Eric Du — Physics 105
HW 04 — Analytic Mechanics — February 18, 2023

Collaborators

Problem 1

Consider a vertical plane in a constant gravitational field (which is also vertical). A particle of mass m moves on the plane, and it is also under the influence of a force $F = -Ar^{\alpha-1}$, where A and α are constants ($\alpha \neq 0, 1$). Here, r is the distance from the origin, which is some fixed point on the plane. After choosing suitable generalized coordinates, find the Lagrangian and the equations of motion. Is the angular momentum conserved? Explain.

Solution: Since we're given the force, we actually also know the potential:

$$-\frac{\partial}{\partial r}\left(\frac{A}{\alpha}r^{\alpha}\right) = F \implies U = \frac{A}{\alpha}r^{\alpha}$$

and so therefore we can write the Lagrangian as:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r\dot{\theta}^2) - mgr\sin\theta - \frac{A}{\alpha}r^{\alpha}$$

So we can just solve using the normal method:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \\ \frac{\partial \mathcal{L}}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0\end{aligned}$$

And so solving this system, we get the following two sets of equations:

$$\begin{aligned}m\ddot{r} + Ar^{\alpha-1} - mg\sin\theta = 0 \\ r(\ddot{\theta} + \cos\theta) - \dot{r}\dot{\theta} = 0\end{aligned}$$

Angular momentum is written as:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr\dot{\theta}$$

And since this quantity is not constant, then we know that angular momentum is not conserved in this model. $\square$

Problem 2

Use the method of Lagrange multipliers to find the tension in the string of the double Atwood machine given in Homework 2 Problem.

Solution: Our constraint equation is $f(x_1, x_2, X) = x_1 + x_2 + X = L$, where we interpret L as the length of the string. From Homework 2, we know that our Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}^2 - m_1gx_1 - m_2gx_2 - Mgx$$

and so now we can use the modified Lagrange equations with constraints to solve:

$$\frac{\partial \mathcal{L}}{\partial x_i} + \lambda(t)\frac{\partial f}{\partial x_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

And so therefore we have the equations:

$$\begin{aligned} -m_1g + \lambda(t) &= m_1\ddot{x}_1 \\ m_2g + \lambda(t) &= m_2\ddot{x}_2 \\ -Mg + 2\lambda(t) = M\ddot{x} \end{aligned}$$

And so solving the last equation, we get:

$$\lambda(t) = \frac{M}{2}(\ddot{x} + g)$$

Substituting this relation into the first equation, we get:

$$\begin{aligned} m_1\ddot{x}_1 + m_1g &= \frac{M}{2}g - \frac{M}{4}\ddot{x}_1 - \frac{M}{4}\ddot{x}_2 \\ \ddot{x}_1\left(m_1 + \frac{M}{4}\right) + \frac{M}{4}\ddot{x}_2 = g\left(\frac{M}{2} - m_1\right) \end{aligned}$$

Likewise for the second equation, we can also obtain

$$\begin{aligned} m_2\ddot{x}_2 + m_2g &= \frac{M}{2}(\ddot{x} + g) \\ &= \frac{M}{2}\left(g - \frac{\ddot{x}_1 + \ddot{x}_2}{2}\right) \\ &= \frac{M}{2}g - \frac{M}{4}\ddot{x}_1 - \frac{M}{4}\ddot{x}_2 \end{aligned}$$

And so therefore we obtain:

$$\ddot{x}_2\left(m_2 + \frac{M}{4}\right) + \frac{M}{4}\ddot{x}_1 = g\left(\frac{M}{2} - m_2\right)$$

To find the tension in the rope, we can then use the relation that

$$\lambda\frac{\partial f}{\partial y} = F^{cstr} = T$$

And so therefore:

$$T = \frac{M}{2}(\ddot{x} + g)$$

□

Problem 3

Consider a particle of mass m_2 constrained to move on the surface of a cone which is placed vertex-up. There is a massless, ideal string of length ℓ connecting this mass to another mass m_1 which hangs inside the cone, as in the figure below. The cone has an opening angle of 2α.

a) Define generalized cylindrical coordinates suitable to this system, and write down the Lagrangian.

