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Collaborators

I worked with **Andrew Binder, Christine Zhang, Teja Nivarthi** on this assignment.

Problem 1

The electric field of a solid sphere with radius R and uniform charge density ρ is given by

$$\mathbf{E} = \begin{cases} \dfrac{\rho \mathbf{r}}{3\epsilon_0} & (r < R) \\ \\ \dfrac{kQ}{r^2}\hat{\mathbf{r}} & (r > R) \end{cases} \tag{1}$$

where Q is the total charge of the sphere. The magnetic field of an infinitely long thick table with radius a is given by

$$\mathbf{B} = \begin{cases} \dfrac{\mu_0 J s}{2}\hat{\phi} & (s < a) \\ \\ \dfrac{\mu_0 I}{2\pi s}\hat{\phi} & (s > a) \end{cases}$$

where the net current I flows in the $+z$-direction. Note that the **E**-field and **B**-field are expressed in spherical and cylindrical coordinates respectively.

(a) Calculate the divergence and curl of **E** with spherical coordinates

Solution: Firstly, there's no θ or ϕ dependence, so we only care about the r part of the divergence, so for $r < R$,

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\rho}{3\epsilon_0}\right) = \frac{1}{r^2}\frac{\rho}{3\epsilon_0} \cdot 2r = \frac{2\rho}{3\epsilon_0 r}$$

Then for $r > R$:

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{kQ}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(kQ) = 0$$

Therefore, we can write:

$$\nabla \cdot \mathbf{E} = \begin{cases} \dfrac{2\rho}{3\epsilon_0 r} & (r < R) \\ 0 & (r > R) \end{cases}$$

The electric field has no θ or ϕ component anywhere, so therefore $\nabla \times \mathbf{E} = 0$ for both regimes $r < R$ and $r > R$. □

(b) Calculate the divergence and curl of **B** in cylindrical coordinates.

Solution: Just like the previous part, we know that since **B** only has a ϕ component for $s < a$:

$$\nabla \cdot \mathbf{B} = \frac{1}{s}\frac{\partial B_\phi}{\partial \phi} = \frac{1}{s}\frac{\partial}{\partial \phi}\left(\frac{\mu_0 J s}{2}\hat{\phi}\right) = 0$$

We also have for $s > a$:

$$\nabla \cdot \mathbf{B} = 0$$

For the curl, we know that the curl takes on the general form:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{z}$$

$\mathbf{B}$ has no θ or ϕ component, so only the $\hat{z}$ component survives. For $r < a$:

$$\nabla \times \mathbf{B} = \frac{1}{s}\left[\frac{\partial}{\partial s}\left(\frac{s\mu_0 J s}{2}\right)\right]\hat{z} = \frac{1}{s}\left[\frac{\mu_0 J}{2}\cdot 2s\right]\hat{z} = \mu_0 J\hat{z}$$

and likewise for $r > a$:

$$\nabla \times \mathbf{B} = \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\cdot\frac{\mu_0 I}{2\pi s}\right)\right]\hat{z} = \frac{1}{s}\left[\frac{\partial}{\partial s}(\mu_0 I)\right]\hat{z} = 0$$

And so therefore:

$$\nabla \times \mathbf{B} = \begin{cases} \mu_0 J\hat{z} & (s < a) \\ 0 & (s > a) \end{cases}$$

□

Independent of the previous part, consider a vector field $\mathbf{V} = s(2 + \cos^2\phi)\hat{s} + s\sin\phi\cos\phi\hat{\phi} + 3z\hat{z}$.

(c) Calculate the divergence and curl of the vector $\mathbf{V}$.

Solution: Here we use the full form of the divergence. We then get:

$$\begin{aligned}\nabla \cdot \mathbf{V} &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\cdot 2\cos^2\phi\right) + \frac{1}{s}\frac{\partial}{\partial \phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z) \\ &= 2\cos^2\phi + (\cos^2\phi - \sin^2\phi) + 3 \\ &= 2\cos^2\phi + \cos(2\phi) + 7\end{aligned}$$

Similarly with the curl:

