Eric Du
Prof. Edward Frenkel

Solutions to Selected Axler Problems



Header styling inspired by CS 70: https://www.eecs70.org/

# 1 Linear Maps

## 1.1 Vector Space of Linear Maps

**Problem:** Suppose $b, c \in \mathbf{R}$. Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$$

Show that $T$ is linear if and only if $b = c = 0$.

*Solution:* We first show that if $b = c = 0$, then $T$ is linear. Recall the facts of linearity:

$$T(u + v) = Tu + Tv \quad T(\lambda v) = \lambda(Tv)$$

for all $v \in V$. If $b = c = 0$, then we can define $T$ as:

$$T(x, y, z) = (2x - 4y + 3z, 6x)$$

Now suppose we have two vectors $u = (x_1, y_1, z_1), v = (x_2, y_2, z_2)$. Then:

$$Tu+Tv = (2x_1 - 4y_1 + 3z_1, 6x_1) + (2x_2 - 4y_2 + 3z_2, 6x_2) = (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2), 6(x_1 + x_2)) = T(u+v)$$

Now for homogeneity:

$$T(\lambda u) = T(\lambda x_1, \lambda y_1, \lambda z_1) = (2\lambda x_1 - 4\lambda y_1 + 3\lambda z_1, 6\lambda x_1) = \lambda(Tu)$$

Therefore, both conditions are satisfied, indeed $T$ is linear. Now we show that if $T$ is linear, then $b = c = 0$ is necessary. Consider what we had earlier:

$$Tu + Tv = (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + 2b, 6(x_1 + x_2) + cx_1y_1z_1 + cx_2y_2z_2)$$

This is only equal to $T(u + v)$ if $2b = 0$ and $c(x_1y_1z_1 + x_2y_2z_2) = 0$, since they are the only nonlinear terms. Thus, if $T$ is linear, then $b = c = 0$. □

---

**Problem:** Suppose that $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in $V$ such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in $W$. Prove that $v_1, \ldots, v_m$ is linearly independent.

*Solution:* We return to the definition of linear independence: a set of vectors $v_1, \ldots v_m$ is linearly independent the solution to the equation:

$$a_1v_1 + \cdots + a_mv_m = 0$$

is that $a_1, \ldots, a_m = 0$. Since we know that the list $Tv_1, \ldots, Tv_m$ is linearly independent in $w$, then the solution to the equation:

$$a_1Tv_1 + \cdots + a_mTv_m = 0$$

is $a_1 = \cdots = a_m = 0$. Now, apply the rules of $T$ being a linear map:

$$a_1Tv_1 + \cdots + a_mTv_m = T(a_1v_1) + \cdots + T(a_mv_m) = T(a_1v_1 + \cdots a_mv_m) = 0$$

Now, we use the fact that since linear maps take 0 to 0, this implies that $a_1v_1+\cdots+a_mv_m = 0$. Further, since the only values of $a_i$ that satisfy this equation is $a_1 = \cdots = a_m = 0$, then this satisfies the condition that $v_1, \ldots, v_m$ is linearly independent. □

**Problem:** Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

*Solution:* Since $V$ is one-dimensional, this implies that there is only one basis vector, $v_1$. Therefore, for all vectors $v \in V$, then $v = \alpha v_1$ for some $\alpha \in \mathbf{F}$. Then, becuase $T \in \mathcal{L}(V)$, then $T$ must map every vector $v \in V$ to another vector in $V$, which must be expressed as a scalar times $v$. Thus, $Tv = \lambda v$ is the only option for a linear map on this space. More precisely:

$$Tv = T(\lambda v_1) = \lambda T v_1 = \alpha \lambda v_1 = \lambda(\alpha v_1) = \lambda v$$

as desired. $\square$

---

**Problem:** Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}$ but $\varphi$ is not linear.

**Problem:** Suppose $U$ is a subspace of $V$ with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T : V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that $T$ is not a linear map on $V$.

