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Physics 110B

HW 04

Electromagnetism and Optics II

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Problem 1

In this problem we will consider the reflection and transmission of EM waves in linear medium with the oblique incidence of a monochromatic sinusoidal plane wave and the polarization of the incidence wave to be *perpendicular* to the plane of incidence. The coordinates are set up in a way as shown in the right figure. (You might need to zoom in with the pdf file to see the angle ϕ_R and ϕ_T). In this problem, you can assume that Snell's law holds, and we define the parameters $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$ and $\alpha \equiv \frac{\cos\theta_2}{\cos\theta_1}$.

(a) Using boundary conditions, show that the reflected and transmitted waves have the same polarization as the incident wave. That is, show that

$$\phi_R = \phi_T = \frac{\pi}{2}$$

Solution: First, we break each field down via vector decomposition:

$$\begin{aligned}
\mathbf{E}_I &= E_I \hat{\mathbf{y}} \\
\mathbf{E}_R &= E_R \left[\sin\phi_R \hat{\mathbf{y}} + \cos\phi_R \sin\theta_1 \hat{\mathbf{z}} + \cos\phi_R \cos\theta_1 \hat{\mathbf{x}}\right] \\
\mathbf{E}_T &= E_T \left[\sin\phi_T \hat{\mathbf{y}} + \cos\phi_T \sin\theta_2 \hat{\mathbf{z}} - \cos\phi_T \cos\theta_2 \hat{\mathbf{x}}\right]
\end{aligned}$$

The $\mathbf{B}$ field can also be found, using $\mathbf{B} = \frac{1}{c}\mathbf{k} \times \mathbf{E}$ to get the direction:

$$\begin{aligned}
\mathbf{B}_I &= B_I(\sin\theta_1 \hat{\mathbf{z}} - \cos\theta_1 \hat{\mathbf{x}}) \\
\mathbf{B}_R &= B_R(\cos\phi_R \hat{\mathbf{y}} + \sin\phi_R \sin\theta_1 \hat{\mathbf{z}} + \sin\phi_R \cos\theta_1 \hat{\mathbf{x}}) \\
\mathbf{B}_T &= B_T(\cos\phi_T \hat{\mathbf{y}} + \sin\phi_T \sin\theta_2 \hat{\mathbf{z}} - \sin\phi_T \cos\theta_2 \hat{\mathbf{x}})
\end{aligned}$$

The boundary conditions are imposed just the same as in lecture:

$$\epsilon_1(\mathbf{E}_I + \mathbf{E}_R)_z = \epsilon_2(\mathbf{E}_T)_z \tag{1}$$

$$(\mathbf{E}_I + \mathbf{E}_R)_{x,y} = (\mathbf{E}_T)_{x,y} \tag{2}$$

$$(\mathbf{B}_I + \mathbf{B}_R)_z = (\mathbf{B}_T)_z \tag{3}$$

$$\frac{1}{\mu_1}(\mathbf{B}_I + \mathbf{B}_r)_{x,y} = \frac{1}{\mu_2}(\mathbf{B}_T)_{x,y} \tag{4}$$

So, imposing all the boundary conditions, we have:

$$1: \quad \epsilon_1 E_R \cos\phi_R \sin\theta_1 = \epsilon_2 E_T \cos\phi_T \sin\theta_2$$

$$2: \quad \begin{cases} E_R \cos\phi_R \cos\theta_1 = -E_T \cos\phi_T \cos\theta_2 & \text{(x direction)} \\ E_I + E_R \sin\phi_R = E_T \sin\phi_T & \text{(y direction)} \end{cases}$$

$$3: \quad \frac{1}{v_1}(E_I \sin\theta_1 + E_R \sin\phi_R \sin\theta_1) = \frac{1}{v_2} E_T \sin\phi_T \sin\theta_2$$

$$4: \quad \begin{cases} \frac{1}{\mu_1 v_1}(-E_I \cos\theta_1 + E_R \sin\phi_R \cos\theta_1) = -\frac{1}{\mu_2 v_2} E_T \sin\phi_T \cos\theta_2 & \text{(x direction)} \\ \frac{1}{\mu_1 v_1} E_R \cos\phi_R = -\frac{1}{\mu_2 v_2} E_T \cos\phi_T & \text{(y direction)} \end{cases}$$

