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HW 11 Analytic Mechanics September 7, 2023

Collaborators

I worked with **Andrew Binder** and **Adarsh Iyer** to complete this assignment.

Problem 1

Determine the Hamiltonian and Hamilton's equations for the double Atwood machine given in Homework 2 Problem 4.

Solution: For this problem, let the height of masses 1 and 2 be m_1 and m_2, and the height of the pulley be X. From the previous homeworks, we know that:

$$\mathcal{L} = T - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 - m_1gx_1 - m_2gx_2 + Mg\left(\frac{x_1 + x_2}{2}\right)$$

Then, since we're in a natural coordinate system, we know that $\mathcal{H} = T + U$, therefore:

$$\mathcal{H} = T + U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + m_1gx_1 + m_2gx_2 - Mg\left(\frac{x_1 + x_2}{2}\right)$$

Then, using $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, we get the system of equations:

$$\begin{aligned} p_1 &= m_1\dot{x}_1 + \frac{M}{4}(\dot{x}_1 + \dot{x}_2) \\ p_2 &= m_2\dot{x}_2 + \frac{M}{4}(\dot{x}_1 + \dot{x}_2) \end{aligned}$$

Multiplying these equations by 4 so we don't have to deal with fractions, we get:

$$\begin{aligned} 4p_1 &= \dot{x}_1(4m_1 + M) + M\dot{x}_2 \\ 4p_2 &= \dot{x}_2(4m_2 + M) + M\dot{x}_1 \end{aligned}$$

Solving this system of equations, we get:

$$\begin{aligned} \dot{x}_1 &= \frac{p_1(4m_2 + M) - Mp_2}{4m_2m_1 + M(m_2 + m_1)} \\ \dot{x}_2 &= \frac{p_2(4m_1 + M) - Mp_1}{4m_1m_2 + M(m_1 + m_2)} \end{aligned}$$

Therefore, the Hamiltonian becomes:

$$\begin{aligned} H = \frac{1}{2}m_1\left[\frac{p_1(4m_2 + M) - Mp_2}{4m_2m_1 + M(m_2 + m_1)}\right]^2 + \frac{1}{2}m_2\left[\frac{p_2(4m_1 + M) - Mp_1}{4m_1m_2 + M(m_1 + m_2)}\right]^2 + \\ \frac{1}{2}M\left[\frac{4(p_1m_2 + p_2m_1) - M(p_2 + p_1)}{2(4m_2m_1 + M(m_1 + m_2))}\right]^2 + m_1gx_1 + m_2gx_2 - Mg\left(\frac{x_1 + x_2}{2}\right) \end{aligned}$$

Now, Hamilton's equations means that we have to find $\frac{\partial \mathcal{H}}{\partial p_i}$. Therefore:

$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = m_1 \left[\frac{p_1(4m_2 + M) - Mp_2}{4m_2 m_1 + M(m_2 + m_1)} \right] \cdot \frac{4m_2 + M}{4m_2 m_1 + M(m_1 + m_2)} -$$

$$m_2 \left[\frac{p_2(4m_1 + M) - Mp_1}{4m_1 m_2 + M(m_1 + m_2)} \right] \cdot \frac{M}{4m_2 m_1 + M(m_1 + m_2)} +$$

$$M \left[\frac{4(p_1 m_2 + p_2 m_1 - M(p_2 + p_1)}{2(4m_2 m_1 + M(m_1 + m_2))} \right] \cdot \frac{4m_2 - M}{2(4m_2 m_1 + M(m_1 + m_2))}$$

With the power of Mathematica, this simplifies to:

$$\dot{x}_1 = \frac{4m_2 p_1 + M(p_1 - p_2)}{4m_1 m_2 + M(m_1 + m_2)}$$

also, the equations here are symmetric in the sense that for $\dot{x}_2$, it's going to be the expression for x_1 except we change every $1 \longrightarrow 2$ and $2 \longrightarrow 1$, so therefore:

$$\dot{x}_2 = \frac{4m_1 p_2 + M(p_2 - p_1)}{4m_1 m_2 + M(m_1 + m_2)}$$

To get the other two equations we can use the relation $\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q}$:

$$\dot{p}_1 = \left(\frac{M}{2} - m_1 \right) g x_1$$

$$\dot{p}_2 = \left(\frac{M}{2} - m_2 \right) g x_2$$

$\square$

Problem 2

Consider a simple plane pendulum of mass m and length ℓ. After the pendulum is set into motion, the length of the string is shortened at a constant rate.

