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HW 06 Quantum Mechanics October 24, 2022

Problem 1

Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$ and $\langle T\rangle$ for the n-th stationary state of the harmonic oscillator, using the method of Example 2.5. Check that the uncertainty relation is satisfied.

Solution: We know the following operator relations:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \qquad \hat{p} = \sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

So therefore, computing $\langle x\rangle$:

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\psi|(a_+ + a_-)\psi\rangle$$

Now notice that since $\psi_n(x)$ is an energy eigenstate, then the raising and lowering operators will raise $\psi(x)$ to $\psi_{n+1}(x)$ or $\psi_{n-1}(x)$. Since energy eigenfunctions are an orthonormal basis, then $\langle\psi_n|\psi_m\rangle = 0$ whenever $n \neq m$. Therefore, the whole expression will actually evaluate to 0. Therefore,

$$\boxed{\langle x\rangle = 0}$$

A basically identical argument exists for p, since it is also a combination of a_+ and a_-, so

$$\boxed{\langle p\rangle = 0}$$

as well. For $\langle x^2\rangle$, we have a slightly different relation. We know that

$$\hat{x}^2 = \frac{\hbar}{2m\omega}(a_+^2 + 2a_+a_- + 1 + a_-^2)$$

This relation is obtained by squaring $\hat{x}$, then applying the relation of the commutator:

$$[a_-, a_+] = 1 \implies a_-a_+ = 1 + a_+a_-$$

Therefore, our expectation value expression becomes:

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left\langle\psi_n\middle|(a_+^2 + 2a_+a_- + 1 + a_-^2)\psi_n\right\rangle$$

Now let's look at $(a_+^2 + 2a_+a_- + 1 + a_-^2)\psi_n$ more closely, and consider what the product of it with $\psi_n^\star(x)$ looks like. For terms like $a_+^2\psi_n(x)$ and $a_-^2\psi_n(x)$, we know that they will be orthogonal to $\psi_n^\star(x)$, so therefore they will vanish. Thus, we only care about the terms which give us back $\psi_n(x)$, namely the terms which contain an equal number of raising and lowering operators. Now, we use the relation that

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1} \qquad a_-\psi_n = \sqrt{n}\psi_{n-1}$$

This gives us the relation

$$a_+a_-\psi_n = 2a_+\sqrt{n}\psi_{n-1} = 2\sqrt{n}\sqrt{n}\psi_{n-1} = 2n\psi_n$$

Therefore, we have:

$$\boxed{\left\langle x^2 \right\rangle = \frac{\hbar}{2m\omega}\left\langle \psi_n | (2n+1)\psi_n \right\rangle = \frac{\hbar}{2m\omega}(2n+1)}$$

Similarly for $\left\langle p^2 \right\rangle$, we have

$$\hat{p}^2 = -\frac{\hbar m\omega}{2}(a_+^2 - 2a_+a_- - 1 + a_-^2)$$

and so just like before, the terms with a_+^2 and a_-^2 cancel. Therefore,

$$\boxed{\left\langle p^2 \right\rangle = -\frac{\hbar m\omega}{2}\left\langle \psi_n | -(2n+1)\psi_n \right\rangle = \frac{\hbar m\omega}{2}(2n+1)}$$

Therefore

$$\boxed{\left\langle T \right\rangle = \frac{\left\langle p^2 \right\rangle}{2m} = \frac{1}{2m}\frac{\hbar m\omega}{2}(2n+1) = \frac{\hbar\omega}{4}(2n+1)}$$

Now we can verify the uncertainty principle:

$$\sigma_x = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 = \sqrt{\frac{\hbar}{2m\omega}(2n+1)}$$
$$\sigma_p = \left\langle p^2 \right\rangle - \left\langle p \right\rangle^2 = \sqrt{\frac{\hbar m\omega}{2}(2n+1)}$$

Therefore:

$$\sigma_x\sigma_p = (2n+1)\hbar = \left(n+\frac{1}{2}\right)\frac{\hbar}{2}$$

And since $n = 0, 1, 2, \ldots$, then

$$\left(n+\frac{1}{2}\right)\frac{\hbar}{2} \geq \frac{\hbar}{2}$$

And so the uncertainty relation is satisfied.

