Collaborators

I worked with **Andrew Binder, Adarsh Iyer, Nathan Song** and **Teja Nivarthi** on this homework assignment.

Problem 1 (25 pts)

In this class (and in pretty much all of physics) it is key to be proficient at computing Gaussian integrals. Consider

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2}$$

a) Show that

$$I^2 = \int_{\mathbb{R}} dx\,dy\, e^{-\frac{1}{2}(x^2+y^2)}$$

Solution: The definition of I^2 is that we multiply two integrals together, with separate integration variables. Therefore, we can write,

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \right)$$

Then notice that since these integrals are independent of each other, this is the same as writing a double integral:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy = \int_{\mathbb{R}^2} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy$$

as desired. ☐

b) Compute I^2 by expressing this integral in polar coordinates. Conclude that

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$$

Solution: Following the hint, we compute this integral in polar coordinates, where our integration bounds now go from $r \in [0, \infty)$ and $\theta \in [0, 2\pi]$. Our integral becomes:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{1}{2}r^2} r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{1}{2}r^2} r\,dr$$

The second integral can be solved via a u-substitution of $u = \frac{r^2}{2}$ so $du = r\,dr \implies dr = \frac{du}{r}$, hence:

$$I^2 = 2\pi \int_0^{\infty} e^{-u} du = 2\pi \left[-e^{-u} \right]_{u=0}^{\infty} = 2\pi$$

Since this is I^2, then we conclude that $I = \sqrt{2\pi}$. ☐

c) Show that

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}$$

Solution: The same steps done to part (b) can solve this problem as well. We split into polar coordinates,

and instead of letting $u = \frac{r^2}{2}$, we can let $u = \frac{ar^2}{2}$, meaning $dr = \frac{du}{ar}$. Therefore, the integral becomes:

$$I^2 = 2\pi \int_0^\infty e^{-u} \frac{du}{a} = \frac{1}{a} \underbrace{\left(2\pi \int_0^\infty e^{-u} du \right)}_{\text{same integral as part (b)}} = \frac{2\pi}{a}$$

Hence, $I = \sqrt{\frac{2\pi}{a}}$, as desired. $\qquad\square$

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d) Show that

$$I = \int_{-\infty}^\infty dx\, e^{-\frac{a}{2}x^2 + bx} = e^{\frac{1}{2a}b^2} \sqrt{\frac{2\pi}{a}}$$

Solution: Here, we complete the square on the exponent. First, factor out $-\frac{a}{2}$ from the exponent:

$$I = \int_{-\infty}^\infty e^{-\frac{a}{2}\left(x^2 + \frac{2b}{a}x\right)} dx$$

And now we complete the square:

$$I = \int_{-\infty}^\infty e^{-\frac{a}{2}\left(\left(x - \frac{b}{a}\right)^2 - \frac{b^2}{a^2}\right)} dx = \int_{-\infty}^\infty e^{-\frac{a}{2}\left(x - \frac{b}{a}\right)^2 + \frac{b^2}{2a}} dx = e^{\frac{b^2}{2a}} \int_{-\infty}^\infty e^{-\frac{a}{2}\left(x - \frac{b}{a}\right)^2} dx$$

This integral gives the same result as the previous integral, since we can perform a u-substitution $u = x + \frac{b}{a}$ so $du = dx$. Therefore:

$$I = e^{\frac{b^2}{2a}} \underbrace{\int_{-\infty}^\infty e^{-\frac{a}{2}u^2} du}_{\text{same as part (c)}} = e^{\frac{b^2}{2a}} \sqrt{\frac{2\pi}{a}}$$

as desired. $\qquad\square$

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e) Show that

$$I = \int_{-\infty}^\infty dx\, e^{-\frac{a}{2}x^2} x^2 = \frac{1}{a}\sqrt{\frac{2\pi}{a}}$$

Hint: Compute

$$\left[\frac{\partial^2}{\partial b^2} \int_{-\infty}^\infty dx\, e^{-\frac{a}{2}x^2 + bx} \right]_{b=0}$$

by both differentiating under the integral sign and explicitly computing its derivatives using the result in d).

