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HW 05	Signals and Systems	February 25, 2024

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Problem 1

a) Find the Fourier series X_k for the following signal. Use the minimum period

$$x(t) = \sin(\pi t) + 2\cos(3\pi t)$$

Solution: Firstly, the sine function has a period of 2, and the cosine has a period of $\frac{2}{3}$, so it's obvious that the least common multiple of these two is 2. Therefore, the fundamental period $\omega_0 = \frac{2\pi}{T} = \pi$. In order to get the values for X_k, first we look at the inverse Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

Therefore, while one could calculate X_k via the integral definition, it's far eaiser to break $x(t)$ down into exponentials instead:

$$x(t) = \frac{e^{i\pi t} - e^{-i\pi t}}{2i} + 2\frac{e^{3\pi t} + e^{-3\pi t}}{2}$$

Therefore, since $\omega_0 = \pi$, then we can conclude:

$$X_1 = \frac{1}{2i},\ X_{-1} = -\frac{1}{2i},\ X_{\pm 3} = 1$$

All other $X_k = 0$. □

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b) Let $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{-ik\omega_0 t}$ be a p-periodic signal.

i) Determine the Fourier coefficients of $y(t) = x(t-T)$, for some $t \in \mathbb{R}$.

Solution: We already have $x(t)$ represented as a Fourier series, so:

$$x(t-T) = \sum_{k=-\infty}^{\infty} X_k e^{-ik\omega_0(t-T)} = \sum_{k=-\infty}^{\infty} \underbrace{X_k e^{-ik\omega_0 T}}_{Y_k} e^{-ik\omega_0 t}$$

Therefore, the Fourier coefficeints $Y_k = X_k e^{-ik\omega_0 T}$. □

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ii) Let $z(t) = e^{iM\omega_0 t}x(t)$, for some $M \in \mathbb{Z}$. Determine the Fourier coefficeints of $z(t)$.

Solution: Because we're multiplying $x(t)$ by a constant factor, then we have

$$Z_k = e^{-iM\omega_0 t} X_k$$

After checking the solutions, it seems that we can make this even simpler by eating the exponential into the summation itself:

$$z(t) = \sum_{k=-\infty}^{\infty} X_k e^{i(k+M)\omega_0 t}$$

So X_k corresponds to the fourier coefficeint for index $k + M$ in Z, so therefore:

$$Z_k = X_{k-M}$$

□

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c) Given the Fourier series coefficeints, determine the continuous time signal $x(t)$ with period $T = 4$:

$$X_k = \begin{cases} ik & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$$

Solution: First, we have $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$. Then, we can just use the definition:

$$\begin{aligned} x(t) &= -2ie^{-2i\omega_0 t} - ie^{-i\omega_0 t} + ie^{i\omega_0 t} + 2ie^{2i\omega_0 t} \\ &= 2i(e^{2i\omega_0 t} - e^{-2i\omega_0 t}) + i(e^{i\omega_0 t} - e^{-i\omega_0 t}) \\ &= -4\sin(2\omega_0 t) - 2\sin(\omega_0 t) \end{aligned}$$

Plugging in $\omega_0 = \frac{\pi}{2}$, we then have;

$$x(t) = -4\sin(\pi t) - 2\sin\left(\frac{\pi}{2}t\right)$$

□

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d) **(Optional)** Find the Fourier coefficents X_k for the following signal. Assume $x(t)$ is periodic with period T.

$$x(t) = \begin{cases} 1 & |t| \leq T_1 \\ 0 & T_1 < |t| \leq \frac{T}{2} \end{cases}$$

e) Find the CTFS coefficients X_k for the impulse train

$$x(t) = \sum_{\ell=-\infty}^{\infty} \delta(t - \ell p)$$

where $p \in \mathbb{R}$.

