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HW 02

Signals and Systems

EE 120
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Collaborators

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Problem 1

Determine whether or not each of the followoing continuous-time or discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

a) $x(t) = 3\cos\left(4t + \frac{\pi}{3}\right)$

Solution: This is a cosine signal, so it is periodic. The fundamental period is $\frac{2\pi}{4} = \frac{\pi}{2}$ $\square$

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b) $x(t) = e^{i(\pi t - 1)}$

Solution: Express $e^{i(\pi t-1)} = \cos(\pi t - 1) + i\sin(\pi t - 1)$. Both parts are periodic, with a period of 2. $\square$

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c) $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)} u(2t - n)$

Solution: For integer values of $2t$, the summation is the same, except shifted by a constant amount because the value of n where the step function $u(t)$ is 1 changes. This doesn't change the value of the infinite sum however, becuase our summation over n goes from $-\infty$ to ∞ so we can just shift by whatever $2t$ was to recover the same sum.

For non-integral values of $2t$, the summation over n doesn't change, but the value of $e^{-(2t-n)}$ changes. This repeats at every integer (because of the earlier argument), so therefore the summation will repeat at every integer. This means that this signal is periodic when $2t$ is an integer, or in other words it has a half-integer period: $\frac{1}{2}$. $\square$

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d) $x[n] = \sin[\frac{n}{8} - \pi]$

Solution: For $n = \{1, 2, \ldots, 7\}$, the argument to the sine function is different, but every 8 inputs the cycle repeats, so therefore the fundamental period is 8. $\square$

Problem 2

For each of the following systems defined by their LCCDE, specify the system is i) linear or not, ii) time-invariant or not, iii) memoryless or not, iv) causal or not, and v) BIBO stable or not. Show your work along with your conclusions to get credit. For all the sub-questions, $x(t)$ or $x[n]$ is the input, and $y(t)$ or $y[n]$ is the system output.

a) $y(t) = x(t-2) + x(2-t)$

Solution:

- Linear, because given two signasl $x_1(t)$ and $x_2(t)$, define a new signal $x'(t) = ax_1(t) + bx_2(t)$, then $y'(t)$ is

$$y'(t) = ax_1(t-2) + bx_2(t-2) + ax_1(2-t) + bx_2(2-t) = ay_1(t) + by_2(t)$$

  another argument for this is that given a signal $x(t) = 0$ we get an output $y(t) = 0$, so it satisfes the zero-input zero-output property.

- Not time invariant, because given a signal $x'(t) = x(t-T)$, the output is:

$$y'(t) = x'(t-2) + x'(2-t) = x(t-2-T) + x(2-t-T)$$

  But $y(t-T) = x(t-2-T) + x(2-t+T)$; they're not equal, so the system is not time invariant.

- This is not memoryless, because the output at time t, requires knowledge of $x(t-2)$.
- It's not causal, because $t = 0$, then the system $y(t)$ depends on the $x(2)$, implying it's not causal.
- This system is BIBO stable, because for a bounded input $x(t)$, the output $y(t)$ takes in two bounded values, hence the output must be bounded.

□

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b) $y[n] = \sum_{k=0}^{4}(-1)^k x[n-k]$

Solution:

- Linear, because all sums by themselves are linear and the terms within the summation are also linear. Also, zero input give zero output.
- Time invariant, because given a signal $x'[n] = x[n-m]$, the output $y[n]$ is:

$$y[n] = \sum_{k=0}^{4}(-1)^k x'[n-k] = \sum_{k=0}^{4}(-1)^k x[n-m-k]$$

  Then, $y[n-m]$ is given by:

$$y[n-m] = \sum_{k=0}^{4}(-1)^k x[n-m-k]$$

  They match, as desired.

- The system is not memoryless, because the output $y[n]$ depends on previous values $x[n-1]$ thorugh $x[n-4]$.
- The system is causal, because it doesn't depend on future values of n.
- This system is BIBO stable, because given a bounded $x[n]$, the output $y[n]$ sums over four bounded values, which gives us a resulting bounded output.

