Eric Du | Physics 137A
HW 08 | Quantum Mechanics | November 4, 2022

Collaborators

I worked with **Andrew Binder** to complete this assignment.

Problem 1

For the most general noramlized spinor χ (Equation 4.139), compute $\langle S_x \rangle, \langle S_y \rangle, \langle S \rangle_Z, \langle S_x^2 \rangle, \langle S_y^2 \rangle, \langle S_z^2 \rangle$. Check that $\langle S_x^2 \rangle + \langle S_y \rangle^2 + \langle S_z^2 \rangle = \langle S^2 \rangle$.

Solution: We know that the general spin state can be written in terms of a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ so therefore applying the definitions of $\langle S_x \rangle$, $\langle S_y \rangle$, and so on onto our vectors, we get:

$$\begin{aligned}
\langle S_x \rangle &= \langle \chi | S_x \chi \rangle \\
&= \frac{\hbar}{2}(a^\star \ b^\star) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \frac{\hbar}{2}(a^\star \ b^\star) \begin{pmatrix} b \\ a \end{pmatrix} = \frac{\hbar}{2}(a^\star b + b^\star a)
\end{aligned}$$

By that same logic, we can compute $\langle S_y \rangle$ and $\langle S_z \rangle$ (I'm skipping all the algebra here but its just a lot of matrix multiplication and I can't really be bothered to write that many matrices):

$$\begin{aligned}
\langle S_y \rangle &= \frac{\hbar}{2} i(ab^\star - a^\star b) \\
\langle S_z \rangle &= \frac{\hbar}{2}(a^\star a - b^\star b)
\end{aligned}$$

To compute $\langle S_x^2 \rangle, \langle S_y^2 \rangle$, and $\langle S_z^2 \rangle$, we first compute S_x^2 itself:

$$\begin{aligned}
S_x^2 &= \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Similarly, using the definitions of S_y and S_z:

$$\begin{aligned}
S_y^2 &= \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
S_z^2 &= \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Now notice that $S_x^2 = S_y^2 = S_z^2$. Let's compute one of $\left\langle S_x^2 \right\rangle$:

$$\begin{aligned}
\left\langle S_x^2 \right\rangle &= \frac{\hbar^2}{4} (a^\star \ b^\star) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \frac{\hbar^2}{4} (a^\star a + b^\star b)
\end{aligned}$$

Since $S_x^2 = S_y^2 = S_z^2$, then we know that the results for all three operators will also be the same. Therefore:

$$\left\langle S_x^2 \right\rangle + \left\langle S_y^2 \right\rangle + \left\langle S_z^2 \right\rangle = \frac{3\hbar^2}{4} (a^\star a + b^\star b)$$

If we then use the definition for S^2 in the book, we also get:

$$S^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \left\langle S^2 \right\rangle = \frac{3\hbar^2}{4} (a^\star a + b^\star b)$$

And so we're done. $\square$

Problem 2

(a) Find the eigenvalues and eigenspinors of S_y.

Solution: We know the matrix representation of S_y from the previous part. Therefore, we can find its eigenvalues by using $\det(A - \lambda I) = 0$:

$$\begin{aligned}
0 &= \det\left(\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \\
&= \lambda^2 - \frac{\hbar^2}{4} \\
\therefore \lambda &= \pm\frac{\hbar}{2}
\end{aligned}$$

To find the eigenspinors, we need to then find spinors such that $S_y\chi = \frac{\hbar}{2}\chi$, so we want to find:

$$S_y\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore:

$$\begin{aligned}
\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \\
\frac{i\hbar}{2}\begin{pmatrix} -b \\ a \end{pmatrix} &= \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}$$

which gives us the equation $-ib = \pm a$. Due to normalization, we know that $|a^2| + |b|^2 = 1$, so therefore this gives us $a^2 + a^2 = 1 \implies a = \frac{1}{\sqrt{2}}$. So this then gives us two possibilites for our spinor:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

□

(b) If you measured S_y on a particle in the general state χ (Equation 4.139), what values might you get, and what is the probabilty of each? Check that the probabilities add up to 1. *Note:* a and b need not be real!

