Physics 137A Homework

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Collaborators

I worked with **Andrew Binder** to complete this homework.

Problem 1

Show that $[Ae^{ikx} + Be^{-ikx}]$ and $[C\cos kx + D\sin kx]$ are equivalent ways of writing the same function of x, and determine the constants C and D in terms of A and B, and vice versa.

Solution: We can rewrite:

$$\begin{cases} Ae^{ikx} = A(\cos kx + i\sin kx) \\ Be^{-ikx} = B(\cos kx - i\sin kx) \end{cases}$$

So we can sum the two:

$$Ae^{ikx} + Be^{-ikx} = (A+B)\cos kx + (A-B)i\sin kx$$

Here we get that $C = A + B$ and $D = (A-B)i$. If we were to define them the other way around, we would get $A = \frac{C-iD}{2}$ and $B = \frac{C+iD}{2}$.

Problem 2

A Gaussian distribution, parametrised by μ and σ, is given by

$$\rho(x) = A \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

(a) Define

$$I = \int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x$$

Show that $I = \sqrt{\pi}$.

Solution: The trick is to write this integral as the product of two identical integrals, using different variables for each:

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\ &= \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \mathrm{d}x \ \mathrm{d}y \end{aligned}$$

Now we introduce a change of variables from cartesian to polar coordinates:

$$\begin{aligned} I^2 &= \int_{-\pi}^{\pi} \int_{0}^{\infty} e^{-r^2} r \ \mathrm{d}r \ \mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \mathrm{d}\theta \int_{0}^{\infty} e^{u} \ \mathrm{d}u \\ &= 2\pi \cdot \frac{1}{2} e^{u} \Big|_{-\infty}^{0} \\ &= \pi \end{aligned}$$

And since $I^2 = \pi$, the it follows that $I = \sqrt{\pi}$.

(b) Hence find the normalization constant A.

Solution: We use our result from part (a) by substituting $x \to \frac{x-\mu}{\sqrt{2}\sigma}$, meaning that our overall integral changes by a factor of $\frac{1}{\sqrt{2}\sigma}$.

But then we need $\int_{-\infty}^{\infty} \rho(x) = 1$ so we have

$$\frac{A\sqrt{\pi}}{\sqrt{2}\sigma} = 1 \implies A = \sqrt{\frac{2\sigma^2}{\pi}}$$

(c) Find $\langle x \rangle$, $\langle x^2 \rangle$, $\sigma(x)$ of the given gaussian.

Solution: We have $\langle x \rangle = \int x\rho(x)\mathrm{d}x$:

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} x \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} (u+\mu) e^{-u^2} 2\sigma^2 \ \mathrm{d}u \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \left[\int_{-\infty}^{\infty} u \exp\left(-\frac{u^2}{2\sigma^2}\right) \ \mathrm{d}u + \mu \int_{-\infty}^{\infty} \exp\left(\frac{-u^2}{2\sigma^2}\right) \ \mathrm{d}u\right]
\end{aligned}
$$

The first integral is equal to zero, a result known from complex analysis. Our second integral is simply a variation of the one in part (a), so we can use the result from there to get:

$$
\langle x \rangle = \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \cdot \frac{\mu\sqrt{\pi}}{\sqrt{2}\sigma} = \mu
$$

A similar approach is applied to get $\langle x^2 \rangle$:

$$
\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 A \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \\
&= \frac{\sqrt{\pi}}{\sqrt{2}\sigma} \int_{-\infty}^{\infty} (u+\mu)^2 \exp\left(-\frac{u^2}{2\sigma^2}\right) \ \mathrm{d}u \\
&= \frac{\sqrt{\pi}}{\sqrt{2}\sigma} \left[\int_{-\infty}^{\infty} u^2 \exp\left(-\frac{u^2}{2\sigma^2}\right) \ \mathrm{d}u + \mu \int_{-\infty}^{\infty} 2u \exp\left(-\frac{u^2}{2\sigma^2}\right) \ \mathrm{d}u + \mu^2 \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2\sigma^2}\right)\right]
\end{aligned}
$$

We use the hint to evaluate the first integral, the second integral is equal to zero just like in $\langle x \rangle$, and the third integral is our original gaussian:

