Non-Linear Dynamics (NLD)

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Outline

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Dynamics

- The study of how a system evolves through time.

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- *What happens to the sequence $\{x_n\}$ if I start at x_0 and iterate with the rule $x_1 = f(x_0)$ for some f of my choosing?*
- One thing we care about is whether the system is *linear*

Nonlinearity

- When a derivative in the differential equation (equation of motion) has a square term or higher, or has a cross term involving two derivatives.
  - More formally, the differential equation is linear if it can be expressed as a linear polynomial:

    $$a_0(x)y + a_1(x)y' + \cdots + a_n(x)y^{(n)} = b(x)$$

    $a_0(x), \ldots, a_n(x)$ and $b(x)$ need not be linear.
- As for the examples in the previous slide:
  - Ball: $y(t) = \dot{y}(0)t - \frac{1}{2}\ddot{y}t^2$

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  - $\{x_n\}$ sequence: depends on f!

What is Chaos?

- **Aperiodic:** Over long time, the behavior appears random and does not settle into a periodic state, where the same state is visited infinitely many times.
- **Deterministic:** Given a specific set of initial conditions, we can predict the evolution of the system through time.
- **Sensitive Dependence on Initial Conditions:** Trajectories which start off with an infinitesimal deviation diverge exponentially quickly (we will come back to this).

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Recursive Sequences

- Characterized by a recursive relation $x_{n+1} = f(x_{n-1}, x_{n-2}, \ldots, x_0)$.
- e.g. The Fibonacci sequence: $f_n = f_{n-1} + f_{n-2}$
- Our recursive equation of interest: $x_{n+1} = f(x_n) = rx_n(1 - x_n)$, sometimes called the logistic equation.
- We control the value of r, and are interested in how the sequence x_n evolves through successive iterations.
- We will choose the highest embedding dimension so that we can capture as much information about the state as possible.

Logistic Equation

What happens if we start at some small $x_0 = 0.01$, and apply the rule $x_n = f(x_{n-1})$, where $f(x) = rx(1-x)$, with $r = 2$?

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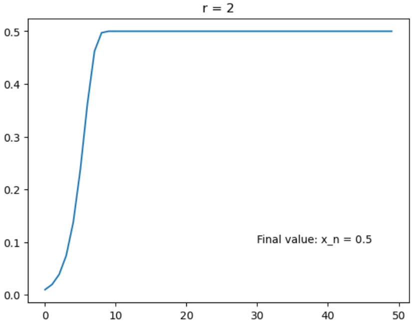

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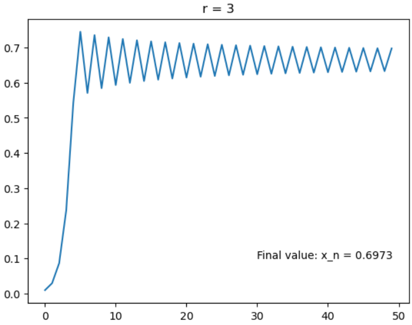
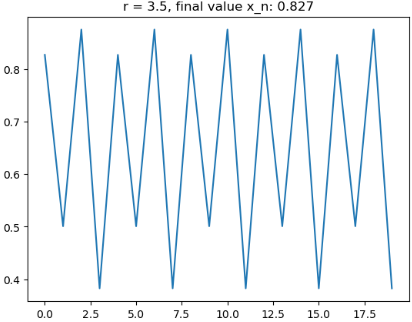


- The first ~ 30 points do not oscillate between the same two points, and are called *transients*
- Because they're not relevant to long-term behavior, we will remove them in future plots.

Logistic Equation

What happens if we start at some small $x_0 = 0.01$, and apply the rule $x_n = f(x_{n-1})$, where $f(x) = rx(1-x)$, with $r = 3.5$?



