PHYSICS 110A NOTES

Typeset Notes for Physics 110A: Electromagnetism and Optics

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Lecture 13

Lecture 13 was held on **Friday, February 17, 2023** and covered the Separation of Variables technique for solving the Laplace equation.

0.1 Separation of Variables

Our goal is to find a general solution to the equation $\nabla^2 V = 0$. Once we find this solution, we can then just fit them to our boundary conditions in order to solve the potential of our specific system. To solve this system, we can assume that V takes on a nice form, specifically a separable form:

$$V = X(x)Y(y)Z(z)$$

Plugging this into $\nabla^2 V = 0$, we get:

$$0 = YZ\frac{\partial}{\partial x^2}x + XZ\frac{\partial}{\partial y^2}Y + XY\frac{\partial}{\partial z^2}z$$

Diving both sides by XYZ, we then get:

$$\underbrace{\frac{1}{X}\frac{\partial}{\partial x^2}X}_{f(x)} + \underbrace{\frac{1}{Y}\frac{\partial}{\partial y^2}Y}_{g(y)} + \underbrace{\frac{1}{Z}\frac{\partial}{\partial z^2}Z}_{h(z)} = 0$$

Insight: Note that this is not the *most* general form for V. We require that V is separable in order to come up with this equation. Generally, we'd have to plug our equation into some form of computer (i.e. Mathematica or MATLAB) in order to solve this equation. A good example of this is in quantum mechanics, where solutions to the Schrödinger equation for spherically symmetric potentials is

$$\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta, \phi)$$

Notice that in this equation, we actually require that f, g and h be constant. If they were not constant, then for a given point (x_0, y_0, z_0) if we have

$$f(x_0) + g(y_0) + h(z_0) = 0$$

and we then proceed to vary x, $f(x) + g(y_0) + h(z_0) = 0$ does not necessarily hold, but it is a requirement of the Laplace equation that it holds over all space, and thus f, g and h must be constant. Therefore, we now have 3 ordinary differential equations:

$$\frac{1}{X}\frac{\partial}{\partial x^2}X = C_1$$
$$\frac{1}{Y}\frac{\partial}{\partial y^2}Y = C_2$$
$$\frac{1}{Z}\frac{\partial}{\partial z^2}Z = -(C_1 + C_2)$$

It turns out that although not all solutions to $\nabla^2 V = 0$ has the form of $X(x)Y(y)Z(z)$, the general solution often forms a complete basis so that any arbitrary solution can written as a linear combination of these separable solutions.

0.1.1 Linear Algebra Aside

Suppose we have a set of functions $\{\phi_i\}$ that form a complete basis. This means that any function f can be written as a linear combination of ϕ_i:

$$f = \sum_i c_i \phi_i$$

Insight: Note that this summation notation can also be converted into an integral if we're working in a continuous basis. This is the case in quantum mechanics when we are working in a continuous basis, such as momentum space, where we can write our wavefunction in terms of a Fourier transform:

$$\psi(x) = \frac{1}{2\pi} \int \phi(k) e^{ikx} dk$$

Regardless of whether this is treated as a sum or integral, we assume that the ϕ_i terms are all orthogonal to one another.

Notation (Orthogonality): A set of functions $\{\phi_i\}$ are orthogonal to one another if the inner product is written as:

$$\langle \phi_i | \phi_j \rangle = N_i \delta_{ij} = \begin{cases} 0 & i \neq j \\ N_i & i = j \end{cases}$$

To compute the inner product, we compute the integral:

$$\langle \phi_i | \phi_j \rangle = \int d^3x \phi_i^*(x, y, z) \phi_j(x, y, z)$$

We can also find the coefficients c_i by taking the inner product of our function and the ϕ_i we're interested in:

$$\langle \phi_i | f \rangle = \left\langle \phi_i \middle| \sum_i c_i \phi_j \right\rangle = \sum_j c_j \langle \phi_i | \phi_j \rangle = c_i N_i$$

And so therefore:

$$c_i = \frac{1}{N_i} \langle \phi | \phi \rangle$$

Insight: Intuitively, this process of finding the inner product is identical to what we would to to find a given component of a vector $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$: we'd take the dot product $v_y = \hat{y} \cdot \vec{v}$. In this case with functions, we are essentially just generalizing this process, but the underlying principles remain the same.

