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HW 08

Electromagnetism and Optics II

Physics 110B

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Collaborators

I worked with **Teja Nivarthi** on this assignment.

Problem 1

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t) = kt$$

for $t > 0$. Find the electric and magnetic fields generated.

Solution: The formula for the vector potential is:

$$\mathbf{A}(s,t) = \frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}\,dz$$

The example has already established for us that we only need to worry about the portion where

$$|z| \leq \sqrt{(ct)^2 - s^2}$$

because of causality, so we will do the same. We will also restrict the integral to be between 0 and this number, due to symmetry. The only difference between this problem and the example is that instead of $I = I_0$, we have $I = kt$. This means that $I(t_r) = kt_r = k(t - \mathscr{r}/c)$. Therefore, the integral we need to calculate is:

$$\mathbf{A} = \frac{\mu_0 k}{2\pi}\hat{\mathbf{z}}\int_0^{\sqrt{(ct)^2-s^2}}\frac{t}{\mathscr{r}} - \frac{1}{c}\,dz$$

We then substitute $\mathscr{r} = \sqrt{s^2 + z^2}$, and compute the integral via Mathematica. This gives:

$$\mathbf{A} = \frac{\mu_0 k}{2\pi}\hat{\mathbf{z}}\left[t\ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right) - \frac{1}{c}\sqrt{(ct)^2 - s^2}\right]$$

Now, from here, we can just compute $\mathbf{E} = -\partial_t\mathbf{A}$ and $\mathbf{B} = \nabla\times\mathbf{A}$. Again, Mathematica does the heavy lifting here:

$$\mathbf{E} = -\frac{\mu_0 k}{2\pi}\ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right)\hat{\mathbf{z}}$$

Then, the $\mathbf{B}$ field is calculated using $\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial s}\boldsymbol{\phi}$, so that just amounts to taking the s-derivative of the expression for $\mathbf{A}$. This gives us:

$$\mathbf{B} = \frac{\mu_0 k\boldsymbol{\phi}}{2\pi sc}\sqrt{(ct)^2 - s^2}$$

□

(b) Do the same for the case of a sudden burst of current:

$$I(t) = q_0\delta(t)$$

Solution: The delta function makes the integrals easier here. Now, the integral becomes $\delta(t_r)$, so the overall integral is:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty} \frac{q\delta(t_r)}{\mathscr{r}}\,dz = \frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty} \frac{q\delta\left(t - \frac{\mathscr{r}}{c}\right)}{\mathscr{r}}\,dz$$

The problem with this integral now is that we need to change our dummy variable to be in terms of $\mathscr{r}$, which we can do with a change of variable. If we let $z = \sqrt{\mathscr{r} - s^2}$, then:

$$dz = \frac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2 - s^2}}$$

So now, with this change of variables, we are now ready to evaluate the integral. The bounds for the integral also change, since a range from $z \in (0, \infty)$ corresponds to an interval of $\mathscr{r} \in [s, \infty)$. Therefore, we have:

$$\mathbf{A} = \frac{\mu_0 q_0 \hat{\mathbf{z}}}{2\pi}\int_{s}^{\infty} \frac{\delta\left(t - \frac{\mathscr{r}}{c}\right)}{\sqrt{\mathscr{r}^2 - s^2}}\,d\mathscr{r} = \frac{\mu_0 q_0}{2\pi}\frac{1}{\sqrt{(ct)^2 - s^2}}$$

Then, $\mathbf{E}$ is the time derivative of this, so:

$$\mathbf{E} = -\frac{\mu_0 q_0}{2\pi}\frac{2c^2 t}{[(ct)^2 - s^2]^{3/2}} \cdot -\frac{1}{2}\hat{\mathbf{z}} = \frac{\mu_0 q_0}{2\pi}\frac{c^2 t}{[(ct)^2 - s^2]^{3/2}}\hat{\mathbf{z}}$$

Then, $\mathbf{B} = -\frac{\partial A_z}{\partial s}\boldsymbol{\phi}$, so:

$$\mathbf{B} = -\frac{\mu_0 q_0}{2\pi}\frac{-2s}{[(ct)^2 - s^2]^{3/2}} \cdot -\frac{1}{2}\boldsymbol{\phi} = -\frac{\mu_0 q_0}{2\pi}\frac{s}{[(ct)^2 - s^2]^{3/2}}\boldsymbol{\phi}$$

$\square$

Problem 2

A piece of wire bent into a loop, as shown in Fig. 10.5, carries a current that increases linearly with time:

$$I(t) = kt \quad (-\infty < t < \infty)$$

Calculate the retarded vector potential $\mathbf{A}$ at the center. Find the electric field at the center. Why does this (neutral) wire produce an *electric* field? Why can't you determine the *magnetic* field from this expression for $\mathbf{A}$?

