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HW 07 | Electrostatics and Optics | August 29, 2024

Collaborators

I worked with **Andrew Binder, Teja Nivarthi, Nathan Song, Christine Zhang** and **Nikhil Maserang** to complete this homework.

Problem 1

Show that the quadrupole term in the multipole expansion can be written as

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j Q_{ij}$$

where

$$Q_{ij} = \int\left[\frac{3}{2}r_i'r_j' - \frac{1}{2}(r')^2\delta_{ij}\right]\rho(\mathbf{r}')d\tau'$$

Note that Q_{ij} is a two-rank tensor, so it is possible to express it as a matrix. Also show that Q_{ij} is traceless.

Solution: Here we will compare this expression to the quadrupole expansion term we are normally used to show equivalence. Recall that the actual way we write the quadrupole term is:

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int(r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(r')d\tau$$

Comparing this with our expression, it's clear that we only need to show that:

$$\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\int\left(\frac{3}{2}r_i'r_j' - \frac{1}{2}(r')^2\delta_{ij}\right)\rho(r')d\tau' = \int(r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(r')d\tau'$$

Moving the integral in the sum, it means we need to show that:

$$\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\left(\frac{3}{2}r_i'r_j' - \frac{1}{2}(r')^2\delta_{ij}\right) = (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) \tag{1}$$

Looking at the first term in particular, we want to show that:

$$\frac{3}{2}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j r_i'r_j' = (r')^2\cos^2\alpha$$

We can then split the left hand side into $\frac{3}{2}(\sum_i \hat{r}_i r_i' \sum_j \hat{r}_j r_j')$ and each term $\sum_i \hat{r}_i r_i' = r'\cos\alpha$, then we get that the total term:

$$\frac{3}{2}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j r_i'r_j' = \frac{3}{2}\sum_{i=1}^{3}\hat{r}_i r_i'\sum_{j=1}^{3}\hat{r}_j r_j' = \frac{3}{2}(r'\cos\alpha)(r'\cos\alpha) = \frac{3}{2}(r')^2\cos^2\alpha$$

Now for the second term, we can see that:

$$\frac{1}{2}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j(r')^2\delta_{ij} = \frac{1}{2}\sum_{i=1}^{3}(\hat{r}_i)^2(r')^2 = \frac{1}{2}(r')^2$$

So therefore, the two terms on either side of equation 1 are the same, and thus this is an equivalent way to write the quadrupole term. Furthermore, to show that Q_{ij} is traceless, notice that when $i = j$, then the term we get is:

$$\sum_{i=1}^{3}\frac{3}{2}(r_i)^2 - \frac{1}{2}(r')^2 = \frac{3}{2}(r')^2 - \frac{3}{2}(r')^2 = 0$$

Since the diagonal elements are zero, Q_{ij} is traceless. $\square$

Problem 2

Show that the quadrupole moment Q_{ij} is independent of origin if the monopole and dipole moments both vanish.

Solution: If the monopole and dipole terms vanish, this means that $p = 0$ and $Q = 0$. Suppose we picked another origin point $O' = O + \mathbf{R}$. Then, we have the following:

$$\begin{aligned}
\mathbf{r_{o'}} &= \mathbf{r_o} + \mathbf{R} \\
\mathbf{r_{i,o'}} &= \mathbf{r_{i,o}} + \mathbf{R_i} \\
\mathbf{r_{j,o'}} &= \mathbf{r_{j,o}} + \mathbf{R_j}
\end{aligned}$$

