Eric Du | EE 120
HW 04 | Signals and Systems | February 21, 2024

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Collaborators

I worked with the following people to complete this assignment:

- Teja Nivarthi: 3036508567
- Nikhil Maserang: 3036978230

Problem 1

An important concept in many communications applications is the correlation between two signals. This problem is meant to serve as a brief introduction to correlation functions and some of their properties.

Let $x(t)$ and $y(t)$ be two signals; then the *correlation function* is defined as

$$r_{xy}(t) = (x \circ y)(t) = \operatorname{corr}(x, y) = \int_{-\infty}^{\infty} x(\tau) y(t + \tau)\, d\tau$$

The function r_{xx} is usually referred to as the *autocorrelation function* of the signal $x(t)$, while $r_{xy}(t)$ is often called a *cross-correlation function*.

a) What is the relationship between $r_{xy}(t)$ and $r_{yx}(t)$?

Solution: We can write $r_{yx}(t)$ as:

$$r_{yx}(t) = \int_{-\infty}^{\infty} y(\tau) x(t + \tau)\, d\tau$$

Now, let $\tau' = t + \tau$, meaning that $\tau = \tau' - t$. Therefore:

$$r_{yx}(t) = \int_{-\infty}^{\infty} x(\tau') y(-t + \tau')\, d\tau'$$

This is the same as $r_{xy}(t)$, except that we now have $-t$ instead of t. Therefore, the relationship is:

$$r_{xy}(t) = r_{yx}(-t)$$

□

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b) Compute the odd portion of $r_{xx}(t)$.

Solution: We will begin by using the property that the odd part of any function can be written as:

$$f_o(t) = \frac{f(t) - f(-t)}{2}$$

Applying this to our autocorrelation function:

$$r_{xx,o}(t) = \frac{r_{xx}(t) - r_{xx}(-t)}{2}$$

Then, from part (a) we know that $r_{xx}(t) = r_{xx}(-t)$, so this actually just becomes 0. □

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c) Suppose that $y(t) = x(t + T)$. Express $r_{xy}(t)$ and $r_{yy}(t)$ in terms of $r_{xx}(t)$.

Solution: We have:

$$\begin{aligned}
r_{xx}(t) &= \int_{-\infty}^{\infty} x(\tau) x(t + \tau)\, d\tau \\
r_{xy}(t) &= \int_{-\infty}^{\infty} x(\tau) y(t + \tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau) x(t + \tau + T)\, d\tau \\
r_{yy}(t) &= \int_{-\infty}^{\infty} y(\tau) y(t + \tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau + T) x(t + \tau + T)\, d\tau
\end{aligned}$$

Notice that for $r_{xy}(t)$, this is the same equation as $r_{xx}(t)$ but with $t + T$ as an input. Therefore, we have $r_{xy}(t) = r_{xx}(t + T)$. For $r_{yy}(t)$, let $\tau' = \tau + T$, then we can write:

$$r_{yy}(t) = \int_{-\infty}^{\infty} x(\tau') x(t + \tau')\, d\tau'$$

which is the same integral as $r_{xx}(t)$, so we conclude that $r_{yy}(t) = r_{xx}(t)$. □

Problem 2

a) **One-sided decaying potential:** The CTFT of the one-sided decaying exponential $x(t) = e^{-at}u(t), \forall t$, where $a > 0 \in \mathbb{R}$ is given by

$$\mathcal{F}\{x(t)\} = X(\omega) = \frac{1}{a + i\omega}$$

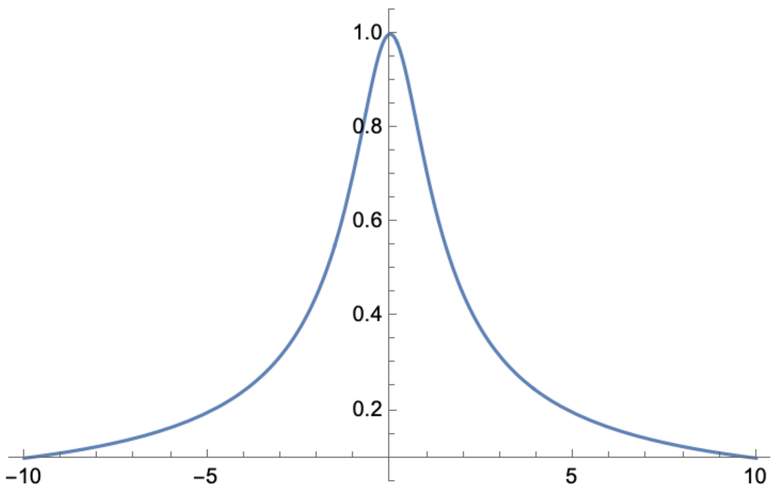
as shown in discussion and lecture. Provide a well-labeled sketch of $|X(\omega)|, \forall\omega$. Describe how the bandwidth of x varies in relation to the decay rate a of the one-sided potential?

