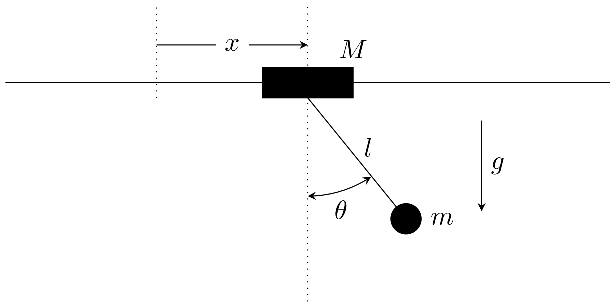
Eric Du | Physics 105
HW 03 | Analytic Mechanics | February 17, 2023

Collaborators

I worked with **Aren Martinian, Andrew Binder** and **Adarsh Iyer** to complete this assignment.

Problem 1

A bob of mass m is suspended by a massless rigid rod of length l that is hinged to a sled of mass M. THe sled slides without friction on a horizontal rail, as shown in the figure.



(a) Write down the Lagrangian for the system and derive the equations of motion.

Solution: We can first express the coordinates of the mass bob in cartesian coordinates:

$$X = x + l\sin\theta \; Y = -l\cos\theta$$

so this leads to the equations:

$$\dot{X} = \dot{x} + l\dot{\theta}\cos\theta \; \dot{Y} = l\dot{\theta}\sin\theta$$

Therefore, the kinetic energy term is (skipping the intermediate algebra):

$$\begin{aligned} T &= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}M\dot{x}^2 \\ &= \frac{\dot{x}^2}{2}(m+M) + \frac{m}{2}\left(2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) \end{aligned}$$

Now, let the sled have zero potential energy. Therefore:

$$U = mgl(1-\cos\theta)$$

And so therefore, the Lagrangian is:

$$\mathcal{L} = T - U = \frac{\dot{x}^2}{2}(m+M) + \frac{m}{2}\left(2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) - mgl(1-\cos\theta)$$

With this now, we recognize that x and θ are our coordinates, so there are two Euler-Lagrange equations.

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} = 0 &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}(m+M) + ml\dot{\theta}\cos\theta\right) \\ 0 &= \ddot{x}(m+M) + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \end{aligned}$$

and likewise for the θ direction:

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 = -m\dot{x}l\dot{\theta}\sin\theta - mgl\dot{\theta}\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}l\cos\theta - ml^2\dot{\theta}\right)$$
$$0 = ml^2\ddot{\theta} + m\ddot{x}l\cos\theta + mgl\sin\theta$$

□

(b) Suppose, for all $t < 0$, the masses are at rest with $\theta = 0$. Then, at $t = 0$, an impulse $\Delta P = F\Delta t$ is applied to the bob over a small timespan Δt from a sharp horizontal tap. Find $\dot{x}$ and $\dot{\theta}$ immediately after the tap. (Hint: Consider both linear and angular momentum)

Solution: Based on the principles of linear and angular momentum, we know that

$$\Delta p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad\qquad l\Delta p = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$

And so therefore we get the following system of equations:

$$\Delta p = \dot{x}(m+M) + ml\dot{\theta}\cos\theta,$$
$$l\Delta p = m\dot{x}l\cos\theta + ml^2\dot{\theta}$$

Multiplying the top equation by l and subtracting the second, we get:

$$\dot{x}l(m+M) + ml^2\dot{\theta} = m\dot{x}l + ml^2\dot{\theta}$$
$$Ml\dot{x} = 0$$
$$\therefore \dot{x} = 0$$

Then, since $\dot{x} = 0$, then we also have $\Delta p = ml\dot{\theta}$ so $\dot{\theta} = \dfrac{\Delta p}{ml}$. □

(c) Suppose the impulse ΔP in (b) is also small, so that the θ stays small for all t. Use the small angle approximation to simplify your equations of motion from (a), and sovle for $x(t)$ and $\theta(t)$ in this approximation.

