Collaborators

I worked with **Andrew Binder** to complete this assignment.

Problem 1

Calculate the total cross-section for scattering from a Yukawa potential, in the Born approximation. Express your answer as a function of E.

Solution: Griffiths solves in problem 10.11 that the scattering amplitdue is given by:

$$f(\theta) = -\frac{2m\beta}{\hbar^2(\mu^2 + \kappa^2}$$

Using this result, we can now calculate the differential cross section using the relation $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2$, so therefore:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4m^2\beta^2}{\hbar^2(\mu^2 + \kappa^2)}$$

Now we integrate this with respect to $d\Omega = \sin\theta d\theta d\phi$ in order to get the total cross-section. Here, we will need to return the substitution $\kappa = 2k\sin\frac{\theta}{2}$, since there is θ dependence in κ. Therefore, we integrate:

$$\sigma = \left(\frac{2m\beta}{\hbar}\right)^2 \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta}{\mu^2 + 4k^2\sin^2\frac{\theta}{2}} d\theta d\phi$$

From here, the integral can be evaluated by hand (the integral by hand is rather tedious in my opinion) or via a computer, eventually we get:

$$\sigma = \left(\frac{4m\beta}{\mu\hbar^2}\right)^2 \frac{\pi}{\mu^2 + \frac{8mE}{\hbar^2}}$$

□

Problem 2

For the potential in Problem 10.4,

(a) calculate $f(\theta)$, $D(\theta)$ and σ, in the low-energy Born approximation;

Solution: The potential in question is the delta function $V(r) = a\delta(r-a)$, and in the low energy Born approximation, we compute:

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int V(r)d^3r$$

So therefore:

$$\begin{aligned} f(\theta) &= -\frac{m}{2\pi\hbar^2}\int_0^\infty\int_0^{2\pi}\int_0^\pi \alpha\delta(r-a)d\theta d\phi dr \\ &= \frac{4\pi\alpha m}{2\pi\hbar^2}\int_0^\infty \delta(r-a)r^2 dr \\ &= -\frac{2\alpha ma^2}{\hbar^2} \end{aligned}$$

In the last step, I used the fact that $\int f(x)\delta(x-a) = f(a)$, a well known result. The differential cross section $D(\theta) = |f(\theta)|^2$, so therefore:

$$D(\theta) = \frac{4\alpha^2 m^2 a^4}{\hbar^4}$$

Thus, the total cross section integrates over $d\Omega = \sin\theta d\theta d\phi$, so:

$$\begin{aligned} \sigma &= \int D(\theta)d\Omega \\ &= \int_0^\pi\int_0^2 \pi\frac{4\alpha^2m^2a^4}{\hbar^4}\sin\theta d\theta d\phi \\ &= \frac{4\alpha^2m^2a^4}{\hbar^4}(4\pi) \\ &= \frac{16\pi\alpha^2m^2a^4}{\hbar^4} \end{aligned}$$

□

(b) calculate $f(\theta)$ for arbitrary energies, in the Born approximation;

Solution: For arbitrary energies, the scattering amplitude is:

$$f(\theta) = -\frac{2m}{\hbar^2\kappa}\int_0^\infty V(r)r\sin(\kappa r)dr$$

So plugging our $V(r)$ in:

$$\begin{aligned} f(\theta) &= -\frac{2m\alpha}{\hbar^2\kappa}\int_0^\infty \delta(r-a)r\sin(\kappa r)dr \\ &= -\frac{2m\alpha}{\hbar^2\kappa}a\sin(\kappa a) \\ &= -\frac{ma\alpha}{\hbar^2 k\sin\frac{\theta}{2}}\sin\left(2k\sin\frac{\theta}{2}a\right) \end{aligned}$$

□

Problem 3

Using the Born approximation, evaluate the differential scattering cross section for scattering of particles of mass m and incident energy E by the repulsive well with potential

$$V(r) = \begin{cases} V_0 & 0 < r < a \\ 0 & r > a \end{cases}$$

Exhibit E and θ dependence.

