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1 The Real and Complex Number Systems

Problem: If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution: Suppose by contradiction that $r + x$ and rx are both rational. Then, since r is rational, then we can write $r = \frac{a}{b}$. Similarly, we write $rx = \frac{c}{d}$, $r + x = \frac{e}{f}$. Then, we have:

$$\frac{a}{b} + x = \frac{e}{f} \quad \frac{a}{b}x = \frac{c}{d}$$

Rearranging for x in both cases:

$$x = \frac{e}{f} - \frac{a}{b} \quad x = \frac{c}{d} \cdot \frac{b}{a}$$

In both cases, since the rationals are closed under multiplication and addition, this implies that x is rational, which is a contradiction. $\square$

Problem: Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution: Let S be the set such that $E \subset S$. Since α is a lower bound of E, then this means that $\alpha \in S$ such that $\alpha \leq x$ for every $x \in E$, and $\beta \geq x$ for every $x \in E$.

Now suppose for the sake of contradiction that $\beta < \alpha$. Then, since α is the lower bound, this implies that for all $x \in E$, $\alpha < x$. But since $\alpha > \beta$, then this means that $\beta < x$ for every x as well. This is impossible however, since this contradicts the fact that β is an upper bound. $\square$

Problem: Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A)$$

Solution: Since A is bounded below, then $\beta = \inf A$ exists, such that $\beta \leq x$ for all $x \in A$. Equivalently, we can write this as $-\beta \geq -x$ for all $x \in A$. Then, since $-x$ for $x \in A$ is the set $-A$, then we conclude that $-\beta$ is an upper bound for the set $-A$. Hence, we have:

$$\sup(-A) = -\beta = -\inf A \implies \inf A = -\sup(-A)$$

as desired. $\square$

Problem: Fix $b > 1$.

a) If m, n, p, q are integers, $n > 0, q > 0$ and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

Solution: We solve this by showing that we can exponentiate both sides to arrive at an equal value. If this is the case, then we know that the original numbers must be the same, becuase Theorem 1.21 guarantees us uniqueness in roots.

Notice that since $m/n = p/q$, then we can write:

$$b^{mq} = \left((b^m)^{1/n}\right)^{nq} \qquad b^{np} = \left((b^p)^{1/q}\right)^{nq}$$

Then, since $m/n = p/q$, we also have $mq = np$, so the exopnents are the same on both sides here. Thus, the two original quantities are the same. $\square$

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b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Solution: Again, we aim to show that these are equal like in the previous case. Since r and s are rational, let $r = \frac{m}{n}$ and $s = \frac{p}{q}$. Then, we have:

$$b^{r+s} = b^{\frac{m}{n}+\frac{p}{q}} = b^{\frac{mq+np}{nq}} \quad b^r b^s = b^{\frac{m}{n}} b^{\frac{p}{q}}$$

Now, exponentiate both of these to the power of nq, which is an integer:

$$(b^{\frac{mq+np}{nq}})^{nq} = b^{mq+np} \quad \left(b^{\frac{m}{n}} b^{\frac{p}{q}}\right)^{nq} = b^{mq+np}$$

Then, since these values are the same, then it must follow that $b^{r+s} = b^r b^s$. $\square$

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c) If x is real, define $B(x)$ to be the set of all numbers b^t where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

Solution: Here, it is immediately clear that b^r is an upper bound of $B(r)$, since r is the largest exponent, but we now need to show that b^r is the least upper bound of $B(r)$.