Now, combining the first part of 2 and second part of 4, we get:

$$-\beta E_T \cos\phi_T \cos\theta_1 = -E_T \cos\phi_T \cos\theta_2 \implies E_T \cos\phi_T(\cos\theta_2 - \beta\cos\theta_1) = 0$$

We know that $\cos\theta_2 - \beta\cos\theta_1$ is not always zero, so this leaves $\cos\phi_T = 0$, which requires $\phi_T = \frac{\pi}{2}$. Plugging this back into the second part of 4, this also implies $\phi_R = \frac{\pi}{2}$. $\square$

(b) Show that the complex amplitudes $\tilde{E}_{0I}$, $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$ are related to each other by

$$\tilde{E}_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0I} \quad \text{and} \quad \tilde{E}_{0T} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{0I}$$

Solution: With the condition on ϕ_R and ϕ_T, it helps to simplify the boundary conditions, so $\sin\phi_R = 1$ and $\cos\phi_R = 0$:

$$\begin{aligned}
&2: \quad E_I + E_R = E_T \\
&3: \quad \frac{1}{v_1}(E_I\sin\theta_1 + E_R\sin\theta_1) = \frac{1}{v_2}E_T\sin\theta_2 \\
&4: \quad \frac{1}{\mu_1 v_1}(-E_I\cos\theta_1 + E_R\cos\theta_1) = -\frac{1}{\mu_2 v_2}E_T\cos\theta_2
\end{aligned}$$

The third equation can be written in terms of α and β:

$$-E_I + E_R = -\alpha\beta E_T$$

Now substitute in the first equation:

$$-E_I + E_R = -\alpha\beta(E_I + E_R) \implies E_R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)E_I$$

as desired. Substituting this back into the first equation, we have:

$$E_I + \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)E_I = E_T \implies E_T = \left(\frac{2}{1+\alpha\beta}\right)E_I$$

as desired. $\square$

(c) In the case where $\mu_1 \simeq \mu_2 \simeq \mu_0$, show that the reflected wave has a π-phase shift with respect to the incident wave then $v_2 < v_1$.

Solution: In this case, then $\beta \approx \frac{v_1}{v_2} > 1$, so all we need to show is that $1 - \alpha\beta < 1$, or in other words $\alpha > 1$. We show this by rewriting α using Snell's law

$$\alpha = \frac{\cos\theta_2}{\cos\theta_1} = \frac{\sqrt{1-\sin^2\theta_2}}{\cos\theta_1} = \left(\frac{1-\left(\frac{v_2}{v_1}\right)^2\sin^2\theta_1}{1-\sin^2\theta_1}\right)^{1/2}$$

Now, take a look at this quantity. We know that $\frac{v_2}{v_1} < 1$, so the quantity in the numerator is larger than that in the denominator. Therefore, the fraction is larger than one, so $\alpha > 1$. Therefore $\alpha\beta > 1$, and as such $1 - \alpha\beta < 1$, so that introduces a minus sign:

$$E_R = -\frac{|1-\alpha\beta|}{|1+\alpha\beta|}E_I$$

which is equivalent to a π phase shift. $\square$

(d) Is there *Brewster's angle* when the incident wave has a polarization perpendicular to the plane of incidence? That is, is there an incident angle such that $\tilde{E}_{0R} = 0$?

