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Lecture Notes	Abstract Linear Algebra	Edward Frenkel

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1 Formal System of Vector Spaces

Arrived late to class, so what came before this is lost to history.

- A vector space over F (a field) is a set V equipped with 2 functions:
  - Addition: $V \times V \mapsto V$
  - Scalar multiplication: $F \times V \mapsto V$

1.1 Axioms of a Vector Space

- The axioms of vector space are as follows:
  - **Commutativity over addition:** $\forall u, v \in V, u + v = v + u$.
  - **Associativity under addition:** $\forall u, v, w \in V, (u + v) + w = u + (v + w)$.
  - **Associativity under Multiplication:** $(ab)v = a(bv)$.
  - **Additive Identity:** There exists a "zero element", such that $v + 0 = v$ for any arbitrary v.
  - **Additive Inverse:** $\forall v \in V, \exists w \in V$ such that $w + v = 0$.
  - **Multiplicative Identity:** There is an element 1 such that $1 \cdot v = v$
  - **Distributive properties:** $(a + b)v = av + bv$, and $a(u + v) = au + av$.

1.2 Theorems

Theorem 1.1. *Let V be a vector space over F. If $0 \in V$ and $0' \in V$ both satisfy Axiom 3, then $0 = 0'$.*

Proof. Our proof consists of a list of sentences:

S1) Use Axiom 3: $v + 0 = v,\ \forall v \in V$.

S2) Set $v = 0' : 0' + 0 = 0'$

S3) Use Axiom 1: $u + v = v + u$

S4) Replace $u = 0, v = 0' : 0' + 0 = 0 + 0'$

S5) Use Axiom 3, but for $0' : v + 0' = v,\ \forall v \in V$

S6) Substitute $v = 0$: $0 + 0' = 0$

S7) Combine S2 and S4: $0 + 0' = 0'$

S8) Combine S7 and S6: $0' = 0$.

□

Note that here, we're not proving that $0 = 0'$, but instead that under the assumption that they both satisfy Axiom 3, then $0 = 0'$. The statement "if" is actually the sentence that provides us the axiom, since it tells us that we're living in a world where that assumption holds true.

Theorem 1.2. *Let v be a vector space over F and $v \in V$. If $w \in V$ and $w' \in V$ both satisfy axiom 4, then $w = w'$.*

Proof. Again, we use sentences, except we'll be a bit more concise this time:

S1) Use Axiom 3 for our specific $v : w + 0 = w$

S2) Substitute $v + w'$ for 0, since we know that w' satisfies Ax. 4: $w + (v + w') = w + 0$

S3) Associativity: $(w + v) + w' = w + 0 = w'$

S4) Hence, $w + v = 0$, so $w = w'$.

□

2 Subsets

Let's look back at what we've defined so far:

- Sets, fields, vector spaces: these are groups of objects that follow a certain pattern, becoming increasingly specific.
- We also defined the notion of a subset: given two sets $T_1 \subset S$ and $T_2 \subset S$, then we also investigated the union $T_1 \cup T_2$ and intersection $T_1 \cap T_2$ of T_1 and T_2.
- Within these definitions, we've implicitly defined the idea of a "sub" – the idea that a set can be contianed within another set.

Sets on their own have no real structure, so it's fairly difficult to really study them. Instead, we will study sets with *some* structure, such as a field. How would we define subsets for a field F? Well, we'd define them to have the same structure as F: being euipped with two operations $+, \cdot$, and that it's closed under these operations.

An example of a subfield is the reals: $\mathbb{R} \subset \mathbb{C}$. Complex numbers are of the form $a + bi$, whereas reals are written as $a + 0i$, a subset of $\mathbb{C}$. The reals also satisfies the notion of a field, since the sum and product of two real numbers is also a real number.

Our interest is to define the notion of a *subspace*, for vector spaces. It's defined as follows:

Definition 2.1. *Let V be a vector space over F. A subset $U \subset V$ is called a subspace of V if it is preserved by the operations of addition and scalar multiplication.*

Explicitly, this means:

$$\forall v, w, \in U, v + w \in U$$
$$\forall v \in U, \lambda \in F, \lambda v \in U$$

Consequently, this means that U is also a vector space over F (the same F), with respect to these operations. For instance, first the set of real tuples $\mathbb{R}^2 = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$. Now, consider the set $U = (x_1, 0) | x_1 \in \mathbb{R}$. It's easy to show U is preserved under addition and scalar multiplication, then U is indeed a vector space. Further, we can also show that $U \subset R$, so therefore U is a valid vector subspace of $\mathbb{R}^2$.

Also, trivially, given a vector space V, then V is also its own subspace.

Theorem 2.1. *A subset $U \subset V$ (V is the same as defined earlier) is a subspace if and only if the following three conditions are satisfied:*

a) The zero element $0 \in V$ is also in U.

b) For any $u, v \in U$, then $u + v \in U$.

c) For all $v \in U$ and $\lambda \in F$, then $\lambda v \in U$.

How is this statement different from the definition stated earlier? The latter two conditions are the same, but the first condition of the zero element is the distinction. Earlier, to prove a vector subspace, what we'd have to do is prove its closedness, then prove all six axioms of a vector space. However, we no longer need to do all that – all we need now is just to show that the zero element exists within U.