Physics W89 - Introduction to Mathematical Physics - Summer 2023
Problem Set - Module 09 - One, Two, Three, Fourier

Last Update: September 14, 2023

Problem 9.1 - The Legendre Polynomials

Relevant Videos: Power Series Solutions; The Method of Frobeneus

The ***Legendre polynomials*** $P_\ell(x)$ are solutions to the second-order ODE

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0. \tag{1}$$

(a) Use the ansatz $y(x) = ax^2 + bx + c$ to try to find solutions for this ODE in the three different cases $\ell = 0$, $\ell = 1$, and $\ell = 2$.
[*Note: You will only get a single solution using this ansatz for each of these cases since the other linearly independent solution is not a polynomial of finite order.*]

Solution: Using the Antsatz, we find $y''(x) = 2a$ and $y'(x) = 2ax + b$. Starting with $\ell = 0$, let's plug this in:

$$\begin{aligned}(1 - x^2)(2a) - 2x(2ax + b) &= 0\\ -3ax^2 - bx + a &= 0\end{aligned}$$

Comparing coefficients, we realize that all the coefficients on the left hand side must be zero, Hence, we get that $a = b = 0$, but c can be anything, so any equation of the form $y = c$ where c is a constant solves the equation.

For $\ell = 1$:

$$\begin{aligned}(1 - x^2)(2a) - 2x(2ax + b) + 2(ax^2 + bx + c) &= 0\\ -2ax^2 + (a + c) &= 0\end{aligned}$$

So from here we get that $a = 0$ and $a + c = 0$, but since $a = 0$ this means that $c = 0$ too. Since this doesn't restrict anything on b, so the equation $y = bx$ for any b would work.

Finally for $\ell = 2$:

$$\begin{aligned}(1 - x^2)(2a) - 2x(2ax + b) + 6(ax^2 + bx + c) &= 0\\ 2bx + 3c + a &= 0\end{aligned}$$

From here we get that $b = 0$, and $a + 3c = 0$, so this implies that $a = -3c$. Thus, any equation of the form $y = ax^2 - \frac{1}{3}a$ would satisfy this differential equation. □

We can solve Eq. 1 by a power series solution using the Frobenius method! Recall that this involves the ansatz

$$y(x) = x^r \sum_{m=0} a_m x^m = \sum_{m=0} a_m x^{r+m},$$

where $r \in \mathbb{R}$ is some unknown initial power and $a_0 \neq 0$.

(b) Plug the power series ansatz into Eq. 1. Find the indical equation (the coefficient of the x^{r-2} terms) to determine the two possible values of r.
Partial Answer (highlight to reveal): [r can be 0 or 1].

Solution: Taking derivatives:

$$y'(x) = \sum_m (r+m) a_m x^{r+m-1}$$
$$y''(x) = \sum_m (r+m)(r+m-1) a_m x^{r+m-2}$$

Therefore our differential equation is:

$$(1-x^2)\sum_m (r+m)(r+m-1)a_m x^{r+m-2} - 2x\sum_m (r+m)a_m x^{r+m-1} + \ell(\ell+1)\sum_m a_m x^{r+m} = 0$$

Expanding out the $1-x^2$:

$$-\sum_m (r+m)(r+m-1)a_m x^{r+m} + \sum_m (r+m)(r+m-1)a_m x^{r+m-2} - 2\sum_m (r+m)a_m x^{r+m} + \ell(\ell+1)\sum_m a_m x^{r+m} = 0$$

For the indical equation, we set $m=0$, giving us:

$$a_0 x^r r(r-1) + r(r-1)a_0 x^{r-2} - 2ra_0 x^r + \ell(\ell+1)a_0 x^r = 0$$

Looking at the coefficient on the x^{r-2} term, we have the equation $r(r-1)a_0 = 0$, but since $a_0 \neq 0$, this requires that $r(r-1) = 0$, so hence r can be either 0 or 1, as confirmed by the spoiler. □

(c) Find the equation for the coefficient of the x^{r-1} terms. Show that in the case (spoilers!) $r = 0$ from part (b) that the coefficient a_1 may be left arbitrary but in the case $r = 1$, we must have $a_1 = 0$.

Solution: Looking at the equation we got from plugging in our Ansatz:

$$-\sum_m (r+m)(r+m-1)a_m x^{r+m} + \sum_m (r+m)(r+m-1)a_m x^{r+m-2} - 2\sum_m (r+m)a_m x^{r+m} + \ell(\ell+1)\sum_m a_m x^{r+m} = 0$$

Here, only the second term in this entire expression has an x^{r-1} exponent, when $m = 1$. Therefore, we get the equation:

$$r(r+1)a_1 x^{r-1} = 0$$

Here, if $r = 0$, then a_1 can be anything, since the product equals zero regardless. However, if $r = 1$, then we have $2a_1 = 0$, meaning that a_1 must equal zero. □

(d) Set the coefficients of the x^{r+m} terms ($m \geq 0$) to zero to get a ***recursion relation*** relating a_{m+2} to a_m. Then, in the case $\ell = 4$, $r = 0$, and $a_1 = 0$, find all the non-zero coefficients in terms of a_0. Write down the solution $P_4(x)$ and check that it solves Eq. 1.

