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1 Introduction: Axioms of Quantum Mechanics

- Also called postulates.
- Typically four axioms:
  1. Quantum states & superposition
  2. Unitary Evolution: deterministic
  3. Measurements: introduces statistical nature to quantum behavior
  4. Observables: quantities we can measure in the real world

1.1 Quantum States

- A quantum state is denoted by $|\psi\rangle$. It's a vector in a complex-valued vector space, with a particular inner product structure. This combination of the vector space with the inner product structure is called a Hilbert space, denoted by $\mathcal{H}$.
- Vectors in $\mathcal{H}$ are denoted by *kets* $|v\rangle$, and because it's a vector space (hence it's linear), we can make other vectors by adding together two vectors: $|w\rangle = |u\rangle + |v\rangle$.

  We also have a null vector $|0\rangle = |u\rangle - |u\rangle$.
- Linearly independent vectors:

$$a_1 |u_1\rangle + a_2 |u_2\rangle + \cdots + a_n |u_n\rangle = 0$$

  if the only solution to this is to set $a_1, a_2, \ldots, a_n$ to 0, then the set of vectors $|u_1\rangle, |u_2\rangle, \ldots, |u_n\rangle$ is linearly independent.

  We will only work with finite dimensional vector spaces, for the sake of quantum information
- If the set of vectors $\{|u_i\rangle\}$ spans the space, then they are referred to as a basis. This means that any vector $|w\rangle$ can be written as a linear combination of some $|u_i\rangle$:

$$|w\rangle = \sum_i a_i |u_i\rangle$$

  It can also be represented as a column vector of n values:

$$|w\rangle = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

- An example where $n = 2$, is *spin projection*, which has two possible values: $\pm\hbar/2$. In this case, the general state $|\psi\rangle$ can be written as $|\psi\rangle = a_1 |+\hbar/2\rangle + a_2 |-\hbar/2\rangle$.

  We'll be dealing with mostly two-state systems in this class, and any other two-state system that we choose is sometimes called "pseudo-spin" since the math is nearly identical.

- In all cases, we should have $\sum_i |a_i|^2 = 1$; we call the states that follow this behavior (and they should) to be **normalized to 1**.

1.2 Inner Product

- Given $|w\rangle = \sum_i a_i |u_i\rangle$ and $v = \sum_i b_i |u_i\rangle$, then the complex-valued inner product $\langle v|w\rangle = \sum_i b_i^* a_i$. It can be real-valued, but in general it's considered complex.

  This gives a way for us to talk about how far apart two vectors are from one another, similar to a dot product.

- If the inner product is 0 and our vectors are not the zero vector themselves, then we call these two vectors **orthogonal**.

- An **orthonormal basis** is one where all the vectors are orthogonal, and also normalized to 1. In other words, we have $\langle u_i|u_j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

- So what is $\langle u|$? $\langle u|$ lives in the *dual space*, and is defined as follows: if $|w\rangle = \sum_i a_i |u_i\rangle$, then $\langle w| = \sum_i a_i^* \langle u_i|$.

  So if $|w\rangle$ is represented as a column vector (earlier), then $\langle w|$ is represented as a row vector:

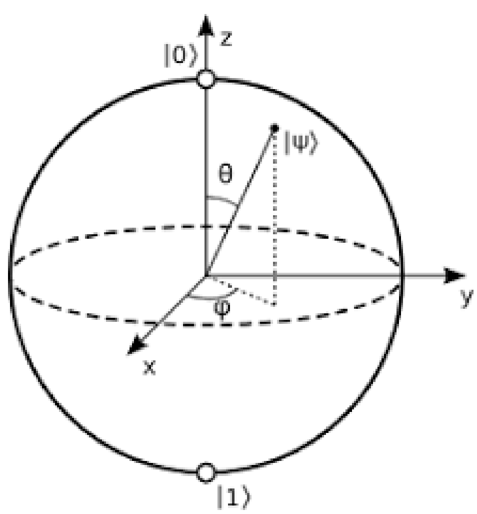
$$\langle w| = \begin{bmatrix} a_1^* & \dots & a_n^* \end{bmatrix} = w^{\top *} = w^\dagger$$

- The properties of the inner product:
  - $\langle u|v\rangle = \langle v|u\rangle^*$
  - Antilinearity: $\langle u|av\rangle = a\langle u|v\rangle$, but $\langle au|v\rangle = a^* \langle u|v\rangle$.
  - Norm of $|v\rangle$: $\langle v|v\rangle = \|v\|^2$. Hence, $\|v\| = \sqrt{\langle v|v\rangle}$.