Solution: We will split the calculation into three portions: the straight portion and the two circular arcs. For the straight portion, we can write this as:

$$\frac{\mu_0}{4\pi}\int_a^b 2\frac{I(\mathbf{r}, t_r)}{\mathscr{r}}\,d\ell = \frac{\mu_0 k}{2\pi}\hat{\mathbf{x}}\int_a^b \frac{t - \frac{r}{c}}{r}\,dr$$

This integral evaluates to:

$$\mathbf{A}_1 = \frac{\mu_0 k\hat{\mathbf{x}}}{2\pi}\left[t\ln\left(\frac{b}{a}\right) - \frac{b-a}{c}\right]$$

Now, consider the circular arc with radius a. Then, the retarded time $t_r = t - \frac{a}{c}$ since the distance to the origin is constant. Further, we only care about the $\hat{\mathbf{x}}$ direction because the $\hat{\mathbf{y}}$ direction cancels out. So, the integral becomes:

$$\mathbf{A}_2 = \frac{\mu_0}{4\pi}\hat{\mathbf{x}}\int_0^\pi \frac{\mathbf{I}(\mathbf{r}, t_r)}{r} r\sin\theta\,d\theta = \frac{\mu_0 k}{4\pi}\hat{\mathbf{x}}\int_0^\pi \left(t - \frac{a}{c}\right)\sin\theta\,d\theta = \frac{\mu_0 k}{2\pi}\left(t - \frac{a}{c}\right)\hat{\mathbf{x}}$$

The third semicircle is the exact same calculation except the radius is b and the direction ks $-\hat{\mathbf{x}}$, so we have:

$$\mathbf{A}_3 = -\frac{\mu_0 k}{2\pi}\left(t - \frac{b}{c}\right)\hat{\mathbf{x}}$$

Put these together, and you get the full expression for $\mathbf{A}$:

$$\mathbf{A} = \frac{\mu_0 k\hat{\mathbf{x}}}{2\pi}\left[t\ln\left(\frac{b}{a}\right) - \frac{b-a}{c}\right] + \frac{\mu_0 k}{2\pi}\left(t - \frac{a}{c}\right)\hat{\mathbf{x}} - \frac{\mu_0 k}{2\pi}\left(t - \frac{b}{c}\right)\hat{\mathbf{x}}$$

The electric field is just the time derivative of this, which is equal to:

$$\mathbf{E} = -\partial_t \mathbf{A} = -\frac{\mu_0 k}{2\pi}\ln\left(\frac{b}{a}\right)\hat{\mathbf{x}}$$

While you can determine the electric field, the magnetic field is not possible to determine since $\mathbf{B} = \nabla \times \mathbf{A}$, and we only have $\mathbf{A}$ at a single point so we don't have full information to calculate the curl. □

Problem 3

Show that the scalar potential of a point charge moving with constant velocity (Eq. 10.49) can be written more simply as

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1 - v^2\sin^2\theta/c^2}}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{v}t$ is the vector from the present(!) position of the particle to the field point $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Note that for nonrelativistic velocities ($v^2 \ll c^2$),

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

Solution: Let's start with the original expression fro $V(\mathbf{r},t)$, equation 10.49, that the problem refers to:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

so all we need to do is just simplify the stuff under the square root. The term in the square root, after expanding, is:

$$c^4t^2 - 2c^2t\mathbf{r}\cdot\mathbf{v} + (\mathbf{r}\cdot\mathbf{v})^2 + c^2r^2 - c^4t^2 - v^2r^2 + v^2c^2t^2 = (\mathbf{r}\cdot\mathbf{v})^2 + r^2(c^2 - v^2) - 2c^2(\mathbf{v}t\cdot\mathbf{r}) + v^2c^2t^2$$

Now, we plug in $\mathbf{v}t = \mathbf{r} - \mathbf{R}$, giving us:

$$(\mathbf{r}\cdot\mathbf{v})^2 + r^2(c^2 - v^2) - 2c^2((\mathbf{r} - \mathbf{R})\cdot\mathbf{r}) + c^2(\mathbf{r} - \mathbf{R})^2$$

Now, expand:

$$(\mathbf{r}\cdot\mathbf{v})^2 + r^2(c^2 - v^2) - 2c^2(r^2 - \mathbf{R}\cdot\mathbf{r}) + c^2(r^2 - 2\mathbf{r}\cdot\mathbf{R} + R^2) = (\mathbf{r}\cdot\mathbf{v})^2 - v^2r^2 + c^2R^2$$

Rewrite $\mathbf{r} = \mathbf{R} + \mathbf{v}t$, we can expand the first two terms:

$$\begin{aligned}
(\mathbf{r}\cdot\mathbf{v})^2 - v^2r^2 &= ((\mathbf{R} + \mathbf{v}t)\cdot\mathbf{v})^2 - (\mathbf{R} + \mathbf{v}t)^2v^2 \\
&= (\mathbf{R}\cdot\mathbf{v} + v^2t)^2 - v^2(R^2 + 2\mathbf{R}\cdot\mathbf{v}t + v^2t^2) \\
&= (\mathbf{R}\cdot\mathbf{v})^2 + 2\mathbf{R}\cdot\mathbf{v}(v^2t) + v^4t^2 - v^2R^2 - 2v^2(\mathbf{R}\cdot\mathbf{v})t - v^4t^2 \\
&= (\mathbf{R}\cdot\mathbf{v})^2 - R^2v^2
\end{aligned}$$

Now, $\mathbf{R}\cdot\mathbf{v} = Rv\cos\theta$, so therefore:

$$(\mathbf{R}\cdot\mathbf{v})^2 - R^2v^2 = R^2v^2\cos^2\theta - R^2v^2 = R^2v^2(\cos^2\theta - 1) = -R^2v^2\sin^2\theta$$

Finally, we can put this back with what we had before to complete the equation:

$$-R^2v^2\sin^2\theta + c^2R^2$$

Therefore:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{R^2c^2 - R^2v^2\sin^2\theta}}$$

So if we factor out R^2c^2 from the square root, we thne get:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1 - v^2\sin^2\theta/c^2}}$$

as desired. $\square$