Physics 5C Homework 7

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0 Introduction and Collaborators

I collaborated with **Eric Du**, **Nikhil Maserang**, **Nathan Song**, **Teja Nivarthi**, and **Christine Zhang** on this homework assignment.

1 Problem 4.1 in F&T (Families of Finite Square Wells)

A particle of mass m has N quantized energy levels in a one-dimensional square well of depth V_0 and width L. N is more than 10.

a) *Approximately how many bound states does the particle have in a well of the same depth but width $2L$?*

When we double the width, we can fit twice as many nodes into this well, which means that we will have twice as many energy levels allowed (given by the Node Theorem). Thus, there are twice as many bound states: $\boxed{2N}$.

b) *Approximately how many states does the particle have in a well of width L but depth $2V_0$?*

For a finite square well, the energy will scale with n^2 rather than with n. This means that doubling the depth of the well won't double the number of energy levels: rather, it will scale it by $\boxed{\sqrt{2}N}$.

c) *Approximately how many states does the particle have in a well of depth $2V_0$ and width $2L$?*

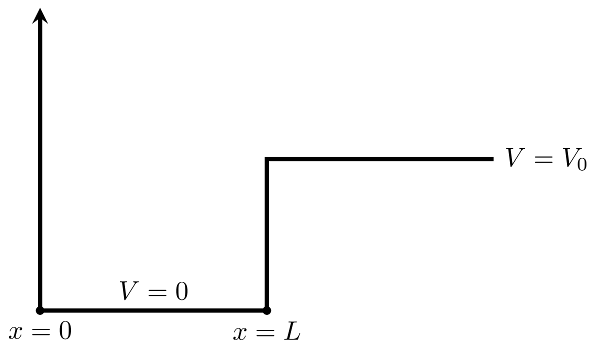
Multiplication is commutative and these two scaling factors are independent of each other, so doubling both width and length will scale by 2 and $\sqrt{2}$, respectively, yielding $\boxed{2\sqrt{2}N}$.

d) *How many bound states does the particle have in a well of depth $\frac{V_0}{4}$ and width $2L$? In a well of depth $4V_0$ and width $\frac{L}{2}$?*

This time, if we perform the algebra, we notice that decreasing the depth and increasing the width in this precise way will perfectly cancel the effect, yielding $\boxed{\text{no change}}$ in the allowed number of energy levels. We observe $\boxed{\text{similar behavior}}$ with increasing the depth and decreasing the width in this precise way.

2 Problem 4.2 in F&T (The 'Half-Infinite' Well)

A particle is bound in a one-dimensional well with one rigid wall whose potential is shown in the figure.



a) *For $E < V_0$, write down and solve the Schrödinger equation for the region inside the well and the region outside the well.*

Inside the well, we have a free particle (since $V(x) = 0$), yielding scattering states:

$$\Psi_W(x) = A\cos(kx) + B\sin(kx), \qquad k = \left(\frac{2mE}{\hbar^2}\right)^{\frac{1}{2}}$$

Now, to the left of the well, since we have an infinite potential, the wave function must vanish completely:

$$\Psi_L(x) = 0$$

To the right of the well, we have evanescent decay (since our energy is below the potential, but the potential difference is nevertheless not infinite):

$$\Psi_R(x) = Ce^{-\alpha x}, \qquad \alpha = \left(\frac{2m(V_0 - E)}{\hbar^2}\right)^{\frac{1}{2}}$$

So, we have our following solutions to the Schrödinger equation:

$$\boxed{\Psi(x) = \begin{cases} 0 & & x \leq 0 \\ A\cos(kx) + B\sin(kx), & k = \left(\frac{2mE}{\hbar^2}\right)^{\frac{1}{2}} & 0 < x < L \\ Ce^{-\alpha x}, & \alpha = \left(\frac{2m(V_0-E)}{\hbar^2}\right)^{\frac{1}{2}} & x \geq L \end{cases}}$$

b) *Apply the boundary conditions at $x = 0$ and $x = L$ to obtain an equation that defines the allowed values of the energy E.*

For simplicity in calculation, let's only consider odd solutions here. In that case, we have

$$\Psi_C(x) = B\sin(kx)$$

inside the well. Automatically, this will match the given boundary condition

$$\Psi(0) = 0$$

Finally, at $x = L$, we must enforce continuity in Ψ and Ψ':

$$\begin{aligned} B\sin(kL) &= Ce^{-\alpha L} && \text{(Continuity in } \Psi) \\ Bk\cos(kL) &= -\alpha Ce^{-\alpha L} && \text{(Continuity in } \Psi') \end{aligned}$$

