Collaborators

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Problem 1

The electric field of a point charge Q is given by $\mathbf{E} = \frac{kQ}{r^2}\hat{\mathbf{r}}$, and the magnetic field of an infinitely long wire with current I is $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$, where the fields are expressed in spherical and cylindrical coordinates respectively. Express the fields in the Cartesian coordinate and calculate the following. You can ignore the singularity at $r = 0$ and $s = 0$ in the problem. (You are asked to calculate the divergence and curl in Cartesian coordinates. Do not simply use the formulas for spherical and cylindrical coordinates.)

(a) the divergence and curl of $\mathbf{E}$

Solution: To convert from spherical to cartesian coordinates, we have:

$$
\begin{aligned}
x &= r\sin\phi\cos\theta \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

Then, we can rewrite the trigonometric ratios:

$$
\begin{aligned}
\sin\phi &= \frac{y}{\sqrt{x^2+y^2}} & \cos\phi &= \frac{x}{\sqrt{x^2+y^2}} \\
\sin\theta &= \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & \cos\theta &= \frac{z}{\sqrt{x^2+y^2+z^2}}
\end{aligned}
$$

From this, we then get that:

$$\hat{i} = \frac{x}{(x^2+y^2+z^2)^{3/2}},\ \hat{j} = \frac{y}{(x^2+y^2+z^2)^{3/2}},\ \hat{k} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

And so we have

$$\vec{E} = \frac{kQ}{(x^2+y^2+z^2)^{3/2}}(x, y, z)$$

We can calculate the divergence from here using the usual method.

$$
\begin{aligned}
\nabla \cdot E &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (E_x, E_y, E_Z) \\
&= \frac{\partial}{\partial x}\frac{kQx}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{kQy}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{kQz}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{kQ}{(x^2+y^2+z^2)^{5/2}}\left[(y^2+z^2-2x^2)+(x^2+z^2-2y^2)+(x^2+y^2-2z^2)\right] \\
&= 0
\end{aligned}
$$

And so the divergence is zero. A similar process is done with the curl, which is:

$$
\nabla \times E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial x}E_y\right) - \hat{j}\left(\frac{\partial}{\partial x}E_z - \frac{\partial}{\partial z}E_x\right) + \hat{k}\left(\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial z}E_x\right)
$$

Here, I'll show one of these terms explicitly:

$$
\begin{aligned}
\frac{\partial}{\partial y}E_z &= \frac{\partial}{\partial y}\frac{kQz}{(x^2+y^2+z^2)^{3/2}} &= -\frac{3kQyz}{(x^2+y^2+z^2)^{5/2}} \\
\frac{\partial}{\partial x}E_y &= \frac{\partial}{\partial x}\frac{kQz}{(x^2+y^2+z^2)^{3/2}} &= -\frac{3kQyz}{(x^2+y^2+z^2)^{5/2}}
\end{aligned}
$$

These two terms are the same, and so the difference is the same. The same thing will happen with the other three terms, and so we can conclude that

$$
\nabla \times E = 0
$$

□

(b) the divergence and curl of **B**.

Solution: Here, we rewrite $\hat{\phi}$ into cartesian coordinates:

$$
\cos\phi = \frac{x}{\sqrt{x^2+y^2}}, \ \sin\phi = \frac{y}{\sqrt{x^2+y^2}}
$$

So therefore:

$$
\begin{aligned}
B_x &= -B_0 r \sin\phi = -\frac{\mu_i I y}{2\pi(x^2+y^2)} \\
B_y &= B_0 r \cos\phi = \frac{\mu_0 I x}{2\pi(x^2+y^2)}
\end{aligned}
$$

