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HW 05 **Electromagnetism and Optics II** March 6, 2025

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Problem 1

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$)) is $(2/\sigma)\sqrt{\epsilon}/\mu$ (independent of the frequency). Find the skin depth (in meters) for (pure) water. (Use the static values of ϵ, μ and σ; your answers will be valid only at relatively low frequencies.)

Solution: The skin depth is given by:

$$d = \frac{1}{\kappa} = \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{-1/2}$$

When $\sigma \ll \omega\epsilon$, then we can make a Taylor approximation and use $(1+x)^n \approx 1 + nx$, so:

$$\begin{aligned} d &= \frac{1}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\left[1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 - 1\right]^{-1/2} \\ &= \frac{1}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\left[\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{-1/2} \\ &= \frac{1}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\sqrt{2}\left(\frac{\epsilon\omega}{\sigma}\right) \\ &= \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}} \end{aligned}$$

as desired. For water, we have $\epsilon = 80.2\epsilon_0$, $\mu = 1.256 \times 10^6$ and $\sigma = \frac{1}{8.3\times 10^3}$. Plugging this in to Mathematica I get 8662 m. □

(b) Show that the skin depth in a good conductor ($\sigma \gg \omega\epsilon$) is $\lambda/2\pi$ (where λ is the wavelength *in the conductor*). Find the skin depth (in nanomoeters) for a typical metal ($\sigma \approx 10^7(\Omega\cdot\text{m})^{-1}$) in the visible range ($\omega \approx 10^{15}/\text{s}$), assuming $\epsilon \approx \epsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

Solution: First, we solve for d in a good conductor. Here, since $\sigma \gg \omega\epsilon$, then the $\frac{\sigma}{\omega\epsilon}$ term dominates so we have:

$$d = \frac{1}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\left(\frac{\epsilon\omega}{\sigma}\right)$$

Now, $\lambda = \frac{2\pi}{k}$, and k is defined as:

$$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{1/2}$$

This is very similar to κ, and under the approximation of a good conductor they become equal (the ± 1 outside the square root becomes negligible). Therefore, the wavelength is:

$$\lambda = \frac{2\pi}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\left(\frac{\epsilon\omega}{\sigma}\right) = 2\pi d$$

Thus, we get $d = \frac{\lambda}{2\pi}$, as desired. For a metal, I get 3.362×10^{-9} m. This shows why metals are opaque, as the skin depth is on the order of nanometers. □

(c) Show that in a good conductor the magnetic field lags the electric field by $45°$, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b).

Solution: In a good conductor, as we've seen in the previous example, we have $k \approx \kappa$, and since $\phi = \tan^{-1}\left(\frac{\kappa}{k}\right)$, then we have $\phi \approx \tan^{-1}(1) = 45°$. The ratio of the amplitudes is given by equation 9.139, which gives:

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$

Using the approximation of a good conductor, this simplifies to:

$$\frac{K}{\omega} = \sqrt{\epsilon\mu\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\sigma\mu}{\omega}}$$

Plugging in the values for the "typical metal", we get 1.12×10^{-7}. $\square$

Problem 2

(a) Calculate the (time-averaged) energy density of an electromagnetic plane wave in a conducting medium (Eq. 9.140). Show that the magnetic contribution always dominates. [*Answer:* $(k^2/2\mu\omega^2)E_0^2 e^{-2\kappa z}$.]

Solution: The total energy is given by:

$$\int_{\mathcal{V}} \frac{1}{2}\epsilon_0 |\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2 \, d\tau$$

so the energy density is just the integrand. The real part of the waves (by Equation 9.138) is given by

$$\mathbf{E}(z,t) = E_0 e^{-\kappa z}\cos(kz - \omega t + \delta_E)\hat{\mathbf{x}}$$
$$\mathbf{B}(z,t) = B_0 e^{-\kappa z}\cos(kz - \omega t + \delta_E + \phi)\hat{\mathbf{y}} = \frac{K}{\omega}E_0 e^{-\kappa z}\cos(kz - \omega t + \delta_E + \phi)\hat{\mathbf{y}}$$

Therefore, the energy density is given by:

$$U = \frac{1}{2}\left(\epsilon E_0^2 e^{-2\kappa z}\cos^2(kz - \omega t + \delta_E) + \frac{K^2}{\omega^2}E_0^2 e^{-\kappa z}\cos^2(kz - \omega t + \delta_E + \phi)\right)$$

