Physics 137A Homework

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October 24, 2022

Collaborators

I worked with **Andrew Binder** to complete this homework.

Problem 1

Linear operators are linear maps from the Hilbert space to itself, i.e. the functions $\hat{L}$ from wavefunctions to wavefunctions such that $\hat{L}(\alpha\Psi + \beta\Phi) = \alpha\hat{L}\Psi + \beta\hat{L}\Phi$, for all wavefunctions Ψ, Φ and all complex numbers α, β. A Hermitian operator $\hat{A}$ is one whose adjoint is equal to itself, $\hat{A}^\dagger = \hat{A}$, i.e. for all Ψ, Φ,

$$\left\langle \Psi, \hat{A}\Phi \right\rangle = \int \Psi^\star(x)[\hat{A}\Phi](x)\mathrm{d}x = \int [\hat{A}\Psi]^\star(x)\Phi(x)\mathrm{d}x = \left\langle \hat{A}\Psi, \Phi \right\rangle$$

Physical measureables are represented by Hermitian operators. We know two such operators, the position and momentum:

$$\hat{x}\ : [\hat{X}\Psi](x) = x\Psi(x), \qquad \hat{p}\ : [\hat{p}\Psi](x) = -i\hbar\frac{\partial\Psi}{\partial x}(x)$$

For the following maps, check whether they are linear operators, and if so, whether they are Hermitian.

Solution: To check for linearity, we check whether $\hat{L}(\alpha\Psi + \beta\Phi) = \alpha\hat{L}\Psi + \beta\hat{L}\Phi$. Then, if they are linear, we proceed to check whether they are Hermitian, using the definition given in the problem.

(a) $\hat{x}$

Solution: It's given in the problem statement that the position operator $\hat{x}$ is linear and is also Hermitian.

(b) $\hat{A}\ : \hat{A}\Psi(x) = \Psi^\star(x)\Psi(x)$

Solution: To check linearity, we check whether $\hat{A}(\alpha f(x) + \beta g(x)) = \hat{A}\alpha f(x) + \hat{A}\beta g(x)$. We compute the left and right hand sides separately:

$$\begin{aligned}
\text{Left: } \hat{A}(\alpha f(x) + \beta g(x)) &= (\alpha f(x) + \beta g(x))(\alpha f(x) + \beta g(x))^\star \\
&= (\alpha f^\star(x) + \beta g^\star(x))(\alpha f(x) + \beta g(x)) \\
&= \alpha^2 f^\star(x) f(x) + \alpha\beta(f^\star(x)g(x) + f(x)g^\star(x)) + \beta^2 g(x)g^\star(x) \\
\text{Right: } \hat{A}\alpha f(x) + \hat{A}\beta g(x) &= \alpha f(x) f^\star(x) + \beta g(x) g^\star(x)
\end{aligned}$$

As we can see, the right hand side is missing the cross term $\alpha\beta(f^\star(x)g(x) + f(x)g^\star(x))$, and thus this is not a linear operator. If it's not linear, then there's no hope that it's Hermitian either.

(c) $\hat{p}$

Solution: This is also given in the problem statement to be a linear and Hermitian operator.

(d) $\hat{B}$: $\hat{B}\Psi = -iL^2 \frac{\partial^2\Psi}{\partial x^2} + i\frac{x^2}{L^2}\Psi$, for some fixed L