Solution: To get the x^{r+m} term, we first shift the indexing of the second term by 2, so that all the exponents on the x are in line with each other:

$$-\sum_{m=0} (r+m)(r+m-1)a_m x^{r+m} + \sum_{m=-2} (r+m+2)(r+m+1)a_{m+2} x^{r+m} - 2\sum_{m=0} (r+m)a_m x^{r+m} + \ell(\ell+1)\sum_{m=0} a_m x^{r+m} = 0$$

Pulling x^{r+m} out of the equation, we notice that all the coefficients must be zero, which leaves us with the following equation (I'm intentionally skipping the algebra here since its a lot to type):

$$-(r+m)(r+m-1)a_m + (r+m+2)(r+m+1)a_{m+2} - 2(r+m)a_m + \ell(\ell+1)a_m = 0$$

Solving for a_{m+2}, this gives:

$$a_{m+2} = \frac{2(r+m) + (r+m)(r+m+1) - \ell(\ell+1)}{(r+m+2)(r+m+1)} a_m$$

Now for our specific case of $\ell = 4$, $r = 0$ and $a_1 = 0$, the recursion relation becomes:

$$a_{m+2} = \frac{2m + m(m-1) - 20}{(m+1)(m+2)} a_m$$

Since $a_1 = 0$, and all odd terms $a_3, a_5, \dots, a_{2n+1}$ all depend on a_1, we conclude that all odd coefficients $a_{2n+1} = 0$. To find a_2, we plug in $m = 0$:

$$a_2 = -\frac{20}{2} = -10a_0$$

Then to find a_4, we plug in $m = 2$:

$$a_4 = \frac{4+2-20}{12} = -\frac{14}{12} = -\frac{7}{6}a_2 = -\frac{7}{6}(-10a_0) = \frac{35}{3}a_0$$

When we look to find a_6, we get:

$$a_6 = \frac{8+12-20}{30}a_4 = 0$$

And since all subsequent even terms depend on a_6 (recursively), then we conclude that all the remaining terms here must be zero. Therefore, we've found all the nonzero coefficients. So writing out $P_4(x)$:

$$P_4(x) = \frac{35}{3}a_0x^4 - 10a_0x^2 + a_0$$

Computing its derivatives:

$$P_4'(x) = \frac{140}{3}a_0x^3 - 20a_0x$$
$$P_4''(x) = 140a_0x^2 - 20a_0$$

Plugging this into the differential equation:

$$(1-x^2)(140a_0x^2 - 20a_0) - 2x\left(\frac{140}{3}a_0x^3 - 20a_0x\right) + 20\left(\frac{35}{3}a_0x^4 - 10a_0x^2 + a_0\right)$$
$$= 140a_0x^2 - 20a_0 - 140a_0x^4 + 20a_0x^2 - \frac{280}{3}a_0x^4 + 40a_0x^2 + \frac{700}{3}a_0x^4 - 200a_0x^2 + 20a_0$$

Grouping terms:

$$a_0x^4\left(\frac{700}{3} - \frac{280}{3} - 140\right) + a_0x^2(140+20+40-200) - a_0(20-20) = 0x^4 + 0x^2 + 0 = 0$$

And since this all equals zero, we conclude that this indeed solves the differential equation. $\square$

Problem 9.2 - Trigonometric Fourier Series

Relevant Videos: The Trigonometric Fourier Series

Consider the space of periodic functions of period 2π. This is a vector subspace of the vector space of functions of the variable θ,

$$\left\{\vec{f} \doteq f(\theta) \,\middle|\, f(\theta + 2\pi) = f(\theta)\right\}.$$

We can introduce the following inner product on the subspace of periodic functions,[1]

$$\vec{f} \cdot \vec{g} \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)^* g(\theta) d\theta.$$

In class we introduced the basis vectors $\{\vec{e}_0, \hat{e}_{c,n}, \hat{e}_{s,n}\}$, with $n \in \mathbb{N}$,

$$\vec{e}_0 \doteq \frac{1}{2}; \qquad \hat{e}_{c,n} \doteq \cos(n\theta); \qquad \hat{e}_{s,n} \doteq \sin(n\theta).$$

(a) **Extra Part** (*Not for Credit*) Show that the space of periodic functions of period 2π is indeed a vector subspace of the vector space of functions.

(b) Show that $\{\vec{e}_0, \hat{e}_{c,n}, \hat{e}_{s,n}\}$ is an orthogonal set of functions under the inner product given and that $\hat{e}_{c,n}$ and $\hat{e}_{s,n}$ are normalized. What is the normalization of $\vec{e}_0$?

Solution: To prove they are orthogonal, we show that the inner product between all three of these equals zero. First, we start with $\hat{e}_0 \cdot \hat{e}_{s,n}$:

$$\begin{aligned}
\vec{e}_0 \cdot \hat{e}_{s,n} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\sin(n\theta) d\theta \\
&= \frac{1}{2\pi n}\left(-\cos(n\theta)\right)_{-\pi}^{\pi} \\
&= \frac{1}{2\pi n}\left[-\cos(n\pi) + \cos(-n\pi)\right]
\end{aligned}$$

Since the cosine function is even, we have $\cos(x) = \cos(-x)$, so the term in the square brackets cancels out, and we get $\hat{e}_0 \cdot \hat{e}_{s,n} = 0$. Now for $\hat{e}_0 \cdot \hat{e}_{c,n}$:

$$\begin{aligned}
\vec{e}_0 \cdot \hat{e}_{c,n} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\cos(n\theta) d\theta \\
&= \frac{1}{2\pi n}(\sin(n\theta))_{-\pi}^{\pi} \\
&= \frac{1}{2\pi n}(2\sin(n\pi)) \\
&= \frac{\sin(n\pi)}{\pi n}
\end{aligned}$$

