Problem 1

Given $f : \mathbb{R} \in \mathbb{R}$, prove

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f\left(\frac{1}{t}\right), \quad \lim_{x\to-\infty} f(x) = \lim_{t\to 0^-} f\left(\frac{1}{t}\right)$$

(assume all the limits exist)

Solution: This problem boils down to showing that

$$\lim_{t\to 0^+} \frac{1}{t} = \lim_{x\to\infty} x$$

since if this is true, then the limits are true as well. To do this, we show that there is no upper bound M for the left hand side. So our goal is this: for all $\epsilon > 0$, we want to find a δ such that $0 < |t - 0| < \delta$, $t \in A$.

From this last condition we get $|t| < \delta$, and since $t \to 0^+$, then we know that t is always positive so we can drop the absolute value: $0 < t < \delta$. Since this is true, then we also know that

$$\frac{1}{t} > \frac{1}{\delta}$$

so we can choose $M = \left\lfloor \frac{1}{\delta} \right\rfloor$, giving the inequality:

$$\frac{1}{t} > M$$

for all M. Therefore, there is no M that bounds our limit, and thus the limit is ∞. The same approach exists for the second case, except we choose $M = \left\lceil \frac{1}{\delta} \right\rceil$ giving us

$$\frac{1}{t} < M$$

for all M, so the limit there would be $-\infty$. $\square$

Problem 2

Prove the following theorem:

Theorem 1. *Suppose that $f, g, h : A \to \mathbb{R}$ and c is an accumulation point of A. If*

$$f(x) \le g(x) \le h(x) \text{ for all } x \in A$$

and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

then the limit of $g(x)$ as $x \to c$ exists and

$$\lim_{x\to c} g(x) = L$$

Solution: From Theorem 3.1 in note 4, we know that if $f(x) \le h(x)$ for all x and the limit as $x \to c$ exists, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} h(x)$$

Therefore, from the left side of the inequality, we have:

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x) \implies L \le \lim_{x\to c} g(x)$$

And from the right hand side:

$$\lim_{x\to c} g(x) \le \lim_{x\to c} h(x) \implies \lim_{x\to c} g(x) \le L$$

Combining both these conditions together, we get:

$$L \le \lim_{x\to c} g(x) \le L$$

which implies $\lim_{x\to c} g(x) = L$. $\square$

Problem 3

Prove the following theorem:

Theorem 2. *If $f : A \to \mathbb{R}$ and $c \in A$ is an accumulation point of A, then f if continuous at c if and only if*

$$\lim_{n\to\infty} f(x_n) = f(c)$$

for every sequence (x_n) in A such that $x_n \to c$ as $n \to \infty$.

Solution: We prove the forward case: f is continuous if $\lim_{n\to\infty} f(x_n) = f(c)$ for every sequence x_n such that $x_n \to c$ as $n \to \infty$.

This result follows directly from Theorem 1.1 in the notes. Since $x_n \to c$, then we can always define some $\delta > 0$ such that $|x_n - c| = \delta$, meaning that $x_n \neq c$ for all x_n. Therefore, the conclusion in Theorem 1.1 holds for our case.

For the reverse case, we need to show that f being continuous means that we can always choose a subsequence that satisfies the property shown above. Since f is continuous, then we know the definition of continuity holds: for all $\epsilon > 0$, we can find some $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

Using this definition, define a monotonically decreasing sequence ϵ_n which approaches 0 as $n \to \infty$. Now suppose $x - c > 0$ for all x (this simply means approaching c from the right), then for every ϵ_n we can find an according δ_n and x_n satisfying $x_n - c < \delta_n \implies x_n < \delta_n + c$, so we can choose, for instance, $x_n = \frac{\delta_n + c}{2}$. Since the ϵ sequence is an arbitrary but strictly decreasing sequence, then this sequence x_n is also arbitrary, meaning that this condition actually holds for *all* sequences x_n where $x_n \to c$ as $n \to \infty$.[1] $\square$

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[1] I'm curious: is it possible to just state that both sides of Theorem 1.1 hold for this problem and leave it at that? It feels too simple so I devised this way of showing the reverse case.