Solution: Checking for linearity:

$$\begin{aligned}
\text{Left: } \hat{B}(\alpha\Phi + \beta\Psi) &= -iL^2 \frac{\partial^2}{\partial x^2} + i\frac{x^2}{L^2}(\alpha\Phi + \beta\Psi) \\
&= -iL^2 \alpha \frac{\partial^2\Psi}{\partial x^2} + i\frac{x^2}{L^2}\alpha\Phi - iL^2\beta\frac{\partial^2\Psi}{\partial x^2} + i\frac{x^2}{L^2}\beta\Psi \\
&= \alpha\left(-iL^2\frac{\partial^2\Phi}{\partial x^2} + i\frac{x^2}{L^2}\Phi\right) + \beta\left(-iL^2\frac{\partial^2\Psi}{\partial x^2} + i\frac{x^2}{L^2}\Psi\right) \\
\text{Right: } \hat{B}\alpha\Phi + \hat{B}\beta\Psi &= \alpha\left(-iL^2\frac{\partial^2\Phi}{\partial x^2} + i\frac{x^2}{L^2}\Phi\right) + \beta\left(-iL^2\frac{\partial^2\Psi}{\partial x^2} + i\frac{x^2}{L^2}\Psi\right)
\end{aligned}$$

Thus the left and right hand sides are equal. Now we check for Hermiticity:

$$\begin{aligned}
\text{Left: } \int \Psi^\star\left(-iL^2\frac{\partial^2\Psi}{\partial x^2} + i\frac{x^2}{L^2}\Psi\right)\mathrm{d}x &= \int -iL^2\frac{\partial^2\Psi}{\partial x^2}\Psi^\star + i\frac{x^2}{L^2}\Psi^\star\Psi\mathrm{d}x \\
\text{Right: } \int\left(-iL^2\frac{\partial^2\Psi^\star}{\partial x^2} + i\frac{x^2}{L^2}\Psi^\star\right)^\star\Psi\mathrm{d}x &= \int -iL^2\frac{\partial^2\Psi^\star}{\partial x^2}\Psi + i\frac{x^2}{L^2}\Psi^\star\Psi\mathrm{d}x
\end{aligned}$$

But since $\frac{\partial^2\Psi}{\partial x^2}\Psi^\star \neq \frac{\partial^2\Psi^\star}{\partial x^2}\Psi$, the left and right side do not equal, and thus the operator is not Hermitian.

(e) $\hat{P}_\Phi$: $\hat{P}_\Phi\Psi(x) = \Phi(x)\int\Phi^\star(y)\Psi(y)\ \mathrm{d}y$, for some fixed $\Phi(x)$

Solution: Checking linearity:

$$
\begin{aligned}
\text{Left:} \quad &= \hat{P}_\Phi(\alpha f(x) + \beta g(x)) \\
&= \Phi(x) \int \Phi^\star(y)[\alpha f(y) + \beta f(y)] \, \mathrm{d}y \\
&= \Phi(x) \left[\alpha \int \Phi^\star f(y) \, \mathrm{d}y + \beta \int \Phi^\star(y) g(y) \, \mathrm{d}y \right] \\
\text{Right:} \quad &= \hat{P}_\Phi \alpha f(x) + \hat{P}_\Phi \beta g(x) \\
&= \alpha \Phi(x) \int \Phi^\star(y) f(y) \, \mathrm{d}y + \beta \Phi(x) \int \Phi^\star(z) g(z) \, \mathrm{d}z \\
&= \Phi(x) \left[\alpha \int \Phi^\star(y) f(y) \, \mathrm{d}y + \beta \int \Phi^\star(z) g(z) \, \mathrm{d}z \right]
\end{aligned}
$$

Since f and g are the same on the left and right hand side, this operator is linear. Now we check for hermiticity:

$$
\begin{aligned}
\text{Left:} \quad &= \int \left[\Phi(x) \int \Phi^\star(y) f(y) \, \mathrm{d}y \right] g(x) \, \mathrm{d}x \\
&= \int \Phi^\star(x) \left[\int \Phi^\star(y) f(y) \right]^\star g(x) \, \mathrm{d}x \\
&= \int \Phi(x) g(x) \int \Phi(y) f^\star(y) \, \mathrm{d}y \mathrm{d}x \\
\text{Right:} \quad &= \int f^\star(x) \Phi(x) \int \Phi^\star(y) g(y) \, \mathrm{d}y \mathrm{d}x
\end{aligned}
$$

Since these integrals do not equal, $\hat{P}_\Phi$ is not Hermitian.

