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HW 01 | Real Analysis | January 19, 2023

Problem 1

Using proof by induction to prove that: For every $n \in \mathbb{N}$, $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Solution: Let $A \subset N$ be the set of naturals which satisfies the above proposition. First, we show that $m = 1 \in A$ $\square$

Problem 2

(a) Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers n.

(b) The princpile of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, P_{n+1} is true whenever P_n is true and $n \geq m$.

  (i) Prove $n^2 > n + 1$ for all integers $n \geq 2$.

  (ii) Prove $n! > n^2$ for all integers $n \geq 4$. [Recall $n! = n(n-1) \cdots 2 \cdot 1$; for example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.]

Problem 3

Prove: $\sqrt{3}$ is not a rational number

Problem 4

Prove: $\sqrt{2} + \sqrt{3}$ is not a rational number.

Problem 5

(a) Show $|b| \leq a$ if and only if $-a \leq b \leq a$.

(b) Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Problem 6

Given a nonempty set $A \subset \mathbb{R}$. Using the definition of supremum/infimum, show that

- $\sup A \geq \inf A$
- If $\max A$ ($\min A$) exists, then $\sup A = \max A$ ($\inf A = \min A$)
- $\inf A = -(\sup(-A))$, where $-A = \{-a | a \in A\}$

Problem 7

Using the completeness axiom theorem to prove the theorem for strong induction:

Theorem 1. *Assume A is a subset of $\mathbb{N}$, if A satisfies the following two properties:*

(1) $1 \in A$

(2) If $\{1, 2, 3, \ldots, n\} = \{x | x \leq n, x \in N\} \subset A$, *then* $n + 1 \in A$

Then $A = \mathbb{N}$

Hint: Use proof by contradiction.