Problem 1

Using proof by induction to prove that: For every $n \in \mathbb{N}$, $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Solution: Let $A \subset N$ be the set of naturals which satisfies the above proposition. First, we show that $m = 1 \in A$:

$$1 = \frac{1(2)}{2} \quad \checkmark$$

Now, suppose that an arbitrary $m \in A$. We show that $m + 1 \in A$:

$$\begin{aligned}\sum_{k=1}^{m+1} k &= \sum_{k=1}^{m} k + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

as desired. $\square$

Problem 2

(a) Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers n.

Solution: Just like the previous problem, let $A \subset \mathbb{N}$ which satisfies the proposition. We show that $m = 1 \in A$:

$$1^3 = \left(\frac{1(2)}{2}\right)^2$$

Now assume that for some $m-1$, we have $\sum_{n=1}^{m-1} n = (1 + 2 + \cdots + m - 1)^2$. Now we show that $P(m-1) \implies P(m)$:

$$\begin{aligned}
\sum_{n=1}^{m} n^3 &= \sum_{n=1}^{m-1} n^3 + m^3 \\
&= \left(\frac{m(m-1)}{2}\right)^2 + m^3 \\
&= \frac{m^4 - 2m^3 + m^2}{4} + \frac{4m^3}{4} \\
&= \frac{m^4 + 2m^3 + m^2}{4} \\
&= \left(\frac{m(m+1)}{2}\right)^2
\end{aligned}$$

as desired. □

(b) The princpile of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, P_{n+1} is true whenever P_n is true and $n \geq m$.

(i) Prove $n^2 > n + 1$ for all integers $n \geq 2$.

Solution: Following the steps of induction, let $A \subset \mathbb{N}$ be the set which satisfies the proposition. we show that $m = 2 \in A$

$$2^2 > 2 + 1 \quad \checkmark$$

Now assume that for some m the proposition holds. Now we show P_{m+1} also holds:

$$\begin{aligned}
(m+1)^2 &> (m+1) + 1 \\
m^2 + 2m + 1 &> m + 2 \\
m^2 + m - 1 &> 0
\end{aligned}$$

This statement is clearly true for $m > 2$, since $m^2 + m > 1$. Therefore, P_{m+1} is true, and so we're done. □

(ii) Prove $n! > n^2$ for all integers $n \geq 4$. [Recall $n! = n(n-1)\cdots 2{\cdot}1$; for example, $5! = 5{\cdot}4{\cdot}3{\cdot}2{\cdot}1 = 120$.]

Solution: Just like the previous problem, let $A \subset \mathbb{N}$ be the set which satisfies the propositn. We prove that $m = 4$ satisfies:

$$4! = 120 > 16 \quad \checkmark$$

Now assume for some m that the proposition holds. Thus from the inductive hypothesis, we get:

$$(m+1)! = (m+1)m! > (m+1)m^2$$

Then from part (a) we know that $m^2 > m + 1$ so we now write:

$$(m + 1)m^2 > (m + 1)(m + 1) = (m + 1)^2$$

as desired. $\square$

Problem 3

Prove: $\sqrt{3}$ is not a rational number

Solution: Let $\sqrt{3}$ be defined as the solution to the polynomial $x^2 - 3 = 0$. Then, by the rational root theorem we know that rational solutions to this polynomial must divide 3, which are going to be $\pm 1, \pm 3$. Since none of these solve the equation, then $\sqrt{3}$ is not a rational number. $\square$

Problem 4

Prove: $\sqrt{2} + \sqrt{3}$ is not a rational number.

Solution: Here, we construct a polynomial where $\sqrt{2} + \sqrt{3}$ is the root. One that comes to mind is:

$$x^2 - (\sqrt{2} + \sqrt{3})^2 = 0$$

However, this gives:

$$x^2 - 5 - 2\sqrt{6} = 0$$

which is not a polynomial with integer coefficients. However, we can remedy this by moving $2\sqrt{6}$ to the right hand side and squaring both sides again:

$$\begin{aligned} (x^2 - 5)^2 &= 24 \\ x^2 - 10x + 1 &= 0 \end{aligned}$$

Now the rational root theorem holds. Any rational solution to this polynomial must divide 1, so therefore our candidates are only $x = \pm 1$, but none of these solve the equation. Therefore, $\sqrt{2} + \sqrt{3}$ is not rational. $\square$

Problem 5

(a) Show $|b| \leq a$ if and only if $-a \leq b \leq a$.

Solution: First we prove that if $|b| \leq a$, then $-a \leq b \leq a$. In this case, we look at the definition of the absolute value:

$$|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

Therefore, if $|b| \leq a$, then we know that if $b > 0$, then $b \leq a$. Otherwise, if $b < 0$, then $-b \leq a \implies b \geq -a$, and so we're done.

Now for the reverse case. If $-a \leq b \leq a$, then we need to prove that $|b| \leq a$. Since $-a \leq b$, then this implies that the distance between 0 and $-a$ is longer than that from 0 and b. Likewise, the same conclusion can be drawn about the statement $b \leq a$ - the distance between 0 and b is less than the distance between 0 and a. Therefore, if we think about this as a distance, then it makes sense that the distance of b (denoted as $|b|$) will be less than the distance from 0 to a, denoted as $|a|$. It's implied $a > 0$ here (otherwise we can choose $-a$), so therefore we can remove the absolute value. Thus, $|b| \leq a$ follows. $\square$

(b) Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Solution: I couldn't solve this problem. $\square$

Problem 6

Given a nonempty set $A \subset \mathbb{R}$. Using the definition of supremum/infimum, show that

- $\sup A \geq \inf A$

  Solution: Suppose for the sake of contradiction that $\inf A > \sup A$. For the inf statement, it means that there exists an x such that for all $a \in A$, $x \leq a$. However, the $\sup A$ implies an existence of X such that for all $a \in A$, $X \geq a$. Since $x > X$, the elements in a must be both less than X and greater than x, but this is impossible since $x > X$. This is a contradiction. Therefore, $\sup A \geq \inf A$. $\square$

- If $\max A$ ($\min A$) exists, then $\sup A = \max A$ ($\inf A = \min A$)

  Solution: We know that $\max A$ is defined as the value of $a_M \in A$ such that for all other $a \in A$, $a \leq a_M$. Notice that this is the exact definition for the supremum: the smallest value X such that for all $a \in A$, $X \geq a$. Therefore, if $\max A$ exists, then $\sup A = \max A$.

  The same logic exists for the infimum. $\min A$ is defined as the value $a_m \in A$ such that for all other $a \in A$, $a_m \leq a$. This is the exact definition for the infimum, and so $\inf A = \min A$. $\square$

- $\inf A = -(\sup(-A))$, where $-A = \{-a | a \in A\}$

  Solution: Given a nonempty set A, we know via the completeness axiom (and its corollary) that $\sup A$ and $\inf A$ exist. We know that here, the $\inf A$ is defined as the value a_m such that $a_m \leq a$ for all $a \in A$.

  Now if we take the negative of both sides, we get $-a_m \geq a$. In other words, $-\inf A$ bounds the set from above! Therefore, we have the relation that $-\inf A = \sup(-A)$, which we can then rearrange this to become $\inf A = -\sup(-A)$, as desired. $\square$

Problem 7

Using the completeness axiom theorem to prove the theorem for strong induction:

Theorem 1. *Assume A is a subset of $\mathbb{N}$, if A satisfies the following two properties:*

(1) $1 \in A$

(2) If $\{1, 2, 3, \ldots, n\} = \{x | x \leq n, x \in N\} \subset A$*, then* $n + 1 \in A$

Then $A = \mathbb{N}$

Hint: Use proof by contradiction.

Solution: We prove that property (2) is always true given proposition (1). Firstly, we know that $1 \in A$ so $2 \in A$ as well. Now suppose that we now have a set $\{1, 2, \ldots, n\}$.

To prove that all the numbers from 1 to n exist within this set, we can take increasing set sizes: $\{1\}, \{1, 2\}$ and in every one of these sets, the completeness axiom says that the $\sup(A)$ exists, in other words using these sets we can show that the numbers $1, 2$ and eventually n also exists, implying the existence of $n + 1$. Thus, this process can repeated ad infinitum, implying that $A = \mathbb{N}$. $\square$