- Conventionally, although we denote $|w\rangle = \sum_i^{n-1} a_i |u_i\rangle$, we generally deal with $n = 2$, so we have $|0\rangle$ and $|1\rangle$ as our states. This is called the **computational basis**.

1.3 Geometric Interpretation

- For $n = 2$, there is a nice geometric interpretation called the **Bloch sphere**:



  The sphere has radius 1, and all points on the sphere represent quantum states. A general state $|\psi\rangle$ is written as

$$|\psi\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right] = \alpha|0\rangle + \beta|1\rangle$$

If we want our state to be normalized, then we want $\|\alpha\|^2 + \|\beta\|^2 = 1$.

- There are also other orthonormal bases we can choose:
  - x-basis: $|+x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle$, $|-x\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
  - y-basis: $|+y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle$, and $|-y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$.

1.4 Unitary Evolution

- All equations we'll deal with are relations in $\mathcal{H}$, and these operations form a group called SU(2). This is called the *Special unitary group*.
- This unitary transformation takes our vectors $|0\rangle$ and $|1\rangle$ and does the following:

$$|0\rangle \xrightarrow{U} a\,|0\rangle + b\,|1\rangle$$
$$|1\rangle \xrightarrow{U} c\,|0\rangle + d\,|1\rangle$$

  In this case, we can write U as a 2x2 matrix:

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \; U^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

  Recall that $U^\dagger$ is the conjugate transpose. If U is a unitary operator, then $U \dagger U = I = UU^\dagger$. This implies that $U\dagger = U^{-1}$

- On a qubit, we will apply many gates throughout this semester. Some of these are listed below:
  - X-gate: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  - Z-gate: $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  - Hadamard gate: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

  All of these operations can be interpreted as a series of rotations on the Bloch sphere.

1.5 Observables

- An operator A, and its Hermitian conjugate is denoted by $A^\dagger = (A^\top)^*$.
- In QM, Hermitian operators are related to real observables we can measure in the lab, and because they are measurable, they must have real eigenvalues.
- They will also have mutually orthogonal eigenvectors.
- As an example, the X gate is Hermitian, with eigenvectors of $|+x\rangle$ and $|-x\rangle$. This is also sometimes called the Hadamard basis, because acting the Hadamard gate on $|0\rangle$ gives us $|+x\rangle$, and acting it on $|1\rangle$ gives $|-x\rangle$.

2 Entanglement & Bell Inequalities

2.1 Projection Operators

- The basic form of an operator is that it takes one vector and spits out another: $|c\rangle = |c\rangle\,\langle a|a\rangle$. So, the outer product $|c\rangle\,\langle a|$ is the operator.

- Consider a state $|w\rangle = \sum_{i=1}^{n} a_i |u_i\rangle$, where $\{|u_i\rangle\}$ form an orthonormal basis. If we want to find any one of the a_j, then we compute $\langle u_j|w\rangle$:

$$\langle u_j|w\rangle = \sum_{i=1}^{n} a_i \underbrace{\langle u_j|u_i\rangle}_{\delta_{ij}} = a_j$$

Alternatively, this allows us to write $|w\rangle$ in terms of:

$$|w\rangle = \sum_{i}^{n} \langle u_i|w\rangle |u_i\rangle = \sum_{i=1}^{n} |u_i\rangle \langle u_i|w\rangle$$

Now, the term $|u_i\rangle \langle u_i|$ is an operator, and is called the **projection operator**. If we act the operator on one of the basis vectors:

$$|u_i\rangle \langle u_i|u_i\rangle = |u_i\rangle$$

whereas if we do it on an arbitrary vector $|w\rangle$:

$$|u_i\rangle \underbrace{\langle u_i|w\rangle}_{a_i} = a_i |u_i\rangle$$

- The projection operator is written as $P_i = |u_i\rangle \langle u_i|$, which has the property that $P_i^2 = |u_i\rangle \langle u_i|u_i\rangle \langle u_i| = P_i$. It also has the property that

$$\sum_{i}^{n} P_i = \sum_{i}^{n} |u_i\rangle \langle u_i| = I$$

2.2 General Operators

- A general operator is defined as $A = IAI$. Now, we're going to express the identity matrices in terms of the projection operators:

$$\begin{aligned} A &= \sum_i \sum_j |u_i\rangle \overbrace{\left\langle u_i|A|u_j\right\rangle}^{A_{ij}} \left\langle u_j\right| \\ &= \sum_{i,j} A_{ij} |u_i\rangle \left\langle u_j\right| \end{aligned}$$