And so therefore we have:

$$\vec{B} = \frac{\mu_0 I}{2\pi(x^2+y^2)}(-y, x.0)$$

Taking the divergence of this:

$$\begin{aligned}
\nabla \cdot B &= \frac{\mu_0 I}{2\pi}\left[\frac{\partial}{\partial x}\frac{y}{(x^2+y^2)} + \frac{\partial}{\partial y}\frac{x}{(x^2+y^2)}\right] \\
&= \frac{\mu_0 I}{2\pi(x^2+y^2)^2}\left[xy - xy\right] \\
&= 0
\end{aligned}$$

And so we conclude that $\nabla \cdot B = 0$. Similarly, we do the curl:

$$\nabla \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & 0 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial x}B_y\right) - \hat{j}\left(\frac{\partial}{\partial x}B_z - \frac{\partial}{\partial z}B_x\right) + \hat{k}\left(\frac{\partial}{\partial x}B_y - \frac{\partial}{\partial z}B_x\right)$$

Based on our equations, we know that B_y and B_x have no z-dependence, and so terms like $\frac{\partial}{\partial z}B_x$ and $\frac{\partial}{\partial z}B_y$ will go to zero. Further, we know that $B_z = 0$, so any partial derivative is also zero. Thus, the only nonzero term is the last one, where we need to calculate:

$$\begin{aligned}
\frac{\partial B_y}{\partial x} &= \frac{\mu_0 I}{2\pi}\frac{\partial}{\partial x}\frac{x}{(x^2+y^2)} \\
&= \frac{\mu_0 I}{2\pi}\frac{y^2-x^2}{(x^2+y^2)^2} \\
\frac{\partial B_x}{\partial y} &= \frac{\mu_0 I}{2\pi}\frac{\partial}{\partial y}\frac{y}{(x^2+y^2)} \\
&= \frac{\mu_0 I}{2\pi}\frac{y^2-x^2}{(x^2+y^2)^2}
\end{aligned}$$

And since these two terms are the same, then we get zero. Therefore, we conclude that

$$\nabla \times B = 0$$

$\square$

Problem 2

Many quantities in vector calculus involves anti-symmetric structures, such as cross products, curl, etc. For these quantities it is easier to derive various properties via Levi-Civita symbol and index notations.

(a) Let $\mathbf{S}$ be a symmetric matrix. That is, $S_{ij} = S_{ji}$. Show tat

$$\epsilon^{ijk} S_{ij} = 0$$

Solution: Since we know that $S_{ij} = S_{ji}$, then:

$$\epsilon^{ijk} S_{ij} = -\epsilon^{jik} S_{ji} = -\epsilon^{ijk} S_{ij}$$

And since we now have that $\epsilon^{ijk} S_{ij} = -\epsilon^{ijk} S_{ij}$, then it follows that

$$\epsilon^{ijk} S_{ij} = 0$$

$\square$

(b) Write $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ in terms of the Levi-Civita symbol and components with indices, then find the relation between $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$. *Note that* $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ *is the volume of the parallelpiped spanned by the three vectors. In other words, the notion of volume involves anti-symmetric structure.*

Solution: First, we compute $A \cdot (B \times C)$. To do so, we first compute the cross product:

$$(B \times C) = \epsilon^{ijk} b_j c_k$$

Therefore, if we now dot this with A we get:

$$A \cdot (B \times C) = \delta_{ij} a_i (B \times C)_j = a_i \epsilon^{ijk} b_j c_k$$

Now looking at $B \cdot (A \times C)$, we get:

$$B \cdot (A \times C) = b_i \epsilon^{ijk} a_j c_k$$

But this is the same as the previous equation, only with $i \to j$ and $j \to i$, so therefore:

$$a_i \epsilon^{ijk} b_j c_k = b_j \epsilon^{jik} a_j c_k$$

And so therefore

$$A \cdot (B \times C) = -B \cdot (A \times C)$$

$\square$

(c) Using Levi-Civita symbol, show that

$$\nabla \cdot (\nabla \times A) = 0$$

Solution: We can write the cross product as $\nabla \times A$ as:

$$\nabla \times A = \epsilon^{ijk}\partial_j A_k$$

And so combining this with the divergence we get:

$$\nabla \cdot (\nabla \times A) = \epsilon^{ijk}\partial_i \partial_j A_k$$

Since $\partial_i \partial_j$ commute, then we can rewrite this as:

$$\begin{aligned}\nabla \cdot (\nabla \times A) &= \epsilon^{ijk}\partial_j \partial_i A_k \\ &= -\epsilon^{jik}\partial_j \partial_i A_k \\ &= -\epsilon^{ijk}\partial_i \partial_j A_k\end{aligned}$$

And so now we've derived the relation that

$$\epsilon^{ijk}\partial_i \partial_j A_k = -\epsilon^{ijk}\partial_i \partial_j A_k$$

And so therefore we must conclude that

$$\epsilon^{ijk}\partial_i \partial_j A_k = 0$$

as desired. □

(d) Show that

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Solution: We can just write the algebra out:

$$\begin{aligned}\nabla \times (A \times B) &= \epsilon^{ijk}\partial_j \epsilon^{klm} a_l b_m \\ &= \partial_j a_l b_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\ &= \partial_j (a_i b_j) - \partial_j (a_j b_i) \\ &= a_i \partial_j b_j + b_j \partial_j a_i + b_j \partial_j a_i - (a_j \partial_j B_i + B_i \partial_j a_j) \\ &= (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A)\end{aligned}$$

□

(e) Show that the cross product of two vectors $\mathbf{V}$ and $\mathbf{W}$, explicitly defined as $\epsilon_{ijk}V^jW^k$, transform as a dual vector under rotation. Note that the Levi-Civita symbol is a symbol here and does not transform under rotation. *Hint:* Remember that or any matrix M^i_j, we have $\epsilon_{lmn}\det(M) = \epsilon_{ijk}M^i_l M^j_m M^k_n$.

Solution: Here, we essentially want to show that the operation of rotation distributes across $V \times W$:

$$R(V \times W) = RV \times RW$$

Computing the i-th component of the right hand side:

$$\begin{aligned}((RV) \times (RW))_i &= \epsilon_{ijk}(RV)^j(R)^k \\ &= \epsilon_{ijk}R_{jk}V_xR_{ky}W_y\end{aligned}$$

And on the left hand side:

$$R(V \times W) = R(\epsilon_{ijk}V^jW^k) = R_{ai}\epsilon_{ijk}V^jW^k$$

Then, we can use the hint given in the problem statement and combining it with the relation $R_{xy}R_{ay} = \delta_{xa}$ to get:

$$\begin{aligned}(RV) \times (RW) &= \epsilon_{ijk}R_{jx}V_xR_{ky}W_y \\ &= \epsilon_{ijk}R_{ba}R_{ab}R_{jk}R_{ky}V_xW_y \\ &= \epsilon_{ajk}\delta_{ia}R_{ba}R_{jx}R_{ky}R_{ab}V_xW_y \\ &= \epsilon_{ijk}\det(R)R_{ab}V_xW_y \\ &= R_{ai}\epsilon_{ixy}V_xW_y\end{aligned}$$

Which equals what we obtained on the right hand side, when we make the substitution that $x \to j$ and $y \to k$. Therefore, we now have

$$R(V \times W) = (RV) \times (RW)$$

as desired. Therefore, the cross product does transform as a dual vector under rotation. □

(f) For a vector-valued function $\mathbf{F}(\mathbf{r})$, show that

$$\nabla \times (\mathbf{F} \times \mathbf{r}) = \mathbf{2F} + r\frac{\partial \mathbf{F}}{\partial r} - \mathbf{r}(\nabla - \mathbf{F})$$

where $r = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Solution: Writing out one of the two components for the left hand side, we get:

$$\begin{aligned}[\nabla \times (\mathbf{F} \times \mathbf{r})]_i &= \epsilon_{ijk}\partial^j(\mathbf{F} \times \mathbf{r})^k \\ &= \epsilon^{ijk}\partial_j\epsilon^{klm}F_lr_m \\ &= \epsilon^{kij}\epsilon^{klm}\partial_jF_lr_m\end{aligned}$$

