Physics W89 - Introduction to Mathematical Physics - Summer 2023
Problem Set - Module 01 - Mathematical Preliminaries

Last Update: May 26, 2023

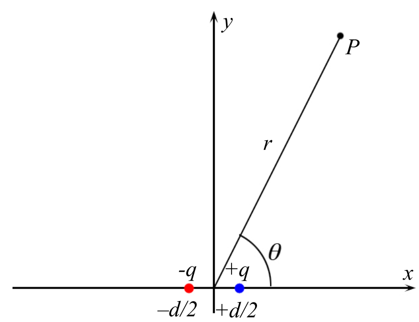
Problem 1.1 - Electric Multipoles

Relevant Videos: Taylor Expansions and Approximations

In lecture we saw how to use Taylor approximations to find the potential along the axis of an ***electric dipole***. In this problem we will expand on this analysis to study ***electric multipoles***. Recall that the potential due to a *single* point charge q a distance r away from the point charge is given by

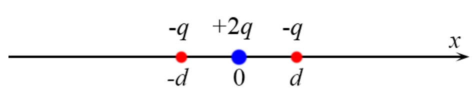
$$V = \frac{kq}{r}.$$

Consider an electric dipole consisting of a positive charge $+q$ on the x-axis at $x = d/2$ and an equal and opposite negative charge $-q$ on the x-axis at $x = -d/2$ as shown.



(a) Find the lowest-order approximation to the potential at a general point $(x, y) = (r\cos\theta, r\sin\theta)$ when $r \gg d$.
[*Spoilers!* Your dimensionless small parameter here will be d/r.]

Next consider a ***linear electric quadrupole***, with a point charge $+2q$ at the origin, a point charge $-q$ at $x = +d$, and a point charge $-q$ at $x = -d$. Consider a point P at position x on the x-axis and a good distance to the right of the quadrupole so $x \gg d$.



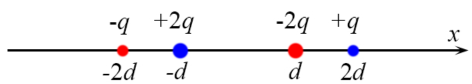
(b) Find the lowest-order approximation to the potential at point P when $x \gg d$.
[*Note: The first step is to get an exact expression for the potential by adding the potentials from the three point-charges.*]
[*Note: Remember, if you get an answer of zero, you are not yet at the lowest order and need to go at least one order higher!*]
[*Spoilers!* The lowest-order term here will be second-order in the small parameter d/x.]

Rather than thinking of the quadrupole from part (b) as a set of three charges, we can consider it a combination of two oppositely-oriented dipoles of the kind seen in part (a), with one dipole centered at $+d/2$ and the other dipole centered at $-d/2$.

(c) Extra Part (*Not for Credit*) Show that approximating the quadrupole potential using the sum of two approximate dipole potentials gives the same lowest-order result as part (b).

Commentary: You <u>don't</u> need the following for this problem but here is some fun supplementary information about this setup. We can encode information about the dipoles by introducing the ***dipole moment vector*** $\vec{p} = q\Delta\vec{r}$*, where* q *is the magnitude of the equal and opposite charges making up the dipole and* $\Delta\vec{r}$ *is the displacement vector pointing from the negative charge to the positive charge. In the linear quadrupole setup, we are considering two equal and opposite dipoles with* $\vec{p} = +qd\hat{x}$ *at* $x = -d/2$ *and* $\vec{p} = -qd\hat{x}$ *at* $x = +d/2$.

How about a linear ***electric octopole*** (a $-q$ charge at $x = -2d$, a $+2q$ charge at $x = -d$, a $-2q$ charge at $x = +d$, and a $+q$ charge at $x = +2d$)?



(d) Extra Part (*Not for Credit*) Find the potential at points on the x-axis the the right of the linear octopole with $x \gg d$. Sensing a trend yet?

Problem 1.2 - This Module is Taylor-Made for Trigonometry[1]

Relevant Videos: Taylor Expansions and Approximations

In class we presented the Taylor expansions for $\sin(x)$ and $\cos(x)$ about the point $x = 0$,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

(a) Let $f(x) = \sin(x)$ and explicitly carry out the Taylor expansion about the point $x = 0$. Show that your answer reproduces the formula for $\sin(x)$ shown above.
[*Spoilers!* You will need to do some *index manipulation* here! I suggest breaking the sum in the original Taylor expansion formula into the even terms and the odd terms and then saying that odd n can be expressed as $n = 2m + 1$.]

