Eric Du		EE 120
Lecture Notes	Signals and Systems	Chunlei Liu

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1 Lecture 1

1.1 Motivations

- Why study this class?
  - Given a "black box" circuit, with input and output leads, we can determine what's within the "black box".
  - In this particular case, if our black box contains a voltage divider, and the output voltage is given by the equation:

    $$v_{\text{out}}(t) = \frac{R_2}{R_1 + R_2} v_{\text{in}}(t)$$

    In principle though, the signal can be anything that we want: for facial recognition software, the input signal could be the configuration of the intensity the camera picks up. There's many more we went over, don't really want to write it all down.
- In essence, there's a lot of systems that can be modeled by a system that takes in a signal $x(t)$, and outputs a signal $y(t) = f(x(t))$.
  - The signals are usually functions of time, location, in any number of dimensions.
  - The systems does some sort of transformation on an input signal. In particular, we will study linear systems, shift-invariant systems, etc.

    We'll talk about mathematical operations that we use to perform these transformations: Fourier, Laplace, Z-transformations, convolutions, correlation, etc.
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1.2 Types of Signals

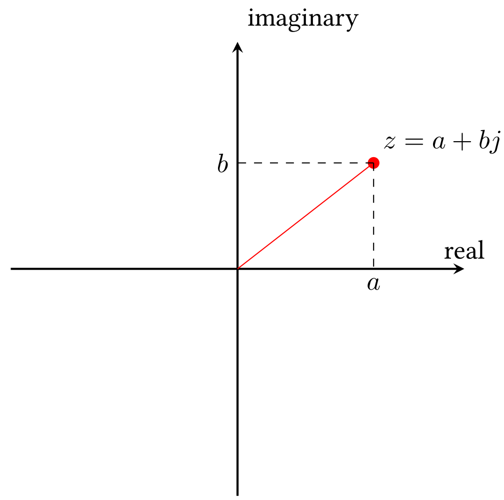
- **Continuous-time:** signals defined over continuous variables (e.g. position, time). For instance, a signal $x(t)$ is continuous for our purposes, since time is a continuous variable.

  Further, because t is continuous, then x must also be continuous. If the signal is differentiable, then the derivative $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ also exists.

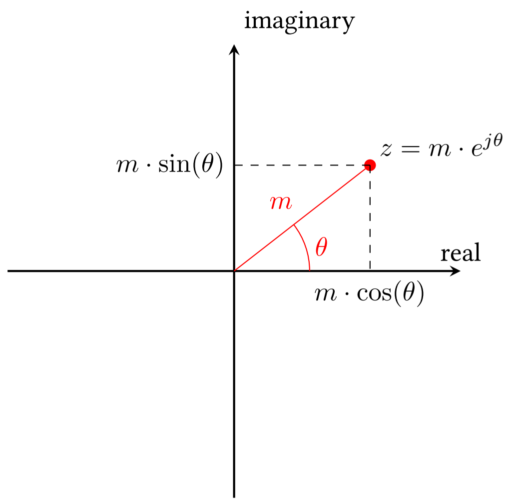
  t being continuous does not imply that $x(t)$ is continuous (e.g. Thomae function), but is it true for this class?
- **Discrete-Time:** These are signals defined over discrete variables. For instance, if we had $x[n]$ as a signal, where n is an integer.

  We don't have a concept of differentiability, but we can compute the difference: $x[n] - x[n-1]$, and talk about that quantity.
- **Real-Valued:** A signal $x(t)$ is real-valued if $x(t) \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of all real numbers.
- **Complex-Valued:** A signal $x(t)$ is complex-valued if $x(t) \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers.
- Note that while we're using the continuous-time notation here, the same concepts apply with discrete-time signals.
  - Quick recap on complex numbers: denoted by $a + bj$ or $a + bi$, where i and j denote the imaginary unit.
  - They are defined as $i^2 = -1$ or $j^2 = -1$.
  - a is the real part, and b is the imaginary part.

- We can plot these values in the complex plane, using the real and imaginary representation:



Or using the magnitude-phase representation:



We represent the magnitude as $m = |z|$, and the phase angle θ is the angle made with the real axis.

- **Periodic Signal:** Two quantities we'll introduce here: the period T is the time it takes for the signal to repeat itself. T is measured in units of time, generally seconds.

  The frequency f is the "inverse" of period, defined by $f = \frac{1}{T}$. We will also use the angular frequency ω, defined by $\omega = \frac{2\pi}{T} = 2\pi f$. Angular frequency is mainly going to be used when we involve complex numbers. We will see:

$$e^{j\omega t} = e^{j(2\pi f t)} = \cos(2\pi f t) + i \sin(2\pi f t)$$

- **Dimensionality:** We will deal with multi-dimensional signals: an example of a 2D signal are images, which determine the color of a pixel based on a row and column. The spaces that we'll be working with are either $\mathbb{R}^n$ or $\mathbb{C}^n$.

1.3 Signal Transformations

- **Shifts:** Essentially just shifts the signal along one dimension: $x(t) \to x(t - T)$. T is some constant. If $T > 0$, then the shift is to the *right*, and if $T < 0$ then the shift is to the *left*.
- **Scaling:** We can multiply a signal $x(t)$ by some constant $a : x(t) \to a \cdot x(t)$. If $a < 1$, then we shrink $x(t)$, and if $a > 1$ then we amplify the signal.
- **Reversal:** Given $x(t)$, we can "reverse time" by adding a negative to the argument: $x(t) \to x(-t)$. Visually, all we do is flip the signal around the y-axis.

1.4 Signal Properties

- **Even:** Functions which satisfy $x(t) = x(-t)$. In other words, if we perform a reversal, the signal stays the same.
- **Odd:** Functions which satisfy $x(t) = -x(-t)$. If we perform a reversal, the signal becomes the negative of itself.
- **Periodic:** If T is the period, then nT is also a period for any $n \in \mathbb{Z}$. However, we will call T the fundamental period; the smallest T for which the function repeats.

  For the function $\sin(2\pi f t)$, the fundamental period is $1/f$.

1.5 Model Functions

- These are called model functions because they're idealized models to analyze.
- **Heaviside Step function:** For the continuous-time case it's usually modeled by:

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

  In the discrete-time case, it's written as:

$$u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

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2.1 Model Functions Continued

- **Ramp Function:** The continuous-time is expressed as:

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

  Similarly in discrete time:

$$\text{ramp}[n] = \begin{cases} 0 & \text{for } n < 0 \\ n & \text{for } n \geq 0 \end{cases}$$

  Note that we can express the ramp function in terms of the step function, in many ways:

  - $r(t) = t \cdot u(t)$
  - $r(t) = \int_{-\infty}^{t} u(t)\, dt$, the discrete case is just a sum over the same bound.

- **Rectangular Function:** In continuous-time:

$$\text{rect}(t) = \sqcap(t) = \begin{cases} 1 & \text{for } |t| \leq 1/2 \\ 0 & \text{else} \end{cases}$$

  In discrete time:

$$\text{rect}\left[\frac{n}{N}\right] = \begin{cases} 1 & \text{for } |n| \leq N \\ 0 & \text{for } |n| > N \end{cases}$$

  We can also express $\text{rect}(t)$ in terms of $u(t)$:

$$\sqcap(t) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

- **Triangle Function:** In continuous-time:

$$\Lambda(t) = \begin{cases} 1 - |t| & \text{for } |t| \le 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

  And in discrete-time:

$$\Lambda\left[\frac{n}{N}\right] = \begin{cases} 1 - \left|\frac{n}{N}\right| & \text{for } |n| \le N \\ 0 & \text{else} \end{cases}$$

- **Delta Function:** In continuous time, it's called the Dirac delta function. It has the property that $\delta(t) = 0$ for all $t \neq 0$, but

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

  So in essence, this is an infinitesimally "thin" function, that extends to infinity. There are also other ways to represent the Delta function:

  - Derivative of the Heaviside step function: $\delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$
  - The integral of a complex exponential:

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft}\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

    The delta function allows us to approximate the integral $\int_{-\infty}^{\infty} \cos(\omega t)\, dt$. We can do the following:

$$\begin{aligned} \int_{-\infty}^{\infty} \cos(\omega t)\, dt &= \mathrm{Re}\left[\int_{-\infty}^{\infty} \cos(\omega t) + i\sin(\omega t)\right]\, dt \\ &= \mathrm{Re}\left[\int_{-\infty}^{\infty} e^{i\omega t}\, dt\right] \\ &= \mathrm{Re}\left[2\pi\delta(\omega)\right] \end{aligned}$$

    Looking at the delta function, we know that when $\omega = 0$, then $\cos(\omega t) = 1$, so the integral diverges, as expected. When $\omega \neq 0$, our integral result implies that the integral evaluates to 0. This is not exactly true since the integral will oscillate between ± 1, which is relatively small compared to $\omega = 0$, so it can effectively be taken as 0.

