Collaborators

I worked with **Andrew Binder, Adarsh Iyer** and **Aren Martinian** to complete this assignment.

Problem 1 - A Driven Oscillator

An oscillator of mass m, spring constant k and damping coefficient b is subject to a time-dependent driving force:

$$F(t) = A\cos(\omega t) + B\cos(2\omega t)$$

(a) Find the long term behavior $x(t)$, the oscillator's position.

Solution: The long term behavior of the oscillator is given by $x_p(t)$, which we saw in class to have solutions of the form:

$$x(t) = A_1\cos(\omega t - \delta_1) + A_2\cos(\omega t - \delta_2)$$

where we also have the definitions that

$$A_1 = \frac{C}{[(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2]}, \quad A_2 = \frac{D}{[(\omega_0^2 - 4\omega^2)^2 + 16\omega^2\beta^2]}$$

and

$$\delta_1 = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \qquad \delta_2 = \tan^{-1}\left(\frac{4\beta\omega}{\omega_0^2 - 4\omega^2}\right)$$

And so therefore we write

$$x(t) = A_1\cos(\omega t - \delta_1) + A_2\cos(\omega t - \delta_2)$$

using $A_1, A_2, \delta_1, \delta_2$ as defined above. □

(b) Now suppose that $(b/2m)^2 < k/m$, so that the oscillator is underdamped. Given initial conditions $x(0) = 0$ and $\dot{x}(0) = v$, explain how to obtain equations that determine $x(t)$ fully for $t \geq 0$. Explain why $x(t)$ tends towards your answer in (a).

Solution: The homogeneous solution gives us the transient motion, and we know from lecture that $x_h(t)$ has the form:

$$x_h(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. Equivalently, we acn also rewrite this as:

$$x_h(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$$

From here, then we can combine this with the particular solution we found above:

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta) + A_1\cos(\omega t - \delta_1) + A_2\cos(\omega t - \delta_2)$$

Then, we can impose the initial conditions that $x(0) = 0$ and $x'(0) = v$ in order to solve for the coefficients A and δ. Unfortunately we cannot do this analytically, but this is what we should do. The solution tends towards the answer in part (a) since the transient motion $x_h(t)$ dies off over time due to the $e^{-\beta t}$ term, leaving only $x_p(t)$ remaining. □

Problem 2 - A Sum from Perseval's Theorem

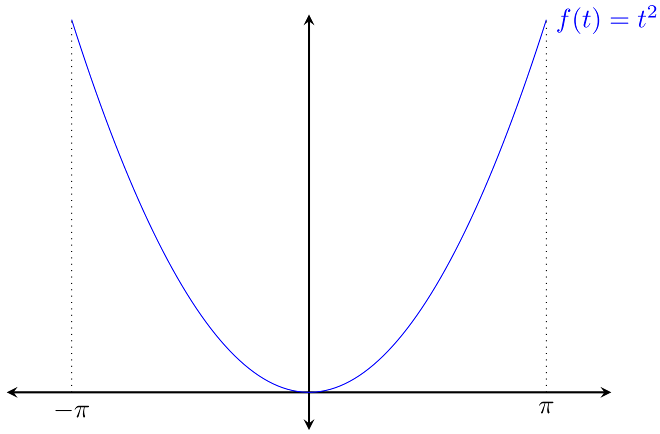
In this problem you'll use Fourier series to deduce the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

To do so, consider the function f defined on $[-\pi, \pi]$ by $f(t) = t^2$.

(a) Draw the graph of f.

Solution: The graph of f looks like the following:



$\square$

(b) Calculate the Fourier coefficients of f. What is the Fourier series in terms of trigonometric functions?

Solution: From Fourier's theorem, we know that the function f can be written as:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nt)$$

a_0 is relatively easy to calculate. We compute:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3}$$

Now onto the a_n terms.

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} t^2 \cos(nt) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos(nt) dt$$

This integral can be solved via integration by parts twice. In order to save my wrists from typing, I'm going to skip the algebra. After the dust settles, we get:

$$a_n = \frac{4}{n^2}(-1)^n$$

And so therefore we can conclude that

$$f(t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nt)$$

$\square$

(c) Use Parseval's theorem to find the value for the given sum.

