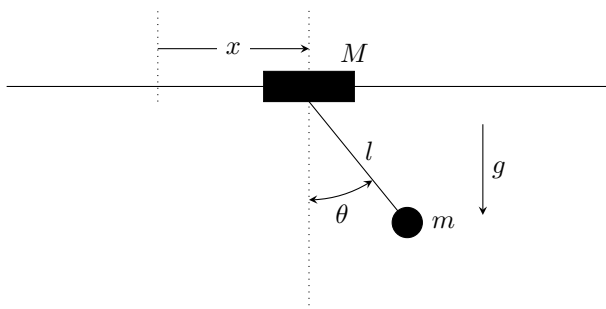


## Collaborators

I worked with **Aren Martinian**, **Andrew Binder** and **Adarsh Iyer** to complete this assignment.

## Problem 1

A bob of mass  $m$  is suspended by a massless rigid rod of length  $l$  that is hinged to a sled of mass  $M$ . The sled slides without friction on a horizontal rail, as shown in the figure.



- (a) Write down the Lagrangian for the system and derive the equations of motion.

*Solution:* We can first express the coordinates of the mass bob in cartesian coordinates:

$$X = x + l \sin \theta \quad Y = -l \cos \theta$$

so this leads to the equations:

$$\dot{X} = \dot{x} + l\dot{\theta} \cos \theta \quad \dot{Y} = l\dot{\theta} \sin \theta$$

Therefore, the kinetic energy term is (skipping the intermediate algebra):

$$\begin{aligned} T &= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}M\dot{x}^2 \\ &= \frac{\dot{x}^2}{2}(m + M) + \frac{m}{2} \left( 2\dot{x}l\dot{\theta} \cos \theta + l^2\dot{\theta}^2 \right) \end{aligned}$$

Now, let the sled have zero potential energy. Therefore:

$$U = mgl(1 - \cos \theta)$$

And so therefore, the Lagrangian is:

$$\mathcal{L} = T - U = \frac{\dot{x}^2}{2}(m + M) + \frac{m}{2} \left( 2\dot{x}l\dot{\theta} \cos \theta + l^2\dot{\theta}^2 \right) - mgl(1 - \cos \theta)$$

With this now, we recognize that  $x$  and  $\theta$  are our coordinates, so there are two Euler-Lagrange equations.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} &= 0 = -\frac{d}{dt} \left( \dot{x}(m + M) + ml\dot{\theta} \cos \theta \right) \\ 0 &= \ddot{x}(m + M) + ml\ddot{\theta} \cos \theta - ml\dot{\theta}^2 \sin \theta \end{aligned}$$

and likewise for the  $\theta$  direction:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= 0 = -m\dot{x}l\dot{\theta} \sin \theta - mgl\dot{\theta} \sin \theta - \frac{d}{dt} (m\dot{x}l \cos \theta - ml^2\dot{\theta}) \\ 0 &= ml^2\ddot{\theta} + m\ddot{x}l \cos \theta + mgl \sin \theta\end{aligned}$$

□

- (b) Suppose, for all  $t < 0$ , the masses are at rest with  $\theta = 0$ . Then, at  $t = 0$ , an impulse  $\Delta P = F\Delta t$  is applied to the bob over a small timespan  $\Delta t$  from a sharp horizontal tap. Find  $\dot{x}$  and  $\dot{\theta}$  immediately after the tap. (Hint: Consider both linear and angular momentum)

*Solution:* Based on the principles of linear and angular momentum, we know that

$$\Delta p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \quad l\Delta p = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$

And so therefore we get the following system of equations:

$$\begin{aligned}\Delta p &= \dot{x}(m + M) + ml\dot{\theta} \cos \theta, \\ l\Delta p &= m\dot{x}l \cos \theta + ml^2\dot{\theta}\end{aligned}$$

Multiplying the top equation by  $l$  and subtracting the second, we get:

$$\begin{aligned}\dot{x}l(m + M) + ml^2\dot{\theta} &= m\dot{x}l + ml^2\dot{\theta} \\ Ml\dot{x} &= 0 \\ \therefore \dot{x} &= 0\end{aligned}$$

Then, since  $\dot{x} = 0$ , then we also have  $\Delta p = ml\dot{\theta}$  so  $\dot{\theta} = \frac{\Delta p}{ml}$ .

