Problem 1

An infinite slab of thickness d with a uniform magnetization $\mathbf{M} = M_x\hat{x} + M_y\hat{y} + M_z\hat{z}$ (M_x, M_y, M_z are constants) extends along the xy-plane. Find the magnetic field $\mathbf{B}$ for a point at a distance L from the infinite slab.

Solution: From the magnetization M, we can calculate the volume and bound currents. Since M is constant, then we know that $J_b = \nabla \times M = 0$. For the surface current, we need $K_b = M \times \hat{n}$. For the top plate, we have $\hat{n} = \hat{z}$, so therefore:

$$K_{\text{b, top}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ 0 & 0 & 1 \end{vmatrix} = M_y\hat{x} - M_x\hat{y}$$

Now, since for the bottom slab we'll get $\hat{n} = -\hat{z}$, then $K_{\text{b, bottom}}$ will flow in the opposite direction of $K_{\text{b, top}}$, so therefore

$$K_{\text{b, bottom}} = -M_y\hat{x} + M_x\hat{y}$$

This means that we can treat our system essentially as if we had two infinitely large parallel plates, with one plate having current flowing in one direction and the other in the opposite direction. Then, we can use Example 5.8 from Griffiths to motivate our solution. In that example, we can see that for a single slab with surface current $\vec{K}$, then we have:

$$\mathbf{B} = \begin{cases} -\frac{\mu_0}{2}K\hat{y} & \text{above} \\ \frac{\mu_0}{2}K\hat{y} & \text{below} \end{cases}$$

Naturally, the signs will be reversed for the bottom plate. Therefore, when summing up the vector components, we find that the magnetic field outside the slabs is 0 everywhere. Therefore, to answer the original problem, we get that $B = 0$ everywhere outside the slab.

Alternatively, this argument can also be made via Ampere's law. Consider an Amperian loop that runs through tne entire slab, to a distance L above and below the slab. We know that here, the enclosed current is 0 (since they flow in exact opposite directions), so therefore we require that

$$\oint \mathbf{B} \cdot dl = 0$$

But in this case, since we expect the B field to be constant in magnitude at a constant distance L above and below and also that their directions should be opposite of each other, this requires that $B = 0$ everywhere as well. $\square$

Problem 2

A long cylinder of radius R carries a magnetization parallel to the axis $\mathbf{M} = ks^2\hat{z}$, where k is a constant and s is the distance from the axis.

a) Find the magnetic field inside and outside the cylinder.

Solution: Just like the previous problem, we calculate the surface and volume currents for this situation. Here, we have $J_b = \nabla \times M$, so therefore:

$$\mathbf{J}_b = \nabla \times M = -2ks\hat{\phi}$$

Similarly, we can calculate the bound current using $K_b = M \times n$:

$$\mathbf{K}_b = M \times \hat{s} = kR^2\hat{\phi}$$

These quantities will be useful when computing the B field inside the cylinder. Outside the cylinder, we can draw an Amperian loop along the vertical axis - here, due to the symmetry in the problem, the B field on opposite sides of the loop are equal. Then, because there is no enclosed current, then naturally we conclude that $B = 0$ outside the cylinder. To calculate the B field inside the cylinder, consider an Amperian loop that penetrates the cylinder to a depth $R - s$.

Using Ampere's law, we get:

$$\begin{aligned}
\mathbf{B} \cdot dl &= \mu_0 I_{enc} \\
\mathbf{B}l &= \mu_0 \left(\int_s^R \mathbf{J}_b \cdot da + K_b l \right) \\
\mathbf{B}l &= \mu_0 \left(-\int_s^R 2ks \cdot lds + K_b l \right) \\
\therefore \mathbf{B} &= \mu_0 \left(-k(R^2 - s^2) + kR^2 \right) \\
&= \mu_0 ks^2
\end{aligned}$$

From the right hand rule, we can then figure out the direction of the B field must be pointing in the $\hat{z}$ direction, so therefore

$$B = \mu_0 ks^2\hat{z}$$

□

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b) Find $\mathbf{H}$ inside and outside the cylinder

Solution: We use the definition of $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$. Since $\mathbf{B} = 0$ outside the cylinder and $\mathbf{M} = 0$ as well, then we know that outside the cylinder, $\mathbf{H} = 0$. Inside the cylinder, we get:

$$\mathbf{H} = \frac{1}{\mu_0}(\mu_0 ks^2\hat{z}) - ks^2\hat{z} = 0$$

Therefore, $\mathbf{H} = 0$ both inside and outside the cylinder. □

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c) Check that the Ampere's law (6.20) in Griffiths is satisfied

Solution: Equation 6.20 reads

$$\oint \mathbf{H} \cdot d\mathbf{l} = \mu_0 I_{f_{enc}}$$

Since $\mathbf{H} = 0$ over the entire space, then we know that the left hand side of the equation evaluates to 0. Furthermore, since there is no free current (since the only current is from the magnetization, which only gives bound currents), then the right hand side also evaluates to 0, regardless of the Amperian loop we choose. $\square$

Problem 3

An infinitely long cylinder of radius R carries a current with a *free* current density $\mathbf{J}_f(s) = J_f(s)\hat{z}$ along its axis, within $s < R$. There is no current outside the cylinder. The cylinder is made by linear material with permeability μ, and the magnetic field is found to be

$$\mathbf{B} = \begin{cases} ks^2\hat{\phi} & \text{for } s < R \\ 0 & \text{for } s > R \end{cases}$$

Find the free current density $\mathbf{J}_f(s)$ in the bulk of the cylinder, as well as the free current density $\mathbf{K}_f$ flowing over the side surface of the cylinder.

Solution: Since we know that $\mathbf{B} = \mu\mathbf{H}$, then we can derive H by dividing both sides by μ:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{ks^2}{\mu}\hat{\phi}$$

Then, we can determine $\mathbf{J}_f$ by using $\mathbf{J}_f = \nabla \times \mathbf{H}$:

$$\nabla \times \mathbf{H} = \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\frac{ks^2}{\mu}\right)\right]\hat{z}$$

$$\therefore \mathbf{J}_f = \frac{3ks}{\mu}\hat{z}$$

To find the surface current, we use the fact that $\mathbf{H}_{\text{above}} - \mathbf{H}_{\text{below}} = \mathbf{K}_f \times \hat{n}$. Since the current outside the cylinder is 0, then this means that $\mathbf{H}_{\text{above}} = 0$. Below the cylinder, we have $\mathbf{H}_{\text{below}} = \frac{kR^2}{\mu}\hat{\phi}$. Therefore:

$$-\frac{kR^2}{\mu}\hat{\phi} = \mathbf{K}_f \times \hat{r}$$

$$\therefore \mathbf{K}_f = \frac{kR^2}{\mu}\hat{z}$$

$\square$