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Graphs

Computer Science Mentors 70

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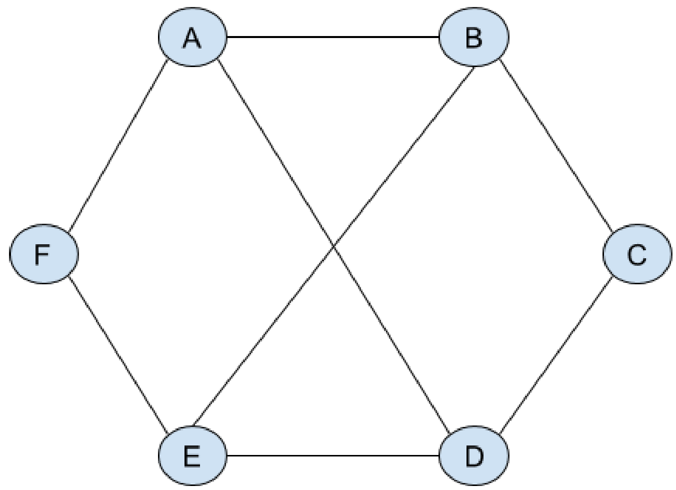
Prepared by: Amogh Gupta, Sylvia Jin, Aekus Bhathal, Abinav Routhu, Debayan Bandyopadhyay, Thomas Chen

1 Graph 101



1. Take a look at the following undirected graph.

   (a) How many vertices are in this graph?

   Solution: 6

   (b) What is the degree of vertex B?

   Solution: 3

   (c) What is the total degree of this graph?

   Solution: 16

   (d) Consider the traversal $A \to B \to C \to D \to A$. How would you categorize it (walk / cycle / simple path / tour)?

   Solution: Walk, cycle, and tour. A simple path does not repeat vertices.

   (e) Give an example of a simple path of length 4.

   Solution: $A \to B \to C \to D \to E$ and many others.

   (f) Is it possible to construct a traversal that is a tour but not a simple cycle from this graph (can go through vertices twice, but not edges)? Why or why not?

Solution: No, because there are no vertices with even degree greater than 2 (a condition which is needed to enter and exit a vertex twice or enter, exit, and also start and end at the same place).

(g) in terms of |E| and |V|, what is the sum of the degrees of the graph?

Solution: 2 * |E|

(h) What is the average degree of the graph?

Solution: 2 * |E| / |V|

2. Which of these graphs have Eulerian tours?

(a) The complete graph on 5 vertices (K_5).

Solution: Yes, every vertex is of degree 4. A graph in which every vertex has even degree has an Eulerian tour.

(b) The complete graph on 6 vertices (K_6).

Solution: No, every vertex is of degree 5.

(c) The complete graph on 7 vertices (K_7).

Solution: Yes, every vertex is of degree 6.

(d) The 3-dimensional hypercube.

Solution: No, every vertex is of degree 3.

(e) The 4-dimensional hypercube.

Solution: Yes, every vertex is of degree 4.

3. In this question we will work through the canonical example of buildup error. Recall that a graph is **connected** iff there is a path between every pair of its vertices.

False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof. We use induction on the number of vertices $n \geq 1$. let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

Base case: A graph with 1 vertex doesn't have any positive-degree vertices so $P(1)$ is true vacuously.

Inductive Hypothesis: Assume $P(n)$ holds. We want to show this implies $P(n+1)$.

Inductive Step: Consider an n vertex graph that has positive degree. By the asumption $P(n)$, this graph is connected and there is a path from every vertex to every other vertex. Now add a new vertex to create an $n+1$ vertex graph. All that remains is to check that there is a path from v to every other vertex. Suppose we add this vertex v to an existing vertex u. Since the graph was previously connected, we already know there is a path from u to every other vertex in the graph. Therefore, when we connect v to u, we know there will be a path from v to every other vertex in the graph. This proves the claim for $P(n+1)$. □

(a) Give a counter-example to show the claim is false.

Solution: Consider 2 pairs of vertices where each pair is connected by an edge. Each vertex has degree 1 but the two pairs are distinct connected components and the graph is disconnected.

(b) Since the claim is false, there must be an error in the proof. Explain the error.

Solution: The proof is actually logically correct until the last sentence. The problem is that for $P(n+1)$ to be true, we must show that every $(n+1)$-vertex positive-degree graph is connected. Instead, the proof shows that every $(n+1)$-vertex positive-degree graph *that can be constructed by adding a vertex of positive degree to an existing (n)-vertex positive-degree graph* is connected. Confirm that there is no way to build your counter-example graph by the method in the proof.

