Eric Du EE120
HW 07 **Signals and Systems** March 17, 2024

Header styling inspired by CS 70: https://www.eecs70.org/

Collaborators

I worked with the following people on this assignment:

- Teja Nivarthi: 3036508567
- Nikhil Maserang: 3036978230

Problem 1

Consider a discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the following properties:

- $x[n + 4l] = x[n],\ \forall l, n \in \mathbb{Z}$
- $\sum_{n=-1}^{2} x[n] = 2$
- $\sum_{n=-1}^{2} (-1)^n x[n] = 4$
- $\sum_{n=-1}^{2} x[n] \cos\left[\frac{\pi}{2} n\right] = \sum_{n=-1}^{2} x[n] \sin\left[\frac{\pi}{2} n\right] = 0$

a) Determine the complex exponential Fourier series coefficients X_{-1}, X_0, X_1 and X_2 for the signal x. From the coefficients, determine and provide a well-labeled plot for the signal x.

Solution: Recall the formula for calculating the Fourier coefficients:

$$X_k = \frac{1}{N_0} \sum_{n=0}^{N_0 - 1} x[n] e^{-jk\omega_0 n} \quad k = 0, 1, 2, \ldots$$

From $x[n] = x[n + 4l]$ we know that $x[n] = x[n + 4]$, implying that $N_0 = 4$. Thus, $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$. Now, from the 4th condition, we can identify:

$$\frac{1}{4} \left(\sum_{n=-1}^{2} x[n] \cos\left[\frac{\pi}{2} n\right] + i \sum_{n=-1}^{2} x[n] \sin\left[\frac{\pi}{2} n\right] \right) = \frac{1}{4} \sum_{n=-1}^{2} x[n] e^{i \frac{2\pi}{4} n} = X_{-1} = 0$$

Same goes for X_1 (so we have $X_{-1} = X_1 = 0$), since the formula is the same up to a minus sign. For X_0, we have:

$$X_0 = \frac{1}{4} \sum_{n=-1}^{2} x[n] = \frac{1}{2}$$

and for X_2:

$$\begin{aligned} X_2 &= \frac{1}{4} \sum_{n=-1}^{2} e^{-i \frac{2\pi}{4} (2n)} \\ &= \frac{1}{4} \sum_{n=-1}^{2} x[n] e^{-i\pi n} \\ &= \frac{1}{4} \sum_{n=-1}^{2} (-1)^n x[n] = 1 \end{aligned}$$

and that completes all of them. Now, we use the synthesis equation to recover $x[n]$:

$$x[n] = \sum_{n=-1}^{2} X_k e^{jk\omega_0 n}$$

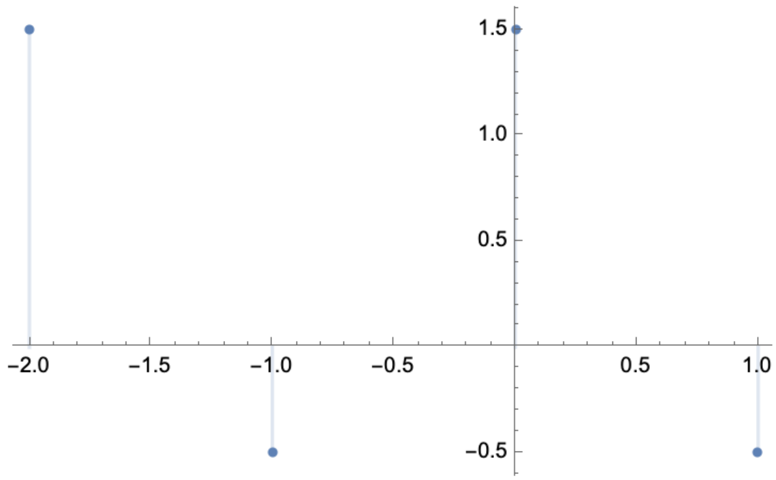
Notice that since $X_{-1} = X_1 = 0$, then only $k = 0, 2$ terms are nonzero. Therefore, we can write:

$$x[n] = \frac{1}{2} + e^{j\pi n} = \frac{1}{2} + (-1)^n$$

Crunching the numbers, this gives us:

$$\begin{aligned} x[-1] &= -\frac{1}{2} \\ x[0] &= \frac{3}{2} \\ x[1] &= -\frac{1}{2} \\ x[2] &= \frac{3}{2} \end{aligned}$$

As for a plot, this is quite easy:



□

b) Based on your results from part (a), determine the fundamental period p and fundamental frequency ω_0 of the signal x. Express x in terms of its DTFS coefficients and complex exponentials of appropriate frequencies, and identify each coefficient and its corresponding harmonic frequency.