We prove this by contradiction. Suppose there is some real number s such that $s < b^r$ and s is still an upper bound on $B(r)$. Then, this implies that $b^t < s$ for all $t < r$ as well. However, this cannot be the case, since $B(x)$ is defined to contain b^r, so s cannot be an upper bound on the entire set. Therefore, b^r is the least upper bound. $\square$

Problem: Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Solution: By proposition 1.18 condition (d), we require that if $x \neq 0$, then $x^2 > 0$. However, this is not true for the complex field, since $i^2 = -1 < 0$. $\square$

Problem: If z is a complex numer, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Solution: This is essentially the construction of the polar coordinates. Let $z = a + bi$. Then, we may express z in terms of $z = |z|\hat{z}$, where $|z| = r$ and $\hat{z} = w$. To find r, one finds the norm of z:

$$r = \sqrt{a^2 + b^2}$$

similarly, w takes the for $e^{i\theta}$, where θ is defined as the angle counterclockwise from the x axis:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

This is not a unique representation, since different ratios of b and a can give rise to the same value of $\tan^{-1}(\theta)$. $\square$

1.1 Challenge Problem from Bergman

On p. 21, Rudin mentions, but does not prove, that any two ordered fields with the least-upper bound property are isomorphic. This exercise will sketch how that fact can be proved. For the benefit of students who have not had a course in Abstract Algebra, I begin with some observations generally included in that course (next paragraph and part (a) below).

If F is any field, let us define an element $n_F \in F$ for each integer n as follows: let 0_F and 1_F be the elements of F called "0" and "1" conditions (A4) and (M4) of the definition of a field. (We add the subscript F to avoid confusion with elements $0, 1 \in \mathbb{Z}$). for $n = 1$, once n_F is defined we recursively define $(n+1)_F = n_F + 1_F$; in this way n_F is defined for all nonnegative integers. Finally, for negative integers n we define $n_F = -(-n)_F$. (Note that in that expression, the "inner" minus is applied in Z, the "outer" minus in F.)

a) Show that under the above conditions, we have $(m+n)_F = m_F + n_F$ and $(mn)_F = m_F n_F$ for all $n, n \in \mathbb{Z}$

b) Show that if F is an *ordered* field, then we laso have $m_F < n_F \iff m < n$

2 Basic Topology

2.1 Notes

Definition: Consider two sets A and B, whose elements may be any objects. Suppose for every element $x \in A$, there is an associated element in B. Then, f is said to be a function from A to B (or a mapping). A is called the domain, and B is called the range of f.

2.2 Problems

Problem: Prove that the empty set is a subset of every set.

Solution: The definition of a subset is that every element of the subset is also a member of the superset. Vacuously, the empty set contains elements of any set, and thus it is a subset of every set. □

Problem: A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$ not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

Prove that the set of algebraic numbers is countable. *Hint:* For every positive integer N, there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N$$

Solution: By the fundamental theorem of algebra, we know that a degree n polynomial with complex coefficients (of which the integers are a subset) has exactly n roots, counting multiplicity. Therefore, for every polynomial of degree n, there are finitely many roots.

We now prove that there are countably many polynomials with integer coefficients. This is easy enough, since for every n, let E_n denote the set of integer coefficient of degree n. Since each coefficient is an integer, this set is countable. Then, the set of polynomials with integer coefficients can be represented as:

$$S = \bigcup_{i=1}^{\infty} E_n$$

which by Theorem 2.12, S is countable. Combining this with the fact that for any given n there are finitely many roots, we conclude that the set of algebraic numbers is countable.

Not sure how to use the hint to solve the problem. □

Problem: Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points (Recall that $\overline{E} = E \cup E'$.). Do E and E' always have the same limit points?

Solution: First, we list off the relevant definitions:

- **Closedness:** A set E is closed if the limit points of E are contained in E.
- **Limit points:** A point p is a limit point of E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.
- **Neighborhood:** The set $N_r(p)$ consisting of all points q such that $d(p,q) < r$ for some $r > 0$. Here, r is called the *radius.*

So, to prove that E' is closed, we aim to show that all the limit points of E' are contained within E'. We prove this by contradiction. Suppose E' is not closed. Then, there exists a point p such that p is a limit point of E' but $p \notin E'$. This implies two things:

- Since p is a limit point of E', it means that for any $r > 0$, there exists a neighborhood $N_r(p)$ that contains a point $q \in E'$.
- Since p is not a member of E', it means that p is not a limit point of E. So, there exists some radius $r_0 > 0$ such that $N_{r_0}(p)$ does not contain any points within E.