Solution: To simplify our equations of motion, we use the simplificaitons that $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. Doing so, we get the equations:

$$\ddot{x}(m+M) + ml\ddot{\theta} - ml\dot{\theta}^2\theta = 0$$
$$ml^2\ddot{\theta} + m\ddot{x}l + mgl\theta = 0$$

From the second equation, we get that $\ddot{x} = -g\theta - l\ddot{\theta}$, which we can then plug into the first equation:

$$(-g\theta - l\ddot{\theta})(m+M) + ml\ddot{\theta} = 0$$
$$Ml\ddot{\theta} + g(m+M) = 0$$
$$\therefore \ddot{\theta} + \frac{g(m+M)}{Ml}\theta = 0$$

This equation just simple harmonic motion, with $\omega^2 = \frac{g(m+M)}{Ml}$, so we can conclude:

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t)$$

And since we know that $\theta(0) = 0$, then we choose $\theta(t) = B\sin(\omega t)$ to simplify our calculations. We can solve for B by finding $\dot{\theta}(0) = \frac{\Delta P}{ml}$, so we get:

$$\dot{\theta}(0) = B\omega\cos(\omega t) = \frac{\Delta P}{ml}$$
$$\therefore B = \frac{\Delta P}{ml\omega}$$

Now we can proceed to find $x(t)$. To do so, we first need to find $\ddot{x}(t)$, which requires $\ddot{\theta}(t)$. So taking two time derivatives, we get:

$$\ddot{\theta}(t) = -\frac{\Delta P}{ml\omega}\omega^2\sin(\omega t)$$

And so now we can plug:

$$0 = \ddot{x}(m+M) + ml \cdot -\frac{\Delta P}{ml}\omega\sin(\omega t)$$
$$\ddot{x} = \frac{\Delta P\omega}{m+M}\sin(\omega t)$$

which we can now integrate:

$$\dot{x} = -\frac{\Delta P\omega}{m+M}\frac{1}{\omega}\cos(\omega t) + C_1$$

C_1 can be determined by using the condition that $\dot{x}(0) = 0$ from part (b), so therefore:

$$C_1 = \frac{\Delta P}{m+M}$$

Now we can integrate again to get $x(t)$:

$$x(t) = -\frac{\Delta P}{\omega(m+M)}\sin(\omega t) + \frac{\Delta P t}{m+M} + C_2$$

Then we can set $x(0) = x_0$, some arbitrary position, which gives:

$$C_2 = x_0$$

And so therefore our full equation for $x(t)$ is:

$$x(t) = -\frac{\Delta P}{\omega(m+M)}\sin(\omega t) + \frac{\Delta P t}{m+M} + x_0$$

$\square$

Problem 2

Consider a particle moving in three dimensions, described by the Lagrangian

$$L = \frac{1}{2}m|\dot{r}|^2 - V(r)$$

Using Cartesian coordinates, where $r = (x, y, z)$ we can rewrite this as

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

It's often useful to use a different choice of coordinates, such as cylindrical coordinates or spherical coordinates. For example, in problems where there's symmetry, we can more easily deal with constraints in a symmetry-appropriate coordinate system. For this problem, re-express the kinetic term of the Lagrangian in

(a) Cylinderical coordinates (ρ, ϕ, z)