□

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c) $y[n] = \cos[x[n] + \frac{\pi}{3}]$

Solution:

- Linear: Given a zero input, $y[n] = \cos[\pi/3] \neq 0$.
- Time invariant: Given an input signal $x'[n] = x[n - n_0]$, the signal output is:

$$y'[n] = \cos\left[x'[n] + \frac{\pi}{3}\right] = \cos\left[x[n - n_0] + \frac{\pi}{3}\right] = y[n - n_0]$$

  so the signal is delayed by a constant n_0, hence it is time invariant.
- Memoryless: The system only depends on n.
- Causal: memoryless implies causal.
- The system is BIBO stable, because given a bounded $x[n]$, $\cos[x[n] + \pi/3]$ returns a bounded value.

□

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d) $y(t) = x(t)v(t)$, where $v(t)$ is a fixed signal such that

- For all $t < 0$, $v(t) = 0$
- For every $B \in \mathbb{R}$, there exists $T \in \mathbb{R}$ such that $v(T) > B$

Solution:

- Linear: Given a zero input, the output is always zero.
- Not time invariant: Given an input signal $x'(t) = x(t - t_0)$, the output is given by:

$$y'(t) = x'(t)v(t) = x(t - t_0)v(t) \neq y(t - t_0)$$

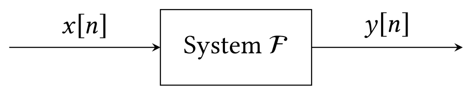
- Memoryless: the signal only depends on t.
- Causal: memoryless implies causality.
- The system is not BIBO stable, becuase given a bounded input $x(t)$, there are signals $v(t)$ (e.g. a signal that is always infinite) which could return an unbounded value for $y(t)$.

  Alternatively, one could argue that given a bounded input $x(t) = 1$, then $y(t) = v(t)$, but $v(t)$ is necessarily unbounded (by its description), so the output signal is necessarily unbounded.

□

Problem 3

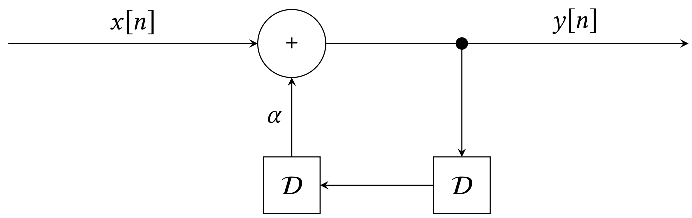
The causal, discrete time LTI filter $\mathcal{F}$ is shown below,



is characterized by the recursive difference equation $y[n] = \alpha y[n-N] + x[n]$, where $|\alpha| < 1$ and $N \in \{1, 2, 3, \ldots\}$. Assume systems start at rest, that is for $n < 0$, $x[n], y[n] = 0$.

a) Draw the block diagram of this system for $N = 2$.

Solution: For $N = 2$, the block diagram is:



$\square$

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b) Determine an expression for $f[n]$, where f is the impulse response of the filter. Express your answer in terms of α and N. *Solution:* The system is definde as:

$$y[n] = \alpha y[n-N] + x[n]$$

Expanding this out further:

$$y[n] = \alpha(y[n-N] + x[n-N]) + x[n] = \alpha(\alpha(y[n-2N] + x[n-2N]) + x[n-N]) + x[n]$$

Therefore, in general:

$$y[n] = \sum_{k=0}^{\lfloor n/N \rfloor} \alpha^k x[n-kN]$$

Now, plugging in $x[n] = \delta[n]$, then we have:

$$f[n] = \sum_{k=0}^{n/N} \alpha^k \delta[n-kN]$$

$\square$

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c) Provide well-labeled sketches of $f[n]$ for