First, since $q \in E'$, then q is a limit point of E. Now, consider the set $N_\alpha(q)$, where $\alpha = r_0 - d(p,q)$. This is the set of points x defined by:

$$\{x \mid d(x,q) < r_0 - d(p,q)\}$$

The fact that q is a limit point of E implies that there exists some $q' \in N_\alpha(q)$ such that $q \in E$. Rearranging the inequality condition, we get $d(x,q) + d(p,q) < r_0$. Now, we use the property that $d(p,q) = d(q,p)$ and the triangle inequality to conclude that every point in $N_\alpha(q)$ satisfies the inequality $d(x,p) < r_0$. This condition is incidentally the same as that of $N_{r_0}(p)$, so the points in $N_\alpha(q)$ form a (potentially proper) subset of $N_{r_0}(p)$. By the second bullet point, we then know no point within $N_\alpha(q)$ is contained within E. This is a contradiction however, since q is defined to be a limit point. Thus, E' must be closed.

Now, we prove that E and $\overline{E}$ have the same limit points. First, the limit points of E is just the set E' itself. Then, since $\overline{E} = E \cup E'$, then we know that the limit points of $\overline{E}$ are either limit points of E or E'. If they're limit points of E, they're contained in E', and if they're limit points of E', the fact that E' is closed means that they are also a part of E'. Therefore, since the limit points of both sets is just E' itself, they have the same set of limit points.

E and E' must always have the same set of limit points. The set E' is the set of limit points of E, and since E' is closed, any limit point of E' is also contained in E', hence the set of limit points for E and E' is just E' itself. $\square$

Problem: Let $A_1, A_2, \ldots, A_n$ be subsets of a metric space.

a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B}_n = \bigcup_{i=1}^n \overline{A}_i$, for $n = 1, 2, 3, \ldots$.

Solution: Assuming that $A_i = A_i \cup A_i'$, where A_i' is the set of limit points of A_i. Then, this equation becomes:

$$\bigcup_{i=1}^n \overline{A}_i = \bigcup_{i=1}^n (A_i \cup A_i') = \bigcup_{i=1}^n A_i \cup \bigcup_{i=1}^n A_i' = B_n \cup B_n' = \overline{B}_n$$

this leverages the associativity property of unions. $\square$

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b) If $B = \bigcup_{i=1}^n A_i$, prove that $\overline{B} \supset \bigcup_{i=1}^\infty \overline{A}_i$

Problem: Let E° denote the set of all interior points of a set E [See Definition 2.18(e); E° is called the *interior* of E.]

a) Prove that E° is always open.

Solution: We can prove that E° is open by showing that every point of E° is an interior point of E°. Suppose for the sake of contradiction that there exists a point p that is an interior point of E°, but $p \notin E^\circ$.

We can prove that E° is open by showing that every point of E° is an interior point of E°. Suppose for the sake of contradiction that there exists a point $p \in E^\circ$ that is not an interior point of E°. Then, this implies that there is no neighborhood N around p such that $N \subset E^\circ$. Then, since $p \in E^\circ$, p is an interior point of E, so there exists a neighborhood N around p of radius r such that $N_r(p) \subset E$.

Now, let $q \in N_r(p)$ and $q \neq p$, and let the distance $d = d(p, q) > 0$. Since $N_r(p) \subset E$, then this also implies that $N_\alpha(q) \subset E$, where $\alpha = r - d(p, q)$, since $N_\alpha(q) \subset N_r(p)$. Then, this implies that q is an interior point of E, and since q was arbitrarily chosen, this is true for every point within $N_r(p)$. Therefore, $N_r(p) \subset E^\circ$ since all points in $N_r(p)$ are interior points of E°, which is a contradiction since we initially assumed that such a neighborhood cannot exist. □

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b) Prove that E is open if and only if $E^\circ = E$.

Solution: Forward case: Since $E^\circ = E$, then since E° is always open (by part a), then E is also open.