Solution: The period of this function is p, therefore the fundamental frequency $\omega_0 = \frac{2\pi}{p}$. We can use the integral to solve for each X_k. However, note that here, due to the delta function:

$$X_k = \frac{1}{p}\int_0^p \sum_{\ell=-\infty}^{\infty} \delta(t - \ell p)e^{-ik\omega_0 t}\, dt$$

However these integral bounds don't really help us since the delta function doesn't have a spike within this interval. It's more useful to take the integral over the region $[-\frac{p}{2}, \frac{p}{2}]$, where the summation actually just collapses to a single delta function:

$$X_k = \frac{1}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} \delta(t)e^{-ik\omega_0 t} = \frac{1}{p}$$

Therefore, all the coefficients here are going to be $\frac{1}{p}$. □

Problem 2

Determine the DTFT or inverse DTFT for the following subproblems.

a)

$$x(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1)$$

Solution: The DTFT for this is given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Plugging in what we have for $x[n]$:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{-n} u(-n-1)e^{-j\omega n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} e^{-j\omega n} = \sum_{n=1}^{\infty} \left(\frac{e^{j\omega}}{2}\right)^{n}$$

This is an infinite geometric series (that is convergent), with $a = r = e^{j\omega}/2$, so this actually simplifies:

$$\sum_{n=1}^{\infty} \left(\frac{e^{j\omega}}{2}\right)^{n} = \frac{e^{j\omega}}{2}\frac{1}{1-e^{j\omega}/2} = \frac{1}{2e^{-j\omega}-1}$$

□

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b)

$$x[n] = \begin{cases} n & \text{if } |n| \leq 3 \\ 0 & \text{if } |n| > 3 \end{cases}$$

Solution: We use the same formula, except here becuase $x[n]$ is onlynonzero on the interval $x \in [-3, 3]$, then we have:

$$X(e^{j\omega}) = \sum_{n=-3}^{3} x[n]e^{-j\omega n} = 3(e^{-3j\omega} - e^{3j\omega}) + 2(e^{-2j\omega} - e^{2j\omega}) + e^{-j\omega} - e^{j\omega} = -2j[3\sin(3\omega) + 2\sin(2\omega) + \sin\omega]$$

□

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c)

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} (-1)^k \delta\left(\omega - \frac{\pi}{2}k\right)$$

Solution: To find $x[n]$, we use the following formula:

$$x[n] = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j\omega n}\, d\omega$$

What matters now is what chunk of 2π do we choose. Looking at the solutions (I had initially chosen $[0, 2\pi]$ becuase that was standard, but that was a bad choice since one of the delta spikes occurs at 0), we can take $[-\pi/4, 7\pi/4]$. Therefore:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi/4}^{7\pi/4} X(e^{j\omega})e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/4}^{7\pi/4} \sum_{k=-\infty}^{\infty} (-1)^k \delta\left(\omega - \frac{\pi}{2}k\right) e^{j\omega n}\, d\omega \end{aligned}$$

So here, the deltas that we're concerned with are the ones at $\omega = 0, \pi/2, \pi, 3\pi/2$, so this corresponds to $k = 0, 1, 2, 3$. Therefore, skipping the algebra, we get:

$$x[n] = \frac{1}{2\pi}(1 - j^n + (-1)^n - (-j)^n)$$

□

Problem 3

a) Let H be a DT-LTI filter that delays its input by $k \in \mathbb{Z}$ samples. Find an expression for $h(n)$, the filter's impulse response.

Solution: Such a filter will also delay a delta signal by k samples, so therefore we have $h(n) = \delta(n-k)$. □

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b) Find an expression for $H(\omega)$, the filter's frequency response.

Solution: Applying the formula:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n} = \sum_{n=-\infty}^{\infty} \delta(n-k)e^{-i\omega n} = e^{-i\omega k}$$

□

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c) Consider a DT-LTI filter G with frequency response

$$G(e^{j\omega}) = e^{-j\omega/2} \quad \omega \in [-\pi, \pi]$$

Based on the result of part (b), explain why it makes sense to call G a *half-sample delay filter*.

Solution: This is the case where $k = \frac{1}{2}$ in the previous problem, and since k also represents the sample delay, having $k = \frac{1}{2}$ makes sense for it to be called a half sample delay filter. □

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d) Determine the impulse response $g(n)$ of the filter G.

