Collaborators

I worked with **Andrew Binder, Adarsh Iyer** and **Aren Martinian** to complete this assignment.

Problem 1

a) Use Kepler's laws to weigh the sun. Use $G = 6.67 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$ and $R = 1.50 \times 10^{11}$ m (the average radius of Earth's orbit)

Solution: Here we use Kepler's relation that

$$\tau^2 = \frac{4\pi^2 a^3}{GM}$$

where M here represents the mass of the sun. Therefore, solving for M, we get:

$$M = \frac{4\pi^2 a^3}{G\tau^2}$$

Computing this with the known values, we get: 1.979×10^{30} kg   □

b) For which n do stable circular orbits exist for the potential

$$V(r) = -\frac{A}{r^n}$$

Think graphically!

Solution: This would imply an effective potential of

$$V_{eff} = -\frac{A}{r^n} + \frac{\ell^2}{\mu r^2}$$

This solution only has a bounded orbit when $n = 1$, when we plot it on a graphing calculator. This also makes sense, since for $n = 1$ our V_{eff} is written as:

$$V_{eff} = -\frac{A}{r} + \frac{\ell^2}{\mu r^2}$$

which is the equation for planetary motion, which we know has bounded orbits. On the other hand, we can also just graph V_{eff} for different values of n:

☐
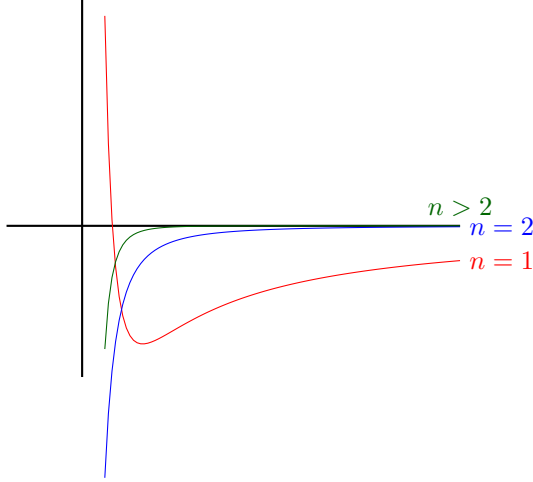
c) (i) A puck of mass m on frictionless ice is attached to a horiziontal string which is fed through a thin hole in the ice of radius R. Initially, the piece of string above the ice has length l. With the free end of the string in place, the puck is given a kick so that it circles around the hole with initial speed v_0 at radius l. But at $t = 0$, the free end of the string is pulled downwards beneath the ice at a constant rate λ, so that the length of string *above the ice* is $l(t) = l - \lambda t$. Eventually, the string has been pulled all the way in, and the puck falls into the hole, upon reaching $r = R$. What quantity is preserved during the motion? What is the puck's speed right before it falls through the hole? Explain how this is consistent with conservation of energy and conservation of angular momentum.

Solution: Angular momentum of the system must be conserved because there is no net torque on the system. Therefore, we have the equation:

$$mv_0 l = mv_f R \implies v_f = \frac{v_0 l}{R}$$

The energy of the system is not conserved here, because we are actively pulling on the string, which does work on the system. While the tangential velocity of the puck does not change, its radial velocity changes (since we are puling it), and thus energy is being put into the system so the energy is not conserved. ☐

(ii) Another puck of mass m on frictionless ice is attached by a horizontal string of length l to a very thin vertical pole of radius R. The puck is given a kick and circles around the pole with initial speed v_0. The string wraps around the pole, and the puck gets drawn in and eventually hits the pole. What quantity is preserved during the motion? What is the puck's speed right before it hits the pole?

Answer the same questions from (i)

Solution: Here, the energy is conserved, because the tension in the string always acts perpendicular to the motion of the puck, so no work is done to the puck. However, the angular momentum here is not conserved, because the path of the puck spirals into the pole, and in order for this to occur a torque must have caused that motion. Therefore, using conservation of energy, we get:

$$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 \implies v_f = v_i$$

Problem 2

Suppose you have a satellite orbiting the earth (mass M) at a radius R. We want to move the satellite to a higher orbit at radius $R' > R$. One way to make such a transfer is by the following maneuver, illustrated and described in the figure overleaf. Calculate the boosts Δv and $\Delta v'$ needed. It'll be useful to consider when energy and momentum are conserved.