Reverse case: If E is open, then every point of E is an interior point of E hence, the set E° must also be open since it's just the set E itself. □

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c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$

Solution: Since G is open, then $G^\circ = G$, and since $G \subset E$, then $G^\circ \subset E^\circ$ (one can easily show this to be true), and hence $G \subset E^\circ$. □

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d) Prove that the complement of E° is the closure of the complement of E.

Solution: The closure of a set E is the set $\overline{E} = E \cup E'$, where E' denotes the set of limit points of E. So, the closure of the complement of E is the union between $E^c \cup \overline{E}^c$.

From part (c), we know that E° is the largest open set contained within E, meaning that it is the union of all open sets contained within E. Therefore, its complement is the intersection of all closed sets that contain the complement of E. In other words, for all sets F in the complement, $E^c \subset F$. Therefore, by Theorem 2.27 (b), we see that $\overline{E^c} \subset F$ for every set F. Then, since we are taking the intersection of all such sets, we get that the intersection leaves us with only $\overline{E^c}$, which is the closure of the complement of E. □

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e) Do E and $\overline{E}$ always have the same interiors?

Solution: No, take any finite set of points: then the interior of E is the null set, whereas the closure of E contains at least E itself and is nonempty. □

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f) Do E and E° always have the same closures?

Solution: No, again take any finite set of points. Then the closure of E contains at least E, whereas $E^\circ = \varnothing$, so $\overline{E^\circ} = \varnothing$. □

Problem: Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Solution: To prove that this is a metric, it just needs to satisfy the three properties:

- Positivity: If $p \neq q$, then $d(p,q) = 1 > 0$, and if $p = q$ then $d(p,q) = 0$, which is satisfied.
- Associativity: $d(p,q) = d(q,p)$, this is rather obvious.
- Triangle inequality: for any $r \in X$, $d(p,q) = 1$, whereas $d(p,r) + d(r,q) = 2$ if $r \neq p$ and $r \neq q$. If $r = p$ or $r = q$, then $d(p,r) + d(r,q) = 1$. In any case, the triangle inequality is satisfied.

In terms of open subsets, we need to find subsets which only contain interior points. Recall that an interior point is one where we can always find a neighborhood N around a point p such that $N \subset E$. Since the defined metric gives a distance of 1 for every point q as long as $q \neq p$, then we know that every set containing only one point is open, since $N_{1/2}(p) = \{p\}$, and is hence a subset of itself. By extension, we can union these sets together, so every subset of X is also open.

Every set is also closed, since the complement of any subset is also a subset of X.

A set is compact if every open cover of K contains a finite subcover. In other words, we basically require that there is a finite number of open subsets G_α such that $K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. Based on this definition, every finite subset satisfies this property, since every subset of X is open. However, the set X itself cannot be written in this way, since we cannot cover an infinitely-sized X with subsets of itself. $\square$

Problem: Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Solution: Recall that a set is compact if every open cover of K has a finite subcover, or in other words, we can find a finite collection $\{G_\alpha\}$ of open subsets of R^1 such that $K \subset \bigcup_\alpha G_\alpha$. In other words, our objective is to find such a collection of open sets.

Consider an open cover $G = \bigcup_\alpha G_\alpha$. Let G_0 be the open set containing 0. Since G_0 is open, this implies there exists some neighborhood $N_\epsilon(0) \subset G$ that exists, where $\epsilon > 0$. Now, let N be the largest value of n such that $\frac{1}{N}$ is not contianed within G_0. Then, let G_k denote the open set containing $1/k$, which we can then union together with G_0 to form the finite subcover of K. $\square$

Problem: Construct a compact set of real numbers whose limit points form a countable set.

Solution: Take the set above: the set of 0 with the numbers $1/n$: 0 is the only limit point in this set, since it is the only point such that for every neighborhood $N_r(0)$ can we find a point $1/n$ that is always in the set.