More generally, this is an example of "build-up error". This error arises from a faulty assumption that every graph of size $n+1$ with some property can be built by adding a vertex to an n vertex graph that also has that property. This assumption is correct in some cases, and incorrect in others.

In this class you will most likely only see build-up error in graph problems, but its important to understand that this can occur anytime you are doing induction on the size of some mathematical structure (i.e. matrices).

(c) How can we avoid this mistake?

Solution: We want to consider all possible $(n+1)$-vertex graphs but we also want to apply our induction hypothesis. The correct way to do this is to use a "shrink down, grow back" approach where we start with a graph on $n+1$ vertices that we assume satisfies the property we care about, remove an arbitrary vertex, and show that the induction hypothesis still holds for the new graph. Then add back the vertex and argue that $P(n+1)$ holds.

(d) What happens in the inductive step when you apply the fix?

Solution: Consider a graph on $n+1$ vertices where every vertex has degree at least 1. Remove an arbitrary vertex leaving an n-vertex graph. To apply the inductive hypothesis to this graph, we need every vertex to have positive degree, but this is not guaranteed to happen when we remove an arbitrary vertex. Thus we are stuck, unable to apply the inductive hypothesis.

4. Consider all complete undirected graphs on an even number of vertices. Prove that such graphs can be partitioned into $\frac{n}{2}$ spanning trees that share no edge with another spanning tree.

Solution: We proceed with a proof by induction on the number of vertices.

Base Case: For a 2-vertex complete undirected graph, there is only one spanning tree, which consists of the edge between the two nodes. $1 = \frac{2}{2} = \frac{n}{2}$

Inductive Hypothesis: for some even $k \geq 2$, the k-complete undirected graph can be partitioned into $\frac{k}{2}$ spanning trees that share no edge with another spanning tree.

Inductive Step:

We now consider the complete undirected graph on $k+2$ vertices. First, let us shrink our $k+2$ graph by removing 2 vertices (denoted as x and y). Now applying the induction hypothesis, we know there are $\frac{k}{2}$ spanning trees that include all but two vertices. We will now try to build new spanning trees by connecting each of these trees to x and y in the following manner.

First, add back the two new vertices and split the pre-existing k vertices into two halves, A and B. To re-construct the existing $\frac{k}{2}$ spanning trees: for each spanning tree, we will add to it an edge from x to a vertex within A, and an edge from y to a vertex within B. Note that in order to prevent shared edges between spanning trees, we need to pick out a new vertex from A and B each time to connect to x and y. Since the $k+2$ graph is complete, there will indeed be enough vertices in A and B for us to do this modification for the first $\frac{k}{2}$ spanning trees.

After we have found the first $\frac{k}{2}$ new spanning trees, we need a last one (total of $\frac{k+2}{2} = \frac{k}{2} + 1$). We know that to be a spanning tree, all vertices in A and B must be a part of it, along with x and y of course. Since we have already used some edges from x to set A (same case for y and B), to construct this last tree we want to instead use all the edges from x to vertices in set B, and y to vertices in set A.

The entire $k+2$ graph is made of vertices either in A, B, or vertices x and y, so the only thing left to connect in this new tree is x and y, which we can do since no previous spanning tree has used this edge.

2 Graph Coloring

1. Show that any tree is 2-colorable.

Solution: We prove the result by induction on the number of vertices.
Base case: $v = 1$
The graph is a single point, which is clearly 2-colorable.
Inductive step: Suppose that the problem statement holds for $v = k$, with $k \geq 1$. We show that the problem statement holds for $v = k + 1$.
Let G be the tree of $k+1$ vertices, and consider any vertex w in G. Remove w and all the edges connected to it, leaving $\leq k$ connected components. These connected components must also be trees, since if the connected component contained cycle, G would also contain a cycle. By the inductive hypothesis, each of these connected components is 2-colorable. Now, add w back into the graph. w is connected to at most k other vertices. We can rotate the colorings in each of the connected components such that each of the $\leq k$ vertices w is connected to is the same color, allowing us to color w the second color. Thus, G is 2-colorable, as desired.
Thus, the induction is complete.

We can also prove the result by induction on the number of edges:
Base case: $e = 0$
The graph is single point, which is clearly k-colorable.
Inductive step: Suppose that the problem statement holds for $e = k$, with $k \geq 1$. We show that the problem statement holds for $e = k + 1$.
Let G be the graph of $k + 1$ edges, and consider any edge e in G. Remove e, leaving two connected components. These connected components must also be trees, since if the connected component contained cycle, G would also contain a cycle. By the inductive hypothesis, each of these connected components is 2-colorable. Now, add e back into the graph. e connects two vertices, one from each connected component. We can rotate the colorings in each of the 2 connected components such that the 2 vertices w connects are different colors, giving a 2-coloring of G. Thus, G is 2-colorable, as desired.
Thus, the induction is complete.