Problem 2

A particle in the harmonic oscillator potential starts out in the state

$$\Psi(x,0) = A[3\psi_0(x) + 4\psi_1(x)]$$

(a) Find A.

Solution: To find A, we need that $\int |\Psi(x,0)| = 1$, so therefore:

$$\begin{aligned}1 &= \int A^2(9|\psi_0(x)|^2 + 16|\psi_1(x)|^2)dx \\ &= 25A^2\end{aligned}$$

So $A = \frac{1}{5}$.

(b) Construct $\Psi(x,t)$ and $|\Psi(x,t)|^2$

Solution: Adding in time dependence means adding a phase to each corresponding energy:

$$\begin{aligned}\Psi(x,t) &= \frac{3}{5}\psi_0(x)e^{iE_0t/\hbar} + \frac{4}{5}\psi_1(x)e^{iE_1t/\hbar} \\ &= \frac{3}{5}\psi_0(x)e^{i\omega t/2} + \frac{4}{5}\psi_1(x)e^{3i\omega t/2}\end{aligned}$$

And now we can calculate $|\Psi(x,t)|^2$:

$$\begin{aligned}|\Psi(x,t)|^2 &= \left(\frac{3}{5}\psi_0^\star e^{-i\omega t/2} + \frac{4}{5}\psi_1^\star e^{-3i\omega t/2}\right)\left(\frac{3}{5}\psi_0 e^{i\omega t/2} + \frac{4}{5}e^{3i\omega t/2}\right) \\ &= \frac{9}{25}\psi_0^2 + \frac{12}{25}\psi_0\psi_1(e^{i\omega t} + e^{-i\omega t}) + \frac{16}{25}\psi_1^2 \\ &= \frac{9}{25}\psi_0^2 + \frac{24}{25}\psi_0\psi_1\cos(\omega t) + \frac{16}{25}\psi_1^2\end{aligned}$$

(c) Find $\langle x\rangle$ and $\langle p\rangle$. Don't get too excited if they oscillate at the classical frequency; what would it have been had I specified $\psi_2(x)$, instead of $\psi_1(x)$? Check that Ehrenfest's theorem (Equation 1.38) holds for this wave function.

Solution: We have the following operator relations[1]:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \qquad \hat{p} = \sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

So therefore,

[1] operators are the best thing ever

$$\langle x \rangle = \langle \Psi | \hat{x} \Psi \rangle$$

So this means we have:

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \Psi | (a_+ + a_-) \Psi \rangle$$

Now let's take a look at $(a_+ + a_-)\Psi$ closely. Since $\Psi(x,t) = \frac{3}{5}\psi_0(x)e^{iE_0 t/\hbar} + \frac{4}{5}\psi_1(x)e^{iE_1 t/\hbar}$, then notice the following: if we act the raising operator on $\psi_1(x)$, then we get $\psi_2(x)$. But since no other term here has $\psi_2(x)$, then by orthogonality all terms containing $\psi_2(x)$ must be zero! Furthermore, acting the lowering operator on $\psi_0(x) = 0$, so the only real terms we care about are when we act the raising operator $\psi_0(x)$ and the lowering operator on $\psi_1(x)$, getting us $a_+\psi_0(x) = \psi_1(x)$ and $a_-\psi_1(x) = \psi_0(x)$ respectively. Therefore:

$$\begin{aligned}
\langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \int \left(\frac{3}{5} e^{i\omega t/2} \psi_0^\star(x) + \frac{4}{5} e^{3i\omega t/2} \psi_1^\star(x) \right) \left(\frac{4}{5} e^{3i\omega t/2} \psi_0(x) + \frac{3}{5} e^{i\omega t/2} \psi_1(x) \right) dx \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int \frac{12}{25} e^{i\omega t} |\psi_0(x)|^2 + \frac{12}{25} e^{-i\omega t} |\psi_1(x)|^2 dx \\
&= \frac{12}{25} \sqrt{\frac{\hbar}{2m\omega}} \left(e^{i\omega t} + e^{-i\omega t} \right) \\
&= \frac{24}{25} \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t)
\end{aligned}$$