And since n is an integer, $\sin(n\pi) = 0$, so therefore $\hat{e}_0 \cdot \hat{e}_{c,n} = 0$. Finally, for $\hat{e}_{c,n} \cdot \hat{e}_{s,n}$:

$$\hat{e}_{c,n} \cdot \hat{e}_{s,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta)\sin(n\theta) d\theta$$

Here we can so a u-substitution of $u = \sin(n\theta)$, so $du = n\cos(n\theta) d\theta$, therefore:

$$\hat{e}_{c,n} \cdot \hat{e}_{s,n} = \int_0^0 \frac{1}{n} du = 0$$

[1] Since the functions are periodic the range of integration can be any 2π interval such as $-\pi$ to π or 0 to 2π.

This integral is equal to zero since we're integrating over a region of zero width. Hence, we've proven that every basis vector is orthogonal to one another. To show that the $\hat{e}_{c,n}$ and $\hat{e}_{s,n}$ are normalized, we take the inner product with themselves, here I used Mathematica to compute these integrals since I couldn't figure these out by hand:

$$\hat{e}_{c,n} \cdot \hat{e}_{c,n} = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(n\theta) d\theta = \frac{1}{\pi} \left(\pi + \frac{\sin(2n\pi)}{2n} \right) = \frac{1}{\pi}\pi = 1$$

Now for sine:

$$\hat{e}_{s,n} \cdot \hat{e}_{s,n} = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(n\theta) d\theta = \frac{1}{\pi} \left(\pi - \frac{\sin(2n\pi)}{2n} \right) = \frac{1}{\pi}\pi = 1$$

As for the normalization of $\hat{e}_0$, all we have to do is find the inner product of $\hat{e}_0$ with itself:

$$\vec{e}_0 \cdot \vec{e}_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{4} d\theta = \frac{1}{4\pi}(2\pi) = \frac{1}{2}$$

As for the "length" of the vector, we can take the square root of this quantity:

$$|\vec{e}_0| = \frac{1}{\sqrt{2}}$$

so if we want to find $\hat{e}_0$, we take $\vec{e}_0$ and divide it by the magnitude:

$$\hat{e}_0 = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}$$

and that completes the problem. $\square$

Any arbitrary real periodic function $f(\theta)$ of period 2π can be written with respect to this basis which results in the ***trigonometric Fourier expansion*** of $f(\theta)$:

$$\vec{f} = a_0 \vec{e}_0 + \sum_{n=1}^{\infty} a_n \hat{e}_{c,n} + \sum_{n=1}^{\infty} b_n \hat{e}_{s,n} \tag{2}$$

$$\implies \quad f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta). \tag{3}$$

(c) Extra Part (*Not for Credit*) Use the expression Eq. 2 given for $\vec{f}$ and the results from part (b) to show that the Fourier coefficients can be determined via the following expressions:

$$a_0 = 2\vec{e}_0 \cdot \vec{f}, \qquad a_n = \hat{e}_{c,n} \cdot \vec{f}, \qquad b_n = \hat{e}_{s,n} \cdot \vec{f}.$$

Given the result of part (c) we can find the coefficients using the integral expression of the dot product. For example, $a_2 = \hat{e}_{c,2} \cdot \vec{f} = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2\theta) f(\theta) dx$.

(d) Use symmetry to determine which coefficients are zero for ***even functions***, $f(-\theta) = f(\theta)$. For ***odd functions***, $f(-\theta) = -f(\theta)$?

Solution: For a_0, we have:

$$a_0 = \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} f(\theta) d\theta$$

Since the integral is over a symmetric interval, we get that an odd $f(\theta)$ will give us zero, while even functions will be nonzero.

For a_n, we have:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\theta) f(\theta) d\theta$$

Since $\cos(n\theta)$ is even, and the product of an even function and an odd function is odd, then we find that odd functions will have all $a_n = 0$, while even functions will be nonzero.

Finally, for b_n:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(n\theta) f(\theta) d\theta$$

We find the exact opposite is the case. Since $\sin(n\theta)$ is an odd function, we find that if $f(\theta)$ is even then $b_n = 0$, while odd functions have nonzero b_n. □

(e) What are the trigonometric Fourier coefficients $\{a_0, a_n, b_n\}$ for the function $f(\theta) = \cos(3\theta)$? $f(\theta) = \sin(4\theta)$? $f(\theta) = e^{2i\theta}$?
[*Spoilers!* You can basically just read these off! For the last one, use Euler's formula.]

Solution: Following the spoiler, we can just read these off. Since the expansion is:

$$f(x) = \frac{1}{2} a_0 + \sum_n a_n \cos(n\theta) + \sum_n b_n \sin(n\theta)$$

the function $f(\theta) = \cos(3\theta)$ has $a_3 = 1$, and all other $a_n, b_n = 0$. Likewise, for $f(\theta) = \sin(4\theta)$ we have $b_4 = 1$, with all other $a_n, b_n = 0$. Finally, with $f(\theta) = e^{2i\theta}$, we use Euler's formula to rewrite this as:

$$e^{2i\theta} = \cos(2\theta) + i \sin(2\theta)$$

So this means that $a_2 = 1$, $b_2 = i$, and all other $a_n, b_n = 0$. □

(f) Find the trigonometric Fourier coefficients for the ***square wave***, a periodic function defined so $f(\theta) = +1$ if $n\pi < \theta < (n+1)\pi$ and n is an even integer and $f(\theta) = -1$ if $n\pi < \theta < (n+1)\pi$ and n is an odd integer.