The term A_{ij} represents a *matrix element*, represented in the $\left|u_j\right\rangle$ basis.

What does the $|u_i\rangle \left\langle u_j\right|$ operator represent?

- One basis that we'll use very frequently is to express A in terms of the eigenbasis. That is, the set $|a_i\rangle$ of vectors such that

$$A |a_i\rangle = a_i |a_i\rangle$$

In this basis, then A is written as:

$$\begin{aligned} A &= IAI \\ &= \sum_{ij} |a_i\rangle \langle a_i|A|a_j\rangle \langle a_j| \\ &= \sum_{i,j} a_j |a_i\rangle \langle a_i|a_j\rangle \langle a_j| \end{aligned}$$

Here we've used the property that $A|a_j\rangle = a_j |a_j\rangle$. Then, if we choose the eigenvectors to be orthogonal (which is okay for a Hermitian A), then $\langle a_i|a_j\rangle = \delta_{ij}$, so:

$$A = \sum_i a_i |a_i\rangle \langle a_i|$$

Why can we choose the $\{|a_i\rangle\}$ to be orthogonal?

- We choose A to be Hermitian (which is the only way we were able to make this simplification). Since they have real eigenvalues, they have mutually orthogonal eigenvectors.

2.3 Measurement Postulate

- An observable A can be measured by a set of operators $\{M_m\}$ with outcomes (observable values) m.
- For example, a qubit (so any 2-level system) with states $|0\rangle$ and $|1\rangle$, we can make a general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with normalization constraint $\|\alpha\|^2 + \|\beta\|^2 = 1$.

  By measuring, we "learn" the value of α and β. Our measurement operators consist of

  $$M_0 = |0\rangle\langle 0|, \quad M_1 = |1\rangle\langle 1|$$

  You'll notice that these are projections onto a given state – this is intentional.
- Upon measuring $|\psi\rangle$, we will get one outcome (either 0 or 1), with probability $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$.
- After measurement, the state "collapses" into the state $\frac{M_m|\psi\rangle}{\sqrt{p(m)}}$. This is a fancy way to say that it will only give us $|0\rangle$ if the outcome was 0. This probabilistic determination of the final state is intrinsic to quantum mechanics.

  As an example, if we have one state $|\psi\rangle$, we either get 0 or 1 but have no information about α or β. However, if we have many identical $|\psi\rangle$, then we get 0 with probability $\|\alpha\|^2$, and we get 1 with probability $\|\beta\|^2$. This is because:

  $$p(m=0) = \langle\psi|0\rangle\langle 0|0\rangle\langle 0|\psi\rangle = \|\alpha\|^2$$
  $$p(m=1) = \langle\psi|1\rangle\langle 1|1\rangle\langle 1|\psi\rangle = \|\beta\|^2$$

  Note that $M_m^\dagger = M$, based on the way we've defined them. If we get 0, then the final state is written as:

  $$\frac{|0\rangle\langle 0|\psi\rangle}{\sqrt{\|\alpha\|^2}} = |0\rangle = e^{i\theta}|0\rangle$$

  the $e^{i\theta}$ is just some overall phase factor.
- We introduce an average over many measurements to be the quantity $\langle A\rangle$, which is calculated as:

  $$\langle A\rangle = \sum_m p(m) a_m$$

  This is also sometimes called the *average value* of an operator. The measurement basis we choose for a Hermitian A is given by the eigenvectors of A, so we have:

  $$M_m = |a_m\rangle\langle a_m|$$

  where $|a_i\rangle$ is the i-th eigenvector of A. Then, this means that $\langle A\rangle = \sum_m p(m)a_m$. Remember that A is represented as:

  $$A = \sum_m a_m |a_m\rangle\langle a_m|$$
- Some cool expansion:

  $$\begin{aligned}\langle A\rangle &= \sum_m p(m)a_m \\ &= \sum_m a_m \langle\psi|M_m^\dagger M_m|\psi\rangle \\ &= \sum_m a_m \langle\psi|a_m\rangle\langle a_m|a_m\rangle\langle a_m|\psi\rangle \\ &= \sum_m a_m \langle\psi|a_m\rangle\langle a_m|\psi\rangle\end{aligned}$$