Averaged over time, the $\cos^2(x)$ terms average out to $\frac{1}{2}$, so we have

$$U = \frac{1}{2}\left(\epsilon E_0^2 e^{-2\kappa z}\cdot\frac{1}{2} + \frac{1}{\mu}\frac{K^2}{\omega^2}E_0^2 e^{-2\kappa z}\cdot\frac{1}{2}\right)$$

Factoring out terms and also replacing K^2 in terms of μ, ω and other terms:

$$U = \frac{1}{4}E_0^2 e^{-2\kappa z}\left(\epsilon + \frac{1}{\mu}\frac{1}{\omega^2}\omega^2\epsilon\mu\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right) = \frac{1}{4}E_0^2 e^{-2\kappa z}\epsilon\left(1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right) \tag{1}$$

Finally, we note that the term in parentheses can be rewritten as $\frac{2k^2}{\omega^2\mu\epsilon}$, so finally:

$$U = \frac{k^2}{2\omega^2\mu}E_0^2 e^{-2\kappa z}$$

exactly as hinted at by the answer. Clearly the magnetic contribution dominates – looking at 1, we see that the contribution is captured by the term in parentheses:

$$1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}$$

The second term represents the magnetic contribution, which is far greater than the electric contribution represented by the first term. □

(b) Show that the intensity is $(k/2\mu\omega)E_0^2 e^{-2\kappa z}$.

Solution: The intensity is given as the average power per unit area, so we have $I = \langle \mathbf{S}\cdot\hat{\mathbf{z}}\rangle$. The Poynting vector is given by:

$$\mathbf{S} = \frac{1}{\mu}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu}\left(E_0\frac{K}{\omega}E_0 e^{-2\kappa z}\cos(kz - \omega t + \delta_E)\cos(kz - \omega t + \delta_E + \phi)\hat{\mathbf{z}}\right)$$

So the intensity is:

$$I = \frac{E_0^2}{\mu}\frac{K}{\omega}e^{-2\kappa z}\langle\cos(kz - \omega t + \delta_E)\cos(kz - \omega t + \delta_E + \phi)\rangle$$

The time average is calculated as:

$$\langle f\rangle = \frac{1}{T}\int_0^T f(t)\,dt$$

so in our case:

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(kz - \omega t + \delta_E)\cos(kz - \omega t + \delta_E + \phi)\, dt$$

Plugging this into Wolfram, we get:

$$\langle \cos(kz - \omega t + \delta_E)\cos(kz - \omega t + \delta_E + \phi) \rangle = \frac{\cos\phi}{2}$$

Thus, we have:

$$I = \frac{1}{2}\frac{E_0^2}{\mu}\frac{K}{\omega}e^{-2\kappa z}\cos\phi$$

Finally, we use the fact that $K\cos\phi = k$, so therefore:

$$I = \frac{k}{2\mu\omega}E_0^2 e^{-2\kappa z}$$

as desired. □

Problem 3

Calculate the reflection coefficient for light at an air-to-silver interface ($\mu_1 = \mu_2 = \mu_0, \epsilon_1 = \epsilon_0, \sigma = 6 \times 10^7 (\Omega \cdot \mathrm{m})^{-1}$), at optical frequencies ($\omega = 4 \times 10^{15}/\mathrm{s}$).

Solution: This problem literally just comes down to plugging in numbers. From equation 9.149, we know that

$$E_{0R} = \left(\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) E_{0I}$$

so the reflection coefficient is given by:

$$R = \left| \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right|^2$$

where $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 v_2} \tilde{k}_2$. Because $\mu_2 = \mu_1 = \mu_0$ and $\epsilon_1 = \epsilon_0$, then we have:

$$\tilde{\beta} = \frac{1}{\sqrt{\mu_1 \epsilon_1} \omega} \tilde{k}_2$$

We have $\tilde{k} = k + i\kappa$, with k and κ as they are in equation 9.128, so from here it's just an exercise of plugging in numbers. I did this in Wolfram, which gave me a reflection of 0.933. Here's the code I used, since I feel like I really haven't shown enough work for this problem:

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In[14]:= σ = 6 * 10^7;
         ϵ = 8.83 * 10^-12;
         μ = 1.256 * 10^-6;
         ω = 4 * 10^15;

In[18]:= k = ω Sqrt[ϵ μ / 2] (Sqrt[1 + (σ / (ϵ ω))^2] + 1)^(1/2);
         ϰ = ω Sqrt[ϵ μ / 2] (Sqrt[1 + (σ / (ϵ ω))^2] - 1)^(1/2);

In[20]:= β = 1 / (Sqrt[μ ϵ] ω) (k + I ϰ);

In[21]:= Abs[(1 - β) / (1 + β)]^2

Out[21]= 0.933671
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□