Combining gives us:

$$\boxed{\tfrac{1}{k}\tan(kL) = \tfrac{1}{\alpha}}$$

The above equation, as desired, defines the allowed energy values.

c) *Show that for V_0 very large, the permitted energies approach those for the infinitely depp square well of width L.*

As V_0 gets very large, α tends to ∞. So, substituting this back into our equation, we get:

$$\tan(kL) = 0$$

Expanding gives:

$$\begin{aligned} kL &= n\pi \\ \therefore \left(\frac{2mE_n}{\hbar^2}\right)^{\frac{1}{2}} &= \frac{n\pi}{L} && \text{(Substitute expression for } k) \\ \therefore \frac{2mE_n}{\hbar^2} &= \frac{n^2\pi^2}{L^2} && \text{(Square both sides)} \\ \therefore E_n &= \boxed{\tfrac{\hbar^2 n^2 \pi^2}{2mL^2}} \end{aligned}$$

And so, we have confirmed that the allowed energy levels here are:

$$\boxed{E_n = \tfrac{\hbar^2 n^2 \pi^2}{2mL^2}},$$

exactly those for an infinite square well of width L with the left edge at $x = 0$.

d) *Introduce dimensionless quantities θ and θ_0 according to Eqs. 4-5 and 4-6 of the text. By comparing the result with Eq. 4-8, show that the permitted energies of the "half-infinite" well of width L are exactly the energies for the odd solutions of the finite well of the same depth V_0 but twice the width, $2L$. Explain how this arises out of the boundary condition at $x = 0$ for the half-infinite case.*

As suggested, let's introduce some dimensionless quantities:

$$\theta = \frac{kL}{2}, \qquad \theta_0 = \frac{L\hbar}{2}\sqrt{2mV_0}$$

With this, let's reciprocate our expression for the energy and write it in terms of our dimensionless quantities and expand out:

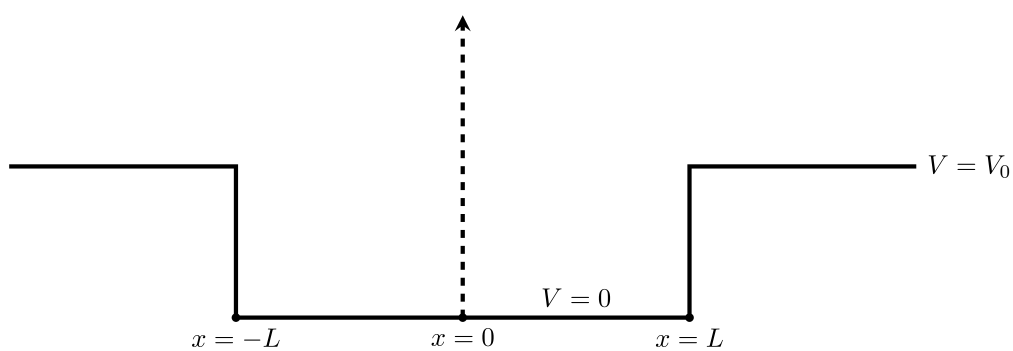
$$\cot(2\theta) = -\frac{\alpha}{k} = -\sqrt{\frac{V_0}{E} - 1} = -\sqrt{\left(\frac{\theta_0}{\theta}\right)^2 - 1}$$

Now, consider Eq. 4-8 from the text:

$$\cot(\theta) = -\sqrt{\left(\frac{\theta_0}{\theta}\right)^2 - 1}$$

This looks almost exactly like what we just found, but instead of θ in the cotangent, we have 2θ. This means that our equation only allows half as many solutions (since our angle increases twice as fast).