Solution: By the quantum postulates, you would get one of the two eigenvalues, those being $\pm\frac{\hbar}{2}$. We can then express a general state χ as:

$$\chi = \left(\frac{a - ib}{\sqrt{2}}\right)\chi_+ + \left(\frac{a + ib}{\sqrt{2}}\right)\chi_-$$

Therefore, the probabilities will be:

$$P(\chi_+) = \left|\frac{a - ib}{\sqrt{2}}\right|^2$$
$$P(\chi_-) = \left|\frac{a + ib}{\sqrt{2}}\right|^2$$

To check the probability equals 1:

$$\begin{aligned}
P(\chi_+) + P(\chi_-) &= \left|\frac{a - ib}{\sqrt{2}}\right|^2 + \left|\frac{a + ib}{\sqrt{2}}\right|^2 \\
&= \frac{1}{2}(|a|^2 - 2i|a||b| + |b|^2 + |a|^2 + 2i|a||b| + |b|^2) \\
&= |a|^2 + |b|^2 = 1
\end{aligned}$$

where we've used the fact that $|a|^2 + |b|^2 = 1$ because of normalization. □

(c) If you measured S_y^2, what values might you get, a with what probabilities?

Solution: Since we can write S_y^2 as:

$$S_y^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} I$$

then it only has one eigenvalue, which is $\frac{\hbar^2}{4}$. Then, because there is only one eigenvalue, then it must also be true that we measure this eigenvalue with probability 1, due to the law of total probability. □

Problem 3

(a) Apply S_- to $|1\ 0\rangle$ (Equation 4.177) and confirm that you get $\sqrt{2}\hbar\,|1\ \ -1\rangle$

Solution: We use the definition of $|1\ 0\rangle$ and apply S_- to it:

$$\begin{aligned}
S_-\,|1\ 0\rangle &= S_-\left(\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\right) = \frac{1}{\sqrt{2}}(S_-^{(1)} + S_-^{(2)})(\uparrow\downarrow + \downarrow\uparrow) \\
&= \frac{1}{\sqrt{2}}\left[(S_-\uparrow)\downarrow + (S_-\downarrow)\uparrow + \uparrow(S_-\downarrow) + \downarrow(S_-\uparrow)\right] \\
&= \frac{1}{\sqrt{2}}2\hbar\downarrow\downarrow && S_-\downarrow = 0 \text{ by definition} \\
&= \sqrt{2}\hbar\downarrow\downarrow = \sqrt{2}\hbar\,|1 - 1\rangle
\end{aligned}$$

as desired. □

(b) Apply $S_\pm$ to $|0\ 0\rangle$ (Equation 4.178), and confirm that you get zero.

Solution: We do the same thing as part (a):

$$\begin{aligned}
S_-\,|00\rangle &= S_-\left(\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\right) = \frac{1}{\sqrt{2}}(S_-^{(1)} + S_-^{(2)})(\uparrow\downarrow - \downarrow\uparrow) \\
&= \frac{1}{\sqrt{2}}\left[(S_-\uparrow)\downarrow + (S_-\downarrow)\uparrow - \uparrow(S_-\downarrow) - \downarrow(S_-\uparrow)\right] \\
&= 0
\end{aligned}$$

also as desired. □

(c) Show that $|1\ 1\rangle$ and $|1\ \ -1\rangle$ (Equation 4.177) are eigenstates of S^2, with the appropriate eigenvalue.

Solution: Using the definition of S^2 and $|1\ 1\rangle = \uparrow\uparrow$ and $|1\ \ -1\rangle = \downarrow\downarrow$, we can do the algebra by brute force:

$$\begin{aligned}
S^2\,|1\ 1\rangle &= S^2(\uparrow\uparrow) = \left((S^{(1)})^2 + (S^2)^2 + 2S^{(1)}\cdot S^{(2)}\right)\uparrow\uparrow \\
&= (S^2\uparrow)\uparrow + \uparrow(S^2\uparrow) + 2\left[(S_x\uparrow)(S_x\uparrow) + (S_y\uparrow)(S_y\uparrow) + (S_z\uparrow)(S_z\uparrow)\right] \\
&= \frac{3}{4}\hbar^2\uparrow\uparrow + \frac{3}{4}\hbar^2\uparrow\uparrow + 2\left(\frac{\hbar}{2}\downarrow\frac{\hbar}{2}\downarrow + \frac{i\hbar}{2}\downarrow\frac{i\hbar}{2}\downarrow + \frac{\hbar}{2}\uparrow\frac{\hbar}{2}\uparrow\right) \\
&= \frac{3}{2}\hbar^2\uparrow\uparrow + 2\left(\frac{\hbar^2 - \hbar^2}{4}\downarrow\downarrow + \frac{\hbar^2}{4}\uparrow\uparrow\right) \\
&= \frac{3}{2}\hbar^2\uparrow\uparrow + \frac{\hbar^2}{2}\uparrow\uparrow \\
&= 2\hbar^2\uparrow\uparrow = 2\hbar^2\,|11\rangle
\end{aligned}$$

