Problem 1

Let f be a continuous function from $(0, 1)$. Assume $f(x) < 1$ for any $x \in (0, 1)$, $\lim_{x\to 0} f(x) < 1$ and $\lim_{x\to 1} f(x) < 1$. Prove $\sup_{x\in(0,1)} f(x) < 1$.

Solution: Since $f(x) < 1$ for any $x \in (0, 1)$ then if we pick a subinterval $(a, b) \subset (0, 1)$, then we know that $f(x) < 1$ for any $x \in (a, b)$, and thus $\sup_{x\in(a,b)} f(x) < 1$ since (a, b) is a subinterval of $(0, 1)$. Now, we need to prove that this also holds for the intervals $(b, 1)$ and $(0, a)$. Let's first do it for $(b, 1)$.

Consider the sequence of intervals:

$$\left(b, b + \frac{1-b}{2}\right), \left(b + \frac{1-b}{2}, b + \frac{1-b}{2} + \frac{1 - \frac{1-b}{2}}{2}\right), \ldots$$

where each next interval is defined recursively as $(b_{n-1}, b_{n-1} + \frac{1-b_{n-1}}{2})$. Focusing just on the upper bound, we have the recursive relation $b_n = b_{n-1} + \frac{1-b_{n-1}}{2}$. If we can prove that as $n \to \infty$ that $b_n \to 1$, then we can prove that in the limit, this limit is equivalent to the interval $(b, 1)$. Now let's prove this claim. This sequence b_n is written as:

$$b_n = b_{n-1} + \frac{1 - b_{n-1}}{2} = \frac{1}{2} + \frac{b_{n-1}}{2}$$

So in the limit as $n \to \infty$, then we can actually write:

$$b_\infty = \frac{1}{2} + \frac{b_\infty}{2} \implies \frac{b_\infty}{2} = \frac{1}{2}, b_\infty = 1$$

so therefore we do conclude that $b_n \to 1$ as $n \to \infty$. Then, since each of these intervals (b_n, b_{n+1}) is a subinterval of $(0, 1)$, we know that $\sup_{x\in(b_n,b_{n+1})} f(x) < 1$. This implies that the supremum of the union of all these intervals:

$$S = \bigcup_{n=1}^{\infty} (b_n, b_{n+1})$$

is also less than 1, or equivalently $\sup_{x\in S} f(x) < 1$. The same logic holds for the interval $(0, a)$, where we construct a sequence $a_n = \frac{a_{n-1}}{2}$, and it's clear that regardless of the choice of a_0 that this sequence goes to 0, so the supremum of the unions:

$$S' = \bigcup_{n=1}^{\infty} (a_{n+1}, a_n)$$

is also less than 1, or $\sup_{x\in S'} f(x) < 1$. Now, we can combine all three intervals together. Since the supremum of each interval S, S' and (a, b) is less than 1, then we know that the supremum of the set $A = S \cup S' \cup (a, b)$ also has the property that $\sup_{x\in A} f(x) < 1$, and since $A = (0, 1)$, this implies that $\sup_{x\in(0,1)} f(x) < 1$, as desired. $\square$

Problem 2

Let f be a continuous function from $[0,1]$ to $[0,1]$. Prove: There exists one (or more) fixed point x such that $f(x) = x$. Hint: Consider $g(x) = f(x) - x$.

Solution: Consider the function $g(x) = f(x) - x$. The minimum difference that $f(x)$ can be from x given the problem statement is -1 (just take the minimum value and maximum values), and the maximum difference is 1. Therefore, $g(x)$ maps to a subset of the interval $[-1,1]$. Now, if this interval that $g(x)$ maps to contains 0, then we know that the fixed point we are looking for is the value x_0 such that $g(x_0) = 0$. Now, we prove that it must always contain 0. We proceed by contradiction.

Suppose that the interval $g(x)$ maps onto does not contain 0. Then, this means that the sign of $g(x)$ is either strictly positive or strictly negative. In the case where $g(x) > 0$ for all $x \in [0,1]$, then this implies that $f(x) > x$ on this interval. However, this is impossible: consider the point $x = 1$. The sign of $g(x)$ implies that $f(x) > 1$ for this specific point. This is a contradiction, since the maximum value of f is 1, as indicated by the question.

