Header styling inspired by CS 70: https://www.eecs70.org/

Problem 1

Let $\Phi = E - TS - \mu N$ be the "grand" potential

a) Derive the thermodynamic identity $d\Phi = -SdT - PdV + Nd\mu$

b) Under what conditions does a system adjust so as to minimize Φ?

c) Suppose you have computed $\Phi(T, V, \mu)$. How could you use it to determine $N(T, V, \mu)$ and $P(T, V, \mu)$?

d) Prove that $\Phi(T, V, \mu) = -kT \ln \mathscr{Z}$, where $\mathscr{Z} = \sum_\alpha e^{-\beta(E_\alpha - \mu N_\alpha)}$

Schroeder 7.19

Each atom in a chunk of copper contributes one conduction electron. Look up the density and atomic mass of copper, and calculate the Fermi energy, the Fermi temperature, the degeneracy pressure, and the contribution of the degeneracy pressure to the bulk modulus. Is room temperature sufficiently low to treat this system as a degenerate electron gas?

Schroeder 7.22

Consider a degenerate electron gas in which essentially all of the electrons are highly relativistic $\epsilon \gg mc^2$, so that their energies are $\epsilon = pc$ (where p is the magnitude of the momentum vector).

a) Modify the derivation given above to show that for a relativistic electron gas at zero temperature, the chemical potential (or Fermi energy) is given by $\mu = hc(3N/8\pi V)^{1/3}$.

b) Find a formula for the total energy of this system in terms of N and μ.

Schroeder 7.23

A **white dwarf** star (see Figure 7.12) is essentially a degenerate electron gas, with a bunch of nuclei mixed in to balance the charge and to provide gravitational attraction that holds the start together. In this problem you will derive a relation between the mass and the radius of a white dwarf star, modeling the star as a uniform-density sphere. White dwarf stars tend to be extremely hot by our standards; nevertheless, it is an excellent approximation in this problem to set $T = 0$.

a) Use dimensional analysis to argue that the gravitational potential energy of a uniform-density sphere (mass M, radius R) must equal

$$U_{\text{grav}} = -(\text{constant})\frac{GM^2}{R}$$

where (constant) is some universal constant. Be sure to explain the minus sign. The constnat turns out to equal 3/5; you can derive it by calculating the (negative) work needed to assemble the sphere, shell by shell, from the inside out.

b) Assuming that the star contains one proton adn one neutron for each electron, and that the electrons are nonrelativistic, show that the total (kinetic) energy of the degenerate electrons equals

$$U_{\text{kinetic}} = (0.0088)\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2}$$

The numerical factor can be expressed exactly in terms of π and cube roots and such, but it's not worth it.

c) The equilibrium radius of the white dwarf is that which minimizes the total energy $U_{\text{grav}} + U_{\text{kinetic}}$. Sketch the total energy as a function of R, and find a formula for the equilibrium radius in terms of the mass. As the mass increases, does the radius increase or decrease? Does this make sense?

d) Evaluate the equilibrium radius for $M = 2 \times 10^{30}$ kg, the mass of the sun. Also evaluate the density. How does the density compare to that of water?

e) Calculate the Fermi energy and the Fermi temperature, for the case considered in part (d). Discuss whether the approximation $T = 0$ is valid.

f) Suppose instead that the electrons in the white dwarf are highly relativistic. Using the result from the previous problem, show that the total kinetic energy of the electrons is now proportional to $1/R$ instead of $1/R^2$. Argue that there is no stable equilibrium radius for such a star.

g) The transition from the nonrelativistic regime to the ultrarelativistic regime occurs approximately where the average kinetic energy of an electron is equal to its rest energy, mc^2. Is the nonrelativistic approximation valid for a one-solar-mass white dwarf? Above what mass would you expect a white dwarf to become relativistic and hence unstable?

Schroeder 7.25

Use the results of this section to estimate the contribution of conduction electrons to the heat capacity of one mole of copper at room temperature. How does this contribution compare to that of lattice vibrations, assuming that these are not frozen out? (The electronic contribution has been measured at low temperatures, and turns out to be about 40% more than predicted by the free electron model used here.)