MATH 104 NOTES

Typeset notes for Math 104: Real Analysis

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Lecture 0

Introduction

INsert syllabus here

Lecture 1 (01/17)

This lecture was held on **January 17th, 2023**. It covered the syllabus content and also an introduction to rational and irrational numbers.

Lecture 2 (01/19)

This lecture was held on **January 19th, 2023**. It covered

2.1 Rational Root Theorem (continued)

Last time we looked at the rational root theorem. As a reminder, it says:

Theorem 2.1

Consider the polynomial equation ($n \geq 1$)

$$x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0$$

where $c_{n-1}, c_{n-2}, \cdots \in \mathbb{Z}$, $c_0 \neq 0$. Then, every rational solution to this equation must be an integer that divides c_0.

Proof. Assume that $x = \frac{p}{q}, p, q \in \mathbb{Z}$ with p, q are relatiely prime. Therefore, we can write:

$$\left(\frac{p}{q}\right)^n + c_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + c_0 = 0$$

Multiplying both sides by q^n we now get

$$p^n + c_{n-1}p^{n-1}q + \cdots + c_0 q^n = 0 \tag{2.1}$$

And so therefore

$$\begin{aligned} p^n &= -(c_{n-1}p^{n-1}q + \cdots + c_1 p q^{n-1}) - c_0 q^n \\ &= -p(c_{n-1}p^{n-2}q + \cdots + c_1 q^{n-1}) - c_0 q^n \\ \therefore p^{n-1} &= -(c_{n-1}p^{n-2} + \cdots + c_1 q^{n-1}) - \frac{c_1 q^n}{p} \end{aligned}$$

Since p^{n-1} is an integer, then it directly follows that $c_0 q^n / p$ is also an integer. Then since we've assumed before that p, q are relatively prime, then we know that p must divide c_0.

Remark: To do this formally, we use the idea that if p, q are relatively prime, then p^n and q^m are also relatively prime for any positive n, m. If you have time, you should attempt to prove this yourself.

Now we need to show that $q = 1$. To do that, we go back to eq. (2.1) and rearrange it to become:

$$\begin{aligned} c_0 q^n &= -(p^n + c_{n-1}p^{n-1}q + \cdots + c_1 p q^{n-1}) \\ &= -p^n - q(c_{n-1}p^{n-1} + \cdots + c_1 p q^{n-2}) \\ \therefore c_0 q^{n-1} &= -\frac{p^n}{q} - (c_{n-1}p^{n-1} + \cdots + c_1 p q^{n-2}) \end{aligned}$$

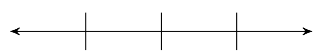
Since p, q are relatively prime, then the only value of q which would satisfy this equation is $q = 1$, which completes the proof.

□

2.2 The Real Numbers

In high school we've covered real numbers before, but we haven't really explored how they are formally generated. Here, we will explore that.

Given a number line, we can define the number 0 to be somewhere on this line. Then, we need to define 1 to be some distance from 0. Then, we can generate the rest of the integers by repeating the same length over the rest of the number line.



Similarly, we can define rationals by dividing our existing number line into equal parts. For instane, we can generate $\frac{1}{3}$ by dividing the segment between 0 and 1 into three equal parts. But what happens when we try to define irrational numbers? We can't really keep dividing to generate numbers like π or $\sqrt{2}$ becuase they cannot be expressed as any fraction.

Generating the reals ends up being a very difficult process, so we will skip it for now. Instead, we will devote our time to exploring the properties of the reals. Namely, we will look at:

(a) Algebraic property: you can add, subtract, multiply and divide two real numbers
  - Metric Property

(b) Order property: we can compare two reals using the symbols $<$, $>$, $\leq$, $\geq$, etc.
  - Supremum, Infimum
  - Completeness Axiom

2.2.1 Algebraic Property of the Reals

The algebraic property is an *axiom* - we cannot prove such a statement but we take it to be factually true.

Definition 2.1 (Addition and Multiplication): There exists binary operations

$$a, m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

call them $a(x, y) = x + y$ and $m(x, y) = xy$

Further, there exists two elements $0, 1 \in \mathbb{R}$ such that

(a) $x + 0 = x$ (or formally, $a(x, 0) = x$)

(b) For any $x \in \mathbb{R}$, we can find $y \in \mathbb{R}$ such that $x + y = 0$. In this formulation, y is also called $-x$.

(c) $x + (y + z) = (x + y) + z$ (associativity)

(d) $x + y = y + x$ (commutativity)

(e) $x \cdot 1 = x$

(f) The inverse of x exists: for any $x \in \mathbb{R}\backslash\{0\}$, then we can find $y \in \mathbb{R}$ such that $xy = 1$. In this formulation, y is also called $\frac{1}{x}$.

(g) $x(yz) = (xy)z$

(h) $xy = yx$

We can then use these properties to generate the division operation, by defining

Definition 2.2 (Division): For any $x, y \in \mathbb{R}$, we define

$$\frac{x}{y} = x \cdot \frac{1}{y}$$

2.2.2 Order Property of the Reals

It's commonly stated as:

Definition 2.3 (Order Property): There exists three relations on $\mathbb{R}$ such that $\forall x, y \in \mathbb{R}$: either $x > y$, $x < y$ or $x = y$.

This order property then has a couple consequences. If $x < y$, then the following consequences hold:

(a) $x + z < y + z$ for all $z \in \mathbb{R} \setminus 0$

(b) $xz < yz$ for all $z > 0, z \in \mathbb{R}$

(c) $-x > -y$

There are many more here, see book for details. Now we move on to the metric property. To do this, we first define the *absolute value* $A : \mathbb{R} \to \mathbb{R}$ as:

$$A(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Then from here, we can define the distance:

Definition 2.4: The distance between two numbers $x, y \in \mathbb{R}$, denoted as $d(x, y)$ is written as

$$d(x, y) = A(x - y) = |x - y|$$

With this definition established, we can look at some of the consequences:

(a) $|x| \geq 0$

(b) $|xy| = |x||y|$

(c) $|x + y| \leq |x| + |y|$ (Triangle Inequality)

Here, we present a proof for the triangle inequality:

Proof. Notice that for any real x, we have $\leq |x|$ and $-x \leq |x|$. Then, this gives us that $\pm x \leq |x|$ and $\pm y \leq |y|$ so therefore the theorem holds. $\square$