□

- (c) Suppose the impulse  $\Delta P$  in (b) is also small, so that the  $\theta$  stays small for all  $t$ . Use the small angle approximation to simplify your equations of motion from (a), and solve for  $x(t)$  and  $\theta(t)$  in this approximation.

*Solution:* To simplify our equations of motion, we use the simplifications that  $\cos \theta \approx 1$  and  $\sin \theta \approx \theta$ . Doing so, we get the equations:

$$\begin{aligned}\ddot{x}(m + M) + ml\ddot{\theta} - ml\dot{\theta}^2\theta &= 0 \\ ml^2\ddot{\theta} + m\ddot{x}l + mgl\theta &= 0\end{aligned}$$

From the second equation, we get that  $\ddot{x} = -g\theta - l\ddot{\theta}$ , which we can then plug into the first equation:

$$\begin{aligned}(-g\theta - l\ddot{\theta})(m + M) + ml\ddot{\theta} &= 0 \\ Ml\ddot{\theta} + g(m + M)\theta &= 0 \\ \therefore \ddot{\theta} + \frac{g(m + M)}{Ml}\theta &= 0\end{aligned}$$

This equation just simple harmonic motion, with  $\omega^2 = \frac{g(m+M)}{Ml}$ , so we can conclude:

$$\theta(t) = A \cos(\omega t) + B \sin(\omega t)$$

And since we know that  $\theta(0) = 0$ , then we choose  $\theta(t) = B \sin(\omega t)$  to simplify our calculations. We can solve for  $B$  by finding  $\dot{\theta}(0) = \frac{\Delta P}{ml}$ , so we get:

$$\begin{aligned}\dot{\theta}(0) &= B\omega \cos(\omega t) = \frac{\Delta P}{ml} \\ \therefore B &= \frac{\Delta P}{ml\omega}\end{aligned}$$

Now we can proceed to find  $x(t)$ . To do so, we first need to find  $\ddot{x}(t)$ , which requires  $\ddot{\theta}(t)$ . So taking two time derivatives, we get:

$$\ddot{\theta}(t) = -\frac{\Delta P}{ml\omega} \omega^2 \sin(\omega t)$$

And so now we can plug:

$$\begin{aligned}0 &= \ddot{x}(m + M) + ml \cdot -\frac{\Delta P}{ml} \omega \sin(\omega t) \\ \ddot{x} &= \frac{\Delta P \omega}{m + M} \sin(\omega t)\end{aligned}$$

which we can now integrate:

$$\dot{x} = -\frac{\Delta P \omega}{m + M} \frac{1}{\omega} \cos(\omega t) + C_1$$

$C_1$  can be determined by using the condition that  $\dot{x}(0) = 0$  from part (b), so therefore:

$$C_1 = \frac{\Delta P}{m + M}$$

Now we can integrate again to get  $x(t)$ :

$$x(t) = -\frac{\Delta P}{\omega(m + M)} \sin(\omega t) + \frac{\Delta P t}{m + M} + C_2$$

Then we can set  $x(0) = x_0$ , some arbitrary position, which gives:

$$C_2 = x_0$$

And so therefore our full equation for  $x(t)$  is:

$$x(t) = -\frac{\Delta P}{\omega(m + M)} \sin(\omega t) + \frac{\Delta P t}{m + M} + x_0$$

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□

## Problem 2

Consider a particle moving in three dimensions, described by the Lagrangian

$$L = \frac{1}{2}m|\dot{r}|^2 - V(r)$$

Using Cartesian coordinates, where  $r = (x, y, z)$  we can rewrite this as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

It's often useful to use a different choice of coordinates, such as cylindrical coordinates or spherical coordinates. For example, in problems where there's symmetry, we can more easily deal with constraints in a symmetry-appropriate coordinate system. For this problem, re-express the kinetic term of the Lagrangian in