We can also show that 0 is the only limit point of the set; consider any point $1/m \in K$. Then, for any neighborhood $N_r(1/m)$ where $r < \frac{1}{m} - \frac{1}{m+1} < \frac{1}{m-1} - \frac{1}{m}$ will have no point within K inside, so it cannot be a limit point. $\square$

Problem: Show that Theorem 2.36 and its Corollary become false (in R^1, for example) if the word "compact" is replaced by "closed" or by "bounded."

Solution: I'll restate Theorem 2.36 and the Corollary for convenience:

> **Theorem:** If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.
>
> **Corollary:** If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \ldots$), then $\bigcap_1^\infty K_n$ is not empty.

First, note that a set that is both closed and bounded is compact. We can prove this as follows: let K be a closed and bounded set, and consider an open cover $\{G_\alpha\}$ of K. Let $p \in K$. Since $\{G_\alpha\}$ is an open cover, let G_1 be the open set containing p. Since G_1 is open, it has nonzero width (since it must contain some epsilon ball). Then, consider points on the boundary of the closure of G_1 that are also in K. These points are not contained in p, so there must be another open set G_2 that contians these points, which also have nonzero width. We repeat this construction until no boundary points are contained within K.

The closedness of K ensures that all limit points of K are covered, and the boundedness ensures that we may complete this process in a finite number of steps. The union of the sets we selected, $\bigcup_1^N G_n$ is a finite subcover of K.

With that in mind, we turn to the corollary: if $\{K_\alpha\}$ is bounded only, consider the set of open sets $\{K_\alpha\} = \{(0, x) \mid x \in \mathbb{R}\}$, then for any subcollection of these sets $\{G_\alpha\}$, there exists a smallest set $K_\epsilon = (0, \epsilon) = \bigcap_\alpha G_\alpha$. However, if we take the infinite intersection, then for every point x there exists at least one set $(0, x - \epsilon)$ that does not contain x, and hence the infinite interseciton is the empty set.

If $\{K_\alpha\}$ is unbounded, then consider the sets $\{K_\alpha\} = \{(x, \infty) \mid x \in \mathbb{R}\}$, and the idea is the same – any finite intersection of K_x will be nonempty, but in the limit the intersection is the empty set, becuase for every point there exists at least one set that does not contain that point.

Because the corollary is a special case of the theorem, the theorem also fails to these two examples, and hence we are done. $\square$

Problem: Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

Solution: E is clearly uncountable – we can generate a decimal on the diagonal following the rule that if it's a 4, then turn it into a 7, and vice versa. E is dense, but not dense in $[0, 1]$, since $E \subset [0.4, 0.8]$. E is closed, since every limit point, which are the points within E, is obviously contained within E. E is also bounded, which means that it is also compact.

Since all the limit points of E are contained within E, this implies that E is also perfect, since every point of E is a limit point of E. $\square$

Problem: Is there a nonempty perfect set in R^1 which contains no rational number?

Solution: I beleive the irrationals are considered a perfect set. Because the irrationals themselves are dense (as the reals themselves are dense), then every point in the irrationals is a limit point, which is all we need in order to ensure that the set is perfect. $\square$

Problem:

a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

   Solution: Two sets A and B are considered separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. Since A and B are closed sets, then $A = \overline{A}$ and $B = \overline{B}$, so therefore $A \cap B = \varnothing$ (disjoint) implies that $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. $\square$

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b) Prove the same for disjoint open sets.

   Solution: Let $p \in A$, and consider the intersection $A \cap \overline{B}$. Assume that $A \cap \overline{B}$ is nonempty. Therefore, we must ahve $p \in \overline{B}$, and since $p \notin B$, then p must be a limit point of B. Since $p \in A$ and p is open, then consider the neighborhood of radius r such that $N_r(p) \subset A$. Since p is a limit point of B, then $N_r(p)$ must contain a point $q \in B$ as well. This is contradictory, however, since this implies that $A \cap B \neq \varnothing$. $\square$

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c) Fix $p \in X, \delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

