PHYSICS 105B NOTES

Typeset notes for Physics 105: Analytic Mechanics

Andrew Binder and Eric Du
University of California, Berkeley

Spring 2023

Lecture 0

Introduction

- Instructor: Prof. Dan McKinsey
- Office Hours: Tuesdays, 2:30pm in 441 Physics South
- Grading:
  - 30% Homework
  - 20% Midterm 1
  - 20% Midterm 2
  - 30% Final

  The grade bins for this class are fixed, and they will not be altered throughout the semester.

Lecture 1 (01/17)

This lecture was held on **January 17, 2023**. It covered equations for simple harmonic motion in one and two dimensions.

1.1 Why Do We Study Oscillations?

We study oscillations because they are very common in physics – they happen any time we have a system with a stable equilibrium point. When we nudge our system away from this point, a restoring force $F_x(x)$ tries to bring our system back to equilibrium.

Although $F_x(x)$ could potentially be functions of more variables, we can let it be x for now. We will also assume that $F_x(x)$ has continuous derivtaives everywhere so that we can expand it as a Taylor series. Thus:

$$F_x(x) = F_0 + x\left(\frac{dF_x}{dx}\right)_0 + \frac{1}{2}x^2\left(\frac{d^2F_x}{dx^2}\right)_0 + \dots$$

Then, since the origin is also an equilibrium point, then F_0 must equal 0 at the equilibrium point. Then, neglecting higher order terms, we get the approximate relation that:

$$F_x(x) = -kx$$

where $k \equiv -\left(\frac{dF_x}{dx}\right)$. Since the restoring force always points toward the equilibrium positio, then dF/dx is alwaysnegative, so k is a positive constant.

Alternatively, we can also write the force in terms of the potential:

$$U(x) = \frac{1}{2}kx^2$$

where $U(x) = U(0)+U'(0)+\frac{1}{2}U''(0)x^2+\dots$, with $U'(0) = 0$ using the same logic as before. These oscillations can be damped or driven, which we will revisit later.

1.2 Simple Harmonic Oscillator

Here we will look at different ways to represent the oscillatory equations of motion for simple harmonic oscillators. To start, let's use Newton's second law to get the differential equation:

$$-kx = m\ddot{x}$$

Then, we can define $\omega^2 \equiv \frac{k}{m}$, then we get the equation

$$\ddot{x} + \omega^2 x = 0$$

which is the standard differential equation for simple harmonic motion. This differential equation has the solutions:

$$
\begin{aligned}
x(t) &= A\sin(\omega t - \varphi) \\
x(t) &= A\cos(\omega t - \delta)
\end{aligned}
$$

where $|\delta - \phi| = \pi/2$. The kinetic energy can also be calculated:

$$
\begin{aligned}
T = \frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m\left[A\omega\cos(\omega t - \phi)\right]^2 \\
&= \frac{1}{2}mA^2\frac{k}{m}\cos^2(\omega t - \varphi) \\
&= \frac{1}{2}kA^2\cos^2(\omega t - \varphi)
\end{aligned}
$$

Since $U(x) = \frac{1}{2}kx^2$, then we get:

$$
U(x) = \frac{1}{2}kA^2\sin^2(\omega t - \varphi)
$$

Adding the two, we get:

$$
T + U = \frac{1}{2}kA^2(\sin^2(\omega t - \varphi) + \cos^2(\omega t - \varphi)) = \frac{1}{2}kA^2
$$

which is a constant for all θ. We expect this result, since we know that the total energy of an isolated oscillatory system doesn't change. The period τ can also be expressed as:

$$
\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}
$$

And we also have the relation that $\nu = 2\pi\omega$.

1.2.1 Method 2: Summation of sines and cosines

Looking at the second solution more closely, we note that

$$
\begin{aligned}
A\cos(\omega t - \delta) &= A\left[\cos(\delta)\cos(\omega t) + \sin(\delta)\sin(\omega t)\right] \\
&= A\left[\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right] \\
&= B_1\cos(\omega t) + B_2\sin(\omega t)
\end{aligned}
$$

with $A = \sqrt{B_1^2 + B_2^2}$. This form of the solution is particularly nice since it allows us to deal with initial conditions very easily. For instance, if the oscillation started at the peak, then we know that $B_2 = 0$ and so we're left with a pure sine wave which is really easy to deal with.