*Solution:* A linear map must satisfy $T(\lambda v) = \lambda(Tv)$ for all $v \in V$. However, consider some $\lambda$ such that $v \in U$ but $\lambda v \notin U$. Then:

$$T(\lambda v) = 0 \quad \lambda(Tv) = \lambda S v \neq 0$$

Hence, $T$ is not linear on $V$. Alternatively, we could define $v \in V$ and $w \in V$ but $w \notin U$, then we have:

$$T(v + w) = 0$$

But:

$$Tv + Tw = Sv \neq 0$$

so this also violates linearity. $\square$

---

**Problem:** Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in $V$. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

*Solution:* Since $v_i$ is linearly dependent, then we can write $v_i = \sum_{j \neq i} a_j v_j$ for some set of $a_j$. Then, consider some nonzero $w \in W$, and set the $w_k$'s as follows:

$$w_k = \begin{cases} w & k = i \\ 0 & \text{else} \end{cases}$$

Then, suppose for all $j \neq i$, that $Tv_j = w_j = 0$. Then, let's write $Tv_i$:

$$Tv_i = T\left(\sum_{j \neq i} a_j v_j\right) = \sum_{j \neq i} a_j T v_j \sum_{j \neq i} a_j w_j = w_i$$

But since all $w_j = 0$, this means that $w_i = 0$, but we set $w_i \neq 0$ purposefully, therefore there is no $T$ that would stasify this. $\square$

---

## 1.2 Null Spaces and Ranges

**Problem:** Suppose $S, T \in \mathcal{L}(V)$ are such that range $S \subseteq$ null $T$. Prove that $(ST)^2 = 0$.

*Solution:* Consider a vector $v \in V$. Then:

$$\begin{aligned}(ST)^2 v &= (STS)(Tv) \\ &= ST(Sv')\end{aligned}$$

Now, $Sv'$ will exist within range $S$, and since we know that range $S \subseteq$ null $T$, then this impleis that $T(Sv') = 0$. Finally, $S(0) = 0$, so hence $(ST)^2 v = 0$, so $(ST)^2 = 0$. □

---

**Problem:** Suppose $v_1, \ldots, v_m$ is a list of vectors in $V$. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m$$

a) What property of $T$ corresponds to $v_1, \ldots, v_m$ spanning $V$?

   *Solution:* If $T$ is surjective, then $v_1, \ldots, v_m$ spans $V$. This is because if the range of $T$ is $V$, then the set of vectors $T$ applies a linear combination to must span $V$. □

---

b) What property of $T$ corresponds to the list $v_1, \ldots, v_m$ being linearly independent?

   *Solution:* The set $v_1, \ldots, v_m$ is linearly independent if and only if $T$ is injective, since linear independence means that there is only one way to express every vector (i.e. the solution to $T(z_1, \ldots, z_m) = 0$ is $z_1 = \cdots = z_m = 0$). □

---

**Problem:** Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in $V$. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in $W$.

*Solution:* The proof of this is very similar to the one we did earlier. If $v_1, \ldots, v_n$ is linearly dependent, then this means that

$$a_1 v_1 + \cdots + a_n v_n = 0$$

is solved by setting $a_i = 0$. Then, now let's consider the list $Tv_1, \ldots, Tv_n$:

$$a_1 Tv_1 + \cdots + a_n Tv_n = T(a_1 v_1 + \cdots + a_n v_n) = 0$$

where the last equality we obtain from the fact that linear maps map 0 to 0. This implies that the only solution to this equation is $a_i = 0$, hence the list $Tv_1, \ldots, Tv_n$ is linearly independent. □

---

**Problem:** Suppose that $V$ is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace $U$ of $V$ such that

$$U \cap \text{null } T = \{0\} \quad \text{and} \quad \text{range } T = \{Tu : u \in U\}.$$

*Solution:* We can define $U = \{0\} \cup (V \setminus \text{null } T)$. This way, null $T \cap U = \{0\}$ by definition, and range $T = \{Tv : v \in V\}$, but since $U$ defines the same set of vectors (since we only get rid of the null space), then range $T = \{Tu : u \in U\}$, as desired. □

---

**Problem:** Suppose $T$ is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that $T$ is surjective.

*Solution:* We know that $T \in \mathcal{L}(\mathbf{F}^4, F^2)$. Because the null space can be determined by two variables only $(x_1, x_3)$, so $\dim \text{null } T = 2$. This implies that since $\dim V = 4$, then $\dim \text{range } T = 2$, and since this equals the dimension that $T$ maps to $\mathbf{F}^2$, then this implies that $T$ is indeed surjective by (3.19). □

**Problem:** Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists an injective map from $V$ to $W$ if and only if $\dim V \leq \dim W$.

*Solution:* Recall the definition of injectivity: a linear map $T \in \mathcal{L}(V, W)$ is injective if and only if $Tu = Tv$ implies that $u = v$, or equivalently that $\operatorname{null} T = \{0\}$.

