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Problem 1

a) Consider a general qubit basis $|v\rangle, |v^\perp\rangle$ where $|v\rangle = a|0\rangle + b|1\rangle$, and $|v^\perp\rangle = b^*|0\rangle - a^*|1\rangle$ are arbitrary normalized vectors. Show that $|v\rangle$ and $|v^\perp\rangle$ are orthogonal.

Solution: To show that they're orthogonal, we can take the inner product of the two:

$$\begin{aligned}\langle v^\perp | v\rangle &= (\langle 0| b - \langle 1| a)(a|0\rangle + b|1\rangle) \\ &= ba\langle 0|0\rangle + b^2\langle 0|1\rangle - a^2\langle 1|0\rangle + ab\langle 1|1\rangle \\ &= ab - ba \\ &= 0\end{aligned}$$

And since the inner product evaluates to 0, then these two vectors are orthogonal. □

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b) Prove that the Bell state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|vv^\perp\rangle - |v^\perp v\rangle)$ with $|v\rangle$ annd $|v^\perp\rangle$ perpendicular normalized vectors, is invariant under rotations of the two qubits (applying the same rotation on both qubits). i.e., taking the form of $|v\rangle$ and $|v^\perp\rangle$ as in (a), show that the state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|vv^\perp\rangle - |v^\perp v\rangle)$ will always be equal to $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$.

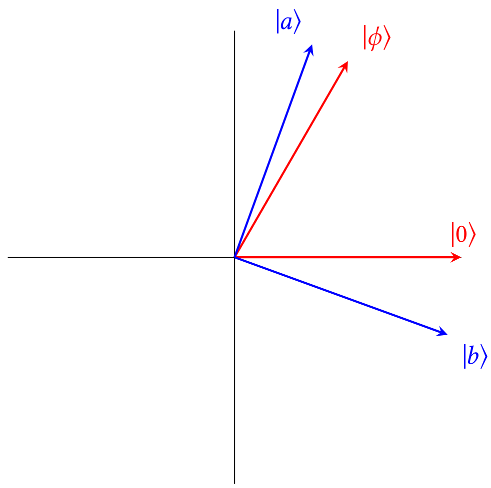
Problem 2

Consider the state $|\phi\rangle = \cos\phi\,|0\rangle + \sin\phi\,|1\rangle$. Suppose that with 1/2 probability you are given the state $|\phi\rangle$ and with 1/2 probability you're given the state $|0\rangle$, but you don't know which one you were given. What measurement basis is optimal to distinguish the two states, i.e., to guess with the greatest likelihood which of the two states you have been give?

In the following you will prove that the basis $\{|a\rangle, |b\rangle\}$ that maximizes the probability of distinguishing the two states is given by

$$|a\rangle = \cos(\pi/4 + \phi/2)\,|0\rangle + \sin(\pi/4 + \phi/2)\,|1\rangle \tag{1}$$
$$|b\rangle = \sin(\pi/4 + \phi/2)\,|0\rangle - \cos(\pi/4 + \phi/2)\,|1\rangle \tag{2}$$

Note that this basis sares the same angle bisector as te one between $|0\rangle$ and $|\phi\rangle$ as shown in the drawing below.



So how do we go about proving this? First, we have to quantify what we mean by distinguishing the two states and also define the probability of doing this successfully. To do this, use basic concepts of probability theory to argue that the probability of guessing the correct state from a measurement with basis $\{|a\rangle, |b\rangle\}$ is

$$\frac{1}{2}|\langle a|\phi\rangle|^2 + \frac{1}{2}|\langle b|0\rangle|^2 \tag{3}$$

where outcome a would imply the state is $|\phi\rangle$ and outcome b would imply the state is $|0\rangle$.

Now show that the optimal basis to distinguish these 2 states is on the real plane. To do so consider a general parametrization of our measurement basis as

$$|a\rangle = \cos(\theta)\,|0\rangle + \sin(\theta)e^{i\gamma}\,|1\rangle \tag{4}$$
$$|b\rangle = \sin(\theta)e^{-i\gamma}\,|0\rangle - \cos(\theta)\,|1\rangle \tag{5}$$

By plugging this in, you should be able to deduce that the guessing probability is maximized when $e^{i\gamma}$ is $+1$ or -1, so the optimal basis is indeed on the real plane.