Now we will illustrate how to utilize these principles in an example:

Example 0.1

Consider two planes bounded by the yz-plane, with one plane sitting at $y = 0$ and the other at $y = a$. The planes are connected together by a thin strip on the yz-plane, with a potential $V(y)$. The top and bottom planes have potential $V = 0$ and the potential along the x-direction V_x goes to 0 as $x \to \infty$. Find the solution to V in the region $y = 0$ and $y = a$.
[INSERT TIKZ HERE]
There is translational symmetry in the $\hat{z}$ direction, so therefore the system has no $\hat{z}$ dependence. Thus, we can write $V = V(x, y)$. Using the method of separation of variables, we can write:

$$\frac{1}{X}\partial_x^2 X = \text{const.}$$
$$\frac{1}{Y}\partial_y^2 Y = \text{const}$$

Assuming real functions for X and Y, these differential equations mean that depending on the sign of the constant, our solutions are either oscillatory or exponential. More specifically, if the constant is negative then we get oscillatory solutions, and exponential ones if the constant is positive.
Note that we want $V = 0$ and $y = 0$ and $y = a$, so therefore it makes sense that the $\hat{y}$ direction should have oscillatory solutions. And since the sum of these two must be zero, it forces the solution to X to be an exponential function. However, this also makes sense intuitively, since we require that $V_x \to 0$ as $x \to \infty$. So now we have:

$$\frac{1}{X}\partial_x^2 X = k^2$$
$$\frac{1}{Y}\partial_y^2 Y = -k^2$$

And so therefore:

$$x = Ae^{kx} + Be^{-kx}$$
$$Y = C\sin(ky) + D\cos(ky)$$

Now we apply the boundary conditions, which will fix the solution to our given problem. We have that $V \to 0$ as $x \to \infty$, so therefore this kills the growing exponential term, so $A = 0$. Likewise, we have $V = 0$ at $y = 0$, so this forces the cosine term to disappear to, so $D = 0$. Therefore, we have the equations:

$$x = Be^{-kx}$$
$$Y = C\sin(ky)$$

So $V(x, y) = BCe^{-kx}\sin(ky)$. Now, we can let $\mathcal{A} = BC$, so we can change our equation into $V(x, y) = \mathcal{A}e^{-kx}\sin(ky)$. Then, we impose the condition that $V = 0$ at $y = a$, so we get:

$$0 = e^{-ka}\sin(ka) \implies k = \frac{n\pi}{a}, n = 1, 2, 3, 4, \ldots$$

And so therefore we can write:

$$V(x, y) = \sum_{n=1}^{\infty} A_n e^{n\pi x/a}\sin\left(\frac{n\pi y}{a}\right)$$

Lecture 14

Lecture 14 was held on **Wednesday Feb 22nd, 2022**. It finished the example problem that we started last lecture and also covered the **Laplace Equation in Spherical Coordinates**

1.1 Last Time: Two Planes

Recall from last time that we have the solution

$$V(x,y) = \sum_n c_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

which we've obtained by imposing the following conditions:

1. $V = 0$ at $y = 0$, getting rid of the cosine term
2. $V = 0$ at $y = a$, giving us the quantization of the wavelength
3. $V \to 0$ as $x \to \infty$, giving us no exponential growth.

Now we will impose our final boundary condition: at $x = 0$, we have $V = V_0(y)$. So plugging in $x = 0$, we see that:

$$V(0,y) = \sum_n c_n \sin\left(\frac{n\pi y}{a}\right)$$

Insight: Note that this is actually only possible because the set of functions form a complete basis, and they are orthonormal, so the sum of sines can represent any arbitrary function $V_0(y)$. Again, recall the orthonormality condition:

$$\left\langle \sin\left(\frac{n\pi y}{a}\right) \middle| \sin\left(\frac{m\pi y}{a}\right) \right\rangle \equiv \int_0^a dy \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) = \begin{cases} 0 & n \neq m \\ \frac{a}{2} & n = m \end{cases}$$

and so we can see from this that we can grab each coefficient by writing:

$$c_n = \frac{2}{a} \left\langle \sin\left(\frac{n\pi y}{a}\right) \middle| V_0(y) \right\rangle = \frac{2}{a} \int_0^a c_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Insight: We also see this sinusoidal behavior with other partial differential equations as well:

$$\left(\partial_x^2 - \frac{1}{v^2}\partial_t^2\right)\phi = 0 \qquad \text{(Wave Equation)}$$

$$\partial_t \phi = \nabla^2 \phi \qquad \text{(Flow Equation)}$$

These equations all share the property that they have a second order differential which equals itself, which generally means that the solution is some form of complex exponential (e^{ikx} of some kind). This is also useful because we know that e^{ikx} terms form a complete orthonormal basis, so we can model any arbitrary function with these exponentials as well.