Here we can invoke the identity that $\epsilon^{kij}\epsilon^{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$ to get:

$$\begin{aligned}\epsilon^{ijk}\epsilon^{klm}\partial_jF_lr_m &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_jF_lr_m \\ &= \partial_j(F_ir_j) - \partial_j(F_jr_i) \\ &= F_i\partial_jr_j + r_j\partial_jF_i - r_i\partial_jF_j - F_j\partial_jr_i\end{aligned}$$

Now we go term by term. Starting with the first, we have:

$$F_i \partial_j r_j = F_i(\nabla \cdot r) = 3F_i \qquad \text{Since } \left(\frac{\partial r_j}{\partial x^j} = 1\right)$$

Again using the fact that $\frac{\partial r_j}{\partial x^j} = 1$, we can also simplify the second term:

$$\begin{aligned} r_j \partial_j F_i &= r_1 \frac{\partial F_i}{\partial x^1} + r_2 \frac{\partial F_i}{\partial x^2} + r_3 \frac{\partial F_i}{\partial x^3} \\ &= r_i \left(\frac{\partial F_i}{\partial r} \underbrace{\frac{\partial r}{\partial x^j}}_{=1} \right) \\ &= r_i \frac{\partial F_i}{\partial r} \end{aligned}$$

The third term $r_i \partial_j F_j$ just simplifies to $r_i(\nabla \cdot F)$. Now for the final term:

$$F_j \partial_j r_i = F_j \left(\frac{\partial r_i}{\partial x^j} \right)$$

But here, notice that $\frac{\partial r_i}{\partial x^j} = 0$ in rectangular coordinates when $i \neq j$, and when $i = j$ then it's equal to 1, using the relation we had from before. Therefore, we can simplify this down to:

$$F_j \left(\frac{\partial r_i}{\partial x^j} \right) = F_1 + F_2 + F_3 = F_i$$

Now, we can put this all together:

$$\begin{aligned} F_i \partial_j r_j + r_j \partial_j F_i - r_i \partial_j F_j - F_j \partial_j r_i &= 3F_i + r_i \frac{\partial}{\partial \mathbf{r}} F_i - r_i(\nabla \cdot \mathbf{F}) - F_i \\ &= 2F_i + r_i \frac{\partial F_i}{\partial r} - r_i(\nabla \cdot F) \end{aligned}$$

As desired. $\square$

Problem 3

The formal definition of divergence $\mathbf{F}$ at the position $\mathbf{r}$ is given by

$$\nabla \cdot \mathbf{F} = \lim_{\nu \to 0} \frac{\oint_S \mathbf{F} \cdot d\mathbf{a}}{\nu}$$

where $\mathcal{S}$ is any surface enclosing $\mathbf{r}$, and ν is the volume bounded by $\mathcal{S}$, as shown in the left figure. Use this definition with a rectangular box enclosing (x, y, z), as shown in the right figure, to prove that in Cartesian coordinates,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Solution: The net flux in the x-direction can be written as the difference between the flux leaving our imaginary cube and the incoming flux, divided by the total cross sectional length we're taking:

$$\begin{aligned} F \cdot dx &= \frac{F(x + dx, y, z) - F(x, y, z)}{dx} dxdydz \\ &= \frac{\partial F}{\partial x} dV \end{aligned}$$

We are working in $\mathbb{R}^3$, so therefore we can extend this to three dimensions pretty easily:

$$\oint_s F \cdot da = V \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \right)$$

And so therefore

$$\nabla \cdot F = \lim_{v \to 0} \frac{\oint_S F \cdot da}{v} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$$

$\square$