Solution: We actually see that even though $x[n] = x[n+4]$, this is not the fundamental period since $x[n]$ actually only oscillates over 2 values. Therefore, the fundamental period is 2, and we have $N_0 = p = 2$ so $\omega_0 = \frac{2\pi}{T} = \pi$. Recalculating X_k:

$$X_k = \frac{1}{2}(x[0] + x[1]e^{-j\pi k})$$

This implies that $X_0 = \frac{1}{2}$ and $X_1 = 1$. Now, recalculating $x[n]$, we have:

$$x[n] = X_0 + X_1 e^{j\pi n} = \frac{1}{2} + (-1)^n$$

this matches exactly what we got from part (a). As for identifying the coefficients, we have X_0 corresponds to $\omega = 0$, and X_1 corresponds to $\omega = \pi$. These coefficients repeat (since x is periodic), so this means that the even numbered X_k implicitly correspond to $\omega = 0$, and the odd ones implicitly correspond to $\omega = \pi$. □

Problem 2

Consider a periodic, discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the discrete-time Fourier series (DTFS) expansion

$$x[n] = \sum_{k=\langle p \rangle} X_k e^{jk\omega_0 n}$$

where ω_0 denotes the fundamental frequency of the signal; if p is the period of x, then $\omega_0 = 2\pi/p$.

Suppose x is the input signal applied to a linear time-invariant (LTI) system characterized by the impulse response $h : \mathbb{Z} \to \mathbb{R}$ and corresponding frequency response H, where

$$H(e^{i\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}, \quad \forall \omega$$

Let y be the corresponding output signal.

a) Prove that the output signal y is periodic; that is, show that if $x[n+p] = x[n]$, then $y[n+p] = y[n]$.

Solution: We know that $y[n] = x[n] * h[n]$, then by the symmetry of the convolution, we also have $y[n] = h[n] * x[n]$. Therefore, for $y[n+p]$, we have:

$$y[n+p] = \sum_{k=-\infty}^{\infty} h[k] x[n+p-k]$$

but since $x[n] = x[n+p]$, then we know $x[n+p-k] = x[n-k]$, hence:

$$y[n+p] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] = h[n] * x[n] = y[n]$$

as desired. □

---

b) Let the DTFS expansion of the output signal y be

$$y[n] = \sum_{k=\langle p \rangle} Y_k e^{jk\omega_0 n}$$

i) Express the output-signal DTFS coefficients Y_k in terms of the input signal DTFS X_k and the frequency response H.

Solution: We know that based on convolution properties:

$$Y_k = X_k H_k$$

where H_k represents $H(e^{i\omega})$ sampled at the point $\omega = k\omega_0$. Therefore, we write:

$$Y_k = X_k H(e^{ik\omega_0})$$

□

---

ii) Suppose the impulse response of the LTI system is given by $h[n] = \delta[n - n_0]$, where $n_0 \in \mathbb{Z}$. Explicitly determine the output-signal DTFS coefficients Y_k in terms of the input-signal DTFS coefficients X_k.

Solution: Based on the given $h[n]$, then we can determine:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

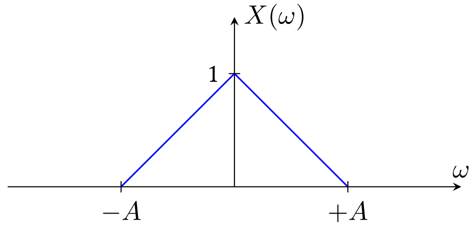
Now, using the previous part, we get:

$$Y_k = X_k H(e^{jk\omega_0}) = X_k e^{-j\omega k n_0}$$

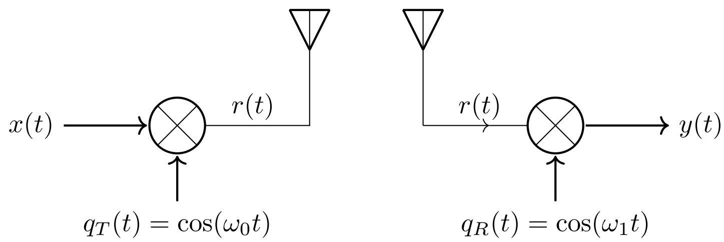
□

Problem 3

A bandlimited continuous-time signal x has the triangular spectrum shown below:



The following diagram shows an amplitude modulation-demodulation scheme to communicate signal x to a receiver. In this problem, you'll explore the effects of frequency mismatch between the transmitter and receiver carriers q_T and q_R respectively.