1.2 Laplace's Equation in Spherical Coordinates

Here, the differential form of Laplace's equation remains the same $\nabla^2\phi = 0$, but the Laplacian is different since we're now working in spherical coordinates:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial\phi^2}$$

For simplicity, we assume azimuthal symmetry, so V does not depend on the ϕ coordinate. If it did, then we'd have to impose the boundary condition that $\phi(0) = \phi(2\pi)$, which would imply a solution of the form $e^{in\phi}$. Because there is no ϕ coordinate, then we know that $\frac{\partial V}{\partial\phi} = 0$. Writing out the other two terms:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$

We now consider the separable solution $V(r,\theta) = R(r)\Theta(\theta)$. Plug this ansatz back into the the equation and multiplying by $r^2/R\Theta$, we get:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\sin^2\theta\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = 0$$

And so just like last time, each one of these terms must be constant, otherwise we can fix one and vary the other which wouldn't give us 0 in general. For later convenience, we will define the constant as $l(l+1)$, so we write:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = l(l+1)$$
$$\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -l(l+1)$$

Looking at the radial equation, we have:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = l(l+1)R$$

This differential equation asks us to take the function R and take the derivative twice, multiply by r^2 and it's supposed to give us R back. This hints at a solution of the form $R = r^m$, since the exponent is returned after differentiating twice. So plugging this back in:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}r^m\right) = m(m+1)r^m$$

Giving us

$$m(m+1)r^m = l(l+1)r^m$$
$$m^2 + m - l(l+1) = 0$$
$$(m-l)(m+(l+1)) = 0 \implies m = l \text{ or } m = -(l+1)$$

And so therefore we can write R as a linear combination of these two values for m:

$$R(r) = \sum_l \left(A_l r^l + B_l\frac{1}{r^{l+1}}\right)$$

Insight: So far, l could still take on any value, so even though this is a summation notation we haven't set any restriction on l. It's the angular equation that will set the restriction on l.

So now let's look at the angular equation:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = -l(l+1)\sin\theta\Theta$$

This differential equation, unlike the previous one, is highly nontrivial to solve. However, this differential equation has been solved previously and is well known as the Legendre equation, which has the form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_n(x)}{\mathrm{d}x}\right]+n(n+1)P_n(x)=0$$

the solution to this is given by Rodrigues' formula, where

$$P_n(x)=\frac{1}{2^n n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n(x^2-1)^n$$

Notation (Legendre Polynomials): We denote $P_n(x)$ as the n-th Legendre polynoimal.

In our case, we have $x=\cos\theta$ and $\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{1}{-\sin\theta}$, so therefore the Legendre equation reads:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\sin\theta\frac{\mathrm{d}P_n(\cos\theta)}{\mathrm{d}\theta}\right]+n(n+1)P_n(\cos\theta)=0$$

which is exactly what our original differential equation looks like! Therefore, we now know that $\Theta(\theta)=P_l(\cos\theta)$.

Insight: As we can see, it is merely the fact that we would like non-singular solutions (i.e. non-constant) solutions between $\theta=0$ and $\theta=\pi$ that constrains l to be a nonnegative integer. Note that 0 is allowable in this case because the Rodrigues' formula allows us to generate $P_0(x)$.