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = -\alpha$$

The suspension point remains fixed. Determine the both Lagrangian and Hamiltonian functions. Compare the Hamiltonian with $E = T + U$, and discuss conservation of energy for this system.

Solution: We write the kinetic energy as $T = \frac{1}{2}m\ell^2\dot{\phi}^2 + \frac{1}{2}m\alpha^2$, and the potential energy is $U = -mg\ell\cos\phi$, so therefore the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}m\ell^2\dot{\phi}^2 + \frac{1}{2}m\alpha^2 + mg\ell\cos\alpha$$

From here, we can get the generalized momenta using $p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$:

$$p_l = \frac{\partial\mathcal{L}}{\partial\dot{\ell}} = 0$$
$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\ell^2\dot{\phi}$$

So now, we can use $\mathcal{H} = \sum_i p_i q_i - \mathcal{L}$ to get:

$$\mathcal{H} = p_\phi\dot{\phi} - \left(\frac{1}{2}m\ell^2\dot{\phi}^2 + \frac{1}{2}m\alpha^2 + mgl\cos\alpha\right) = \frac{1}{2}m\ell^2\dot{\phi}^2 - \frac{1}{2}m\alpha^2 - mgl\cos\phi$$

Here, we can see that the Hamiltonian isn't the total energy, which is given by $T + U = \frac{1}{2}m\alpha^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mgl\cos\phi$. This is because the Hamiltonian $\mathcal{H}$ has explicit time dependence, namely in the parameter l and the fact that l changes in time as $\ell(t) = \ell_0 - \alpha t$, so thus $\frac{\partial\mathcal{H}}{\partial t} \neq 0$. □

Problem 3

Consider a particle of mass m constrained to move on a frictionless cylinder of radius R, described in cylindrical coordinates by the surface $\rho = R$. The mass is attached to a spring (spring constant k and equilibrium length zero) whose other end is fastened at the origin. Using cylindrical coordinates, (ρ, ϕ, z), find the Hamiltonian H. Then write down and solve Hamilton's equations and describe the motion.

Solution: Since cylindrical coordinates are natural, we know that the Hamiltonian is just going to be the kinetic plus potential energy. Setting the axis to be the origin, we can now write down the kinetic and potential energies in cylindrical coordinates:

$$\begin{aligned} T &= \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{2}(R^2\dot{\phi}^2 + z^2) \\ V &= \frac{1}{2}kr^2 = \frac{1}{2}k(R^2 + z^2) \end{aligned}$$

We can cancel the $\dot{\rho}$ terms since the mas sis constrained to move on the cylinder. Therefore, we have:

$$\mathcal{H} = T + V = \frac{m}{2}(R^2\dot{\phi}^2 + \dot{z}^2) + \frac{k}{2}(R^2 + z^2)$$

Computing the generalized momenta using $p_i = \frac{\mathrm{d}T}{\mathrm{d}q_i}$, since we're working in natural coordinates :

$$\begin{aligned} p_z &= \frac{\mathrm{d}T}{\mathrm{d}\dot{z}} = m\dot{z} \\ p_\phi &= \frac{\mathrm{d}T}{\mathrm{d}\dot{\phi}} = mR^2\dot{\phi} \end{aligned}$$

So Hamilton's equation then becomes:

$$\mathcal{H} = \frac{m}{2}\left(R^2\frac{p_\phi^2}{(mR^2)^2} + \frac{p_z^2}{m^2}\right) + \frac{k}{2}(R^2 + z^2) = \frac{p_\phi^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{k}{2}(R^2 + z^2)$$