Note: Consider as the bandwidth frequency ω_B the threshold where $|X(\omega_B) = \frac{1}{\sqrt{2}} \max_\omega |X(\omega)|$. This bandwidth frequency (or cutoff frequency) is chosen by convention to correspond to the "3 dB (decibel)" drop in magnitude. Notice that $20\log_{10}(\frac{1}{\sqrt{2}}) \approx -3$ dB. This has historical roots in circuit theory and design.

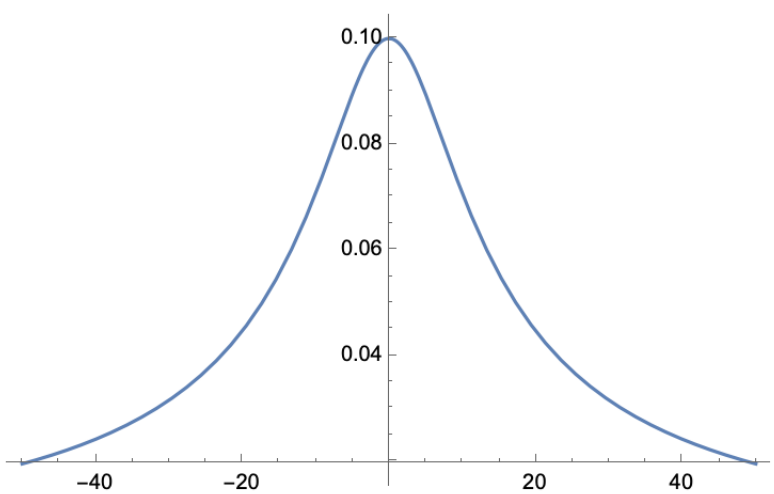
Solution: We can write:

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

For $a = 1$, the plot looks like this:



For $a = 10$, the plot looks like this:



What we can see is that the peak of $|X(\omega)|$ shrinks considerably, but the signal also takes much longer to decay. This is evidenced by the fact that at for $a = 1$, $|X(\omega)|$ is almost zero at $\omega = 10$, whereas it is roughly 0.0075 when $a = 10$. The bandwidth therefore increases as a increases.

Intuitively this also makes sense. As a increases, then $|X(\omega)|$ depends less on ω for small values of ω. In other words, when a is larger, then ω needs to also be larger in order for us to see the effects of ω in $|X(\omega)|$. Thus, there is less ω

dependence at small ω, therefore the bandwidth increases. $\square$

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b) For this and all following signals, determine $X(\omega)$, their corresponding continuous-time Fourier transform (CTFT).
Two-sided decaying exponential: $x(t) = e^{-a|t|}, \forall t$, where $a > 0$.

Solution: We solve this in almost the identical way to problem 1.2b:

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at} e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt \\
&= \frac{1}{a - i\omega} e^{(a-i\omega)t} \Big| -\infty^{0} - \frac{1}{a + i\omega} e^{-(a+i\omega)t} \Big|_{0}^{\infty} \\
&= \frac{1}{a - i\omega} + \frac{1}{a + i\omega} \\
&= \frac{2a}{a^2 + \omega^2}
\end{aligned}$$

This aligns exactly with the solution to 1.2b, since we have te condition that $a > 0$ here (since we have $e^{-a|t|}$ instead of $e^{a|t|}$ that we had in 1.2b.) $\square$

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c) **Sinusoid:** $x(t) = \sin(\omega_0 t), \forall t$.
Hint: It may be helpful to decompose the signal into complex exponentials using Euler's formula.