Similarly, we have

$$\langle p \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \langle \Psi | (a_+ - a_-) \Psi \rangle$$

And the same rule applies here when we evaluate $(a_+ + a_-)\Psi$. We only care about the terms when we act the raising operator on $\psi_0(x)$ and the lowering operator on $\psi_1(x)$, other terms will evaluate to 0. Therefore:

$$\begin{aligned}
\langle p \rangle &= i\sqrt{\frac{\hbar m\omega}{2}} \int \left(\frac{3}{5} e^{-i\omega t/2} \psi_0^\star(x) + \frac{4}{5} e^{-3i\omega t/2} \psi_1^\star(x) \right) \left(\frac{4}{5} e^{3i\omega t/2} \psi_0(x) - \frac{3}{5} e^{i\omega t/2} \psi_1(x) \right) dx \\
&= i\sqrt{\frac{\hbar m\omega}{2}} \int \frac{12}{25} e^{i\omega t} |\psi_0(x)|^2 + \frac{12}{25} |\psi_1(x)|^2 dx \\
&= i\sqrt{\frac{\hbar m\omega}{2}} \frac{12}{25} (e^{i\omega t} - e^{-i\omega t}) \\
&= i\sqrt{\frac{\hbar m\omega}{2}} \cdot \frac{24}{25} i \sin(\omega t) \\
&= -\sqrt{\frac{\hbar m\omega}{2}} \cdot \frac{24}{25} \sin(\omega t)
\end{aligned}$$

Ehrenfest's theorem says that

$$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x} \right\rangle$$

This is a harmmonic oscillator, so

$$-\left\langle \frac{\partial V}{\partial x} \right\rangle = -m\omega^2 x$$

Now calculating the left hand side:

$$\begin{aligned}
\frac{d\langle p\rangle}{dt} &= \frac{d}{dt}\left(m\frac{d\langle x\rangle}{dt}\right) \\
&= \frac{d}{dt}\left(m\frac{d}{dt}\left(\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\right)\right) \\
&= \frac{d}{dt}\left(-\frac{24m}{25}\sqrt{\frac{\hbar}{2m\omega}}\omega\sin(\omega t)\right) \\
&= -m\omega^2\left(\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\right)
\end{aligned}$$

Notice that $\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) = x$, so therefore

$$\frac{d\langle p\rangle}{dt} = -m\omega^2 x$$

And so Ehrenfest's theorem is verified.

(d) If you measured the energy of this particle, what values might you get, and with what probabilities?

Solution: We know that the probability associated with each energy is $|c_n|^2$, where c_n refers to the coefficient associated with ψ_n, and thus an energy E_n. In our wavefunction $\Psi(x)$, we only have it in terms of $\psi_0(x)$ and $\psi_1(x)$, so we only have probabiliy of measuring E_0 and E_1. Therefore, we have:

$$\begin{aligned}
c_0 &= \frac{3}{5}e^{i\omega t/2} \quad \text{for } E_0 = \frac{\hbar\omega}{2} \\
c_1 &= \frac{4}{5}e^{3i\omega t/2} \quad \text{for } E_1 = \frac{3\hbar\omega}{2}
\end{aligned}$$

Therefore, we get:

$$\begin{aligned}
P(E_0) &= \left|\frac{3}{5}e^{i\omega t/2}\right|^2 = \frac{9}{25} \\
P(E_1) &= \left|\frac{4}{5}e^{3i\omega t/2}\right|^2 = \frac{16}{25}
\end{aligned}$$

Therefore, the probability of measuring $E_0 = \frac{\hbar\omega}{2}$ is $\frac{9}{25}$, and for $E_1 = \frac{3\hbar\omega}{2}$ is $\frac{16}{25}$.

Problem 3

Among the stationary states of the harmonic oscillator ($|n\rangle = \psi_n(x)$, Equation 2.67) only $n = 0$ hits the uncertainty limit ($\sigma_x \sigma_p = \hbar/2$); in general, $\sigma_x \sigma_p = (2n+1)\hbar/2$, as you found in Problem 2.12. But certain *linear combination* (known as **coherent states**) also minimize the uncertianty product. They are (as it turns out) *eigenfunctions of the lowering operator*

$$a_- |\alpha\rangle = \alpha |\alpha\rangle$$

(the eigenvalue can be any complex number)

(a) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$ in the state $|\alpha\rangle$. *Hint:* Use the technique in Example 2.5, and remember that a_+ is the hermitian conjugate of a_-. Do *not* assume α is real.