Solution: m_1 is constrained to move in the $\hat{z}$-direction, so we can write the coordinates of m_1 in terms of strictly z_1. For m_2, we can write its position in terms of (x_2, ϕ, z_2), so therefore we can write the kinetic energy as:

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}_2^2 + x_2^2\dot{\phi}^2 + \dot{z}_2^2) + \frac{1}{2}m\dot{z}_1^2 \\ &= \frac{1}{2}mz^2 + \frac{1}{2}m\left(\dot{x}^2 + x^2\dot{\phi}^2 + \frac{\dot{x}^2}{\tan^2\alpha}\right) \end{aligned}$$

Here, I've dropped the indices since none of the variables interfere with one another. We can also write the potential as:

$$U = mgz + mgx\cos\alpha$$

And so our full Lagrangian $\mathcal{L}$ is:

$$\mathcal{L} = T - U = \frac{1}{2}m_1z^2 + \frac{1}{2}m_2\left(\dot{x}^2 + x^2\dot{\phi}^2 + \frac{\dot{x}^2}{\tan^2\alpha}\right) - m_1gz - m_2g\frac{x}{\tan\alpha}$$

□

---

b) Find the equations of motion; use the constraint, so that you only have two equations.

Solution: Right now we have three coordinates: x, z and ϕ. The constraint tells us that

$$x = (\ell - z)\sin\alpha$$

So now we can substitute and get:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m_1\dot{z}^2 + \frac{1}{2}m_2\left[\dot{z}^2\sin^2\alpha + (\ell - z)^2\sin^2\alpha\dot{\phi}^2 + \dot{z}^2\cos^2\alpha\right] - m_1gz - m_2g(l - z)\cos\alpha \\ &= \frac{1}{2}m_1\dot{z}^2 + \frac{1}{2}m_2\left[\dot{z}^2 + (\ell - z)^2\sin^2\alpha\dot{\phi}^2\right] - m_1gz - m_2g(\ell - z)\cos\alpha \end{aligned}$$

Now we have an equation in terms of two variables, for which we can solve using the standard Euler-Lagrange equations, which gives us:

$$\begin{aligned} 0 &= (\ell - z)^2\ddot{\phi} - 2(\ell - z)\dot{\phi}\dot{z} \\ 0 &= -2(\ell - z)\sin^2\alpha\dot{\phi}^2 - m_1g + m_2g\cos\alpha - \ddot{z}(m_1 + m_2) \end{aligned}$$

□

---

c) Notice that the equation for φ can be separated and integrated to obtain a solution for $\dot{\varphi}$. Do so, and plug this equation in to the equation for z

Solution: Notice that our equation for ϕ can be written as

$$\frac{\ddot{\phi}}{\dot{\phi}} = \frac{2\dot{z}}{\ell - z}$$

which we can integrate to get:

$$\int \frac{1}{\dot{\phi}} \frac{\mathrm{d}\dot{\phi}}{\mathrm{d}t}\, dt = \int \frac{2\dot{z}}{\ell - z}\, dt$$
$$\ln \dot{\phi} = -2 \ln(\ell - z)$$
$$\therefore \dot{\phi} = \frac{1}{(\ell - z)^2}$$

So now we can plug this into the equation for z:

$$\begin{aligned} 0 &= -2(\ell - z)\sin^2\alpha \left(\frac{1}{(\ell - z)^2}\right)^2 - \ddot{z}(m_1 + m_2) \\ &= -\frac{2\sin^2\alpha}{(\ell - z)^3} - m_1 g + m_2 g \cos\alpha - \ddot{z}(m_1 + m_2) \end{aligned}$$

□

---

d) You should now have a differential equation for z only. Show that this equation of motion corresponds to a potential of the form

$$U(z) = \frac{A}{(\ell - z)^2} + Bz$$

Solution: So looking at the potential, we can get the force by taking the derivative:

$$\begin{aligned} F &= -\frac{\mathrm{d}U}{\mathrm{d}z} \\ &= -\left(\frac{2A}{(\ell - z)^3} + B\right) \\ &= -\frac{2A}{(\ell - z)^3} - B \end{aligned}$$