We do the same for $|1\ \ -1\rangle$:

$$
\begin{aligned}
S^2\,|1\ \ -1\rangle &= S^2(\downarrow\downarrow) = \left((S^{(1)})^2 + (S^2)^2 + 2S^{(1)}\cdot S^{(2)}\right)\downarrow\downarrow \\
&= (S^2\downarrow)\downarrow + \uparrow(S^2\downarrow) + 2\left[(S_x\downarrow)(S_x\downarrow) + (S_y\downarrow)(S_y\downarrow) + (S_z\downarrow)(S_z\downarrow)\right] \\
&= \frac{3}{4}\hbar^2\downarrow\downarrow + \frac{3}{4}\hbar^2\downarrow\downarrow + 2\left(\frac{\hbar}{2}\uparrow\frac{\hbar}{2}\uparrow + \left(-\frac{i\hbar}{2}\uparrow\right)\left(-\frac{i\hbar}{2}\uparrow\right) + \left(-\frac{\hbar}{2}\downarrow\right)\left(-\frac{\hbar}{2}\downarrow\right)\right) \\
&= \frac{3}{2}\hbar^2\uparrow\uparrow + 2\left(\frac{\hbar^2-\hbar^2}{4}\downarrow\downarrow + \frac{\hbar^2}{4}\uparrow\uparrow\right) \\
&= \frac{3}{2}\hbar^2\uparrow\uparrow + \frac{\hbar^2}{2}\downarrow\downarrow \\
&= 2\hbar^2\downarrow\downarrow = 2\hbar^2\,|1-1\rangle
\end{aligned}
$$

And so we're done. $\square$

Problem 4

In this problem, we would like to compute the probability distribution of measurements J_x, J_y and J_z for particles with $J = 1$.

(a) We will do this problem in the $|j = 1, m_z = 1\rangle$, $|j = 1, m_z = 0\rangle$, $|j = 1, m_z = -1\rangle$ basis. How does the $\hat{J}_z$ operator look in this basis? Note that we will have a 3×3 matrix. (Hint: Think about which eigenvalues the matrix should have.)

(b) Recall that $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$. Show that $\hat{J}_+^\dagger = \hat{J}_-$.

Solution: We know that $J_+ = J_x + iJ_y$, so therefore $J_+^\dagger = J_x - iJ_y = J_-$, simply by the definition of the ladder operators J. □

(c) Show that $\hat{J}_\mp \hat{J}_\pm = \hat{J}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z$.

Solution: We use J_x and J_y and compute via brute force:

$$\begin{aligned}
\hat{J}_\mp \hat{J}_\pm &= (\hat{J}_x \mp i\hat{J}_y)(\hat{J}_x \pm i\hat{J}_y) \\
&= \hat{J}_x^2 + \hat{J}_y^2 \pm i\hat{J}_x\hat{J}_y \mp i\hat{J}_y\hat{J}_x \\
&= (\hat{J}_x^2 + \hat{J}_y^2) \pm i(\hat{J}_x\hat{J}_y \mp \hat{J}_y\hat{J}_x) \\
&= (\hat{J}^2 - \hat{J}_z^2) + i(\pm i\hbar J_z) = \hat{J}^2 - \hat{J}_z^2 \mp \hbar J_z
\end{aligned}$$

as desired. □

(d) Determine the form of the raising and lowering operators in the $\{|j = 1, m_z\rangle\}$ basis. (Hint: First determine which elements are non-zero by recalling how the raising and lowering operators act on the basis states. Then use the above facts to determine exactly what the non-zero elements are.)

(e) Use the raising and lowering operaors to construct the representations of $\hat{J}_x$ and $\hat{J}_y$ in the $\{|j = 1, m_z\rangle\}$ basis.

