Physics 5C Homework

Yutong Du

October 24, 2022

Collaborators

I worked primarily with **Andrew Binder** to complete this homework. Credit to him for the
$\sum P(x) = e^{-m} + me^{-m} + \frac{e^{-m}m^2}{2!} + \dots \quad = e^{-m}\left(1 + m + \frac{m^2}{2} + \dots\right)$

What we notice is that the term in the parentheses is the taylor expansion for e^m, so we have $\sum P(x) = e^{-m}e^m = 1$.

A quick comment on why this result is important: if it had not been the case that $\sum P(x) = 1$, then it would mean that the probability for any of the possibilities to happen is greater than 1, which is nonphysical.

Show that the mean value of the probability distribution is $\langle x \rangle = \sum_{x=0}^{\infty} = m$

Again, it helps to expand here:

$$\begin{aligned}\sum_{x=0}^{\infty} xP(x) &= me^{-m} + 2\frac{m^2e^{-m}}{2!} + \dots + n\frac{m^ne^{-m}}{n!} \\ &= e^{-m}\left(me^{-m} + \frac{2m^2}{2!} + \dots + \frac{nm^n}{n!}\right)\end{aligned}$$

Here we can rewrite the general term $\frac{nm^n}{n!} = m\frac{m^{n-1}}{(n-1)!}$, so we can factor m:

$$\begin{aligned}\sum_{x=0}^{\infty} xP(x) &= me^{-m}\left(1 + m + \frac{m^2}{2} + \dots\right) \\ &= me^{-m}e^m = m\end{aligned}$$

Just like part (a), the term in the parentheses is just the taylor expansion for e^m, which justifies the last step.

The text was too long for me to type it out, but here's my solution:

From the table, we know that the mean is

$$\langle x \rangle = \frac{\sum x_i f_i}{\sum x_i} = \frac{122}{200} = 0.61$$

Since we know that the mean of the distribution must be m, then we know that $m = 0.61$. Now that we have $P(x)$ we need to multiply $P(x)$ by the total frequency in order to get our counts:

x	$P(x)$
0	108.7
1	66.29
2	20.22
3	4.11
4	0.63
5	0.07
6	0

In reality the value we get at 6 is 0.00003, but we can effectively take this as equalling zero. Overall, the theoretical poisson distribution matches the observed values quite well.

Problem 2

This question is about a continuous probability distribution known as the **exponential distribution**. Let x be a continuous random variable that can take any value $x \geq 0$. A quantity is said to be exponentially distributed if it takes values between x and $x + \, \mathrm{d}x$ with probability

$$P(x)dx = Ae^{-x/\lambda} \, \mathrm{d}x$$

where λ and A are constants.

(a) Find the value of A that makes $P(x)$ a well defined continuous probability distribution so that $\int_0^\infty P(x) \, \mathrm{d}x = 1$.

Taking the integral:

$$\begin{aligned} 1 &= \int_0^\infty P(x) \, \mathrm{d}x = A \int_0^\infty e^{-x/\lambda} \, \mathrm{d}x \\ \therefore A &= \frac{1}{\int_0^\infty e^{-x/\lambda} \, \mathrm{d}x} \\ A &= \frac{1}{-\lambda e^{-x/\lambda}\Big|_0^\infty} \end{aligned}$$

Now evaluating the bounds, we have $x = 0 \to -\lambda$, and $x = \infty \to 0$, so we have:

$$A = \frac{1}{\lambda}$$

(b) Show that the mean value of the probability distribution is $\langle x \rangle = \int_0^\infty xP(x) \, \mathrm{d}x = \lambda$.

We compute the integral, using the A that we found before:

$$\begin{aligned} \int_0^\infty xP(x) &= \int_0^\infty x \cdot \frac{1}{\lambda} e^{-x/\lambda} \, \mathrm{d}x \\ &= \frac{1}{\lambda} \int_0^\infty xe^{-x/\lambda} \, \mathrm{d}x \\ &= \frac{1}{\lambda} \left[\underbrace{x(-\lambda)e^{-x/\lambda}\Big|_0^\infty}_{=0} + \lambda e^{-x/\lambda} \, \mathrm{d}x \right] && \text{integration by parts} \\ &= \int_0^\infty e^{-x/\lambda} \, \mathrm{d}x \\ &= \lambda && \text{from part (a)} \end{aligned}$$

(c) Find the variance and standard deviation of this probability distribution.

