square well" of width L (a "particle in a box"), the wave function is defined between $x = 0$ and $x = L$. We can treat the space of wave functions as a vector space. The inner product of two "state vectors" $\vec{\psi}_\alpha \doteq \psi_\alpha(x)$ and $\vec{\psi}_\beta \doteq \psi_\beta(x)$ is given by

$$\vec{\psi}_\alpha \cdot \vec{\psi}_\beta \equiv \int_0^L \psi_\alpha^*(x)\psi_\beta(x)\, dx.$$

A convenient orthonormal basis of wave functions in this space is

$$\vec{\psi}_n \doteq \psi_n(x) \equiv \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \qquad n = 1, 2, 3, \cdots$$

[2] I haven't decided yet!

Recall that with an orthonormal basis, we can uniquely write any vector as a linear combination of the basis vectors. That is, given a vector/state/wave function $\vec{\psi}_\gamma \doteq \psi_\gamma(x)$, we can find a set of coefficients c_n such that

$$\vec{\psi}_\gamma = \sum_{n=1}^{\infty} c_n \vec{\psi}_n \quad \Longrightarrow \quad \psi_\gamma(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}.$$

(a) [*5 points*] Show that the space of "particle in a box" wave functions of functions that are 0 for $x \leq 0$ and $x \geq L$ is a vector subspace of the vector space of all functions.

(b) [*6 points*] Find the components c_n of the wave function $\vec{\psi}_\gamma \doteq \psi_\gamma(x) = \sqrt{\frac{2}{L}} \cos(\pi x/L)$ with respect to the basis.
[*Note: Feel free to use an integral solver to help you with the integrals as needed.*]

Consider the linear transformation that "reflects" the wave function about the center of the box. That is,

$$\mathsf{P}\vec{\psi} \doteq -\psi(L-x)$$

(c) [*6 points*] Show or argue that P is indeed a linear transformation on wave functions and keeps them in the vector space of wave functions in a box. Then show or demonstrate that the function $\mathcal{M}$ that gives the magnitude-squared of the wave function $\mathcal{M}(\vec{\psi}) \doteq |\psi(x)|^2$ is *not* a linear transformation on wave functions.

Consider the three wave functions $\psi_a(x) = x^2$, and $\psi_b(x) = e^{\pi i x/L}$, and $\psi_c(x) = \sin \frac{2\pi x}{L}$, .

(d) [*5 points*] Show that these three wave functions are linearly independent.
[*Hint: One way would be to use the Wronskian, but you can use any method you'd like.*]

When constructing the wave functions for states of the quantum simple harmonic oscillator, a special set of polynomials called the ***Hermite polynomials***[3] are used. This is a set of polynomials that are orthogonal with respect to the inner product

$$\vec{f} \cdot \vec{g} \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f^*(x) g(x) e^{-x^2} dx.$$

We can algorithmically construct the Hermite polynomials by using the Gram-Schmidt procedure on the set of monomials $\{1, x, x^2, x^3, \cdots\}$. Consider the set of the first three monomials,

$$\{\vec{f}_0 \doteq 1, \vec{f}_1 \doteq x, \vec{f}_2 \doteq x^2\}$$

(e) [*7 points*] Apply the Gram-Schmidt procedure to the set of three vectors $\{\vec{f}_0, \vec{f}_1, \vec{f}_2\}$ to come up with a set of three functions that are orthonormal with respect to this inner product.
[*Note: For partial credit, at least describe the Gram-Schmidt procedure and how it might apply to this system. Remember that normalization is an important part of the Gram-Schmidt process. There are a bunch of integrals to do in this problem but you don't have to do them by hand! Rather, feel free to use an integral solver (just remember to cite your sources if you do). Once you've gotten your three functions, we recommend doing a verification that your results are indeed orthogonal, though this is not required for credit.*]

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[3] https://en.wikipedia.org/wiki/Hermite_polynomials. The Hermite polynomials discussed in this problem are what Wikipedia calls the "physicist's Hermite polynomials". Due to a different normalization convention, your answers will not exactly equal the polynomials in the list but they *will* be proportional to them!

There's another way of creating the Hermite polynomials that will become useful in upper-division classical mechanics. This is through the use of ***generating functions***. Consider the following generating function for the Hermite polynomials:

$$G(x, y) = e^{-y^2+2yx}.$$

We can perform a Taylor expansion of $G(x, y)$ with respect to the variable y (treating x like a constant) about the point $y = 0$. The coefficients of the expansion will be functions of x which happen to coincide with the Hermite polynomials! That is,

$$G(x, y) = e^{-y^2+2yx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n.$$

(Again, due to different normalization conventions, you might not get the exact answers you saw in (e) or in a list of the Hermite polynomials, but your answers should again be proportional to them).

(f) [*7 points*] Use the definition of the Taylor series to expand out $G(x, y)$ to second order in y about the point $y = 0$ and use your result to determine the first three Hermite polynomials $H_0(x)$, $H_1(x)$, and $H_2(x)$.

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Wallace the Brave ©Will Henry, 5/26/22

Honor Pledge

Prior to receiving this exam, you submitted and signed a copy of the honor pledge. The following is a post-exam statement, stating that you abided by the terms of the honor pledge. Please include a copy of this pledge (you do not have to copy it by hand), ***sign*** and ***date*** it, and include it as the last page of your exam submission.

As a member of the UC Berkeley Community, I act with honesty, integrity, and respect for others. I, ________, pledge the following:

- I did not discuss any aspect of the exam with anyone other than the instructor or GSI during the exam period.
- I did not collaborate on any aspect of the exam.
- I cited all my sources and did not plagiarize.
- I did not copy, transcribe, upload, or post any portion of the exam or my solutions to any service other than Gradescope.
- I limited myself to the materials and resources discussed at the start of the exam.

Signature :

Date :

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