Solution: We can calculate the initial velocity of the satellite can be calculated as

$$\frac{mv^2}{R} = \frac{GMm}{r^2} \implies v_1 = \sqrt{\frac{GM}{R}}$$

I will stick to using v_1 for now, to simplify the algebra. We know that the energy right after the boost is:

$$E_1 = \frac{1}{2}m(v_1 + \Delta v)^2 - \frac{GMm}{R}$$

And since angular momentum is conserved, then we also know that $m(v_1 + \Delta v)R = mv_2 R'$, where v_2 is the velocity of the satellite just before the second boost. This gives:

$$v_2 = \frac{(v + \Delta v)R}{R'}$$

Writing out the energy at this location:

$$E_1 = \frac{1}{2}mv_2^2 - \frac{GMm}{R'}$$

And since energy is conserved, these two quantities are equal to each other (I've suggestively labeled them as I've used E_1 in both equations, but the real principle is that energy is conserved). Therefore:

$$\frac{1}{2}m\left[\frac{(v_1 + \Delta v)R}{R'}\right]^2 - \frac{GMm}{R'} = \frac{1}{2}m(v_1 + \Delta v)^2 - \frac{GMm}{R}$$

$$\frac{(v_1 + \Delta v)^2 R^2}{R'^2} - \frac{2GM}{R'} = (v_1 + \Delta v)^2 - \frac{2GM}{R}$$

$$(v_1 + \Delta v)^2 = \frac{2GM\left(\frac{1}{R'} - \frac{1}{R}\right)}{\frac{R^2}{R'^2} - 1}$$

$$= \frac{2GMR'}{R(R + R')}$$

$$\therefore \Delta v = \sqrt{\frac{2GMR'}{R(R + R')}} - \sqrt{\frac{GM}{R}}$$

Since the second boost is circular, then we can also write:

$$v_f = \sqrt{\frac{GM}{R'}}$$

And since we know $\Delta v' = v_f - v_2$, we can just solve for Δv explicitly:

$$\Delta v = v_f - v_2$$

$$= \sqrt{\frac{GM}{R'}} - \frac{R}{R'}\left(\sqrt{\frac{2GMR'}{R(R + R')}}\right)$$

$\square$

Problem 3

For a particle subject to an inverse square force, written $\mathbf{F} = -\frac{km}{r^2}\hat{\mathbf{r}}$, we define the vectors

$$\mathbf{h} = \mathbf{r} \times \hat{\mathbf{r}}$$

and

$$\mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{k} - \hat{\mathbf{r}}$$

The vector $\mathbf{e}$ is known as the *Runge-Lenz vector*. It can be used to show that particles follow conic sections when acted upon by an inverse square force. For this problem, it may be useful to consult `https://en.wikipedia.org/wiki/Triple_product`

a) Show that $\mathbf{h}$ is constant, and hence that under arbitrary conditions the particle will be confined to move in a plane passing through the origin.

Solution: To show that $\mathbf{h}$ is constant, we show instead that $\dot{\mathbf{h}} = 0$. Taking the derivative, we get:

$$\dot{\vec{h}} = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}$$
$$= \vec{r} \times \left(-\frac{k}{r^2}\hat{r}\right)$$
$$= -\frac{k}{r^2}(\vec{r} \times \hat{r}) = 0$$

□

b) Show that $\mathbf{e}$ is also constant.

Solution: We do the same thing as the previous problem:

$$\dot{\vec{e}} = \frac{1}{k}\left(v\ddot{e}cr \times \vec{h} + \dot{\vec{r}} \times \dot{\vec{h}}\right) - \dot{\hat{r}}$$
$$= \frac{1}{k}\left(-\frac{km}{r^2}\hat{r} \times \vec{h}\right)$$
$$= \frac{1}{r^2}(\hat{r} \times (\vec{r} \times \dot{\vec{r}})) - \dot{\theta}\hat{\theta}$$
$$= -\frac{1}{r^2}\left(\vec{r}(\hat{r} \cdot \dot{\vec{r}}) - (\hat{r} \cdot \vec{r})\dot{\vec{r}}\right) - \dot{\theta}\hat{\theta}$$
$$= -\frac{1}{r^2}\left(\hat{r}\dot{r} - \dot{\vec{r}}\right) - \dot{\theta}\hat{\theta}$$
$$= -\frac{1}{r}\left(\hat{r}\dot{r} - (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})\right) - \dot{\theta}\hat{\theta}$$
$$= \dot{\theta}\hat{\theta} - \dot{\theta}\hat{\theta} = 0$$