  How does this compare with the definition we use in physics that $\delta(t)$ is defined as the function which satisfies:

$$\int_{-\infty}^{\infty} f(t)\delta(t) = f(0)$$

  Do both work?

  See below bullet point, the definition allows you to derive this property.

- Let's explore some properties of the Delta function:
  - **Scaling:**

$$\int_{-\infty}^{\infty} \delta(\alpha t) dt = \int_{-\infty}^{\infty} \delta(t)\frac{\mathrm{d}\tau}{\mathrm{d}\alpha} = \frac{1}{|\alpha|}$$

    In other words, $\delta(\alpha t) = \frac{\delta(t)}{|\alpha|}$
  - **Sifting:** If we have $f(t)$ and multiply it by a Delta function:

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\, dt = f(T)$$

  - **Delta function of a function:** We can take the delta function of a function as well:

$$\delta(f(t)) = \frac{\delta(t - t_0)}{|f'(t_0)|},\ f(t_0) = 0$$

    We take the derivative in the denominator.

- In discrete time, the delta function is represented as the Kronecker delta:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

The function attempts to model the Dirac delta but for discrete time intervals:

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-10] = x[10]$$

- **Shah function:** It's basically a bunch of Dirac deltas:

$$\text{III}(t) = \sum_{k=0}^{\infty} \delta(t-k)$$

In discrete time, it also is a sum of all deltas:

$$\text{III}[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

2.2 System Characterization

- Systems perform operations on input signals, like functions $F : x \to y$. For instance, the following is a moving average filter:

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$

- We will be particularly interested in **linear systems**, systems which satisfy the following two properties:
  - **Scaling:** If for any input-output pair $x(t) \to y(t)$, then for any constant a, $ax(t) \to ay(t)$
  - **Addition:** Given any two input-output pairs

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$

    Then it's also true that $x_1(t) + x_2(t) \to y_1(t) + y_2(t)$

  Combining these two properties, given two general signals $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, then $ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$.

  Note that a function like $y(t) = x(t) + b$ is not a linear function, becuase it doesn't satisfy the second property. Even though the function is linear, doesn't mean that the transformation is linear.

- **Shift Invariant:** A shift-invariant system is a system where if we shift the input, the output is also shifted. Given $x(t)$ and its output $y(t)$, then $x(t-T)$ should produce $y(t-T)$ for any T.

- **Memoryless:** A function whose output at any given time only depends on the input at that time. For instance, machine learning algorithms are not memoryless, since their output depends on previous inputs.

  Does "current" here refer to a given input, or does it refer to past inputs? For instance, is the moving average function considered memoryless?

  Most systems that take time to react are not considered memoryless, since

- **Causality:** A system is causal if the output depends on the input at the present and past times only, not on future times. A system defined by:

$$y[n] = \frac{1}{3}(x[n] + x[n+1])$$

  is not considered causal, because $y[n]$ depends on the $n+1$-th input.

- **Stability:** There are many different ways to define stability, here are some of them:
  - A system is called BIBO stable if bounded inputs generate boudned outputs. Mathematically, this means:
  $$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$
  And in discrete time:
  $$\sum_{-\infty}^{\infty} |x[n]| < \infty$$
  - We can also look at the energy and power:
  $$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \;\; P = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$
- **System Response function:** These are particular outputs for systems when given an impulse response of a delta function. They are calculated by substituting $x(t) = \delta(t)$ in the continuous case, and $x[n] = \delta[n]$ in the discrete case. Watch lecture for this.

  To calculate the impulse response for the moving average filter:
  $$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$
  To find the impulse response, we substitute $x[n] = \delta[n]$ to get $h[n]$. Here, notice that for $n < -2$, then $h[n] = 0$, since $x[-2+1] = x[-1] = 0$, and same goes for the other terms. Then, refer to the following table:

n	$h[n]$
-2	0
-1	1/3
0	1/3
1	1/3
2	0

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3.1 Step Response

- The step response function is the function $y_{\text{step}}(t)$ when a step function $u(t)$ is fed into the system. In discrete-time: we feed $u[n]$ into the system, and get $y_{\text{step}}[n]$ as an output.
- For instance, for the moving average filter defined earlier, we have the following result:

n	$y_{\text{step}}[n]$
-2	0
-1	1/3
0	2/3
1	1
2	1

  Note that this resembles a ramp function, and is called a ramp-step function.
- **Harmonic Response:** The harmonic response is the response by the system when presented with a harmonic function, of the form $Ae^{i\omega t}$.

  In discrete time, we feed in $Ae^{i\omega n}$ where n is an integer.

- For the moving average filter, let's write out $y[n]$:

$$\begin{aligned} y[n] &= \frac{1}{3}\left(Ae^{i\omega(n-1)} + Ae^{i\omega n} + Ae^{i\omega(n+1)}\right) \\ &= \frac{1}{3}\left(e^{-i\omega} + 1 + e^{i\omega}\right) \\ &= \frac{1}{3}(2\cos\omega + 1)Ae^{i\omega n} \end{aligned}$$

The interesting thing here is that when given a harmonic function, the system response just scales the signal by a constant amount!

3.2 LCCDE

- In this class, we will deal with lots of differential equations, so it's going to be very useful to look at their form, and how to solve them.
- There are two solutions to any differential equation:
  - **Particular Solution:** $y_p(t)$ is called a particular solution if it satisfies:

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y_p(t)}{\mathrm{d}t^k} = \sum_{k=0}^{N} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

  - **Homogeneous Solution:** $y_h(t)$ is called a homogeneous solution if it satisfies:

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y_p(t)}{\mathrm{d}t^k} = 0$$

- In general, the solution will be a linear combination of the two:

$$y(t) = y_p(T) + ay_h(t)$$

the value of a is generally going to be given by some initial condition.

- For the homogeneous solution, an ansats of the form Ae^{st} where s is an undetermined constant will solve the differental equation. We can then determine the value of s by solving the resulting polynomial.

  To determine the value of A, these are determined by the initial conditions, and depending on the number of initial conditions given, that would correspond directly to the number of distinct values of A.

3.2.1 Example

- Given a first order LCCDE, with a step function input (this means that the right hand side is a step function). This means we're solving for a solution $y(t)$ to:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) = bx(t) = bu(t)$$

- First we look for the homogeneous solution, which will give us:

$$\frac{\mathrm{d}y_h(t)}{\mathrm{d}t} + ay_h(t) = 0$$

This is a separable DE, so we move $ay_h(t)$ to the right hand side and integrate, which gives us a solution of the form:

$$y_h(t) = Ae^{-at}$$

- The particular solution is the function $y_p(t)$ that satisfies:

$$\frac{\mathrm{d}y_p(t)}{\mathrm{d}t} + ay_p(t) = bu(t)$$

Now, we break this up into what values $u(t)$ takes along the real line. For $t < 0$, we have $u(t) = 0$, which gives us a solution of $y_p(t) = 0$. Note that even though the right hand side is zero doesn't mean this is homogeneous, since the homogeneous solution has 0 on the right hand side *for all* t.

- For $t > 0$, the differential equation becomes:

$$\frac{\mathrm{d}y_p(t)}{\mathrm{d}t} + ay_p(t) = b$$

We can rearrange this slightly:

$$\frac{\mathrm{d}\left(y_p(t) - \frac{b}{a}\right)}{\mathrm{d}t} + a\left(y_p(t) - \frac{b}{a}\right) = 0$$

You can verify that this is actually the same differential equation, since the derivative of a constant is zero. But now, we can define $z(t) = y_p(t) - \frac{b}{a}$, and solve instead a homogeneous differential equation for $z(t)$. Going through the same steps as before, we have $y_p(t) = \frac{b}{a} + Be^{-at}$.

- We can combine both $t < 0$ and $t > 0$ with a nice analytical form:

$$y_p(t) = \left(\frac{b}{a} + Be^{-at}\right)u(t)$$

Is this necessary when possible?