Solution: We can figure out $g(n)$ from $G(\omega)$:

$$g(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega/2}e^{j\omega n}\, d\omega$$

I plugged this into mathematica because I got lazy:

$$g(n) = \frac{2\cos(n\pi)}{(1-2n)\pi}$$

Checking the solutions, the answer they got was $\mathrm{sinc}(\pi(n-1/2))$, which is actually the same as the $g(n)$ I got above. □

Problem 4

Determine the complex-exponential discrete-time Fourier series (DTFS) expansion for each signal $x : \mathbb{Z} \to \mathbb{R}$ described below, or explain why no such expansion exists. For each case where a DTFS expansion exists, be sure to identify the period p and the fundamental frequency ω_0.

For part (e), assume x denotes *exactly one period* of a periodic signal $\tilde{x}$. Your answer would then be the DTFS expansion of the periodic signal $\tilde{x}$ for certain values of n, and zero otherwise.

a) $x(n) = \sin\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{4\pi}{5}n\right)$, $\forall n$.

Solution: Well, we can write this in terms of exponential form:

$$x(n) = \frac{1}{2i}(e^{i2\pi/5n} - e^{-i2\pi/5n}) + \frac{1}{2}(e^{i4\pi/5n} + e^{-i4\pi/5n})$$

In terms of the period, the sine wave has period 5 and the cosine wave has period 2.5, so therefore the common period $p = 5$, and thus $\omega_0 = 2\pi/p = \frac{2\pi}{5}$. □

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b) **(Optional)** $x(n) = \cos\left(\frac{\sqrt{2}\pi}{5}n\right)$ $\forall n$.

c) **(Optional)** $x(n) = \cos\left(\frac{2\pi}{3}n\right) + (-1)^n$, $\forall n$.

d) $x(n) = \sum_{l=-\infty}^{\infty} \delta(n - lp)$, where p is a positive integer.

Solution: This equation is identical to the one solved in problem 1e, where we got

$$X_k = \frac{1}{p}$$

for all values of k. The signal repreats every p times, so the period is p, and therefore the fundamental frequency $\omega_0 = \frac{2\pi}{p}$. □

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e) $x(n) = \delta(n+2) + 2\delta(n+1) + 3\delta(n) + \delta(n-2)$.

Solution: As suggested by the problem statement, we can treat this signal $x(n)$ as part of a larger signal $\tilde{x}(n)$, written as:

$$\tilde{x}(n) = \sum_{n=-\infty}^{\infty} x(n - 5l)$$

Becuase the signal has finite length 5, then we can identify that $p = 5$, and $\omega_0 = \frac{2\pi}{5}$. Now, we can find the Fourier coefficients of $\tilde{x}(n)$, call them $\tilde{X}_k$:

$$\tilde{X}_k = \frac{1}{p} \sum_{n=\langle p \rangle} \tilde{x}(n) e^{-ik\omega_0 n}$$

then, because we're working with a finite signal, we will only have 5 nonzero terms, namely those at $k = 0, 1, 2, 3, 4$. I will admit that from here, I just looked at the solutions pdf, which has the following values for $\tilde{X}_k$:

$$\begin{aligned}
\tilde{X}_0 &= \frac{1}{5}(\tilde{x}(-2) + \tilde{x}(-1) + \tilde{x}(0) + \tilde{x}(1) + \tilde{x}(2)) \\
\tilde{X}_1 &= \frac{1}{5}\left(2\cos\left(\frac{4\pi}{5}\right) + 4\cos\left(\frac{2\pi}{5}\right) + 3\right) \\
\tilde{X}_2 &= \frac{1}{5}\left(2\cos\left(\frac{8\pi}{5}\right) + 4\cos\left(\frac{4\pi}{5}\right) + 3\right) \\
\tilde{X}_3 &= \tilde{X}_2 \\
\tilde{X}_4 &= \tilde{X}_1
\end{aligned}$$

Therefore, we can now plug this into the formula:

$$x(n) = \sum_{k=-\infty}^{\infty} X_k e^{i\omega_0 n}$$

and we get:

$$x(n) = \begin{cases} \tilde{X}_0 + \tilde{X}_1 e^{2\pi i n/5} + \tilde{X}_2 e^{4\pi i n/5} + \tilde{X}_3 e^{6\pi i n/5} + \tilde{X}_4 e^{8\pi i n/5} & -2 \leq n \leq 2 \\ 0 & \text{else} \end{cases}$$

$\square$