Solution: Continuing the math from the previous section, we know that we can write:

$$\alpha = \frac{1}{\beta}\frac{\sqrt{\beta^2 - \sin^2\theta_1}}{\cos\theta_1}$$

For $\tilde{E}_{0R} = 0$, then we require $1 - \alpha\beta = 0$, so we are looking for an angle θ_1 such that:

$$1 - \frac{\sqrt{\beta^2 - \sin^2\theta_1}}{\cos\theta_1} = 0 \implies \cos\theta_1 = \sqrt{\beta^2 - \sin^2\theta_1}$$

This eventually simplifies to $\beta = 1$, which is the case where the two media are identical (i.e. no boundary exists). Therefore, there is no Brewster's angle. $\square$

(e) Consider the z-component of the Poynting vector.

$$\langle|\mathbf{S}_I \cdot \hat{\mathbf{z}}|\rangle = \frac{|\tilde{E}_{0I}|^2}{2\mu_1 v_1}\cos\theta_1, \quad \langle|\mathbf{S}_R \cdot \hat{\mathbf{z}}|\rangle = \frac{|\tilde{E}_{0R}|^2}{2\mu_1 v_1}\cos\theta_1, \quad \text{and} \quad \langle|\mathbf{S}_T \cdot \hat{\mathbf{z}}|\rangle = \frac{|\tilde{E}_{0T}|^2}{2\mu_2 v_2}\cos\theta_2$$

where $\langle \dots \rangle$ means time average. The reflection and transmission coefficients are defined as

$$R \equiv \frac{\langle|\mathbf{S}_R \cdot \hat{\mathbf{z}}|\rangle}{\langle|\mathbf{S}_I \cdot \hat{\mathbf{z}}|\rangle} \quad \text{and} \quad T \equiv \frac{\langle|\mathbf{S}_T \cdot \hat{\mathbf{z}}|\rangle}{\langle|\mathbf{S}_I \cdot \hat{\mathbf{z}}|\rangle}$$

Explicitly check that $R + T = 1$.

Solution: We use the formulas that we have from part (b):

$$R = \frac{\frac{|E_R|^2}{2\mu_1 v_1}\cos\theta_1}{\frac{|E_I|^2}{2\mu_1 v_1}\cos\theta_1} = \frac{|E_R|^2}{|E_I|^2} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 \quad T = \frac{\frac{|E_T|^2}{2\mu_2 v_2}\cos\theta_2}{\frac{|E_I|^2}{2\mu_1 v_1}\cos\theta_1} = \alpha\beta\frac{|E_T|^2}{|E_I|^2} = \alpha\left(\frac{2}{1+\alpha\beta}\right)^2$$

Adding these two up:

$$R + T = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 + \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2 = \frac{(1-\alpha\beta)^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1$$

as desired. $\square$

Problem 2

According to Snell's law, when light passes from an optically dense medium into a less dense one ($n_1 > n_2$), the propagation vector $\mathbf{k}$ bends *away* from the normal (Fig. below). In particular, if the light is incident at the **critical angle**

$$\theta_c \equiv \sin^{-1}(n_2/n_1)$$

then $\theta_T = 90°$, and the transmitted ray just grazes the surface. If θ_I *exceeds* θ_c, there is no refracted ray at all, only a reflected one (this is the phenomenon of **total internal reflection**, on which light pipes and fiber optics are based). But the *fields* are not zero in medium 2; what we get is a so-called **evanescent wave**, which is rapidly attenuated and transports no energy into medium 2.

A quick way to construct the evanescent wave is simply to quote the results of Section 9.3.3, with $k_T = \omega n_2/c$ and

$$\mathbf{k}_T = k_T(\sin\theta_T\hat{\mathbf{x}} + \cos\theta_T\hat{\mathbf{z}})$$

the only change is that

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$$

is now greater than 1, and

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = i\sqrt{\sin^2\theta_T - 1}$$

is imaginary. (Obviously, θ_T can no longer be interpreted as an angle!)

(a) Show that

$$\mathbf{E}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}$$

where

$$\kappa \equiv \frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - n_2^2} \quad \text{and} \quad k \equiv \frac{\omega n_1}{c}\sin\theta_I$$

this is a wave propagating in the x direction (*parallel* to the interface!), and attenuated in the z direction.