Now suppose that $g(x) < 0$ for all $x \in [0,1]$. This would then imply that $f(x) < x$ for all x. Now, consider $x = 0$. In this case, the sign of $g(x)$ implies that $f(x) < 0$, which is also impossible since the minimum value of f is 0, given in the problem. Therefore, since neither case works, then $g(x)$ must contain 0, and therefore must contain at least one fixed point. $\square$

Problem 3

Prove that a polynomial function f of odd degree has at least one real root. *Hint:* It may help to consider the first case of a cubic, i.e. $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_3 \neq 0$.

Solution: Since the polynomial is an odd degree, then our goal is to show that regardless of the coefficients, we can choose a value x_1 such that $f(x_1) < 0$ and another point x_2 such that $f(x_2) > 0$. Since f is continuous[1], if we can prove the existence of these two points then we can invoke the intermediate value theorem to prove that there's some $x_1 < c < x_2$ such that $f(c) = 0$.

Now, we aim to prove the existence of x_1 and x_2. To do so, we look at the sign of the leading coefficient. If $a_3 > 0$, then we know that $\lim_{x\to\infty} f(x) = \infty$ (see below for proof). Specifically, this means that for any positive $M > 0$, we know that there exists some x_2 such that $f(x_2) > M$. Therefore, there exists x_2 such that $f(x_2) > 0$. Likewise, we know that $\lim_{x\to-\infty} f(x) = -\infty$, so therefore for any $M < 0$, there exists a value x_1 such that $f(x_1) < M$, and thus $f(x_1) < 0$. Finally, since x_1 and x_2 exist, then by IVT there exists some real value c where $x_1 < c < x_2$ such that $f(c) = 0$.

The same logic works for the second case where $a < 0$. In this case, we know that $\lim_{x\to\infty} = -\infty$, so for any $M < 0$ there exists some $f(x_2) < M$, and hence $f(x_2) < 0$. Likewise, since $\lim_{x\to-\infty} f(x) = \infty$, then for any $M > 0$, there exists some x_1 such that $f(x_1) > M$, hence $f(x_1) > 0$. Therefore, since $f(x_1) > 0$ and $f(x_2) < 0$, and f is continuous, there must exist a point c such that $f(c) = 0$, implying that f has a real root.

I'm going to now prove that the limits as $x \to \pm\infty$, we have $f(x) \to \pm\infty$. Consider the function

$$g(x) = \frac{\text{sgn}(a_n)}{x^{n-1}} f(x) = \text{sgn}(a_n)(a_{n-1} + a_n x)$$

From here, it is clear that depending on the sign of a_n, that the limit as $x \to \pm\infty$ means that $g(x) \to \pm\infty$, and thus $f(x) \to \pm\infty$ as well. □

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[1] I assume that polynomials are continuous, this is also something that's assumed in the lecture notes

Problem 4

Suppose f is continuous on $[0,2]$ and $f(0)=f(2)$. Prove there exist x, y in $[0,2]$ such that $|y-x|=1$ and $f(x)=f(y)$. *Hint:* Consider $g(x)=f(x+1)-f(x)$ on $[0,1]$.

Solution: Following the hint, consider $g(x)=f(x+1)-f(x)$. Now we proceed to show that $g(x)=0$ for some x via contradiction.

Suppose that $g(x) \neq 0$ for all $x \in [0,1]$. Since g is continuous, then this means that either $g(x)>0$ for all $x \in [0,1]$ or $g(x)<0$ for all $x \in [0,1]$. Otherwise, if $g(x)$ flips sign in the interval $[0,1]$, then the intermediate value theorem would tell us that there exists some x_0 such that $g(x_0)=0$, and hence $f(x)=f(y)$ when $|y-x|=1$.

First, suppose $g(x)>0$ for all x. Then, this means that for $x=0$, this gives the relation that $f(1)>f(0)$, and plugging in $x=1$ gives $f(2)>f(1)$. Combining these two inequalities, this gives $f(2)>f(0)$, which is a contradiction.

Likewise, suppose that $g(x)<0$ for all x. Then, this gives $f(1)<f(0)$ for $x=0$ and plugging in $x=1$ gives $f(2)<f(1)$, so combining these two this gives $f(2)<f(0)$, which is also a contradiction.