Finally, we can write Hamilton's equations:

$$\begin{aligned} \dot{\phi} &= \frac{\partial\mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{2mR^2} \\ \dot{z} &= \frac{\partial\mathcal{H}}{\partial p_z} = \frac{p_z}{m} \\ \dot{p}_\phi &= -\frac{\partial\mathcal{H}}{\partial\phi} = 0 \\ \dot{p}_z &= -\frac{\partial\mathcal{H}}{\partial z} = -kz \end{aligned}$$

These then give us our equations of motion. Since $\dot{p}_\phi = 0$, this implies that p_ϕ is a constant, and thus $\dot{\phi}$, which is defined in terms of p_ϕ, is also a constant. In the z direction, we have

$$\dot{z} = \frac{p_z}{m} \implies \ddot{z} = -\frac{k}{m}z$$

And so we get $z(t) = A\cos(\omega t - \delta)$, where A, δ are some constants given by initial conditions.

In terms of the motion of the particle, we see that it oscillates sinusoidally in the z direction, and moves around in the $\hat{\phi}$ direction with a uniform velocity. This makes intuitive sense as well, since the spring force always acts perpendicular to $\hat{\phi}$, so we don't expect any change in velocity in that direction. The same cannot be said for the z direction, which is why we sinusoidal oscillations. $\square$

Problem 4

Consider the modified Atwood machine shown in Figure 13.11. The two weights on the left have equal mass m and are connected by a massless spring of force constant k. The weight on the right has mass $M = 2m$, and the pulley is massless and frictionless. The coordinate x is the extension of the spring from its equilibrium length; that is, the length of the spring is $l_e + x$, where l_e is the equilibrium length (with all the weights in position M held stationary).

a) Show that the total potential energy (spring plus gravitational) is just $U = \frac{1}{2}kx^2$ (plus a constant that can be taken to zero).

Solution: Using the position of the big wheel as zero potential energy, we can write the gravitational potential energy in the system as:

$$U_g = -mgy - mg(y + l_e + x) - Mg(L - y) = mg(y + y + l_e + x + 2(L - y) = -mgx + \text{const.}$$

Then, to express spring potential energy, let l_0 denote the natural length of the spring. From balancing of forces, we see that:

$$k(le - l_0) = mg$$

Then, we can now calculate the spring potential energy:

$$U_s = \frac{1}{2}k(x + le - l_0)^2 = \frac{1}{2}k\left(x + \frac{mg}{k}\right)^2 = \frac{1}{2}kx^2 + mgx + \frac{1}{2}k^3$$

Combining the two expressions now:

$$U_g + U_s = \frac{1}{2}kx^2 + mgx + \frac{1}{2}k^3 - mgx - \text{const.}$$

Since $\frac{1}{2}k^3$ is also a constant and the mgx terms cancel, then we see that the total energy in the system is indeed equal to $\frac{1}{2}kx^2$ plus a constant. □

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b) Find the two momenta conjugate to x and y. Solve for $\dot{x}$ and $\dot{y}$ and write down the Hamiltonian. Show that the coordinate y is ignorable.

Solution: Here, since the coordinate system is natural, we can find p_i by taking $\frac{\mathrm{d}T}{\mathrm{d}\dot{q}_i}$. First, calculating T:

$$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}my^2 + \frac{1}{2}m(\dot{x} + \dot{y})^2 = \frac{3}{2}m\dot{y}^2 + \frac{1}{2}m(\dot{x} + \dot{y})^2$$

therefore, we can now get p_i:

$$p_x = \frac{\partial T}{\partial \dot{x}} = m(\dot{x} + \dot{y})$$
$$p_y = \frac{\partial T}{\partial \dot{y}} = 3m\dot{y} + m(\dot{x} + \dot{y})$$

From these equations, we can then backsolve for $\dot{y}$ and $\dot{x} + \dot{y}$

$$\dot{x} + \dot{y} = \frac{p_x}{m}$$
$$\dot{y} = \frac{p_y - p_x}{3m}$$

Therefore, the Hamiltonian written in these coordinates is:

$$\mathcal{H} = T + U = \frac{1}{2}m\left[\frac{p_x^2}{m^2} + \frac{(p_y - p_x)^2}{m^2}\right] + \frac{1}{2}kx^2$$

Since $\frac{\partial \mathcal{H}}{\partial y} = 0$, then this means that y is an ignorable coordinate. □

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c) Write down the four Hamilton equations and solve them for the following initial conditions: You hold the mass M fixed with the whole system in equilibrium and $y = y_0$. Still holding M fixed, you pull the lower mass m a distance x_0, and at $t = 0$ you let go of both masses. [*Hint:* Write down the initial values x, y and their momenta. You can solve the x equations by combining them into a second-order equation for x. Once you know $x(t)$, you can quickly write down the other three variables.] Describe the motion. In particular, find the frequency with which x oscillates.

