Eric Du
HW 11 Physics 137B
Quantum Mechanics II April 22, 2023

Collaborators

I worked with **Andrew Binder** to complete this assignment.

1 Problem 1

Derive the Optical theorem: $\sigma = \frac{4\pi}{k}\text{Im}(f(0))$

Solution: We know the existence of the following form for the optical theorem:

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

To prove the optical theorem, we plug in $\theta = 0$ into the normal scattering amplitude:

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l$$

Notice that this will cause $\cos\theta = 1$, and since $P_l(1) = 1$ for all Legendre polynomials, then this will kill off all the Legendre polynomial terms. Then, taking the imaginary part, we take the $\sin\delta_l$ component of $e^{i\delta_l}$. Therefore:

$$\text{Im}(f(0)) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)(\sin\delta_l)\sin\delta_l = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Then, multiplying this by $\frac{4\pi}{k}$ in order to get the original expression for σ, we get:

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

As desired. □

Problem 2

a) At a center-of-mass energy of 5 MeV, the phase shifts describing the elastic scattering of a neutron by a certain nucleus have the following values:

$$\delta_0 = 32.5^\circ, \quad \delta_1 = 8.6^\circ \quad \delta_2 = 0.4^\circ$$

Assuming all other phase shifts to be negligible, plot $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ as a function of scattering angle. what is the total cross section σ? For simplicity take the reduced mass of the system to be that of the neutron.

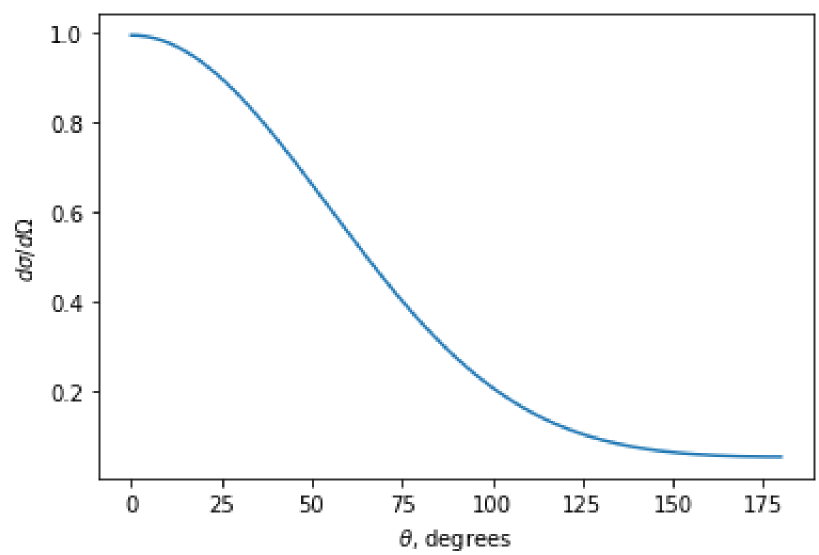
Solution: We use the expansion of $f(\theta)$ from the partial wave analysis:

$$f(\theta) = \sum_{l=1}^{\infty}(2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l$$

Then, since the problem tells us that the contribution from $l > 2$ is negligible, we can just take the sum from $l = 0, 1, 2$ terms. Since there are only three terms here, I just plugged each δ_i into python, obtained $f(\theta)$ in this way, then used the relation:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2$$

to calculate the differential cross section. Doing so, we get the following plot using the δ_i given in the problem:



To calculate the total cross section, we have to take this integral over all θ. I did this part numerically (since I plotted in python), so I'll show the cross section there. □