Problem 4

Prove each of the following functions is continuous at x_0 by verifying the $\epsilon - \delta$ property of Theorem 17.2.

a) $f(x) = x^2$, $x_0 = 2$

Solution: We want to show that for any $\epsilon > 0$, we can find a δ such that $0 < |x-2| < \delta$ implies that $|f(x) - f(2)| = |x^2 - 4| < \epsilon$. Then we need to consider two cases: $x^2 - 4 > 0$ and $x^2 - 4 < 0$.

Case 1: Suppose $x^2 - 4 > 0$. Then, we can drop the absolute value symbols so we have $x^2 - 4 < \epsilon$, so we can then write $x < \sqrt{\epsilon + 4}$ for any ϵ we choose. Intuitively, this statement means that for any ϵ, if we choose $x < \sqrt{\epsilon + 4}$ then $|f(x) - f(2)|$ is satisfied. Given this knowledge, then, we can choose $\delta > \sqrt{\epsilon + 4} - 2$ to satisfy the δ condition. Since this condition holds for all δ, then we're done.

Case 2: Suppose $x^2 - 4 < 0$. In this case, we get the inequality that $x > \sqrt{4 + \epsilon}$. Here, we will use $\epsilon < 4$[2] Note that for the condition of $x^2 - 4$ to hold, it must also be true that $x < 2$, so we have the inequality $\sqrt{4 - \epsilon} < x < 2$, so we can choose $\delta = 2 - \sqrt{4 - \epsilon}$ to satisfy the δ condition. [3] □

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b) $f(x) = \sqrt{x}$, $x_0 = 0$

Solution: We take a very similar approach to the previous problem: if we have $|f(x) - f(0)| < \epsilon$ then we can simplify it to $\sqrt{x} < \epsilon$ so $x < \epsilon^2$. Therefore, if we choose $\delta = \epsilon^2$, then we have satisfied the δ condition. □

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c) $f(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$, $x_0 = 0$

Solution: Here, $|f(x) - f(0)| < \epsilon$ means $|x \sin\left(\frac{1}{x}\right)| < \epsilon$. Here, if we choose some $|x| < \delta$, then we derive the equation:

$$\left| x \sin\left(\frac{1}{x}\right) \right| < \left| \delta \sin\left(\frac{1}{x}\right) \right| < \delta < \epsilon$$

This confirms that for any arbitrary ϵ, we can set δ to be some value less than ϵ □

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d) $g(x) = x^3$, x_0 is arbitrary.

Hint for (d): $x^3 - x_0^3 = (x - x_0)(x^2 + x_0 x + x_0^2)$

[2] This is actually a general statement: if we have some $\epsilon' > 4$, then the δ we find from $\epsilon < 4$ would hold for this ϵ' as well, so choosing $\epsilon < 4$ can be made basically without loss of generality.

[3] Sorry if this solution is very word heavy – I did this solution intuitively instead of following what was done in the textbook, this is the only way $\epsilon - \delta$ proofs make sense to me.

Problem 5

Consider the function

$$h(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that h is continuous at 0 but at no other point.

Solution: We do this in three cases: if $x \in \mathbb{Q}$, $x \in \mathbb{R} \setminus Q$ and $x = 0$.

Case 1: if $x \in \mathbb{Q} \setminus \{0\}$, then $h(x) = x > 0$, so for any δ we choose there will be irrational points y such that $|h(y) - h(x)| = x$ since $h(y) = 0$, so there does not exist a δ for every $\epsilon > 0$. Therefore, the function is discontinuous at the rationals.

Case 2: if $x \in \mathbb{R} \setminus \mathbb{Q}$, then $h(x) = 0$. Since the rationals are dense (I shall take this as an axiom), then just like the previous part, for any δ we choose we can always find a rational y such that $h(y) = y$ so $|h(x) - h(y)| = y$, meaning we also can't find a δ for every $\epsilon > 0$. Therefore, the function is discontinuous at the irrationals.

Case 3: if $x = 0$, then for any $\delta > 0$, we can find an irrational point y such that $h(y) = 0$, so we can always find a delta such that $|h(y) - h(x)| = 0 < \epsilon$, therefore h is continuous at $x = 0$ $\square$