Solution: The four Hamilton equations come from just taking derivatives:

$$\begin{aligned}
\dot{x} &= \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{3m}(4p_x - p_y) \\
\dot{y} &= \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y - p_x}{3m} \\
\dot{p}_x &= -\frac{\partial \mathcal{H}}{\partial x} = -kx \\
\dot{p}_y &= -\frac{\partial \mathcal{H}}{\partial y} = 0
\end{aligned}$$

Since we hold M fixed, then this also means that we're holding the mass on the other side to be fixed as well. This gives the initial conditions: $\dot{x}(0) = 0$ and $\dot{y}(0) = 0$, implying that $p_x(0) = p_y(0) = 0$. Further, from Hamilton's equations, we can find $\ddot{x}$

$$\ddot{x} = \frac{1}{3m}(4\ddot{p}_x - \ddot{p}_y) = \frac{4k}{3m}x$$

This is the same equation of motion as a harmonic oscillator, so therefore x oscillates with frequency $\omega = \sqrt{\frac{4k}{3m}}$. Therefore:

$$x(t) = x_0 \cos(\omega t)$$

As for the other coordinates, since we know that $\dot{p}_x = kx$, then:

$$p_x(t) = \frac{kx_0}{\omega} \sin(\omega t)$$

Finally, we can get $y(t)$ using $\dot{y} = -\frac{p_x}{3m}$:

$$y(t) = \frac{1}{3m}\frac{x_0}{\omega} \cos(\omega t) + y_0$$

$\square$

Problem 5

A spherical pendulum consists of a mass m attached to a massless, rigid rod of length ℓ. The end of the rod opposite the mass can pivot freely (hence “spherical" pendulum) in all directions about some fixed point.

a) Set up the Hamiltonian function (in spherical coordinates!). Check that if $p_\phi = 0$, the result is the same as that for a plane pendulum.

Solution: To set up the Hamiltonian, we again use the fact that our coordinate system is natural, so the Hamiltonian in this case will just be the sum of our potential and kinetic energies. Writing this out in spherical coordinates, we get:

$$T = \frac{1}{2}m(\dot{r}^2 + \ell^2\dot{\theta}^2 + \ell^2\sin^2\theta\dot{\phi}^2) = \frac{m}{2}\left(\ell^2\dot{\theta}^2 + \ell^2\sin^2\theta\dot{\phi}^2\right)\; V = mg\cos\theta$$

We drop the $\dot{\ell}$ term because the rod is rigid, so there is no change radially. Now writing out the generalized momentum using $p_i = \frac{\mathrm{d}T}{\mathrm{d}\dot{q}_i}$ (again, we can do this because our coordinate system is natural):

$$\begin{aligned} p_\theta &= \frac{\partial T}{\partial\dot{\theta}} = m\ell^2\dot{\theta} \implies \dot{\theta} = \frac{p_\theta}{m\ell^2} \\ p_\phi &= \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\sin^2\theta\dot{\phi} \implies \dot{\phi} = \frac{p_\phi}{m\ell^2\sin^2\theta} \end{aligned}$$

So therefore, we can rewrite T:

$$T = \frac{1}{2m\ell^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right)$$

In the case where $p_\phi = 0$, then the second term in the parentheses goes to zero, so therefore we're left with:

$$\mathcal{H} = \frac{p_\theta^2}{2m\ell^2} + mg\cos\theta$$

Which is the exact equation for a plane pendulum. □

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b) Combine the term that depends on p_ϕ with the ordinary potential energy term to define an effective potential $V(\theta, p_\phi)$. Sketch V as a function of θ for several values of p_ϕ, making sure to include $p_\phi = 0$.