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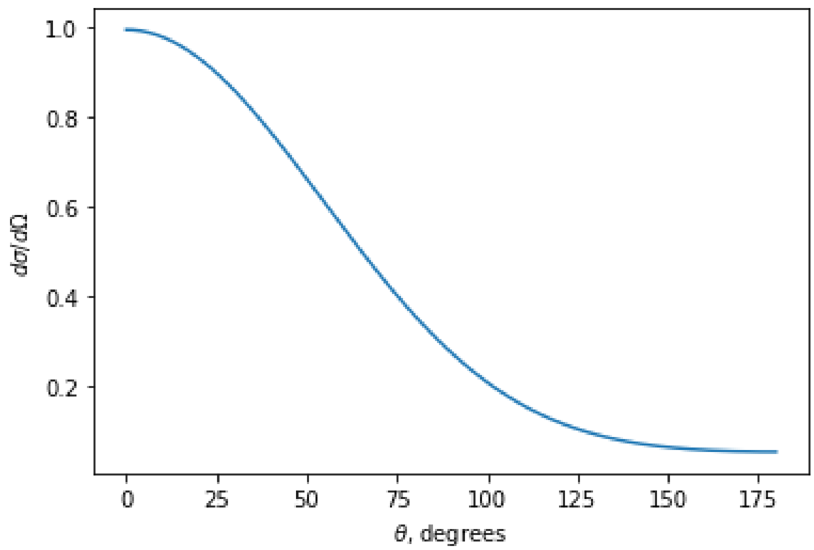
b) the same if the algebraic sign of all three phase shifts is reversed.

Solution: All we have to do here is alter the initial θ given. This plot looks like:



Notice that this plot looks the same as the one from part (a). □

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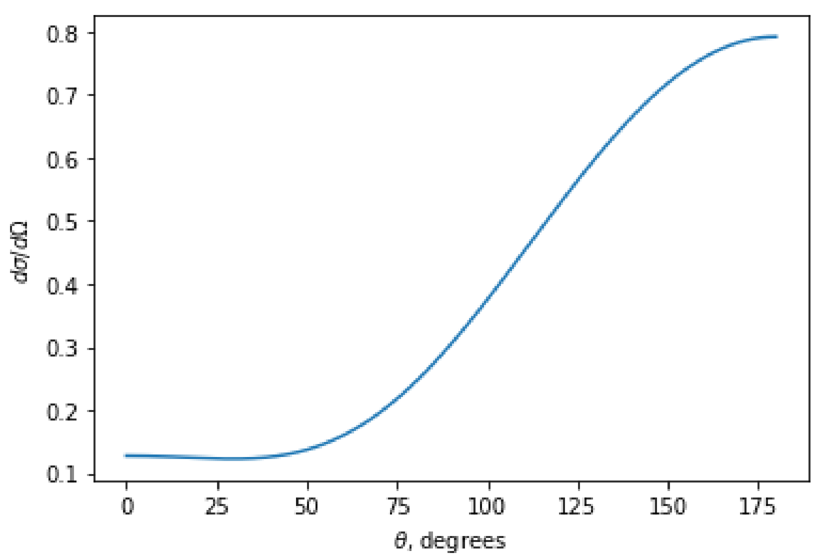
c) The same if the sign of only δ_0 is reversed.

Solution: Again, we do the same thing as part (a) but plug in different δ_i values:



□

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d) Using the results of part (a), calculate the *total* number of neutrons scattered per second out of a beam of 10^{10} neutrons per cm^2 per sec, of cross sectional area 2 cm^2, incident upon a foil containing 10^{21} nuclei per cm^2 How many neutrons per second would be scattered into a counter at 90° to the incident beam and subtending a solid angle of 2×10^{-5} ster-radians?

Solution: Here, we use the relation that $N_{sc} = N_{inc} n_{tar} \sigma$, so all we have to do is compute σ. Computing the numerical integral, we get the cross section $\sigma = 8.34 \times 10^{-24}$. Therefore, this means that the total number of neutrons scattered:

$$N_{sc} = (2 \times 10^{10})(10^{21})(3.253 \times 10^{-25}) = 650600$$

As for $\theta = \frac{\pi}{2}$, we get that the scattering cross section over an angle of 2×10^{-5} ster-radians is $\sigma = 2.263 \times 10^{-31}\text{cm}^2$, so therefore:

$$N_{sc} = (2 \times 10^{10})(10^{21})(2.263 \times 10^{-31}) = 4.526$$

So we can see that about one sixth of the incident particles scatter and hit the detector at an angle of $90°$. □

Problem 3

Analysis of the scattering of particles of mass m and energy E from a fixed scattering center with characteristic length a finds the phase shifts

$$\delta_l = \sin^{-1}\left[\frac{(iak)^l}{\sqrt{(2l+1)l!}}\right]$$

a) Derive a closed expression for the total cross section as a functi of incident energy E.