(f) $\hat{Q}_\Phi \ : \hat{Q}_\Phi \Psi(x) = \Phi(x) \int \Psi^\star(y) \Phi(y) \, \mathrm{d}y$, for some fixed $\Phi(x)$

Solution: Checking for linearity:

$$
\begin{aligned}
\text{Left:} \quad &= \hat{Q}_\Phi \left(\alpha f(x) + \beta g(x) \right) \\
&= \Phi(x) \int \left[\alpha^\star f^\star(y) + \beta^\star g^\star(y) \right] \Phi(y) \, \mathrm{d}y \\
&= \alpha^\star \Phi(x) \int f^\star(y) \Phi(y) \, \mathrm{d}y + \beta^\star \Phi(x) \int g^\star(y) \Phi(y) \, \mathrm{d}y \\
\text{Right:} \quad &= \hat{Q}_\Phi \alpha f(x) + \hat{Q}_\Phi \beta g(x) \\
&= \alpha \Phi(x) \int f^\star(y) \Phi(y) \, \mathrm{d}y + \beta \Phi(x) g^\star(y) \Phi(y) \, \mathrm{d}y
\end{aligned}
$$

And since $\alpha^\star \neq \alpha$ and $\beta^\star \neq \beta$, then this operator is not linear. If it's not linear, it cannot be Hermitian either.

(g) $\hat{T}_a \;: \hat{T}_a\Psi(x) = \Psi(x+a)$, for some fixed a

Solution: We check for linearity:

$$\hat{T}_a(\alpha f(x) + \beta g(x)) = \alpha f(x) + \beta g(x) + a$$
$$\hat{T}_A \alpha f(x) + \hat{T}_A \beta g(x) = \alpha f(x+a) + \beta g(x+a)$$

And thus they are not linear, so they cannot be Hermitian either.

(h) $\hat{x}\hat{p}$

Solution: $\hat{x}\hat{p}$ are linear since partial derivatives are linear. However, due to the uncertianty principle, they cannot be Hermitian since the order in which we apply $\hat{x}$ and $\hat{p}$ matters.

(i) $\hat{x}\hat{p} + \hat{p}\hat{x}$

Solution: First, we can rewrite $\hat{x}\hat{p} + \hat{p}\hat{x} = x\left(-xi\hbar\frac{\partial}{\partial x} - i\hbar\frac{\partial}{\partial x}x\right)$. Now we check for linearity:

$$\begin{aligned}
(\hat{x}\hat{p} + \hat{p}\hat{x})(\alpha f(x) + \beta g(x)) &= \left(-xi\hbar\frac{\partial}{\partial x} - i\hbar\frac{\partial}{\partial x}\right)(\alpha f(x) + \beta g(x)) \\
&= -xi\hbar\frac{\partial}{\partial x}f(x) - xi\hbar\frac{\partial}{\partial x}\beta g(x) - i\hbar\frac{\partial}{\partial x}\alpha f(x) - i\hbar\frac{\partial}{\partial x}x\beta g(x) \\
&= -xi\hbar\alpha\frac{\partial f}{\partial x} - i\hbar\alpha\frac{\partial x f(x)}{\partial x} - xi\hbar\beta\frac{\partial g}{\partial x} - i\hbar\beta\frac{\partial}{\partial x}xg(x) \\
&= \left(-xi\hbar\alpha\frac{\partial}{\partial x} - i\hbar\alpha\frac{\partial}{\partial x}x\right)f(x) + \left(-xi\hbar\alpha\frac{\partial}{\partial x} - i\hbar\alpha\frac{\partial}{\partial x}x\right)g(x)
\end{aligned}$$