- The general solution is then:

$$y(t) = y_p(t) + y_h(t) = \frac{b}{a} + Be^{-at} + Ae^{-at}$$

We need an initial condition to solve for A and B :

  - Initial rest condition: at $t = 0$, no input, so the output should also be 0. This gives the equation:

$$A + B = -\frac{b}{a}$$

4 Lecture 4

4.1 System Block Diagram

- Now we'll look at how to convert an LCCDE into a block diagram.
- Suppose we're given a system of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

This implies the equation:

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] - \sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k]$$

4.2 Linear Time Invariant (LTI), Linear Shift Invariant (LSI)

- What is an LTI system? Firstly, it's linear, so it satisfies the superposition rule: given two signals $x_1(t)$ and $x_2(t)$, then an input of $ax_1(t) + bx_2(t)$ will generate an output of $ay_1(t) + by_2(t)$.
- An LSI is also a linear system, and given an input signal $x(t)$ with an output $y(t)$, then we can shift the system $x(t-T)$ to generate an output $y(t-T)$, but $y(t-T) = y(t)$. In other words, the output will look like $y(t)$, except shifted by T.

- As an example, the continuous LCCDE is a linear time invariant system. This is because the derivative is linear:
$$\sum_{k=0}^{M} \frac{\mathrm{d}^k(ax_1(t)+bx_2(t))}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t-T)}{\mathrm{d}(t-T)^k}$$
And then we can substitute $u = t - T$:
$$\sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(u)}{\mathrm{d}u^k} = \sum_{k=0}^{M} \frac{\mathrm{d}^k y(t-T)}{\mathrm{d}(t-T)^k}$$
- The same principle also holds for discrete time signals.
- The most important property of an LTI system is that **the system response is fully characterized by an impulse response function.**

  What this means is that if we feed the system a $\delta(t)$ or $\delta[n]$, it gives us an impulse response function $h(t)$ or $h[n]$, and this gives us enough information to characterize the entire system.
- In the continuous time case, suppose we had the following:
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$
Then, $y(t)$, the signal generated by an arbitrary $x(t)$ is generated by:
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ d\tau$$
In discrete time, the formula is:
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
This is called a *convolution*, we will come back to this later.

4.2.1 Why a convolution?

- Again, consider the diagram:
$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = h(t)$$
If we send a signal $\delta(t-\tau)$ into the system, then due to linear time invariance, the system should output $y(t-\tau) = h(t-\tau)$.
- If we now send the signal $x(\tau)\delta(t-\tau)$, then because $x(\tau)$ is a constant, then we invoke linearity to get that the output is $x(\tau)h(t-\tau)$.
- Now, consider what happens when we send in the signal that is just a combination of all possible τ. Each $x(\tau)$ is a constant, so the output signal is of the form
$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) \mapsto \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ d\tau$$
But now notice that this signal can also be written as:
$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\ d\tau = x(t)$$
And so if we're sending in a signal $x(t)$, then the output should be $y(t)$! Thus, we've proven that the impulse response is all we need in order to characterize $y(t)$.
- For future reference, a convolution, denoted by $x(t) * h(t)$, is defined as:
$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\ d\tau$$
this last equality shows that convolution is a commutative operation.

4.2.2 Impulse Response of 1st order LCCDE

- Recall the step response to LCCDE:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) = bx(t) = bu(t) \implies y_{\text{step}}(t) = \left(\frac{b}{a}(1 - e^{-at})\right) u(t)$$

- Given an impulse, which in this case can be written as:

$$\delta(t) = \lim_{\epsilon \to 0} \frac{u(t) - u(t-\epsilon)}{\epsilon}$$

  This implies that the response $h(t)$ is given by:

$$h(t) = \lim_{\epsilon \to 0} \frac{y_{\text{step}}(t) - y_{\text{step}}(t-\epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\frac{b}{a}\left(e^{-a(t-\epsilon)}u(t-\epsilon) - e^{-at}u(t)\right)}{\epsilon} = be^{-at}u(t)$$

  (verify this at home, the simplification makes use of the fact that $e^{a\epsilon} \approx 1 + a\epsilon + a^2\frac{\epsilon^2}{2} + \cdots$, but the higher order terms die).

4.3 Harmonic Response of an LTI system

- The response of an LTI system to a complex signal $x(t) = Ae^{j\omega t}$ is always going to be another complex exponential signal $y(t) = H(\omega)Ae^{j\omega t}$
- Given the input signal $x(t) = Ae^{j\omega t}$, we can write:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} Ae^{j\omega t}h(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} Ae^{j\omega(t-\tau')}h(\tau')\,d\tau' \\ &= Ae^{j\omega t}\underbrace{\int_{-\infty}^{\infty} e^{-j\omega\tau'}h(\tau')\,d\tau'}_{H(\omega)} \\ &= H(\omega)Ae^{j\omega t} \end{aligned}$$

- By definition:

$$H(\omega) \equiv \int_{-\infty}^{\infty} e^{-j\omega\tau'}h(\tau')\,d\tau' \quad H(f) \equiv \int_{-\infty}^{\infty} e^{-j2\pi ft}h(t)\,dt$$

  You'll recognize $H(\omega)$: it's the Fourier transform equation.

  When given an harmonic input, and we're asked to measure it, are we measuring the real part of the signal?

4.3.1 Example: Frequency response of an RC Circuit

- Given the following circuit:
- The impulse response is given by the differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

  This is a first order LCCDE, so therefore the impulse response $h(t)$ is given by $h(t) = be^{-at}u(t)$.

- For the frequency response, we have a function of the form $x(t) = e^{j\omega t}$, which we know has an output signal of the form $y(t) = H(\omega)e^{j\omega t}$. So all that remains now is to find $H(\omega)$:

$$\begin{aligned} y(t) &= e^{j\omega t} \int_{-\infty}^{\infty} e^{-j\omega t} h(\tau)\, d\tau \\ &= e^{j\omega t} \int_{-\infty}^{\infty} b e^{-a\tau} u(\tau) e^{-j\omega\tau}\, d\tau \\ &= b e^{j\omega t} \int_{0}^{\infty} e^{-a\tau} e^{-j\omega\tau}\, d\tau \\ &= \left(-\frac{1}{a + j\omega} e^{-a\tau} e^{-j\omega t} \Big|_0^{\infty} \right) b e^{j\omega t} \\ &= \frac{b}{a + j\omega} e^{j\omega t} \end{aligned}$$

Now, if we impose that $a = b = \frac{1}{RC}$, then we get the equation:

$$\frac{\frac{1}{jc\omega}}{\frac{1}{jc\omega} + R} e^{j\omega t}$$

Now, $\frac{1}{jc\omega}$ is the impedance of a capacitor, and this overall equation takes the form of a voltage divider for a circuit with known impedance:

$$y(t) = \frac{z(\omega)}{z(\omega) + R} e^{j\omega t}$$

4.4 Sinusoidal Input

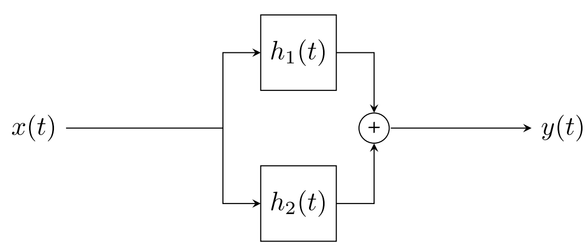
- With the harmonic response tools, we can now evaluate the system response when given a sinusoidal input, since we know that

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Does the same work with sine, where there's a complex number is in the denominator?

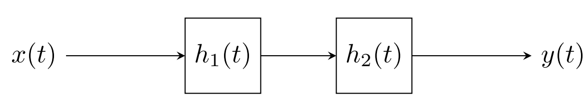
4.5 LTI systems in Parallel and Series

- For a basic system with a single input and output, we've already discussed that $y(t) = x(t) * h(t)$. Now, what if we connect these systems in parallel?



then, the result $y(t)$ is given by $y(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$.

- If we connect them in series:



then

5 Lecture 5

- Recall that last time, given a system equation $y[n] - ay[n-1] = x[n]$, an impulse response function $h[n] = a^n u[n]$ and a step input signal $x[n] = u[n]$, then we can calculate the general signal response $y[n] = x[n] * h[n]$.
- see lecture for notes

5.1 Causality and BIBO stability

- An LTI system is causal if and only if the impulse response function is a causal function. This means that $h(t) = 0$ for all $t < 0$. Under this condition, then the system will be causal.
- For BIBO stability, the same thing applies: it's BIBO stable if and only if its impulse response $h(t)$ is absolute integrable:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

- The same thing applies for discrete-time: it's causal iff $h[n] = 0$ for all $n < 0$.

  Proof. First, we prove the forward case: if $h[n] = 0$ for all $n < 0$, then:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

  But then this means that $y[n]$ only depends on the present and the past, hence it's a causal system.