Solution: Based on the definition, $A = N_\delta(p)$, which is open. Now, it is clear that $B = (\overline{A})^c$, since the closure is the set such that $d(p,q) \leq \delta$, so B is also an open set. These two sets are also disjoint (obviously), so by part (b) we conclude that they are separated. □

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d) Prove that any connected metric space with at least two points is uncountable. *Hint: Use (c).*

Solution: Consider a metric space X. Let a, b be two points in X, and $\delta = d(a,b)/2$, and let A and B be defined as in part (c) with this value of δ. Since X is connected, it cannot be the union of two (nonempty) separated sets, and since we know from part (c) that A and B are separated, then X cannot be the union of these two. Therefore, the points $e \in X$ such that $d(a,e) = \delta$ must be in X as well.

However, we can choose different values of δ as well. For every value $r \in (0,1)$, one can choose $\delta_r = r \cdot d(a,b)$, to which there is a correspodning $e_r \in X$. Since there are uncountably many r, there are uncountably many points in X. □

Problem: Let A and B be separated subsets of some R^k, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $p(t) \in A$.]

a) Prove that A_0 and B_0 are separated subsets of R^1.

Solution: Suppose for contradiction that A_0 and B_0 are not separated subsets of R^1. WLOG, this i=mplies that $A_0 \cap \overline{B}_0$ is nonempty. Consider a point x in this intersection. Since $x \in A_0$, then we know that $p(x) \in A$. We also have $x \in \overline{B}_0$, and we know that x must exist in the closure of B_0, since if $x \in B_0$, then this implies that $p(x) \in B$, but we know that A and B are separated subsets of R^1.

Since $x \in A$, then $p(x)$ is an interior point of A, therefore there exists a neighborhood $N_r(p(x)) \subset A$. Since $x \in B_0'$ then x is a limit point of B_0, so using the same neighborhood $N_r(p(x))$ there exists another point $q \neq p(x)$ such that $q \in B$. This last point then implies that $A \cap B$ is nonempty, which is a contradiction. □

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b) Prove that there exists $t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$.

Solution: Suppose such a t_0 doesn't exist. Then, for every $t \in (0,1)$, it must be the case that $p(t) \in A \cup B$. Since A and B are separated subsets and $A \cap B = \varnothing$, then this means that each point $t \in (0,1)$ can be placed into A_0 and B_0. Now, consider a point t on the boundary of B_0, and consider any neighborhood $N_r(t)$. t is a limit point of B_0 by definition, but t must also be a limit point of A_0, since any point $t - \epsilon \in A_0$ for every $\epsilon > 0$.

Therefore, one of $A_0 \cap \overline{B}_0$ or $\overline{A}_0 \cap B_0$ is nonempty, contradicting the assumption that A_0 and B_0 are separated. □

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c) Prove that every convex subset of R^k is connected.

Solution: What is a convex subset? □

Problem: A metric space is called *separable* if it contains a countable dense subset. Show that R^k is separable. *Hint:* Consider the set of points which have only rational coordinates.

Solution: Before we begin, we should also establish a different definition of density: given an open set $S \subset X$, the requirement that the set S be nonempty is equivalent to density. Part (j) of Definition 2.18 gives the following definition of density:

E is *dense* in X if every point of X is a limit point of E, or a point of E (or both).

All we need to do is to show that the given definition implies that S is nonempty. Consider a point $x \in X$ and an open neighborhood $N_r(x)$. Based on the given definition, either $x \in E$ or x is a limit point of E. In the former case, then $N_r(x)$ is obviously nonemtpy since it contains x. In the latter, then $N_r(x)$ by definition must contain a point $q \neq x$ such that $q \in E$, also satisfying that $N_r(x)$ is nonempty.