Solution: We did this portion in lecture. Consider first that the transmitted wave can be written as:

$$\mathbf{E}_T = \mathrm{Re}\left\{E_{0T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\right\}$$

Here, we write

$$\mathbf{k}_T = k_T\cos\theta_T\hat{\mathbf{z}} + k_T\sin\theta_T\hat{\mathbf{x}}$$

From the problem statement, we know that $\cos\theta_T$ is purely imaginary, so we can write $i\kappa$ as:

$$i\kappa = k_T\cos\theta_T = k_T i\sqrt{\sin^2\theta_T - 1} = i\frac{k_T}{n_2}\sqrt{(n_1\sin\theta_I)^2 - n_2^2} = i\frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - n_2^2}$$

Now, we can do the same thing for the $\hat{\mathbf{x}}$ direction, using Snell's law once again:

$$k = k_T\sin\theta_T = k_T\frac{n_1}{n_2}\sin\theta_I = \frac{\omega}{c}n_1\sin\theta_1$$

Combining these two equations, we get the desired result:

$$\mathbf{E}_T = \mathbf{E}_{0T}e^{i(i\kappa z + kx - \omega t)} = \mathbf{E}_{0T}e^{-\kappa z}e^{i(kx-\omega t)}$$

as desired. □

(b) Noting that α is now imaginary, use Eq. 9.110 to calculate the reflection coefficient for polarization parallel to the plane of incidence. [Notice that you get 100 percent reflection, which is better than a conducting surface (see, for example, prob. 9.23).]

Solution: Equation 9.110 reads:

$$\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \tilde{E}_{0I}$$

Now that α is purely imaginary, then we can write $\alpha = ia$ for $a \in \mathbb{R}$, so therefore:

$$R = \left|\frac{\alpha - \beta}{\alpha + \beta}\right|^2 = \left|\frac{-\beta + ia}{\beta + ia}\right|^2 = \left|\frac{a^2 + \beta^2}{a^2 + \beta^2}\right| = 1$$

as desired. ☐

(c) Do the same for polarization perpendicular to the plane of incidence (use the results of Prob. 9.17).

Solution: We will take the results from problem 1 of the homework. None of the math actually changes except for the fact that now α is again imaginary, so we have:

$$R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right) E_I, \quad T = \frac{2}{1 + \alpha\beta} E_I$$

Again, let $\alpha = ia$ where $a \in \mathbb{R}$, so:

$$R = \left|\frac{1 - ia\beta}{1 + ia\beta}\right|^2 = \frac{1 + a^2\beta^2}{1 + a^2\beta^2} = 1$$

so this is the same. ☐

(d) In the case of polarization perpendicular to the plane of incidence, show that the (real) evanescent wave fields are

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= E_0 e^{-\kappa z} \cos(kx - \omega t)\hat{\mathbf{y}} \\ \mathbf{B}(\mathbf{r}, t) &= \frac{E_0}{\omega} e^{-\kappa z} \left[\kappa \sin(kx - \omega t)\hat{\mathbf{x}} + k \cos(kx - \omega t)\hat{\mathbf{z}}\right] \end{aligned}$$

Solution: From part (a) we know that:

$$\mathbf{E}_T = \mathbf{E}_{0T} e^{-\kappa z} e^{i(kx - \omega t)}$$

From problem 1, we know that $\mathbf{E}_{0T}$ points strictly in the $\hat{\mathbf{y}}$ direction, so taking the real part:

$$\mathbf{E}_T = E_{0T} e^{-\kappa z} \cos(kx - \omega t)\hat{\mathbf{y}}$$

For the $\mathbf{B}$ field, we know from problem 1 that:

$$\mathbf{B}_T = B_T(\mathbf{r}, t)(\sin\theta_T \hat{\mathbf{z}} - \cos\theta_T \hat{\mathbf{x}}) = \frac{E_T(\mathbf{r}, t)}{v}(\sin\theta_T \hat{\mathbf{z}} - \cos\theta_T \hat{\mathbf{x}})$$

From here, it will be useful to define a new constant a:

$$a = \frac{1}{n_2}\sqrt{\sin^2\theta_T - 1}$$

so we may write $\cos\theta_T = ia$. Likewise, we can substitute $\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$. This leaves us with:

$$\mathbf{B}_T(\mathbf{r}, t) = \frac{E_{0T}}{v} e^{-\kappa z}\, \mathrm{Re}\left\{(\cos(kx - \omega t) + i\sin(kx - \omega t))\left(\frac{n_1}{n_2}\sin\theta_I \hat{\mathbf{x}} - ia\hat{\mathbf{z}}\right)\right\}$$