  Now for the reverse case by contrapositive: for any $m < 0$, if $h[m] \neq 0$, then there is one $n-k < 0$ such that $h[n-k] \neq 0$. If this is true, then the system depends on at least one value $k > n$, hence it's non-causal. □

- For BIBO stability, the equivalence here that the impulse response is absolute summable, so:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

  Proof. First, we show that if $h[n]$ is absolute summable, then given a bounded input:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \leq c \sum_{k=-\infty}^{\infty} |h[k]| \leq \infty$$

  Now the reverse case: if the impulse response by contraposition: if our system is not absolute summable, then our system is not BIBO stable. To do this, let $x[n] = \operatorname{sgn}(h[-n])$, then since:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  then for instance, we can evaluate $y[0]$:

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$

  And since $x[k] = \operatorname{sgn}(h[-k])$, then:

$$y[0] = \sum_{k=-\infty}^{\infty} |h[-k]| = \infty$$

  This is equal to infinity, because this is this is precisely the fact that h is not absolute absolute sumamble. □

5.2 Convolution and Correlation

- As we've seen already, the convolution is essential to an LTI system. Given a signal $x(t)$ and system $h(t)$ and output $y(t)$, then we can calcualte $y(t)$ based on the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

- In discrete-time:

$$y[n] = \sum_{k=\infty}^{\infty} x[k]h[n-k]$$

- A **convolution** takes in two signals x, y and computes:

$$\text{cov}(x,y) = (x * y)(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\, d\tau$$

  Notice that we're integrating with respect to τ, meaning that the resulting function should be a function of t.

- A **cross-correlation** is defined as r_{xy}, and is computed as:

$$\text{corr}(x,y) = (x \circ y)(t) = x(t) \circ y(t) = \int_{-\infty}^{\infty} x(\tau)y(t+\tau)\, d\tau$$

  The main difference between the two is that with convolution, we're flipping the second function and computing the integral of the product. With cross correlation, we're not flipping anything, and instead just "sliding" the product across.

  To be clear, it's the second function that we're sliding across the first one.

- Because the two equations are so similar, the convolution and cross correlation are related:

$$y(t) \circ x(t) = x(t) * y(-t)$$

  The proof is fairly trivial; too lazy to write it down. Note that the order matters though.

- See the lectures for a graphical representation of what a convolution does.

5.2.1 Example Functions

- Given two rectangle functions $x(t) = \sqcap(t-0.5), y(t) = \sqcap(t-0.5)$, their convolution is a triangle function $\wedge(t-1)$, and their correlation is $\wedge(t)$. Notice that the convolution is shifted over by 1, and the correlation is not. This is because the convolution flips the sign of one of them, so they intersect at a later time.
- The convolution of the delta function with itself is the delta function: $(\delta(t) * \delta(t)) = \delta(t)$.

5.3 Convolution Identities

- Convolution follows the distributive property: $x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$
- The convolution is commutative: $x(t)*h(t) = h(t)*x(t)$. This is different from the cross-correlation, where the resulting signal is time-reversed.
- The unit impulse is the identity element of the convolution:

$$x(t) * \delta(t) = x(t)$$

- The convolution of a signal with a shifted impulse shifts the signal:

$$x(t) * \delta(t-T) = x(t-T), \;\; x[n] * \delta[n-N] = x[n-N]$$

- The convolution of two time-shifted signals will produce a signal that accounts for the shift of both functions:

$$x(t - T_1) * h(t - T_2) = y(t - T_1 - T_2)$$

- When a function of time width T_1 is convolved with a function of width T_2, the resulting function has a width of $T_1 + T_2$.

  This makes sense, since the convolution computes the overlap – so if we don't have any overlap, then the result will be 0.

- Convolution also follows an area property: given a convolution $x(t) * h(t)$, then the area of $y(t)$ is given by the product of the area of $x(t)$ and $h(t)$ individually.

6 Lecture 6

6.1 Clarification on BIBO Stability

- When we say a "bounded" signal, we mean that the amplitude of the signal is bounded at all times:

$$|x(t)| < \infty \; \forall t \in \mathbb{R}$$

  The same definition follows for discrete-time signals.

- For LTI systems, we call the system BIBO stable if and only if its impulse $h(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$$

6.2 Cross-Correlation

- The cross correlation between two signals $r_{xy}(t) = r_{yx}(-t)$. To show this explicitly, we look at the cross-correlation equation:

$$\begin{aligned} r_{xy}(t) &= \int_{-\infty}^{\infty} x(\tau)y(t+\tau) \, d\tau \\ r_{yx}(t) &= \int_{-\infty}^{\infty} y(\tau)x(t+\tau) \, d\tau \end{aligned}$$

  But for the second equation, we can define a $\tau' = t + \tau$, so then we get:

$$r_{yx}(t) = \int_{-\infty}^{\infty} y(\tau' - t)x(\tau') \, d\tau' = \int_{-\infty}^{\infty} x(\tau')y(-t+\tau') \, d\tau'$$

  This looks like the first equation except we have $-t$ instead of t. Therefore, we have $r_{xy}(t) = r_{yx}(-t)$. The same works for discrete time: $r_{xy}[n] = r_{yx}[-n]$.

6.3 More Convolution Properties

- **Differentiation property:** Given $y(t) = x(t) * h(t)$, then:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = x(t) * \frac{\mathrm{d}h(t)}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} * h(t)$$

- **Intergration Property:** Given $y(t) = x(t) * h(t)$, we have:

$$\int_{-\infty}^{t'} y(t) \, dt = x(t) * \int_{-\infty}^{t'} h(\tau) \, d\tau$$

6.4 Fourier Transform

- The fourier transform came from the study of the heat equation, written as:

$$c\rho\frac{\partial}{\partial t}u(x,y,z,t) = k\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u(x,y,z,t)$$

  Fourier then claimed that the solution can be expanded in a series of sines with multiples of the variable. In other word,s the solution is of the form:

$$f(x) = \frac{1}{2a_0} + (a_1\sin(x) + b_2\cos(x)) + (a_2\sin(2x) + b_2\cos(2x)) + \cdots$$

- Recall the frequency response of an LTI system:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

  Recall that we can characterize $y(t)$ via a convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

  If we do this with our input $e^{j2\pi ft}$, then we get:

$$y(t) = H(f)e^{j2\pi ft} = H(\omega)e^{j\omega t}$$

  Here, $H(\omega)$ is defined to be the Fourier transfrm of the impusle response $h(t)$:

$$H(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}h(t)\,dt$$

  Alternatively, written in frequency language:

$$H(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft}h(t)\,dt$$

- Formally, the Fourier transform is defined as:

$$H(f) \equiv \mathcal{F}\{h(t)\} \equiv \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt$$

  This transforms the signal $h(t)$ from the time domain into the frequency domain. The reason for this is becuase the Fourier transform is a definite integral, which kills off any t dependence entirely. In terms of angular frequency, we have:

$$H(\omega) \equiv \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

- The inverse Fourier transform is:

$$h(t) = \mathcal{F}^{-1}\{H(f)\} = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,df$$

  Since the Fourier transform takes objects from the time domain to the frequency domain, the inverse Fourier transform takes things from the frequency domain to the time domain.

  In terms of angular frequency, we have:

$$h(t) = \mathcal{F}^{-1}\{H(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega$$

  This is also sometimes called the "synthesis equation", since we basically create $x(t)$ out of $H(\omega)$.

- We can also provably show that the Inverse fourier transform does indeed invert the Fourier transform, albeit with a lot of algebra. See lecture slides for the full derivation.