Solution: Recall the equation for the cross section scattering:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2}{4\pi^2\hbar^4} | \langle k_f | V(r) | k_i \rangle |^2$$

So we need to compute the matrix element $\langle k_f | V(r) | k_i \rangle$. Under the born approximation, we assume that the wavefunctions are free particles, so this matrix element is the integral:

$$\begin{aligned} \langle k_f | V(r) | k_i \rangle &= \int e^{-i\vec{k_f}\cdot\vec{r}} V(r) e^{i\vec{k_i}\cdot\vec{r}} d^3r \\ &= \int e^{-i(\vec{k}_f - \vec{k}_i)\cdot r} V(r) d^3r \end{aligned}$$

To compute this integral, we first perform the usual coordinate transform, so this integral becomes:

$$\langle k_f | V(r) | k_i \rangle = V_0 \int e^{-i(2k\sin\frac{\theta}{2})r'\cos\theta'} r;^2 \sin\theta dr' d\theta' d\phi'$$

where $k = k_f - k_i$. Applying bounds to our integral, this gives:

$$\langle k_f | V(r) | k_i \rangle = 2\pi V_0 \int_0^a \int_0^\pi \int_0^{2\pi} e^{-2ik\sin\frac{\theta}{2}r'\cos\theta'} r'^2 \sin\theta d\phi d\theta dr$$

The ϕ integral evaluates to 2π since no term has ϕ dependence, then the other two integrals are taken by a calculator. When evaluated, they give:

$$\langle k_f | V(r) | k_i \rangle = 4\pi \frac{V_0}{k'} \frac{\sin(ak') - ak'\cos(ak')}{k'^2}$$

Here, $k' = 2k\sin\frac{\theta}{2}$. Therefore, the scattering cross section is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2}{4\pi^2\hbar^4} \frac{16\pi^2 V_0^2}{k'^6} \left[\sin(ak') - ak'\cos(ak')\right]^2 = \frac{4m^2V_0^2}{\hbar^4 k'^6} \left[\sin(ak') - (ak')\cos(ak')\right]^2$$

$\square$

Problem 4

Using the Born approximation, obtain an integral expression for the total cross section for scattering of particles of mass m under the attractive Gaussian potential

$$V(r) = -V_0 \exp\left[-\left(\frac{r}{a}\right)^2\right]$$

Solution: We are just asked to come up with the integral expression, which essentially just asks for $\langle k_f|V(r)|k_i\rangle$. Therefore, the matrix element becomes:

$$\langle k_f|V(r)|k_i\rangle = -V_0 \int e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}} e^{-(r/a)^2} d^3r$$

Just like the previous problem, this means that we're solving:

$$\int_0^\infty \int_0^{2\pi} \int_0^\pi e^{-2ik\sin\frac{\theta}{2}r'\cos\theta'} e^{-r'^2/a^2} r'^2 \sin\theta' d\theta' d\phi' dr'$$

The ϕ integral drops out as usual, and after evaluating the θ term, we're left with the integral:

$$\frac{2\pi}{k\sin\frac{\theta}{2}} \int_0^\infty e^{-(r'/a)^2} r' \sin\left(2r'\sin\frac{\theta}{2}k\right) dr'$$

I tried plugging this into WolframAlpha, and it didn't give me an expression out. So unfortunately, we're going to have to leave this integral as is. Since $d\Omega = \sin\theta d\phi d\theta$, the total cross section becomes:

$$\begin{aligned}\sigma &= \frac{m^2}{4\pi^2\hbar^4} \int_0^\pi \int_0^{2\pi} \left[\frac{2\pi}{k\sin\frac{\theta}{2}} \int_0^\infty e^{-(r/a)^2} r' \sin\left(2r'\sin\frac{\theta}{2}k\right) dr'\right]^2 \sin\theta d\phi d\theta \\ &= \frac{m^2}{k\hbar^4} \int_0^\pi \frac{\sin\theta}{\sin^2\frac{\theta}{2}} \left[\int_0^\infty e^{-(r/a)^2} r' \sin\left(2r'\sin\frac{\theta}{2}k\right) dr'\right]^2 d\theta\end{aligned}$$

$\square$