A little more qualitatively, we can think of a finite square well as two of these "half-infinite" square wells stitched together in the middle:



And so, we have a finite square well with the same depth but with twice the width. Now, the boundary condition of the half-infinite well tells us that $\Psi(0) = 0$, and we can enforce this here too (since odd solutions in the finite square well here will simply vanish in the middle), so we can enforce continuity in Ψ and Ψ' by simply equating those values from either side. Now, the only solutions in the finite square well that vanish in the middle are the odd solutions, those are the only solutions that will be valid on the RHS for the half-infinite well that we actually are interested in. And so, we have shown (both qualitatively and quantitatively) that the solutions to our half-infinite well with given width and depth correspond exactly to the odd solutions of the finite square well with the same depth but twice the width, as desired. ■

e) *For the case $\theta_0 = \pi$, or $V_0 = \frac{h^2}{2mL^2}$, graphically solve the equation for the lowest permitted energy. Show that this is about three-quarters of the energy $\frac{h^2}{8mL^2}$ that one would have for the infinitely deep well of width L.*

For the case of $\theta_0 = \pi$, we can solve for the lowest value of θ numerically to get $\boxed{\theta = 1.3488}$. Now, we can solve for the energy:

$$\frac{\sqrt{2mE_1}}{\hbar}\frac{L}{2} = 1.3488$$
$$\therefore E_1 = \boxed{\tfrac{2(1.3488)^2\hbar^2}{mL^2}}$$

Now, lets, use $\frac{3}{4}$ of the energy given in the problem statement:

$$E_1 = \frac{3}{4}\frac{\pi^2\hbar^2}{8mL^2}$$

We can approximate here. See that $2(1.3488)^2 \approx 3.59$ and $\frac{2\pi^2}{8} \approx 3.7$. In other words, they are fairly similar coefficients, meaning that these two energies are essentially the same, as desired. ■

f) *How many bound states are there altogether for the case $\theta_0 = \pi$?*

If we solve numerically for this case, we get a total of $\boxed{2}$ bounds state solutions for θ.

g) *Show graphically that there are <u>no</u> bound states if $\theta_0 < \frac{\pi}{4}$ or $V_0 < \frac{h^2}{32mL^2}$.*

When graphing, notice that the first x-intercept for $\cot(2\theta)$ is at $\theta = \frac{\pi}{4}$. Since the right-hand side will always be negative, this means that $\cot(2\theta)$ will be strictly nonnegative for $\theta < \frac{\pi}{4}$, meaning we will have no solutions, as desired. ■

3 Problem 6.2 in F&T (Diagnosis Using an xy Analyzer)

A calcite xy analyzer is placed in various beams of monochromatic photons (all photons of the same energy). The analyzer is rotated about the beam as axis.

a) *For beam A, there is one orientation of the analyzer for which the output of channel y has intensity I_0 and the output of channel x is zero. Predict the intensities of the outputs of both channels as the analyzer is rotated about the beam as axis.*

Malus' Law tells us that

$$\boxed{I_x = I_0^2 \sin^2(\theta)} \quad , \quad \boxed{I_y = I_0^2 \cos^2(\theta)}$$

b) *For beam B both output beams of the xy abalyzer have equal intensities for all orientations of the analyzer. What conclusion(s) can you draw about the beam incident on the analyzer?*

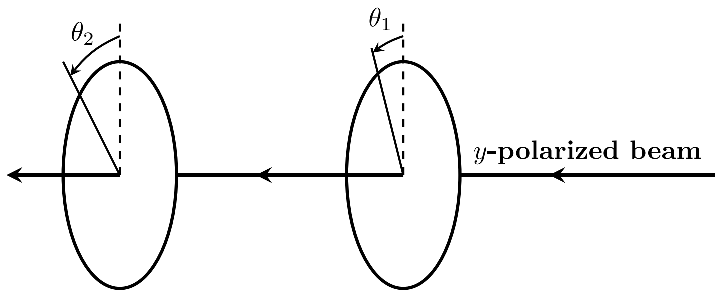
Notice that this light could either be $\boxed{\text{completely unpolarized}}$ or $\boxed{\text{circularly polarized}}$ (since these behave the same way when passing through this analyzer). Regardless of the initial polarization, we would obtain light of $\boxed{\text{uniform intensity}}$ when rotating our analyzer.

c) *For beam C the outputs of the x and y channels each vary with orientation of the analyzer, but there is no orientation for which the output of either channel is zero. What conclusion(s) can you draw about the beam incident on the analyzer?*

In this case, the light could either be $\boxed{\text{elliptically polarized}}$ or $\boxed{\text{non-uniformly linearly polarized}}$. In either case, there will be some nonzero intensity at all angles, and so the intensity will differ based on the rotation angle.