Solution: We first use the fact that we have the following two relations:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \qquad \hat{p} = \sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

I'm going to drop the operator symbols here because there's too many of them. First, let's calculate $\langle x \rangle$:

$$\begin{aligned} \langle x \rangle &= \langle \alpha | \hat{x} \alpha \rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha | (a_+ + a_-) \alpha \rangle \end{aligned}$$

Now we act $a_+ + a_-$ on α:

$$\begin{aligned} (a_+ + a_-) |\alpha\rangle &= a_+ |\alpha\rangle + a_- |\alpha\rangle \\ &= a_+ |\alpha\rangle + \alpha |\alpha\rangle \end{aligned}$$

To calculate $a_+ |\alpha\rangle$ we, make use of the fact that $a_+ |\alpha\rangle = (a_- \langle\alpha|)^\dagger = (a_- |\alpha\rangle)^\dagger$ since $|\alpha\rangle$ can be taken to be real. Now we have $(\alpha |\alpha\rangle)^\dagger = \alpha^\star |\alpha\rangle$, so we've derived the relation:

$$a_+ |\alpha\rangle = \alpha^\star |\alpha\rangle$$

This is a useful relation that we will use over and over again to calculate the remaining values. Now returning to our expression we have to evaluate:

$$a_+ |\alpha\rangle + \alpha |\alpha\rangle = (\alpha^\star + \alpha) |\alpha\rangle$$

And so now the expectation value becomes:

$$\begin{aligned} \langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}} (\alpha^\star + \alpha) \underbrace{\langle \alpha | \alpha \rangle}_{=1} \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\alpha^\star + \alpha) \end{aligned}$$

Similarly, we calculate $\langle p \rangle$

$$\begin{aligned}
\langle p \rangle &= i\sqrt{\frac{\hbar m\omega}{2}} \langle \alpha | (a_+ - a_-)\alpha \rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}} \langle \alpha | (a_+ - a_-)\alpha \rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}} (\alpha^\star - \alpha)
\end{aligned}$$

Now $\left\langle x^2 \right\rangle$, we need to expand out the operator first:

$$\begin{aligned}
\hat{x}^2 &= \frac{\hbar}{2m\omega}(a_+ + a_-)^2 \\
&= \frac{\hbar}{2m\omega}(a_+^2 + a_+a_- + a_-a_+ + a_-^2)
\end{aligned}$$

Now to make our lives a bit easier, we can use the commutator relation

$$[a_-, a_+] = 1 \implies a_-a_+ = 1 + a_+a_-$$

So therefore

$$\hat{x}^2 = \frac{\hbar}{2m\omega}(a_+^2 + 2a_+a_- + 1 + a_-^2)$$

Now we can calculate $\left\langle x^2 \right\rangle$. Note that $a_+^2 \left|\alpha\right\rangle = (\alpha^\star)^2 \left|\alpha\right\rangle$ and $a_-^2 \left|\alpha\right\rangle = \alpha^2 \left|\alpha\right\rangle$, so therefore:

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \frac{\hbar}{2m\omega} \left\langle \alpha \middle| (a_+^2 + 2a_+a_- + 1 + a_-^2)\alpha \right\rangle \\
&= \frac{\hbar}{2m\omega} \left\langle \alpha \middle| ((\alpha^\star)^2 + 2\alpha^\star\alpha \left|\alpha\right\rangle + 1 + \alpha^2 \left|\alpha\right\rangle) \right\rangle \\
&= \frac{\hbar}{2m\omega} \left\langle \alpha \middle| ((\alpha^\star + \alpha)^2 + 1)\alpha \right\rangle \\
&= \frac{\hbar}{2m\omega}((\alpha^\star + \alpha)^2 + 1) \langle \alpha | \alpha \rangle \\
&= \frac{\hbar}{2m\omega} \left[(\alpha^\star + \alpha)^2 + 1\right]
\end{aligned}$$