So the measurement basis parametrization simplifies to

$$|a\rangle = \cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle$$
$$|b\rangle = \sin(\theta)\,|0\rangle - \cos(\theta)\,|1\rangle$$

Now you can write out the corresponding probability of successfully distinguishing the states, Eq. (3), in terms of θ. Find the maximum value of this with respect to the angle θ that defines the optimal measurement basis vectors $|a\rangle$ and $|b\rangle$. This should give you an equation defining one or more possible values of θ in the relevant range $[0, 2\pi)$. Insert each of your solutions back into the guessing function, eq. (3) to identify the value of θ that gives the maximum probability.

Solution:

$\square$

Problem 3

This problem will have you explore basic quantum operations on states. Consider the following situation: you start with the two qubit state

$$|\psi_0\rangle = \frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle$$

Next you apply a Hadamard gate to the first qubit then a CNOT gate with the first qubit as the control and the second as the target.

a) Verify that the initial state is normalized.

Solution: To verify that it's normalized, we have:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{i}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$$

Since the total sum is 1, then the state is fully normalized. □

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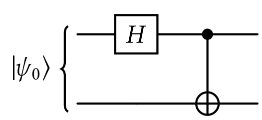
b) Draw a quantum circuit diagram representing this series of operations.

Solution: The quantum cricuit can be drawn as:



□

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c) Write the intermediate state after application of the Hadamard gate. Argue that it is also noramlized (hint: this can be done without explicit calculation using properties of unitary operations).

Solution: The Hadamdard gate does the following operation on the qubits:

$$|0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Therefore, we have:

$$\begin{aligned}|\psi_1\rangle &= \frac{1}{2}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|0\rangle + \frac{i}{2}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|1\rangle - \frac{1}{\sqrt{2}}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|1\rangle \\ &= \frac{1}{2\sqrt{2}}|00\rangle + \left(\frac{1}{2\sqrt{2}} - \frac{1}{2}\right)|01\rangle - \frac{1}{2\sqrt{2}}|10\rangle + \left(\frac{i}{2\sqrt{2}} + \frac{1}{2}\right)|11\rangle\end{aligned}$$

This state must be normalized becuase a unitary gate is a norm-preserving transformation. □

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d) What is the final state at the end of the circuit. Is it normalized?

Solution: We then send the bit through a CNOT, so the final state can be written as:

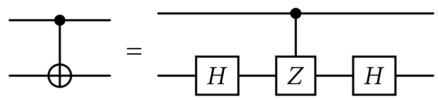
$$|\psi_2\rangle = \frac{1}{2\sqrt{2}}|00\rangle + \left(\frac{i}{2\sqrt{2}} - \frac{1}{2}\right)|01\rangle - \frac{1}{2\sqrt{2}}|11\rangle + \left(\frac{i}{2\sqrt{2}} + \frac{1}{2}\right)|10\rangle$$

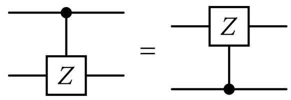
The factors in front of all the states are the same, and since the previous part is normalized, then so is this one. □

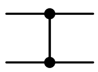
Problem 4

Circuit identities are mathematical equivalences between different operations on quantum registers. They can be useful in converting between desired algorithmic opreations and the restricted set of possible operations on a given processor, i.e., for compiling. This question will have you explore several different circuit idnetities.

a) Prove the following 2-qubit identities:





The second identity shows that the control and target in the controlled-Z gate are symmetric, and so the gate is often denoted as: 

Solution: We can do this by showing that the resulting matrices from the tensor product of the two result in the same matrix: ☐

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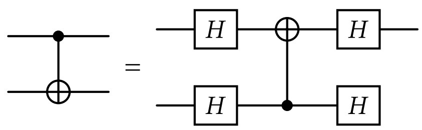
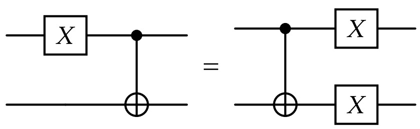
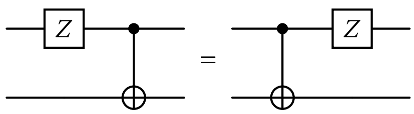
b) Prove that the direction of the CNOT is reversed in the Hadamard basis, i.e., show the following circuit identity (hint: use the prior part):

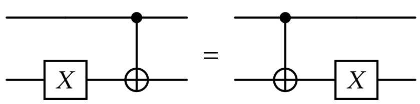


Problem 5

The following circuit identities will be useful when we study quantum error correction. The point is that single qubit operations can be "pushed through" conditional gates. Prove each one.

a) 

b) 

c) 

d)