But now let's throw a $\langle\psi|$ to the left:

$$\langle\psi| \sum_m a_m |a_m\rangle \langle a_m|\psi\rangle = \langle\psi|A|\psi\rangle$$

This is the matrix element we've come across earlier.

2.3.1 Specific Examples

- Suppose we want to measure Z for a qubit. Recall that $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This has eigenvalues ± 1, with eigenvectors $|0\rangle, |1\rangle$.
- Now, we compute $\langle Z\rangle$ for a general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

$$\begin{aligned}\langle Z\rangle &= p(+1)(+1) + p(-1)(-1) \\ &= \|\alpha\|^2 - \|\beta\|^2\end{aligned}$$

Remember that the equation is (probability of obtaining state) × (eigenvalue of that state).

- Now let's measure $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the same state $|\psi\rangle$. It's eigenvalues are ± 1, with eigenvectors $|+\rangle, |-\rangle$. Recall that

$$\begin{aligned}|+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\end{aligned}$$

This means that we can solve for $|0\rangle$ and $|1\rangle$:

$$\begin{aligned}|0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |0\rangle) \\ |1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\end{aligned}$$

Therefore, the average $\langle X\rangle$:

$$\langle X\rangle = p_m(+1)(+1) + p_m(-1)(-1)$$

Then, we expand the probabilities:

$$\begin{aligned}p_m(+1) &= \langle\psi|+\rangle \langle +|+\rangle \langle +|\psi\rangle = \langle\psi|+\rangle \langle +|\psi\rangle \\ p_m(-1) &= \langle\psi|-\rangle \langle -|-\rangle \langle -|\psi\rangle = \langle\psi|-\rangle \langle -|\psi\rangle\end{aligned}$$

To complete the computation, we have to express $|\psi\rangle$ in the $|\pm\rangle$ basis:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \frac{\alpha}{\sqrt{2}}(|+\rangle + |-\rangle)$$

3 Multiple Qubits, Entanglement

3.1 Multiple Qubits

- Suppose we have two qubits $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $|\psi_2\rangle = x|0\rangle + y|1\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$

- Then the combined state, if the two qubits live on their own, is given by $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. The $\otimes$ symbol denotes a tensor product.
$$|\psi_1\rangle \otimes |\psi_2\rangle = (\alpha |0\rangle + \beta |1\rangle) \otimes (x |0\rangle + y |1\rangle)$$
In matrix form, this is represented as:
$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \beta x \\ \beta y \end{bmatrix}$$
the resulting vector lives in $\mathbb{C}^4$, with the basis states $|00\rangle = |0\rangle_1 |0\rangle_2$, $|10\rangle = |1\rangle_1 |0\rangle_2$, $|01\rangle = |0\rangle_1 |1\rangle_2$, $|11\rangle = |1\rangle_1 |1\rangle_2$

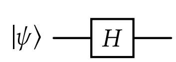
- In general given n qubits, there are 2^n basis states, and hence we will be working with superpositions over these 2^n basis states. This fact underscores the power of quantum computers, since they scale much more efficiently than classical computers. This is also sometimes referred to as "quantum parallelism".

- If we measure all qubits, then the outcome is just some sort of bitstring, so we have to be clever about how we are measuring to get the information we want.

- With multiple qubits, operators are also tensor products. Given the two operators:
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$
Then $A \otimes B$, the operator that acts on the multi-qubit state, is given by
$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}b \\ a_{21}B & a_{22}B \end{pmatrix}$$
Note that $A \otimes B$ is not the same as $B \otimes A$.

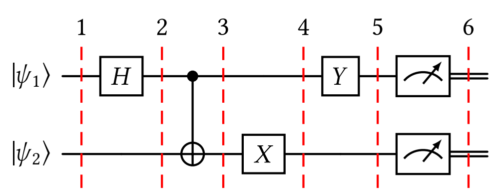
3.2 Quantum Circuits

- A generic quantum circuit is written as:



the box with an H denotes a gate (in this case, a Hadamard gate), which corresponds to a rotation on the Bloch sphere.