- (a) Cylindrical coordinates  $(\rho, \phi, z)$

*Solution:* In cylindrical coordinates, we have the equations

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

and so therefore our kinetic term is

$$\begin{aligned} T &= \frac{m}{2} \left[ \left( \dot{\rho} \cos \phi - \rho \sin \phi \dot{\phi} \right)^2 + \left( \dot{\rho} \sin \phi + \rho \cos \phi \dot{\phi} \right)^2 + \dot{z}^2 \right] \\ &= \frac{m}{2} \left[ \dot{\rho}^2 \cos^2 \phi - 2\dot{\rho} \cos \phi \rho \sin \phi \dot{\phi} + \rho^2 \sin^2 \phi \dot{\phi}^2 + \dot{\rho}^2 \sin^2 \phi + 2\dot{\rho} \sin \phi \rho \cos \phi \dot{\phi} + \rho^2 \cos^2 \phi \dot{\phi}^2 + \dot{z}^2 \right] \\ &= \frac{m}{2} \left[ \dot{\rho}^2 (\cos^2 \phi + \sin^2 \phi) + \rho^2 \dot{\phi}^2 (\cos^2 \phi + \sin^2 \phi) + \dot{z}^2 \right] \\ &= \frac{m}{2} \left[ \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \right] \end{aligned}$$

□

- (b) spherical coordinates  $(r, \phi, \theta)$

*Solution:* In spherical coordinates, we have:

$$x = r \sin \theta \cos \phi$$

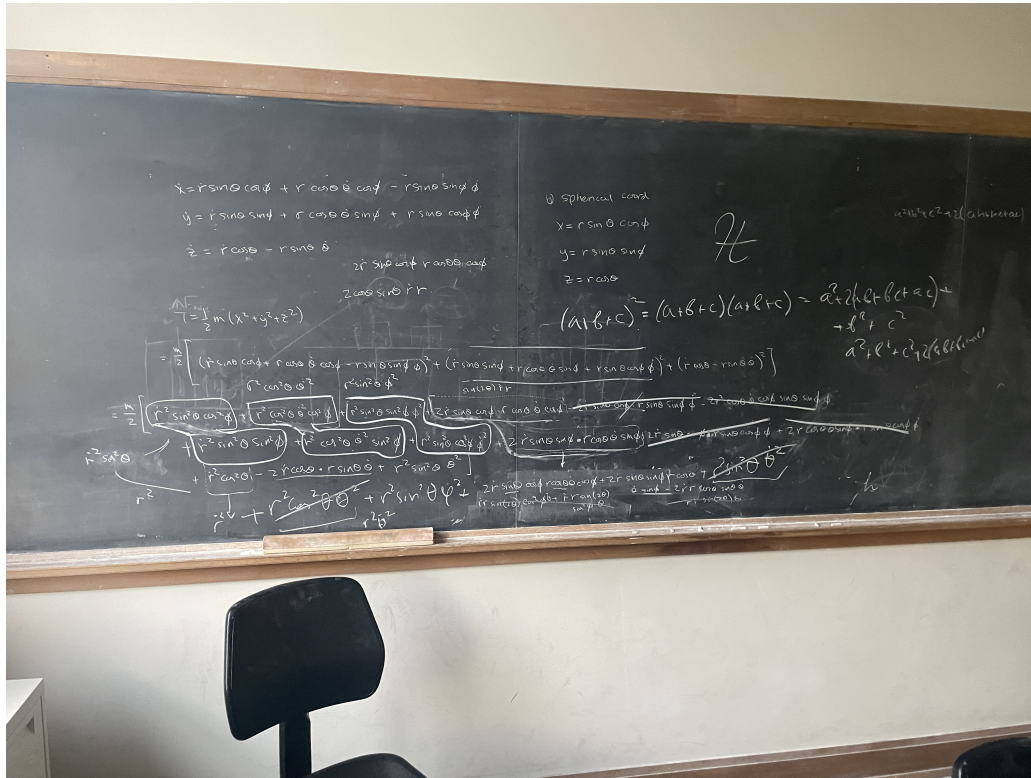
$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

So our process to do this is to find  $\dot{x}$ ,  $\dot{y}$  and  $\dot{z}$  then take

$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

This process involves doing triple product rule and gets us many many terms. There are too many terms in this expansion so I won't write it out, but in replacement here's an image of our work on the blackboard:



In any case, we get many cancellations and nice combinations of terms, all of them using the nice identity that  $\sin^2 \phi + \cos^2 \phi = 1$ , which eventually (albeit after a lot of struggle) gets us:

$$T = \frac{m}{2} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2$$

□

### Problem 3

Consider a ring of wire of radius  $R$ , mounted vertically as in the figure below. A frictionless bead of mass  $m$  is threaded on this wire. The ring is forced to rotate around the indicated axis at constant angular velocity  $\Omega$ . The position of the bead is specified by  $\theta$ . Gravity acts downwards and has magnitude  $g$ .