7 Lecture 7

7.1 DTFT and Convergence

- Not all functions have a Fourier transform, and the problem of whether a function has a Fourier integral is an incredibly complex problem with no simple statement.
- However, we know that there are several sufficient (but not necessary) conditions. Firstly, we know that $x(t)$ must be absolutely integrable. That is,
$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

7.2 Fourier Transform Pairs

- There are several pairs of Fourier transforms that are useful to memorize.
- The Delta function:
$$x(t) = \delta(t - t_0) \leftrightarrow X(f) = e^{-j2\pi f t_0}$$
This actually has strong implications about the nature of the Fourier transform – there is an "uncertainty principle" that manifests itself here. A signal cannot be both localized in time and frequency at the same time.
- Complex exponentials:
$$x(t) = e^{j\omega_0 t} = e^{j2\pi f_0 t} \leftrightarrow X(f) = \delta(f - f_0)$$
This is the same as the previous point, except now we're going backwards.
- Cosine functions:
$$x(t) = \cos(2\pi f_0 t) \leftrightarrow X(f) = \frac{1}{2}\delta(f - f_0) + \delta(f + f_0))$$
This makes sense: a plane wave is a composition of a left and right travelling wave.
- Sine functions:
$$x(t) = \sin(2\pi f_0 t) \leftrightarrow X(f) = \frac{1}{2j}(\delta(f - f_0) - \delta(f + f_0))$$
Note that the only difference here is the minus sign, as a result of the conversion of sine into complex exponentials.
- Shah function:
$$x(t) = III(t) \leftrightarrow X(f) = III(f) \text{ or } X(\omega) = \frac{1}{2\pi} III(\omega)$$
- Rect function:
$$x(t) = \sqcap(t) \leftrightarrow \text{sinc}(f)$$

8 Fourier Transforms

8.1 Discrete Fourier Transform

- Recall the Shah function: $x(t) = \text{III}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k)$. Its fourier transform in ordinary frequency is:
$$X(f) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(t - k) e^{-j2\pi f t}\, dt = \sum_{n=-\infty}^{\infty} \delta(f - n) = III(f)$$
So the Fourier transform of a Shah function is itself a shah function, in frequency space.
- In Angular frequency (ω), then it's writtne as:
$$X(\omega) = X(f) = 2\pi \sum_{n=-\infty}^{\infty} \delta(2\pi f - 2\pi n) = 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n)$$

8.2 Convolutions and Fourier Transform

- There is an identity between the Fourier transform and convolution, called the convolution theorem:

$$x_1(t) * x_2(t) \leftrightarrow X_1(f)X_2(f)$$

The right hand side can also be replaced by $X_1(\omega)X_2(\omega)$, up to a normalization factor. This equation basically says that the Fourier transform of the convolution of two function is also equal to the pointwise product of the Fourier transforms of x_1 and x_2.

Proof. Start with the Fourier transform of $x_1(t) * x_2(t)$:

$$\int_{-\infty}^{\infty} (x_1(t) * x_2(t))e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau \right) e^{-j2\pi ft}\, dt$$

Now, we take the t integral first becuase it's easier:

$$\int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(t-\tau)e^{-j2\pi ft}\, dt\, d\tau = \int_{-\infty}^{\infty} x_1(\tau)e^{-j2\pi f\tau} X_2(f)\, d\tau = X_1(f)X_2(f)$$

□

- For example, consider the rect function $\sqcap(t)$ that is 1 on the interval $[-1/2, 1/2]$. If we do the Fourier transform of this, we get a sinc function. At $f = 0$, we have $\mathrm{sinc}(f) = 1$.

If we wanted to take the FT of the convolution of two rect functions, then we can use the convolution theorem to conclude that $\mathcal{F}(\sqcap(t)) = \mathrm{sinc}^2(f)$. Conversely, we know that the inverse Fourier transform of $\mathrm{sinc}^2(f)$ would be the triangular function, becuase we know that it's the convolution of two rect functions.

8.3 Fourier Series

- Consider a periodic function, where $x(t) = x(t + nT)$. One way to write $x(t)$ and also to reflect its period, we can write it as a sum of delta functions:

$$x(t) = \sum_{n=-\infty}^{\infty} \left(x(t) \sqcap \left(\frac{t}{T} \right) \right) * \delta(t - nT)$$

Basically the rect function restricts the domain to only one period of $x(t)$, and we convolve this with a series of Delta functions in order to generate the roiginal fnction back. We can also do some simplifications on this:

$$x(t) = \left(x(t) \sqcap \left(\frac{t}{T} \right) \right) * \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Then we can use an identity to get:

$$x(t) = \left(x(t) \sqcap \left(\frac{t}{T} \right) \right) \frac{1}{T} \mathrm{III} \left(\frac{t}{T} \right)$$

How did we get the simplification of the delta sum into the Shah function?

- Turns out, if we perform a Fourier transform of $x(t)$, then we get a discrete set of values, also called a Fourier series. Taking the Fourier transform of the above function, and defining $g(t) = x(t) \sqcap (t/T)$, then we can write:

$$\begin{aligned} \mathcal{F}\{x(t)\} &= \mathcal{F} \left\{ g(t) * \frac{1}{T} \mathrm{III} \left(\frac{t}{T} \right) \right\} \\ &= \mathbf{G}(f)\mathbf{III}(Tf) \end{aligned}$$

where $\mathbf{G}(f)$ is the Fourier transform of $g(t)$. Expanding the Shah function out, we have:

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} \mathbf{G}(f)\delta \left(f - \frac{n}{T} \right)$$

The delta function will select $f = \frac{n}{T} = nf_0$, so we get:

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} G(nf_0)\delta \left(f - \frac{n}{T} \right)$$

- So, from here we can conclude that the Fourier transform of a periodic function is just a series over that function.
- The inverse Fourier transform used to be:

$$x(t) = \int_{-\infty}^{\infty} \mathbf{X}(f) e^{j2\pi ft} \, df$$

Where $\mathbf{X}(f)$ is the (continuous-time) Fourier transform of $x(t)$. Since x is periodic, then we can write

$$\begin{aligned} x(T) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathbf{G}(n)\delta\left(f - \frac{n}{T}\right) e^{j2\pi ft} \, df \\ &= \frac{1}{T} \sum_{n=-\infty}^{\infty} \mathbf{G}(n) \int_{-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) e^{j2\pi ft} \, df \\ &= \frac{1}{T} \sum_{n=-\infty}^{\infty} \mathbf{G}(n) e^{2\pi \frac{n}{T} t} \\ &= \frac{1}{T} \sum_{n=-\infty}^{\infty} \mathbf{G}(n) e^{j2\pi n f_0 t} \end{aligned}$$

Note that the delta function only picks out the frequency $\frac{n}{T}$, which allows us to get a discrete sum of frequencies. This is called the Discrete Fourier Series.

9 More on Fourier Transforms

9.1 Discrete Time Fourier Transforms (DTFT)

- Recall that for continuous time signals, the fourier transform is written as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

- For discrete time signals, the way to do this is to write the signal as a continuous time signal via delta functions, and then apply CTFT:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-n)$$

Then, we can write:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-n) e^{-j\omega t} \, dt = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

So this actually just means that in discrete time, the Fourier transform of $x[n]$ is just the amplitude of the signal at that particular n, multiplied by a sinusoid of a corresponding frequency specified by n.

- In general, we have:

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn} \qquad x[n] = \int_{-\infty}^{\infty} X(e^{j2\pi f}) e^{j2\pi fn} \, df$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

Sometimes textbooks use Ω insetad of ω, but we will use the latter.

- See lectures for worked examples on how to do this.

9.2 Characteristics of DTFT

- DTFT is generally a continuous function over f or ω, even though we know that $x[n]$ is a discrete time signal. This is due to the presence of the delta functions.
- Further, DTFT is periodic with a period of 1 in frequency or 2π in angular frequency:

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f(n+1)}$$

This is because $e^{-j2\pi f(n+1)} = e^{-j2\pi fn}e^{-j2\pi n}$ and the latter term is 1.

9.3 Common DTFTs

- For a delta function $x[n] = \delta[n]$, its Fourier transform $X(e^{j\omega}) = 1$.
- For a constant function $x[n] = 1$, its Fourier transform is the Shah function:

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- For complex exponentials $x[n] = e^{j\omega_0 n}$, its Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(f - f_0 - k)$$

So this is basically this is the comb function, but shifted over by some constant amount ω_0. This is incredibly useful for applications such as signal modulation and other applications, where we talk about a "linear phase" addition.

For instance, consider sending a signal to a receiver that only accepts a specific frequency. Then, in order for a source to be able to send an appropriate signal, we can "modulate" the signal by a constant factor ω_0 instead of completely modifying our signal.

- For sinusoids, recall the identities:

$$\cos[\omega_0 n] = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n})$$

and sicne we've expressed it in terms of exponentials, we can use the earlier bullet point to find DTFT here. This gives us:

$$X(\omega) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi k)$$

For the sine function, we have

$$\sin[\omega_0 n] = \frac{1}{2j}(e^{j\omega_0 n} - e^{-j\omega_0 n})$$

and we can do the same trick.