1.2.2 Method 3: Exponentials

We can also formulate the solution as

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

This form is useful since integrals and derivatives are especially easy. You can also check that this solution satisfies the differential equation by plugging in $x(t)$ into our differential equation. To show that it's consistent with our previous form, we use Euler's identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$:

$$\begin{aligned} x(t) &= C_1 e^{i\omega t} + C_2 e^{-i\omega t} \\ &= (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\sin(\omega t) \\ &= B_1 \cos(\omega t) + B_2 \sin(\omega t) \end{aligned}$$

And so naturally in this form we assume $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$.

1.2.3 Real Part of Expoonentials

Since $x(t)$ is a real quantity, then we can actually make a couple simplifications to our solution in the previous section. Firstly, we note that since $x(t)$ is real, then B_1, B_2 must also be real. Therefore, this enforces $C_1 = C_2^*$, so we now have

$$x(t) = C_1 e^{i\omega t} + C_1^* e^{-i\omega t}$$

And since we know that $z + z^* = 2\,\mathrm{Re}(z)$, then letting $z = C_1 e^{i\omega t}$, we get:

$$x(t) = 2\,\mathrm{Re}\left(C_1 e^{i\omega t}\right)$$

Then as one final simplification, if we let $C = 2C_1$, then $C = B_1 - iB_2 = Ae^{i\delta}$ so we can write:

$$x(t) = \mathrm{Re}(Ce^{i\omega t}) = A\cos(\omega t - \delta)$$

As an illustration of this expresion, we can imagine moving around a unit circle:
[INSERT TIKZ HERE]

1.2.4 Summary

To summarize, we have the following equivalent ways of writing solutions to these oscillations:

$$\begin{aligned} x(t) &= A\cos(\omega t - \delta) \\ &= B_1 \cos(\omega t) + B_2 \sin(\omega t) \\ &= C_1 e^{i\omega t} + C_2 e^{i\omega t} \\ &= C_1 e^{i\omega t} + C_1^* e^{i\omega t} \\ &= \mathrm{Re}(Ce^{i\omega t}) \\ &= \mathrm{Re}(Ae^{i\omega t - \delta}) \end{aligned}$$

These solutions are all equivalent, each having its own benefits when it comes to solving problems. It's our job to figure out which form is the most convenient for our problem at hand.

1.3 Oscillations in 2 Dimensions

How do our equations for oscillations generalize in 2 dimensions? Well, now our restoring force is slightly more complicated to account for a new dimension:

$$\vec{F} = -k\vec{r}$$

And we can split this up into component form:

$$\begin{aligned} F_x &= -kr\cos\theta = -kx \\ F_y &= -kr\sin\theta = -ky \end{aligned}$$

This then generates the same set of differential equations as before, and they are independent so we can solve them separately. Therefore, we generate the equations:

$$\begin{aligned} x(t) &= A_x\cos(\omega t - \delta_x) \\ y(t) &= A_y\cos(\omega t - \delta_y) \end{aligned}$$

Here, we can "zero out" one of these phases (by simply starting at one of the phases), so this changes our equations to:

$$\begin{aligned} x(t) &= A_x\cos(\omega t) \\ y(t) &= A_y\cos(\omega t - \delta) \end{aligned}$$

where δ now refers to some *relative phase* between the two oscillations. Now we ask ourselves, what is the path of this particle? To do this we eliminate t. First, we can expand out $y(t)$ without introducing the relative phase:

$$\begin{aligned} y(t) &= A_y\cos(\omega t - \delta_x + (\delta_x - \delta_y)) \\ &= A_y\cos(\omega t - \delta_x)\cos(\delta_x - \delta_y) - A_y\sin(\omega t - \delta x)\sin(\delta_x - \delta_y) \\ &= A_y\cos(\omega t - \delta_x)\cos(\delta_x - \delta_y) + A_y\sin(\omega t - \delta x)\sin(\delta_y - \delta_x) \end{aligned}$$