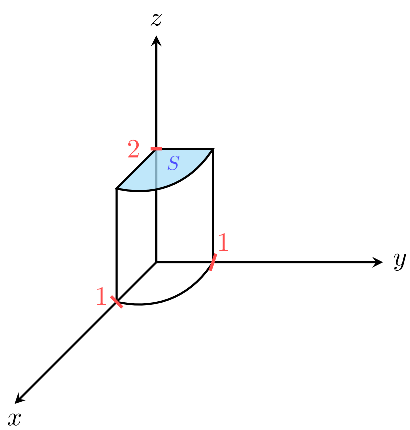
$$\begin{aligned}\nabla \times \mathbf{V} = &\left(\frac{1}{s}\frac{\partial}{\partial \phi}(3z) - \frac{\partial}{\partial z}(s\sin\phi\cos\phi)\right)\hat{s} + \left(\frac{\partial}{\partial z}\left(s\cdot 2\cos^2\phi\right) - \frac{\partial}{\partial s}(3z)\right)\hat{\phi} \\ &+ \frac{1}{s}\left(\frac{\partial}{\partial s}\left(s^2\sin\phi\cos\phi\right) - \frac{\partial}{\partial \phi}\left(s\cdot 2\cos^2\phi\right)\right)\hat{z}\end{aligned}$$

The $\hat{s}$ and $\hat{\phi}$ components here evaluate to zero. Then, we can simplify the $\hat{z}$ direction:

$$\begin{aligned}\nabla \times \mathbf{V} &= \frac{1}{s}\left[2s\sin\phi\cos\phi + 2\sin\phi\cos\phi\right]\hat{z} \\ &= 2\sin(2\phi)\hat{z}\end{aligned}$$

□

(d) Verify the divergence theorem holds true using the quarter-cylinder of radius 1 and height 2. shown in the figure below.



Solution: The divergence theorem says:

$$\iint_S \mathbf{V}\cdot\hat{n}\ dA = \iiint_E \nabla\cdot\mathbf{V} d\mathcal{V}$$

For the right hand side, we can compute the volume integral:

$$\begin{aligned}\iiint_E (\nabla\cdot\mathbf{V})d\mathcal{V} &= \int_0^1\int_0^{\pi/2}\int_0^2 \left(2\cos^2\phi+\cos(2\phi)+7\right)\cdot sdzd\phi ds\\ &= \int_0^1\int_0^{\pi/2} s(4\cos^2\phi+2\cos(2\phi)+7)d\phi ds\\ &= \int_0^1 s(4\frac{\pi}{4}+2(0)+7\pi)ds\\ &= \frac{8\pi}{2} = 4\pi\end{aligned}$$

To compute the surface integral, we have to break it up into 5 segments: the two caps, the curved surface, and the two surfaces along the axes. First, we calculate the two caps.

Notice that for the caps, $\hat{n} = (0,0,1)$, so we only take the $\hat{z}$ comopnent of $\mathbf{V}$. Our bounds of integration are from $\phi\in[0,\pi/2]$ and $s\in[0,1]$. First, we calculate the bottom cap:

$$s_1 = \int_0^1\int_0^{\pi/2} 3sz\ d\phi ds$$

But $z=0$, so we get $s_1 = 0$.

For the top cap, we have $z=2$, so therefore

$$s_2 = \int_0^1\int_0^{\pi/2} 6s\ d\phi ds = \int_0^1 3\pi s\ ds = \frac{3\pi}{2}$$

Now we compute the curved surface s_3. Here, we have $\phi\in[0,\pi/2]$ and $z\in[0,2]$, $s=1$ and $\hat{n}=(1,0,0)$ is $(1,0,0)$, so we only take the $\hat{s}$ component of $\mathbf{V}$. Therefore:

$$s_3 = \int_0^2\int_0^{\pi/2}(2+\cos^2\phi)s\ d\phi dz = \int_0^2 2\cdot\frac{\pi}{2}+\frac{\pi}{4} = \frac{5\pi}{2}$$

Now we take a look at the two sides on the axes. For the plane on the xz-plane, we have $s\in[0,1]$, $z\in[0,2]$, $\hat{n}=(0,-1,0)$ and $\phi=0$. Therefore:

$$s_4 = -\int_0^1\int_0^2 s^2\sin\phi\cos\phi\ dzds$$

But notice that since $\phi = 0$, then $\sin\phi = 0$ and so therefore $s_4 = 0$.