- For the rectangular function $x[n] = \sqcap[n]$, its Fourier transform is bascially a restricted comb function:

$$X(e^{j\omega}) = e^{j\omega} + 1 + e^{j\omega} = 2\cos(\omega) + 1$$

Note that here we're using the standard rectangular function that is only nonzero over $n \in [-1, 0, 1]$. For the general rectangular function: $x[n] = \sqcap\left[\frac{n}{N}\right]$, then we have:

$$X(e^{j\omega}) = \sum_{n=-N}^{N} e^{-j\omega n}$$

So it is a restricted sum over $-N$ to N. But this is just a geometric series, so this will give us the formula:

$$X(e^{j\omega}) = e^{j\omega N}\frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}} = \frac{e^{j\omega N} - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} = \frac{\sin(\omega(N + 1/2))}{\sin(\omega/2)}$$

This is very similar to a sinc function.

10 Continuous-Time Fourier Transform Properties

10.1 Why CTFT?

- Recall the definition of the Fourier transform:

$$X(f) \equiv \mathcal{F}\{x(T)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \qquad x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$$

$$X(\omega) \equiv \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

- Why would we do a Fourier transform? For instance, NASA's Perseverance rover carries two microphones that record the audio on the Martian surface. There are many sources of the sound; one of them is instrumentation sound, which we would like to remove. The only to remove sources like this is to perform a Fourier transform, get rid of the unwatned frequencies, then do an IFT to get back to the original signal.

10.2 CTFT Properties, DTFT Properties

- Linearity: given $ax(t) + by(t)$, then the Fourier transform would be $aX(\omega) + bY(\omega)$. An easy way to see this is the fact that the Fourier transform relies on an integral, which is indeed linear.

  So, computing the Fourier transform of $\cos(2\pi f_0 t)$ can be simplified using linearity:

$$\cos(2\pi f_0 t) = \frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0)) = \frac{1}{2}(2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0))$$

- Time shift property: If a signal is shifted in time (i.e. $x(t - t_0)$), then the Fourier transform picks up a phase of $e^{-j\omega t_0}$. That is:

$$x(t - t_0) \quad \leftrightarrow \quad e^{-j\omega t_0}X(\omega) = e^{-j2\pi f t_0}X(f)$$

  where $X(\omega) = \mathcal{F}\{x(t)\}$. The proof of this property is in the lecture notes, it's also not very hard to prove; just use a change of variables.

- Frequency shift property: Given a signal $e^{j\omega_0 t}x(t)$, then the Fourier transform would be shifted by that frequency:

$$e^{j\omega_0 t} \quad \leftrightarrow \quad X(\omega - \omega_0)$$

  An application of this is with AM waves, where we have a signal to be transmitted $x(t)$ that serves as the envelope of some *carrier signal*, gemerally of the form $\cos(\omega_0 t)$. Therefore, we can write the transmitted signal as:

$$y(t) = x(t)\cos(\omega_0 t)$$

  In Fourier space, then this signal transmits as:

$$Y(\omega) = \frac{1}{2}X(\omega - \omega_0) = \frac{1}{2}X(\omega + \omega_0)$$

  with the shift due to the frequency shift property.

- Complex conjugate symmetry: the Fourier transform of a signal $x^*(t)$ transforms as:

$$x^*(t) \quad \leftrightarrow \quad X^*(\omega)$$

  Again, proof in lecture notes. If $x(t)$ is a real function, then we have $x^*(t) = x(t)$, so $X^*(-\omega) = X(\omega)$.

  The magnitude is an even function, and the phase will be an odd function.

- Differentiation Property: The Fourier transform of the derivative of $x(t)$ is given by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \quad \leftrightarrow \quad j\omega X(\omega) = j2\pi f X(f)$$

  Further, we have:

$$-jtx(t) \quad \leftrightarrow \quad \frac{\mathrm{d}X(\omega)}{\mathrm{d}\omega}$$

- Time and Frequency scaling: Given a signal $x(at)$, then the Fourier transform is

$$x(at) \quad \leftrightarrow \quad \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

as long as $a \neq 0$.

- Multiplication Property: Given two signals $x_1(t)x_2(t)$, then in frequency space this is a convolution:

$$x_1(t)x_2(t) \quad \leftrightarrow \quad \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$$

- Parseval's relation: A theorem about the signal $x(t)$:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

This is basically a statement of energy conservation between the two domains: the energy of the wave in temporal space is expressed as the integral of $|x(t)|^2$, and intuitively by taking a fourier transform that energy shouldn't change, so the integral $|X(f)|^2$ should be the same.

- Energy spectral density: intuitively, this is a measure of the "energy per unit" of frequency interval. Mathematically, this is written as:

$$\epsilon(f_0) = \lim_{\delta f \to 0} \frac{E(f_0 + \delta f)}{\delta f}$$

So basically the slope of the energy curve. By Parseval's theorem, then we can write:

$$\epsilon(f_0) = \lim_{\delta f \to 0} \frac{\int_{f_0}^{f_0+\delta f} |X(f)|^2 \, d}{\delta f} = \lim_{\delta f \to 0} \frac{|X(f)|^2 \delta f}{\delta f} = |X(f_0)|^2$$

In terms of ω, we have:

$$\epsilon(\omega_0) = \frac{1}{2\pi}\lim_{\delta \omega \to 0} \frac{|X(\omega)|^2 \delta\omega}{\delta\omega} = \frac{1}{2\pi}|X(\omega)|^2$$

11 Laplace Transform

- Laplace transforms concern the equation:

$$: tabn X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt$$

The basic idea is that instead of solving for $x(t)$, we solve instead for $X(s)$, which is much easier to do sometimes than $x(t)$.

- Suppose you have an LTI system with impulse response $h(t)$, and we input a harmonic exponential $e^{j2\pi ft}$, then we get an output $y(t) = H(f)e^{j2\pi ft} = H(\omega)e^{j\omega t}$. (this is the eigenfunction property.)

- What about an input e^{st}? Then, we can express this as a convolution, which happens to be written as: $y(t) = H(s)e^{st}$, where $H(s)$ is the Laplace transform of $h(t)$.

- Consider an input $x(t) = e^{st}$, and since $s \in \mathbb{C}$, then we can write $s = \sigma + j\omega$. Then we convolve:

$$y(t) = \int_{-\infty}^{\infty} h(t)e^{s(t-\tau)} \, d\tau = \int_{-\infty}^{\infty} h(t)e^{st}e^{-s\tau} \, d\tau = e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau} \, d\tau}_{H(s)}$$

where the integral we defined earlier as the Laplace transform $H(s)$. Here, we call it specifically the bilateral Laplace transform of $h(t)$. This distinction just has to do with the integration bounds; a unilateral Laplace transform is defined as

$$H_{\text{uni}}(s) = \int_0^{\infty} h(\tau)e^{-s\tau} \, d\tau$$

but we will normally consider bilateral Laplace transforms in this class.

- Note also that the Laplace transform is also the more general case of the Fourier transform: the Fourier transform occurs when $\text{Re}(s) = 0$.

11.1 Notation, Terms

- With Laplace transforms, we say that they transform between the time domain and the s-domain, and it's written as:

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\, dt$$

  Because s is a complex number, the Laplace transform takes our 1-dimensional signal $x(t)$ and turns it into a two-dimensional signal in s-space.

11.2 Laplace Transform Pairs

- Given a signal $x(t) = \delta(t)$, then the Laplace transform $X(s) = 1$. This is seen easily from the integral itself:

$$X(s) = \int_{-\infty}^{\infty} \delta(t)e^{-st}\, dt = 1$$

  As a 2D diagram, we can imagine that over the entire complex plane, $X(s)$ takes on a value of 1.

- Given a unit step function $x(t) = u(t)$, then the Laplace transform $X(s) = \frac{1}{s}$. Again, just do the integral:

$$X(s) = \int_{-\infty}^{\infty} u(t)e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt = \frac{1}{s}$$

  Note that this only works if $\mathrm{Re}(s) > 0$, otherwise we have an unbounded integral. This condition is also called the *region of convergence* for the integral. This actually means that the Laplace transform may not be defined for all values of s!

  Note that $\mathrm{Im}(s)$ doesn't matter here, since all it does is give us an overall phase factor of $e^{j\omega t}$ that has magnitude 1 all the time.

- Let $x(t) = e^{-at}u(t)$. Then, let's look at its Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\, dt = \int_{0}^{\infty} e^{-(a+s)t}\, dt = \frac{1}{a+s}$$

  which again, only holds true when $\mathrm{Re}(a+s) > 0$. Since a is real, then the condition simplifies to $\mathrm{Re}(s) > -a$, so this is our region of convergence.