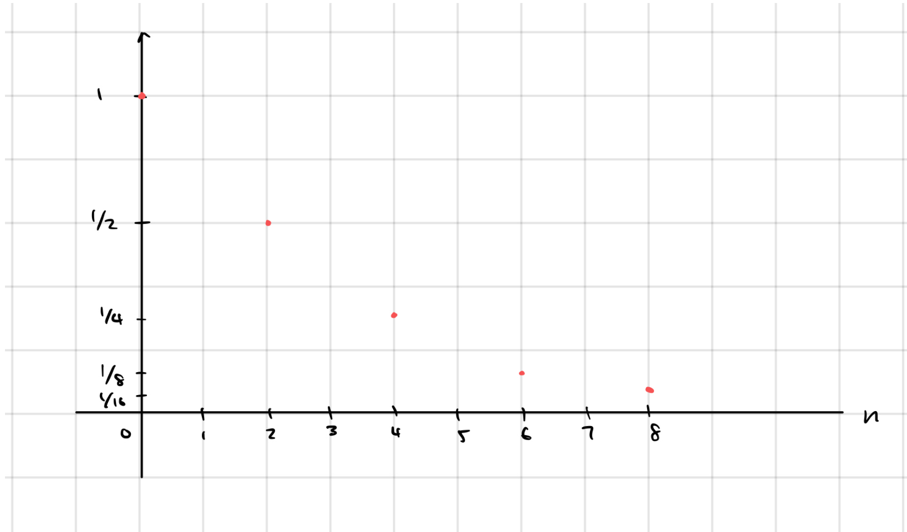
i) $\alpha = \frac{1}{2}$ and $N = 2$

ii) $\alpha = \frac{1}{\sqrt{2}}$ and $N = 2$.

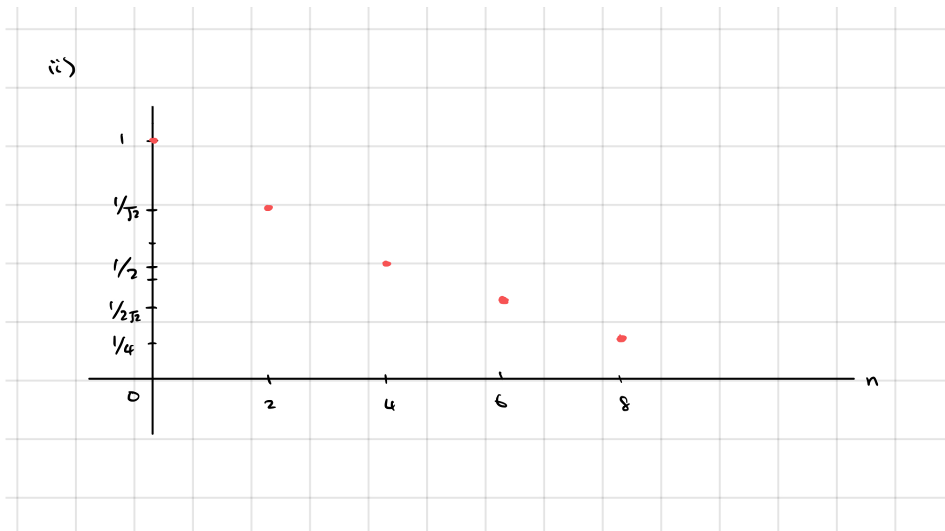
iii) $\alpha = \frac{1}{2}$ and $N = 4$

Describe, in simple terms, the respective effects of α and N on the qualitative features of the impulse response f.

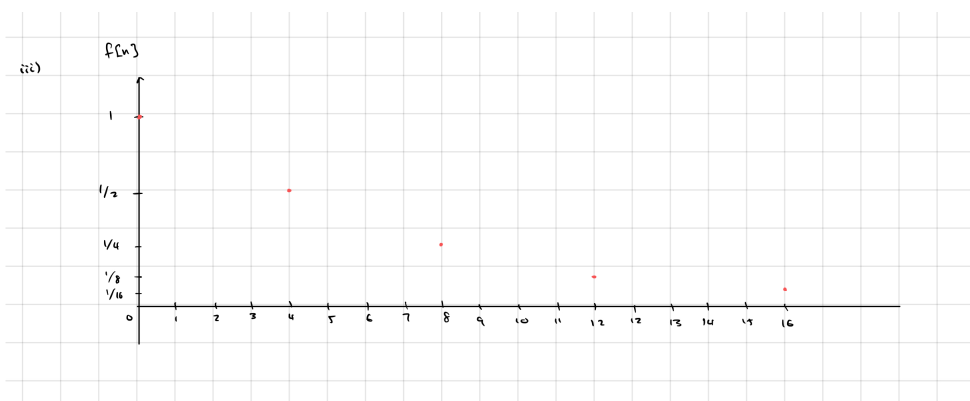
Solution: I give up on drawing this in LaTeX, here's the plots I drew on my ipad:


1
1/2
1/4
1/8
1/16
0
1
2
3
4
5
6
7
8
n

i)


ii)
1
1/√2
1/2
1/2√2
1/4
0
2
4
6
8
n

ii)


iii)
f[n]
1
1/2
1/4
1/8
1/16
0

iii)

☐

Problem 4

For the following system with $x(t) = e^{-t}u(t)$ as input, $y(t)$ as output

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{3}y(t) = x(t)$$

Assume the system is at rest.

a) Determine the homogeneous solution to the output of this system.
Hint: Assume the solution is of the form Ae^{st} where $A \neq 0$.

Solution: The homogeneous solution is when $x(t) = 0$. We use the Ansatz of $y = Ae^{st}$, so $\frac{\mathrm{d}y}{\mathrm{d}t} = Ase^{st}$:

$$Ase^{st} + \frac{A}{3}e^{st} = 0 \implies s + \frac{1}{3} = 0 \implies s = -\frac{1}{3}$$

so the homogeneous solution is $Ae^{-t/3}$. □

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b) Determine the particular solution to the output of this system
Hint: Assume the solution is of the form $Ke^{bt}u(t)$ where $K \neq 0$

Solution: Here, we're trying to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{3}y(t) = e^{-t}u(t)$$

For $t < 0$, the right hand side is 0, so we have $y(t) = 0$ for $t < 0$. For $t \geq 0$, $u(t) = 1$, so our Ansatz is Ke^{bt}, and we have:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{3}y(t) = e^{-t}$$

Therefore:

$$Kbe^{bt} + \frac{1}{3}Ke^{bt} = e^{-t} \implies Ke^{bt}\left(b + \frac{1}{3}\right) = e^{-t}$$

Now, we match the exponents: we want the exponent to be e^{-t}, implying that $b = -1$. Then, we have:

$$K\left(-1 + \frac{1}{3}\right) = 1 \implies K = -\frac{3}{2}$$

□

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c) Determine the output of this system, when $x(t) = e^{-t}u(t)$ is the input.

Solution: The output of the system is going to be a superposition of the homogeneous and particular solution, so:

$$y(t) = Ae^{-t/3} - \frac{3}{2}e^{-t}u(t)$$

The condition of the system being at rest is the condition that $y(0) = 0$, implying that

$$A - \frac{3}{2} = 0 \implies A = \frac{3}{2}$$

Therefore:

$$y(t) = \frac{3}{2}\left(e^{-t/3} - e^{-t}u(t)\right)$$

□

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d) Now let the input be $\hat{x} = x(t-T) = e^{-(t-T)}u(t-T)$, determine the input $\hat{y}(t)$ of this system. Show that $\hat{y}(t) = y(t-T)$.

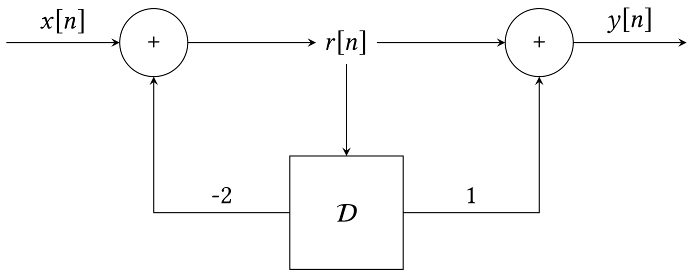
Solution: All we need to do is show that the system is time invariant. Given a signal $\hat{x} = x(t-T)$, then the differential equation looks like:

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}t} + \frac{1}{3}\hat{y}(t) = \hat{x}(t) = x(t-T)$$

And since the differential equation has the same form (i.e. we can substitute $t' = t - T$ and we get the exact same DE, then this implies that $\hat{y}(t) = y(t-T)$. □

Problem 5

Consider the following system according to its diagram. Assume the system is initially at rest, that is for $n < 0$, $x[n], y[n] = 0$.



a) Express the output $y[n]$ with respect to $x[n]$ with a recursive equation.