Then by Newton's second law, we can write:

$$\begin{aligned} -\frac{2A}{(\ell - z)^3} - B &= (m_1 + m_2)\ddot{z} \\ -\frac{2A}{(\ell - z)^3} - (m_1 + m_2)\ddot{z} &= B \end{aligned}$$

Comparing this with our solution from above, we can identify:

$$A = \sin^2\alpha \qquad\qquad B = m_1 g - m_2 g \cos\alpha$$

□

---

e) When is B positive or negative? What does this correspond to physically? Analyze the motion in both cases, $B < 0$ and $B > 0$.

Solution: From the previous part, we know that $B = m_1 g - m_2 g \cos\alpha$, so B is positive when $m_1 g > m_2 g \cos\alpha$ and $B < 0$ when $m_1 g < m_2 g \cos\alpha$. Physically, from a force perspective, it compares whether the downward force exerted by m_1 is larger or that of m_2 or the other way around.

Therefore, B is positive when the downward force from m_1 exceeds that of m_2, and so m_2 moves up the cone. Likewise, if B is negative then the force from m_2 exceeds that of m_1, and so m_2 moves down the cone.

To analyze the motion, we first make the substitution that $u = \ell - z$, so therefore we have the equation:

$$-\frac{2A}{u^3} + (m_1 + m_2)\ddot{u} = \frac{B}{m_1 + m_2}$$

And so therefore we can write:

$$\ddot{u} - \frac{2A}{(m_1 + m_2)u^3} = \frac{B}{m_1 + m_2}$$

I'm not really sure where to go from here, since this differential equation is nearly impossible to solve.

$\square$

---

f) Use the method of Lagrange multipliers to determine the tension in the string in terms of z (and the initial conditions)

Solution: I wasn't able to carry through the previous part so I couldn't explicitly get an expression, but what I can do is describe how I would do it. Essentially, we just need to setup the expression for Lagrange multipliers:

$$\frac{\partial \mathcal{L}}{\partial x_i} + \lambda \frac{\partial f}{\partial x_i} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

And once we solve for λ, we can then write:

$$F^{cstr} = T = \lambda \frac{\partial f}{\partial x_i}$$

which would get us the expression for the tension. $\square$

Problem 4

Consider a uniform chain of length ℓ and mass m which is free to move without friction on a triangular block as in the figure below. Suppose that at $t = 0$, the chain has length x_0 lying over the edge of the side of the prism. Find and solve the equation of motion.

Solution: First, let $\lambda = \frac{m}{\ell}$. Then we can find the potential. To do so, we split up the integral based on which side of the block our chain is:

$$U(x) = -\int_0^x (\lambda d\ell) g\ell \sin\alpha - \int_x^\ell \lambda \, d\ell g\ell \sin\beta = \lambda g \left(\sin\alpha \frac{x^2}{2} - \sin\beta \left(\frac{l^2 - x^2}{2} \right) \right)$$

and so our Lagrangian can be written as:

$$\mathcal{L} = \frac{1}{2} m\dot{x}^2 - \lambda g \sin\alpha \frac{x^2}{2} - \lambda g \sin\beta \left(\frac{\ell^2 - x^2}{2} \right)$$

And so solving for the equation of motion, we get:

$$x(\lambda g \sin\beta - \lambda g \sin\alpha) - m\ddot{x} = 0$$
$$\ddot{x} - \frac{\lambda g(\sin\beta - \sin\alpha)}{m} x = 0$$

This gives solutions of the form:

$$x(t) = Ae^{kt} + Be^{-kt}, \quad k = \sqrt{\frac{\lambda g(\sin\beta - \sin\alpha)}{m}}$$

We can then solve for the initial condition that $x(0) = x_0$, which gives us

$$x(0) = x_0 = A + B$$

We can't really solve for A and B explicitly without knowing more information (such as knowing $\dot{x}(0)$, so this is the most general solution that we can have:

$$x(t) = Ae^{kt} + Be^{-kt}$$

where $k = \sqrt{\frac{\lambda g(\sin\beta - \sin\alpha)}{m}}$ □