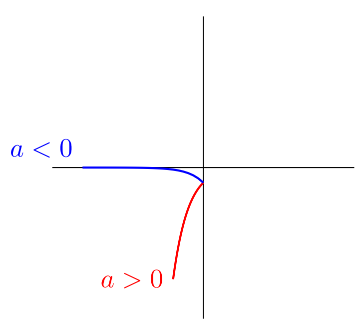
  We could be a little more specific with the integration:

$$X(s) = \int_{0}^{\infty} e^{-(a+\sigma)t}e^{-j\omega t}\, dt = \frac{1}{j\omega(+a+\sigma)} = \frac{1}{s+a}$$

  but this simplifies in the same way.

  If we didn't have the $u(t)$, then would the integral still converge?

- Now consider $x(t) = -e^{-at}u(-t)$. This is just the time-flipped version of the previous signal. Visually, for different values of a, this is what it looks like:



Doing the Laplace transform:

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}\,dt = \int_{-\infty}^{0} e^{-(a+s)t}\,dt = \frac{1}{a+s}$$

As before, this only holds when $\mathrm{Re}(a+s) > 0$.

Mathematica gives $-\frac{1}{a+s}$, why?

11.3 Why Laplace Transform?

- Laplace transform has many of the same properties as the Fourier transform, and due to its wider scaope (i.e. having $X(s)$ be a two-dimensional function), it allows us to take the transform of functions that the Fourier transform cannot handle.
- It's also useful for integral differential equations. Consider the following circuit: The voltage readout is given by:

$$Ri(t) + \left[\frac{1}{\mathcal{L}}\int_{0^-}^{t} i(t')\,dt' + v_c(0^-)\right]$$

we will see later on how the Laplace transform simplifies the calculation of this integral.

11.4 Region of Convergence

- The Laplace transform is also related to the Fourier transform, by a multiplication of a decaying exponential:

$$\mathcal{L}\{x(t)\} = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

This also tells us why the Laplace transform is more general – by multiplying by an $e^{-\sigma t}$, we can actually modify what $x(t)$ is. This gives us more control, and makes it a more powerful tool.

The region of convergence is defined as the set of $s \in \mathbb{C}$ suhc that $x(t)e^{-st}$ has a Fourier transform. Mathematically:

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-at}e^{-j\omega t}\,dt < \infty$$

We've already talked a lot about the region of convergence in the previous section, so just refer to that instead.

- Consider the signal $x(t) = 3d^{-2t}u(t) - 2e^{-t}u(t)$. What is its region of convergence? Since the Laplace transform is linear, we can basically just find the region of convergence for both these terms, and find the common ROC to get the ROC for $X(s)$. Doing so, we get:

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{(s+2)(s+1)}$$

Therefore, the region of convergence is $\mathrm{Re}(s) > -1$.

11.5 Properties of Laplace Transform

- These are very similar to the Fourier transform properties. We will denote the relationships as $x(t) \overset{\mathcal{L}}{\Leftrightarrow} X(s)$.
- **Linearity:** Given $ax_1(t)+bx_2(t)$, then the Laplace transform gives $aX_1(s)+bX_2(s)$. The region of convergence contains $R_1 \cap R_2$, but it can be larger. For example, if you take a signal with some limited ROC and consider $x_1(t) - x_1(t)$, then the ROC would now turn into the entire plane!
- **Time shift:** Given $x(t-t_0)$, then we have $e^{-st_0}X(s)$. Very similar to the Fourier transform! The proof is relatively easy, just do a change of variables.
- **S-domain shift:** Given $e^{s_0 t}x(t)$, then this gives $X(s-s_0)$. Again, just do the integral via a change of variables. The ROC will be shifted by $\mathrm{Re}(s_0)$.

  More formally, if we let $s' = s - s_0$ and it has a ROC of R, then the ROC of s is going to be $R + \mathrm{Re}(s_0)$.

- **Time scaling:** For a signal $x(at)$, then the Laplace transform gives $\frac{1}{a}X(s)$. So if we shrink in time domain, then we expand in the s-domain, just like the Fourier transform. The new ROC is now $a \cdot R$. As for the proof:

$$\begin{aligned}\int_{-\infty}^{\infty} x(at)e^{-st}\,dt &= \int_{-\infty}^{\infty} x(\tau)e^{-s\tau/a}\frac{1}{a}\,d\tau \\ &= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau/a}\,d\tau = \frac{1}{|a|}X\left(\frac{s}{a}\right)\end{aligned}$$

  study this proof later.

  There is also the case where $a = -1$, in which case we have $x(-t) \overset{\mathcal{L}}{\leftrightarrow} X(-s)$.

- Given a signal $\cos(\omega_0 t)u(t)$, then we get $X(s) = \frac{s}{s^2+\omega_0^2}$. The strategy is basically the same thing as the Fourier transform, where we split the cosine into complex exponential form. The region of convergence is $\text{Re}(s) > 0$.

  For $\cos(-\omega_0 t)u(-t)$, then we have $X(s) = -\frac{s}{s^2+\omega_0^2}$.

  check this.

- **Conjugation:** Suppose we have $x(t)$ and take the complex conjugate $x^*(t)$, then just like the Fourier transform where $x^*(t)$ corresponds to $X^*(-\omega)$, here we it gives us $X^*(s^*)$.

  This doesn't change the ROC, since the ROC only depends on the real component.

  Isn't this different from the Fourier property, where the input is not conjugated?

  Because the Fourier transform deals with a purely imaginary s, then swapping to a negative is the same as conjugating a general complex value $s \in \mathbb{C}$.

- **Convolution Theorem:** We have $x_1(t) * x_2(t)$ transforms as $X_1(s)X_2(s)$. The region of convergence here is $R_1 \cap R_2$, since we need a place where the Laplace transform always exists.

  The mathematical steps to prove this are nearly identitcal to what we had for the Fourier transform.

- **Differentiation:** The derivative $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ transforms as $sX(s)$. This makes solving differential equations to be much more approachable. The ROC will contain R, but can sometimes include more than just R. An example where R increases is the signal $x(t) = \sin(\omega_0 t)u(t)$.

  We know that $x(t)$ transforms to $X(s) = \frac{\omega_0}{s^2+\omega_0^2}$, then $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ will transform as $\frac{\omega_0 s}{s^2+\omega_0^2}$.

- **Differentiation in s-domain:** The function $-tx(t)$ transforms as $\frac{\mathrm{d}X(s)}{\mathrm{d}s}$, and the region of convergence does not change. In general, for a signal $t^n e^{-at}u(t)$, this will transform as

$$X(s) = \frac{(-1)^n n!}{(s+a)^n}$$

12 More Laplace Transform

12.1 Poles and Zeros of the Laplace Transform

- If we can write $X(s) = \frac{N(s)}{D(s)}$, then the zeros of $N(s)$ are the *zeros* of $X(s)$, and the zeros of $D(s)$ are the *poles* of $X(s)$.
- As an example, consider $x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$. Then, the Laplace transform is given by:

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} = \frac{3s^2-6s+3}{3(s+1)(s-2)} = \frac{(s-1)^2}{(s+1)(s-2)}$$

  so in this factored form, we see $s = 1$ is a zero, and $s = -1, 2$ are the poles.

12.2 Inverse Laplace Transform by Partial Fraction Expansion

- Suppose you have a Laplace transform $X(s) = X(\sigma + j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\}$. As a Fourier transform, this is written as:

$$\int_{-\infty}^{\infty} x(t)e^{-at}e^{-i\omega t}\,dt$$

we already know what the inverse FT is for a given σ, so this allows us to also discover the Laplace transform:

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}X(\sigma + j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

moving the exponential to the right side, we have:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}\,d\omega$$

and recall the definition of $s = \sigma + j\omega$, so we have:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j(\infty)}^{\sigma + j(\infty)} X(s)e^{st}\,ds$$

the factor of j comes from the fact that $ds = j\,d\omega$. So it's almost the same as the Fourier transform, except for the extra factor of j and also the integration bounds.

How do we deal with the integration bounds? Is this going to be a double integral?

Formally, we actually write this as:

$$x(t) = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{\sigma - iT}^{\sigma + iT} e^{st}X(s)\,ds$$

- To evaluate ILT, one strategy we can employ is called *partial fraction expansion*, where we take an expression $X(s)$ into a summation of simpler rational functions. For a rational $X(s)$:

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

There are three categories of $X(s)$:

  - $m < n$: this is called a strictly proper rational function.
  - $m = n$: this is called a proper rational function.
  - $m > n$: this is called an improper rational function.

This is literally partial fraction decomposition, check the notes for an example.