And we use a very similar approach for $\left\langle p^2 \right\rangle$. Here, squaring the commutator gives:

$$\begin{aligned}
\left\langle p^2 \right\rangle &= -\frac{\hbar m\omega}{2}(a_+ - a_-)^2 \\
&= -\frac{\hbar m\omega}{2}(a_+^2 - a_+a_- - a_-a_+ + a_-^2) \\
&= -\frac{\hbar m\omega}{2}(a_+^2 - 2a_+a_- - 1 + a_-^2)
\end{aligned}$$

So therefore:

$$\begin{aligned}
\left\langle p^2 \right\rangle &= -\frac{\hbar m\omega}{2}\left\langle \alpha \middle| (a_+^2 - 2a_+a_- - 1 + a_-^2)\alpha \right\rangle \\
&= -\frac{\hbar m\omega}{2}\left\langle \alpha \middle| ((\alpha^\star)^2 - 2\alpha^\star\alpha + \alpha^2\alpha - 1)\alpha \right\rangle \\
&= -\frac{\hbar m\omega}{2}\left[(\alpha^\star - \alpha)^2 - 1\right]
\end{aligned}$$

(b) Find σ_x and σ_p; show that $\sigma_x\sigma_p = \hbar/2$.

Solution: Now to check that the uncertainty relation is satisfied, we can write:

$$\begin{aligned}
\sigma_x^2 &= \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 \\
&= \frac{\hbar}{2m\omega}\left[1 + (\alpha + \alpha^\star)^2\right] - \frac{\hbar}{2m\omega}(\alpha^\star + \alpha)^2 \\
&= \frac{\hbar}{2m\omega} \implies \sigma_x = \sqrt{\frac{\hbar}{2m\omega}}
\end{aligned}$$

And similarly,

$$\begin{aligned}
\sigma_p^2 &= \left\langle p^2 \right\rangle - \left\langle p \right\rangle^2 \\
&= -\frac{\hbar m\omega}{2}\left[(\alpha^\star - \alpha)^2 - 1\right] + \frac{\hbar m\omega}{2}(\alpha^\star - \alpha)^2 \\
&= \frac{\hbar m\omega}{2} \implies \sigma_p = \sqrt{\frac{\hbar m\omega}{2}}
\end{aligned}$$

So now we can calculate $\sigma_x\sigma_p$:

$$\begin{aligned}
\sigma_x\sigma_p &= \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\frac{\hbar m\omega}{2}} \\
&= \frac{\hbar}{2}
\end{aligned}$$

And so the uncertainty relation is satsified.

(c) Like any other wave function, a coherent state can be expanded in terms of energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Show that the expansion coefficients are

$$c_n = \frac{\alpha^n}{\sqrt{n!}}c_0$$

Solution: Here, we can use the relation that α is an eigenfunction of the lowering operator, so $a_- |\alpha\rangle = \alpha |\alpha\rangle$. Writing this in terms of the expansion:

$$a_- \sum_{n=0}^{\infty} c_n |n\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

We also know that the lowering operator has the following relation:

$$a_- |n\rangle = \sqrt{n} |n-1\rangle$$

So therefore,

$$\sum_{n=1}^{\infty} c_n \sqrt{n} |n\rangle = \alpha \sum_{n=0}^{\infty} \alpha c_n |n\rangle$$

So now our expansion becomes (I dropped the bounds here becuase all we care about is the summation terms themselves):

$$\sum c_n \sqrt{n} |n-1\rangle = \sum \alpha c_n |n\rangle$$

Now, since the eigenstates $|n\rangle$ are an orthonormal basis, we can compare coefficients of the same eigenfunction with each other to obtain the following relation:

$$c_n \sqrt{n} = \alpha c_{n-1}$$

And so

$$c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}} c_{n-2}$$

Notice we can continue this recurrence relation all the way down to c_0. We can do this n times, so therefore our final relation becomes:

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

As desired.