In particular, assume that

$$0 < \epsilon \ll A < \omega_0 = \omega_1 + \epsilon$$

a) Determine reasonably simple expressions for the signals r and y. You may find the following trigonometric identity useful:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

Solution: This is an AM modulation scheme, so we have:

$$r(t) = x(t)q_T(t) = x(t)\cos(\omega_0 t)$$

Then, we since we multiply by $q_R(t)$, then we have:

$$y(t) = x(t)q_T(t)q_R(t) = x(t)\cos(\omega_0 t)\cos(\omega_1 t) = \frac{x(t)}{2}[\cos((\omega_0+\omega_1)t) + \cos((\omega_0-\omega_1)t)$$

to find $x(t)$, we have to take the inverse Fourier transform of $X(\omega)$. The functional form of $X(\omega)$ is:

$$X(\omega) = \begin{cases} \frac{\omega}{A} + 1 & \omega < 0 \\ -\frac{\omega}{A} + 1 & \omega > 0 \end{cases}$$

Performing the Fourier transform, we ultimately get:

$$x(t) = \frac{1-\cos(At)}{A\pi t^2}$$

So fnally, we have:

$$r(t) = \frac{1-\cos(At)}{A\pi t^2}\cos(\omega_0 t)$$
$$y(t) = \frac{1-\cos(At)}{2A\pi t^2}[\cos((\omega_0+\omega_1)t) + \cos((\omega_0-\omega_1)t)]$$

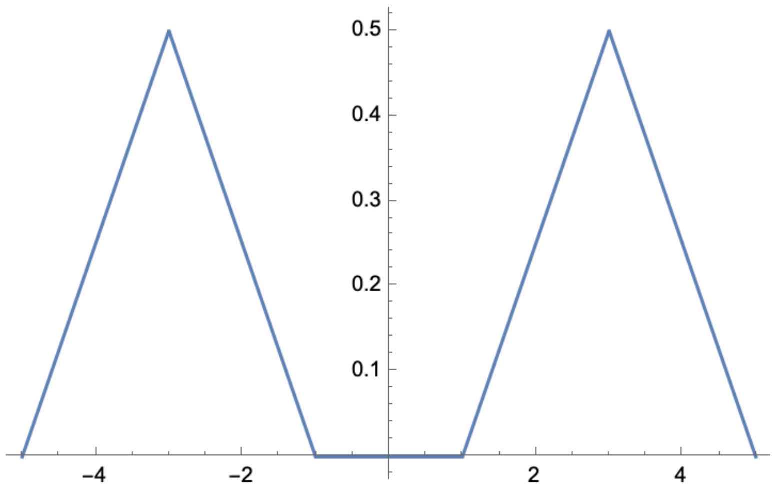
$\square$

b) Provide well-labeled plots of $R(\omega)$ and $Y(\omega)$, the spectra of the signals r and y, respectively.

Solution: For $R(\omega)$, we take the Fourier transform, and from lecture we know that:

$$R(\omega) = \frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)]$$

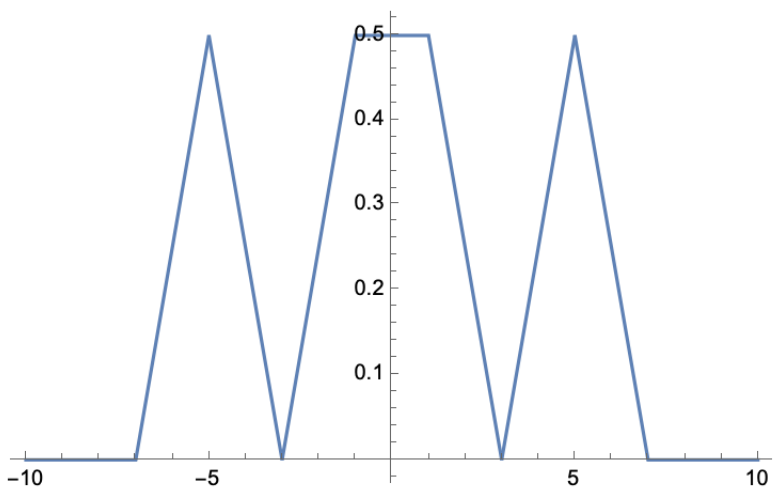
so this will look like two triangles that are symmetric about $\omega = 0$. A plot is given below, plotted in Mathematica using $A = 2, \omega_0 = 3$:



For $Y(\omega)$, we can use the same argument, giving us:

$$Y(\omega) = \frac{1}{2}\left[X(\omega - (\omega_1 + \omega_0)) + X(\omega + (\omega_1 + \omega_0)) + X(\omega - (\omega_1 - \omega_0)) + X(\omega + (\omega_1 - \omega_0))\right]$$

For the same parameters above and $\omega_1 = 2$, the plot looks like this:



□

c) Explain why the information-bearing signal x is irrecoverable, even if we send the signal y through an ideal low-pass filter.