Solution: Parseval's theorem says:

$$\langle f^2 \rangle = \langle t^4 \rangle = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2$$

First we compute $\langle t^4 \rangle$ as we would compute it for any other interval:

$$\begin{aligned}\langle t^4 \rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^4 dt \\ &= \frac{\pi^4}{5}\end{aligned}$$

Now we compute the other side:

$$\begin{aligned}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2 &= \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)^2 \\ &= \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^2}\end{aligned}$$

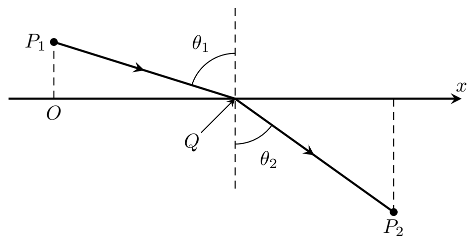
And now we can make the two equations equal each other:

$$\begin{aligned}\frac{\pi^4}{5} &= \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^2} \\ \frac{4\pi^4}{45} &= 8\sum_{n=1}^{\infty}\frac{1}{n^2} \\ \therefore \sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^4}{90}\end{aligned}$$

$\square$

Problem 3 - Snell's Law

Fermat's principle says that the travel time of a ray of light moving from point A to point B is stationary.



The ray of light is emitted at P_1, in a medium with index of refraction n_1. It travels through this medium until it hits a boundary at $y = 0$, after which it propagates through a medium with index of refraction n_2 until it arrives at P_2. Using Fermat's principle, derive Snell's law, which states that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

It will be useful to know that light travels at speed $v = c/n$ in a medium with index of refraction n.

Solution: We know that the time of travel can be written as:

$$T = \int_{P_1}^{P_2} \frac{n(x,y)}{c} ds$$

Then, since we are going through two mediums, we can write:

$$T = \frac{n_1}{c} \int_{P_1}^{Q} ds + \frac{n_2}{c} \int_{Q}^{P_2} ds$$

Here we do two things. First, let $P_1 = (x_1, y_1)$, $Q = (x, 0)$ and $P_2 = (x_2, y_2)$. Then, we compute the integral $\int ds$. From the textbook, we know that for a constant index n (which is the case along path PQ_1 and QP_2), that the shortest path is a straight line. Therefore, if we write $P_1 = (x_1, y_1)$, $Q = (x, 0)$ and $P_2 = (x_2, y_2)$, then we have:

$$T = n_1\sqrt{(x - x_1)^2 + y_1^2} + n_2\sqrt{(x_2 - x)^2 + y_2^2}$$

Now we aim to find the location of x where the time of travel is minimized, so thus we are trying to find $\frac{\mathrm{d}T}{\mathrm{d}x} = 0$. Therefore,

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}x} = 0 &= \frac{n_1}{c}\frac{1}{2\sqrt{(x - x_1)^2 + y_1^2}} \cdot 2(x - x_1) + \frac{n_2}{c}\frac{1}{2\sqrt{(x_2 - x)^2 + y_2^2}} \cdot 2(x_2 - x)(-1) \\ &= \frac{n_1}{c}\underbrace{\frac{x - x_1}{\sqrt{(x - x_1)^2 + y_1^2}}}_{\sin\theta_1} - \frac{n_2}{c}\underbrace{\frac{x_2 - x}{\sqrt{(x_2 - x)^2 + y_2^2}}}_{\sin\theta_2}\end{aligned}$$

And so therefore we get

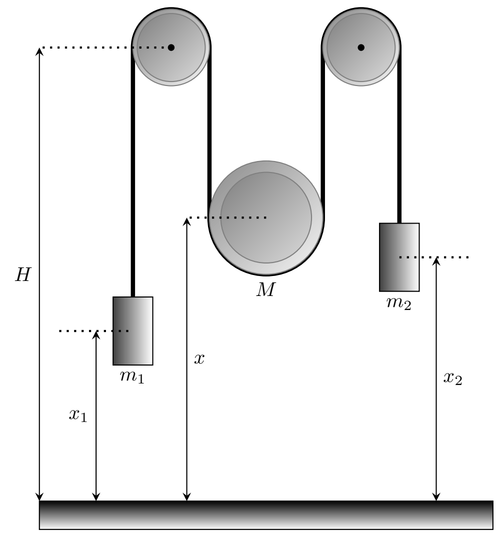
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$\square$

Problem 4 - Double Atwood Machine

The figure below shows a double Atwood machine. The center pulley is free to move verticaly and has mass M. The (ideal, massless) string connecting the three masses shown is weightless. Masses m_1 and m_2 hang on the left and right respectively of the fixed pulleys. The acceleration of gravity is g. All three pulleys are frictionless, so that the string slides freely over them.