Therefore, since $g(x)$ cannot strictly be either positive or negative, there must exist a point where $g(x)$ switches sign. Therefore, there exists a point where $g(x)=0$, and thus there exists $x, y \in [0,2]$ such that $|y-x|=1$ and $f(x)=f(y)$. $\square$

Problem 5

Prove the Cauchy condition for the limits of a function: Given $f : A \to \mathbb{R}$ and $c \in A$ is an accumulation point of A. Then, $\lim_{x\to c} f$ exists if and only if the following Cauchy condition holds. For any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x_1, x_2 \in (c - \delta, c + \delta)$, we have

$$|f(x_1) - f(x_2)| < \epsilon$$

Solution: First we prove the forward direction: that $\lim_{x\to c} f(x)$ exists if the Cauchy condition holds. In other words, we prove that $\lim_{x\to c}$ exists if for any $x_1, x_2 \in (c - \delta, c + \delta)$ then $|f(x_1) - f(x_2)| < \epsilon$. In this case, since x_1, x_2 are chosen from the interval $(c - \delta, c + \delta)$, then we know that $0 < |x_1 - c| < \delta$ and $0 < |x_2 - c| < \delta$. Then by the triangle inequality:

$$|f(x_1) - f(c) - f(x_2) + f(c)| \leq |f(x_1) - f(c)| + |f(x_2) - f(c)| < \epsilon$$

This implies that $|f(x_1) - f(c)| < \epsilon$ and $|f(x_2) - f(c)| < \epsilon$, which is the standard statement for continuity.

Now for the reverse direction: if the limit exists, we prove the Cauchy condition holds. Recall the standard definition of the limit: $\forall \epsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta$ means that $|f(x) - f(c)| < \epsilon$. Therefore, the Cauchy condition holds if we just choose two x_1, x_2 such that $x_1 \neq x_2$ that satisfy this continuity statement since x_1, x_2 are within the interval $(c - \delta, c + \delta)$, and again by the triangle inequality,

$$|f(x_1) - f(c)| + |f(c) - f(x_2)| < 2\epsilon \implies |f(x_1) - f(x_2)| < 2\epsilon$$

Since ϵ is arbitrary, we can just choose $\epsilon' = 2\epsilon$ and we get the Cauchy condition. $\square$

Problem 6

Prove: A set A is compact (bounded and closed) if and only if A is sequentially compact, meaning that for any sequences (x_n) of A, there exists a subsequence (x_{n_k}) of (x_n) such that x_{n_k} converges to some point $a \in A$.

Solution: We prove the forward direction: A is sequentially compact if A is compact. Because A is bounded, then every sequence $x_n \in A$ has a convergent subsequence (By Bolzano-Weierstrass). Furthermore, since A is closed, then every sequence *must* converge to a point $a \in A$, since A contains all its limit points.

Now the reverse direction: we prove that if A is sequentially compact, then A is compact. Using the hint from the notes, suppose that A is not compact. This means that either A is not bounded, not closed, or both.

Firstly, if A is not closed, then this implies the existence of some sequence x_n such that it converges to a point $a \notin A$. This contradicts the statement that A is sequentially compact, because we can just choose the subsequence x_{n_k} to be the sequence x_n itself, which does not converge to $a \in A$.

Secondly, if A is not bounded, then this implies that A is not bounded either from above or below. Suppose without loss of generality that A is not bounded from above. Now, define a sequence x_n which is strictly increasing. Then, every subsequence of x_n is also strictly increasing, and therefore there will not exist an M which bounds *any* sequence, because for any M that bounds x_{n_k} up to some n_k, we know that there exists some x_{n_m} where $n_m > n_k$ such that $x_{n_m} - x_{n_k} > M - x_{n_k}$. This implies that if we take the sequence up to x_{n_m}, M no longer bounds the sequence. Therefore, since every subsequence of x_n is unbounded, it must diverge, contradicting the fact that there exists a sequence x_{n_k} that converges to a value $a \in A$. This is a contradiction, hence A must be bounded.

Therefore, A must be closed and bounded, and thus compact by definition. $\square$