Solution: An ideal low-pass filter will filter out all signals above a certain frequency $\omega = \omega_c$. However, we see that $Y(\omega)$ has overlapping spikes, meaning that we won't be able to discern our signal $x(t)$ out clearly. □

Problem 4

Consider an LTI system with frequency response as follows

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a}{1 - ae^{-j\omega}}$$

where a is a real number such that $|a| < 1$.

a) Show that $|H(e^{j\omega})| = 1$ at all frequencies. What kind of filter is this?

Solution: We just do the algebra:

$$|H(e^{j\omega})|^2 = \frac{e^{-j\omega} - a}{1 - ae^{-j\omega}} \frac{e^{j\omega} - a}{1 - ae^{j\omega}} = \frac{1 - ae^{-j\omega} - ae^{j\omega} + a^2}{1 - ae^{j\omega} - ae^{-j\omega} + a^2} = 1 \implies |H(e^{j\omega})| = 1$$

Because $M(\omega) = 1$ for all ω, this is considered an all-pass filter. $\square$

---

b) Derive an expression for the phase $\angle H(e^{j\omega})$.

Solution: The algebra in this question is relatively long, so I'm going to skip most of it. Firstly, we know that

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{\mathrm{Im}(H)}{\mathrm{Re}(H)}\right)$$

So to find these two expressions, we first express H in terms of its real and imaginary parts:

$$H(e^{j\omega}) = \frac{\cos(\omega) - i\sin(\omega) - a}{1 - a(\cos\omega - i\sin(\omega)} = \frac{\cos(\omega) - a - i\sin(\omega)}{1 - a\cos(\omega) - ia\sin(\omega)}$$

Now we multiply to make the denominator a real quantity:

$$\frac{(\cos(\omega) - a - i\sin(\omega))(1 - a\cos(\omega) + ia\sin(\omega)}{(1 + a\cos(\omega))^2 + a^2\sin^2(\omega)}$$

From here, we expand the numerator, and collect the real and imaginary part. Because they're being divided together, the denominator doesn't actually matter here. Therefore, our final expression (again, skipping the algebra because I can't be bothered to type it out):

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{a\sin(\omega)\cos(\omega) - a^2\sin(\omega) - \sin(\omega) + \cos(\omega)}{\cos(\omega) - a\cos^2(\omega) - a + a^2\sin(\omega)\cos(\omega) + a\sin^2(\omega)}\right]$$

yeah. Not a nice expression, but oh well. $\square$

---

c) Take $a = \frac{1}{\sqrt{3}}$ and determine the output $y[n]$ for the input

$$x[n] = \cos\left[\frac{\pi}{6}n\right] + \cos[\pi n]$$

Solution: Following what I saw on Ed, we write $x[n]$ in terms of exponentials:

$$x[n] = \frac{1}{2}(e^{i\frac{\pi}{6}n} + e^{-i\frac{\pi}{6}n} + e^{i\pi n} + e^{-i\pi n})$$

Now, we use the fact that given a signal $x[n] = Ae^{i\omega n}$, then the output $y[n] = H(\omega)Ae^{i\omega n}$. Using this fact, we get:

$$y[n] = \frac{1}{2}\left(H\left(\frac{\pi}{6}\right)e^{i\frac{\pi}{6}n} + H\left(-\frac{\pi}{6}\right)e^{-i\frac{\pi}{6}n} + H(\pi)e^{i\pi n} + H(-\pi)e^{-i\pi n}\right)$$

I'm not convinced that plugging in the values for H would give you a much nicer expression so I'm going to leave it at this. $\square$

d) Write a difference equation that implements an LTI system with the frequency response above.

