## Problem 1 (25 pts)

In this class (and in pretty much all of physics) it is key to be proficient at computing Gaussian integrals. Consider

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2}$$

a) Show that

$$I^2 = \int_{\mathbb{R}} dx dy e^{-\frac{1}{2}(x^2+y^2)}$$

b) Compute $I^2$ by expressing this integral in polar coordinates. Conclude that

$$I = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$$

c) Show that

$$I = \int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}$$

d) Show that

$$I = \int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2+bx} = e^{\frac{1}{2a}b^2}\sqrt{\frac{2\pi}{a}}$$

e) Show that

$$I = \int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2} x^2 = \frac{1}{a}\sqrt{\frac{2\pi}{a}}$$

Hint: Compute

$$\left[\frac{\partial^2}{\partial b^2}\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2+bx}\right]_{b=0}$$

by both differentiating under the integral sign and explicitly computing its derivatives using the result in d).

## Problem 2 (20 pts)

Consider a system of $N$ particles with spin. Label the particles using an index $i = 1, \ldots, N$ so that the $i$-th particle has spin $s_i$. Unlike the systems we have encountered so far, in this system the possible values for each spin are $s_i = -1, 0, 1$. The energy of this system is given by

$$E = D \sum_{i=1}^{N} s_i^2$$

In other words, when the spin is $\pm 1$, this costs the system an energy $D$, while whenever the spin is $0$, this spin does not contribute to the energy of the system.

a) Explain why the number of accessible states that the system when it has an energy $E$ is

$$\Omega(N, E) = \binom{N}{E/D} 2^{E/D}$$

b) Compute the entropy of the system $S(N, E)$ in the limit of many particles and high energies $E \gg D$.

c) Compute the temperature of this system as a function of the energy and the number of particles. Can the temperature of this system be negative?

d) Obtain the energy for the system as a function of temperature. Discuss the low and high temperature limits. What happens to the entropy i these limits? What is the physical meaning of this?

## Problem 3 (20 pts)

Consider an ideal gas undergoing a process described by the fact that $pV^2$ is a constant. What is the molar heat capacity of this process?