

Collaborators

I worked with **Andrew Binder**, **Teja Nivarthi**, **Nathan Song**, **Christine Zhang** and **Nikhil Maserang** to complete this homework.

Problem 1

Show that the quadrupole term in the multipole expansion can be written as

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j Q_{ij}$$

where

$$Q_{ij} = \int \left[\frac{3}{2} r'_i r'_j - \frac{1}{2} (r')^2 \delta_{ij} \right] \rho(\mathbf{r}') d\tau'$$

Note that Q_{ij} is a two-rank tensor, so it is possible to express it as a matrix. Also show that Q_{ij} is traceless.

Solution: Here we will compare this expression to the quadrupole expansion term we are normally used to show equivalence. Recall that the actual way we write the quadrupole term is:

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int (r')^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) \rho(r') d\tau'$$

Comparing this with our expression, it's clear that we only need to show that:

$$\sum_{i,j=1}^3 \hat{r}_i \hat{r}_j \int \left(\frac{3}{2} r'_i r'_j - \frac{1}{2} (r')^2 \delta_{ij} \right) \rho(r') d\tau' = \int (r')^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) \rho(r') d\tau'$$

Moving the integral in the sum, it means we need to show that:

$$\sum_{i,j=1}^3 \hat{r}_i \hat{r}_j \left(\frac{3}{2} r'_i r'_j - \frac{1}{2} (r')^2 \delta_{ij} \right) = (r')^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) \quad (1)$$

Looking at the first term in particular, we want to show that:

$$\frac{3}{2} \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j r'_i r'_j = (r')^2 \cos^2 \alpha$$

We can then split the left hand side into $\frac{3}{2} (\sum_i \hat{r}_i r'_i \sum_j \hat{r}_j r'_j)$ and each term $\sum_i \hat{r}_i r'_i = r' \cos \alpha$, then we get that the total term:

$$\frac{3}{2} \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j r'_i r'_j = \frac{3}{2} \sum_{i=1}^3 \hat{r}_i r'_i \sum_{j=1}^3 \hat{r}_j r'_j = \frac{3}{2} (r' \cos \alpha) (r' \cos \alpha) = \frac{3}{2} (r')^2 \cos^2 \alpha$$

Now for the second term, we can see that:

$$\frac{1}{2} \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j (r')^2 \delta_{ij} = \frac{1}{2} \sum_{i=1}^3 (\hat{r}_i)^2 (r')^2 = \frac{1}{2} (r')^2$$

So therefore, the two terms on either side of equation 1 are the same, and thus this is an equivalent way to write the quadrupole term. Furthermore, to show that Q_{ij} is traceless, notice that when $i = j$, then the term we get is:

$$\sum_{i=1}^3 \frac{3}{2}(r_i)^2 - \frac{1}{2}(r')^2 = \frac{3}{2}(r')^2 - \frac{3}{2}(r')^2 = 0$$

Since the diagonal elements are zero, Q_{ij} is traceless. □

Problem 2

Show that the quadrupole moment Q_{ij} is independent of origin if the monopole and dipole moments both vanish.

Solution: If the monopole and dipole terms vanish, this means that $p = 0$ and $Q = 0$. Suppose we picked another origin point $O' = O + \mathbf{R}$. Then, we have the following:

$$\begin{aligned}\mathbf{r}_{o'} &= \mathbf{r}_o + \mathbf{R} \\ \mathbf{r}_{i,o'} &= \mathbf{r}_{i,o} + \mathbf{R}_i \\ \mathbf{r}_{j,o'} &= \mathbf{r}_{j,o} + \mathbf{R}_j\end{aligned}$$