Solution: We follow the hint, and use the fact that $\sin(\omega_0 t) = \frac{1}{2i}(e^{i\omega_0 t} - e^{-i\omega_0 t})$. Therefore:

$$\begin{aligned}
X(\omega) &= \frac{1}{2i} \left[\int_{-\infty}^{\infty} e^{i\omega_0 t - i\omega t} - e^{-i\omega_0 t - i\omega t}\, dt \right] \\
&= \frac{1}{2i} (\delta(\omega - \omega_0) - \delta(\omega + \omega_0))
\end{aligned}$$

This makes sense – the Fourier transform of a pure sine wave is just a linear combination of the left-travelling and right-travelling wave at frequency ω_0. $\square$

Problem 3

a) Calculate the CTFT of a finite-duration rectangular signal of arbitrary width and length.

$$x(t) = \begin{cases} b & |t| \leq a \\ 0 & \text{otherwise} \end{cases}$$

Describe how the CTFT of this signal changes as the parameters a and b change. You can assume that the parameters a and b have appropriately-chosen positive values. Experss your final answer as an expression involving a sinc function defined as

$$X(\omega) = \text{sinc}(\omega) = \begin{cases} \frac{\sin(\pi\omega)}{\pi\omega} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Note: As a sanity check and a bit of background for part (b), the solution to this problem should demonstrate a phenomenon known as the time-frequency uncertainty principle. This principle states that a given function cannot be arbitrarily compact in both time and frequency. Practically what this means is that as you shrink (define using fewer timesteps) the time-domain representation, the number of frequencies defining your impulse response will increase (frequency spectrum will widen), and vice versa.

Solution: Here we just take the Fourier transform:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\, dt \\ &= \int_{-a}^{a} be^{-i\omega t}\, dt \\ &= -\frac{b}{i\omega}\left[e^{-i\omega t}\right]_{-a}^{a} \\ &= -\frac{b}{i\omega}\left[e^{i\omega a} - e^{-i\omega a}\right] \\ &= \frac{2b}{\omega}\sin(\omega a) \end{aligned}$$

Then, using the property of the sinc function, we can see that:

$$\text{sinc}\left(\frac{\omega a}{\pi}\right) = \frac{\sin(\omega a)}{\omega a}$$

So we can write:

$$X(\omega) = 2ab\text{sinc}\left(\frac{\omega a}{\pi}\right)$$

□

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b) We call a signal bandlimited (in frequency) if $\exists B_1 \leq B_2 \in \mathbb{R}$ such that $\forall \omega \leq B_1, \omega \geq B_2, X(\omega) = 0$. Likewise, we call a signal finite duration (in time) if there $\exists T_1 \leq T_2 \in \mathbb{R}$ such that $\forall t \leq T_1, t > T_2, x(t) = 0$. Show that if a signal has finite duration, then it is not bandlimited. You can ignore the trivial case of $x(t) = 0$. Explain why this is a manifestation of the time-frequency uncertainty principle.

Hint: It is possible to re-express **any** finite-duration or band-limited signal in terms of the familiar box we have been working with.

Solution: We can always write a finite duration signal as a product of a rectangular box of height 1 and a (potentially infinite) signal $f(t)$:

$$x(t) = f(t) \cdot \sqcap(t)$$

Then, we have to take the Fourier transform of this. Using the convoluiton theorem, we have:

$$X(\omega) = \mathcal{F}\{f(t)\} * \mathcal{F}\{\sqcap(t)\}$$

We know the convolution of the square pulse is a sinc function, which is not band limited (part a). Then, since the width of functions is additive in a convolution, then since $\mathcal{F}\{\sqcap(t)\}$ is not band limited then the entire transform $X(\omega)$ is also not band limited.

This is a manifestation of the time-frequency uncertainty principle, since it says that if we have a localized function in time, then taking the Fourier transform gives us a function that isn't localized in frequency. $\square$

Problem 4

Consider the following impulse responses of two LTI systems:

$$h(t) = \delta(t-2)$$
$$g(t) = e^{-\alpha|t|}, \alpha \in \mathbb{R}+$$

a) Find $H(\omega)$ and $G(\omega)$, the frequency responses (CTFT) of the systems.