(Credit to Andrew Binder for this TikZ diagram)



(a) By considering carefully the forces acting on each mass and pulley, and the constraint imposed by the string's length being fixed, show that the equations of motion are:

$$\begin{cases} \left(m_1 + \frac{M}{4}\right)\ddot{x}_1 + \frac{M}{4}\ddot{x}_2 = g\left(\frac{M}{2} - m_1\right), \\ \frac{M}{4}\ddot{x}_1 + \left(m_2 + \frac{M}{4}\right)\ddot{x}_2 = g\left(\frac{M}{2} - m_2\right) \end{cases}$$

Solution: First, we write down the equations of motion:

$$\begin{aligned} m_1\ddot{x}_1 &= -m_1 g + T_1 \\ m_2\ddot{x}_2 &= -m_2 g + T_2 \\ M\ddot{x} &= -Mg + T_1 + T_2 \end{aligned}$$

Now becuase the rope is frictionless, we know that $T_1 = T_2$. Thus, we now have the equations:

$$m_1\ddot{x}_1 = -m_1 g + T \tag{1}$$

$$m_2\ddot{x}_2 = -m_2 g + T \tag{2}$$

$$M\ddot{x} = -Mg + 2T \tag{3}$$

Now, let H be defined as in the diagram above. We can now find the total length of the rope (excluding the πr terms denoting the length of the rope on the pulleys themselves):

$$(H - x_1) + (H - x_2) + 2(H - x) = L$$

And since the length of the rope doesn't change in time, we have $\frac{\mathrm{d}L}{\mathrm{d}t} = 0$, so therefore:

$$\begin{aligned}\dot{x}_1 + \dot{x}_2 + 2\dot{x} &= 0 \\ \ddot{x}_1 + \ddot{x}_2 + 2\ddot{x} &= 0 \\ \therefore \ddot{x} &= -\left(\frac{\ddot{x}_1 + \ddot{x}_2}{2}\right)\end{aligned}$$

Now, we can subtract twice the first equation by the third to get

$$2m_1\ddot{x}_1 - M\ddot{x} = -2m_1 g + Mg$$

And so now we can apply the constraint and solve:

$$\begin{aligned}2m_1\ddot{x}_1 + M\left(\frac{\ddot{x}_1 + \ddot{x}_2}{2}\right) &= -2m_1 g + Mg \\ m_1\ddot{x}_1 + M\left(\frac{\ddot{x}_1 + \ddot{x}_2}{4}\right) &= -m_1 g + \frac{M}{2}g \\ \therefore \ddot{x}_1\left(m_1 + \frac{M}{4}\right) + \ddot{x}_2\frac{M}{4} &= g\left(\frac{M}{2} - m_1\right)\end{aligned}$$

The exact same algebra exists for the other equation:

$$\begin{aligned}2m_2\ddot{x}_2 + M\left(\frac{\ddot{x}_1 + \ddot{x}_2}{2}\right) &= -2m_2 g + Mg \\ m_2\ddot{x}_2 + M\left(\frac{\ddot{x}_1 + \ddot{x}_2}{4}\right) &= -m_2 g + \frac{M}{2}g \\ \therefore \ddot{x}_2\left(m_2 + \frac{M}{4}\right) + \ddot{x}_1\frac{M}{4} &= g\left(\frac{M}{2} - m_2\right)\end{aligned}$$

□

(b) Solve this problem again, this time by writing down the Lagrangian, and finding the equations of motion.