With this established, consider the set $Q^k \subset R^k$, the set of points with only rational coordinates. Then, consider two points in this set, and consider these two points to the the boundary of our open set. That is, given $A = (a_1, a_2, \dots, a_k)$ and $B = (b_1, b_2, \dots, b_k)$, define the open set to be the points $X = (x_1, x_2, \dots, x_k)$ such that

$$a_1 < x_1 < b_1$$
$$a_2 < x_2 < b_2$$
$$\vdots$$
$$a_k < x_k < b_k$$

this set is clearly open (proof is very easy), and since the rationals are dense in R, then we are guaranteed to find an X that satisfies this inequality. Hence, the subset is dense. Further, this subset is also countable, since the rationals Q are countable. Combining these two allows us to conclude that R^k is separable. $\square$

Problem: A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* of X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

Solution: The metric space X is separable, so it contains a countable dense subset. Then, following the hint, we take all neighborhoods with rational radius and center within such a countable dense subset of X. Let the set $S = \{x_1, x_2, \dots\}$ represent this countable dense subset, and consider neighborhoods with rational radius and center just like the hint says.

Then, consider a point $x \in G$. Because G is an open set, then $N_r(x) \subset G$ exists for some $r \in \mathbb{R}$. Now, since S is dense in X, then based on the previous problem we know that every open set in X must contain an element of S. Therefore, there exist some $x_k \in N_{\frac{r}{2}}(x)$. Now, let $\alpha = d(x, x_k)$, and since the rationals are dense in $\mathbb{R}$, then we know that there exists some rational r' such that $\alpha < r' < \frac{r}{2}$. Then, $x \in N_{r'}(x_k)$, and $N_{r'}(x_k)$ is a neighborhood with rational radius and center as required by our countable dense subset. $\square$

Problem: Let X be a metric space for which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, pick $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ ($n = 1, 2, 3, \dots$), and consider the centers of the corresponding neighborhoods.

Solution: To prove separability, we need to prove that it has a countable dense subset.

We approach this problem using the hint. Fix $\delta > 0$, and pick $x_1 \in X$, and upon choosing x_j, we choose $x_{j+1} \in X$ such that $d(x_i, x_{j+1}) \geq \delta$ for all $i = 1, \dots, j$. Our aim is to show that this process can only proceed a finite number of times.

Suppose for contradiction that this sequence does go on infinitely, then the set $S_\delta = \{x_1, x_2, \dots\}$ is a countably infinite subset of X, and therefore must have a limit point. That is, there exists a point $x' \in X$ such that there exists some x_k in every neighborhood $N_r(x')$. Now, consider the neighborhood $N_{\frac{\delta}{2}}(x')$. This neighborhood can only contain one point x_k, and therefore if $\gamma = d(x', x_k)$, then $N_\gamma(x')$ contains no points in S_δ. The only way that $N_\gamma(x')$ contains a point is if $x' \in S_\delta$, but then $N_{\frac{\delta}{2}}(x')$ cannot contain any other points in S, so we are left with the same result. In either case, x' cannot be a limit point, and therefore this process is guaranteed finite.

Then, we've proven that X can therefore be covered by finitely many neighborhoods of radius δ. Now, we take $\delta = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$, which generates a countably infinite set of points $S = \bigcup_n S_{\frac{1}{n}}$. Now, we need to show that this subset is dense, which is the same as claiming that every open set $G \subset X$ contains at least one point x_k.

Consider a point $p \in G$. G is an open set, so there exists a neighborhood $N_r(p) \subset G$, but this neighborhood must also contain a point x_k, generated by $\delta < r$, so incidentally we prove that G must always contain a point $x_k \in S$. Thus, S is dense and countable, therefore X is separable.

Perhaps this last argument can be made slightly more rigorous. □

Problem: Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.

Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint:* Let $\{V_n\}$ be a countable base of R^k, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Solution: Again, referring back to Definition 2.18, a set P is perfect if P is closed and if every point of P is a limit point of P. We first show that the set P defined in the problem is closed. For a set to be closed, it means that every limit point of P is contained within P.