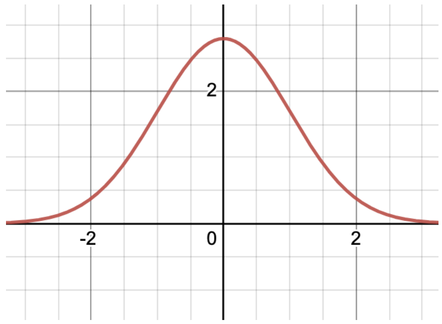
$$
\begin{aligned}
\langle x^2 \rangle &= -\frac{\partial}{\partial(\frac{1}{2\sigma^2})} \frac{\sqrt{\pi}}{\sqrt{2}\sigma} + \mu^2 \frac{\sqrt{\pi}}{\sqrt{2}\sigma} \\
&= \frac{\sqrt{\pi}\sigma}{\sqrt{2}} \left(\mu^2 - 1\right)
\end{aligned}
$$

First, we can calculate $\sigma^2(x) = \langle x^2 \rangle - \langle x \rangle^2$:

$$
\begin{aligned}
\sigma^2(x) &= \frac{\sqrt{\pi}\sigma}{\sqrt{2}} \left(\mu^2 - 1\right) - \mu^2 \\
\therefore \sigma(x) &= \sqrt{\frac{\sqrt{\pi}\sigma}{\sqrt{2}} \left(\mu^2 - 1\right) - \mu^2}
\end{aligned}
$$

(d) Sketch the gaussian. Indicate μ and σ on your sketch or describe the effect of changing them.

Solution: Changing μ shifts the gaussian so that its peak is at $x = \mu$, and shifting σ affects the spread of the distribution. As σ increases, the peak of the gaussian decreases and its width increases, and the opposite occurs when σ decreases. Here's the image:


2
-2
0
2

Problem 3

This problem is designed to guide you through a "proof" of Plancherel's theorem, by starting with the theory of ordinary Fourier series on a *finite interval*, and allowing that interval expand to infinity.

(a) Dirichlet's theorem says that "any" function $f(x)$ on the interval $[-a,+a]$ can be expanded as a fourier series:

$$f(x)=\sum_{n=0}^{\infty}[a_n\sin(n\pi x/a)+b_n\cos(n\pi x/a)]$$

Show that this is equivalently written as

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{in\pi x/a}$$

What is c_n, in terms of a_n and b_n?

Solution: We know from Euler's formula that $e^{ikx}=\cos(kx)+i\sin(kx)$, and by extension from problem 1, we can see that:

$$\frac{e^{ikx}+e^{-ikx}}{2}=\cos kx,\;\frac{e^{ikx}-e^{-ikx}}{2i}=\sin kx$$

And thus we can rewrite our original expression:

$$\begin{aligned}f(x)&=\sum_{n=0}^{\infty}\frac{a_n}{2i}\left(e^{in\pi x/a}-e^{-in\pi x/a}\right)+\frac{b_n}{2}\left(e^{in\pi x/a}+e^{-in\pi x/a}\right)\\&=\sum_{n=0}^{\infty}\left(\frac{a_n}{2i}+\frac{b_n}{2}\right)e^{inx/a}+\sum_{n=0}^{\infty}\left(\frac{b_n}{2}-\frac{a_n}{2i}\right)e^{-in\pi x/a}\end{aligned}$$

To reconcile the indices, we can let the latter take on values of $-n$, and as a result we can combine both terms:

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{b_n-ia_n}{2}e^{in\pi x/a}$$

And thus giving us $c_n=\frac{b_n-ia_n}{2}$.

(b) Show (by appropriate modification of Fourier's trick) that

$$c_n=\frac{1}{2a}\int_{-a}^{+a}f(x)e^{-in\pi x/a}\mathrm{d}x$$

Solution: We do discrete fourier transform on a:

$$\begin{aligned}\int_{-a}^{a} f(x) &= \int_{-a}^{a} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}\mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} \int_{-a}^{a} c_n e^{in\pi x/a}\mathrm{d}x\end{aligned}$$

Note that thsi integral is equal to zero for any $n \neq 0$, so our sum effectively disappears, and we're left with the integral where $n = 0$:

$$\int_{-a}^{a} f(x) = c_0 \cdot 2a \implies c_0 = \frac{1}{2a}\int_{-a}^{a} f(x)\mathrm{d}x$$

Thus, our general term becomes

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}\mathrm{d}x$$

(c) Eliminate n and c_n in favor of the new variables $k = (n\pi/a)$ and $F(k) = \sqrt{2/\pi}ac_n$. Show that (a) and (b) now become

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(k)e^{ikx}\Delta k : F(k) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x)e^{-ikx}\mathrm{d}x$$

where Δk is the increment from one n to the next.