Solution: We have the kinetic and potential energies:

$$\begin{aligned}T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}M\dot{x}^2 \\ U &= mgx_1 + mgx_2 + mgx\end{aligned}$$

Therefore the Lagrangian is:

$$\mathcal{L} = T - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + \frac{1}{2}M\dot{x}^2 - m_1gx_1 - m_1gx_2 - Mgx$$

Here we can now use the constraint equation $2\dot{x} + \dot{x}_1 + \dot{x}_2 = 0$ in order to get:

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + \frac{1}{2}M\left(\frac{\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2}{2}\right)^2 - m_1gx_1 - m_2gx_2 - Mg\left(\frac{x_1 + x_2}{2}\right)$$

This is a lagrangian with two coordinates, x_1 and x_2, and so therefore we can do them separately. Writing down the Euler-Lagrange equation for x_1:

$$\frac{\partial \mathcal{L}}{\partial x_1} = -m_1 g + \frac{1}{2}Mg = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_1\dot{x}_1 + \frac{2}{8}\dot{x}_1 M + \frac{2\dot{x}_2}{8}M\right)$$

We have three terms on the rightmost side here because in the expansion of $\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2$ we have two terms containing $\dot{x}_1$. Now, we can rearrange:

$$-m_1 g + \frac{1}{2}Mg = m_1\ddot{x}_1 + \frac{1}{4}M\ddot{x}_1 + \frac{1}{4}M\ddot{x}_2$$
$$\therefore \ddot{x}_1\left(m_1 + \frac{M}{4}\right) + \frac{M}{4}\ddot{x}_2 = g\left(\frac{M}{2} - m_1\right)$$

The exact same algebra (in fact, the equations are symmetric for x_1 and x_2) follows for x_2:

$$\frac{\partial \mathcal{L}}{\partial x_2} = -m_2 g + \frac{1}{2}Mg = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_2\dot{x}_2 + \frac{2}{8}\dot{x}_2 M + \frac{2\dot{x}_1}{8}M\right)$$
$$-m_2 g + \frac{1}{2}Mg = m_2\ddot{x}_2 + \frac{1}{4}M\ddot{x}_2 + \frac{1}{4}\ddot{x}_1$$
$$\therefore \frac{M}{4}\ddot{x}_1 + \ddot{x}_2\left(m_2 + \frac{M}{4}\right) = g\left(\frac{M}{2} - m_2\right)$$

$\square$

Problem 5

Consider the Lagrangian of some physical system $L(q, \dot{q}, t)$, where q is a generalized coordinate and t is time. The action is given by $S[L] = \int L dt$. Suppose we add a function to the Lagrangian $f(q, \dot{q}, t)$. Show that if f is a total time derivative, then the equations of motion are unchanged. Try to think of other modifications to the Lagrangian that preserve the equations of motion.

Solution: If f is a total time derivative, then it means we can write:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial q}\dot{q}(t) + \frac{\partial f}{\partial t}$$

Then, we can now add this onto $\mathcal{L}$ to get $\mathcal{L}' = \mathcal{L} + f$, and then now we can write down the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial}{\partial q}\frac{\mathrm{d}f}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}} = 0$$

We know that the first two terms equal zero since they are part of the "original" Lagrangian. Therefore we only need to look at the last two terms:

$$\frac{\partial}{\partial q}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)$$

And we can substitute the first relation into here to get:

$$\begin{aligned}
\frac{\partial}{\partial q}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}\right)\right] &= \frac{\partial}{\partial q}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}f}{\mathrm{d}q}\dot{q} + \frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}f}{\mathrm{d}t}\right] \\
&= \frac{\partial}{\partial q}\frac{\mathrm{d}f}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}f}{\mathrm{d}q}\dot{q} + \frac{\partial f}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}f}{\mathrm{d}t}\right] \\
&= \frac{\partial}{\partial q}\frac{\mathrm{d}f}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q} = 0
\end{aligned}$$

where in the last step we've used the relation that $\frac{\partial}{\partial q}\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q}$ - namely the fact that we can switch the order in which we take the derivatives. Since they equal zero, then it means that we've verified our equations of motion, and so therefore the equations of motion are preserved.

In general, we can see that any modification that alters our coordinate axes (in other words, transforming $q(t)$ into some arbitrary $q'(t)$) should preserve the Lagrangian, since the motion of a system should not be dependent on the way we look at it. These transformations should also not alter the Lagrangian at all, and so the equation of motion remains the same. $\square$