Instructor's Manual to accompany *Classical Mechanics*

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INSTRUCTOR'S MANUAL

for

Classical Mechanics

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Introduction

This manual is to help people teaching from my book *Classical Mechanics.* This Introduction begins with a brief outline of the manual itself, followed by a discussion of some of the choices that I had to make in writing the book and that you will have to make using it.

Outline of the Manual

This manual contains sixteen chapters corresponding exactly to the sixteen chapters of the book. Each chapter of the manual consists of a brief overview of the corresponding chapter of the book, followed by a complete set of solutions to all the problems at the end of that chapter.

At the start of each overview, I give the number of lectures it took me to cover the chapter in class. Obviously this is only a rough guide, but I hope it will be helpful in planning your course. Then, I explain any unusual aspects of the chapter. For example, in the introduction to the chapter on Hamiltonian mechanics, I explain why I put this chapter of the book so late (Chapter 13), rather than immediately after the chapter on Lagrangian mechanics (Chapter 7), and emphasize that you can if you wish (and if your students are well prepared) move straight to Chapter 13 as soon as you've finished Chapter 7.

One of the hardest choices that any author of a textbook faces is deciding what topics can be omitted. As soon as you decide that topic X could be omitted, you immediately find several colleagues who say that topic X is essential and that without topic X the book could certainly not be used in their department. The result is that all textbooks (including this one) are longer than their authors would like and contain more material than could possibly be covered in a standard course. Thus, choices have to be made, and an important function of the brief overviews in this manual is to offer a little help with these choices. Certain sections of the book are flagged with an asterisk as "omittable" and I try here to explain the sense in which this is the case — usually that the material is not needed to understand anything later in the book. Occasionally, the material of a section that is designated "omittable" *is* needed in one or two later sections; when this is the case, I always mention it at the relevant point of the manual. Again, some of the earlier chapters could be skipped, or treated as quick reviews, by students who are well prepared, and I try to offer guidance on these possibilities.

In teaching the course for which this book was designed (our junior mechanics course), I realized that our students missed the demonstration experiments that are such a feature of many freshman physics courses. I realized that there are, in fact, quite a lot of such experiments that are appropriate at the present level. When I took these into class they were usually a big hit. Therefore, from time to time in this manual, I offer a few suggestions of such experiments, mainly in the hope that this will remind you to think of some yourself.

To conclude the chapter overviews, I mention any of the end-of-chapter problems that seem particularly interesting or which give further coverage to material in the text.

The solutions to the problems need little comment. They are fairly complete, with diagrams when appropriate, and include at least nearly as much explanation as one would give the students. The only exception is that to save space I sometimes omit a few steps in long but straightforward calculations; for example, I often skip the details of an integration, mentioning just the method needed to do it (change of variables, integration by parts, etc.). I follow no hard and fast rule about significant figures, but try to give as many digits as seem consistent with the nature of the problem. Sometimes, when an answer is needed for a later calculation, I keep an extra digit or two. The pictures were all drawn using *Mathematica* and should therefore be mathematically accurate. On the other hand, I never did learn to control the fonts as the pictures moved from *Mathematica* to Latex, and I apologize that the fonts in the pictures are not consistent with those in the text; in particular, there is no distinction in the pictures between ordinary and boldface type.

Choices

As I have said, there is more material in the book than could possibly be taught in a one-semester course, and you are going to have to make choices. The choices you make will depend in part on the state of preparation of your students. The book was written to serve a course in "intermediate mechanics" — meaning a course that comes after the mechanics of "freshman physics" and before a graduate course on the subject, usually in the junior or senior year at an American university. Unfortunately, this still leaves a fairly wide range of levels of preparation. At Colorado, our freshman physics is taught from a text such as the classics by Halliday, Resnick, and Walker or Sears and Zemansky. Thus students begin our intermediate mechanics with a moderate aquaintance with Newton's laws and the concepts of energy and momentum. They certainly know how to handle a projectile moving in the vacuum, but they almost certainly don't know what to do if air resistance is important. Similarly, they have certainly studied simple harmonic motion, but they probably can't solve for the motion of a driven damped oscillator. The opening chapters of my book were written with this sort of preparation in mind. However, some departments offer a more advanced introductory mechanics, using a text such as the excellent *Introduction to Mechanics* by Kleppner and Kolenkow. If your students have taken a course like this, then your first choice will be how much of the first five chapters of my book really need to be covered and by what means. Before we take this up, let me review the structure of my book.

