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HW 07

Quantum Mecahnics II

Physics 137B
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Collaborators

I worked with **Andrew Binder** to complete this assignment.

Problem 1

A particle of mass m is initially in the ground state of the (one-dimensional) infinite square well. At time $t = 0$ a "brick" is dropped into the well, so that the potential becomes

$$V(x) = \begin{cases} V_0 & 0 \le x \le a/2 \\ 0 & a/2 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

where $V_0 \ll E_1$. After a time T, the brick is removed, and the energy of the particle is measured. Find the probability (in first-order perturbation theory) that the result is now E_2.

Solution: Our job here is to calculate $|c_2(t)|^2$ for this perturbation. To do so, we use the formula for a constant perturbation (Griffiths 11.120):

$$|c_2(t)|^2 = 4|H'_{12}|^2 \frac{\sin^2\left[(E_1 - E_2)T/2\hbar\right]}{(E_1 - E_2)^2}$$

We have $E_1 - E_2 = \frac{-3\pi^2\hbar^2}{2ma^2}$, so all we need to do now is calculate H'_{12}. Note that the perturbation happens only on $0 \le x \le a/2$, so we actually only need to integrate over that region (more specifically, the integral will evaluate to 0 in the region $a/2 \le x \le a$).

$$\begin{aligned} H'_{12} &= \frac{2V_0}{a} \int_0^{a/2} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) dx \\ &= \frac{2V_0}{a} \cdot \frac{2a}{3\pi} \\ &= \frac{4V_0}{3\pi} \end{aligned}$$

Putting this all together, we get:

$$|c_2(t)|^2 = 4 \cdot \left(\frac{4V_0}{3\pi}\right)^2 \sin^2\left(-\frac{3\pi^2\hbar T}{4ma^2}\right)\left(\frac{2ma^2}{3\pi^2\hbar^2}\right)^2$$

$\square$

Problem 2

A harmonic oscillator of mass m, charge $e$and classical frequency ω is in its ground state.

a) A uniform electric field $\mathscr{B}$ is turned on at $t = 0$ and is then turned off at $t = \tau$. Use first-order time dependent perturbation theory to estimate the probability that the system is excited to the n-th state.

   Solution: Here, we use equation 11.120, since the perturbation is constant:

$$P_{N \to M} = 4|H'_{MN}|^2 \frac{\sin^2\left[(E_N - E_M)T/2\hbar\right]}{(E_N - E_M)^2}$$

   Now, we have $N = 0$ since the particle is in the ground state. Further, calculating H'_{MN}:

$$\begin{aligned} H'_{MN} &= \langle M|H'|0\rangle \\ &= \langle M|eEx|0\rangle \\ &= eE\,\langle M|\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)|0\rangle \\ &= eE\sqrt{\frac{\hbar}{2m\omega}}\,\langle M|1\rangle \end{aligned}$$

   From this calculation, we actually see that $M = 1$ is the only state where probability can flow. Since $E_0 - E_1 = -\hbar\omega$, we have:

$$P_{0\to 1} = |c_1(t)|^2 = 4\frac{e^2E^2\hbar}{2m\omega}\frac{\sin^2(\omega T/2)}{\hbar^2\omega^2} = \frac{2e^2E^2}{m\omega}\frac{\sin^2(\omega T/2)}{\hbar\omega^2}$$

$\square$

Problem 3

Suppose that an electron in a one-dimensional harmonic-oscillator potential $\frac{1}{2}m\omega_0 x^2$ is subjected to an oscillating electric field $\mathscr{E} = \mathscr{E}(0)\cos\omega t$ in the x direction.

a) If the electron is initially in the ground state, what is the probability that the electron will be the n-th excited state at time t?

Solution: For a sinusoidal electric field and a particle originating in the ground state, we have the equation:

$$\begin{aligned}|c_m(t)|^2 &= |H'_{m0}|^2 \frac{\sin^2\left[(E_m - E_0 - \hbar\omega)t/2\hbar\right]}{(E_m - E_0 - \hbar\omega)^2} \\ &= |H'_{m0}|^2 \frac{\sin^2((m\omega_0 - \omega)t/2)}{(m\hbar\omega_0 - \hbar\omega)^2}\end{aligned}$$

Again just like the previous problem, only the first state ($m = 1$)is affected since the electric field is $H' = eEx$. Therefore, we can calculate:

$$|c_1(t)|^2 = \frac{e^2\mathscr{E}(0)\hbar}{2m\omega_0} \frac{\sin^2\left[(\omega_0 - \omega)t/2\right]}{(\omega_0 - \omega)^2}$$

□

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b) If $\omega = \omega_0$, perturbation theory will fail at some time t. What is the critical time?

Solution: My interpretation of the "critical time" is the time at which perturbation theory fails. We can rewrite $|c_1(t)|^2$ in terms of the sinc function, which is nicer since it doesn't blow up, so we can focus on t:

$$\begin{aligned}|c_1(t)|^2 &= \frac{e^2\mathscr{E}(0)^2\hbar}{2m\omega_0} \frac{\sin^2\left[(\omega_0 - \omega)t/2\right]}{(\omega_0 - \omega)^2} \\ &= \frac{e^2\mathscr{E}(0)^2\hbar}{2m\omega_0} \left[\frac{\sin\left((\omega_0 - \omega)t/2\right)}{(\omega_0 - \omega)t/2}\right]^2 \left(\frac{t}{2}\right)^2 \\ &= \frac{e^2\mathscr{E}(0)^2\hbar t^2}{8m\omega_0} \operatorname{sinc}^2\left[(\omega_0 - \omega)t/2\right]\end{aligned}$$