Solution: Here, we'll integrate the function from 0 to 2π instead of $-\pi$ to π. The function is odd over this interval, so from part (d), we know that $a_0 = 0$ and $a_n = 0$, since for them to be nonzero we require an even function $f(\theta)$. All that remains is to calculate b_n:

$$\begin{aligned} b_n &= \hat{e}_{s,n} \cdot \vec{f} \\ &= \frac{1}{\pi} \left[\int_0^{\pi} \sin(n\theta)(1) d\theta + \int_{\pi}^{2\pi} \sin(n\theta)(-1) d\theta \right] \end{aligned}$$

After some algebra, we get:

$$b_n = \frac{2}{n\pi} \left[1 - (-1)^n \right]$$

This means that if n is even, then $b_n = 0$, the term in the square brackets evaluates to zero. However, if b_n is odd, then

$$b_n = \frac{4}{n\pi}$$

To write this more concisely:

$$b_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

And that completes the problem. □

(g) **Extra Part** (*Not for Credit*) Graph the square wave function and the approximations taken by only keeping the terms up to $n = 1$, $n = 3$, and $n = 10$.

(h) **Extra Part** (*Not for Credit*) What does taking the derivative of a function do to the trigonometric Fourier coefficients? That is, if $f(x)$ has coefficients $\{a_0, a_n, b_n\}$, what are the coefficients $\{\tilde{a}_0, \tilde{a}_n, \tilde{b}_n\}$ of $f'(\theta) = \frac{df}{d\theta}$?
[*Spoilers!* Start with the expansion in Eq. 3 and take the derivative. Then you can just read off the coefficients of the cosines and signs.]

Problem 9.3 - The Exponential Fourier Series

Relevant Videos: The Complex Fourier Series

Consider the space of periodic functions of period $L = 2\pi/k_0$ or, equivalently, the space of functions defined on the interval $x_0 \le x \le x_0 + L$ so that the range of interest has width L. This is again a vector subspace of the vector space of functions. We can again make an inner product on this space,[2]

$$\vec{f} \cdot \vec{g} = \frac{1}{L} \int_{x_0}^{x_0+L} f^*(x) g(x) dx.$$

With respect to this inner product we have an orthonormal set of vectors $\{\hat{e}_n\}$, $n \in \mathbb{Z}$ for this subspace,

$$\hat{e}_n \doteq e^{ink_0 x},$$

The ***exponential Fourier series*** for a function is then the expansion of that function with respect to this basis,

$$\vec{f} = \sum_{n=-\infty}^{\infty} c_n \hat{e}_n \quad \Longrightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ink_0 x}. \tag{4}$$

We can find the expansion coefficients using orthonormality of these basis vectors,

$$c_n = \hat{e}_n \cdot \vec{f} = \frac{1}{L} \int_{x_0}^{x_0+L} e^{-ink_0 x} f(x)\, dx. \tag{5}$$

(a) Plug Eq. 4 into the right-side of Eq. 5 and use orthonormality to show that the right-hand side of Eq. 5 does indeed evaluate to c_n.
[*Spoilers!* You may assume orthonormality. Be careful! You will have to change the summation index in Eq. 4 since we are already using the index n in Eq. 5]

Solution: Substituting the relevant equations in:

$$\begin{aligned} c_n &= \frac{1}{L} \int_{x_0}^{x_0+L} e^{-ink_0 x} \sum_{n'} c_{n'} e^{in' k_0 x} dx \\ &= \sum_{n'} c_{n'} \underbrace{\frac{1}{L} \int_{x_0}^{x_0+L} e^{-ink_0 x} e^{in' k_0 x} dx}_{=\hat{e}_n \cdot \hat{e}_{n'}} \end{aligned}$$

Now, notice that since our basis functions are exponentials $\hat{e}_n = e^{ink_0 x}$, then this integral is actually just the inner product of two basis vectors, or $\hat{e}_n \cdot \hat{e}_{n'}$. Hence, this is equal to the delta function, so we have:

$$\sum_{n'} c_n \delta_{nn'} = c_n$$

So the expression indeed gives us c_n. □

(b) **Extra Part** (*Not for Credit*) Show that if $f(x)$ is a real function then $c_{-n} = c_n^*$, and if $f(x) = \pm f(-x)$ then $c_{-n} = \pm c_n$.

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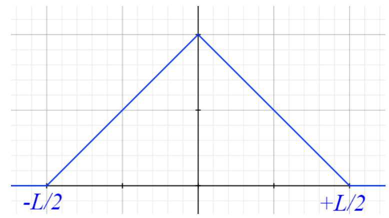
[2] It turns out that if $f(x)$ and $g(x)$ are periodic with period L then this choice of starting point is irrelevant and doesn't appear in any answers.