(d) Determine c_0 by normalizing $|\alpha\rangle$

Solution: To normalize, we want

$$\begin{aligned} 1 &= \sum_n |c_n|^2 \\ &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} c_0 \\ &= |c_0|^2 \left(\alpha + \frac{(\alpha^2)^2}{\sqrt{2}} + \frac{(\alpha^2)^3}{2} + \cdots \right) \end{aligned}$$

Note that this is in fact the taylor expansion of $e^{\alpha^2/2}$, so therefore

$$\begin{aligned} |c_0| &= \left(\frac{1}{e^{\alpha^2}}\right)^{1/2} \\ &= e^{-\alpha^2/2} \end{aligned}$$

And since we have α^2 then it's guaranteed to be positive, so we can put absolute value around α:

$$c_0 = e^{-|\alpha|^2/2}$$

(e) Now put in the time dependence:

$$|n\rangle \to e^{-iE_n t/\hbar} |n\rangle$$

and show that $|\alpha(t)\rangle$ remains an eigenstate of a_-, but the *eigenvalue* evolves in time

$$\alpha(t) = e^{-i\omega t}\alpha$$

So a coherent state *stays* coherent, and continues to minimize the uncertainty product.

Solution: Adding the time dependence means adding a phase, as suggested by the question. Therefore,

$$|\alpha\rangle = \sum \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle$$

To find the eigenvalues, we act the lowering operator:

$$\begin{aligned} a_- |\alpha\rangle &= \sum \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle \\ &= e^{-iE_n t/\hbar} a_- \sum c_n |n\rangle \\ &= e^{-iE_n t/\hbar} \alpha |\alpha\rangle \end{aligned}$$

And so therefore we have the expression:

$$\alpha(t) = e^{-iE_n t}\alpha$$

The energies do not evolve in time at all, so therefore the only time-dependent term is the t in the exponent.

(f) Is the ground state ($|n = 0\rangle$) itself a coherent state? If so, what is the eigenvalue?

Solution: $|0\rangle$ is a coherent state, since acting the lowering operator on it:

$$a_- |0\rangle = 0 |0\rangle$$

And so it's eigenvalue is 0.

(g) Are the coherent state orthogonal? Compute $\langle\alpha|\beta\rangle$.

Solution: Suppose that α and β are two coherent states. Then, if we compute the product:

$$\begin{aligned}\langle\alpha|\beta\rangle &= \int \sum_n c_n^\star \langle n| \cdot \sum_m c_m |m\rangle \, dx \\ &= \int \sum c_n^\star c_m \langle n|m\rangle\end{aligned}$$

Since $|m\rangle$ and $|n\rangle$ are an orthonormal basis, then $\int \langle n|m\rangle$ is the dirac delta function, or in other words:

$$\langle\alpha|\beta\rangle = \sum c_n^\star c_m \delta(n-m)$$

And so therefore $\langle\alpha|\beta\rangle$ are orthogonal.

Problem 4

(a) For a function $f(x)$ that can be expaned in a Taylor series, show that

$$f(x+x_0) = e^{i\hat{p}x_0/\hbar} f(x)$$

(where x_0 is any constant distance). For this reason, $\hat{p}/\hbar$ is called the **generator of translations in space**. *Note:* The exponential of an *operator* is defined by the power series expansion: $e^{\hat{Q}} \equiv 1 + \hat{Q} + (1/2)Q^2 + \cdots$.

Solution: First we can write out the Taylor expansion for the right hand side centered at $x = x_0$:

$$f(x+x_0) = \sum \frac{1}{n!} f^n(x) x_0^n = \sum \frac{1}{n!} \frac{d^n f(x)}{dx^n} x_0^n$$

Recall that we have: $\hat{p} = -i\hbar \frac{\partial}{\partial x}$, so:

$$f(x+x_0) = \sum \frac{x_0^n \hat{p}^n}{n!} \frac{1}{(-i\hbar)^n} f(x)$$

Note that the exponentiated terms are just the taylor series of $e^{\hat{p}x_0/(-i\hbar)}$ so therefore:

$$f(x+x_0) = e^{\hat{p}x_0/(-i\hbar)} f(x) = e^{i\hat{p}x_0/\hbar}$$

As desired.