A similar story exists with the other side: the integration bounds are the same and $\hat{n} = (0, 1, 0)$, $\phi = \pi/2$. Therefore:

$$s_5 = \int_0^1 \int_0^2 s^2 \sin\phi \cos\phi \; dzds$$

But since $\phi = \pi/2$, then $\cos\phi = 0$, and so therefore $s_5 = 0$ as well. Now we can finally take the sum of all of them:

$$\iint_S \mathbf{V} \cdot ndA = s_1 + s_2 + s_3 + s_4 + s_5 = \frac{3\pi}{2} + \frac{5\pi}{2} = 4\pi$$

which equals what we calculated on the right hand side, as desired. □

(e) Verify that Stoke's theorem holds true using the surface S shown in the figure below.

Solution: Stokes' theorem says

$$\oint_S \mathbf{V} \cdot dl = \iint_S (\nabla \times \mathbf{V}) \; dA$$

Computing the right hand side of this integral, the region is defined by $s \in [0, 1], \phi \in [0, \pi/2]$ and $z = 2$, with $\hat{n} = (0, 0, 1)$, so therefore:

$$\iint_S \nabla \times \mathbf{V} \; dA = \int_0^1 \int_0^{\pi/2} 2\sin(2\phi)s \; d\phi ds = 2\int_0^1 sds = 1$$

To compute the left hand side, we split up the integral into three different parts. We split this into three segments, revolving counterclockwise around the surface.

The first segment has $r \in [0, 1]$ with $\phi = 0$ and $z = 2$, so this gives:

$$s_1 = \int_0^1 s(2 + \cos^2\phi)ds = \int_0^1 3s \; ds = \frac{3}{2}$$

The second segment has $r = 1, \phi \in [0, \pi/2], z = 2$, so

$$s_2 = \int_0^{\pi/2} \sin\phi\cos\phi \cdot s \; d\phi = \frac{1}{2}$$

The final segment has $r \in [0, 1]$, $\phi = \pi/2$ and $z = 2$, but we have to take the integral from $r = 1 \to r = 0$ because we need to preserve direction:

$$s_3 = \int_1^0 s(2 + \cos^2\phi) \; ds = \int_1^0 2s \; ds = -1$$

And so the total is:

$$S = \frac{3}{2} + \frac{1}{2} - 1 = 1$$

which is what we obtained on the right hand side, so we're done. □

Problem 2

The vector field

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) - \frac{\mathbf{p}}{3\epsilon_0}\delta^3(r)$$

where p is a constant in the z-direction, can be written as a gradient of some scalar function $V(r)$. Find the scalar functi $V(r)$ for $r \neq 0$. *Note:* The second term including the delta function is added for completeness, but you do not need to worry aboutit here. I do NOT recommend you using the Helmholtz theorem where

$$V(r) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{E}}{\mathscr{r}} d\tau'$$

because the divergence at the origin is very tricky to deal with as it's not mathematically well-defined. Instead, thikn of this problme as solving the differential equation $\mathbf{E} = -\nabla V$ for $r \neq 0$

Solution: Again, we know that the gradient of a scalar function T is written as

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

We don't have a ϕ component in this case, so we instead just solve:

$$-\frac{\partial V}{\partial r}\hat{r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}\hat{r} = \frac{p\cos\theta}{2\pi\epsilon_0 r^3}$$

Integrating this with respect to r, we get:

$$V = \frac{1}{2}\frac{p\cos\theta}{2\pi\epsilon_0 r^2}\hat{r} + g(\theta)$$

We have to insert $g(\theta)$ here for compleness, since its partial derivative with respect to r is 0. Now taking the derivative with respect to θ:

$$-\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p}{4\pi\epsilon_0 r^3} \cdot (-\sin\theta) + g'(\theta) \implies g'(\theta) = 0$$

And so therefore:

$$V(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

For the $\hat{z}$ direction, we have:

$$-\frac{\partial V}{\partial z} = -\frac{\mathbf{p}}{3\epsilon}\delta^3(r)$$

so integrating we get:

$$V_z = \frac{\mathbf{p}z}{3\epsilon_0}\delta^3(r)$$

So finally we have

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r} + \frac{\mathbf{p}z}{3\epsilon_0}\delta^3(r)$$

$\square$

Problem 3

Show the following integral theorems:

(a) $\displaystyle\int_{\mathcal{V}} (\nabla T) d\tau = \oint_{S} T d\mathbf{a}$

(b) $\displaystyle\int_{\mathcal{V}} (\nabla \times \mathbf{V}) d\tau = -\oint_{\mathcal{S}} \mathbf{V} \times d\mathbf{a}$

(c) $\displaystyle\int_{\mathcal{V}} (T\nabla^2 U - U\nabla^2 T) d\tau = \oint_{S} (T\nabla U - U\nabla T) \cdot d\mathbf{a}$

Here $\mathcal{V}$ is a three-dimensional region in 3D flat space and $\mathcal{S}$ is its boundary. T, U are scalar fields, while $\mathbf{V}$ is a vector field. For (a), you can use the divergence theorem but with the vector field to be $\mathbf{c}T$ where $\mathbf{c}$ is a constant vector field. For (b), you can again consider divergence theorem but with the vector field to be $\mathbf{V} \times \mathbf{c}$ where again $\mathbf{c}$ is a constant vector field.