The real part is given by:

$$\mathbf{B}_T(\mathbf{r}, t) = \frac{E_{0T}}{v} e^{-\kappa z}\left(\frac{n_1}{n_2}\sin\theta_I \cos(kx - \omega t)\hat{\mathbf{z}} + a\sin(kx - \omega t)\hat{\mathbf{x}}\right)$$

Rewriting a in terms of κ and other constants, we find that $a = \frac{\kappa c}{\omega n_2}$. So, we have:

$$\begin{aligned}
\mathbf{B}_T(\mathbf{r}, t) &= \frac{E_{0T}}{v} e^{-\kappa z} \left(\frac{n_1}{n_2} \sin\theta_I \cos(kx - \omega t)\hat{\mathbf{z}} + \frac{\kappa c}{\omega n_2} \sin(kx - \omega t)\hat{\mathbf{x}} \right) \\
&= \frac{E_{0T}}{v_2} e^{-\kappa z} \frac{1}{n_2} \frac{c}{\omega} \left(n_1 \sin\theta_I \frac{\omega}{c} \cos(kx - \omega t)\hat{\mathbf{z}} + \kappa \sin(kx - \omega t)\hat{\mathbf{x}} \right) \\
&= \frac{E_{0T}}{\omega} e^{-\kappa z} \left(k \cos(kx - \omega t)\hat{\mathbf{z}} + \kappa \sin(kx - \omega t)\hat{\mathbf{x}} \right)
\end{aligned}$$

as desired. Note that $E_{0T} = E_0$ defined in the problem statement. It took me while to realize that this isn't the incident magnitude, but instead just another way of denoting the magnitude of the transmitted wave. □

(e) Check that the fields in (d) satisfy all of Maxwell's equations (Eq. 9.68).

Solution: We'll check these one by one. Starting with $\nabla \cdot \mathbf{E} = 0$:

$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial y} E_0 e^{-\kappa z} \cos(kx - \omega t) = 0$$

Now for $\nabla \cdot \mathbf{B} = 0$:

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= \frac{E_{0T}}{\omega} \left(\frac{\partial}{\partial x} e^{-\kappa z} \kappa \sin(kx - \omega t) + \frac{\partial}{\partial z} e^{-\kappa z} \cos(kx - \omega t) \right) \\
&= \frac{E_{0T}}{\omega} \left[e^{-\kappa z} \kappa k \cos(kx - \omega t) - \kappa e^{-\kappa z} k \cos(kx - \omega t) \right] = 0
\end{aligned}$$

Now, $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$, starting with $\nabla \times \mathbf{E}$:

$$\nabla \times \mathbf{E} = \epsilon^{ijk} \partial_j E_k = \frac{\partial}{\partial x} E_y \hat{\mathbf{z}} - \frac{\partial}{\partial z} E_y \hat{\mathbf{x}} = E_{0T} \kappa e^{-\kappa z} \cos(kx - \omega t)\hat{\mathbf{x}} - E_{0T} e^{-\kappa z} \kappa \sin(kx - \omega t)\hat{\mathbf{z}}$$

Now for $-\partial_t \mathbf{B}$:

$$\begin{aligned}
-\partial_t \mathbf{B} &= -\frac{E_{0T}}{\omega} e^{-\kappa z} \left(-\kappa\omega \cos(kx - \omega t)\hat{\mathbf{x}} + k\omega \sin(kx - \omega t)\hat{\mathbf{z}} \right) \\
&= E_0 e^{-\kappa z} (\kappa \cos(kx - \omega t)\hat{\mathbf{x}} - k \sin(kx - \omega t)\hat{\mathbf{z}})
\end{aligned}$$