Solution: Following the hint, we know the result of the integral from the previous problem, so we're basically just left with

$$I = \frac{\partial^2}{\partial b^2} \left[e^{\frac{b^2}{2a}} \sqrt{\frac{2\pi}{a}} \right]_{b=0} = \sqrt{\frac{2\pi}{a}} \frac{\partial}{\partial b} \left[\frac{b}{a} e^{\frac{b^2}{2a}} \right]_{b=0} = \sqrt{\frac{2\pi}{a}} \left[\frac{1}{a} e^{\frac{b^2}{2a}} + \frac{b^2}{a^2} e^{\frac{b^2}{2a}} \right]_{b=0} = \frac{1}{a}\sqrt{\frac{2\pi}{a}}$$

as desired. $\qquad\square$

Problem 2 (20 pts)

Consider a system of N particles with spin. Label the particles using an index $i = 1, \ldots, N$ so that the i-th particle has spin s_i. Unlike the systems we have encountered so far, in this system the possible values for each spin are $s_i = -1, 0, 1$. The energy of this system is given by

$$E = D \sum_{i=1}^{N} s_i^2$$

In other words, when the spin is ± 1, this costs the system an energy D, while whenever the spin is 0, this spin does not contribute to the energy of the system.

a) Explain why the number of accessible states that the system when it has an energy E is

$$\Omega(N, E) = \binom{N}{E/D} 2^{E/D}$$

Solution: Since each particle contributes an energy D to the system, if we have a total system with energy E then we can divide this up into $\frac{E}{D}$ quanta of energy, and we need to count the number of ways to allocate $\frac{E}{D}$ quanta of energy among N particles. This is a problem that we've done countless times before, where $q = \frac{E}{D}$, where we have $\binom{N}{q}$ possible ways.

Further, since a spin of ± 1 contributes the same toward the total energy, then we have 2 choices for the spin of each particle among those that we've selected to carry energy. This explains the $2^{E/D}$ term. Putting these two together, we get:

$$\Omega(N, E) = \binom{N}{E/D} 2^{E/D}$$

□

b) Compute the entropy of the system $S(N, E)$ in the limit of many particles and high energies $E \gg D$.

Solution: We use the definition of $S = k \ln \Omega$, so we have:

$$S = k \ln \left[\binom{N}{E/D} \right] + k \frac{E}{D} \ln 2$$

For this, we expand the binomial coefficient into an expression involving factorials, and use Stirling's formula of the form $\ln x! = x \ln x - x$, which gives us:

$$S = k \left[N \ln N - N - \frac{E}{D} \ln \frac{E}{D} + \frac{E}{D} - \left(N - \frac{E}{D} \right) \ln \left(N - \frac{E}{D} \right) + \left(N - \frac{E}{D} \right) + \frac{E}{D} \ln 2 \right]$$
$$= k \left[N \ln N - \frac{E}{D} \ln \frac{E}{D} - \left(N - \frac{E}{D} \right) \ln \left(N - \frac{E}{D} \right) + \frac{E}{D} \ln 2 \right]$$

We can't simplify this further, so that's where we'll stop. In the limit where $E \gg D$, we can replace all $\frac{E}{D} = E$, so this gives us:

$$S = k \left[N \ln N - E \ln E - (N - E) \ln(N - E) + E \ln 2 \right]$$

□

c) Compute the temperature of this system as a function of the energy and the number of particles. Can the temperature of this system be negative?

Solution: To calculate temperature, we use the definition that $\frac{1}{T} \equiv \frac{dS}{dU} = \frac{dS}{dE}$. Therefore, computing this derivative:

$$\frac{1}{T} = k \left[-\frac{1}{D} \ln \frac{E}{D} - \frac{E}{D} \cdot \frac{1}{\frac{E}{D}} \frac{1}{D} - \left(-\frac{1}{D} \right) \ln \left(N - \frac{E}{D} \right) - \left(N - \frac{E}{D} \right) \frac{1}{(N - \frac{E}{D})} \left(-\frac{1}{D} \right) + \frac{1}{D} \ln 2 \right]$$
$$= \frac{k}{D} \left[\ln \left(N - \frac{E}{D} \right) + \ln 2 - \ln \frac{E}{D} \right]$$

We can combine this into a single logarithm, giving us:

$$\frac{1}{T} = \frac{k}{D} \ln \left[\frac{2 \left(N - \frac{E}{D} \right)}{\frac{E}{D}} \right]$$

From here, we can see that it is indeed possible for T to be negative, and precisely it is negative when the logarithm returns a negative number, which occurs when:

$$\frac{2 \left(N - \frac{E}{D} \right)}{\frac{E}{D}} < e$$

Specifically in terms of E, this occurs when $E > \frac{2}{3} ND$. $\qquad \square$

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d) Obtain the energy for the system as a function of temperature. Discuss the low and high temperature limits. What happens to the entropy in these limits? What is the physical meaning of this?