For the sake of contradiction, assume that P is not closed; that is, there exists some $q \notin P$ such that q is a limit point of P, or every neighborhood of q contains at least one point in P. Consider a neighborhood $N_r(q)$, and a point $p \in N_r(q)$ such that $p \in P$. Then, since $p \in P$, then every neighborhood around p contains uncountably many points, so now let $\delta = d(p, q)$, then $N_{r-\delta}(p)$ is contained entirely in $N_r(q)$ (this was done in earlier problems), which implies that $N_r(q)$ contains uncountably many points. This can be done for every neighborhood, so we are forced to conclude that $q \in P$.

Further, we need to show that every point of P is a limit point of P. Again, we consider the contradiction be the fact that there exists some $p \in P$ such that p is not a limit point of P. Since p is not a limit point of P, it means that while $N_r(p)$ is uncountable, the neighborhoods of other points $q \in N_r(p)$ are countable. Now consider other points $q \in N_r(p)$, and consider an infinite union of neighborhoods around q such that the union of such neighborhoods form $N_r(p)$. By Theorem 2.12, this set is countable, but then this contradicts the fact that $N_r(p)$ is uncountable. Therefore, there must exist some neighborhood $N_{r'}(q)$ that is uncountable, and therefore $q \in P$, and p is a limit point of P.

Now we prove that there are countably many points of E that are not in P, leveraging the hint. Let $\{V_n\}$ be a countable base of R^k (this exists because R^k is separable), and let W be the union of those V_n for which $E \cap V_n$ is at most countable. Our goal is to show that $P = W^c$.

Based on the construction of W, we can compute

$$E \cap W = E \cap \left(\bigcup_{\alpha} V_\alpha \right) = \bigcup_{\alpha} (E \cap V_\alpha)$$

where α is an index set of n such that $E \cap V_n$ is countable. This implies that $E \cap W$ is countable, and since E is uncountable, then W is countable. Based on this, it is then impossible for any point $p \in P$ to exist in W, since we can define a neighborhood $N_r(p) \subset W$ that is countable. Thus, every point in P exists in W^c. This is good enough for me, I'm not sure how to show explicit equality. □

Problem: Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in R^k has isolated points.) *Hint:* use the preivous part.

Problem: Imitate the proof of Theorem 2.43 to obtain the following result:

> If $R^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of R^k, then at least one F_n has a nonempty interior.
> *Equivalent statement:* If G_n is a dense open subset of R^k, for $n = 1, 2, 3, \ldots$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in R^k.

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

3 Chapter 3

Problem: Let X be a metric space.[1]

a) Call two Cauchy sequences $\{p_n\}, \{q_n\}$ in X *equivalent* if

$$\lim_{n\to\infty} d(p_n, q_n) = 0$$

Prove that this is an equivalence relation.

Solution: For an equivalence relation, we have three satisfying properties: transitivity, reflexive, and symmetric. Consider three sequences $\{p_n\}, \{q_n\}$ and $\{r_n\}$, where $\{p_n\} \equiv \{q_n\}$ and $\{q_n\} \equiv \{r_n\}$. Then, we have $\lim_{n\to\infty} d(p_n, q_n) = 0$ and $\lim_{n\to\infty} d(q_n, r_n) = 0$. Then, we can write:

$$d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n)$$

due to the triangle inequality. Applying limits to both sides:

$$\lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) = 0$$

We can see this also by letting $\{x_n\} = d(p_n, q_n)$ and $\{y_n\} = d(q_n, r_n)$, allowing us to separate the limit as above. Since the metric is bounded below by zero, then we know that $\lim_{n\to\infty} d(p_n, r_n) = 0$, satisfying transitivity. The limit $\lim_{n\to\infty} d(p_n, p_n) = 0$ is immediately obvious, so reflexivity is satisfied. Finally, $\lim_{n\to\infty} d(p_n, q_n) = 0$ if and only if $\lim_{n\to\infty} d(q_n, p_n) = 0$ since $d(a, b) = d(b, a)$, so symmetry is also satisfied. □

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b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*, Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function on X^*.

c) Prove that the resulting metric space X^* is complete.

d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

[1] Persy wants me to suffer.