Header styling inspired by CS 70: https://www.eecs70.org/

Schroeder 7.55

Suppose that the concentration of infrared-absorbing gases in earth's atmosphere were to double, effectively creating a second "blanket" to warm the surface. Estimate the equilibrium surface temperature of the earth that would result from this catastrophe. (Hint: First show that the lower atmospheric blanket is warmer than the upper one by a factor of $2^{1/4}$. The surface is warmer than the lower blanket by a smaller factor.)

Problem 2

While the Debye results were obtained assuming a linear dispersion $\omega = v|\mathbf{k}|$, a more accurate description of a phonon in a cubic crystal is the frequency relation

$$\omega(\mathbf{k}) = \frac{v}{a}\sqrt{6 - 2\cos(k_x a) - 2\cos\left(k_y a\right) - 2\cos(k_z a)}$$

where a is the lattice spacing of the crystal.

a) Sketch $\omega(k_x, 0, 0)$ across the Broullin zone, and use a Taylor expansion to show the phonon velocity is v.

b) Within the Debye approximation developed in lecture / Schroeder, what is the debye temperature T_D and the expected heat capacity as $T \to 0$ and $T \to \infty$?

c) The heat capacity of the phonons takes the general form $C_V = 3\sum_{\mathbf{k}} f(\mathbf{k})$. What is f in terms of $\hbar\omega(\mathbf{k})$ and $k_B T$?

d) When the system is placed on a cube of linear dimension $L = aN$, there are N^3 terms in $\sum_{\mathbf{k}}$. Using the result of the previous question, write a Python script to compute $C_V(T, N)$ as such as sum, working in units where $a = v = \hbar = k_B = 1$.

e) Use the script to plot $C_V(T, N)/N^3$ for $N = 40$, $0 < T < 5$. Annotate the graph with your prediction of T_D and the high low limits of C_V. Do they agree?

f) Strictly speaking, Debye's T^3 law only holds when $L \to \infty$. For finite L, for what $T < T_L$ do you expect to see deviations? Can you see this effect in your result for the previous part?

Problem 3

Consider a gas of non=interacting spin 1 bosons, each subject to a Hamiltonian

$$\mathcal{H}_1(\mathbf{p}, s_z) = \frac{p^2}{2m} - \mu_0 s_z B$$

where $\mu_0 = e\hbar/mc$ and s_z takes three possible values of $(-1, 0, 1)$. (The orbital effect, $\mathbf{p} \to \mathbf{p} - \epsilon\mathbf{A}$ has been ignored.) Denote $n = N/V$ to be the total gas density.

a) In a grand canonical ensemble of chemical potential μ, what are the average occupation numbers $\{\overline{n}_+(\mathbf{k}), \overline{n}_0(\mathbf{k}), \overline{n}_-(\mathbf{k})\}$ of one-particle states of wavenumber $\mathbf{k}/\hbar$?

b) Calculate the average total numbers $\{N^+, N^0, N^-\}$ of bosons with the three possible values of s_z.

c) Wrie down the expression for the magnetization $M(T, \mu) = \mu_0(N_+ - N_-)$, and by expanding the result for small B find the zero field suscptibility $\chi(T, \mu) = \partial M/\partial B|_{B=0}$

d) For $B = 0$, find the high temperature expansion for $z(\beta, n) = e^{\beta u}$, correct to second order in n. Hence obtain the first correction from quantm statistics to $\chi(T, n)$ at high temperatures.

e) Find the temperuatre $T_c(n, B = 0)$ of Bose-Einstein condensation. What happens to $\chi(T, n)$ on approaching $T_c(n)$ from the high=temperature side?

f) What is the chemical potential μ for $T < T_c(n)$, at a small but finite value of B? Which one-particle state has a macroscopic occupation number?

g) Find the spontaneous magnetization

$$M(T, n) = \lim_{B\to 0} M(T, n, B)$$

Problem 5

Electromagnetic radiation at temperature T_i fills a cavity of volulme V. If the volume of the thermally insulated cavity is expanded quasistatically to a volume $8V$, what is the final temperature T_f? Neglect the heat capacity of the cavity walls.

Problem 7

a) Write the integral for the number of bosons in the excited energy states N_e in a one-dimensional gas of non-interacting bosons with the usual dispersion $\epsilon = \frac{p^2}{2m}$.

b) Argue that your result implies the absence of a BEC in 1D. Hint: Show that in this case the number equation can always be satisfied with “fugacity” $e^{\beta\mu} < 1$, and explain how this implies the absence of a BEC.

Problem 8

Consider bosons moving in a 3D harmonic potential, with single particle energies $E = \frac{p^2}{2m} + \frac{kr^2}{2} = \hbar\omega(n_x + n_y + n_z + 3/2)$ with $n_{x/y/z} = 0, 1, 2, \ldots$. The integers n_i then replace the momenta $\mathbf{k}$ when summing over single-particle states. For simplicity, for the rest of this problem we will subtract off $\frac{3}{2}\hbar\omega$ from E, so that the ground state has energy $E_0 = 0$. As we'll see, the nice thing about this version of the BEC problem is that it is straightforward to compute the thermodynamics via a summation, so we can skip the approximation inherent in replacing sums by integrals $\sum_{E_n} \approx \int dE g(E)$.

a) Defining $n = n_x + n_y + n_z$, give a pictoral (or rigorous) argument that the degeneracy of level $E_n = \hbar\omega n$ is $g(n) = (n+2)(n+1)/2$. It will be sufficient to show it is true just for e first couple n.

b) Write a Python or Mathematica function to evaluate $N(T, \mu) = \sum_{n=0}^{\infty} \frac{g(n)}{e^{\beta(E_n - \mu)} - 1}$. To keep things simple, henceforth we'll choose units in which $\hbar\omega = k_B = 1$.

   Hint: To evaluate the sum, in practice you'll cutoff the series at some large enough n_*, $\sum_{n=0}^{\infty} \approx \sum_{n=0}^{n_*}$. Include some logic to determine a "good enough" value of $n_*(T, \mu)$.

c) Now write a Python or etc. function whihc evaluates $\mu(T, N = 2000)$. Plot the result for $1 \leq T \leq 20$. As a check of your result, compare with the classical expectation for $T \to \infty$ (Midterm Problem 2) and the low-T expectation $\mu \approx -k_B T/N$.

   Hint: I would do it like this. μ is implicitly defined by the condition $N(T, \mu) - 2000 = 0$. To find the μ which satisfies this condition, you can apply `sp.optimize.root_scalar` to the function $f(\mu) = N(T, \mu) - 2000$. This function requires a "bracket", which means an interval $\mu \in [a, b]$ in which the zero exists. $a = -40$ will be sufficient for this problem and for b use your knowledge of the low-T limit.

d) Now that we know $\mu(T, N = 2000)$, use the Bose distribution to plot the occupation of the $n = 0, 1, 2, 3$ states (not including $g(n)$) for $1 \leq T \leq 20$. Do you find evidence for a BEC transition? At approximately what T?