Variance is defined as $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$. We know that $\langle x \rangle^2 = \lambda^2$ from our previous result, so all we have to do is compute $\langle x^2 \rangle$:

$$
\begin{aligned}
\langle x^2 \rangle &= \int_0^\infty x^2 P(x)\,\mathrm{d}x \\
&= \frac{1}{\lambda}\int_0^\infty x^2 e^{-x/\lambda}\,\mathrm{d}x \\
&= \frac{1}{\lambda}\left[\underbrace{x^2(-\lambda)e^{-x/\lambda}\Big|_0^\infty}_{=0} + \lambda\int_0^\infty 2xe^{-x/\lambda}\,\mathrm{d}x\right] && \text{integration by parts} \\
&= 2\int_0^\infty xe^{-x/\lambda}\,\mathrm{d}x
\end{aligned}
$$

We've computed this integral before in part (a), but with a prefactor of $-\frac{1}{\lambda}$, thus, rearranging, we get

$$
2\int_0^\infty xe^{-x/\lambda}\,\mathrm{d}x = \boxed{2\lambda^2}
$$

Thus, we now complete the problem: $\sigma_x^2 = 2\lambda^2 - \lambda^2 = \lambda^2$, and the standard deivation is $\sqrt{\sigma_x^2} = \lambda$.

Problem 3

If θ is a continuous distribution random variable which is uniformly distributed between 0 and π, write down an expression for $P(\theta)$. Hence find the value of the following averages:

Since θ is distributed between 0 and π, we know that

$$\int_0^\pi P(\theta)\ \mathrm{d}\theta = 1$$

And since the distribution is uniform, then $P(\theta)$ should be a constant for all θ. Therefore we have $P(\theta) = \frac{1}{\pi}$ for all θ.

(a) $\langle\theta\rangle$

$$\begin{aligned}\langle\theta\rangle &= \int_0^\pi \theta P(\theta)\ \mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^\pi \theta\ \mathrm{d}\theta \\ &= \frac{1}{\pi}\frac{\theta^2}{2}\bigg|_0^\pi \\ &= \frac{\pi}{2}\end{aligned}$$

This makes sense, since this value is exactly in between 0 and π.

(b) $\langle\theta - \frac{\pi}{2}\rangle$

By linear transformation we have $\langle\theta - \frac{\pi}{2}\rangle = \langle\theta\rangle - \frac{\pi}{2}$ so using our result from the previous part

$$\langle\theta - \frac{\pi}{2}\rangle = \boxed{0}$$

This makes sense, by linearity.

(c) $\langle\theta^2\rangle$:

Since $\langle f(x)\rangle = \int f(x)P(x)\ \mathrm{d}x$, then we have

$$\begin{aligned}\langle\theta^2\rangle &= \int_0^\pi \theta^2\frac{1}{\pi}\ \mathrm{d}\theta \\ &= \frac{1}{\pi}\frac{\theta^3}{3}\bigg|_0^\pi \\ &= \frac{\pi^2}{3}\end{aligned}$$

This result makes sense since σ_θ^2 is nonzero, so this value must be different than $\langle\theta\rangle^2$. I cannot justify why the precise value is $\frac{\pi^2}{3}$ aside from saying that it comes from the integral.

(d) $\langle \theta^n \rangle$

Same thing as part (c) here:

$$\begin{aligned}\langle \theta^n \rangle &= \int_0^\pi \theta^n \frac{1}{\pi} \, \mathrm{d}\theta \\ &= \frac{1}{\pi} \frac{\theta^{n+1}}{n+1} \\ &= \frac{\pi^n}{n+1}\end{aligned}$$

This is a natural generalization of part (c).

(e) $\langle \cos\theta \rangle$

$$\begin{aligned}\langle \cos\theta \rangle &= \frac{1}{\pi} \int_0^\pi \cos\theta \, \mathrm{d}\theta \\ &= \frac{1}{\pi} \sin\theta \Big|_0^\pi \\ &= 0\end{aligned}$$

This makes sense since on the interval from 0 to π, 0 is the average value of cosine over that interval.