- (a) Write down the Lagrangian (Hint: use the result of the previous problem)

*Solution:* We have symmetry along the  $\hat{z}$  axis, so we will choose cylindrical coordinates. The kinetic energy is:

$$T = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$

Note here that in our problem, we have  $\dot{\phi} = \Omega$ , so therefore we have:

$$T = \frac{m}{2} \left[ (R\dot{\theta} \cos \theta)^2 + (R \sin \theta)^2 \Omega^2 + R^2 \dot{\theta}^2 \sin^2 \theta \right] = \frac{m}{2} R^2 \left[ \dot{\theta}^2 + \sin^2 \theta \Omega^2 \right]$$

The potential energy term is just  $U = mgz = mgR(1 - \cos \theta)$  so therefore:

$$\mathcal{L} = \frac{m}{2} R^2 \left[ \dot{\theta}^2 + \sin^2 \theta \Omega^2 \right] - mgR(1 - \cos \theta)$$

□

- (b) Derive the equation of motion

*Solution:*  $\theta$  is the only coordinate here, so we derive the equation of motion:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= 0 = mR^2 \Omega^2 \sin \theta \cos \theta - mgR \sin \theta - mR^2 \ddot{\theta} \\ R\ddot{\theta} &= R\Omega^2 \sin \theta \cos \theta - g \sin \theta \\ \ddot{\theta} &= \Omega^2 \sin \theta \cos \theta - \frac{g}{R} \sin \theta \end{aligned}$$

□

- (c) Use your equation from (c) to determine which angles are equilibria. For what values of  $\Omega$  are these stable? (Hint 1: do all of your equilibria exist for all values of  $\Omega$ ? Hint 2: to investigate stability, write  $\theta = \theta_{eq} + \epsilon$ , where  $\epsilon$  is a small deviation from equilibrium, and find the equation of motion for  $\epsilon$ .)

*Solution:* A point is only at equilibrium if  $\ddot{\theta} = 0$ . Therefore, from our equations of motion, we can derive:

$$\Omega^2 \sin \theta \cos \theta = \frac{g}{R} \sin \theta$$

So we can get:

$$\Omega^2 \cos \theta = \frac{g}{R}$$

In order for this equation to have a real solution for  $\theta$ , we require that

$$\frac{g}{R\Omega^2} \leq 1$$

or equivalently:

$$\Omega \geq \sqrt{\frac{g}{R}}$$

This will admit solutions  $0 \leq \theta < \pi/2$ . The upper bound is not an equality since  $\theta = \pi/2$  is clearly not a stable equilibrium. Now we ask, what happens when  $\Omega < \sqrt{\frac{g}{R}}$ ? In this case, we need to look back at our original equation:

$$\Omega^2 \sin \theta \cos \theta = \frac{g}{R} \sin \theta$$

If  $\Omega^2 < \frac{g}{R}$  in this equation, then we'd require that  $\cos \theta > 1$ , which is impossible. However,  $\theta = 0$  is a solution here, since  $\sin \theta = 0$ . Therefore, we have the solutions:

$$\theta = \begin{cases} \cos^{-1} \left( \frac{g}{R\Omega^2} \right) & \left( \Omega^2 \geq \frac{g}{R} \right) \\ 0 & \left( \Omega^2 < \frac{g}{R} \right) \end{cases}$$

Now we look for stability. To do so, we consider the hint: for a given  $\theta_{eq}$ , we consider  $\theta = \theta_{eq} + \epsilon$ . Doing so, we get:

$$\ddot{\theta}_{eq} + \ddot{\epsilon} = \Omega^2 \sin(\theta_{eq} + \epsilon) \cos(\theta_{eq} + \epsilon) - \frac{g}{R} \sin(\theta_{eq} + \epsilon)$$