Solution:

(a) From the divergence theorem, we know that

$$\int_{\mathcal{V}} \nabla \cdot T = \int_{\partial \mathcal{V}} T dS$$

Now, take the vector field to be $\mathbf{c}T$ where $\mathbf{c}$ is a constant vector field. Therefore:

$$\begin{aligned}
\mathbf{c} \cdot \int_{\mathcal{V}} (\nabla T) d\tau &= \int_{\mathcal{V}} \mathbf{c} \nabla T \; d\tau \\
&= \int_{\mathcal{V}} \nabla (\mathbf{c}T) - \int_{\mathcal{V}} T(\nabla \cdot \mathbf{c}) d\tau \\
&= \int_{\partial \mathcal{V}} T\mathbf{c} \cdot d\mathbf{a} \\
&= \mathbf{c} \cdot \int_{\partial \mathcal{V}} T \; d\mathbf{a} = \mathbf{c} \cdot \oint_{S} T d\mathbf{a}
\end{aligned}$$

here $\mathbf{c}$ is on both sides of the equation, so we can safely say that

$$\int_{\mathcal{V}} (\nabla T) \cdot T = \oint_{\mathcal{S}} T \cdot d\mathbf{a}$$

as desired.

(b) Using the hint again, we let $T = \mathbf{V} \times \mathbf{c}$ where $\mathbf{c}$ is a constant vector field:

$$\begin{aligned}
\int_{\mathcal{V}} (\nabla \cdot (\mathbf{V} \times \mathbf{c})) \; d\tau &= \oint_{\mathcal{S}} (\mathbf{V} \times \mathbf{c}) d\mathbf{a} \\
&= \int_{\mathcal{V}} (\nabla \times \mathbf{V}) \cdot \mathbf{c} - \mathbf{V} \cdot \underbrace{(\nabla \times \mathbf{c})}_{=0} \; d\tau \\
&= \int_{\mathcal{V}} (\nabla \times \mathbf{V}) \cdot \mathbf{c} \; d\tau \\
&= \oint_{\mathcal{S}} (\mathbf{V} \times \mathbf{c}) \cdot d\mathbf{a} \\
&= \oint_{\mathcal{S}} \mathbf{c} \cdot (d\mathbf{a} \times \mathbf{V}) \\
&= \mathbf{c} \cdot \left(-\oint_{\mathcal{S}} \mathbf{V} \times d\mathbf{a} \right)
\end{aligned}$$

And so now we've derived the relation:

$$\mathbf{c} \cdot \int_{\mathcal{V}} (\nabla \times \mathbf{V}) \ d\tau = \mathbf{c} \cdot (-\oint_{S} \mathbf{V} \cdot d\mathbf{a})$$

and since $\mathbf{c}$ exists on both sides, thne we can say:

$$\int_{\mathcal{V}} (\nabla \times \mathbf{V}) d\tau = -\oint_{\mathcal{S}} \mathbf{V} \times d\mathbf{a}$$

As desired.

(c) From the first part, we know that

$$\int_{\mathcal{V}} \nabla F \ d\tau = \oint_{\mathcal{S}} F \ d\mathbf{a}$$

So in this problem, we will let $F = T\nabla U - U\nabla T$ so we can use this theorem and obtian the left hand side. Computing the gradient:

$$\nabla F = T\nabla^2 U - \nabla T \nabla U - \nabla U \nabla T - U\nabla^2 T$$

and since gradients commute, then

$$\nabla F = T\nabla^2 U - U\nabla^2 T$$

So therefore:

$$\int_{\mathcal{V}} (T\nabla^2 U - U\nabla^2 T) \ d\tau = \oint_{\mathcal{S}} (T\nabla U - U\nabla T) \cdot d\mathbf{a}$$

as desired.

$\square$