Therefore, the operator is linear. Now we check for Hermiticity. For simplicity, call $\hat{x}\hat{p} + \hat{p}\hat{x} = \hat{O}$:

$$\begin{aligned}
\left\langle f(x), \hat{O}g(x)\right\rangle &= \int f^\star(0\left[\left(-xi\hbar\frac{\partial}{\partial x} - i\hbar\frac{\partial}{\partial x}x\right)g(x)\right]\mathrm{d}x \\
&= \int f^\star(x)\left(-xi\hbar\frac{\partial g}{\partial x} - i\hbar\frac{\partial}{\partial x}xg(x)\right)\mathrm{d}x \\
\left\langle \hat{O}f(x), g(x)\right\rangle &= \int g(x)\left[-xi\hbar\frac{\partial f}{\partial x} - i\hbar\frac{\partial x f(x)}{\partial x}\right]^\star \mathrm{d}x
\end{aligned}$$

We can stop here with the derivation, since on one hand we have $\frac{\partial f}{\partial x}$ and on the other we have $\frac{\partial g}{\partial x}$, which need not be equal. Therefore, this operator is not Hermitian.

Problem 2

A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-ax^2}$$

where A and a are constants (a is real and positive).

(a) Normalize $\Psi(x,0)$.

Solution: To normalize $\Psi(x,0)$, we compute $\int |\Psi(x,0)|^2 \mathrm{d}x$ and set it equal to 1:

$$\begin{aligned}
1 = \int_{-\infty}^{\infty} |\Psi|^2 \mathrm{d}x &= A^2 \int_{-\infty}^{\infty} e^{-2ax^2} \mathrm{d}x \\
&= \frac{A^2}{2a} \int_{-\infty}^{\infty} e^{-u^2}\, \mathrm{d}u \\
1 &= \frac{A^2\sqrt{\pi}}{\sqrt{2a}} \\
A^2 &= \sqrt{\frac{2a}{\pi}} \\
\therefore A &= \sqrt[4]{\frac{2a}{\pi}}
\end{aligned}$$

(b) Find $\Psi(x,t)$. *Hint:* Integrals of the form

$$\int_{-\infty}^{\infty} e^{(-ax^2+bx)} \mathrm{d}x$$

can be handled by "completing the square": Let $y \equiv \sqrt{a}[x + (b/2a)]$, and note that $(ax^2 + bx) = y^2 - (b^2/4a)$.

Solution: To find $\Psi(x,t)$, we first find the weights $\phi(k)$ by applying the Fourier transform:

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} \mathrm{d}x
\end{aligned}$$

The hint that's given in the problem is perfectly fine to use, but we'll use another one, given in lecture:

$$\int e^{-\alpha u^2} e^{-\beta u}\, \mathrm{d}u = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} e^{\frac{\beta^2}{4\alpha}}$$

Doing so gives us:

$$\phi(k) = \left(\frac{1}{2\pi a}\right)^{\frac{1}{4}} e^{-k^2/4a}$$

Now we plug this into the free particle equation:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx+iEt/\hbar} \,\mathrm{d}k$$

We have $E = \frac{-\hbar k^2}{2m}$, so now we can do the integral. The computation is relatively long, so I won't write out all the steps:

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2\pi a}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-k^2/4a} e^{ikx - \frac{i\hbar k^2}{2m} t} \,\mathrm{d}k \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2\pi a}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-k^2\left(\frac{1}{4a} + \frac{i\hbar t}{2m}\right)} e^{ikx} \,\mathrm{d}k
\end{aligned}$$

Now we can use our integration trick again to get:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2\pi a}\right)^{\frac{1}{4}} \left(\frac{\pi}{\frac{1}{4a} + \frac{i\hbar t}{2m}}\right)^{\frac{1}{2}} e^{\frac{(-ix)^2}{4\left(\frac{1}{4a} + \frac{i\hbar t}{2m}\right)}}$$

Note that $\frac{1}{4a} + \frac{i\hbar t}{2m} = \frac{m + 2i\hbar a t}{4ma} = \frac{1 + 2\hbar a t/m}{4a}$ so we can simplify all the fractions, as well as combine the prefactor:

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{e^{-ax^2/(1+2i\hbar a t/m)}}{\sqrt{1 + 2i\hbar a t/m}}$$

(c) Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{\frac{a}{1 + (2\hbar a t/m)^2}}$$

Sketch $|\Psi|^2$ (as a function of x) at $t = 0$, and again for some very large t. Qualitatively, what happens to $|\Psi|^2$, as time goes on

Solution: Let $\beta = 2\hbar a t/m$. Then, we can rewrite:

$$\begin{aligned}
|\Psi(x,t)|^2 &= \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{e^{-ax^2/[1+\beta] - ax^2/[1-\beta]}}{\sqrt{(1+i\beta)(1-i\beta)}} \\
&= \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{e^{-2ax^2} 1 + \beta^2}{1 + \beta^2}
\end{aligned}$$

Now we can use the definition given in the question to get:

$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}} we^{-2x^2w^2}$$

As time goes on, ω increases and the wavefunction will spread out. In other words, the amplitude will be smaller and our uncertnty will be larger.

(d) Find $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, σ_x and σ_p.

Solution: A lot of this is brute force algebra:

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx = 0$$

This is true since x is an odd function, and $|\Psi(x,t)|^2$ is even. Now we calculate $\langle p \rangle$

$$\langle p \rangle = m \frac{\mathrm{d} \langle x \rangle}{\mathrm{d}t} = 0$$

Now we calculate $\langle x^2 \rangle$:

$$\begin{aligned}
\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\Psi(x,t)|^2 &= \int_{-\infty}^{\infty} x^2 \sqrt{\frac{2}{\pi}} we^{-2w^2x^2} dx \\
&= \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} dx \\
&= \sqrt{\frac{2}{\pi}} w \left(\sqrt{\frac{\pi}{2w^2}} \right) && \text{we've done this integral before} \\
&= \frac{1}{4w^2}
\end{aligned}$$

Now we compute $\langle p^2 \rangle$. To do this we first rewrite $\Psi(x,t) = A'e^{-\alpha x^2}$, where $A' = \frac{A}{\sqrt{1+i\beta}}$ and $\alpha = \frac{a}{1+i\beta}$. Now we have to do some algebra:

$$\begin{aligned}
\frac{\partial^2 \Psi}{\partial x^2} &= A' \frac{\partial}{\partial x} \left(-2\alpha e^{-\alpha x^2} \right) = -2\alpha A'(1 - 2\alpha x^2)e^{-\alpha x^2} \\
\therefore \Psi^\star \frac{\partial^2 \Psi}{\partial x^2} &= -2\alpha |A'|^2 (1 - 2\alpha x^2) e^{-(\alpha + \alpha^\star)x^2} = \\
&= -2\alpha |A'|^2 (1 - 2\alpha x^2) e^{-\frac{2a}{1+\beta^2}x^2} \\
&= -2\alpha |A'|^2 (1 - 2\alpha x^2) e^{-2w^2x^2} \\
&= -2\alpha \left[\sqrt{\frac{2}{\pi}} w \right] (1 - 2\alpha x^2) e^{-2w^2x^2} \\
&= -2\alpha \sqrt{\frac{2}{\pi}} w (1 - 2\alpha x^2) e^{-2w^2x^2}
\end{aligned}$$

Now that we have the partial derivative computed, we can compute $\left\langle p^2 \right\rangle$:

$$\begin{aligned}
\left\langle p^2 \right\rangle &= \int_{-\infty}^{\infty} p^2 |\Psi(x,t)|^2 dx \\
&= 2\alpha\hbar^2 \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} (1 - 2\alpha x^2) e^{-2w^2x^2} dx \\
&= 2\alpha\hbar^2 \left[1 - \frac{\alpha}{2w^2}\right] \\
&= 2\alpha\hbar^2 \left[1 - \left(\frac{a}{1+i\beta}\right)\left(\frac{1+\beta^2}{2a}\right)\right] \\
&= 2\alpha\hbar^2 \left[1 - \frac{1-i\beta}{2}\right] \\
&= 2\alpha\hbar^2 \left[\frac{a}{2\alpha}\right] \\
&= \boxed{a\hbar^2}
\end{aligned}$$