Since this is an equilibrium point, we have  $\ddot{\theta}_{eq} = 0$ . We then use the angle summation formulas and combine them with the fact that  $\epsilon$  is small to get:

$$\begin{aligned} \ddot{\epsilon} &= \Omega^2 (\sin \theta_{eq} + \epsilon \cos \theta_{eq}) (\cos \theta_{eq} - \epsilon \sin \theta_{eq}) - \frac{g}{R} (\sin \theta_{eq} + \epsilon \cos \theta_{eq}) \\ &= \Omega^2 (\sin \theta_{eq} \cos \theta_{eq} - \epsilon \sin^2 \theta_{eq} + \epsilon \cos^2 \theta_{eq} - \epsilon^2 \sin \theta_{eq} \cos \theta_{eq}) - \frac{g}{R} (\sin \theta_{eq} + \epsilon \cos \theta_{eq}) \\ &= \Omega^2 \sin \theta_{eq} \cos \theta_{eq} + \Omega^2 \epsilon \cos(2\theta_{eq}) - \epsilon^2 \sin \theta_{eq} \cos \theta_{eq} - \frac{g}{R} \sin \theta_{eq} - \epsilon \frac{g}{R} \cos \theta_{eq} \end{aligned}$$

First, we neglect the  $\epsilon^2$  term since  $\epsilon$  is small. Then, we have  $\Omega^2 \sin \theta_{eq} \cos \theta_{eq} - \frac{g}{R} \sin \theta_{eq} = 0$  from the equilibrium condition so therefore our equation simplifies to:

$$\ddot{\epsilon} = \Omega^2 \epsilon \cos(2\theta_{eq}) - \epsilon \frac{g}{R} \cos \theta_{eq} = -\epsilon \left( -\Omega^2 \cos(2\theta_{eq}) + \frac{g}{R} \cos \theta_{eq} \right)$$

And so we can get the equation:

$$\ddot{\epsilon} + \left( \frac{g}{R} \cos \theta_{eq} - \Omega^2 \cos(2\theta_{eq}) \right) \epsilon = 0$$

The term in the parentheses here is constant, so this is just the equation for simple harmonic motion. As this is the case, this means that  $\epsilon$  is a stable equilibrium. In fact, we can calculate the angular frequency of this oscillation about  $\theta_{eq}$ :

$$\omega = \sqrt{\frac{g}{R} \cos \theta_{eq} - \Omega^2 \cos \theta_{eq}}$$

This also shows that  $\theta = \pi/2$  is not a stable point, since substituting  $\theta_{eq} = \pi/2$  gives  $\omega = 0$ , which we can interpret as having no simple harmonic motion - in other words, the bead doesn't oscillate, it just falls. □

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## Problem 4

The Lagrangian for a relativistic point particle moving in one dimension is

$$L = -mc^2\sqrt{1 - \dot{x}^2/c^2} - V(x)$$

where  $c$  is the speed of light. Derive the equation of motion and show that it reduces to Newton's equation in the limit  $\dot{x} \ll c$ .

*Solution:* We can just compute the equation of motion:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) &= 0 = -\frac{\partial V}{\partial x} - \frac{d}{dt} \left( -\frac{mc^2}{2\sqrt{1 - \dot{x}^2/c^2}} \cdot -\frac{2\dot{x}}{c^2} \right) \\ &= -\frac{\partial V}{\partial x} - \frac{d}{dt} \left( \frac{m\dot{x}}{\sqrt{1 - \dot{x}^2/c^2}} \right) \\ &= -\frac{\partial V}{\partial x} - \left( \frac{m\ddot{x}\gamma - m\dot{x}\frac{1}{2\gamma} \cdot \frac{2\dot{x}}{c^2}}{\gamma^2} \right) \\ &= -\frac{\partial V}{\partial x} - m\ddot{x} \left[ \frac{1 - \frac{\dot{x}^2}{\gamma^2 c^2}}{\gamma} \right] \end{aligned}$$

In the limit where  $\dot{x} \ll c$ , then we have  $\dot{x}^2/c^2 \rightarrow 0$  and  $\gamma \rightarrow 1$ , so therefore we get:

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

which is exactly Newton's equation. □

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