Solution: Here, we use the Optical theorem to make our lives easier:

$$\sigma = \frac{4\pi}{k}\text{Im}(f(0))$$

Just computing the imaginary part:

$$\begin{aligned}\text{Im}(f(0)) &= \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\delta_l^2 \\ &= \frac{1}{k}\sum_{l=0}^{\infty}\frac{(-1)^l(ak)^{2l}}{l!}\end{aligned}$$

Notice that this is the Taylor expansion of e^{-x}, where $x = (ak)^2$ in this case. Therefore:

$$\text{Im}(f(0)) = \frac{1}{k}e^{-(ak)^2}$$

Therefore, the total cross section is:

$$\sigma = \frac{4\pi}{k^2}e^{-(ak)^2}$$

Using the fact that $k^2 = \frac{2mE}{\hbar^2}$, we get:

$$\sigma = \frac{4\pi\hbar^2}{2mE}e^{-a^2\cdot\frac{2mE}{\hbar^2}}$$

□

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b) At what values of E does S-wave scattering give a good estimate of σ?

Solution: The S-wave scattering is the $l = 0$ term, and plugging $l = 0$ into δ_l it's clear that $\delta_0 = 1$. Therefore:

$$\sigma = \frac{4\pi}{k^2}\delta_1^2 = \frac{4\pi}{k^2} = \frac{4\pi\hbar^2}{2mE}$$

In order for this to be a good approximation, we require the higher order terms in the exponential to be small, therefore:

$$e^{-a^2\cdot\frac{2mE}{\hbar^2}} \approx 1$$

The only way this occurs is if E is small, which makes sense for an approximation. □

Problem 4

In the scattering of particles $E = \hbar^2 k/2m$ by a nucleus, an experimenter finds a differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{k^2}(0.86 + 2.55\cos\theta + 2.77\cos^2\theta)$$

a) What partial waves are contributing to the scattering, and what are their phase shifts at the given energy?

Solution: Since $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2$, then we know that there are only contributions from the $l = 0$ and $l = 1$ terms (since $l = 2$ terms would give us $\cos^3$ and higher order terms, which aren't present in the expansion). The expression for $f(\theta)$ is then:

$$f(\theta) = \frac{1}{k}\sum_{k=0}^{1}(2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l = \frac{1}{k}\left(e^{i\delta_0}\sin\delta_0 + 3\cos\theta e^{i\delta_1\sin\delta_1}\right)$$

Computing $|f(\theta)|^2$:

$$\begin{aligned}
|f(\theta)|^2 &= \frac{1}{k^2}(e^{-i\delta_0}\sin\delta_0 + 3e^{-i\delta_1}\sin\delta_1\cos\theta)(e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cos\theta) \\
&= \frac{1}{k^2}(\sin^2\delta_0 + 3e^{i(\delta_1-\delta_0)}\sin\delta_0\sin\delta_1\cos\theta + 3e^{-i(\delta_1-\delta_0)}\sin\delta_1\sin\delta_0\cos\theta + 9\sin^2\delta_1\cos^2\theta) \\
&= \frac{1}{k^2}(\sin^2\delta_0 + 3\sin\delta_0\delta_1\cos\theta\left(e^{i(\delta_1-\delta_0)} + e^{-i(\delta_1-\delta_0)}\right) + 9\sin^2\delta_1\cos^2\theta) \\
&= \frac{1}{k^2}(\sin^2\delta_0 + 3\sin\delta_0\sin\delta_1\cos\theta\cdot 2\cos(\delta_0-\delta_1) + 9\sin^2\delta_1\cos^2\theta)
\end{aligned}$$

Now matching coefficients, we see that $\sin^2\delta_0 = 0.86$, so $\delta_0 \approx 68.03°$. Similarly, we get $9\sin^2\delta_1 = 2.77$, so $\delta_1 \approx 33.7°$. □

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b) What is the total cross section?

Solution: We use the expression derived in class:

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Of course, since we have $l = 0, 1$ only here, this sum reduces to just a summation of two terms:

$$\sigma = \frac{4\pi}{k^2}(\sin^2\delta_0 + 3\sin^2\delta_1)$$

Since we have δ_0 and δ_1 from the previous problem, we can compute σ:

$$\sigma = \frac{7.1\pi}{k^2}$$

□