(b) Graph $\sin(x)$, and the first, third, and fifth order Taylor series approximations to $\sin(x)$ in the range $-3\pi/2 \leq x \leq 3\pi/2$.
[*Note: For any graphing problems you should use a graphing program or a coding application like Python, Mathematica, etc.*]

Some functions can't be Taylor expanded about certain points. For example, if we try to Taylor-expand $1/x$ about the point $x = 0$ we would fail since $1/x = \infty$! In such cases, it may be possible to expand the function using both positive *and negative* powers of x. For example, we can express the function $\sin(x)/x^2$ as,

$$\frac{\sin x}{x^2} \approx \frac{x - x^3/3! + x^5/5! + \mathcal{O}(x^7)}{x^2} = \frac{1}{x} - \frac{x^2}{3!} + \frac{x^3}{5!} + \mathcal{O}(x^5).$$

[1] Yes. There will be puns.

Such a series is a generalization of the Taylor series called the ***Laurent series***, which is an enormously useful tool in complex analysis.

(c) Use the two series for $\sin(x)$ and $\cos(x)$ to find the Laurent series/small-angle approximation for $\cot(x)$ to the lowest two non-zero terms. Check your approximation by finding $\cot(0.1)$ and comparing with your approximation. How about for $\cot(1)$? $\cot(\pi/2)$?
[*Supplementary Part (Not for Credit): For a challenge, also find the third non-zero term!*]
[*Note: See the lecture notes for a similar example showing the expansion of* $\tan(x)$.]
[*Spoilers!* First expand the denominator and write the expansion as $x(1+\text{terms})$. Rather than play around with dividing polynomials, treat "terms" as small and do an expansion of $1/(1+\text{terms})$ to second order in "terms"! Then you just have to worry about multiplying polynomials, which is much easier.]

(d) Use the Taylor series for e^x, $\sin(x)$, and $\cos(x)$ to show ***Euler's formula***,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Commentary: Using the Taylor series for e^x is, in a way, how we <u>define</u> *the exponential of a complex number!*

Problem 1.3 - Some Simple Complex Problems[2]

Relevant Videos: Complex Numbers and the Complex Plane

(a) Let $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ and $z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}$. Find the real part and imaginary part of the product $z_1 z_2$ and the quotient z_1/z_2 in terms of the Cartesian components x_1, x_2, y_1, and y_2. Then find the real and imaginary part of the product $z_1 z_2$ and the quotient z_1/z_2 in terms of the polar components r_1, r_2, θ_1, and θ_2.

(b) Let $z = x + iy = re^{i\theta}$. Find $|z|^2$ and z^2 first in terms of x and y and then in terms of r and θ. Under what circumstances does $|z|^2 = z^2$?

(c) Use Euler's formula on both sides of $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ to derive the formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.[3]

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

(d) Show that $|z_1 + z_2| = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}$.

(e) **Extra Part** (*Not for Credit*) Use the result from part (d) to show the ***triangle inequality*** $|z_1 + z_2| \leq |z_1| + |z_2|$. Under what conditions is the inequality ***saturated***.
[*Note: We say an inequality is saturated when it becomes an equality. That is, the inequality $a \geq b+c$ is saturated when $a = b+c$,*]

[2] Complaints about puns can be sent to mailto:aphysicist28@berkeley.edu to fuel my maniacal laughter.
[3] As long as you are comfortable manipulating complex numbers, Euler's formula lets you re-derive all of the messy trigonometric identities easily!

(f) **Extra Part** (*Not for Credit*) Show ***De Moivre's formula***, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Let $n \in \mathbb{N}$ (that is, n is a positive whole number). The ***n-th root*** of a complex number z is $z^{1/n}$. When dealing with roots we have to be careful since the n-th root function is ***multi-valued***. In particular, there are n different possible values of w such that $w^n = z$, *all* of which could properly be considered n-th roots of z. Let's explore this.