Consider a guitar string of length L stretched from $-L/2$ to $+L/2$. Let x be the length along the string and let $y(x)$ be the height of the string. Initially, the guitar string is "plucked" to a triangular shape,



$$y(x) = \begin{cases} +x + \frac{L}{2}, & -\frac{L}{2} \le x \le 0 \\ -x + \frac{L}{2}, & 0 \le x \le \frac{L}{2} \end{cases} .$$

(c) Find the Fourier expansion coefficients c_n for this function.
[*Note: Our period here is L and our interval starts at* $x_0 = -L/2$.]
[*Spoilers!* Hmm... this seems awfully familiar...]

Solution: Here we just use the definition that $c_n = \hat{e}_n \cdot \vec{f}$:

$$\begin{aligned} c_n &= \frac{1}{L}\int_{-L/2}^{L/2} e^{-ink_0x} y(x) dx \\ &= \frac{1}{L}\int_{-L/2}^{0} e^{-ink_0x}\left(x + \frac{L}{2}\right) dx + \frac{1}{L}\int_{0}^{L/2} e^{-ink_0x}\left(-x + \frac{L}{2}\right) dx \end{aligned}$$

Now we can use Mathematica to help us out with this integration:

$$c_n = \frac{L(1 - e^{-in\pi} - in\pi)}{4n^2\pi^2} + \frac{L(1 - e^{in\pi} + in\pi)}{4n^2\pi^2}$$

Adding these two up and simplifying, this gets us:

$$c_n = \frac{L(1 - \cos(\pi n))}{2n^2\pi^2}$$

Then, after plugging in that $L = \frac{2\pi}{k}$, we get:

$$c_n = \frac{1 - \cos(\pi n)}{n^2 \pi k_0}$$

□

(d) What does taking the derivative of a function do to the exponential Fourier coefficients? That is, if $f(x)$ has coefficients $\{c_n\}$, what are the coefficients $\{\tilde{c}_n\}$ of $f'(x) = \frac{df}{dx}$?
[*Spoilers!* Start with the expansion in Eq. 4 and take the derivative. Then you can just read off the coefficients of the exponentials.]

Solution: We can just take the derivative starting with the expression

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ink_0x}$$

Simply taking the derivative (making use of its linearity):

$$f'(x) = \sum_{n=-\infty}^{\infty} c_n (ink_0) e^{ink_0x}$$

So we get that $c_n' = c_n(ink_0)$. □

Problem 9.4 - The Dirac Delta

Relevant Videos: The Fourier Transform; The Fourier Transform Completeness Relations (for a discussion of the Dirac Delta Function)

Recall the following two integral relationships involving the Dirac delta function

$$\int_a^b f(u)\delta(u-u_0)du = \begin{cases} f(u_0), & a < u_0 < b \\ 0, & \text{else} \end{cases};$$

$$\int_{-\infty}^{\infty} e^{iau} du = 2\pi\delta(a). \tag{6}$$

Also recall that the Fourier transform of a function of space (or time) $f(x)$ is given by a function of wavenumber (or frequency) $c(k)$ (also written $\mathcal{F}[f](k)$ or $F(k)$ or $\tilde{f}(k)$), where

$$\text{Inverse Fourier Transform:} \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk,$$

$$\text{Fourier Transform:} \qquad c(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx.$$

Warning: *There are many different conventions for the definition of the Fourier transform, including how the* $1/2\pi$ *prefactor of the integrals is split up and whether the cyclic or angular frequencies/wavenumbers are used. This makes comparing between sources a pain.*[3] *See the Problem Set Supplement for a discussion of this.*

(a) Find the Fourier transforms of $\delta(x-x_0)$, e^{ik_0x}, and $\sin(k_0x)$. Be sure to show all your work.
[*Note: Please try to do this one without looking at our lecture notes, which partially contain the solutions.*]

Solution: I'll do this in a list:

- $f(x) = \delta(x - x_0)$:

$$c(k) = \int_{-\infty}^{\infty} e^{-ikx}\delta(x-x_0) = e^{-ikx_0}$$

- $f(x) = e^{ik_0x}$:

$$c(k) = -\int_{-\infty}^{\infty} e^{-ikx}e^{ik_0x} = \int_{-\infty}^{\infty} e^{i(k_0-k)x} = 2\pi\delta(k_0-k) = 2\pi\delta(k-k_0)$$

- $f(x) = \sin(k_0x)$

$$\begin{aligned} c(k) &= \int_{-\infty}^{\infty} e^{-ikx}\sin(k_0x)dx \\ &= \frac{1}{2i}\int_{-\infty}^{\infty} e^{ikx}\left(e^{ik_0x} - e^{-ik_0x}\right) \\ &= \frac{1}{2i}\left[2\pi\delta(k_0-k) - 2\pi\delta(k+k_0)\right] \\ &= \frac{\pi}{i}\left[\delta(k-k_0) - \delta(k+k_0)\right] \end{aligned}$$

□

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[3] Our convention puts the factor of $1/2\pi$ on the Inverse Fourier transform formula, matching what Wikipedia calls the "non-unitary convention" with angular frequency. Boaz notably uses a different convention, with $1/2\pi$ on the Fourier transform formula instead.

(b) **Extra Part** (*Not for Credit*) Use integration to prove the identity $\delta\big(a(x-x_0)\big) = \frac{1}{|a|}\delta(x-x_0)$.
[*Note: We did this in lecture. Recall that a u-substitution is key to this proof.*]

If there is more than one place where the argument of the delta function is zero, we have to be careful. For example, we have the following identity,

$$\delta\big((x-x_1)(x-x_2)\big) = \frac{\delta(x-x_1)+\delta(x-x_2)}{|x_1-x_2|}.$$

Let's prove this identity by integrating! Let $x_1 < x_2$ and consider some constant c between x_1 and x_2 so $x_1 < c < x_2$. Note that these are all strict inequalities.