The book is divided into two main parts:

Part I	Essentials	Chapters 1–11
Part II	Further Topics	Chapters 12–16

The division between the two parts is certainly not about importance; the topics of Part II are just as important as those of Part I. Rather, the chapters of Part I were designed to be read in sequence, while those of Part II are mutually independent and can be read in any order. To read any chapter of Part II, a student should know much of the material of Part I, but it is not necessary to know about any of the other chapters of Part II.

If your students have taken only the traditional introductory mechanics (in the style of Halliday and Resnick, for example), you will probably need to start with Chapter 1 of my book and, in one-semester, you might cover all of Part I plus perhaps one chosen chapter of Part II. If your students have been exposed to a more advanced introductory mechanics, you could go quickly through Chapters 1–5, or assign them as reading, or even skip them entirely; you should then have time for two or three of the chapters of Part II. When I taught the course, I felt it was necessary to go fairly carefully through the early chapters, so I didn't cover any of Part II in class. Instead, each student was required to read and write a report on one of the chapters of Part II as a term project. (These projects proved quite enjoyable. Students listed their preferences for a chapter and then a lottery decided who got his or her first choice. They then had a month or so to write a two or three page report and do ten of the end-of-chapter problems.)

The chapters of Part I are fairly standard fare for a book of this type. Chapters 1–5 all treat Newtonian mechanics. Much of this should be review for many of your students, but some will probably be new. (For example, the material on air resistance in Chapter 2 and on damped and driven oscillators in Chapter 5.) Chapter 6 is a very short chapter on the calculus of variations, to allow me to use Hamilton's principle in the derivation of Lagrange's equations from Newon's in Chapter 7. I regard Lagrangian mechanics as the single most important thing for our students to learn, and from then on I make liberal use of both Newtonian and Lagrangian ideas in treating the several standard topics (central force problems, noninertial frames, rotational motion, and coupled oscillators) that round out Part I.

One difficult decision was where to introduce Hamiltonian mechanics. Some colleagues argued that it is at least as important as Lagrangian mechanics and should be introduced soon after (perhaps in Chapter 8). However, I feel strongly that Lagrange's approach is already a huge intellectual leap for our students, and one that they should have time to assimilate before they take the next leap to Hamilton's. In the end, I put Hamiltonain mechanics as Chapter 13 among the "further topics" of Part II. However, I was careful to write it so that you could jump straight to Chapter 13 after Chapter 7 if you or they wish.

Besides Hamiltonain mechanics (Chapter 13), the other chapters of Part II are chaos (12), collision theory (14), special relativity (15), and continuum mechanics (16). The chapter on chaos (12) focuses narrowly on the driven damped pendulum, because this system is sufficiently uncomplicated that the students can really grasp what is going on. The chapter on collision theory is a little unusual, in that many texts at this level relegate collision theory to a couple of sections at the end of the chapter on the two-body problem. My feeling was that the subject is difficult and important enough that, if it is going to be included at all, it deserves a chapter to itself. Relativity (15) is included for the benefit of the minority of departments that teach relativity in their classical mechanics course. This chapter is a long one because it begins at the beginning and takes the subject through four-dimensional space-time, tensors, and even a little electrodynamics. Finally, several colleagues in geophysics urged me to include a chapter on contiuum mechanics, and that is the subject of Chapter 16, which is a very brief introduction, covering one-dimensional waves, stress and strain tensors, and waves in solids and inviscid fluids. The important thing to remember about Part II is that it is there for you to pick and choose from. No chapter depends on any other.

One of the great things about mechanics at this level is that it is a splendid opportunity for our students to hone their facility with mathematics — vectors, coordinate systems, vector calculus, hyperbolic functions, differential equations, Fourier series, matrices, eigenvalue problems, and a host more. I am certainly not suggesting that we *teach* these topics,[1] but we can say enough to reinforce the students' understanding of them and we can insist that our students do exercises using them. My hope would be that they come away from a lecture or homework set saying "Now I really see the point of such and such a mathematical technique." This is the attitude that I have tried to foster.