(g) Consider the three different cube roots of $z = i$ (which, using Euler's formula, we can also write as $z = e^{i\pi/2}$). Show that $w = e^{i\pi/6}$, $e^{5i\pi/6}$, and $e^{3i\pi/2}$ all cube to $z = i$. Plot these roots on the complex plane and demonstrate graphically how $e^{5i\pi/6}$ cubes to i.
[*Note: Remember that multiplying by a phase $e^{i\theta}$ essentially "rotates" a point in the complex plane counterclockwise by an angle θ.*]

(h) **Extra Part** (*Not for Credit*) Show that the n distinct values of the n-th root of $z = re^{i\theta}$ are $w_k = r^{1/n}e^{i(\theta+2\pi k)/n}$, with $k = 0, \cdots, n-1$. As a check, test that your formula implies that if w is a square root of z then so is $-w$.
[*Spoilers!* Recall that $z^p = r^p e^{ip\theta}$. Also recall that we can always add an integer multiple of 2π to the phase angle θ and leave z unchanged...]

The multi-valued nature of the n-th root ultimately comes from the multi-valued nature of the logarithm. Recall that

$$\ln z = \ln|z| + i\,\mathrm{Arg}(z) + 2\pi i k, \tag{1}$$

where $\mathrm{Arg}(z) = \tan^{-1}(y/x)$ is the ***principal value*** of the argument, so $-\pi < \mathrm{Arg}(z) \leq \pi$, and $k \in \mathbb{Z}$ is the ***branch index***. Before solving part (i), be sure to convince yourself that the exponential of the right-hand side of Eq. 1 does indeed produce $z = |z|\,e^{i\,\mathrm{Arg}(z)}$ for any integer k.

(i) Use Eq. 1 to solve the formula $w^n = z$ for w by first taking the logarithm of both sides, solving for $\ln w$, and then exponentiating. You should get back your formula from part (h)!

Problem 1.4 - A Damped Harmonic Oscillator

Relevant Videos: Complex Numbers and the Complex Plane

Consider a mass m at the end of an ideal spring with spring constant k (so the spring force is $F_{\text{spring}} = -kx$). In introductory physics when studying this system, you find that the mass undergoes *oscillatory* motion with angular frequency $\omega_0 = \sqrt{k/m}$. Now let's make the physical system a little more physically accurate by adding in the effects of ***damping***, a type of friction force (it removes mechanical energy from the system). The damping force is proportional to the velocity, $F_{\text{damping}} = -2m\gamma v$, where $\gamma \geq 0$ is the ***damping coefficient***. Therefore,

$$ma = -m\omega_0^2 x - 2m\gamma v. \tag{2a}$$

Our problem is ironically made less complicated by ***complexifying***.[4] That is, we replace the real position $x \in \mathbb{R}$ with a "complex position" $z \in \mathbb{Z}$ such that $\mathrm{Re}\, z \equiv x$. We intuitively know or can

[4] We did this for the undamped harmonic oscillator in lecture.

guess what our damped spring will behave like - it should oscillate with a decreasing amplitude.[5] Therefore, we will make an ***ansatz*** (a guess) at what the complex position will look like,

$$z(t) = Ae^{i\Omega t}, \tag{2b}$$

where $\Omega \in \mathbb{C}$ is a complex number with units of $[\Omega] = \mathrm{s}^{-1}$ and $A \equiv A_0 e^{i\varphi_0}$ is some initial complex amplitude.[6] We define $\omega \in \mathbb{R}$ and $\Gamma \in \mathbb{R}$ to be the real and imaginary parts of Ω,

$$\Omega \equiv \omega + i\Gamma.$$

[*Note: Technically we're not going to get to differential equations until the second half of the semester, but complex numbers are so cool and useful, I just* have *to give you an application here! Don't worry, I will provide the solution to the one simple differential equation that you have to actually solve when the smoke clears in this problem.*]

(a) Express $z(t) = Ae^{i\Omega t}$ in polar form $r(t)e^{i\phi(t)}$, with $r(t)$ and $\phi(t)$ are *real* functions expressed in terms of the real constants ω, Γ, A_0, and φ_0. Using your physical intuition (think about the actual motion of a damped oscillator or think about energy), what condition must Γ satisfy for our solution to be physically reasonable?

