Collaborators

I worked on this homework with **Andrew Binder, Nikhil Maserang, Teja Nivarthi** and **Adarsh Iyer**.

Problem 1

Assume that gases behave according to a law given by $pV = f(T)$ where $f(T)$ is a function of temperature. Show that this implies

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{1}{V}\frac{\mathrm{d}f}{\mathrm{d}T}$$
$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{p}\frac{\mathrm{d}f}{\mathrm{d}T}$$

Show also that

$$\left(\frac{\partial Q}{\partial V}\right)_p = C_p\left(\frac{\partial T}{\partial V}\right)_p$$
$$\left(\frac{\partial Q}{\partial p}\right)_V = C_V\left(\frac{\partial T}{\partial p}\right)_V$$

In an adiabatic change, we have that

$$\mathrm{d}Q = \left(\frac{\partial Q}{\partial p}\right)_V \mathrm{d}p + \left(\frac{\partial Q}{\partial V}\right)_p \mathrm{d}V = 0$$

Hence show that pV^γ is a constant.

We can rearrange to get $p = \frac{f(T)}{V}$ and $v = \frac{f(T)}{p}$, so differentiating each we get:

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{1}{V}\frac{\mathrm{d}f}{\mathrm{d}t}$$
$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{p}\frac{\mathrm{d}f}{\mathrm{d}T}$$

We know that $C_p = \left(\frac{\partial Q}{\partial T}\right)_p$ and $C_v = \left(\frac{\partial Q}{\partial T}\right)_V$ so we can apply chain rule:

$$C_p \left(\frac{\partial T}{\partial V}\right)_p = \left(\frac{\partial Q}{\partial T}\right)_p \cdot \left(\frac{\partial T}{\partial V}\right)_p = \left(\frac{\partial Q}{\partial V}\right)_p$$
$$C_V \left(\frac{\partial T}{\partial p}\right)_V = \left(\frac{\partial Q}{\partial T}\right)_V \cdot \left(\frac{\partial T}{\partial p}\right)_V = \left(\frac{\partial Q}{\partial p}\right)_V$$

And so we've confirmed the two subsequent equations.

In an adiabatic change, we have:

$$\begin{aligned}
C_V \left(\frac{\partial T}{\partial P}\right)_V \frac{\mathrm{d}p}{\mathrm{d}V} &= -C_p \left(\frac{\partial T}{\partial v}\right)_p \\
\frac{\mathrm{d}p}{\mathrm{d}V} &= -\frac{C_p}{C_v} \left(\frac{\partial p}{\partial V}\right)_V \cdot \left(\frac{\partial T}{\partial V}\right)_p \\
&= -\gamma \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_p \\
&= -\gamma \left(\frac{1}{v}\frac{\mathrm{d}f}{\mathrm{d}T}\right) \left(p\frac{1}{\mathrm{d}f/\,\mathrm{d}T}\right) \\
\therefore -\gamma &= \frac{V}{p}\frac{\mathrm{d}p}{\mathrm{d}V}
\end{aligned}$$

So now we can do simple separation of variables:

$$\begin{aligned}
-\frac{\gamma}{V}\,\mathrm{d}V &= \frac{1}{p}\,\mathrm{d}p \\
-\gamma \ln V + c &= \ln p \\
c &= \ln pV^\gamma
\end{aligned}$$

And so pV^γ is a constant.

Problem 2

Two thermally insulated cylinders, A and B, of equal volume, both equipoped with pistons, are connected by a valve. Initially A has its piston fully withdrawn and contains a perfect monoatomic gas at temperature T, while B has its piston fully inserted, and the valve is closed. Calculate the final temperature of the gas after the following operations, which each start with the same initial arrangement. The thermal capacity of the cylinders is to be ignored.

(a) The valve is fully opened and the gas slowly drawn to B by pulling out the piston B; piston A remains stationary

We use the ideal gas equation:

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2} \implies T_2 = \frac{P_2 V_2}{P_1 V_1} T_1$$

Now our goal is to calculate thsi ratio $\frac{P_2 V_2}{P_1 V_1}$. We know for an adiabatic chagne, pV^γ is a constant, so we know:

$$\begin{aligned}\frac{p_2 V_2^\gamma}{p_1 V_1^\gamma} &= 1 \\ \frac{p_2}{p_1} &= \frac{V_1^\gamma}{V_2^\gamma} \\ \therefore T_2 &= \frac{V_1^\gamma V_2}{V_2^\gamma V_1} = \frac{V_1^{\gamma-1}}{V_2^\gamma - 1}\end{aligned}$$

Since piston A remains stationary, then we know that $V_2 = 2V_1$, so plugging this in (as well as $\gamma = \frac{5}{3}$) it yields:

$$T_2 = \frac{T_1}{2^{2/3}}$$

(b) Piston B is fully withdrawn and the valve is opened slightly; the gas is then driven as far as it will go into B by pushing home piston A at such a rate that the pressure in A remains constant: the cylinders are in thermal contact.