□

c) Show that

$$er\cos\theta = \frac{h^2}{k} - r$$

where $e = |\mathbf{e}|$ and $h = |\mathbf{h}|$ and θ is the angle between $\mathbf{r}$ and $\mathbf{e}$. Since $\mathbf{h}$ and $\mathbf{e}$ are constant, we can solve for $r(\theta)$ to deduce that the particle will follow a conic section trajectory.

Solution: We compute $\vec{e} \cdot \vec{r}$ so we get:

$$\vec{e} \cdot \vec{r} = er\cos\theta = \left(\frac{\dot{\vec{r}} \times \vec{h}}{k} - \hat{r}\right) \cdot \vec{r}$$

$$= \frac{(\dot{\vec{r}} \times \vec{h}) \cdot \vec{r}}{k} - \frac{\vec{r}}{r} \cdot \vec{r}$$

$$= \frac{(\vec{r} \times \dot{\vec{r}}) \cdot \vec{h}}{k} - \frac{r^2}{r}$$

$$= \frac{\vec{h} \cdot \vec{h}}{k} - r$$

$$= \frac{h^2}{k} - r$$

as desired. $\qquad\qquad\square$

Problem 4

Suppose we have a particle in a central force potential

$$V(r) = -\frac{A}{r} + \frac{B}{2mr^2}$$

where A and B are constants, and m is the mass of the particle (the factor of $2m$ appears for convenience only).

a) Show that the equation of orbit has an exact solution that can be put in the form

$$\frac{r_0}{r} = 1 + \epsilon \cos(\alpha\theta)$$

(Hint: repeat the orbit derivation with the new potential)

Solution: As per the hint, we solve the orbit equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u(\phi) = -\frac{m}{\ell^2}\frac{1}{u^2}F(u)$$

We find the force using

$$
\begin{aligned}
F(r) &= -\frac{\mathrm{d}u}{\mathrm{d}r}\\
&= -\frac{A}{r^2} + \frac{B}{r^3}\\
&= -Au^2 + \frac{B}{m}u^3
\end{aligned}
$$

So now we solve the differential equation:

$$
\begin{aligned}
\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u(\phi) &= -\frac{m}{\ell^2}\frac{1}{u^2}\left(-Au^2 + \frac{B}{m}u^3\right)\\
-\frac{\ell^2 u^2}{m}\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} &= -Au^2 + u^3\left(\frac{\ell^2 + B}{m}\right)\\
-\frac{\ell^2}{m}u''(\phi) &= -A + u\left(\frac{\ell^2 + B}{m}\right)\\
\therefore u''(\phi) &= \frac{Am}{\ell^2} - u\left(1 + \frac{B}{l^2}\right)
\end{aligned}
$$

Now let $k^2 = 1 + \frac{B}{\ell^2}$:

$$u''(\phi) + k^2 u = \frac{Am}{\ell^2}$$

Now let $w(\phi) = k^2 u - \frac{Am}{\ell^2}$, this gets us $w''(\phi) = k^2 u''(\phi)$, so this gets us:

$$w''(\phi) + k^2 w(\phi) = 0$$

which is the equation for simple harmonic motion. Therefore, we get the solution

$$w(\phi) = C\cos(k\phi - \delta)$$

Here, we can let $\delta = 0$ for an appropriate choice for $\phi = 0$, so we get

$$w(\phi) = C \cos(k\phi)$$

As the general solution. Substituting back u, we get:

$$k^2 u(\phi) - \frac{Am}{\ell^2} = C \cos(k\phi)$$

$$\therefore u(\phi) = \frac{C}{k^2} + \frac{Am}{k^2 \ell^2}$$

Finally, we return $u = \frac{1}{r}$:

$$\frac{1}{r} = \frac{C}{k^2} \cos(k\phi) + \frac{Am}{k^2 \ell^2}$$

$$= \frac{Am}{k^2 \ell^2} \left(1 + \frac{C\ell^2}{Am} \right)$$

We let $k = \alpha$ to match with the problem, so we get:

$$\frac{1}{r} = \frac{Am}{\alpha \ell^2} \left(1 + \frac{C\ell^2}{Am} \cos(\alpha\phi) \right)$$

setting $\alpha = \frac{k^2 l^2}{Am}$ and $\epsilon = \frac{C\ell^2}{Am}$, we get the equation:

$$\frac{r_0}{r} = 1 + \epsilon \cos(\alpha\phi)$$

$\square$

b) The trajectory in (a) is an ellipse of $\alpha = 1$, but if $\alpha \neq 1$, the ellipse *precesses*, which may be described in terms of the precession of the apsides (turning points). For B small, derive an approximate expression for the precession rate. You may express your answer in terms of the period of the orbit where $B = 0$ - call it T.

Solution: The apsides are the furthest and nearest point, which are described by $\cos(\alpha\phi) = 1$, or equivalently when $\alpha\theta = 2\pi n$. This allows us to solve for θ:

$$\theta = \frac{2\pi n}{\alpha}$$

So the deviation per orbit can be calculated is

$$\Delta\theta = \frac{2\pi(n+1)}{\alpha} - \frac{2\pi n}{\alpha} = \frac{2\pi}{\alpha}$$

So per one orbit, the deviation from θ from a singular orbit is:

$$\delta\theta = 2\pi - \Delta\theta = 2\pi \left(1 - \frac{1}{\alpha} \right) = 2\pi \left[1 - \left(1 + \frac{B}{\ell^2} \right)^{-1/2} \right]$$

Taylor expanding this quantity, we get:

$$2\pi \left[1 - \left(1 + \frac{B}{\ell^2} \right)^{-1/2} \right] = 2\pi \left[1 - \left(1 - \frac{B}{2\ell^2} \right) \right]$$

$$= 2\pi \left(\frac{B}{2\ell^2} \right)$$

$$= \frac{\pi B}{\ell^2}$$

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For small B, where we've dropped the higher order terms. Therefore, the period of precession is:

$$\frac{\delta\theta}{T} = \frac{\pi B}{T\ell^2}$$

$\square$

Example Notebook problems

Question 1: Example Notebook

Learning objectives

In this question you will:

- Demonstrate the basic functionality of Jupyter Notebooks for compuational problems

The purpose of the notebook is to provide students with an opportunity to verify that they have access to the Berkeley Data Hub and that they can answer basic questions using Markdown cells and python scripts. If you are unfamiliar with python, or would like a refresher, please take a look at:

https://datahub.berkeley.edu/hub/user-redirect/git-pull?
repo=https%3A%2F%2Fgithub.com%2Fberkeley-physics%2FPython-
Tutorials&branch=master

1a.

Markdown can be used to put text, equations and tables into a Jupyter Notebook. A basic introduction to Markdown can be found at:

https://jupyter-
notebook.readthedocs.io/en/stable/examples/Notebook/Working%20With%20Markdown%20Cel

Verify that you can fill a Markdown cell correctly by reproducing the text below:

![Drawing]

This is an example of an itemized list in Markdown

- Item 1
  - Subitem a
  - Subitem b
- Item 2
- Item 3

This is an example of an enumerated list in Markdown:

1. Item
   a. Subitem a

  b. Subitem b
2. Item 2
3. Item 3

This is an example of a LaTex equation in Markdown:

$$\int_{-1}^{1} (1 + \cos^2\theta)\, d\cos\theta = \left(\cos\theta + \frac{\cos^3\theta}{3}\right)\Big|_{-1}^{1}$$
$$= \left(\frac{4}{3} - \frac{-4}{3}\right)$$
$$= \frac{8}{3}$$

1b.

