Graphs

Computer Science Mentors 70

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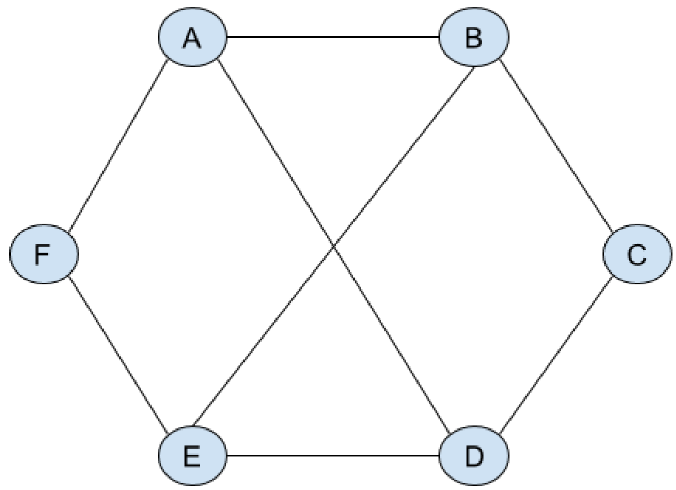
1 Graph 101



1. Take a look at the following undirected graph.
   (a) How many vertices are in this graph?
   (b) What is the degree of vertex B?
   (c) What is the total degree of this graph?
   (d) Consider the traversal $A \to B \to C \to D \to A$. How would you categorize it (walk / cycle / simple path / tour)?
   (e) Give an example of a simple path of length 4.
   (f) Is it possible to construct a traversal that is a tour but not a simple cycle from this graph (can go through vertices twice, but not edges)? Why or why not?
   (g) in terms of |E| and |V|, what is the sum of the degrees of the graph?
   (h) What is the average degree of the graph?

2. Which of these graphs have Eulerian tours?
   (a) The complete graph on 5 vertices (K_5).
   (b) The complete graph on 6 vertices (K_6).
   (c) The complete graph on 7 vertices (K_7).
   (d) The 3-dimensional hypercube.
   (e) The 4-dimensional hypercube.

3. In this question we will work through the canonical example of buildup error. Recall that a graph is **connected** iff there is a path between every pair of its vertices.

   False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

   Proof. We use induction on the number of vertices $n \geq 1$. let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

   Base case: A graph with 1 vertex doesn't have any positive-degree vertices so $P(1)$ is true vacuously.

   Inductive Hypothesis: Assume $P(n)$ holds. We want to show this implies $P(n+1)$.

   Inductive Step: Consider an n vertex graph that has positive degree. By the asumption $P(n)$, this graph is connected and there is a path from every vertex to every other vertex. Now add a new vertex to create an $n+1$ vertex graph. All that remains is to check that there is a path from v to every other vertex. Suppose we add this vertex v to an existing vertex u. Since the graph was previously connected, we already know there is a path from u to every other vertex in the graph. Therefore, when we connect v to u, we know there will be a path from v to every other vertex in the graph. This proves the claim for $P(n+1)$. $\square$

   (a) Give a counter-example to show the claim is false.

   (b) Since the claim is false, there must be an error in the proof. Explain the error.

   (c) How can we avoid this mistake?

   (d) What happens in the inductive step when you apply the fix?

4. Consider all complete undirected graphs on an even number of vertices. Prove that such graphs can be partitioned into $\frac{n}{2}$ spanning trees that share no edge with another spanning tree.

2 Graph Coloring

1. Show that any tree is 2-colorable.

2. You are hosting a very exclusive party such that a guest is only allowed to come in if they are friends with you or someone else already at the party. After everyone has showed up, you notice that there are n people (including yourself); each person has at least one friend (of course), but no one is friends with everyone else. It is still quite a sad party, because among all the possible pairs of people, there are only a total of $n-1$ friendships. You want to play a game with two teams, and in order to kindle new friendships, you want to group the people (including yourself) such that within each team, no one is friends with each other. Is this possible? *(Hint: How might the previous question be useful?)*

3. Two knights are placed on diagonally opposite corners of a chessboard, one white and the other black. The knights take turn moving as in standard chess, with white moving first.

   (a) Let every square on a chessboard represent a vertex of a graph, with edges between squares that are a knight's move away. Describe a 2-coloring of this graph.

   (b) Show that the black knight can never be captured, even if it cooperates with the white knight.

3 Planarity

We say a graph is **planar** if it can be drawn on the plane without any edges crossing each other.
If the graph is planar, we can use Euler's planar formula:

$$v + f = e + 2$$

Some other (less important) results follow from Euler's formula if a graph is planar:

$$\text{Corollary 1: } e \leq 3v - 6$$

$$\text{Corollary 2: } f \leq 2v - 4$$

1. Try to prove all three of the above results.

   Hint 1: What is the minimum number of edges per face? What is the number of faces any edge can tough?
   Hint 2: Take the graph with the most number of edges or faces per vertex. What does it look like? look at the first hint!
   Hint 3: You should come up with a relation between e, f from the previous hints. Sub it into the expression!

2. Consider a group of 6 friends sitting at a round table. Any one person is friends with the two individuals next to them and the person sitting directly across from them. Consider the graph where each individual is a vertex and an edge exists between person u and v if and only if u and v are friends. Prove that this graph is non-planar.

4 Hypercubes

1. Austin has $n \geq 2$ lightbulbs in a row, all turned off. Every second Austin performs a **move**, where Austin either turns a lightbulb on or turns a lightbulb off. Show that for any n, there is a a sequence of moves that Austin can make such that each possible configuration of lightbulbs being on or off has occurred exactly once, and such that the last move causes all the lightbulbs to be off.

   (a) Each of the n lightbulbs is either on or off. How should we represent the lightbulb states mathematically?

   (b) Frame the problem in terms of hypercubes.

   (c) Solve the problem by showing a property of hypercubes.

2. You wish to color the *edges* of a n-dimension hypercube, such that edges that share a vertex are different colors. (Note: This isn't the problem in disc2b, where you colored vertices so that vertices that share an edge are different colors!) Prove that n colors is necessary and sufficient. (You can do this with n colors but not $n - 1$.)