Now that we have all these values, we can compute the standard deviations:

$$\begin{aligned}
\sigma_x^2 &= \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 = \frac{1}{4w^2} \\
\sigma_x &= \sqrt{\sigma_x} = \sqrt{\frac{1}{4w^2}} = \frac{1}{2w}
\end{aligned}$$

And similarly,

$$\begin{aligned}
\sigma_p^2 &= \left\langle p^2 \right\rangle - \left\langle p \right\rangle^2 = a\hbar^2 \\
\sigma_p &= \sqrt{\sigma_p^2} = \sqrt{a\hbar^2} = \sqrt{a}\hbar
\end{aligned}$$

(e) Does the unceertinaty principle hold? At what time t does the system come closest to the uncertainty limit?

Solution: To check if the uncertainty principle holds, we must multiply out the standard deviations, since this contains our uncertainty:

$$\begin{aligned}
\sigma_x\sigma_p &\geq \frac{\hbar}{2} \\
\sigma_x\sigma_p &= \left(\frac{1}{2w}\right)\left(\sqrt{a}\hbar\right) \\
&= \frac{\hbar}{2}\sqrt{1+\beta^2} \\
&= \frac{\hbar}{2}\sqrt{1 + \left(\frac{2\hbar at}{m}\right)^2} \geq \frac{\hbar}{2}
\end{aligned}$$

And thus the uncertianty principle holds. Further, we can see that this value equals $\frac{\hbar}{2}$ at $t = 0$, so that is the moment of minimum uncertainty.

Solution:

(f) Consider a microscopic particle with th emass of an electron localised in a space of 10^{-10}m, about the size of an atom. How long does it take for σ_x to double its initial value? Compare with a macroscopic particle of mass 1 g localised in space of 10^{-6} m.

Solution: Consider $t = 0$:

$$\sigma_x(0) = \frac{1}{2\sqrt{\frac{a}{1+\left(\frac{2\hbar a t}{m}\right)^2}}} = \frac{1}{2\sqrt{a}} = 10^{-10}$$
$$\therefore a = 2.5 * 10^{19}$$

Now let's double σ_x and solve for the time at which that occurs:

$$\sigma_x(T) = \frac{1}{2\sqrt{\frac{a}{1+\left(\frac{2\hbar a T}{m}\right)^2}}} = 2 * 10^{-10}$$

This gives a value of $T \approx 3.1 \times 10^{-16}$ seconds. Now let's do the same for the macroscopic particle:

$$\sigma_x(0) = \frac{1}{2\sqrt{a}} = 10^{-6}$$
$$\therefore a = 2.5 * 10^{11}$$

Now, solving for $t = T$ again:

$$\sigma_x(T) = \frac{1}{2\sqrt{\frac{a}{1+\left(\frac{2\hbar a T}{m}\right)^2}}} = 2 * 10^{-10}$$

We tried throwing this value into a calculator, and it did not give us a specific result, likely because the value of T is too large to compute. This makes sense, since we expect macroscopic particles to behave classically - that is, their uncertianty in time should not change rapidly on the scale of microseconds.

Problem 3

Prove the following three theorems:

(a) For normalizable solutions, the separation constant E must be *real*. *Hint:* Write E (in equation 2.7) as $E_0 + i\Gamma$ (with E_0 and Γ real), and show that if equation 1.20 is to hold for all t, Γ must be zero.