Solution: Based on the previous problems we've had, we know that for an LCCDE equation of the form:

$$a_0 y[n] + \cdots + a_k y[n-k] = b_0 x[n] + \cdots + b_l x[n-l]$$

the frequency response $H(\omega)$ is given by:

$$H(\omega) = \frac{\sum_{m=0}^{k} b_m e^{-i\omega m}}{\sum_{n=0}^{k} a_n e^{-i\omega n}}$$

Based on our $H(\omega)$ we have above, this implies that $b_0 = -1, b_1 = 1$ and $a_0 = 1, a_1 = -a$. Therefore, we have:

$$y[n] - ay[n-1] = -x[n] + x[n-1]$$

□

Problem 5

The impulse response of a real, discrete time FIR filter A is described by

$$a[n] = a_0\delta[n] + a_1\delta[n-1] + \cdots + a_N\delta[n-N] \quad \text{where } N \in \{1, 2, 3, \ldots\}$$

a) Show that the frequency response $A(e^{j\omega})$ of the filter is a polynomial, in terms of $e^{-j\omega}$, whose coefficients have a simple relationship with the impulse response value $a_0, \ldots, a_N$.

Solution: Since the impulse response is given by replacing $x[n]$ with $\delta[n]$, we can just flip it back to get the filter equation:

$$\mathcal{A}[n] = a_0 x[n] + a_1 x[n-1] + \cdots + a_N x[n-N]$$

Now, using the same approach as question 4d, we can use the expression:

$$H(\omega) = \frac{\sum_{m=0}^{k} b_m e^{-i\omega m}}{\sum_{n=0}^{k} a_n e^{-i\omega n}}$$

to get our value for $A(e^{j\omega})$. Here, this comes out to be:

$$A(e^{j\omega}) = \sum_{m=0}^{N} a_m e^{-j\omega m}$$

this is a polynomial in terms of $e^{-j\omega}$, whose coefficients are exactly the impulse response values $a_0, \ldots, a_N$. □

---

b) Suppose the filter $\mathcal{A}$ is plaed in a cascade (series) interconection with another discrete-time FIR filter $\mathcal{B}$ whose impulse response is describd by

$$b[n] = b_0\delta[n] + b_1\delta[n-1] + \cdots + b_M\delta[n-M], \quad \text{where } M \in \{1, 2, 3, \ldots\}$$

The cascade structure which we call $\mathcal{C}$ is shown in the figure below. Let $c[n]$ denote the impulse response of hte cascade interconnection.

a) Expression $c[n]$ in terms of $a[n]$ and $b[n]$.

Solution: Because the systems are being placed in series with each other, then we know that

$$c[n] = a[n] * b[n]$$

□

---

b) Express the frequency response $C(e^{j\omega})$ of the system $\mathcal{C}$ in terms of the frequency responses $A(e^{j\omega})$ and $B(e^{j\omega})$ of the cascaded system $\mathcal{A}$ and $\mathcal{B}$.

Solution: $C(e^{j\omega})$ is the Fourier transform (DTFT) of the impulse response $c[n]$, and by the convolution theorem, we know that:

$$C(e^{j\omega}) = A(e^{j\omega})B(e^{j\omega})$$

□

---

c) Explain why $\mathcal{C}$ must be an FIR filter.

Solution: Since the impulse response $c[n]$ is related to $\mathcal{A}$ and $\mathcal{B}$ by a convolution, the impulse response $c[n]$ will have support over a width which is the sum of the support from $a[n]$ and $b[n]$. Since both of these supports are finite, then $c[n]$ must be finite as well. □

---

c) Consider two polynomials $A(z)$ and $B(z)$ described as follows:

$$A(z) = a_0 + a_1 z + \cdots + a_M z^N \quad B(z) = b_0 + b_1 z + \cdots + b_M z^M$$

where M and N are positive integers. Let $C(z) = A(z)B(z)$,

$$C(z) = c_0 + c_1 z + \cdots + c_{N+M} z^{n+M}$$

Show that multiplying polynomials is tantamount to convolving their coefficients; in particular, explain how $c[n] = (a * b)[n] = \sum_m a_m b_{n-m}$, where $n = 0, 1, \ldots, N + M$.

Solution: When multiplying $A(z)B(z)$, we multiply every term in $A(z)$ with $B(z)$, and since $a_n z^n b_m z^m = a_n b_m z^{n+m}$, this implies that the sum of the coefficient indices gives us the order of the term they multiply to. This is exactly what the convolution is describing:

$$(a * b)[n] = \sum_m a_m b_{n-m}$$

meaning that the coefficient c_n is a summation over all products of $A(z)$ and $B(z)$ whose index coefficients sum to n, which is what we're summing over on the right. We can see this by the fact that the sum of the indices on the right $m + n - m = n$, meaning that it corresponds to the n-th coefficient of $C(z)$, exactly as desired. $\square$
