Eric Du | Physics 137B
HW 01 | Quantum Mechanics II | January 30, 2023

Collaborators

I worked with **Andrew Binder** to complete this assignment.

Problem 1

(a) If ψ_a and ψ_b are orhogonal, and both are normalized, what is the constant A in equation 5.17?

Solution: Equation 5.17 is:

$$\psi_\pm = A\left[\psi_a(r_1)\psi_b(r_2) + \psi_b(r_1)\psi_a(r_2)\right]$$

So therefore if ψ_a and ψ_b are orthogonal, therefore:

$$\begin{aligned}
|\psi_\pm(r_1, r_2)|^2 = 1 &= A^2 \iint |\psi_a(r_1)\psi_b(r_2)|^2 + |\psi_b(r_1)\psi_a(r_2)|^2 dr_1 dr_2 \\
&= A^2\left[\int |\psi_a(r_1)|^2 dr_1 \int |\psi_b(r_2)|^2 dr_2 + \int |\psi_b(r_1)|^2 dr_1 \int |\psi_b(r_2)|^2 dr_2\right] \\
&= A^2[1\cdot 1 + 1\cdot 1] \\
\therefore A &= \frac{1}{\sqrt{2}}
\end{aligned}$$

□

(b) If $\psi_a = \psi_b$ (and it is normalized), what is A? (This case, of course, occurs only with bosons)

Solution: Here, since $\psi_a = \psi_b$, then our equation simplifies to:

$$\psi_\pm = 2\psi_a(r_1)\psi_b(r_2) = 2\psi_a(r_1)\psi_a(r_2)$$

And so therefore if we normalize this:

$$\begin{aligned}
1 &= 4A^2 \iint |\psi_a(r_1)\psi_a(r_2)|^2 \; dr_1 dr_2 \\
&= 4A^2 \int |\psi_a(r_1)|^2 \; dr_1 \int |\psi_a(r_2)|^2 dr_2 \\
&= 4A^2(1\cdot 1) \\
\therefore A &= \frac{1}{2}
\end{aligned}$$

□

Problem 2

Find the next two excited states (beyond the ones given in the example) − wave functions and energies − for each of the three cases (distinguishable, identical bosons, identical fermions)

Solution: Since the energies scale with n^2, then we know that the next two states are going to be $(n_1, n_2) = (2,2), (1,3)$. This is the case except for fermions, as two fermions cannot exist in the same state. Therefore, we are forced to choose $(n_1, n_2) = (1,3), (2,3)$ instead. So our wavefunctions look like:

$$\text{Distinguishable: } \begin{cases} \psi_{22} = \frac{2}{a}\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{2}\right),\ E_{22} = \frac{8\pi^2\hbar^2}{2ma^2} \\ \psi_{13} = \frac{2}{a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right),\ E_{13} = \frac{10\pi^2\hbar^2}{2ma^2} \end{cases}$$

And now for the indistinguishable cases:

$$\text{Fermions: } \begin{cases} \psi_{13} = \frac{\sqrt{2}}{a}\left(\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_1}{a}\right) - \sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{3\pi x_1}{2}\right)\right),\ E_{13} = \frac{10\pi^2\hbar^2}{2ma^2} \\ \psi_{23} = \frac{\sqrt{2}}{a}\left(\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{3\pi x_1}{a}\right) - \sin\left(\frac{2\pi x_2}{a}\right)\sin\left(\frac{3\pi x_1}{2}\right)\right),\ E_{23} = \frac{13\pi^2\hbar^2}{2ma^2} \end{cases}$$

$$\text{Bosons: } \begin{cases} \psi_{13} = \frac{\sqrt{2}}{a}\left(\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_1}{a}\right) + \sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{3\pi x_1}{2}\right)\right),\ E_{13} = \frac{10\pi^2\hbar^2}{2ma^2} \\ \psi_{22} = \frac{\sqrt{2}}{a}\left(\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right) + \sin\left(\frac{2\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{2}\right)\right),\ E_{13} = \frac{4\pi^2\hbar^2}{2ma^2} \end{cases}$$

□

Problem 3

Suppose you had *three* particles, one in state $\psi_a(x)$, one in state $\psi_b(x)$, and one in state $\psi_c(x)$. Assuming ψ_a, ψ_b and ψ_c are orthonormal, construct the three-particle states (analogous to Equations 5.19, 5.20, 5.21) representing: (a) distinguishable particles, (b) identical bosons, and (c) identical fermions. Keep in mind that (b) must be completely symmetric, under exchange of *any* pair of particles, and (c) must be completely *anti*-symmetric, in the same sense. *Comment:* There's a cute trick for constructing completely antisymmetric wave functions: Form the **Slater determinant**, whose first row is $\psi_a(x_1), \psi_b(x_1), \psi_c(x_1)$ etc., whose second row is $\psi_a(x_2), \psi_b(x_2), \psi_c(x_2)$, etc., and so on (this device works for any number of particles).

Solution: For part (a) with distinguishable particles, then the simplest one is the product state:

$$\psi_{MB} = \psi_a(x_1)\psi_b(x_2)\psi_c(x_3)$$

For part (b) with indistinuishable bosons, we can form the 3x3 slater determinant, and use the rule for bosons:

$$\begin{aligned}\psi_{MB} &= \frac{1}{\sqrt{6}}\begin{vmatrix}\psi_a(x_1) & \psi_a(x_2) & \psi_a(x_3)\\ \psi_b(x_1) & \psi_b(x_2) & \psi_c(x_3)\\ \psi_c(x_1) & \psi_c(x_2) & \psi_c(x_3)\end{vmatrix}\\ &= \frac{1}{\sqrt{6}}\Big\{\psi_a(r_1)\left[\psi_b(r_2)\psi_c(r_3) + \psi_b(r_3)\psi_c(r_2)\right] + \psi_a(r_2)\left[\psi_b(r_1)\psi_b(r_3) + \psi_b(r_3)\psi_b(r_1)\right]\\ &\qquad + \psi_a(r_3)\left[\psi_b(r_1)\psi_c(r_2) + \psi_b(r_2)\psi_c(r_1)\right]\Big\}\end{aligned}$$

And for part (c) with indistinguishable fermions, we now take the determinant normally:

$$\begin{aligned}\psi_{MB} &= \frac{1}{\sqrt{6}}\begin{vmatrix}\psi_a(x_1) & \psi_a(x_2) & \psi_a(x_3)\\ \psi_b(x_1) & \psi_b(x_2) & \psi_c(x_3)\\ \psi_c(x_1) & \psi_c(x_2) & \psi_c(x_3)\end{vmatrix}\\ &= \frac{1}{\sqrt{6}}\Big\{\psi_a(r_1)\left[\psi_b(r_2)\psi_c(r_3) - \psi_b(r_3)\psi_c(r_2)\right] - \psi_a(r_2)\left[\psi_b(r_1)\psi_b(r_3) - \psi_b(r_3)\psi_b(r_1)\right]\\ &\qquad + \psi_a(r_3)\left[\psi_b(r_1)\psi_c(r_2) - \psi_b(r_2)\psi_c(r_1)\right]\Big\}\end{aligned}$$

$\square$