(f) Use these matrices to find the representations of the eigenstates of both $\hat{J}_z$ and $\hat{J}_y$ in the $\{|j = 1, m_z\rangle\}$ basis. and their corresponding eigenvalues.

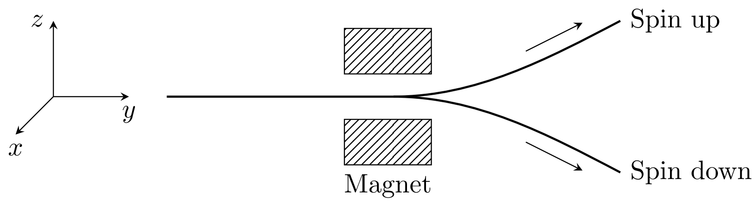
(g) A particle is prepared in the state $|j = 1, m_z = 1\rangle$ and then J_z is measured. What are the possible J_z measurement results, i.e. states, and their respective probaiblities? What is the expectation value of the angular momentum in the x-direction of $|j = 1, m_z = 1\rangle$?

(h) If we measure $J_z = \hbar$ and then we measure J_y, what is the expectation value of the angular momentum in the y-direction?

(i) If we instead measured J_z again after measuring $J_z = \hbar$, what is the probability that we get the original state $|j = 1, m_z = 1\rangle$? You should find that simply making the measurement of J_x changes the state; you can have the value of J_z change just by measuring J_x!

Problem 5

Imagine you have a beam of spin 1/2 particles moving in the y-direction. We can set up an inhomogeneous magnetic field to interact with the particles, separating them according to thier spin component in the direction of the magnetic field, $\mathbf{B} \cdot \hat{\mathbf{S}}$. This is the Stern-Gerlach experiment, depicted in Fig. 1



(a) You set up a magnetic field in the z-direction. As the beam of particles passes through it, it splits in two equal beams: one goes up, corresponding to the spin-up particles (those whose $\hat{S}_z$ eigenvalue was $\frac{\hbar}{2}$), and the other goes down, corresponding to the spin-down particles. Now, you take the beam that went up and pass it through another magnetic field in the z-direction. Does the beam split? If so, what fraction of the particles go to each side?

Solution: Because we are measuring the spin in the z direction after having just measured it in the z-direction, the beam will not split, and will only spit out particles that go up. $\square$

(b) Instead, you pass the beam through a z-field, take the beam that went up, and pass it thorugh the magnetic field in the x-direction. Does the beam split? If so, what fraction of the particles go to each side?

Solution: The beam does split into particles that have spin up or down in the x-direction, with half the particles going up and the other half going down. $\square$

(c) You select one of the beams from part b above, and pass it through another magnetic field in the z-direction. Does the beam split? If so, what fraction of the particles go to each side? Compare with part a and explain.

Solution: The beam now splits in the z-direction with an equal amount going up and down, because we've made an intermediate measurement of the spin in the x-direction. Specifically, the fact that we've made an intermediate measurement in the x-direction is what causes the beam to split.

We can think of this as the fact that when we measure along the z-axis, then the beam splits along the basis in the z-direction. Then, once we measure the spins in the x-direction, we now change bases into the x-direction, which effectively "erases " the information we had about the z-direction, and therefore when we try to measure the z-direction again the beam will split. $\square$

(d) Suppose we start with N particles. We first pass them through a magnetic field in the z-direction, and block the beam that goes down. After this process, you find that only $\frac{N}{2}$ particles remain. they they go thugh a magnetic field in the $x-z$ plane, an angle θ from the z-axis, and the beam that goes against the direction of the field is blocked. Then you have a magnetic field in the z-direction again, and block the beam that goes up this time. How many particles come out? Compare with the case without the middle magnetic field.

Solution: The number of particles that exit the experiment having spin up along $x-z$ plane is going to be $\cos^2(\theta/2)$ (with the magnetic field)and $\sin^2(\theta/2)$ for spin down (against the magnetic field), therefore the number of particles is $N\cos^2(\theta/2)$. Then, when we feed the beam that goes with the magnetic field along the z-axis, this now has an angle $\pi-\theta$ relative to the z-axis, so therefore the amount that goes with the beam now is $N'\cos^2((\pi-\theta)/2)$, where $N' = N\cos^2(\theta/2)$. Therefore, the total number of particles that come out is:

$$N' = N\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\pi-\theta}{2}\right)$$

□