Solution: We simply take the expression we got in the previous part and solve for E, which gives us the following expression:

$$E = \frac{ND}{1 + \frac{1}{2} e^{D/kT}}$$

Now to analyze the bounds. At high temperature, then $e^{D/kT} \to e^0 = 1$, so we have $E \to \frac{2}{3} ND$ in this limit. At low temperature, $e^{D/kT} \to \infty$ so $E \to 0$ in this limit.

Physically, the low temperature limit makes intuitive sense, since $E \to 0$ at low temperature, matching our intuitive relationship between energy and temperature . At high temperature, we expect every single bit to be randomly distributed, so $\frac{2}{3}$ of them will be in the states ± 1, which contributes to the total energy of the system. So, with this analysis, the high temperature limit works as well.

As for the entropy, we can substitute $E = \frac{2}{3} ND$ into the expression we derived in part (b) for the high temperature limit, which gives us:

$$S = kN \ln 3 = k \ln 3^N$$

At the low temperature limit, $E \to 0$, so all the terms with E immediately disappear (note that $\lim_{x \to 0} x \ln x = 0$). This gives us that at low temperature,

$$S = k \left[N \ln N - N \ln N \right] = 0$$

The physical interpretation of the high temperature limit is that at high temperatures, each particle has three possibilities for its spin, hence there are 3^N total microstates, which matches our expression of $S = k \ln \Omega = k \ln 3^N$. At low temperatures, this limit also makes sense (kind of), where we can think of the low temperature limit as a configuration where we have very little information, hence S approaches zero. $\qquad \square$

Problem 3 (20 pts)

Consider an ideal gas undergoing a process described by the fact that pV^2 is a constant. What is the molar heat capacity of this process?

Solution: The heat capacity is defined as $C = \frac{Q}{\Delta T}$, and using the first law of Thermodynamics gets us

$$C = \frac{Q}{\Delta T} = \frac{\Delta U - W}{\Delta T} = \frac{\Delta U}{\Delta T} - \frac{W}{\Delta T}$$

We assume that we're working with a monoatomic ideal gas so $f = 3$ (from Ed), so we have:

$$U = \frac{3}{2}NkT = \frac{3}{2}nRT \implies \frac{\Delta U}{\Delta T} = \frac{dU}{dT} = \frac{3}{2}nR$$

Meanwhile, we can use $W = -P\Delta V$ to simplify the second term:

$$\frac{W}{\Delta T} = -\frac{P\Delta V}{\Delta T} = -P\frac{dV}{dT}$$

To do this, we use the fact that PV^2 is constant, so $\frac{d(PV^2)}{dT} = 0$:

$$\frac{d(PV^2)}{dT} = P\frac{dV^2}{dT} + V^2\frac{dP}{dT}$$
$$0 = 2PV\frac{dV}{dT} + V^2\frac{dP}{dT}$$
$$V^2\frac{dP}{dT} = -2PV\frac{dV}{dT}$$

Rearranging this to match the expression we have for $\frac{W}{\Delta T}$:

$$-P\frac{dV}{dT} = \frac{V}{2}\frac{dP}{dT}$$

To calculate $\frac{dP}{dT}$, we use the ideal gas law:

$$\frac{dP}{dT} = \frac{d}{dT}\left(\frac{nRT}{V}\right) = \frac{nR}{V} - \frac{nRT}{V^2}\frac{dV}{dT}$$

Therefore, we have:

$$-P\frac{dV}{dT} = \frac{V}{2}\left(\frac{nR}{V} - \frac{nRT}{V^2}\frac{dV}{dT}\right)$$
$$= \frac{nR}{2} - \frac{P}{2}\frac{dV}{dT}$$

Noticing that we have the same term on the left hand side, we get:

$$-\frac{P}{2}\frac{dV}{dT} = \frac{nR}{2} \implies -P\frac{dV}{dT} = nR = \frac{W}{\Delta T}$$

Therefore, putting them both together:

$$C = \frac{Q}{\Delta T} = \frac{\Delta U - W}{\Delta T} = \frac{3}{2}nR - nR = \frac{1}{2}nR$$

Finally, computing the molar heat capacity:

$$C_n = \frac{C}{n} = \frac{R}{2}$$

$\square$