Indeed they are equal. Finally, for $\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu\epsilon \partial_t \mathbf{E}$, we first note that $\mathbf{J} = 0$ here, so we need to show $\nabla \times \mathbf{B} = \mu\epsilon \partial_T \mathbf{E}$. Starting with $\nabla \times \mathbf{B}$:

$$\nabla \times \mathbf{B} = \epsilon^{ijk} \partial_j B_k = \frac{\partial}{\partial z} B_x \hat{\mathbf{y}} - \frac{\partial}{\partial y} B_x \hat{\mathbf{z}} + \frac{\partial}{\partial y} B_z \hat{\mathbf{x}} - \frac{\partial}{\partial x} B_z \hat{\mathbf{y}}$$

Note that neither component of $\mathbf{B}$ has y-dependence, so this simplifies to:

$$\begin{aligned}
\nabla \times \mathbf{B} = \frac{\partial}{\partial z} B_x \hat{\mathbf{y}} - \frac{\partial}{\partial x} B_z \hat{\mathbf{y}} = \frac{E_{0T}}{\omega} (-\kappa) e^{-\kappa z} \kappa \sin(kx - \omega t) - \frac{E_{0T}}{\omega} e^{-\kappa z} (-k^2 \sin(kx - \omega t) \\
= \frac{E_{0T}}{\omega} e^{-\kappa z} \sin(kx - \omega t)(k^2 - \kappa^2)
\end{aligned}$$

Now, we compute $\mu\epsilon \partial_t \mathbf{E}$:

$$\mu\epsilon \partial_T \mathbf{E} = E_{0T} e^{-\kappa z} \omega \sin(kx - \omega t)$$

To match these, we simplify $k^2 - \kappa^2$, using their definitions listed in the problem outline:

$$k^2 - \kappa^2 = \left(\frac{\omega}{c} n_1 \right) \sin^2\theta_I - \left[\left(\frac{\omega}{c} \right)^2 (n_1 \sin\theta_I)^2 - \left(\frac{\omega n_2}{c} \right)^2 \right] = \left(\frac{\omega n_2}{c} \right)^2$$

Therefore:

$$\nabla \times \mathbf{B} = \frac{E_{0T}}{\omega} e^{-\kappa z} \sin(kx - \omega t) \left(\frac{\omega n_2}{c} \right)^2 = E_{0T} e^{-\kappa z} \omega \sin(kx - \omega t) \left(\frac{n_2}{c} \right)^2$$

Using the fact that $\frac{n_2}{c} = \frac{1}{v_2} = \sqrt{\mu\epsilon}$, then we get the desired equality:

$$\nabla \times \mathbf{B} = \mu\epsilon E_{0T} e^{-\kappa z} \omega \sin(kx - \omega t) = \mu\epsilon \partial_t \mathbf{E}$$

□

(f) For the fields in (d), construct the Poynting vector, and show that, on average, no energy is transmitted in the z direction.

Solution: The Poynting vector is defined as:

$$\mathbf{S} = \frac{1}{\mu}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu}\epsilon^{ijk} E_j B_k = \frac{1}{\mu}(E_y B_z \hat{\mathbf{x}} - E_y B_x \hat{\mathbf{z}})$$

Expanding this out, we get:

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu}\left(E_{0T} e^{-\kappa z}\cos(kx - \omega t)\frac{E_{0T}}{\omega}e^{-\kappa z} k \cos(kx - \omega t)\hat{\mathbf{x}} - E_{0T} e^{-\kappa z}\cos(kx - \omega t)\frac{E_{0T}}{\omega}e^{-\kappa z}\kappa \sin(kx - \omega t)\hat{\mathbf{z}}\right) \\ &= \frac{1}{\mu\omega}\left(E_{0T} e^{-\kappa z}\right)^2 \left(k\cos^2(kx - \omega t)\hat{\mathbf{x}} - \kappa \cos(kx - \omega t)\sin(kx - \omega t)\hat{\mathbf{z}}\right)\end{aligned}$$

Averaged over time, we see that the Poynting vector in the $\hat{\mathbf{z}}$ direction has a $\sin\theta\cos\theta$ term, and we know that

$$\langle \sin\theta \cos\theta \rangle = \left\langle \frac{1}{2}\sin(2\theta) \right\rangle = 0$$

Therefore, there is no energy transmitted in the z direction. □