Solution: We know that $k = \frac{n\pi}{a}$ and $F(k) = \sqrt{2/\pi}ac_n$ so thus $c_n = \frac{F(k)}{\sqrt{2/\pi a}}$ and $n = \frac{ka}{\pi}$. First we can do $f(x)$:

$$\begin{aligned}f(x) &= \sum_{n=-\infty}^{\infty} \frac{F(k)}{\sqrt{2/\pi}a}e^{-ikx} \\ &= \sqrt{\frac{\pi}{2}} \sum_{n=-\infty}^{\infty} \frac{F(k)}{a}e^{ikx}\end{aligned}$$

Note that since in order to get the desired prefactor, we can multiply and divide by $\frac{\pi}{a}$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(k)e^{-ikx}\frac{\pi}{a}$$

Coincidentally, the step in n is $\frac{\pi}{a}$, so therefore $\Delta k = \frac{\pi}{a}$. Thus:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(k)e^{ikx}\Delta k$$

As desired. Now we do $F(k)$:

$$\begin{aligned}
\frac{F(k)}{\sqrt{2/\pi a}} &= \frac{1}{2a}\int_{-a}^{a} f(x)e^{-i\left(\frac{ka}{\pi}\right)\pi x/a}\mathrm{d}x \\
F(k) &= \frac{\sqrt{2/\pi}}{2}\int_{-a}^{a} f(x)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x)e^{-ikx}\mathrm{d}x
\end{aligned}$$

Which is what we wanted.

(d) Take the limit $a \to \infty$ to obtain Plancharel's theorem.

Solution: Now we take $a \to \infty$, so that means that k becomes a continuous variable, so our sum becomes an integral, thus:

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\ \mathrm{d}k \\
F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}\mathrm{d}x
\end{aligned}$$

Which is precisely Plancahrel's theorem.

Problem 4

A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-a|x|}$$

where A and a are positive real constants.

(a) Normalize $\Psi(x,0)$.

Solution: We know that the integral of the probability distribution must equal 1:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} \left| e^{-a|x|} \right|^2 \mathrm{d}x \\
&= A^2 \left[\int_{-\infty}^{0} e^{2ax} \mathrm{d}x + \int_{0}^{\infty} e^{-2ax} \mathrm{d}x \right] \\
&= A^2 \left[\frac{1}{2a} + \frac{1}{2a} \right] \\
&= \frac{A^2}{a} \\
\therefore A &= \sqrt{a}
\end{aligned}$$

(b) Find $\phi(k)$.

Solution: Performing a continuous fourier transform:

$$\phi(k) = \frac{\sqrt{a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx} dx = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos(kx) - i\sin(kx)) dx$$

We can change the integral bounds because one of these functions will always be odd:

$$\begin{aligned}
\phi(k) &= 2\sqrt{\frac{2}{2\pi}} \int_{0}^{\infty} e^{-ax} \cos kx \mathrm{d}x \\
&= \sqrt{\frac{a}{2\pi}} \int_{0}^{\infty} \left[e^{(ik-a)x} + e^{-(ik+a)x} \right] \mathrm{d}x \\
&= \sqrt{\frac{a}{2\pi}} \left[\frac{-ik - a + ik - a}{-k^2 - a^2} \right] \\
&= \sqrt{\frac{a}{2\pi}} \frac{2a}{k^2 + a^2}
\end{aligned}$$

(c) Construct $\Psi(x,t)$, in the form of an integral.

Solution: To get $\Psi(x,t)$, we combine what we've just got with the integral we found in lecture:

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(\frac{i(p_x x - E(p_x)t)}{\hbar}\right)\phi(p_x)\ \mathrm{d}p_x \\ &= \frac{\sqrt{2}}{\sqrt{2\pi}}\sqrt{\frac{a^3}{2\pi}}\int_{-\infty}^{\infty}\frac{1}{k^2+a^2}\ \exp\left(i\left(kx - \frac{\hbar k^2}{2m}t\right)\right)\end{aligned}$$

(d) Discuss the limiting cases (a very large, a very small)

Solution: For very large a, the amplitude of our wave function will grow very large, so we'll have large spikes in $\Psi(x,t)$, meaning that we will have meaning that we will have high certainty in our position but very little certainty of our momentum. The opposite would be true for small a.