Solution: In cylindrical coordinates, we have the equations

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}$$

and so therefore our kinetic term is

$$\begin{aligned} T &= \frac{m}{2}\left[\left(\dot{\rho}\cos\phi - \rho\sin\phi\dot{\phi}\right)^2 + \left(\dot{\rho}\sin\phi + \rho\cos\phi\dot{\phi}\right)^2 + \dot{z}^2\right] \\ &= \frac{m}{2}\left[\dot{\rho}^2\cos^2\phi - 2\dot{\rho}\cos\phi\rho\sin\phi\dot{\phi} + \rho^2\sin^2\phi\dot{\phi}^2 + \dot{\rho}\sin^2\phi + 2\dot{\rho}\sin\phi\rho\cos\phi\dot{\phi} + \rho^2\cos^2\phi\dot{\phi}^2 + \dot{z}^2\right] \\ &= \frac{m}{2}\left[\dot{\rho}^2(\cos^2\phi + \sin^2\phi) + \rho\dot{\phi}^2\left(\cos^2\phi + \sin^2\phi\right) + \dot{z}^2\right] \\ &= \frac{m}{2}\left[\dot{\rho}^2 + \rho\dot{\phi}^2 + \dot{z}^2\right] \end{aligned}$$

$\square$

(b) spherical coordiantes (r, ϕ, θ)

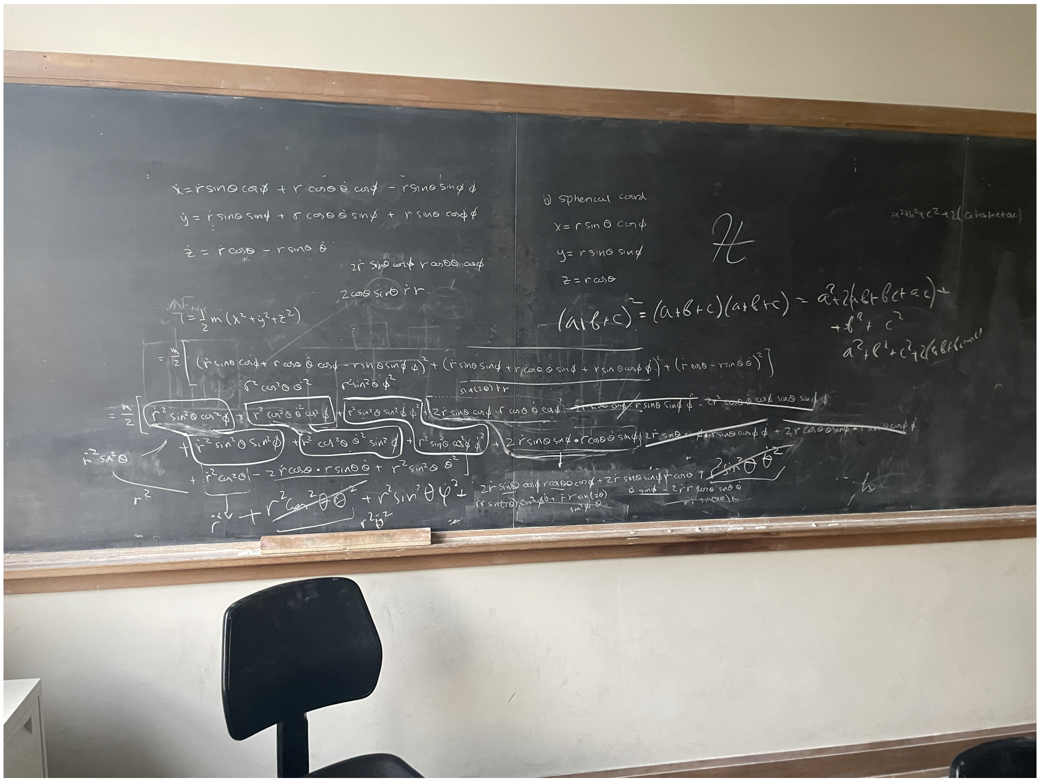
Solution: In spherical coordinates, we have:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

So our process to do this is to find $\dot{x}$, $\dot{y}$ and $\dot{z}$ then take

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

This process involves doing triple product rule and gets us many many terms. There are too many terms in this expansion so I won't write it out, but in replacement here's an image of our work on the blackboard:



In any case, we get many cancellations and nice combinations of terms, all of them using the nice identity that $\sin^2\phi + \cos^2\phi = 1$, which eventually (albeit after a lot of struggle) gets us:

$$T = \frac{m}{2}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$$

$\square$

Problem 3

Consider a ring of wire of radius R, mounted vertically as in the figure below. A frictionless bead of mass m is threaded on this wire. The ring is forced to rotate around the indicated axis at constant angular velocity Ω. The position of the bead is specified by θ. Gravity acts downwards and has magnitude g.