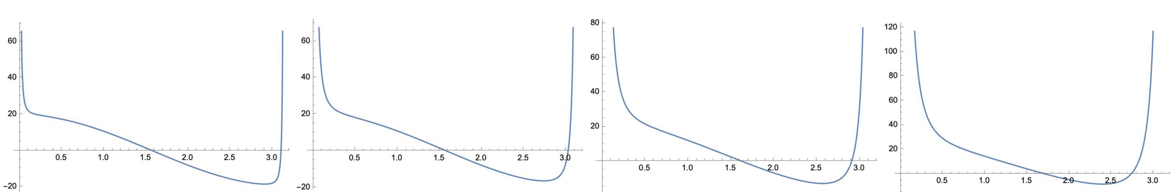
Solution: Rewirting the Hamiltonian a bit:

$$\mathcal{H} = \frac{p_\theta^2}{2m\ell^2} + \frac{p_\phi^2}{2m\ell^2\sin^2\theta} + mg\cos\theta$$

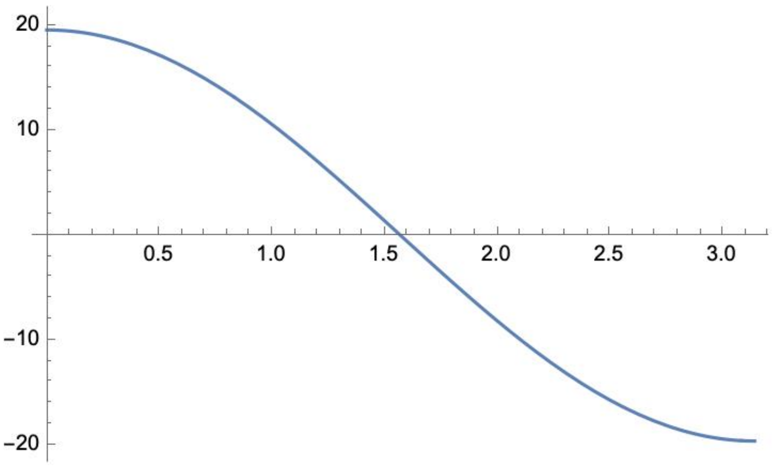
Then it's clear that if we want only the p_θ term to remain, then we would want the effective potential to be

$$V(\theta, p_\phi) = \frac{p_\phi^2}{2m\ell^2\sin^2\theta} + mg\cos\theta$$

The sketches for $V(\theta, p_\phi)$ for $\theta \in [0, \pi]$ are shown below, first for $p_\phi = 1, 3, 6, 10$ in that order, from left to right:



For $p_\phi = 0$, this is what we get over the same interval $\theta \in [0, \pi]$:



□

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c) Discuss the features of the motion, pointing out the differences between $p_\phi = 0$ and $p_\phi \neq 0$. Discuss the limiting case of the conical pendulum, $\theta = C$, a constant, with reference to the $V - \theta$ diagram.

Solution: Firstly, the motion looks very different for $p_\phi = 0$ as opposed to $p_\phi \neq 0$. Perhaps the most noticeable difference is the fact that with the case where $p_\phi \neq 0$, we get a global minimum at a $\theta \in (0, \pi)$, whose specific value is determined by p_ϕ itself. In the case where $p_\phi = 0$, we get a global minimum at $\theta = \pi$ – this makes sense, since there is no angular component to "carry" the mass around the sphere.

In terms of the motion itself: when $p_\phi \neq 0$, then we see that this leads to $\theta \neq 0$, implying that the mass travels around the sphere. For larger values of p_ϕ, we can see that the local minimum of the curve approaches $\frac{\pi}{2}$, this corresponds to the point when the mass rotates around the axis horizontally.

To analyze the conical pendulum, it's useful to first notice that $\frac{\partial \mathcal{H}}{\partial \phi} = 0$, so therefore p_ϕ is a constant. Because of this, it means that the pendulum will circulate the vertical axis about which the pendulum is rotating with both a constant polar angle and constant azimuthal angular velocity In other words, $\dot{\phi}$ is a constant throughout the motion. □