Let V_a be the final volume of chamber A and V be the initial volume of chamber A. Since the whole system is thermally insulated, this process can be considered adiabatic. This means that $\mathrm{d}Q = 0$, and thus

$$\Delta U = -p\Delta V = -p(V - V_a) = \frac{3}{2} nR(T_2 - T_1)$$

From the ideal gas equation, we have $V = nR\frac{T_i}{p}$ and $V_a = \frac{T_f - T_i}{p}$. Therefore,

$$\begin{aligned}
\frac{3}{2}nR(T_f - T_i) &= pnR\left(\frac{T_i}{p} - \frac{T_f - T_i}{p}\right) \\
3T_f - 3T_i &= 4T_i - 2T_f \\
5T_f &= 7T_i \\
\therefore T_f &= \frac{7}{5}T_i
\end{aligned}$$

Problem 3

In Rüchhardt's method of measuring γ, illustrated in Fig. 12.2, a ball of mass $4m$ is placed snugly inside a tube (cross-sectional area A) connected to a container of gas (volume V). The pressure p of the gas inside the container is slightly greater than the atmospheric pressure p_0 because of the downwards force of the ball, so that

$$p = p_0 + \frac{mg}{A}$$

Show that if the ball is given a slight downwards displacement, it will undergo simple harmonic motion with period τ given by

$$\tau = 2\pi\sqrt{\frac{mV}{\gamma p A^2}}$$

Since the system can be treated as adiabatic (there is no heat flow in the system), then

$$\frac{\mathrm{d}p}{p} = -\gamma\frac{\mathrm{d}V}{V}$$

When the ball oscillates harmonically, we can let $\mathrm{d}V = A\ \mathrm{d}x$, which will generate a force: $m\ddot{x} = A\ \mathrm{d}p$ so we get:

$$m\ddot{x} + kx = 0 \implies \ddot{x} + \frac{k}{m}x = 0$$

where $k = \left(\frac{\gamma p A^2}{V}\right)$, so we let $\omega^2 = k$ giving us:

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{mV}{\gamma p A^2}}$$

As desired. The gravitational potential energy of the system is equal to the potential energy of the system, and thus:

$$U = \frac{1}{2}k\left(\frac{L}{2}\right)^2 = \frac{\gamma p A^2 L^2}{8V}$$

Problem 4

Show that the efficiency of the standard Otto cycle (shown in Fig. 13.12) is $1 - r^{1-\gamma}$, where $r = V_1/V_2$ is the compression ratio. The **Otto cycle** is the four-stroke cycle in internal combustion engines in cars, lorries, and electrical generators.

From the diagram, $Q_1 = C_V(T_3 - T_2)$ and $Q_2 = C_V(T_4 - T_1)$. Furthermore, on the regions where the process is adiabatic, $TV^{\gamma-1}$ is a constant. We also have the relationsihp that $V_2 = V_3$ and $V_1 = V_4$ so we have:

$$(T_3 - T_2)V_2^{\gamma-1} = (T_4 - T_1)V_1^{\gamma-1}$$

Now notice that we can write:

$$\frac{T_4 - T_1}{T_3 - T_2} = \frac{V_2^{\gamma-1}}{V_1^{\gamma-1}} = \frac{V_1^{1-\gamma}}{V_2^{1-\gamma}}$$

Now we have sufficient knowledge to calculate efficiency $\eta = 1 - \frac{Q_2}{Q_1}$:

$$\eta = 1 - \frac{C_V(T_4 - T_1)}{C_V(T_3 - T_2)} = 1 - \frac{T_4 - T_1}{T_3 - T_2} = 1 - \left(\frac{V_1}{V_2}\right)^{1-\gamma}$$

As desired. ■

Problem 5

An ideal air conditioner operating on a Carnot cycle absorbs heat Q_2 from a house at temperature T_2 and discharges Q_1 to the outside at temperature T_1, consuming electrical energy E. Heat leakage into the house follows Newton's law,

$$Q = A[T_1 - T_2]$$

where A is a constant. Derive an expression for T_2 in terms of T_1, E, and A for continuous operation when the steady state has been reached.

The air conditioner is controlled by a thermostat. the system is designed so that with the thermostat set at 20°C and outside temperature 30°C the system operates at 30% of the maximum electrical energy input. Find the highest outside temperature for which the house may be maintained inside at 20°C.

We can equate the energy change as $E = -Q = A[T_2 - T_1]$ so rearranging for T_2:

$$T_2 = \frac{AT_1 + E}{A} = T_1 + \frac{E}{A}$$

Now the system is operating at 30% efficiency, so we have $E = 0.3E_{max}$:

$$293.15 = 303.15 + \frac{0.3E_{max}}{A}$$
$$E_{max} = \frac{(293 - 303)A}{0.3}$$

So the highest outside temperature difference that is allowable is 33 degrees, so this gives us: $20°\mathrm{C} + 33°\mathrm{C} = 53°\mathrm{C}$ as the highest temperature.

Problem 6

Two identical bodies of constant heat capacity C_p at temperatures T_1 and T_2 respectively are used as reservoirs for a heat engine. If the bodies remain at constant pressure, show that the amount of work obtainable is

$$W = C_p(T_1 + T_2 - 2T_f)$$

where T_1 is the final temperature attained by both bodies. Show that if the most efficient engine is used, then $T_f^2 = T_1 T_2$.

We use a Carnot engine, since it is the most efficient engine possible. The total amount of energy in the first body is

$$E_1 = C_p(T_f - T_1)$$

and similarly

$$E_2 = C_p(T_2 - T_f)$$

Therefore, the total amount of work possible would be the difference between the two "values":

$$W = E_2 - E_1 = C_p(T_1 + T_2 - 2T_1)$$

Clausius' theorem states that that $\oint \frac{\mathrm{d}Q}{T} = 0$. so therefore we get:

$$\begin{aligned}
0 &= \int_{T_1}^{T_f} C_p \frac{\mathrm{d}T}{T} + \int_{T_2}^{T_f} C_p \frac{\mathrm{d}T}{T} \\
C_p \int_{T_1}^{T_f} \frac{\mathrm{d}T}{T} &= C_p \int_{T_f}^{T_2} \frac{\mathrm{d}T}{T} \\
\ln\left(\frac{T_f}{T_1}\right) &= \ln\left(\frac{T_2}{T_f}\right) \\
\therefore T_f^2 &= T_1 T_2
\end{aligned}$$

As desired. ■