(a) Write down the Lagrangian (Hint: use the result of the previous problem)

Solution: We have symmetry along the $\hat{z}$ axis, so we will choose cylindrical coordinates. The kinetic energy is:

$$T = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$

Note here that in our problem, we have $\dot{\phi} = \Omega$, so therefore we have:

$$T = \frac{m}{2}\left[(R\dot{\theta}\cos\theta)^2 + (R\sin\theta)^2\Omega^2 + R^2\dot{\theta}^2\sin^2\theta\right] = \frac{m}{2}R^2\left[\dot{\theta}^2 + \sin^2\theta\Omega^2\right]$$

The potential energy term is just $U = mgz = mgR(1-\cos\theta)$ so therefore:

$$\mathcal{L} = \frac{m}{2}R^2\left[\dot{\theta}^2 + \sin^2\theta\Omega^2\right] - mgR(1-\cos\theta)$$

□

(b) Derive the equation of motion

Solution: θ is the only coordinate here, so we derive the equation of motion:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 0 &= mR^2\Omega^2\sin\theta\cos\theta - mgR\sin\theta - mR^2\ddot{\theta} \\
R\ddot{\theta} &= R\Omega^2\sin\theta\cos\theta - g\sin\theta \\
\ddot{\theta} &= \Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta
\end{aligned}$$

□

(c) Use your equation from (c) to determine which angles are equilibria. For what values of Ω are these stable? (Hint 1: do all of your equilibria exist for all values of Ω? Hint 2: to investigate stability, write $\theta = \theta_{eq} + \epsilon$, where ϵ is a small deviation from equilibrium, and find the equation of motion for ϵ.)

Solution: A point is only at equilibrium if $\ddot{\theta} = 0$. Therefore, from our equations of motion, we can derive:

$$\Omega^2\sin\theta\cos\theta = \frac{g}{R}\sin\theta$$

So we can get:

$$\Omega^2\cos\theta = \frac{g}{R}$$

In order for this equation to have a real solution for θ, we require that

$$\frac{g}{R\Omega^2} \leq 1$$

or equivalently:

$$\Omega \geq \sqrt{\frac{g}{R}}$$

This will admit solutions $0 \leq \theta < \pi/2$. The upper bound is not an equality since $\theta = \pi/2$ is clearly not a stable equilibrium. Now we ask, what happens when $\Omega < \sqrt{\frac{g}{R}}$? In this case, we need to look back at our original equation:

$$\Omega^2 \sin\theta\cos\theta = \frac{g}{R}\sin\theta$$

If $\Omega^2 < \frac{g}{R}$ in this equation, then we'd require that $\cos\theta > 1$, which is impossible. However, $\theta = 0$ is a solution here, since $\sin\theta = 0$. Therefore, we have the solutions:

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{g}{R\Omega^2}\right) & \left(\Omega^2 \geq \dfrac{g}{R}\right) \\ \\ 0 & \left(\Omega^2 < \dfrac{g}{R}\right) \end{cases}$$

Now we look for stability. To do so, we consider the hint: for a given θ_{eq}, we consider $\theta = \theta_{eq} + \epsilon$. Doing so, we get:

$$\ddot{\theta}_{eq} + \ddot{\epsilon} = \Omega^2 \sin(\theta_{eq} + \epsilon)\cos(\theta_{eq} + \epsilon) - \frac{g}{R}\sin(\theta_{eq} + \epsilon)$$