2. You are hosting a very exclusive party such that a guest is only allowed to come in if they are friends with you or someone else already at the party. After everyone has showed up, you notice that there are n people (including yourself); each person has at least one friend (of course), but no one is friends with everyone else. It is still quite a sad party, because among all the possible pairs of people, there are only a total of $n - 1$ friendships. You want to play a game with two teams, and in order to kindle new friendships, you want to group the people (including yourself) such that within each team, no one is friends with each other. Is this possible? *(Hint: How might the previous question be useful?)*

Solution: Represent people as vertices and friendships between them as edges on a graph G. Based on the circumstances surrounding the party's guests, we know that G is connected (every new friend/vertex needs to be connected to a vertex within the existing graph) and there are a total of $n - 1$ edges (friendships). G is, in fact, a tree!

Now, we want to separate the partygoers into 2 groups such that no two people from the same group are friends. This can be modelled in G by partitioning the vertices into 2 groups such that the only edges are between the groups, not within. Thus, we have reduced the problem to showing a tree G is bipartite. But, we just showed that trees can always be 2-colored! We can just divide the vertices in 2 sets based on what color they are. Thus, all trees are also bipartite.

3. Two knights are placed on diagonally opposite corners of a chessboard, one white and the other black. The knights take turn moving as in standard chess, with white moving first.

   (a) Let every square on a chessboard represent a vertex of a graph, with edges between squares that are a knight's move away. Describe a 2-coloring of this graph.

Solution: Color white squares white, and black squares black, in a checkerboard pattern. Since knights always go from a white square to a black one and vice versa, this forms a 2-coloring.

(b) Show that the black knight can never be captured, even if it cooperates with the white knight.

Solution: At the end of the black knight's move, the two knights are on the same color. Thus, when the white knight moves, it lands on the opposite color of the black knight, and so can never capture the black knight.

3 Planarity

We say a graph is **planar** if it can be drawn on the plane without any edges crossing each other.
If the graph is planar, we can use Euler's planar formula:

$$v + f = e + 2$$

Some other (less important) results follow from Euler's formula if a graph is planar:

Corollary 1: $e \leq 3v - 6$

Corollary 2: $f \leq 2v - 4$

1. Try to prove all three of the above results.

   Hint 1: What is the minimum number of edges per face? What is the number of faces any edge can tough?
   Hint 2: Take the graph with the most number of edges or faces per vertex. What does it look like? look at the first hint!
   Hint 3: You should come up with a relation between e, f from the previous hints. Sub it into the expression!

   Solution:
   Euler's Planar formula: We perform induction on the edges, e. The induction is clear for $e = 0$ beacuse in this case $v = 1$ and $f = 1$, a single vertex with a single region surrounding it. Towards our inductive hypothesis assume that the formula is true for all connected plane graphs with fewer that e edges, here e is greater than or equal to 0, and suppose that our graph G has e edges. If G is a tree then $v = e + 1$, by definition, and $f = 1$ so Euler's formula indeed holds. On the other hand, if G is not a tree, suppose e is an edge that is part of a cycle of G and consider the graph without this edge, $G - e$. The connected graph $G - e$ has v vertices, $e - 1$ edges, and $f - 1$ faces (because a cycle separates a plane into two regions and removing an edge on that cycle merges the two regions), but now since $G - e$ has $e - 1$ edges we can apply our inductive hypothesis to it specifically that $v - (e - 1) + (f - 1) = 2$, but this just simplifies to $v - e + f = 2$ as required.
   Corollary 1: For a graph G with f faces it follows from the handshaking lemma that $2e \geq 3f$, specifically this is because the degree of each face of a simple graph is at least 3, so $f \leq (2/3)e$. Since $v - e + f = 2$ we can plug in our bound on f to get $e - v + 2 \leq (2/3)e$ which simplifies to $e \leq 3v - 6$
   Corollary 2: For G with f faces, it follows from the handshaking lemma for planar graphs that $2e \geq 4f$, because the degree of each face of a simple graph without triangles is at least 4, so we can bound $f \leq (1/2)e$. Combining this bound with Euler's formula we have $e - v + 2 \leq (1/2)e$ therefore $e \leq 2v - 4$.

2. Consider a group of 6 friends sitting at a round table. Any one person is friends with the two individuals next to them and the person sitting directly across from them. Consider the graph where each individual is a vertex and an edge exists between person u and v if and only if u and v are friends. Prove that this graph is non-planar.