(f) $\langle \sin\theta \rangle$

$$\begin{aligned}\langle \sin\theta \rangle &= \frac{1}{\pi} \int_0^\pi \sin\theta \, \mathrm{d}\theta \\ &= \frac{1}{\pi} (-\cos\theta) \Big|_0^\pi \\ &= -\frac{1}{\pi}(-1-1) \\ &= \frac{2}{\pi}\end{aligned}$$

This result makes sense in more or less the same way the previous result makes sense. The mean value of $\sin\theta$ over $[0, \pi]$ is nonzero, and so we expect a nonzero result.

(g) $\langle |\cos\theta| \rangle$

$$\begin{aligned}\langle \sin\theta \rangle &= \frac{1}{\pi} \int_0^\pi |\cos\theta| \, \mathrm{d}\theta \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos\theta \, \mathrm{d}\theta \\ &= \frac{2}{\pi} \sin\theta \Big|_0^{\frac{\pi}{2}} \\ &= \frac{2}{\pi}\end{aligned}$$

We expect this result to be the same as that of $\sin\theta$, since the curves are the same, since the negative part of $\cos\theta$ is reflected about the x axis, becoming positive.

(h) $\langle\cos^2\theta\rangle$

$$\begin{aligned}\langle\cos^2\theta\rangle &= \frac{1}{\pi}\int_0^\pi \cos^2\theta \ \mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^\pi \frac{\cos(2\theta)+1}{2} \ \mathrm{d}\theta \\ &= \frac{1}{2\pi}\left[\frac{1}{2}\sin(2\theta)+\theta\right]_0^\pi \\ &= \frac{1}{2}\end{aligned}$$

This result makes sense when combined with part (i) and (j), since we should get that $\langle\cos^2\theta+\sin^2\theta\rangle = \langle\cos^2\theta\rangle + \langle\sin^2\theta\rangle = 1$.

(i) $\langle\sin^2\theta\rangle$

$$\begin{aligned}\langle\sin^2\theta\rangle &= \frac{1}{\pi}\int_0^\pi \sin^2\theta \ \mathrm{d}\theta \\ &= \frac{1}{\pi}\left(\int_0^\pi 1 \ \mathrm{d}\theta - \int_0^\pi \cos^2\theta \ \mathrm{d}\theta\right) \\ &= \frac{1}{\pi}\left(\pi - \frac{\pi}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

This result makes sense, since we know that $\langle\cos^2\theta+\sin^2\theta\rangle = 1$, and knowing that $\langle\cos^2\theta\rangle = \frac{1}{2}$, it must follow that $\langle\sin^2\theta\rangle = \frac{1}{2}$.

(j) $\langle\cos^2\theta+\sin^2\theta\rangle$

Since this quantity is always equal to 1, then $\langle\cos^2\ \theta+\sin^2\theta\rangle = 1$. This makes sense, since this value is always equal to 1 over the interval $[0,\pi]$.

Problem 4

In experimental physics, it is important to repeat measurements. Assuming that errors are random, show that if the error in making a single measurement of quantity X is Δ, the error obtained after using n measurements $\Delta/\sqrt{n}$

Let $Y = X_1 + \cdots + X_n$. To calculate the error after n measurements, we calculate σ_Y. To do that, we calculate variance:

$$\sigma_Y^2 = \langle Y^2 \rangle - \langle Y \rangle^2$$

We can calculate each of the terms on the right separately:

$$\begin{aligned}
\langle Y^2 \rangle &= \langle (X_1 + \cdots + X_n)^2 \rangle \\
&= \langle X_1 \rangle^2 + \langle X_2 \rangle^2 + \cdots + \langle X_n \rangle^2 + \langle X_1 \rangle \langle X_2 \rangle + (\text{very many cross terms}) \\
\langle Y \rangle^2 &= \langle \sum_n X_i \rangle^2 \\
&= \langle X_1^2 + X_2^2 + \cdots + X_n^2 + X_1 X_2 + \text{more cross terms} \rangle^2 \\
&= \langle X_1 \rangle^2 + \langle X_2 \rangle^2 + \cdots + \langle X_1 \rangle \langle X_2 \rangle + \ldots
\end{aligned}$$