Since this is an equilibrium point, we have $\ddot{\theta}_{eq} = 0$. We then use the angle summation formulas and combine them with the fact that ϵ is small to get:

$$\begin{aligned}\ddot{\epsilon} &= \Omega^2(\sin\theta_{eq} + \epsilon\cos\theta_{eq})(\cos\theta_{eq} - \epsilon\sin\theta_{eq}) - \frac{g}{R}(\sin\theta_{eq} + \epsilon\cos\theta_{eq}) \\ &= \Omega^2(\sin\theta_{eq}\cos\theta_{eq} - \epsilon\sin^2\theta_{eq} + \epsilon\cos^2\theta_{eq} - \epsilon^2\sin\theta_{eq}\cos\theta_{eq}) - \frac{g}{R}(\sin\theta_{eq} - \epsilon\cos\theta_{eq}) \\ &= \Omega^2\sin\theta_{eq}\cos\theta_{eq} + \Omega^2\epsilon\cos(2\theta_{eq}) - \epsilon^2\sin\theta_{eq}\cos\theta_{eq} - \frac{g}{R}\sin\theta_{eq} - \epsilon\frac{g}{R}\cos\theta_{eq}\end{aligned}$$

First, we neglect the ϵ^2 term since ϵ is small. Then, we have $\Omega^2\sin\theta_{eq}\cos\theta_{eq} - \frac{g}{R}\sin\theta_{eq} = 0$ from the equilibrium condition so therefore our equation simplifies to:

$$\ddot{\epsilon} = \Omega^2\epsilon\cos(2\theta_{eq}) - \epsilon\frac{g}{R}\cos\theta_{eq} = -\epsilon\left(-\Omega^2\cos(2\theta_{eq}) + \frac{g}{R}\cos\theta_{eq}\right)$$

And so we can get the equation:

$$\ddot{\epsilon} + \left(\frac{g}{R}\cos\theta_{eq} - \Omega^2\cos(2\theta_{eq})\right)\epsilon = 0$$

The term in the parentheses here is constant, so this is just the equation for simple harmonic motion. As this is the case, this means that ϵ is a stable equilibrium. In fact, we can calculate the angular frequency of this oscillation about θ_{eq}:

$$\omega = \sqrt{\frac{g}{R}\cos\theta_{eq} - \Omega^2\cos\theta_{eq}}$$

This also shows that $\theta = \pi/2$ is not a stable point, since substituting $\theta_{eq} = \pi/2$ gives $\omega = 0$, which we can interpret as having no simple harmonic motion - in other words, the bead doesn't oscillate, it just falls. $\square$

Problem 4

The Lagrangian for a relativistic point particle moving in one dimension is

$$L = -mc^2\sqrt{1 - \dot{x}^2/c^2} - V(x)$$

where c is the speed of light. Derive the equation of motion and show that it reduces to Newton's equation in the limit $\dot{x} \ll c$.

Solution: We can just compute the equation of motion:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 &= -\frac{\partial V}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{mc^2}{2\sqrt{1 - \dot{x}^2/c^2}} \cdot -\frac{2\dot{x}}{c^2}\right) \\
&= -\frac{\partial V}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{x}}{\sqrt{1 - \dot{x}^2\ddot{x}/c^2}}\right) \\
&= -\frac{\partial V}{\partial x} - \left(\frac{m\ddot{x}\gamma - m\dot{x}\frac{1}{2\gamma} \cdot \frac{2\dot{x}}{c^2}}{\gamma^2}\right) \\
&= -\frac{\partial V}{\partial x} - m\ddot{x}\left[\frac{1 - \frac{\dot{x}^2}{\gamma^2 c^2}}{\gamma}\right]
\end{aligned}$$

In the limit where $\dot{x} \ll c$, then we have $\dot{x}^2/c^2 \to 0$ and $\gamma \to 1$, so therefore we get:

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

which is exactly Newton's equation. $\square$