Solution: Consider the subgraph of this graph on 6 vertices, partitioned into the left side of the three vertices $0, 2, 4$ and the right side of $1, 3, 5$. Notice that there is an edge from every node on the left side to the right side and vice versa, creating a $K_{3,3}$ subgraph. Since this graph contains a $K_{3,3}$ subgraph.

4 Hypercubes

1. Austin has $n \geq 2$ lightbulbs in a row, all turned off. Every second Austin performs a **move**, where Austin either turns a lightbulb on or turns a lightbulb off. Show that for any n, there is a a sequence of moves that Austin can make such that each possible configuration of lightbulbs being on or off has occurred exactly once, and such that the last move causes all the lightbulbs to be off.

   (a) Each of the n lightbulbs is either on or off. How should we represent the lightbulb states mathematically?

   Solution: A n-length bitstring. The i^{th} bit represents if the i^{th} bulb is on or off.

   (b) Frame the problem in terms of hypercubes.

   Solution: In a n dimensional hypercube H, we can label each vertex with a n-length bitstring where vertices are connected only if their strings differ by exactly one bit. Each move switches exactly one lightbulb, or one bit. This corresponds to walking across an edge from one vertex to another. Starting from no lit lightbulbs and making moves to go through all configurations before ending at no lit lightbulbs corresponds to a **Hamiltonian cycle** on H, a cycle that visits every vertex (and only once).

   (c) Solve the problem by showing a property of hypercubes.

   Solution: We show that a Hamiltonian cycle exists for all $n \geq 2$ dimensional hypercubes by induction.
   Base case: $n = 2$
   The graph is a square with vertices $V = \{00, 01, 11, 10\}$. Note that $\{(00, 01), (01, 11), (11, 10), (10, 00)\}$ is a Hamiltonian cycle.
   Inductive step: Suppose that the problem statement holds for $n = k$, with $k \geq 1$. We show that the problem statement holds for $n = k + 1$.
   Let H be our $k + 1$-dimensional hypercube. Define H_0 to be the k-dimensional hypercube constructed by all the vertices in H with first bit 0, and H_1 to be the k-dimensional hypercube formed by all the vertices in H with first bit 1. By the inductive hypothesis, there is a Hamiltonian cycle of H_0 starting at $00 \ldots 0$. Let $00 \ldots 010 \ldots 0$ be the last vertex in the Hamiltonian cycle before $00 \ldots 0$. Similarly, by the inductive hypothesis, there is a Hamiltonian cycle of H_1 starting at $10 \ldots 0$, where $10 \ldots 010 \ldots 0$ is the last vertex in the Hamiltonian cycle before $10 \ldots 0$.
   Now we construct a new cycle of H. Our new Hamiltonian cycle takes the path from $00 \ldots 0$ to $00 \ldots 010 \ldots 0$ as defined by the Hamiltonian cycle in H_0, traverses the edge from $00 \ldots 010 \ldots 0$ to $10 \ldots 010 \ldots 0$ and , takes the path from $10 \ldots 010 \ldots 0$ to $10 \ldots 0$ as defined by traversing the Hamiltonian cycle in H_1 in reverse, then traversing the edge from $10 \ldots 0$ to $00 \ldots 0$. This new cycle traverses all the vertices (by the inductive hypothesis), so it is a Hamiltonian cycle of H, as desired.
   Thus, the induction is complete.

2. You wish to color the *edges* of a n-dimension hypercube, such that edges that share a vertex are different colors. (Note: This isn't the problem in disc2b, where you colored vertices so that vertices that share an edge are different colors!) Prove that n colors is necessary and sufficient. (You can do this with n colors but not $n - 1$.)

Solution: Consider the bitwise representation of each vertex. Every vertex is connected to n others, so it is impossible to use less than n colors; otherwise each vertex will have multiple edges with the same color.

Color all edges which go between vertices $0xxxx\ldots$ and $1xxxx..$ one color, with $xxxx\ldots$ ranging over all sequences of $n-1$ bits. None of these edges share a vertex, so our requirement is preserved. Furthermore, we can do the same with each "position" in the bitwise representation; thus, edges connecting vertices $x0xxx\ldots$ to $x1xxx\ldots$ will share a second color, edges connecting vertices $xx0xx\ldots$ to $xx1xx\ldots$ will share a third color, and so on. This overall uses n colors, and encompasses all edges in the hypercube, while ensuring that edges that share a vertex are different colors. Thus, this forms an n coloring of the edges.

Do you see the proof by induction hidden in this proof? Doing this was the same as making a n dimension hypercube by starting with a $n-1$ dimension hypercube and joining it to an exact copy of itself (with all the same edge colorings) using edges of a new color.