12.3 Solving the Integral DE

- Just to remind us of the differential equation, it's given by an RLC circuit in series, where we found:

$$Ri(t) + \left[\frac{1}{\mathcal{L}}\int_{0^-}^{t} i(t')\,dt' + v_c(0^-)\right] + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = V_0 u(t)$$

In the s-domain, this differential equation is:

$$RI(s) + \frac{1}{\mathcal{L}}\left[\frac{I(s)}{s}\right] + \frac{v_c(0^-)}{s} + L[sI(s) - i(0^-)] = \frac{v_0}{s}$$

Now, we solve for $I(s)$, then take the inverse Laplace transform to find $i(t)$!

$$I(s) = \frac{v_0 - v_c(0^-)}{L\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]}$$

If we plug in values, then we have:

$$I(s) = \frac{3}{(s+a)^2}$$

from which we can then take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^2}\right] = te^{-at}u(t) \implies i(t) = 3te^{-5t}u(t)$$

The reason we get the $i(0^-)$ term comes from the integration and the fact that our $i(t)$ is only nonzero for $t > 0$. So, the only way we guarantee that we take the right integral is to take 0^-.

13 Transfer Function of LTI Systems

- For an LTI system with the standard input and output pairs, we know that since $y(t) = x(t) * h(t)$, then this means that

  $$Y(s) = X(s)H(s)$$

  where $H(s)$ denotes the Laplace transform of the impulse response,

  $$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$

- For an LCCDE of the form:

  $$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

  then we can take the Laplace transform of both sides,

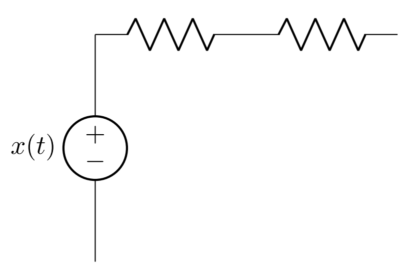
  $$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

  So, this means that:

  $$H(s) = \frac{Y(S)}{X(s)} = \frac{\sum_{k=0}^{\infty} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

13.1 RLC Circuit

- Consider the following circuit:



13.2 Effect of Poles

- For non-repeated poles (as in, the poles aren't degenerate), then the transfer function can be written generically as:

  $$H(s) = \sum_{i=1}^{N} \frac{A_i}{s - a_i}$$

  If the system is causal, then recall that for an LTI system this means that $h(t) = 0$ for $t < 0$

  $$h(t) = \sum_{i=1}^{N} A_i e^{-a_i t} u(t)$$

- If α_i is repeated m times, then the system response will include these terms:

$$t^{m-1}e^{\alpha_i t}, \cdots, te^{\alpha_1}t, e^{\alpha_i t}$$

13.3 Stability of Causal System

- Given a causal system described by $H(s)$, we saw earlier that $Y(s) = H(s)X(s)$. There's also a theorem that says that the system is stable if and only if all poles of $H(s)$ have strictly negative real parts. This is because of two reasons:
  1. This is because if $H(s)$ is rational, then the causality is equivalent to the region of convergence to the right of the rightmost pole (on the real line)
  2. The absolute integrability condition $\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$ means that the imaginary axis is within the ROC.

13.4 Connected LTI Sytems

- For systems connected in series, then $H(s) = H_1(s)H_2(s)$, and in parallel then $H(s) = H_1(s) + H_2(s)$. This is the exact same as the Fourier transform $H(\omega)$ properties.
- For feedback control systems (e.g. a compensator), then we define an error function $E(s) = X(s) - H_2(s)Y(s)$. The subtraction comes from the fact that the feedback is fed as a minus sign. Then, $Y(s) = H_1(s)E(s)$, and solving both equations gives:

$$Y(s) = H_1(s)X(s) - H_1(s)H_2(s)Y(s) \implies H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

13.5 Z-transform

- Very much the same as the Laplace transform, but for discrete-time signals.
- This is very similar to the DTFT formula:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  This is periodic because the signal is discrete in time. The z-transform is the generalized version of this formula, written as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, z \in \mathbb{C}$$

  so really, the only difference is that we swapped $e^{-j\omega}$ for a general complex number z. Because z also has a magnitude term, there's also a corresponding region of convergence for z.

13.6 Pairs

- Given $x[n] = \delta[n]$, then:

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = 1$$

  as long as $z \neq 0$. With that caveat in mind, the ROC is the entire complex plane.
- Given $x[n] = a^n u[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n]z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-(az^{-1})}$$

  the last step simplifies due to geometric series. The condition for this to converge is that $|az^{-1}| < 1$, since that's when the geometric series converges. So this simplifies to $|z| > |a|$, or $|z| > a$ since a is real.

  Visually, this corresponds to a boundary at infinity and one that's a circle at radius a. Consequently, if the unit circle exists within the region of convergence, then we know that the DTFT of the signal exists.

- Given $x[n] = -a^n u[-n-1]$:

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = 1 - \sum_{n=0}^{\infty} (-a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z}$$

  The region of convergence is defined similarly: $|za^{-1}| < 1$, so $|z| < |a|$. This is just a filled circle up to a radius of $|a|$. Of course, the DTFT exists only when the radius of this circle is larger than 1.

- Given $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$, since the z-transform is linear, then:

$$X(z) = \frac{1}{1-2z} + \frac{1}{1+z^{-1}/3} = \cdots = \frac{6z^2-6z-1}{(3z+1)(2z-1)}$$

13.7 Z-transform properties

- The ROC for z-transforms are circles or rings instead of planes, and that's just becuase we're converting $e^{j\omega}$ to z.
- See the lecture slides for a table on the properties.

13.8 Inverse Z-transform

- We can compute the inverse Z-transform using partial fraction expansion. The reason we keep going back to this is because many Z-transforms are characterized by rational functions, so if we can find a way to split the fractions up then we can find the inverse from linearity.

14 2D Image Processing

- To start, we first characterize an image as a grid with M rows and N columns, which we denote as living in $\mathbb{R}^{M\times N}$.
- So, a digital image can be represented as a matrix, generally written as $f[m,n] \in \mathbb{R}^{M\times N}$. The value of the entry can symbolize an intensity (in grayscale).
- With color images, you can define them in RGB colors, then this is a $M \times N \times 3$ dimensional matrix. Here, each pixel $f[m,n]$ is represented by a vector $(r,g,b) \in [0,1]$,
- Images can also be complex-valued, where each entry $f[m,n]$ now has a magnitude and phase.

$$f[m,n] = \text{mag}[m,n]e^{i\text{phase}[m,n]}$$

  We can now plot two different maps: one that just shows magnitude, and another that shows phase.

14.1 Sampling

- We can downsample in two different directions:

$$x[m,n] \to x[2m,3n]$$

  so this means that we down sampled by a factor of 2 in rows and a factor of 3 in columns. So, the number of rows reduces by a factor of 2, and the columns by a factor of 3. This is one way that we can reduce the amount of information contained in an image, so that it may be sent over channels that can't handle that much capacity.

- We can also upsample, in the same way we defined it for one-dimensional signals.

$$x[m,n] = \begin{cases} x\left[\frac{m}{2}, \frac{n}{3}\right] & m \text{ is a multiple of 2 and } n \text{ is a multiple of 3} \\ 0 & \text{else} \end{cases}$$

  so this is the way we increase the length of the signal, by basically sandwiching zeros between our known values. The strategy to do this is to do the rows first, then do the columns.

- The issue with upsampling is that because we fill in the matrix with zeros, it introduces undesirable lines into the final image, which we can fix using interpolation.

14.2 Fourier space

- Recall with discrete 1D sampling that in Fourier space, because upon upsampling we increase the width in time, we decrease the width in frequency. What this means is that over the range $[-\pi, \pi]$, which is what we originally had, we now will have multiple copies of the original image.
- Then, the way to fix this we pass image through an ideal 2D low-pass filter, which will give us back a single copy of the image.
- We can also realize the same thing in "time", by interpolating with a sinc function. Is this convoluiton here?

14.3 2D Convolution

- Here, we have a 2D continuous-time signal $f(x, y)$ and $g(x, y)$, and convolving them together $f * *g$:

$$f * *g = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y')g(x - x', y - y')\, dx',\ dy'$$

  This is exactly the formula that we derived a while back.
- In discrete-time, it's basically the same:

$$f * *g = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k, l]g[m - k, n - l]$$

  Here, just like the other case, m, n is the coordinate, and the summation is over k and l.
- Visually, there's a couple ways to see what this actually is doing:
  1. Flip the second matrix in two dimensions, shift in 2D, then multiply them element-by-element and sum them together. So, we basically overlay everything on top of each other.