Solution: We can express $r[n]$ as:

$$r[n] = x[n] - 2r[n-1]$$

then, we can express $y[n]$ as:

$$y[n] = r[n] + r[n-1]$$

This implies:

$$y[n] = x[n] - 2r[n-1] + r[n-1] = x[n] - r[n-1] = x[n] - x[n-1] + 2(x[n-2] - 2r[n-3])$$

So this simplifies to:

$$y[n] = x[n] + \sum_{k=0}^{n} (-2)^k x[n-k-1]$$

The summation stps at n becuase for $n < 0$, $x[n] = 0$. $\square$

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b) For $x[n] = \delta[n]$, find $r[n]$ for all n.

Solution: For $x[n] = \delta[n]$, then we have $r[n] = \delta[n] - 2r[n-1] = \delta[n] - 2(\delta[n-1] - 2r[n-2])$. Writing this out as a summation, we have:

$$r[n] = \sum_{k=0}^{n} (-2)^k \delta[n-k]$$

So for positive $n > 0$, only $k = n$ matters, since that's the only term that makes $\delta[n-k]$ nonzero. For $n < 0$, the system should always be 0, so therefore we have:

$$r[n] = (-2)^n u[n]$$

$\square$

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c) Find this system's impulse response.

Solution: The impulse response is given by:

$$y[n] = \delta[n] + \sum_{k=0}^{n} (-2)^k \delta[n-k-1]$$

For $n \geq 0$, we have $y[n] = 1$ for $n = 0$, then for other values of $y[n]$ we have:

$$y[n] = (-2)^{n-1} u[n]$$

The exponent of $n-1$ is here becuase within the summation only $k = n-1$ matters, and the step function $u[n]$ exists only to suppress all outputs for $n < 0$. $\square$

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d) Find this system's step response.
Hint: Look up the Jacobsthal numbers.

Solution: Instead of working with the form of $y[n]$, which is quite difficult to do, it helps to look at $r[n]$ instead. Becuase $x[n] = u[n]$, then for all intents and purposes, we can write:

$$r[n] = 1 - 2r[n-1]$$

since for $n > 0$ this formula holds, and that's also the only regime of n that we care about. Then, we write out $r[n-1]$ recursively:

$$r[n] = 1 - 2r[n-1] = 1 - 2(1 - 2r[n-2] = -1 + 4r[n-2]$$

Taking these two forms of $r[n]$, we have:

$$r[n] + r[n] = -2r[n-1] + 4r[n-2] \implies r[n] = -r[n-1] + 2r[n-2]$$

Now, from the block diagrm, we can infer that $r[-1] = 0$ and $r[0] = 1$. Then, it helps to list out the values generated by this recursion:

n	$r[n]$
-1	0
0	1
1	-1
2	3
3	-5
4	11
5	-21
6	43

these are the Jacobsthal numbers, except they're alternating in sign. First, we prove that every second term has the same sign. To do that, notice that the first two terms alternate in sign, meaning that $r[n-1]$ and $r[n-2]$ have opposite signs. This then means that $-r[n-1]$ and $r[n-2]$ have the same sign, so their sum adds constructively. Therefore, looking only at its magnitude, this recurrence formula is the same as $r[n] = r[n-1] + r[n-2]$, if $r[n-1]$ and $r[n-2]$ had the same sign, which is the exact formula for the Jacobsthal numbers. This explains the pattern in the sequence of $r[n]$ so we can write $r[n] = (-1)^n J[n+1]$, if we let $J[0] = 0, J[1] = 1$. Then, since we know that $y[n] = r[n] - r[n-1]$, then $y[n]$ is determined by taking the difference of each $r[n]$. Thus, if $J[n]$ is the n-th Jacobsthal number, one could write:

$$y[n] = (-1)^n (J[n+1] - J[n])$$

the $(-1)^n$ exists to account for the alternating nature of $r[n]$. $\square$