4 Problem 6.5 in F&T (Sequential Projections)

A beam of y-polarized photons is incident on two ideal linear polarizers in sequence, as shown in the figure. The first polarizer has its transmission axis oriented at an angle θ_1 with respect to the y axis and the second at an angle θ_2.



a) *What is the transmission probability through the first polarizer for $\theta_1 = 0$? For $\theta_1 = \frac{\pi}{2}$?*

Here, all light will pass through for $\theta_1 = 0$, because our polarizer is perfectly aligned with such light. This gives us a probability of $\boxed{1}$.

When $\theta_1 = \frac{\pi}{2}$, since all of our light is initially y-polarized, we will get no light passing through, giving us a probability of $\boxed{0}$.

b) *What is the net transmission probability through the system of two polarizers:*

i) *for $\theta_1 = 0$ and $\theta_2 \neq 0$?*

In this case, we should get that the probability is $\boxed{\cos^2(\theta_2)}$, because all of the light will pass through the first lens, meaning the only part that will actually end up getting filtered through will be through second lens.

ii) *for $\theta_1 = \theta_2$?*

In this case, the probability will be $\boxed{\cos^2(\theta_1)}$, since any light that will pass through the first polarizer will be guaranteed to pass through the second polarizer.

iii) *for $\theta_1 = \frac{\theta_2}{2}$?*

Finally, for this case, the probability will be the product of the two transmission probabilities:

$$T = \cos^{(}\theta_1)\cos^2(\theta_2 - \theta_1) = \underbrace{\boxed{\cos^4(\theta_1)}}_{\theta_2 = 2\theta_1}$$

c) *Find an expression for the net transmission probability through the system as a function of θ_1 for a given fixed value of θ_2. For what value(s) of θ_1 is the transmission a maximum?*

As before, our transmission probability is

$$T = \cos^2(\theta_1)\cos^2(\theta_2 - \theta_1)$$

Differentiating and setting to 0 gives the maximum transmission:

$$\frac{\partial T}{\partial \theta_1} = 0 = 2\cos(\theta_1)(-\sin(\theta_1)\cos^2(\theta_2 - \theta_1) + 2\cos(\theta_2 - \theta_1)\sin(\theta_1 - \theta_1)$$

Now, we can solve by rearranging:

$$\begin{aligned}\sin(\theta_1)\cos(\theta_2 - \theta_1) = \cos(\theta_1) &= \cos(\theta_1)\sin(\theta_2 - \theta_1)\\ \therefore \tan(\theta_1) &= \tan(\theta_2 - \theta_1)\\ \therefore \theta_1 &= \boxed{\tfrac{\theta_2}{2}}\end{aligned}$$

And so, we get that transmission is maximum at $\boxed{\theta_1 = \frac{\theta_2}{2}}$.

5 Problem 6.8 in F&T (Verifying a Polarization State)

IN a low-intensity counting experiment a weak beam of right-circularly polarized light enters an xy analyzer. In a given experimental run 54 photons are counted in the x-output channel and 46 in the y-output channel. The experimenter concludes that the incident beam cannot be R polarized since, he says, an R-polarized beam should yield equal intensities in the x and y channels.

a) *Make a brief but, if possible, quantitative argument that the experimental result is not inconsistent with the incident beeam being in the R-polarized state.*

Notice that the beam is weak. This means that we can instead picture this beam as a transmission of individual photons. Thus, the photons here can be thought of as individual discrete particles with certain oscillation periods.

And so, with this in mind, the beam can still be R-polarized, because we could potentially be measuring the light at a point where the polarization is in some superposition of x- and y- polarization. This means that the results won't be inconsistent, as desired. ■

b) *What alternative experiment would you propose in which detection of 100 photons would provide more convincing evidence that the incident beam is R polarized? What outcome would you expect for your proposed experiment? Would such an outcome make it <u>certain</u> that the incident beam is R polarized?*

If we want more convincing evidence of R polarization, we can instead measure the light beam at $\boxed{\text{relatively prime distances}}$. This means that we should expect to obtain a different distribution for every measurement, assuming our light is truly R polarized (since there won't be any overlap).

Unfortunately, this still $\boxed{\text{wouldn't make it certain}}$ that our incident beam is R polarized, because we can always find a wavelength of light for which the experiment will output the same distribution for all measurements, given that we have a finite number of wavelengths. We would need an infinite number of measurements to be truly certainm, which is not something physically possible to obtain.