

## Physics W89 - Introduction to Mathematical Physics - Summer 2023

### Problem Set - Module 02 - What's Your Vector, Vector?

*Last Update: May 26, 2023*

## Problem 2.1 - The Dual Vector Space

*Relevant Videos: Introduction to Vectors and Vector Spaces*

In lecture we expanded our notion of what a vector can be by defining a vector space as a set of vectors that are closed under **vector addition** and **scalar multiplication**, where vector addition and scalar multiplication were defined via a set of eight vector space axioms. We saw that the familiar “arrows in 3D space” vectors that were used all throughout intro physics were indeed vectors (they provided the template for defining a vector!) and we also saw that things like the shape of waves on a string could act as vectors. In this problem we explore another important set of vectors called the **dual vector space**.

In short, a **dual vector** is a machine that *eats a vector and returns a scalar in a linear fashion*. That is, a dual vector  $\underline{\alpha}$  is a **linear function** from a vector space  $V$  to the real (or complex) numbers,  $\underline{\alpha} : V \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ),

$$\underline{\alpha}(a\vec{v}_1 + b\vec{v}_2) = a\underline{\alpha}(\vec{v}_1) + b\underline{\alpha}(\vec{v}_2). \quad (1)$$

Note that we have a notational decoration for dual vectors (an underline) just like we have a notational decoration for vector (the “arrow hat”).

Given a vector space  $V$ , the **dual space**  $V^*$  is abstractly<sup>1</sup> defined as the set of all linear functions from the vector space to the scalars. In this problem we will take our scalars to be the real numbers. Another way of expressing this definition is,

$$V^* \equiv \{\underline{\alpha} : V \rightarrow \mathbb{R} \mid \underline{\alpha} \text{ is linear}\}.$$

For this problem the vector space  $V$  will be  $\mathbb{R}^3$ , the three-dimensional vectors (like force, velocity, displacement, etc.), which we will write in component form as a column of three numbers,  $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ .

First let's explore what these dual vectors are and aren't by looking at some examples of what is and isn't in  $V^*$ .

**(a)** Show that the function that picks out the  $x$ -component of a vector  $\underline{\beta}_x(\vec{v}) \equiv v_x$  is an element of  $V^*$ . Then argue that the same holds for the function that picks out the  $y$ -component and  $z$ -component of a vector.

[Note: When I ask you to “show” something, you are basically providing a short proof. In this case, to show that  $\underline{\beta}_x$  is an element of  $V^*$ , you need to test whether it satisfies the defining property of linearity, Eq. 1. When I ask you to “argue” something, it doesn't need to be a formal or detailed proof. It should be something like a sentence or two explaining why the result follows. This will usually be a reference to a previous proof or an appeal to symmetry.]

<sup>1</sup>Oh, boy! I'll bet you all love seeing *this* word pop up!

(b) Show that the “length” function  $\delta(\vec{v}) \equiv |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$  is *not* an element of  $V^*$  by coming up with an explicit example of the failure of linearity, Eq. 1.

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We may define the vector addition and scalar multiplication of dual vectors via the action of the resulting map on vectors,

$$\begin{aligned}\text{Vector Addition: } (\underline{\alpha} + \underline{\beta})(\vec{v}) &\equiv \underline{\alpha}(\vec{v}) + \underline{\beta}(\vec{v}), \\ \text{Scalar Multiplication: } (a\underline{\alpha})(\vec{v}) &\equiv a(\underline{\alpha}(\vec{v})).\end{aligned}$$

Define  $\underline{\zeta}$  to be the “zero dual vector,” which spits out 0 no matter what vector is fed to it,  $\underline{\zeta}(\vec{v}) = 0 \forall \vec{v}$ .

(c) Show that vector addition and scalar multiplication of dual vectors satisfy the eight axioms for a vector space.

[Note: This is the only time (including exams) that I’ll explicitly ask you to check these eight axioms, but it’s good to go through it once. An example of showing the commutativity of vector addition (the first axiom) is given as a supplement. Don’t overthink this part! A lot of the axioms will be pretty trivially satisfied.]

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Finally, for  $V^*$  to be a vector space we need it to be **closed** under addition and scalar multiplication. That is, we need to show that the sum of two dual vectors is itself a dual vector and the scaling of a dual vector is a dual vector.

(d) **Extra Part (Not for Credit)** Show that  $V^*$  is closed under vector addition and scalar multiplication and is thus a vector space!

(e) Use the results from (a) and (d) to argue that the function that picks out a specific linear combination of components,  $\underline{\beta}(\vec{v}) = b_x v_x + b_y v_y + b_z v_z$  is an element of  $V^*$ , where  $b_x, b_y, b_z$  are all real constants.

*Commentary: We won’t prove it here, but in fact all dual vectors of  $\mathbb{R}^3$  can be interpreted in this way! Once we introduce matrices, we will be able to write these dual vectors as **row vectors**, at which point the action of the dual vector simply becomes matrix multiplication.*

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## Problem 2.2 - Vector Identities in 3D

*Relevant Videos: 3D Vectors and Index Notation*

In lecture we introduced the Kronecker delta  $\delta_{ij}$  and the Levi-Civita symbol  $\epsilon_{ijk}$  which let us express the dot product and cross product of vectors in our usual three-dimensional space in terms of components. You should use Einstein summation convention (any repeated index is summed over) throughout this problem.

(a) Derive the vector triple product identity  $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$  using the Kronecker delta and Levi-Civita symbol. Carefully explain and/or justify each step you take (e.g. “by the antisymmetry of  $\epsilon_{ijk}$ ”).

[Spoilers! Start with the left-hand side. Since both sides are vectors, we really want to show the component-version of the identity,  $(\vec{A} \times (\vec{B} \times \vec{C}))_i = (\vec{A} \cdot \vec{C})B_i - (\vec{A} \cdot \vec{B})C_i$ .]

(b) **Extra Part** (*Not for Credit*) Using index notation, show **Lagrange's identity**,

$$(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = (\vec{A} \cdot \vec{A})(\vec{B} \cdot \vec{B}) - (\vec{A} \cdot \vec{B})^2.$$

We can treat the nabla/del symbol from vector calculus in components as:  $\vec{\nabla} \mapsto \partial_i \equiv \frac{\partial}{\partial x_i}$ . Using this, the gradient, divergence, and curl can be expressed in index notation:

$$\begin{aligned} \text{Gradient:} \quad & (\vec{\nabla} f)_i = \partial_i f, \\ \text{Divergence:} \quad & \vec{\nabla} \cdot \vec{v} = \delta_{ij} \partial_i v_j, \\ \text{Curl:} \quad & (\vec{\nabla} \times \vec{v})_i = \epsilon_{ijk} \partial_j v_k. \end{aligned}$$

(c) **Extra Part** (*Not for Credit*) Write out the **Laplacian** of a scalar function  $\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f$  in index notation and then carry out the sum.

(d) Prove that the curl of the gradient is zero:  $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$ .

(e) Prove that the curl of the curl is given by  $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$ .

## Problem 2.3 - Subspace Communication

*Relevant Videos: Linear Independence and Span; Bases*

A **subset** of a vector space  $W \subset V$  is any collection of objects from the vectors space. In some cases the subset is itself a vector space, in which case we call the subset a **vector subspace**. Since vector addition and scalar multiplication carry over from  $V$  to  $W$ , to determine whether a subset is a subspace, we only need to check that it is closed under vector addition and scalar multiplication<sup>2</sup> - that is, we need to check that an arbitrary linear combination of elements of the subset is itself an element of the subset. Note that *by definition*, the span of a set of vectors  $\{\vec{v}_i \in V\}$  is a subspace of  $V$ . Also note that for a given vector space  $V$ , the full space  $V$  and the subset  $\{0\}$  just containing the zero vector are always subspaces. The zero subspace is zero-dimensional.

(a) Consider a vector space  $V$  and a subset  $W \subset V$ . Show or argue that if the zero vector is not part of a subset ( $\vec{0} \notin W$ ) then  $W$  *can't* be a vector subspace.

[Note: We say that the zero vector being a part of  $W$  is a necessary but not a sufficient condition for  $W$  to be a subspace. That is, for  $W$  to be a subspace it needs to contain the zero vector but  $W$  containing the zero vector isn't enough to claim  $W$  is a subspace.]

[Supplementary Part (*Not for Credit*): Try to come up with an example of a subset  $W$  that contains the zero vector but is not a vector subspace of  $V = \mathbb{R}^2$ .]

Consider the collection of all polynomials (with real coefficients) of degree less than or equal to  $N$  in the real variable  $x$ , which we may write as

$$P_N \equiv \left\{ \sum_{j=0}^N a_j x^j \mid a_j \in \mathbb{R} \right\}. \quad (2)$$

We can take as our starting point that the space of *all* polynomials (with no upper limit on the degree) is a vector space.

<sup>2</sup>See, for example, Problem 2.1(d).

(b) For each of the following subsets, state whether or not the set constitutes a vector space. If so, suggest a convenient basis and give the dimension of the space. If not, explain or show why not. You do not need to explicitly or rigorously *prove* that your suggested basis is a basis, but you should *argue* that the elements of your basis are indeed linear independent and do in fact span the subset.

- The full set  $P_N$  given by Eq. 2.
- The subset of  $P_N$  such that the leading coefficient of the polynomial is 1 (that is, the set such that  $a_N = 1$ ).
- The subset of  $P_N$  such that the polynomials have the value 0 at  $x = 1$ .
- *Supplementary Part (Not for Credit)*: The subset of  $P_N$  such that the polynomials have the value 1 at  $x = 0$ .

## Problem 2.4 - A Wave on a String

*Relevant Videos: Bases; The Inner Product*

Next let's look at our “waves on a string” vector space - the space of continuous real functions  $f(x)$  between  $x = 0$  and  $x = L$  with  $f(0) = f(L) = 0$  (the string is fixed at zero at the endpoints). We can abstractly refer to one of our shapes as a vector  $\vec{g}$  in this space or “in components” as a function  $g(x)$ . Consider the set of vectors  $\{\hat{f}_n\}$ ,

$$\hat{f}_n \doteq f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}$$

with  $n$  a positive integer ( $n = 1, 2, 3, \dots$ ). We define the inner product between two vectors  $\vec{g}$  and  $\vec{h}$  in this space of functions as

$$\vec{g} \cdot \vec{h} \equiv \int_0^L g(x)h(x) dx. \quad (3)$$

(a) Show that the set of vectors  $\{\hat{f}_n\}$  is an orthonormal set. That is, show  $\hat{f}_n \cdot \hat{f}_m = \delta_{nm}$ .  
*[Supplementary Part (Not for Credit): Show that this implies the set  $\{\hat{f}_n\}$  is linearly independent.]*  
*[Note: This set also turns out to span our vector space (this is called **Fourier's theorem**) and thus it serves as an orthonormal basis!]*

(b) **Extra Part (Not for Credit)** Argue that the given inner product does indeed satisfy the three defining features of the inner product (conjugate symmetry, linearity in the second term, and positive-definiteness).

Since we have an orthonormal basis, we can expand any vector  $\vec{g}$  in our vector space (any “shape on our guitar string”) as

$$\vec{g} = \sum_{n=1}^{\infty} c_n \hat{f}_n, \quad (4)$$

where  $c_n$  are a set of real coefficients, the “components” of the function.

(c) Show that we can find the coefficients  $c_n$  in Eq. 4 by taking the inner product  $\hat{f}_n \cdot \vec{g}$ .

*[Note: This is an extremely important proof and result!]*

*[Spoilers! Be careful with indices. We are already using the index  $n$  in the expression  $\hat{f}_n \cdot \vec{g}$ . When you substitute in equation Eq. 4 for  $\vec{g}$ , you will need to use a different summation index (e.g.  $m$ ).]*

Suppose we pluck the string so that the shape is described by the following “triangle” function,

$$\vec{g} \doteq g(x) = \begin{cases} x, & 0 \leq x \leq L/2, \\ (L - x), & L/2 \leq x \leq L. \end{cases} \quad (5)$$

(d) Find the components  $c_n$ . Try finding  $c_n$  for  $n$  odd first, and then find  $c_n$  for  $n$  even.

[*Spoilers!* Remember our result from part (c)! Exploit symmetry and think before you integrate! One of your expressions should be zero.]

If this string were stretched taut on a guitar, then releasing the string from the pluck described in Eq. 5 makes a sound. The value  $c_1$  tells you about the amplitude of the **fundamental frequency** of the sound and the other values of  $c_n$  can be interpreted as the amplitudes of each of the **harmonics**. The total energy produced by the sound wave is proportional to the magnitude-squared of the vector  $\vec{g}$ , and the energy contained in any harmonic is proportional (with the same proportionality constant) to the magnitude-squared of the coefficient  $c_n$ . Letting  $\eta$  be the proportionality constant,

$$E_{\text{wave}} = \eta \vec{g} \cdot \vec{g}; \quad E_{n\text{-th harmonic}} = \eta |c_n|^2.$$

Therefore, the fraction of energy contained in a harmonic is given by  $|c_n|^2 / |\vec{g}|^2$ .

(e) Given the triangle wave  $\vec{g}$  from the previous part, what fraction of energy is in the fundamental ( $n = 1$ ) frequency? The first harmonic ( $n = 2$ )?

[*Spoilers!* You already found  $c_1$  and  $c_2$  in part (d)!]

(f) **Extra Part (Not for Credit)** Physics refresher/reminder! Suppose the string had a lineal mass density of  $\mu$  and was under tension  $T$ . What is the wave speed? What is the fundamental frequency  $\omega_1$ ? The frequency of the  $n$ -th harmonic  $\omega_n$ ?

(g) Let  $L = 1$  and graph<sup>3</sup> the triangle function  $g(x)$  and the partial sums

$$\sum_{n=1}^{n_{\max}} c_n f_n(x),$$

of the expansion for  $n_{\max} = 3, 5, 10, 50$  to see how the series converges to the true function  $g(x)$ .

## Problem 2.5 - A Block Sliding Down an Incline

*Relevant Videos: The Inner Product; The Gram-Schmidt Procedure*

You thought you’d never see these first-semester physics “block slides down a slope” problems again? Consider a triangular piece of plane whose vertices are positions  $\vec{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 2.4 \end{pmatrix}$ ,  $\vec{V}_2 = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$ ,  $\vec{V}_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ . These vectors are written with respect to the unit vectors  $\hat{x}, \hat{y}, \hat{z}$ , with  $\hat{z}$  being the “vertically up” direction. Consider the three “edge vectors” which form the edges of the plane,

$$\vec{A} = \vec{V}_2 - \vec{V}_1 = \begin{pmatrix} 0 \\ 4 \\ -2.4 \end{pmatrix}, \quad \vec{B} = \vec{V}_3 - \vec{V}_2 = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}, \quad \vec{C} = \vec{V}_1 - \vec{V}_3 = \begin{pmatrix} -3 \\ 0 \\ 2.4 \end{pmatrix},$$

<sup>3</sup>Go-go-gadget Python/Mathematica/WolframAlpha/MatLab/etc.

(a) Explicitly show that  $\{\vec{A}, \vec{B}, \vec{C}\}$  is *not* a linearly independent set and does *not* span  $\mathbb{R}^3$ .  
*[Spoilers! This means explicitly constructing a linear combination of two of the vectors to create the third to show linear dependence and showing that there is a vector in  $\mathbb{R}^3$  that cannot be expressed as a linear combination of the three vectors.]*

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The span of the edge vectors creates a plane that passes through the origin,  $P \equiv \text{Span}(\vec{A}, \vec{B}, \vec{C})$ . We want to build an orthonormal basis to describe objects moving around on our plane. Since these three vectors are linearly dependent, we may ignore one of them when building a basis.

(b) Use the Gram-Schmidt procedure on the ordered pair  $\{\vec{B}, \vec{A}\}$  to form an orthonormal basis  $\{\hat{e}_1, \hat{e}_2\}$  for the plane  $P$ .

*[Note: We can do this for any ordered pair of the three vectors  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$ . We choose to use  $\vec{B}$  as our first vector since it has no  $\hat{z}$ -component and we can be assured that our first basis vector will be “horizontal”. Thus we expect a block sliding down this incline will have a constant velocity in the  $\hat{e}_1$ -direction. I chose to use  $\vec{A}$  as our second vector since the answer comes out a lot nicer compared to using  $\vec{C}$ .]*

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Recall that the projection of a vector  $\vec{v}$  onto a space spanned by a set of vectors  $\{\hat{e}_1, \dots, \hat{e}_m\}$  is

$$\vec{v}_{\text{projected}} = \sum_{i=1}^m (\hat{e}_i \cdot \vec{v}) \hat{e}_i.$$

A mass placed on the plane will experience a net force  $\vec{F}_{\text{net}}$  given by the **projection** of the gravitational force  $\vec{F}_g = -mg\hat{z}$  onto the plane. Consider the basis for vectors-in-the-plane  $\{\hat{e}_1, \hat{e}_2\}$  that we found in part (b). The projection is thus

$$\vec{F}_{\text{net}} = \sum_{i=1}^2 (\hat{e}_i \cdot \vec{F}_g) \hat{e}_i.$$

(c) What is the component of  $\vec{F}_g$  that points in the  $\hat{e}_1$ -direction? In the  $\hat{e}_2$ -direction? Use these answers to find  $\vec{F}_{\text{net}}$  in the usual  $\{\hat{x}, \hat{y}, \hat{z}\}$  basis of three-dimensional space.

*[Spoilers! Once you have the components in the  $\hat{e}_1$  and  $\hat{e}_2$  directions, you can solve the last part of the problem by just replacing the two symbols  $\hat{e}_1$  and  $\hat{e}_2$  with their three components. You already found these in part (b).]*

(d) The direction of the vector you found in part (c) represents the “most downward” direction within the plane. What angle  $\theta$  does this direction make with the vertically downward direction  $-\hat{z}$ ?

*[Note: We can consider  $\phi = \pi/2 - \theta$  to define the slope of the plane so  $\phi = 0$  is horizontally flat, for example]*

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The actual plane of the triangle is the plane  $P$  but displaced from the origin. Points on the plane can be described by position vectors  $\vec{r} = \alpha\vec{e}_1 + \beta\vec{e}_2 + \vec{V}_1$ , where  $\alpha$  and  $\beta$  are arbitrary real constants. Note that the top vertex of the triangle is described by  $\alpha = \beta = 0$ .

(e) A block of mass  $m$  is released at time  $t = 0$  from rest at position  $\vec{r}(0) = \vec{V}_1$ . Find the coefficients  $\alpha(t)$  and  $\beta(t)$ . Then find  $\vec{r}(t)$  as a column vector in the usual  $\{\hat{x}, \hat{y}, \hat{z}\}$ -basis.

(f) **Extra Part** (*Not for Credit*) What point on the plane is the closest to the origin? (That is, find the shortest vector  $\vec{r}$  that points from the origin to a point on the plane.) Show that the corresponding vector  $\vec{r}$  is perpendicular to the plane.

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In Module 3, we will be introduced to ***Gauss-Jordan Reduction***, a powerful technique of simplifying matrices that will allow us to extract a lot of information from the matrices. In particular, armed with row reduction, we can determine whether or not a set of vectors is linearly independent and whether or not one vector is in the span of other vectors. This next part is *not due*, but you should look back on it when we get to row reduction as part of your study strategy for the exam!

Define the matrix  $P$  whose columns are composed of the three vectors,

$$P = \begin{pmatrix} \vec{A} & \vec{B} & \vec{C} \end{pmatrix} = \begin{pmatrix} 0 & 3 & -3 \\ 4 & -4 & 0 \\ -2.4 & 0 & 2.4 \end{pmatrix}.$$

**(Future Study Part (g)) Extra Part (Not for Credit)** Reduce the matrix  $P$  to row echelon form and determine  $\text{rank } P$ . Using the rank, explain how you can tell that these vectors are not linearly independent and how you can tell that these vectors don't span  $\mathbb{R}^3$ .

[Note: You do not need to fully reduce to reduced row echelon form - just the simple row echelon form that will allow us to determine the rank. For the explanation, I just want a one-sentence description: "Because (this aspect) is (less than/equal to/greater than) (this other aspect), the vectors (are/are not/do/do not) (property we are interested in)."]




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## Spoilers! An Example Axiom for Problem 2.1(c)

One of the properties of a vector space is the commutativity of vector addition. For the dual vector space this means that we want to show

$$\underline{\alpha} + \underline{\beta} = \underline{\beta} + \underline{\alpha}. \quad (6)$$

We do this by using the fact that the dual vectors are functions on vectors. For this property to hold, the left side of Eq. 6 must be equal to the right side when acting on any arbitrary vector  $\vec{v}$ ,

$$\text{Left: } (\underline{\alpha} + \underline{\beta})(\vec{v}) = \underline{\alpha}(\vec{v}) + \underline{\beta}(\vec{v}) \qquad \text{Right: } (\underline{\beta} + \underline{\alpha})(\vec{v}) = \underline{\beta}(\vec{v}) + \underline{\alpha}(\vec{v}).$$

But  $\underline{\alpha}(\vec{v})$  and  $\underline{\beta}(\vec{v})$  are, by definition, just two real numbers! We know that addition of real numbers commute, so we can claim that the LHS and RHS are the same for an arbitrary vector  $\vec{v}$  and commutativity of vector addition has been demonstrated!