(c) Evaluate the integral $\int_{-\infty}^{\infty} f(x)\delta\big((x-x_1)(x-x_2)\big)\,dx$ by breaking up the range of the integral to $-\infty$ to c followed by c to $+\infty$. Show that this gives the same answer as the integral $\int_{-\infty}^{\infty} f(x)\frac{1}{|x_1-x_2|}\big(\delta(x-x_1)+\delta(x-x_2)\big)\,dx$.
[*Spoilers!* The result from part (b) is key to this proof! Note that in the region around $x = x_1$, we can effectively treat the term $(x-x_2)$ as the constant (x_1-x_2).]

Solution: Splitting the integral we get:

$$\int_{-\infty}^{\infty} f(x)\delta((x-x_1)(x-x_2))dx = \int_{-\infty}^{c} f(x)\delta((x-x_1)(x-x_2))dx + \int_{c}^{\infty} f(x)\delta((x-x_2)(x-x_1))dx$$

Since x_2 is outside the bound of the first integral, the term $x-x_2$ can be treated as a constant, and in the second integral, x_1 is outside the bound so $x-x_1$ is treated as a constant value. Therefore, we can use the relation:

$$\delta(a(x-x_0)) = \frac{1}{|a|}\delta(x-x_0)$$

to simplify the expression. Doing so, we get:

$$\begin{aligned}\int_{-\infty}^{c} \frac{f(x)}{|x-x_2|}\delta(x-x_1)dx + \int_{c}^{\infty} \frac{f(x)}{|x-x_1|}\delta(x-x_2)dx &= \frac{f(x_1)}{|x_1-x_2|} + \frac{f(x_2)}{|x_2-x_1|} \\ &= \frac{f(x_1)+f(x_2)}{|x_1-x_2|}\end{aligned}$$

Now the right hand side:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\frac{1}{|x_1-x_2|}(\delta(x-x_1)+\delta(x-x_2))dx &= \frac{1}{|x_1-x_2|}\left[\int_{-\infty}^{\infty} f(x)\delta(x-x_1)dx + \int_{-\infty}^{\infty} f(x)\delta(x-x_2)dx\right] \\ &= \frac{f(x_1)+f(x_2)}{|x_1-x_2|}\end{aligned}$$

which exactly matches the left hand side. Hence, we conclude that they are equal. □

The ***Heaviside step function*** (really another example of a distribution rather than a proper function) $\theta(x)$ may be defined as[4]

$$\theta(x) \equiv \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}.$$

That is, the function is "heavy" on one side (1, when $x > 0$) and "light" on the other (0, when $x < 0$).[5] The Heaviside step function gives us a really convenient way to write some piecewise functions. For example, the ***rectangle***

[4] There are different conventions for the value of $\theta(0)$. In our definition it's 1, in others it's 1/2.

[5] This is an example of an ***aptronym*** - a name that is amusingly appropriate to the application. The Heaviside step function is actually named after mathematician and physicist Oliver Heaviside.

function $\Pi(x)$ and ***ramp*** function $R(x)$ can be expressed as

$$\Pi(x) = \theta\left(x + \frac{1}{2}\right) - \theta\left(x - \frac{1}{2}\right) = \begin{cases} 1, & -\frac{1}{2} \le x < \frac{1}{2}, \\ 0, & \text{else} \end{cases}$$

$$R(x) = x\theta(x) = \begin{cases} x, & x \ge 0 \\ 0, & x < 0 \end{cases}.$$

(d) Argue that the Heaviside step function can be used to "encode" the limits of integration into the integrand,

$$\int_{-\infty}^{\infty} f(x)\theta(x-a)dx = \int_{a}^{\infty} f(x)dx, \qquad \int_{-\infty}^{\infty} f(x)\theta(b-x)dx = \int_{-\infty}^{b} f(x)dx.$$

Solution: For the first equation, notice that due to the definition of the Heaviside function, that the integrand is only nonzero when $x > a$, so anything with $x < a$ will immediately equal zero, and hence can be taken out of the integral. Therefore:

$$\int_{-\infty}^{\infty} f(x)\theta(x-a)dx = \int_{a}^{\infty} f(x)dx$$

For the second equation, it's the same deal except backwards. The integrand is only nonzero when $b - x > 0$, or $x < b$, so this serves as an upper bound instead:

$$\int_{-\infty}^{\infty} f(x)\theta(b-x)dx = \int_{-\infty}^{b} f(x)dx$$

□

(e) Show or argue that the integral of the Dirac delta is the Heaviside step function, $\theta(x) = \int_{-\infty}^{x} \delta(x')dx'$ and, conversely, that the Dirac delta is the derivative of the Heaviside step function, $d\theta(x)/dx = \delta(x)$.
[*Supplementary Part (Not for Credit): Show that the integral of the Heaviside step function is the Ramp function* $R(x) = \int_{-\infty}^{x} \theta(x')dx'$ *and, conversely, that the Heaviside step function is the derivative of the Ramp function,* $dR(x)/dx = \theta(x)$.]

