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HW 03 Real Analysis February 17, 2023

Problem 1

Let $s_n = n!/n^n$. Prove that $s_n \to 0$ as $n \to \infty$.

Solution: First, we rewrite s_n:

$$\frac{n!}{n^n} = \frac{n(n-1)\cdots 1}{n \cdot n \cdots n} = \frac{n-1}{n} \cdot \frac{(n-2)(n-3)\cdots 1}{n \cdot n \cdots n} \leq \frac{n-1}{n}$$

We also know that this sequence is bounded below by 0 since n is positive, so if we can prove that

$$\lim_{n\to\infty} \frac{n-1}{n} = 0$$

Then we've solved the problem. To do this, we look for a value of N such that for all $\epsilon > 0$:

$$\begin{aligned} |\frac{n-1}{n} - 0| &< \epsilon \\ \frac{n-1}{n} &< \epsilon \\ n - 1 &< n\epsilon \\ \therefore n &> \frac{1}{1-\epsilon} \end{aligned}$$

So therefore if we let $N = \frac{1}{1-\epsilon}$ then we satisfy this inequality for all $\epsilon > 0$. Therefore, we've proven the limit, so we now have

$$0 \leq \lim_{n\to\infty} s_n \leq 0$$

which implies that $\lim_{n\to\infty} s_n = 0$. □

Problem 2

Let (t_n) be a bounded sequence, i.e. there exists M such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove $\lim(s_n t_n) = 0$.

Solution: Since t_n is bounded, then we know that our sequence satisfies:

$$-Ms_{\leq}t_n s_n \leq Ms_n \ \forall n \in \mathbb{N}$$

□