Notice that the cross terms in $\langle Y^2 \rangle$ and $\langle Y \rangle^2$ cancel out pefectly after subtraction. Likewise, notice that the remaining terms can be grouped as $\langle X_i^2 \rangle - \langle X_i \rangle^2 = \Delta^2$, and we have n of these terms so we obtain:

$$\sigma_Y^2 = n\sigma_X^2$$

Now note that since $Y = X_1 + \cdots + X_n$, we need to divide this by n to obtain the quantity we want. Since each term is then divided by n, we simply have an overall factor of $\frac{1}{n^2}$ that is factored out. Thus:

$$\sigma_{Y/n}^2 = \frac{n\sigma_X^2}{n^2}$$

And thus

$$\sigma_{Y/n} = \frac{\sigma_X}{\sqrt{n}} = \frac{\Delta}{\sqrt{n}}$$

Problem 5

A system comprises of N states, which can have energy 0 or Δ. Show that the number of ways $\Omega(E)$ of arranging the total system to have energy $E = r\Delta$ (where r is an integer) is given by

$$\Omega(E) = \frac{N!}{r!(N-r)!}$$

Now remove a small amount of energy $s\Delta$ from the system, where $s \ll r$. Show that

$$\Omega(E-\epsilon) \approx \Omega(E)\frac{r^s}{(N-r)^s}$$

and hence show that the system has temperature T given by

$$\frac{1}{k_B T} = \frac{1}{\Delta}\ln\left(\frac{N-r}{r}\right)$$

Sketch $k_B T$ as a function of r from $r=0$ to $r=N$ and explain the result.

We have N particles, and we want to select r of them. The number of ways we can do this is $\frac{N!}{(N-r)!}$. However, since this selection should not care about the order in which the particles are arranged, we need to divide by a further $r!$ in order to remove this excessive counting. Thus, our formula becomes:

$$\Omega(E) = \frac{N!}{r!(N-r)!}$$

For the second part of the problem: we make the argument that in the case when these values get large, we can simplify the factorial to a simple exponential. Specifically, we can approximate $n! \approx n^n$, arguing that the cross terms are substantially smaller than the leading term. Thus,

$$\Omega(E) \approx \frac{N^N}{r^r(N-r)^{N-r}}$$

Now we remove s from r, which represents removing $s\Delta$ from the energy:

$$\Omega(E-\epsilon) = \frac{N!}{(r-s)!(N-(r-s))!} = \frac{N!}{(r-s)!(N-r+s)!}$$

Applying our exponential approximation we get:

$$\Omega(E-\epsilon) = \frac{N^N}{(r-s)^{r-s}(N-r+s)^{N-r+s}}$$

Notice that in the denominator of this expression, since $s \ll r$, then $r - s \approx r$. Thus, we have r^{r-s} as a simplification. Now we compare this term with that in $\Omega(E)$, and we notice that if we multiply by r^s then we get r^{r-s} in the denominator. A similar simplification can be done with $(N-r+s)^{N-r+s}$, where this can be simplified to $(N-r)^{N-r+s}$, so in this case it makes sense to divide by an extra term of $(N-r)^s$ to reach equality. Thus, we have

$$\Omega(E-\epsilon) \approx \Omega(E)\frac{r^s}{(N-r)^s}$$

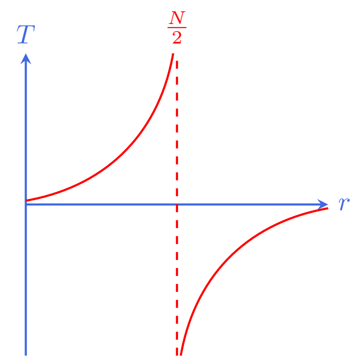
To obtain the final expression, we use the old-fashioned definition of a limit:

$$\frac{\partial \ln \Omega(E)}{\partial E} = \frac{1}{k_B T} = \lim_{\epsilon \to 0} \frac{\ln \Omega(E) - \ln \Omega(E-\epsilon)}{h}$$

So simplifying the right side:

$$\begin{aligned}
\frac{1}{k_B T} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \ln \left(\frac{\Omega(E)}{\Omega(E-\epsilon)} \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{s\Delta} \ln \left(\frac{(N-r)^s}{r^s} \right) \\
&= \lim_{s \to 0} \frac{1}{\Delta} \left(\frac{(N-r)^s}{r^s} \right) \\
&= \frac{1}{\Delta} \left(\frac{(N-r)}{r} \right)
\end{aligned}$$

And thus the result is proven. Sketching $k_B T$ from $r = 0$ to $r = N$, we get:



Thanks to Andrew Binder for sharing me his