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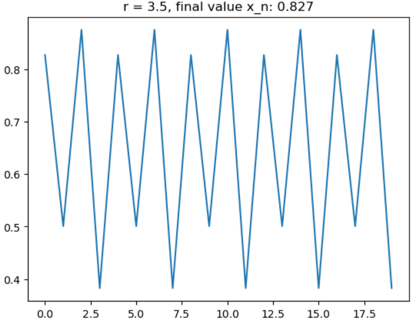
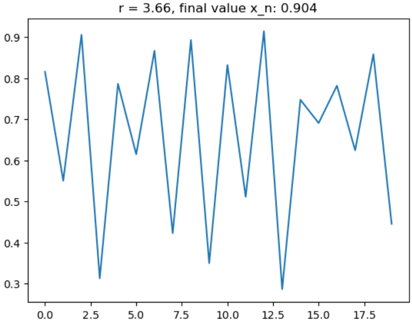


- Now x_n oscillates between 4 values $\implies$ 4-cycle.

Logistic Equation

Now, let's try $r = 3.66$:



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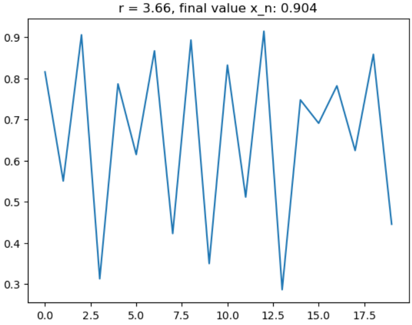

- Is there any pattern here?

Logistic Equation

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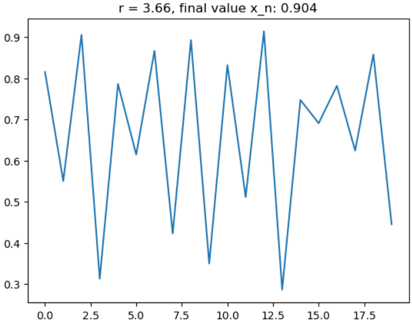
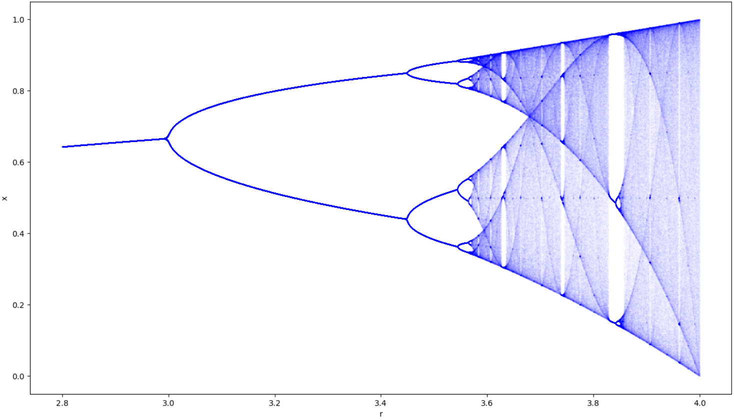


- Is there any pattern here?
- Chaos!

The Logistic Map



- Notice the period doublings!

Properties of the Logistic Map

- First Feigenbaum constant

$$\delta = \lim_{n \to \infty} \frac{d_{n+1} - d_n}{d_n - d_{n-1}} \approx 4.669\ldots$$

d_i is the difference between the i-th and the $(i+1)$-th period doubling.

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  d_i is the difference between the i-th and the $(i+1)$-th period doubling.
- Second Feigenbaum constant:

$$\alpha = \frac{a_n}{a_{n+1}} \approx 2.502\ldots$$

  a_i is the width of the forks
- Not much is known about either of these two numbers – it is unknown if either of these are even irrational!
- They are both universal constants: any chaotia equation of the form $x_{n+1} = \omega f(x_n)$ will have α, δ equal these values.

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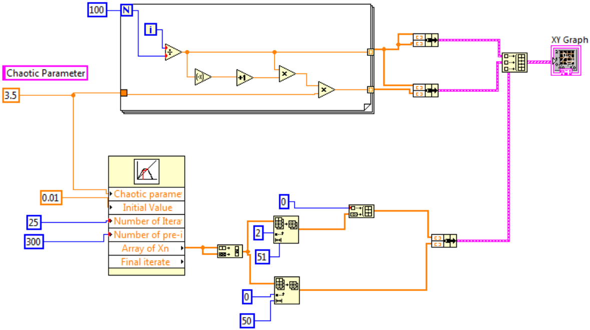
Cobweb.VI

- Another way to visualize the progression of x_n as a function of n.
- From a sequence of values $(m_0, \ldots, m_n)$, we interleave the array with itself to get $(m_0, m_0, \ldots, m_n, m_n)$, then:
  - x_values: $x_i = \{m_0, m_0, \ldots, m_{n-1}, m_{n-1}\}$
  - y_values: $y_i = \{0, m_1, m_1, \ldots, m_{n-1}, m_n\}$
- Interleaving the x_values with y_values, we get the set of coordinates:

$$(x_i, y_i) = \{(x_0, 0), (m_0, m_1), \ldots, (m_{n-1}, m_{n-1}), (m_{n-1}, m_n)\}$$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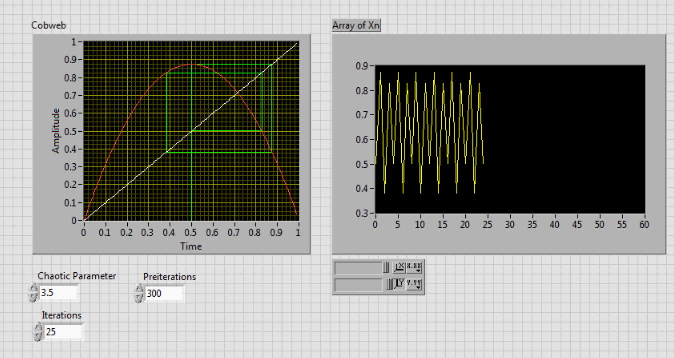
0
50

Visualization



Lyapunov Exponent

- **Sensitive Dependence on Initial Conditions:** Trajectories which start off with an infinitesimal deviation diverge exponentially quickly.
- Suppose you have two initial conditions, x_0 and $x_0 + \delta_0$ where $\delta_0 > 0$ is the initial separation, and is very small.
- Let δ_n be the separation after n iterations. The Lyapunov exponent is defined as the value of λ such that $|\delta_n| = |\delta_0|e^{n\lambda}$. If $\lambda > 0$, then the trajectory is chaotic.

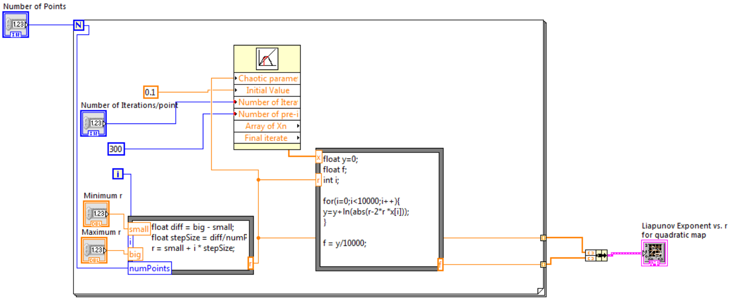
$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

- This formula comes from a linear approximation of the effect of δ_{i+1} from δ_i via linearization, hence the cumulative effect is a product, which decomposes to a sum over logarithms.

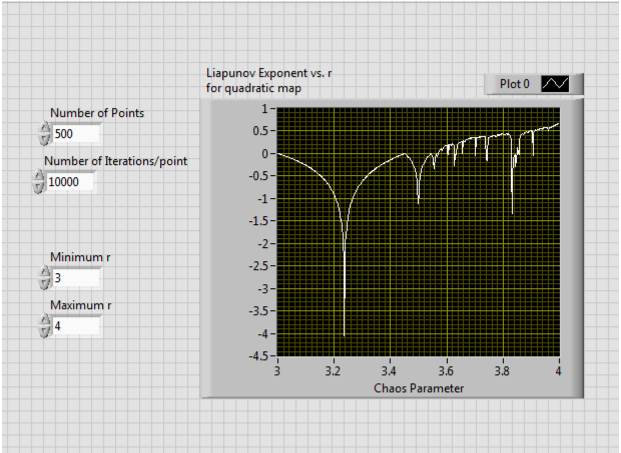
Procedure

- For a fixed value of r, allow the transients to die out, then compute a large number of iterates (10000).
- Compute $\ln |f(x_i)| = \ln |r - 2rx_n|$, and add all of them together.
- Divide the final result by 10000 to get λ.

LabVIEW Circuit

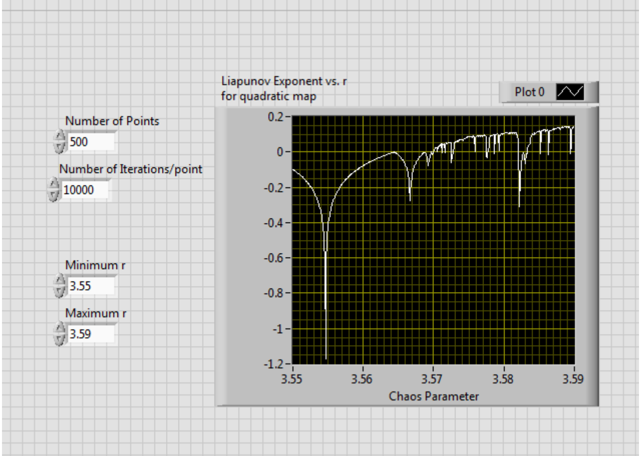


$r = 3$ to $r = 4$



- Notice the increase from negative to positive at around 3.57 – an indicator of chaos.

$r = 3.55$ to $r = 3.59$



- Note that each spike should be infinitely deep, but are not due to numerical imprecision.

Return Map

- Very similar to the Cobweb.VI, we plot (x_n, x_{n+1}) on a 2D plane.
- The **embedding dimension** is the dimension of phase space we choose to use. For a dimension 2, we only plot (x_n, x_{n+1}).
- For $d = 3$, we would plot (x_n, x_{n+1}, x_{n+2}) in 3D.
- Due to its qualitative nature, there is no real benefit to plot in higher dimensions for visual diagrams.

Short Aside: Fourier Series

- A procedure that allows us to decompose any given signal into a set of sinusoids. In our case, because we are sampling a discrete set of points x_n, then we need the Discrete Fourier Transform, generating a set of points:

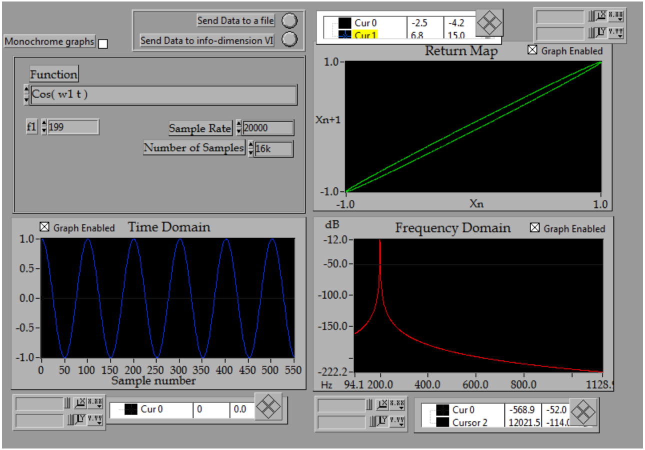
$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi ikn}{N}}$$

  Here, N refers to the number of points.
- Conceptually, it allows us to transform a signal in time into a signal in frequency.
- For a sine wave, because it's a pure frequency, we expect the frequency domain plot to be a spike exactly at the frequency of the input wave.

NLSim.VI

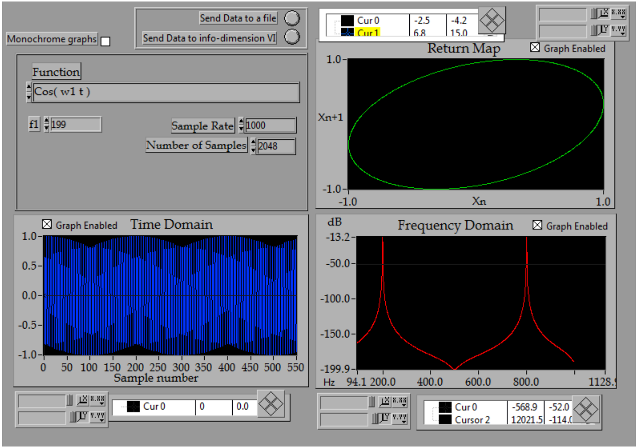
- Input is a sine wave $x(t) = \cos(\omega_1 t)$, we sample at $f = f_s$ for N points.
- We plot the sampled points vs. time, take the Fourier Transform, and also plot the *return map*
  - **Return map:** A plot of x_n vs. x_{n+1} in 2D space.
- Depending on the sampling rate, the sampled points may not give us back $x(t)$, due to aliasing.
- Aliasing occurs when the highest frequency of the signal is larger than half the sampling frequency, or equivalently the sampling frequency is less than twice the bandwidth.
  - For a sine wave at frequency f, its Fourier transform is two Delta functions at $\pm f$, so $\delta(f - f_0) + \delta(f + f_0)$.
  - The bandwidth is then $2f_0$.
- The sampling rate also cannot be a multiple of the signal frequency in order for the return map to display properly. Otherwise, it will only show several points.

LabVIEW Diagrams



- $f_s = 20000$ Hz, $f = 199$ Hz, so no aliasing.

LabVIEW Diagrams



- $f_s = 1000$ Hz, $f = 199$ Hz, so we do have aliasing (as apparent in the frequency plot)

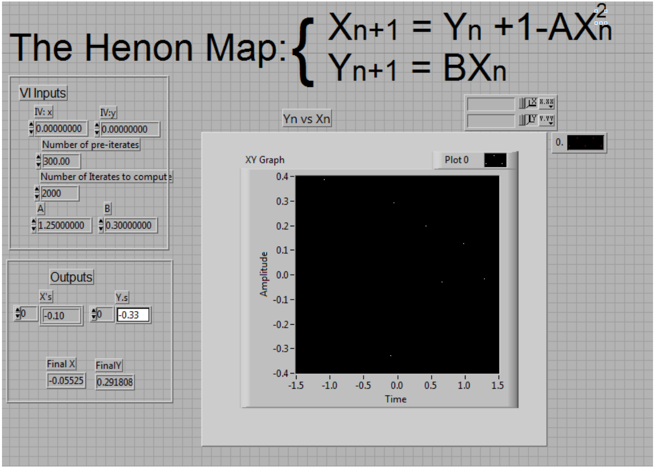
Overview

- The Henon map is another iterative system, except with a different equation.

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + y_n \\ y_{n+1} = bx_n \end{cases}$$

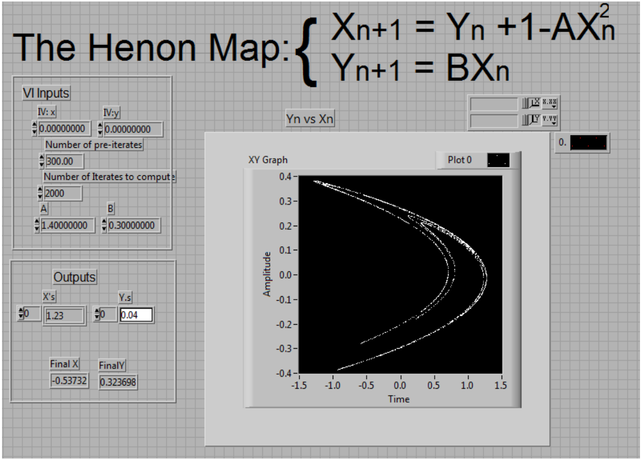
  There are two parameters a, b but a is the only *nonlinear* parameter. We will keep b fixed and vary a.
- We throw away the first 300 iterates, and compute 2000 iterates to generate the return map.

Return Map, $a = 1.25$



- Notice that there are only 7 points, meaning that this motion is periodic.

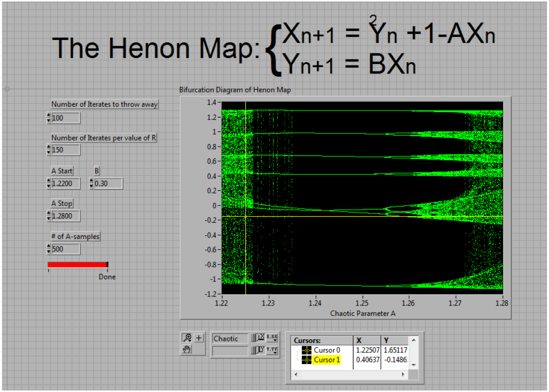
Return Map, $a = 1.4$



- The pattern doesn't seem to repeat this time, instead we see a *strange attractor*.

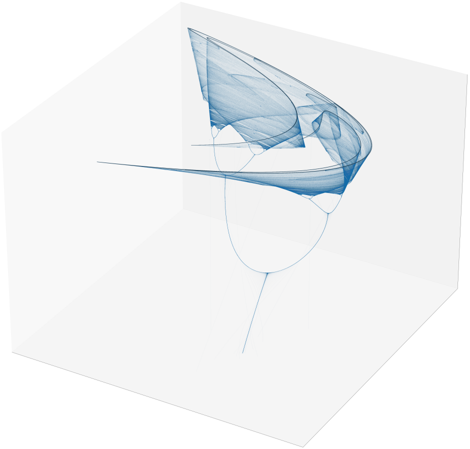
Bifurcation Diagram

- Another way to visualize the Henon map is to plot the periodicity for multiple values of a simultaneously, giving a 2D bifurcation diagram.



- Here we decided to zoom in on the region $a = 1.22$ to $a = 1.28$, but in principle one can do this for any range.

3D Version



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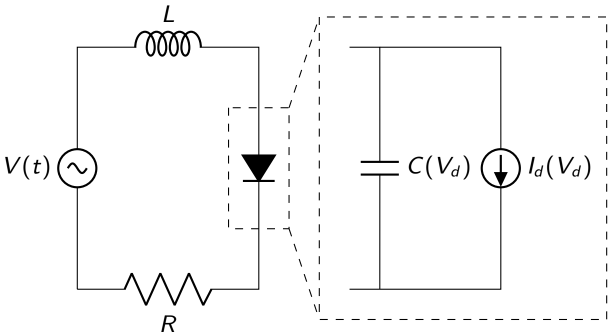
Circuit Diagram



- The nonlinear element is the diode, as it acts as an infinite resistor below V_d and as a current source above V_d.

Equations of Motion

The equations of motion for the PN junction circuit are provided to us:

$$\dot{I} = \frac{V_0 - RI - V_d}{L}$$
$$\dot{V}_d = \frac{I - I_d(V_d)}{C(V_d)}$$
$$\dot{\theta} = \omega$$

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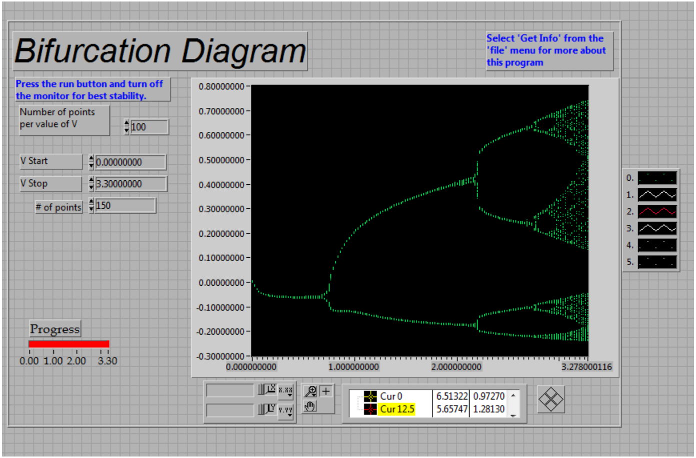
These equations do not depend explicitly on time, and rather the future system only depends on the current state. Therefore, we if $\vec{p}$ describes the current state in phase space, then we may write:

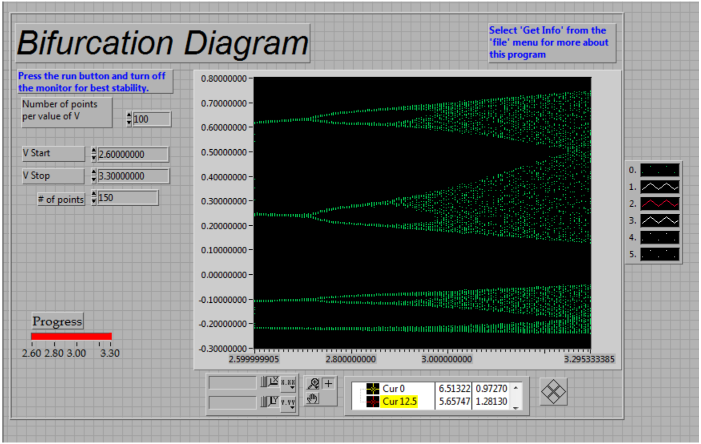
$$\dot{\vec{p}} = F(\vec{p})$$

this is an iterative equation!

Bifurcation Diagram, $V = 0$ to $V = 3.3$



Bifurcation Diagram, $V = 2.6$ to $V = 3.3$



- $1 \to 2$ cycle: $a = 0.786$
- $2 \to 4$ cycle: $a = 2.394$
- $4 \to 8$ cycle: $a = 3.092$

Feigenbaum Ratio

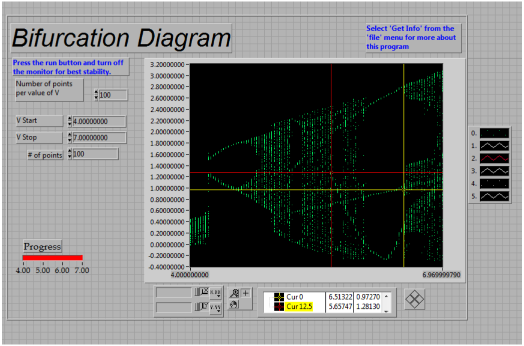
- With three values, there is only a single value of δ we can calculate:

$$\delta = \frac{2.394 - 0.786}{3.092 - 2.394} \approx 2.303$$

- This is a far cry from the expected $\delta \approx 4.669$, but remember that δ is a *limit*, so we don't expect the first few δ_n to match.
- We could have gone further with higher precision, unfortunately we did not consider doing this during the lab.

Bouncing Ball Circuit

- Circuit simulates bouncing a ball on a table that is oscillating vertically according to a sine wave.
- Actual circuitry is a black box and not provided to us :(



- Unlike the PN Junction, there are no real indicators for period doubling.
- Expected, since the motion of the ball is more complex and heavily depends on when the previous "bounce" occurred, so we cannot immediately write $\dot{\vec{p}} = F(\vec{p})$

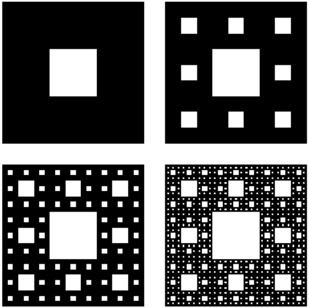
Box Dimension

- How do we extend our intuitive understanding of dimension for fractals?
- Consider a square of width ϵ. Let $N(\epsilon)$ be the number of ϵ-squares required to cover the entire fractal. Then:

$$d = \lim_{n\to\infty} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$$

- Conceptually, I think of this in the same way that the Gamma function $\Gamma(z)$ extends our normal understanding of the factorial.

Example



- The first level is covered by 8 squares of length $\frac{1}{3}$
- The second level is covered by $8^2 = 64$ squares of side length $\frac{1}{3^2}$.
- The n-th level is covered by 8^n squares of side length $\frac{1}{3^n}$.

$$d = \lim_{n \to \infty} \frac{\ln 8^n}{\ln(\frac{1}{3^n})} = \frac{\ln 8}{\ln 3} = 1.892\ldots$$

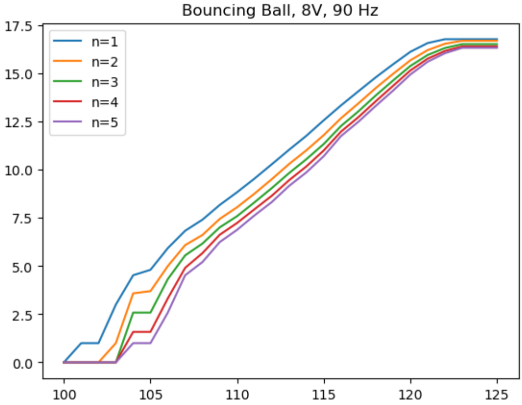
Information Dimension

- Box dimension is nice when you can iteratively generate the fractal, and that the covering with squares is nice. How does one even define a minimal covering?
- Information dimension is an alternative to this: take a point x in phase space, let $N_x(\epsilon)$ be the number of points within an ϵ-ball around x. Conceptually, $N_x(\epsilon)$, measures the number of points which are a distance ϵ from x.
- $N_x(\epsilon) \propto \epsilon^d$, and taking the average over many points x we get the *correlation dimension*

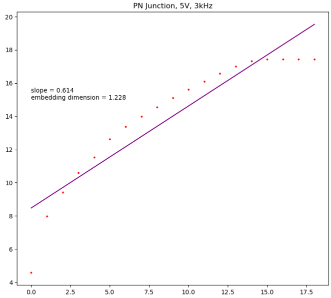
$$C(\epsilon) \propto \epsilon^d$$

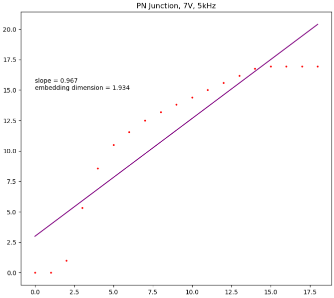
- $C(\epsilon)$ is calculated using `CALCCORR.VI`, then we use Python to fit a slope to them.
- In order to capture the maximum correlation, we want to use the highest embedding dimension possible, in our case we have $n = 5$.

Plot of $C(\epsilon)$

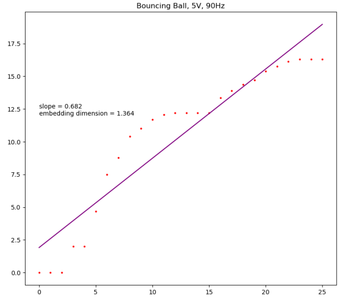


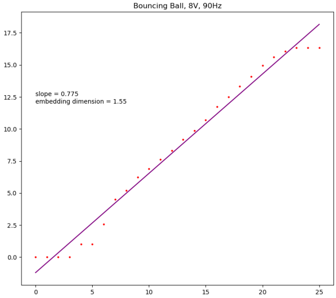
PN Junction


PN Junction, 5V, 3kHz
slope = 0.614
embedding dimension = 1.228


PN Junction, 7V, 5kHz
slope = 0.967
embedding dimension = 1.934

Bouncing Ball





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Summary

- For systems that exhibit more chaotic features, the embedding dimension seems to be higher.
  - This makes sense, since chaotic systems by nature get arbitrarily close to a point x in phase space without being periodic.
  - For conservative systems, the information dimension should also be integral.
- The information dimension for the Bouncing ball is lower than that of the PN junction.
- The PN Junction circuit is an example of an iterative chaotic system, so we get a bifurcation diagram very similar to that of the logistic map.
- We didn't really go into possible error sources in this process since most of it was electronic and numerical, but the two main sources of error would be the circuit elements themselves and numerical imprecision of numerical operations in computers.

Overall Thoughts on the Lab

- This was a fun lab!
- We missed a couple things: like writing down whether the configurations we were using for information dimension were chaotic or not.
  - The periodicity (or lack thereof) plays a large role in information dimension.
- Enjoyed playing around with a physical oscilloscope for the first time, in 111A we used Waveforms only.
- LabVIEW was a pain at times.

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