Solution: We start with the definition of the Dirac delta:

$$\int_{a}^{b} f(u)\delta(u-u_0)du = \begin{cases} f(u_0) & a < u_0 < b \\ 0 & \text{else} \end{cases}$$

Now, we'll let $f(u) = 1$, let $b = x$, $a = -\infty$ and $u_0 = 0$ so that this equation assumes the form given in the problem statement. Then, we get the equation:

$$\int_{-\infty}^{x} \delta(u)du = \begin{cases} 1 & -\infty < 0 < x \\ 0 & \text{else} \end{cases}$$

where the right hand side can be rewritten as:

$$\int_{-\infty}^{x} \delta(u)du = \begin{cases} 1 & x > 0 \\ 0 & \text{else} \end{cases} = \theta(x)$$

which is exactly the definition of the Heaviside step function. On the other side, we can see that the Heaviside function is constant when $x < 0$ and $x > 0$ and hence has zero slope in these regions, whereas at $x = 0$ the function immediately jumps up to 1, which roughly translates to a slope of infinity. This matches exactly the Dirac delta definition, where it's zero everywhere except at $x = 0$, where its value can be interpreted as infinite. □

Problem 9.5 - The Fourier Transform

Relevant Videos: The Fourier Transform; Convolution

For this problem, let's look at time-domain functions and their Fourier-transforms into frequency-domain functions,

$$\mathcal{F}^{-1}[c(\omega)](t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\omega)e^{i\omega t}\,d\omega, \tag{7}$$

$$\mathcal{F}[f(t)](\omega) = c(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt. \tag{8}$$

(a) Show that, given the expression for the Fourier transform $c(\omega)$, the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} c(\omega)e^{i\omega t}d\omega$ does indeed evaluate to $f(t)$.
[*Note: Be careful about labeling integration variables in these problems! There is already a t in our initial integral so when we plug in our expression for c(ω) we will need to use a different integration variable (such as s).*]
[*Spoilers!* Eq. 6 will be helpful here.]

Solution: Algebra time:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} c(\omega)e^{i\omega t}d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-i\omega t'} f(t')dt'\right] e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t') \underbrace{\int_{-\infty}^{\infty} e^{-i\omega(t-t')}d\omega}_{=2\pi\delta(t-t')}\, dt' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t')2\pi\delta(t-t')dt' \\
&= \int_{-\infty}^{\infty} f(t')\delta(t-t')dt' \\
&= f(t)
\end{aligned}$$

as desired. □

(b) Extra Part (*Not for Credit*) Show that, if $c(\omega)$ is the Fourier transform of $f(t)$, then the Fourier transform of $f(t-t_0)$ is $e^{-i\omega t_0}c(\omega)$. Conversely, show that the Fourier transform of $e^{i\omega_0 t}f(t)$ is $c(\omega-\omega_0)$.

(c) Extra Part (*Not for Credit*) Show that the Fourier transform of a real, odd function of t is an imaginary function of ω.

(d) Show that the Fourier transform of the Gaussian wave packet $f(t) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-t^2/2\sigma^2}$ is itself a Gaussian wave packet $c(\omega) = e^{-\sigma^2\omega^2/2}$.
[*Spoilers!* $\int_{-\infty}^{\infty} e^{-(ax^2+bx)}dx = \sqrt{\frac{\pi}{a}}e^{b^2/4a}$]

Solution: Again, more algebra:

$$c(\omega) = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} e^{-t^2/2\sigma^2}e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-t^2/4a-i\omega t}dt$$

And now we use the relation in the spoiler, where we have $a = \frac{1}{2\sigma^2}$ and $b = i\omega$:

$$c(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}}\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}}e^{-\omega^2/4(1/2\sigma^2)} = e^{-\sigma^2\omega^2/2}$$

as desired. □

Commentary: Note that the product of the width σ in the time-domain is the inverse of the width in the frequency-domain. This turns out to be intimately related to the Heisenberg uncertainty principle.

Recall that the ***convolution*** of two functions $f(t)$ and $g(t)$ is given by the integral expression

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(s)g(t-s)ds = \int_{-\infty}^{\infty} f(t-s)g(s)ds.$$

Consider a function $f(t)$ with Fourier transform $F(\omega)$ and another function $g(t)$ with Fourier transform $G(\omega)$. Then, the ***convolution theorem*** says that the Fourier transform of the product is the convolution of Fourier transforms and the Fourier transform of the convolution is proportional to the product of Fourier transforms,

$$\mathcal{F}[f(t)g(t)](\omega) = \frac{1}{2\pi}(F * G)(\omega), \qquad \mathcal{F}[(f * g)(t)](\omega) = F(\omega)G(\omega).$$

(e) Show that the convolution of the Heaviside step function with itself is the Ramp function, $(\theta * \theta)(t) = R(t)$. [*Spoilers!* The result from Problem 9.5(b) may help you here. Consider the cases $t < 0$ and $t > 0$ separately.]

Solution: Let's first write out the convolution:

$$(\theta * \theta)(t) = \int_{-\infty}^{\infty} \theta(s)\theta(t-s)ds$$

First, notice that in order for this integrand to be nonzero, we require that $s \geq 0$, and $t \geq s \geq 0$.