Now we can expand the quadrupole term:

$$\begin{aligned}
Q'_{ij} &= \int \left[\frac{3}{2}(r_{i,o} + Ri)(r_{j,o} + R_j) - \frac{1}{2}(\mathbf{r_o} + \mathbf{R})^2\delta_{ij}\right]\rho(\mathbf{r_0} + \mathbf{R})d\tau' \\
&= \int \left[\frac{3}{2}r_{i,o}r_{j,o} - \frac{1}{2}(\mathbf{r}_o)^2\right]\rho d\tau + \frac{3}{2}R_j\int r_{i,o}\rho d\tau - \mathbf{r}_0 \cdot \mathbf{R}\delta_{ij}\int \rho d\tau \\
&\qquad + \frac{3}{2}R_i\int r_{i,o}\rho d\tau - \int \mathbf{r}_o \cdot \mathbf{R}\rho d\tau + \int \left[\frac{3}{2}R_iR_j - \frac{1}{2}|\mathbf{R}|^2\delta_{ij}\right]\rho d\tau \\
&= Q_{ij} + \frac{3}{2}R_jp_i + \frac{3}{2}R_ip_j - 2\delta_{ij}\mathbf{r}_o \cdot \mathbf{R}Q + \left[\frac{3}{2}R_iR_j - \frac{1}{2}|\mathbf{R}|^2\delta_{ij}\right]Q
\end{aligned}$$

And since $Q = 0$ and $p = 0$, then all the terms vanish except the first term, implying that

$$Q'_{ij} = Q_{ij}$$

And thus the quadrupole moment is independent of the origin. $\square$

Problem 3

A circular disk has a radius R and uniform charge density σ. The disk is lying on the $x - y$ plane, with its center fixed at the origin. Find the potential $V(\mathbf{r})$ of the disk for large r, up to the $1/r^3$ term.

Solution: This problem essentially asks for the multipole expansion for this disk of radius R. Considering the monopole term, we have:

$$V_{\text{mon}}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} = \frac{1}{4\pi\epsilon_0}\frac{\pi R^2\sigma}{r}$$

For the dipole term, we have:

$$V_{\text{dip}}(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' \cos\alpha \rho(r') d\tau'$$

Since the origin is placed at the center of the disc, then this integral becomes:

$$\begin{aligned}\int r' \cos\alpha \sigma(r') dA &= \sigma \cos\alpha \int_0^R \int_0^{2\pi} r' r' d\phi dr' \\ &= 2\pi\sigma\cos\alpha\frac{R^3}{3} \\ &= \frac{2\pi\sigma\cos\alpha R^3}{3}\end{aligned}$$

Giving us:

$$V_{\text{dip}}(r) = \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma\cos\alpha R^2}{3r^2}$$

Similarly for the quadrupole, we need to solve:

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\sigma(r') da$$

Again, computing the integral:

$$\begin{aligned}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\sigma(r') dr' &= \sigma\left[\frac{3}{2}\cos^2\alpha\int (r')^2 r' d\phi dr' - \frac{1}{2}\sigma\int (r')^2 r' d\phi dr'\right] \\ &= \frac{3\pi\sigma\cos^2\alpha R^4}{4} - \frac{\pi\sigma R^4}{4}\end{aligned}$$

So therefore the quadrupole term becomes:

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{3\pi\sigma\cos^2\alpha R^4}{4r^3} - \frac{\pi\sigma R^4}{4r^3}\right)$$

Finally, since α is defined as the angle between the plane and the radius vector $\mathbf{r}$, we can write $\alpha = \frac{\pi}{2} - \theta$, therefore we can express the potential in terms of the natural spherical coordinates $V(r, \theta)$. Combining all three terms, we get:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\pi R^2\sigma}{r} + \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma}{3r^2}\cos\left(\frac{\pi}{2} - \theta\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{3\pi\sigma R^4}{4r^3}\cos^2\left(\frac{\pi}{2} - \theta\right) - \frac{\pi\sigma R^4}{4r^3}\right)$$

We can cancel off the π terms:

$$V(r,\theta) = \frac{R^2\sigma}{4\epsilon_0 r} + \frac{\sigma}{6\epsilon_0 r^2}\cos\left(\frac{\pi}{2} - \theta\right) + \frac{1}{4\epsilon_0 r^3}\left(\frac{3\sigma R^4}{4}\cos^2\left(\frac{\pi}{2} - \theta\right) - \frac{\sigma R^4}{4}\right)$$

$\square$