In the last line we've used the identity that $\sin(-x) = -\sin(x)$. Now, we can use the relative phase $\delta \equiv \delta_y - \delta_x$ and $\cos(\omega t - \delta_x) = \frac{x}{A_x}$ to get:

$$y = \frac{A_y}{A_x}x\cos\delta + A_y\sqrt{1 - \left(\frac{x^2}{A_x^2}\right)}$$

We can alternatively write this as

$$A_x y - A_y x\cos\delta = A_y\sin\delta\sqrt{A_x^2 - x^2}$$

So squaring this, and simplifying, we get:

$$A_y^2 x^2 - 2A_x A_y xy\cos(\delta) + A_x^2 y^2 = A_x^2 A_y^2 \sin^2\delta$$

Then, if $\delta = \pm\frac{\pi}{2}$, then we get an ellipse:

$$\frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1$$

If $\delta = 0$ (i.e. no phase), then we get $(A_y x - A_x y)^2 = 0 \implies y = \frac{A_y}{A_x}x$, which is a straight line! Visually, it looks like this:

[INSERT TIKZ HERE]

1.3.1 Lissajous Curves

Note that in the previous derivation we used the same k in both the x and y directions. However, in the most general case this isn't actually required! Therefore, the general set of equations are:

$$\begin{aligned} x(t) &= A_x \cos(\omega_x t) \\ y(t) &= A_y \cos(\omega_y t - \delta) \end{aligned}$$

If $\frac{\omega_x}{\omega_y}$ is rational, then the motion is periodic, called a Lissajous figure (or a Lissajous curve). If it is irrational, then the curve will eventually fill out a rectangle over time.

Lecture 2 (01/)

This lecture was held on **January 19th, 2023**. It covered damped and driven oscillators.

2.1 Last time: The Free Oscillator

On Tuesday we explored oscillatory mechanics where there were no other forces besides the restoring force. However, in most systems, we will always have some kind of *daming force* which impedes motion. This doesn't always have to be the case, but we will first explore a damping force which is proportional to the velocity:

$$\vec{f} = b\vec{v}$$

Under this, we now have the restoring force and the damping force, so Newton's second law now reads:

$$m\ddot{x} + b\dot{x} + kx = 0$$

And so if we let $\beta = \frac{b}{2m}$ (we'll see later why this substitution is useful), then we can write

$$\ddot{x} + 2\beta x + \omega_0^2 x = 0$$

The nature of these differential equations is that due to their linearity, if we find two independent solutions $x_1(t)$ and $x_2(t)$, then in general their solution will be a linear combination of the two:

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

We saw that exponentials worked before, let's have that as our main guess. Let

$$x(t) = e^{rt} \implies \dot{x}(t) = re^{rt}, \ddot{x}(t) = r^2 e^{rt}$$

Plugging this in, we get:

$$\begin{aligned} r^2 e^{rt} + 2\beta r e^{rt} + \omega_0^2 e^{rt} &= 0 \\ \therefore r^2 + 2\beta r + \omega_0^2 &= 0 \end{aligned}$$

This is quadratic in r, so therefore we have solutions $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$. Now, we can then write

$$\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned}$$

so the general solution now becomes:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2 - \omega_0^2} t} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2} t}\right)$$

This equation makes sense intuitively, since a large value of β generates a faster decay, which makes sense since β refers to the damping constant.

Now we have 3 cases that we want to analyze:

(a) Underdamped: $\omega_0^2 > \beta^2$

(b) Critical damping: $\omega_0^2 = \beta^2$

(c) Overdamped: $\omega_0^2 < \beta^2$

As it will turn out, only the overdamping will give us oscillatory motion.

2.1.1 Case 1: Underdamped Oscillation

Here we look at the case where $\omega_0^2 > \beta^2$. Note that if $\beta = 0$, then we exactly recover the solution that we got last time:

$$x(t) = C_1 e^{\sqrt{-\omega_0^2} t} + C_2 e^{-\sqrt{-\omega_0^2} t} = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$