(b) If $\Psi(x,t)$ satisfies the (time-independent) Schrödinger equation, show that

$$\Psi(x, t+t_0) = e^{-i\hat{H}t_0/\hbar} \Psi(x,t)$$

Solution: We do the same thing as the previous problem, Taylor expand $\Psi(x, t+t_0)$:

$$\Psi(x, t+t_0) = \sum \frac{1}{n} \frac{\partial^n}{\partial t^n} \Psi(x,t) t_0^n$$

Since we know that $\hat{H} = i\hbar \frac{\partial}{\partial t}$ in the time-dependent Schrödinger equation, then

$$\Psi(x, t+t_0) = \sum \frac{1}{n!} \hat{H}^n \Psi(x,t) \cdot \frac{t_0^n}{(i\hbar)^n}$$

Just like the previous part, the terms which are exponentiated give us $e^{\frac{\hat{H}t_0}{i\hbar}} = e^{-i\hat{H}t_0/\hbar}$ so therefore

$$\Psi(x, t+t_0) = e^{-i\hat{H}t_0/\hbar} \Psi(x,t)$$

As desired.

(where t_0 is any constant time); $-\hat{H}/\hbar$ is called the **generator of translations in time**

(c) Show that the expectation value of a dynamical variable $Q(x,p,t)$ at time $t+t_0$ can be written

$$\langle Q \rangle_{t+t_0} = \langle \Psi(x,t)| \, e^{i\hat{H}t_0/\hbar}\hat{Q}(\hat{x},\hat{p},t+t_0)e^{-i\hat{H}t_0/\hbar} \, |\Psi(x,t)\rangle$$

Use this to recover Equation 3.71. *Hint:* Let $t_0 = dt$, and expand to first order in dt.

Solution: Let's consider the form of this equation. From the previous problem, we know that

$$|\Psi(x,t+t_0)\rangle = e^{-i\hat{H}t_0/\hbar} \, |\Psi(x,t)\rangle$$

Therefore, the adjoint operation would be:

$$\langle \Psi(x,t+t_0)| = \langle \Psi(x,t)| \, (e^{-i\hat{H}t_0/\hbar})^\dagger = \langle \Psi(x,t)| \, e^{i\hat{H}t_0/\hbar}$$

Knowing these two relations, we can rewrite the expectation value as:

$$\langle Q \rangle_{t+t_0} = \langle \Psi(x,t+t_0)| \, \hat{Q}(\hat{x},\hat{p},t+t_0) \, |\Psi(x,t+t_0)\rangle$$

which is the standard relation for the expectation value at $t+t_0$. To derive Equation 3.71, we use the product rule to derive the following expression (I omitted what Ψ and Q are functions of becuase otherwise there would be too many parentheses):

$$\begin{aligned}
\frac{\partial \langle Q \rangle}{\partial t} &= \frac{\partial}{\partial t} \langle \Psi| \, \hat{Q} \, |\Psi\rangle \\
&= \left(\frac{\partial}{\partial t} \langle \Psi|\right) \hat{Q} \, |\Psi\rangle + \langle \Psi| \, \frac{\partial \hat{Q}}{\partial t} \, |\Psi\rangle + \langle \Psi| \, \hat{Q} \left(\frac{\partial}{\partial t} \, |\Psi\rangle\right)
\end{aligned}$$

The term in the middle evaluates to $\left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$. Now let's since $\hat{H} = i\hbar\frac{\partial}{\partial t} = -\frac{\hbar}{i}\frac{\partial}{\partial t}$ from the time-dependent Schrödinger equation, then we have:

$$\begin{aligned}
\frac{\partial \langle Q \rangle}{\partial t} &= \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle + \langle \Psi| \, \frac{i}{\hbar}\hat{H}\hat{Q} \, |\Psi\rangle + \langle \Psi| - \frac{i}{\hbar}\hat{Q}\hat{H} \, |\Psi\rangle \\
&= \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle + \frac{i}{\hbar}\left(\hat{H}\hat{Q} - \hat{Q}\hat{H}\right) \langle \Psi|\Psi\rangle \\
&= \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle + \frac{i}{\hbar}\left\langle [\hat{H},\hat{Q}] \right\rangle
\end{aligned}$$

As desired.