- Let's analyze the following quantum circuit:



Let's analyze this in stpes:

  - Initially, we have $|\psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$ and $|\psi_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$, whose combination can be written as:
$$|\psi_{12}^{(1)}\rangle = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

- At step 2, we run the first qubit through a Hadamard gate, and leave the second qubit untouched. This means we act the operator $H \otimes I$ on the state:

$$|\psi_{12}^{(2)}\rangle\, H \otimes I\, |\psi_{12}\rangle = \alpha_1\alpha_2 \left(\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |10\rangle \right) + \alpha_1\beta_2 \left(\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |11\rangle \right) + \beta_1\alpha_2 \left(\frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |10\rangle \right) + \beta_1\beta_2 \left(\frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |11\rangle \right)$$

- At step 3, we apply a CNOT gate, which flips the state of the second bit if the value of the first bit is 1. As a truth table:

Input	Output
00	00
01	01
10	11
11	10

  As a matrix, it's written as;

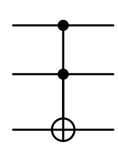
$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

  We then apply this CNOT gate to each component of $|\psi_{12}^{(2)}\rangle$ to get $|\psi_{12}^{(3)}\rangle$.

- Apply $I \otimes X$ to $|\psi_{12}^{(3)}\rangle \to |\psi_{12}^{(4)}\rangle$
- Apply $Y \otimes I$ tp $|\psi_{12}\rangle^{(4)} \to |\psi_{12}^{(5)}\rangle$
- Measurement in the Z basis, by applying projection operators to the final resulting state.

3.2.1 Other Common Gates

- There are many quantum gates that we'll study, here's a list of them that will be useful:
- CPHASE, or controlled Z gate
- Swap gate: swaps the
- S-phase: rotation by 90 degrees, $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
- P-phase: a general phase gate $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$
- Toffoli gate: controlled-controlled NOT gate:



- T-gate: $\begin{pmatrix} 1 & 0 \\ 9 & e^{i\pi/4j} \end{pmatrix}$

3.2.2 Universal Gate Sets

- A set G of quantum gates is considered universal if for $\epsilon > 0$ and for any unitary matrix U on n qubits, there is a sequence of gates from G such that

  $$\|U - U_{g_\ell} \cdots U_{g_2} U_{g_1}\| < \epsilon$$

  In this definition, we define $U_g = V \otimes I$, where V is an operator acting on k qubits, and I acts on the remaining $n - k$ qubits. The double bar represnets an operator norm, defined as:

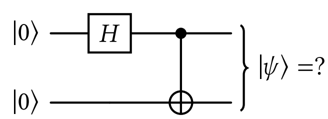
  $$\|U - U'\| = \max_{|v\rangle \text{ unit vectors}} \|(U - U')\,|v\rangle\,\|$$

  where $\|w\| = \sqrt{\langle w|w\rangle}$.

- Examples of universal gate sets:
  - Barenco et al. (1995): CNOT and all single qubit (continuous) gates.
  - CNOT, H, S, T gates
  - Rotation operators $R_x(\theta), R_y(\theta), R_z(\theta)$, the phase operator P_ϕ and CNOT.

3.3 Entanglement

- Consider 2 qubits:

  Well, we first start with the state $|00\rangle$, and after passing the first bit through a Hadamard gate, we get the state

  $$\frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

  Then, running it through the CNOT, then we have:

  $$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \Phi^+$$

  This is one of four states called the "Bell states", because there is no way to express this state as a product state of two individual qubits.

4 More on Multiple Qubits

- Last time, we looked at multiple-qubit states, and talked about how the combination is the tensor product, written like this:

  $$|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$$

- We also talked about how an entangled state is defined as a state where we cannot express as a (tensor) product state. In other words, the state is not separable.

- There are an infinite number of entangled states, called the Bell states:

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

- To quantify entanglement, we use a Schmidt decomposition for qubits: ($d = 2$ for qubits)

$$|\psi_{AB}\rangle = \sum_{i=0}^{d-1} c_i |i\rangle_A \; |i\rangle_B$$

  This state ψ_{AB} is separable if only one $c_i \neq 0$. The number of nonzero c_i is called the schmidt rank, and it's what we use to quantify how entangled a state is. If all c_i are equal, then the state is maximally entangled.