Now we can expand the quadrupole term:

$$\begin{aligned}Q'_{ij} &= \int \left[\frac{3}{2}(r_{i,o} + R_i)(r_{j,o} + R_j) - \frac{1}{2}(\mathbf{r}_o + \mathbf{R})^2 \delta_{ij} \right] \rho(\mathbf{r}_o + \mathbf{R}) d\tau' \\ &= \int \left[\frac{3}{2}r_{i,o}r_{j,o} - \frac{1}{2}(\mathbf{r}_o)^2 \right] \rho d\tau + \frac{3}{2}R_j \int r_{i,o} \rho d\tau - \mathbf{r}_o \cdot \mathbf{R} \delta_{ij} \int \rho d\tau \\ &\quad + \frac{3}{2}R_i \int r_{j,o} \rho d\tau - \int \mathbf{r}_o \cdot \mathbf{R} \rho d\tau + \int \left[\frac{3}{2}R_i R_j - \frac{1}{2}|\mathbf{R}|^2 \delta_{ij} \right] \rho d\tau \\ &= Q_{ij} + \frac{3}{2}R_j p_i + \frac{3}{2}R_i p_j - 2\delta_{ij} \mathbf{r}_o \cdot \mathbf{R} Q + \left[\frac{3}{2}R_i R_j - \frac{1}{2}|\mathbf{R}|^2 \delta_{ij} \right] Q\end{aligned}$$

And since $Q = 0$ and $p = 0$, then all the terms vanish except the first term, implying that

$$Q'_{ij} = Q_{ij}$$

And thus the quadrupole moment is independent of the origin. □

Problem 3

A circular disk has a radius R and uniform charge density σ . The disk is lying on the $x - y$ plane, with its center fixed at the origin. Find the potential $V(\mathbf{r})$ of the disk for large r , up to the $1/r^3$ term.

Solution: This problem essentially asks for the multipole expansion for this disk of radius R . Considering the monopole term, we have:

$$V_{\text{mon}}(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} = \frac{1}{4\pi\epsilon_0} \frac{\pi R^2 \sigma}{r}$$

For the dipole term, we have:

$$V_{\text{dip}}(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int r' \cos \alpha \rho(r') d\tau'$$

Since the origin is placed at the center of the disc, then this integral becomes:

$$\begin{aligned} \int r' \cos \alpha \sigma(r') dA &= \sigma \cos \alpha \int_0^R \int_0^{2\pi} r' r' d\phi dr' \\ &= 2\pi \sigma \cos \alpha \frac{R^3}{3} \\ &= \frac{2\pi \sigma \cos \alpha R^3}{3} \end{aligned}$$

Giving us:

$$V_{\text{dip}}(r) = \frac{1}{4\pi\epsilon_0} \frac{2\pi \sigma \cos \alpha R^2}{3r^2}$$

Similarly for the quadrupole, we need to solve:

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int (r')^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) \sigma(r') da$$

Again, computing the integral:

$$\begin{aligned} \int (r')^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) \sigma(r') dr' &= \sigma \left[\frac{3}{2} \cos^2 \alpha \int (r')^2 r' d\phi dr' - \frac{1}{2} \sigma \int (r')^2 r' d\phi dr' \right] \\ &= \frac{3\pi \sigma \cos^2 \alpha R^4}{4} - \frac{\pi \sigma R^4}{4} \end{aligned}$$

So therefore the quadrupole term becomes:

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\pi \sigma \cos^2 \alpha R^4}{4r^3} - \frac{\pi \sigma R^4}{4r^3} \right)$$

Finally, since α is defined as the angle between the plane and the radius vector \mathbf{r} , we can write $\alpha = \frac{\pi}{2} - \theta$, therefore we can express the potential in terms of the natural spherical coordinates $V(r, \theta)$. Combining all three terms, we get:

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{\pi R^2 \sigma}{r} + \frac{1}{4\pi\epsilon_0} \frac{2\pi \sigma}{3r^2} \cos \left(\frac{\pi}{2} - \theta \right) + \frac{1}{4\pi\epsilon_0} \left(\frac{3\pi \sigma R^4}{4r^3} \cos^2 \left(\frac{\pi}{2} - \theta \right) - \frac{\pi \sigma R^4}{4r^3} \right)$$

We can cancel off the π terms:

$$V(r, \theta) = \frac{R^2 \sigma}{4\epsilon_0 r} + \frac{\sigma}{6\epsilon_0 r^2} \cos \left(\frac{\pi}{2} - \theta \right) + \frac{1}{4\epsilon_0 r^3} \left(\frac{3\sigma R^4}{4} \cos^2 \left(\frac{\pi}{2} - \theta \right) - \frac{\sigma R^4}{4} \right)$$

□