Combining this with our solution to the radial equation, we get the full solution as:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+B_l\frac{1}{r^{l+1}}\right)P_l(\cos\theta)$$

This is the solution to the electrostatic potential of a charge distribution, given azimuthal symmetry. As a recap of Legendre polynomials, we have:

Definition: The Legendre polynomials are solutions to the Legendre equation, and are given by the formula:

$$P_l(x)=\frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l(x^2-1)^l$$

The first few Legendre polynomials are:

$$\begin{aligned}
P_0(x)&=1\\
P_1(x)&=x\\
P_2(x)&=\frac{1}{2}(3x^2-1)\\
P_3(x)&=\frac{1}{2}(5x^3-3x)\\
P_4(x)&=\frac{1}{8}(35x^4-30x^2+3)
\end{aligned}$$

A cool thing to note is that when l is odd, we only get odd exponents and we also only get even ones when l is even. This means that odd values of l correspond to antisymmetric solutions and even ones correspond to symmetric solutions, a property we will explore further in next lecture.

Lecture 15

2.1 Last time: Solution to electrostatic potential

Last time, we saw that the solution to the electrostatic potential $V(r,\theta)$ assuming azimuthal symmetry is of the form:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+B_l\frac{1}{r^{l+1}}\right)P_l(\cos\theta)$$

Let's see how this works with an example.

Example

Suppose we have a conductor placed in a uniform field $\vec{E}=E_0\hat{z}$. We want to find the electrostatic potential everywhere in this system.
To start, we must first identify the boundary conditions:

- We expect the electric field $\vec{E}\to\vec{E}_0$ as $r\to\infty$. So, if we set $V=0$ at $z=0$, then we get that $V\to -E_0 z=-E_0 r\cos\theta$ as $r\to\infty$. This can be shown explicitly by considering a line integral $V=\int\vec{E}\cdot dl$, starting from $z=0$ going to $z=z_0$ for some z.
- We require $V=0$ at $r\le R$, due to the properties of a conductor.

Now, we are ready to impose the boundary conditions onto our general solution. Inside the sphere, we don't expect V to blow up, so therefore all $B_l=0$ in the limit that $r\to\infty$. Further, since $V\to E_0 r\cos\theta$ as $r\to\ \infty$, then we only keep the first-order $\cos\theta$ term, so we only keep $A_1=-E_0$. Now matching the second boundary condition;

$$A_l R^l+B_l\frac{1}{R^{l+1}}=0\implies B_l=-A_l R^{2l+1}$$

And since we know that $A_l=0$ except A_1, then we only keep B_1 as a result. Solving, we get $B_1=E_0R^{2l+1}$. Therefore, after matching boundary conditions we get the solution

$$V(r,\theta)=\left(-E_0 r+E_0\frac{R^3}{r^2}\right)\cos\theta$$

2.2 Multipole Expansion

For simplicity, let's only consider two charges q placed on the z axis, at a distance d away from each other. Now, we could write the potential explicitly:

$$V=\frac{q}{4\pi\epsilon_0}\left(\frac{1}{\mathscr{r}_+}-\frac{1}{\mathscr{r}_-}\right)$$

While this form is ok, we are often interested in the limit where we are very far from the source charges, or in other words when $\mathscr{r}_+$ and $\mathscr{r}_-$ are much larger than d. In this limit, a simplification can be made. First, we write

$$\mathscr{r}_+=\sqrt{r^2+\left(\frac{d}{2}\right)^2\mp dr\cos\theta}=r\sqrt{1\mp\frac{d}{r}\cos\theta+\left(\frac{d}{2r}\right)^2}$$

And a similar expression for $\mathscr{r}_-$. Therefore,

$$\frac{1}{\mathscr{r}_\pm} = \frac{1}{r}\left[1 \mp \frac{d}{r}\cos\theta + \left(\frac{d}{r}\right)^2\right]^{1/2} \approx \frac{1}{r}\left[1 \mp \left(-\frac{1}{2}\frac{d}{r}\cos\theta\right) + O\left(\frac{d^2}{r^2}\right)\right] \approx \frac{1}{r}\left[1 \pm \frac{d}{2r}\cos\theta\right]$$

where in the last step we use the expansion $(1+x)^n \approx 1 + nx + \ldots$. Now, applying this for our potential, we get:

$$\begin{aligned} V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right) &\approx \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r}\left[1 + \frac{d}{2r}\cos\theta - \left(1 - \frac{d}{2r}\cos\theta\right)\right] \\ &= \frac{q}{4\pi\epsilon_0}\frac{d}{r^2}\cos\theta \\ &= \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \end{aligned}$$

Here, we use the definition $p = qd$, where p represents the **dipole moment**. We will explore more about the dipole moment next time.