From the complex position z we can define a complex velocity $\tilde{v} = \frac{dz}{dt}$ and a complex acceleration $\tilde{a} = \frac{d\tilde{v}}{dt}$, in which case Eq. 2a becomes

$$m\tilde{a} = -m\omega_0^2 z - 2m\gamma\tilde{v}. \tag{2c}$$

At the end of the day, we get our physical solutions by taking the real part of the complex solution.

(b) Consider the ansatz from Eq. 2b. Find the complex velocity $\tilde{v}(t)$ and complex acceleration $\tilde{a}(t)$ in terms of the complex constants A and Ω.

(c) Plug your results from (b) into Eq. 2c. Use the fact that $e^{i\Omega t}$ is never zero to simplify your answer to a quadratic equation for Ω.

We have a quadratic equation and we want to solve for Ω! Gut instinct should tell you to use the quadratic formula to solve for Ω, but is that valid for complex numbers?[7]

(d) Extra Part (*Not for Credit*) Show that the quadratic formula works to solve quadratic equations $az^2 + bz + c = 0$, even if the coefficients are complex!
[*Note: The fundamental theorem of algebra tells that that our quadratic equation will* always *have two solutions, with those two solutions coinciding iff* $b^2 - 4ac = 0$.]

(e) Solve the quadratic equation from part (c) for Ω. Does this Ω satisfy the condition you found in part (a)? Under what conditions (if any) on γ and/or ω_0 does Ω become purely real ($\Gamma = 0$)? Purely imaginary ($\omega = 0$)? Neither purely real nor imaginary ($\omega, \Gamma \neq 0$)?
[*Note:* ω_0 *is a real, positive frequency since we have a non-trivial spring and mass.*]

It's time for some reasonability checks! In the middle of a physics calculation it's always good to see if your answer is physically on the right track.

[5] Unless the damping is too large. We will analyze that case later on.
[6] This is again just like what we did in lecture, but now we are letting the "frequency" be complex.
[7] Spoilers: Yes, it is!

(f) Extra Part (*Not for Credit*) First, using Eq. 2c, determine the units/dimensions of the constant γ. Next, make sure your expression for Ω is dimensionally consistent (we can only add quantities if they have the same dimensions). What are the units of Ω? It doesn't make sense to take the exponential (or sine or cosine) of a dimensionful quantity. Is this consistent for our solution $z = Ae^{i\Omega t}$?

For the rest of this problem, let $\varphi_0 = 0$ so that A is real and positive.

(g) Extra Part (*Not for Credit*) Find $x(t) = \text{Re}(z(t))$ when $\gamma = 0$. This is the ***undamped*** oscillator. Does this oscillator behave in the way you would expect?

(h) For *one* of the two solutions for Ω you found in part (e), find $x(t) = \text{Re}(z(t))$ in the case $0 < \gamma < \omega_0$. This is the ***underdamped*** oscillator. Qualitatively sketch the solution $x(t)$.

(i) For *one* of the two solutions for Ω you found in part (e), find $x(t) = \text{Re}(z(t))$ in the case $\omega_0 < \gamma$. This is the ***overdamped*** oscillator. Qualitatively sketch the solution $x(t)$.

(Challenge - Not for Credit) Think you're pretty hot stuff, eh? Mastering all of this complex stuff, defeating my damped oscillator problem? Okay, let's add a snag. Consider adding a ***driving force*** to the oscillator, $F_{\text{drive}} = F_0 \cos(\omega t)$. Again, make everything complex (in particular, the driving force becomes $F_0 e^{i\omega t}$) and consider solutions of the form $z(t) = Ae^{i\Omega t}$. Find what Ω and A have to be to solve this system. Graph the amplitude $|A|$ vs. ω when $\gamma < \omega_0$ and comment.

♦