Solution: We compute $H(\omega)$ first:

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} \delta(t-2)e^{-i\omega t}\, dt \\ &= e^{-2i\omega} \end{aligned}$$

For $G(\omega)$, this is the same thing as problem 2b, so we can just use that result:

$$G(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}$$

where $\alpha > 0$. □

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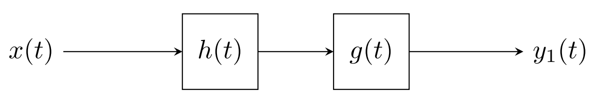
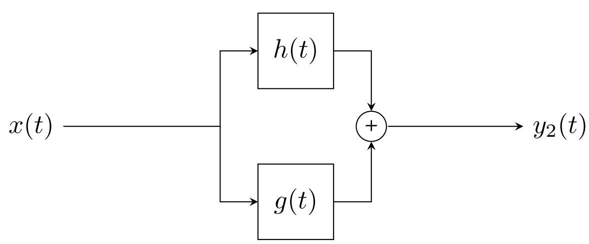
b) Consider interconnected LTI systems with impulse responses $h(t)$ and $g(t)$: System 1, with overall impulse response of $s_1(t)$:



System 2, with overall impulse response of $s_2(t)$:



Find the impulse responses $s_1(t)$ and $s_2(t)$. Also find their respective frequency responses $S_1(\omega)$ and $S_2(\omega)$.

Solution: Starting with system 1, since the impulse responses are connected in series, then we know that the impulse response of the entire sytsem is given by the convolution of the two:

$$s_1(t) = h(t) * g(t) = \int_{-\infty}^{\infty} \delta(\tau - 2)e^{-\alpha|t-\tau|}\, d\tau = e^{-\alpha|t-2|}$$

For system 2, the impulse response is the sum of h and g, since the systems are connected in parallel:

$$s_2(t) = h(t) + g(t) = \delta(t-2) + e^{-\alpha|t|}$$

Now for $S_1(\omega)$ and $S_2(\omega)$. Since $s_1(t)$ is a convolution of two functions, then the convolution theorem says that the Fourier transform of it is going to be a product of the two transformed functions. That is,

$$S_1(\omega) = \mathcal{F}\{s_1(t)\} = \mathcal{F}\{h(t) * g(t)\} = H(\omega) \cdot G(\omega) = \frac{2\alpha e^{-2i\omega}}{\alpha^2 + \omega^2}$$

For $S_2(\omega)$, we know that the Fourier transform is linear, so:

$$S_2(\omega) = \mathcal{F}\{s_2(t)\} = \mathcal{F}\{h(t) + g(t)\} = \mathcal{F}\{h(t)\} + \mathcal{F}\{g(t)\} = e^{-2i\omega} + \frac{2\alpha}{\alpha^2 + \omega^2}$$

□

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c) Now suppose

$$x(t) = e^{i\omega_0 t}$$

Calculate the output $y(t)$ of the two systems S_1 and S_2 when given $x(t)$ as an input.

Hint: Convert $x(t)$ to the frequency domain, and feed it through the composite system before taking the inverse transform of the result. As a further hint, note that the convolution in the time domain is equivalent to multiplication in the frequency domain.

Solution: For system 1, we apply the hint and convert into the frequency domain. The Fourier transform of $x(t)$ is $X(\omega) = \delta(\omega - \omega_0)$. Then, since we know that the system output is characterized by a convolution, then the convolution theorem makes this especially easy:

$$Y(\omega) = \mathcal{F}\{y(t)\} = \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{h(t)\} \cdot \mathcal{F}\{g(t)\} = \delta(\omega - \omega_0)\frac{2\alpha e^{-2i\omega}}{\alpha^2 + \omega^2}$$

Then, we have to take the inverse Fourier transform of this in order to get back into temporal space:

$$y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\frac{2\alpha e^{-2i\omega}}{\alpha^2 + \omega^2} e^{i\omega t}\, d\omega = \frac{\alpha e^{-2i\omega_0}}{\pi(\alpha^2 + \omega^2)} e^{i\omega_0 t}$$

We'll use the same trick for system 2:

$$Y(\omega) = \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{s_2(t)\} = \delta(\omega - \omega_0)\left(e^{-2i\omega} + \frac{2\alpha}{\alpha^2 + \omega^2}\right)$$

Then, we can take the inverse Fourier transform to get $y(t)$:

$$y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\left(e^{-2i\omega} + \frac{2\alpha}{\alpha^2 + \omega^2}\right) e^{i\omega t}\, d\omega = \left(e^{-2i\omega_0} + \frac{2\alpha}{\alpha^2 + \omega_0^2}\right) e^{i\omega_0 t}$$

□