Solution: Let $E = E_0 + i\Gamma$, with E_0 and *Gamma* real. Equation 2.7 in the textbook gives:

$$\Psi(x,t) = \psi(x)e^{-i\frac{Et}{\hbar}} = \psi(x)e^{\frac{\Gamma t}{\hbar}}e^{-i\frac{E_0 t}{\hbar}}$$

So, calculating probability density gives us:

$$|\Psi(x,t)|^2 = |\psi|^2 e^{\frac{2\Gamma t}{\hbar}}$$

Now, we know that this probability density must be normalized:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = e^{\frac{2\Gamma t}{\hbar}} \int_{-\infty}^{\infty} |\psi|^2 dx = e^{\frac{2\Gamma t}{\hbar}} = 1$$
$$\therefore e^{\frac{2\Gamma t}{\hbar}} = 1 \implies \Gamma = 0$$

The last part is true since this wavefunction must be normalized for all t, so the only option would be to let $\Gamma = 0$, as desired. ■

(b) The time-independent wave function $\psi(x)$ can always be taken to be *real* (unlike $\Psi(x,t)$, which is necessarily complex). This doesn't mean that every solution to the time-independent Schrodinger equation *is* real; what it says is that if you've got one that is *not*, it can always be expressed as a linear combination of solutions (with the same energy) that *are*. So you *might as well* stick to ψ's that are real. *Hint:* If $\psi(x)$ satisfies equation 2.5, for a given E, so too does its complex conjugate, and hence also the real linear combinations of $(\psi + \psi^\star)$ and $i(\psi - \psi^\star)$.

Solution: Using the hint, suppose that $\psi(x)$ satisfies the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

If this is true, then the conjugate $\psi^\star$ also satisfies the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi^\star}{dx^2} + V\psi^\star = E\psi^\star$$

Now since ψ and $\psi^\star$ satisfy the Schrödinger equation, then any linear combination of these two will also be solutions to the Schrödinger equation. Thus, $(\psi + \psi^\star)$ and $i(\psi - \psi^\star)$ will also be solutions to the Schrödinger equation. Since these solutions will be real, then we can always write it as a linear combination of real solutions, and hence $\psi(x)$ can always be taken to be real. ■

(c) If $V(x)$ is an **even function** (that is, $V(-x) = V(x)$) then $\psi(x)$ can always be taken to be either even or odd. *Hint:* If $\psi(x)$ satisfies equation 2.5, for a given E, so too does $\psi(-x)$, and hence also the even and odd linear combinations of $\psi(x) \pm \psi(-x)$.

Solution: Now, suppose $V(x)$ is an even function, so$V(-x) = V(x)$. Using the hint, we know that $\psi(x)$ satisfies the Schrödinger equation. Now we plug in $\psi(-x)$:

$$\frac{\partial^2}{\partial(-x)^2} = \frac{\partial^2}{\partial x^2}$$
$$\therefore -\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{\partial x^2} + V(-x)\psi(x) = E\psi(-x)$$

Now, if $V(x) = V(-x)$, we get:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{\partial x^2} + V(x)\psi(-x) = E\psi(-x)$$

In other words, $\psi(-x)$ also satisfies our Schrödinger equation. Now, since $\psi(x)$ and $\psi(-x)$ are both solution, then any linear combination of these wil also be a solution. So we can write:

$$\phi = \psi(x) + \psi(-x) \implies -\frac{\hbar^2}{2m}\frac{\partial^2\phi(-x)}{\partial x^2} + V(x)\phi(-x) = E\phi(x)$$
$$\phi' = \psi(x) - \psi(-x) \implies -\frac{\hbar^2}{2m}\frac{\partial^2\phi'(-x)}{\partial x^2} + V(x)\phi'(-x) = E\phi'(x)$$

Where ϕ is even in the first equation, and ϕ' is odd in the second equation. Since we can write ϕ in this way, then we can always take ψ to be either even or odd. ■

Problem 4

Let $P_{ab}(t)$ be the probability of finding a particle in the range $(a < x < b)$ at time t.