Then, we can use the result from 9.4d twice to change our integration bounds. The first $\theta(s)$ bounds our integral from below by 0, and the second $\theta(t-s)$ bounds our integration from above at t. Therefore, we have:

$$(\theta * \theta)(t) = \int_{0}^{t} \theta(s)\theta(t-s)ds$$

Under the condition that $t \geq 0$ (we don't really need the s condition anymore since it's encoded in the integral), we get:

$$(\theta * \theta)(t) = \int_{0}^{t} 1 ds = t$$

However, if $t < 0$, then notice that in order for $\theta(t-s) = 1$, then $t - s > 0$, so $t > s$, but since t is negative, this means that s is also negative. But then s being negative means that $\theta(s) = 0$, so the integrand is zero for all s if $t < 0$. Therefore, we conclude:

$$(\theta * \theta)(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

which is exactly the ramp function. $\square$

(f) Extra Part (*Not for Credit*) Prove that, given $F(\omega)$ and $G(\omega)$ with inverse Fourier transforms $f(t)$ and $g(t)$, the inverse Fourier transform of the product $H(\omega) = F(\omega)G(\omega)$ is indeed $h(t) = (f * g)(t)$.

The Fourier transform of the Heaviside step function is

$$\mathcal{F}[\theta(t)](\omega) = \frac{1}{i\omega} + \pi\delta(\omega). \tag{9}$$

Let $g(t)$ be a function with Fourier transform given by

$$G(\omega) = \frac{1}{i\omega + c},$$

where c is a constant.

(g) Use the properties of the Fourier transform and Eq. 9 to find $g(t)$, the inverse Fourier transform of $G(\omega)$. Then use convolution to find the inverse Fourier transform of $e^{-i\omega t_0}G(\omega)$.
[*Spoilers!* You don't have to use Eqs. 7 or 8 at all for this part! Start with Eq. 9. Then use the "shifting" property we derived in part (b). For the last part, use the convolution theorem.]

Solution: Starting with Equation 9, we can take the inverse Fourier transform of both sides:

$$\theta(t) = F^{-1}\left[\frac{1}{i\omega}\right] + \pi F^{-1}[\delta(\omega)]$$

Now, we can first compute the inverse Fourier transform of the delta function:

$$\begin{aligned}
F^{-1}[\delta(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\delta(\omega)d\omega \\
&= \frac{1}{2\pi}e^{i(0)t} \\
&= \frac{1}{2\pi}
\end{aligned}$$

Therefore, our equation simplifies to:

$$\theta(t) = F^{-1}\left[\frac{1}{i\omega}\right] + \pi\frac{1}{2\pi} = F^{-1}\left[\frac{1}{i\omega}\right] + \frac{1}{2}$$

Rearranging for $F^{-1}[1/i\omega]$:

$$F^{-1}\left[\frac{1}{i\omega}\right] = \theta(t) - \frac{1}{2}$$

Now we look at $G(\omega)$, and rewrite it a bit:

$$G(\omega) = \frac{1}{i\omega + c} = \frac{1}{i(\omega - ic)}$$

which is the same expression except shifted over by ic. Then, we use the relation in part b to get that:

$$F^{-1}\left[\frac{1}{i\omega + c}\right] = e^{i(ic)t}F^{-1}\left[\frac{1}{i\omega}\right]$$

Substituting our earlier result in, we get:

$$g(t) = F^{-1}\left[\frac{1}{i\omega + c}\right] = e^{-ct}\left(\theta(t) - \frac{1}{2}\right)$$

Now let's analyze this function. For $t > 0$, $\theta(t) = 1$, so the expression is:

$$g(t) = e^{-ct}\frac{1}{2}$$

When $t < 0$, $\theta(t) = 0$, so the expression is:

$$g(t) = -e^{-ct}\frac{1}{2}$$

We see that everything is the same except for a positive and negative sign, which we can encode with a $\text{sgn}(t)$ function. Therefore, we can write:

$$g(t) = \frac{1}{2}\,\text{sgn}(t)e^{-ct}$$

As for the second part of this problem, we use the convolution theorem:

$$\mathcal{F}^{-1}[F(\omega)G(\omega)](t) = (f * g)(t)$$

Using $F(\omega) = e^{i\omega t_0}$, we know that:

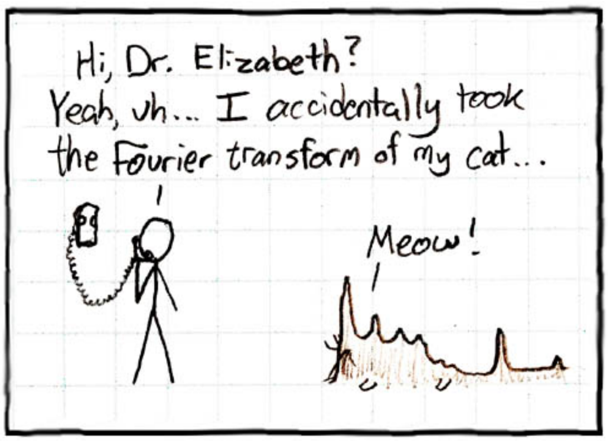
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t_0}e^{i\omega t}d\omega = \delta(t - t_0)$$

Therefore, the right hand side is:

$$f * g = \int_{-\infty}^{\infty} \delta(s - t_0)\frac{1}{2}\operatorname{sgn}(t - s)e^{-c(t-s)}ds = \frac{1}{2}\operatorname{sgn}(t - t_0)e^{-c(t-t_0)}$$

□

♦



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