  The bell states $\Phi^\pm$ are easily seen to be maximally entangled, since $|00\rangle$ and $|11\rangle$ are the basis states, and they each have a coefficient of $1/\sqrt{2}$.

4.1 Measurement

- Given a state $|00\rangle$ and we measure the first qubit in the Z basis, what happens?
- Recall our measurement operator is a projection operator:

$$M_1 = |1\rangle\langle 1|$$
$$M_2 = |0\rangle\langle 0|$$

- Then, applying the measurement operators, we get an outcome of measuring 0 with probability 1. The state after measurement is given by $|00\rangle$. Note that the second qubit is not affected by this measurement.

  Are these two states identical?

- Now suppose we had a state of the form

$$|\psi\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

  This is the state that results when the second qubit is passed through a Hadamard gate. Now, if we measure the first state, we again certainly get a result of 0, so the measurement is given by: =

$$|\psi\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

  If we measure the second qubit (in the Z basis), then we get the state $|0\rangle \otimes |0\rangle$ with probability $\frac{1}{2}$, and $|0\rangle \otimes |1\rangle$ also with probability $\frac{1}{2}$.

- Another example, given the state:

$$|\psi\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

  And now we measure the first qubit, we get 0 and 1 with probability $\frac{1}{2}$, and we get the resulting states:

$$|\psi'\rangle = |0 \text{ or } 1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

4.2 Measurement with Entangled States

- Suppose we have a qubit in the state $|\Psi^-\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Now, we send the first qubit to Alice, and the second one to Bob.
- Alice will measure the first qubit in the Z basis, which will give her 0 or 1 with probability $\frac{1}{2}$.

  The thing is, if alice measures 0, then it means that the state now collapses to the first term in the superposition: $|\psi'\rangle = |01\rangle$, so Bob must get a result of 1 upon measurement. The flip is also true.
- This is an example where the outcomes of the measurements are now correlated!
- Now suppose we change our measurement basis: if we measure in the X basis, where measurements are given by $M_1 = |+\rangle\langle+|$ and $M_2 = |-\rangle\langle-|$.

  The same correlation follows: if Alice measures $|+\rangle$, then Bob will certainly get $|-\rangle$, and if Alice gets $|-\rangle$, Bob will certainly get $|+\rangle$.

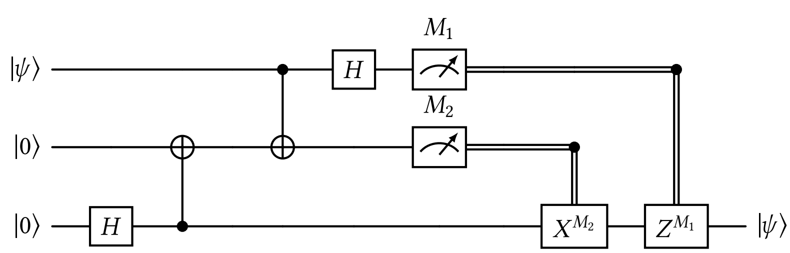
  How is the maesurement carried out? Do we express the state $|\Psi^-\rangle$ in terms of the $|\pm\rangle$ basis, and then carry out the probabilities?
- This idea that you can glean information about a quantum state without making a full measurement was problematic, and led Einstein, Podolsky and Rosen to speculate the presence of "hidden variables".

  John Bell proposed a set of inequalities (now called Bell inequalities) that would tell us for sure whether these hidden variables actually exist. He proposed a set of measurements that can be made called g, and if the systems were truly classical, then we would be able to determine that $\langle g\rangle \leq 2$. Otherwise, $\langle g\rangle > 2$ was possible.

  What we found through experiment was that $\langle g\rangle > 2$ was indeed possible, which leads us to the conclusion that there are no hidden variables are present.

4.3 Quantum Teleportation

- Consider the following circuit:



  Initially, the state is in $|\psi\rangle|0\rangle|0\rangle$. After the third qubit passes through the Hadamard gate, the state is

$$|\psi_2\rangle = |\psi\rangle|0\rangle\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$