(a) Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)$$

where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi \frac{\partial \Psi^\star}{\partial x} - \Psi^\star \frac{\partial \Psi}{\partial x}\right)$$

What are the units of $J(x,t)$? *Comment:* J is called the **probability current**, because it tells you the rate at which probability is "flowing" past the point x. If $P_{ab}(t)$ is increasing, then even more probability is flowing into the region at one end than flows out at the other.

Solution: The probability that we find the partial between (a, b) is:

$$P_{ab} = \int_a^b |\Psi(x,t)|^2 \, \mathrm{d}x = \int_a^b \Psi(x,t)\Psi^\star(x,t)\mathrm{d}x$$

So now taking the derivative of this

$$\begin{aligned}
\frac{\mathrm{d}P_{ab}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \Psi(x,t)\Psi^\star(x,t)\ \mathrm{d}t \\
&= \int_a^b \frac{\partial}{\partial t}\Psi(x,t)\Psi^\star(x,t) \\
&= \int_a^b \frac{\partial \Psi}{\partial t}|\Psi(x,t)|^2\mathrm{d}x
\end{aligned}$$

Now let's look at the derivative more closely:

$$\frac{\partial}{\partial t}|\Psi(x,t)|^2 = \frac{i\hbar}{2m}\left(\Psi^\star \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^\star}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^\star \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^\star}{\partial x}\Psi\right)\right]$$

Thus, we can write:

$$\begin{aligned}
\therefore \frac{dP_{ab}}{dt} &= -\int_a^b \frac{\partial}{\partial x}J(x,t)\mathrm{d}x \\
&= -\left[J(b,t) - J(a,t)\right] = J(a,t) - J(b,t)
\end{aligned}$$

As desired. In terms of the units of $J(x,t)$, we know that P_{ab} should be dimensionless so therefore the probability current should have dimensions of time^{-1}.

(b) Show that if at any time t, $\Psi(x,t)$ is real or has spatially constant phase, i.e. $\Psi(x,t) = e^{i\theta(t)} f(x,t)$ for real functions θ, f then $J(x,t) = 0$. What does this imply for energy eigenstates?

Solution: If $\Psi(x,t)$ is real, then this case is fairly trivial to analyze, since $\Psi(x,t) = \Psi^\star(x,t)$:

$$\begin{aligned} J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial \Psi^\star}{\partial x} - \Psi^\star\frac{\partial \Psi}{\partial x}\right) \\ &= 0 \end{aligned}$$

And so, for real wavefunctions $\Psi(x,t)$, we get that the probability current is 0. Now, suppose $\Psi(x,t) = e^{i\theta(t)} f(x,t)$ for real functions θ, f. Then, we calculate our probability current:

$$\begin{aligned} J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial \Psi^\star}{\partial x} - \Psi^\star\frac{\partial \Psi}{\partial x}\right) \\ &= \frac{i\hbar}{2m}\left(e^{i\theta(t)} f(x,t)\left[e^{-i\theta(t)}\frac{\partial f}{\partial x}\right] - e^{-i\theta(t)} f(x,t)\left[e^{i\theta(t)}\frac{\partial f}{\partial x}\right]\right) \\ &= \frac{i\hbar}{2m}\left(f(x,t)\frac{\partial f}{\partial x} - f(x,t)\frac{\partial f}{\partial x}\right) \\ &= 0 \end{aligned}$$

As desired. ■

(c) Calculate $J(x,0)$ for a Gaussian wavepacket $\Psi(x,t)$.

Solution: We solved earlier that a gaussian wavepacket has the form:

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{e^{-ax^2/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}}$$

So substituting in $t = 0$ we get:

$$\Psi(x,0) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{-ax^2}$$

Since this gaussian is real-valued, then we have $\Psi(x,0) = \Psi^\star(x,0)$, so $\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi^\star}{\partial x}$, so if we compute the probability current:

$$J(x,0) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial \Psi^\star}{\partial x} - \Psi^\star\frac{\partial \Psi}{\partial x}\right) = 0$$

This means that at time $t = 0$, the probability current of the Gaussian wavepacket is 0!