Our coding will be done in python. Start by executing the cell below:

```
In [1]:  a=10
         b=20

         a_list = [1,2,3,4,5]
         b_list = [2,4,6,8,10]

         import numpy as np
         a_array = np.array(a_list)
         b_array = np.array(b_list)
```

Verify that you can use python to do simple arithmetic and to do math using numpy arrays by calculating:

- $a + b$
- The sum of the third out of five elements of a_list and b_list
- $(a\_array)^2 + (b\_array)^2$ (the answer should be given as an array of 5 numbers)

```
In [2]:  #Write your answer here
         print(a+b) # 10 + 20 = 30
         print(a_array[2] + b_array[2]) # 3+6 = 9
         print(a_array**2 + b_array**2)

         30
         9
         [  5  20  45  80 125]
```
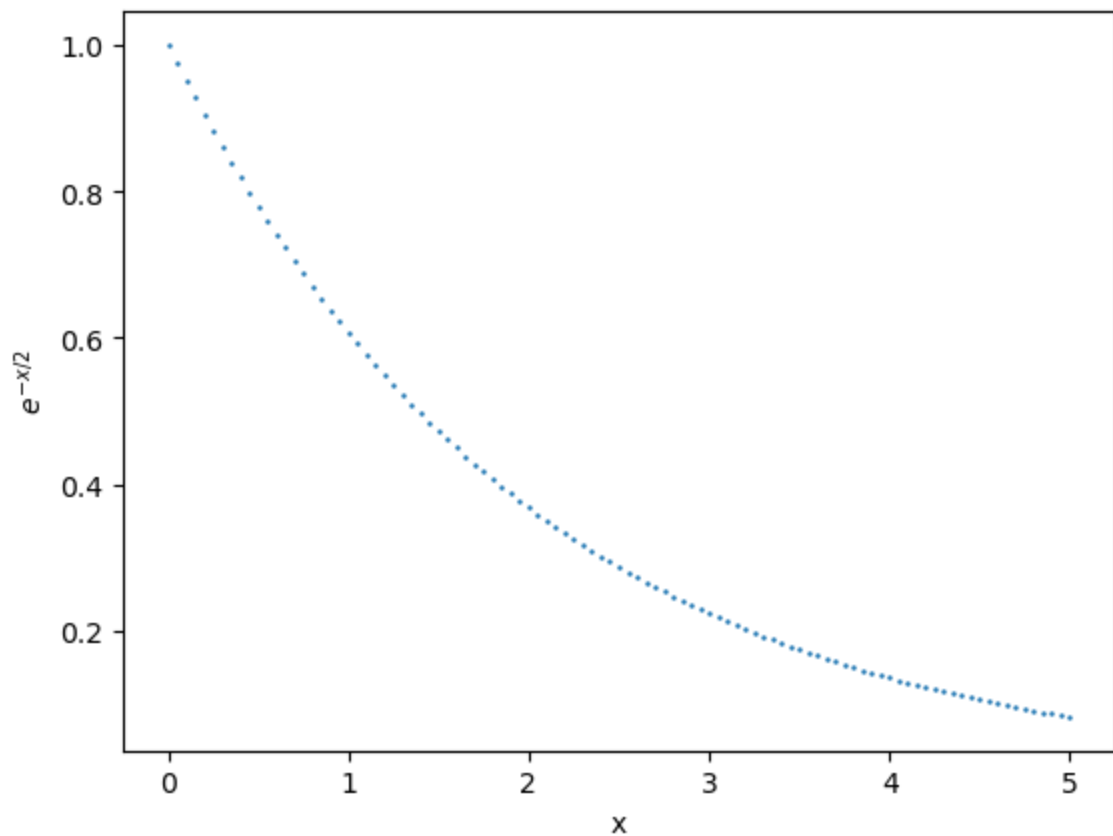
1c.

One useful package to make plots is matplotlib. The code fragment below loads the plotting library and sets up a numpy array of 101 equally spaced numbers with values between 0 and 5

In [3]:
```python
import matplotlib.pyplot as plt
x_array = np.linspace(0,5,101)
```

Make a scatter plot where where the x-axis values are x_array and where the y-axis values are $y = e^{-x/2}$. Make sure to put labels on the x and y axes

In [4]:
```python
#Write your answer here
y_array = np.e**(-x_array/2)

plt.scatter(x_array, y_array, s = 0.6)
plt.xlabel('x')
plt.ylabel('$e^{-x/2}$')
plt.show()
```


In [ ]: