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Lecture Notes | Edward Frenkel

Abstract Linear Algebra

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1 Formal System of Vector Spaces

Arrived late to class, so what came before this is lost to history.

- A vector space over F (a field) is a set V equipped with 2 functions:
  - Addition: $V \times V \mapsto V$
  - Scalar multiplication: $F \times V \mapsto V$

1.1 Axioms of a Vector Space

- The axioms of vector space are as follows:
  - **Commutativity over addition:** $\forall u, v \in V, u + v = v + u$.
  - **Associativity under addition:** $\forall u, v, w \in V, (u + v) + w = u + (v + w)$.
  - **Associativity under Multiplication:** $(ab)v = a(bv)$.
  - **Additive Identity:** There exists a "zero element", such that $v + 0 = v$ for any arbitrary v.
  - **Additive Inverse:** $\forall v \in V, \exists w \in V$ such that $w + v = 0$.
  - **Multiplicative Identity:** There is an element 1 such that $1 \cdot v = v$
  - **Distributive properties:** $(a + b)v = av + bv$, and $a(u + v) = au + av$.

1.2 Theorems

Theorem 1.1 (Uniqueness of Additive Identity)**.** *Let V be a vector space over F. If $0 \in V$ and $0' \in V$ both satisfy Axiom 3, then $0 = 0'$.*

Proof. Our proof consists of a list of sentences:

S1) Use Axiom 3: $v + 0 = v, \ \forall v \in V$.

S2) Set $v = 0' : 0' + 0 = 0'$

S3) Use Axiom 1: $u + v = v + u$

S4) Replace $u = 0, v = 0' : 0' + 0 = 0 + 0'$

S5) Use Axiom 3, but for $0' : v + 0' = v, \ \forall v \in V$

S6) Substitute $v = 0$: $0 + 0' = 0$

S7) Combine S2 and S4: $0 + 0' = 0'$

S8) Combine S7 and S6: $0' = 0$.

□

Note that here, we're not proving that $0 = 0'$, but instead that under the assumption that they both satisfy Axiom 3, then $0 = 0'$. The statement "if" is actually the sentence that provides us the axiom, since it tells us that we're living in a world where that assumption holds true.

Theorem 1.2 (Uniqueness of Additive Inverse)**.** *Let v be a vector space over F and $v \in V$. If $w \in V$ and $w' \in V$ both satisfy axiom 4, then $w = w'$.*

Proof. Again, we use sentences, except we'll be a bit more concise this time:

S1) Use Axiom 3 for our specific $v : w + 0 = w$

S2) Substitute $v + w'$ for 0, since we know that w' satisfies Ax. 4: $w + (v + w') = w + 0$

S3) Associativity: $(w + v) + w' = w + 0 = w'$

S4) Hence, $w + v = 0$, so $w = w'$.

$\square$