This way, when $\omega_0 = \omega$, the sinc function goes to 1. The critical time is when the probability exceeds 1, so calculating the time that this occurs:

$$\begin{aligned}\frac{e^2\mathscr{E}(0)^2\hbar t^2}{8m\omega_0} &> 1 \\ \therefore t &> \frac{2}{e\mathscr{E}(0)}\sqrt{\frac{2m\omega_0}{\hbar}}\end{aligned}$$

Therefore, the critical time is:

$$t = \frac{2}{e\mathscr{E}(0)}\sqrt{\frac{2m\omega_0}{\hbar}}$$

□

Problem 4

At $t < 0$, an electron is assumed to be in the $n = 3$ eigenstate of an infinite square potential well, which extends from $-a/2 < x < a/2$. At $t = 0$, an electric field is applied, with the potential $V = Ex$. The electric field is then removed at time τ. Determine the probability that the electron is in any other state at $t > \tau$.

Solution: Here, we consider two cases: stimulated absorption and stimulated emission. For absorption, we have the equation:

$$|c_m(t)|^2 = |H'_{m3}|^2 \frac{\sin^2\left(\frac{\pi^2\hbar^2}{2ma^2}(m^2-9)\tau/2\hbar\right)}{\left(\frac{\pi^2\hbar^2}{2ma^2}(m^2-9)\right)^2}$$

So now we need to calculate H'_{m3}. Recall that the eigenstates for an infinite square well are:

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right) & m \text{ odd} \\ \sqrt{\frac{2}{a}}\cos\left(\frac{m\pi x}{a}\right) & \text{m even} \end{cases}$$

Therefore, for odd m, we have the integral:

$$H'_{m3} = \frac{2E}{a}\int_{-a/2}^{a/2} \sin\left(\frac{m\pi x}{a}\right) x \sin\left(\frac{3\pi x}{a}\right) dx$$

And notice that we are taking an integral of an odd function over an even interval, this integral evaluates to 0 for odd m. For even m, we have (using WolframAlpha):

$$H'_{m3} = \frac{2E}{a}\int_{-a/2}^{a/2} \cos\left(\frac{m\pi x}{a}\right) x \sin\left(\frac{3\pi x}{a}\right) dx = -\frac{a^2(\pi m(m^2-9)\sin\left(\pi\frac{m}{2}\right) + 2(m^2+9)\cos\left(\pi\frac{m}{2}\right))}{\pi^2(m^2-9)^2}$$

Since m is even, we can simplify this down a bit. First, $\sin\left(\pi\frac{m}{2}\right) = 0$, and $\cos\left(\pi\frac{m}{2}\right) = (-1)^{m/2}$, so we have:

$$H'_{m3} = -\frac{a^2(2(m^2+9)(-1)^{m/2})}{\pi^2(m^2-9)^2}$$

So:

$$|H'_{m3}|^2 = \frac{4a^4(m^2+9)^2}{\pi^4(m^2-9)^4}$$

Therefore, for stimulated absorption, we have the following result:

$$|c_m(t)|^2 = \begin{cases} 0 & m \text{ odd} \\ \frac{4a^4(m^2+9)^2}{\pi^4(m^2-9)^4} \cdot \frac{\sin^2\left[\frac{\pi^2\hbar^2}{2ma^2}(m^2-9)\tau/2\hbar\right]}{\left(\frac{\pi^2\hbar^2}{2ma^2}(m^2-9)\right)^2} & m \text{ even} \end{cases}$$

For stimulated absorption, the integrals are the same so we obtain the same result, but notice that there is only one even number between 1 and 3 (that being 2), so therefore we can get an exact result instead. Therefore:

$$\begin{aligned} H'_{23} &= \frac{2E}{a}\int_{-a/2}^{a/2} \cos\left(\frac{2\pi x}{a}\right) x \sin\left(\frac{3\pi x}{a}\right) dx \\ &= \frac{52Ea}{25\pi^2} \end{aligned}$$

which completes the solution $\square$

Problem 5

Justify the following version of the energy-time uncertainty principle (due to Landau): $\Delta E \Delta t \geq \hbar/2$, where Δt is the time it takes to execute a transition involving energy change ΔE, under the influence of a constant perturbation. Explain more precisely what ΔE and Δt mean in this context. (Δt is the time it takes for $P(t)$ to reach a peak in its oscillation.)

Solution: The equation for $P(t)$ is:

$$P(t) = 4|H'_{MN}|^2 \frac{\sin^2 [\Delta E \Delta t / 2\hbar]}{\Delta E^2}$$

So we are looking for maxima in the $\sin^2$ function. We know that the peaks of this function occur at $(2n-1)\pi/2$, so therefore:

$$\frac{\Delta E \Delta t}{2\hbar} = \frac{\pi}{2}(2n-1)$$

or equivalently,

$$\begin{aligned} \frac{\Delta E \Delta t}{2\hbar} &\geq \frac{\pi}{2} \\ \Delta E \Delta t &\geq \pi\hbar \\ \therefore \Delta E \Delta t &\geq \frac{h}{2} \end{aligned}$$

Here, Δt represents the time interval between the peaks in the probability distribution of $|c_m(t)|^2$ over time, and ΔE represents the energy difference between the two states in question.

This is as far as I could get with this problem. I'm not sure where the other factor of 2π comes from to bring the inequality down to $\frac{\hbar}{2}$. With this relation, the best I can do is:

$$\Delta E \Delta t > \frac{\hbar}{2}$$

but I cannot prove that an equality condition can exist. □