We prove the forward case: if $\dim V \leq \dim W$, we show that there exists an injective map from $V$ to $W$. Let $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_m$ be the basis vectors of $V$ and $W$ respectively, and $n \leq m$. Then, define a map $Tv_i = w_i$ for all $i = 1, \dots, n$. Then, $\dim \operatorname{range} T = \dim V$, implying that $\dim \operatorname{null} T = 0$ from FTLM, as desired.

Now we prove the reverse: we want to show that if there exists an injective map from $V$ to $W$, then $\dim V \leq \dim W$. This is trivial by contradiction: if $\dim V > \dim W$, then by (3.22) this is impossible, so we're done. □

---

**Problem:** Suppose $V$ and $W$ are finite-dimensional and $U$ is a subspace of $V$. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

*Solution:* We prove the forward case: there exists a $T$ such that $\operatorname{null} T = U$ if $\dim U \geq \dim V - \dim W$. Let $\{u_i\}$ be a basis of $U$, and $\{w_i\}$ be a basis of $W$. Then, define a linear map $T$ as follows:

$$Tv_i = \begin{cases} \vec{0} & v_i \in \{u_i\} \\ w_i & v_i \notin \{u_i\} \end{cases}$$

One can check very easily that this is linear, with $\operatorname{null} T = U$. Here, $\dim \operatorname{null} T = \dim U$ and $\dim \operatorname{range} T = \dim W - \dim U$, since the basis vectors that map to a nonzero vector in $W$ are those that do not form a basis of $U$. Therefore, $\dim V = \dim U + (\dim W - \dim U) = \dim W \leq \dim U + \dim W$, so the inequality is satisfied.

Now we prove the reverse case: if such a $T$ exists, then $\dim U \geq \dim V - \dim W$. From the fundamental theorem of linear maps, we know that $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. Now suppose we have a $T$ such that $\operatorname{null} T = U$. Then, we have $\dim V = \dim U + \dim \operatorname{range} T$, and since $\dim \operatorname{range} T \leq \dim W$, then $\dim V \leq \dim U + \dim W$, which is the inequality we wanted to satisfy. □

---

**Problem:** Suppose $W$ is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T$ is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $ST$ is the identity operator on $V$.

*Solution:* We show the forward case: $T$ is injective if there exists an $S \in \mathcal{L}(W, V)$ such that $ST$ is the identity. Suppose $T$ is not injective, so there exists two vectors $u \neq v$ such that $Tu = Tv$. Then, acting $S$ on the left of both sides gives: $STu = STv$ and since $ST$ is the identity, then we're left with $u = v$, which is a contradiction.

Now we show the reverse case: if $T$ is injective, there exists an $S \in \mathcal{L}(W, V)$ such that $ST$ is the identity. Since $T$ is injective, then we know that for any two vectors $v_1, v_2 \in V$, if $Tv_1 = Tv_2$ then $v_1 = v_2$. Now, let $\{w_1, w_2, \dots, w_m\}$ be a basis for $W$ and $\{v_1, v_2, \dots, v_n\}$ be a basis for $V$. Then, let $S \in \mathcal{L}(W, V)$ be defined as:

$$Sw_i = \begin{cases} v_i & i \in \{1, 2, \dots, n\} \\ 0 & \text{else} \end{cases}$$

$S$ is clearly linear, and take any vector $v = \sum_i \alpha_i v_i$. Then:

$$STv = S\left(\sum_i \alpha_i w_i\right) = \sum_i \alpha_i v_i = v$$

so therefore $ST$ is indeed the identity.

As an aside, we also should prove that $T$ transforms in a way such that $Tv_i = w_i$, or in other words $T$ transforms each basis vector in $V$ to a basis vector in $W$. This needs to be true since FTLM says that $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$, and since $\dim \operatorname{null} T = 0$, then $\dim \operatorname{range} T = \dim V$. $\square$

---

**Problem:** Suppose $\phi \in \mathcal{L}(V, \mathbf{F})$ and $\phi \neq 0$. Suppose $u \in V$ is not in $\operatorname{null} \phi$. Prove that

$$V = \operatorname{null} \phi \oplus \{au : a \in \mathbf{F}\}$$

## 1.3 Matrices

**Problem:** Suppose $T \in \mathcal{L}(V, W)$. Show that with respect ot each choice of bases of $V$ and $W$, the matrix of $T$ has at least $\dim \operatorname{range} T$ nonzero entries.

**Problem:** Suppose $v_1, \ldots, v_n$ is a basis of $B$ and $w_1, \ldots, w_m$ is a basis of $W$.

a) Show that if $S, T \in \mathcal{L}(V, W)$ then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$

b) Show that if $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$, then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.