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HW 10 **Introduction to Analysis** April 24, 2023

1 Problem 1

Suppose f is a continuous function on $[a, b]$, and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

Solution: Let $U(f; P)$ be the upper Riemann sum on a partition P of $[a, b]$. We know that since $\int_a^b f = 0$, then $\int_a^b f = \inf_{P\in\Pi} U(f; P) = \sup_{P\in\Pi} L(f; P)$. Using the definition of the lower Riemann sum, we have:

$$\sup_{P\in\Pi} \left(\sum_{k=1}^{n} (x_k - x_{k-1}) \min_{I_k} f \right) = 0$$

where I_k is the interval $[x_{k-1}, x_k]$ and $P = \{a = x_1 < x_2 < \cdots < x_n = b\}$. Using the definition of a supremum, we get:

$$\sum_{k=1}^{n} (x_k - x_{k-1}) \min_{I_k} f \leq 0$$

Since we know that for any partition $x_k - x_{k-1} > 0$, then for this to be satisfied we must have that $\min_{I_k} f \leq 0$. So this means that on every interval, we have $\min_{I_k} f = 0$.

Furthermore, for the same partition, we know that:

$$\inf_{P\in\Pi} \left(\sum_{k=1}^{n} (x_k - x_{k-1}) \max_{I_k} f \right) = 0$$

So using the definition of supremum and the same logic as before, we can deduce that $\max_{I_k} f \geq 0$.

□

Problem 2

Construct an example of a function where $f(x)^2$ is integrable on $[0, 1]$ but $f(x)$ is not.

Solution: We use the Dirichlet function given in class, with one modification. Let it be defined instead as:

$$f = \begin{cases} 1 & \mathbb{Q} \\ -1 & x \in \mathbb{R} \setminus Q \end{cases}$$

Just like how the Dirichlet function is not integrable (the proof for this follows the same way, except here we have $\inf_{I_k} f = -1$ for all partitions instead of 0, and $\sup_{I_k} f = 1$ remains the same here. However, if we take f^2, then this function becomes:

$$f = \begin{cases} 1 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \setminus Q \end{cases}$$

In this case, the lower Riemann sum reads:

$$L(f; P) = \sum_{k=1}^{n} (a_k - a_{k-1}) \inf_{I_k} f = 1$$

and the upper Riemann sum:

$$U(f; P) = \sum_{k=1}^{n} (a_k - a_{k-1}) \sup_{I_k} f = 1$$

And since they agree, then this function is Riemann integrable. Another way we could have done this was to notice that $f = 1$ for all $x \in \mathbb{R}$ in this interval, so clearly this function is Riemann integrable, since f is a constant. $\square$

2 Problem 3

Let f be a bounded function on $[a, b]$, so there exists $B > 0$ such that $|f(x)| < B$ for all $x \in [a, b]$.

a) Show

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

For all partitions P of $[a, b]$. *Hint:* $f(x)^2 - f(y)^2 = [f(x) + f(y)][f(x) - f(y)]$

b) Show that if f is integrable on $[a, b]$, then f^2 also is integrable on $[a, b]$.

Problem 4

Let f and g be integrable functions on $[a, b]$.

a) Show fg is integrable on $[a, b]$. *Hint:* Use exercise 33.7 and $4fg = (f + g)^2 - (f - g)^2$.

b) Show $\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$. *Hint:* Exercise 17.8.

Problem 5

a) For any two numbers $u, v \in \mathbb{R}$, prove that $uv \leq (u^2 + v^2)/2$. Let f and g be two integrable functions on $[a, b]$. Prove that if $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$, then

$$\int_a^b fg \leq 1$$

Solution: Rearranging the inequality, we get:

$$\begin{aligned} 2uv &\leq u^2 + v^2 \\ \therefore (u - v)^2 &\geq 0 \end{aligned}$$

which is a true statement for all u, v.

With the integral, we can write the integral of the product as:

$$\begin{aligned} \int_a^b fg &\leq \int_a^b \frac{f^2 + g^2}{2} \\ &\leq \frac{1}{2} \left(\int_a^b f^2 + \int_a^b g^2 \right) \\ &\leq 1 \end{aligned}$$

as desired. □

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b) Prove the Schwarz inequality, that for any two integrable functions f and g on the interval $[a, b]$,

$$\left| \int_a^b fg \right| \leq \left(\int_a^b f^2 \right)^{1/2} \left(\int_a^b g^2 \right)^{1/2}$$

Solution:

□