As with all branches of physics, few students will learn much without doing lots of homework problems. I have included many problems at the end of every chapter (more than 740 in all), and it is crucial that you assign several regularly. I assigned 5 or 6 problems regularly once a week, making sure that several were of the simple, "one star," variety, but at least one of which was "three star." I also included at least one computer problem every week. These computer problems are not specific to any particular system; on the contrary, they are all phrased in a neutral way and can be (and have been) done using several different systems. Several can be done using a spreadsheet, such as Excel; all could be done using a programming language such as C; but I had in mind that most would be done using one of the sophisciated systems such as Mathematica, Maple, or Matlab. At Colorado our students have easy access to Mathematica, and, at the start of semester, I gave a couple of voluntary sessions to introduce the students to this system, and the first two homework assignments had some simple exercises to give the students practice using it.

I had a great time teaching this class, and the best I can hope for you is that you will enjoy it as much as I did. I would be most grateful to hear of any comments and criticisms, large or small.

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[1]The one exception is the calculus of variations, which seems not to be taught in most lower division math courses, and the sole purpose of Chapter 6 is to teach this subject.

Chapter 1

Newton's Laws of Motion

I covered this chapter in 3.5 fifty-minute lectures.

I wrote this chapter on the assumption that most of it would be review for most of our students. Thus, some of the professors using the preliminary version of the book were able to skip this chapter entirely or assign it to be read outside class. Nevertheless, I found that most of my students needed to go through the chapter fairly carefully. Several ideas, such as the concept of an inertial frame, were still quite hazy in the minds of most students, and others, such as the proof that Newton's third law is equivalent to conservation of momentum and the expression (1.47) for acceleration in polar coordinates, were new to many.

Another important role for Chapter 1 is to establish the notations used in the book. I have found that many students are distressingly conservative in the matter of notation. Many students brought up on the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, are surprisingly unwilling to accept any alternatives. Thus, I have tried to push for a little openness to the many other notations that they are bound to meet as they continue in physics. In particular, I have opted mostly for $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in simple situations, and $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ when the going gets tougher and there is more need to use summations, such as $\mathbf{a} = \sum_{i=1}^{3} a_i \mathbf{e}_i$.

Without belaboring the point too much, I tried to give fairly rigorous definitions of mass and force, and then to discuss the three laws of Newton. Section 1.6 shows how beautifully simple Newton's second law is in Cartesian coordinates, and 1.7 how distressingly more complicated it is in polar coordinates.

In Chapter 1 an unusually high proportion of the end-of-chapter problems are devoted to refreshing the students' memories about the relevant mathematics. For example, Problems 1.1 through 1.25 can all be seen as a refresher course in vector algebra, and Problems 1.47 and 1.48 introduce cylindrical coordinates.

Solutions to Problems for Chapter 1

1.1 ⋆ $\mathbf{b}+\mathbf{c} = 2\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}$, $\quad 5\mathbf{b}+2\mathbf{c} = 7\hat{\mathbf{x}}+5\hat{\mathbf{y}}+2\hat{\mathbf{z}}$, $\quad \mathbf{b}\cdot\mathbf{c} = 1$, $\quad \mathbf{b}\times\mathbf{c} = \hat{\mathbf{x}}-\hat{\mathbf{y}}-\hat{\mathbf{z}}$.

1.2 ⋆ $\mathbf{b}+\mathbf{c}=(4,4,4)$, $5\mathbf{b}-2\mathbf{c}=(-1,6,13)$, $\mathbf{b}\cdot\mathbf{c}=10$, $\mathbf{b}\times\mathbf{c}=(-4,8,-4)$.

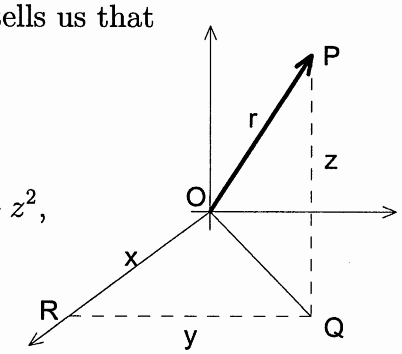
1.3 ⋆ Let P be the point with position vector $\mathbf{r}=(x,y,z)$. Let Q be the projection of P onto the xy plane at $(x,y,0)$, and let R be the projection of Q onto the x axis at $(x,0,0)$. Applied to the right triangle ORQ, Pythagoras' theorem tells us that

$$(OQ)^2 = x^2+y^2,$$

and from the right triangle OQP we find

$$r^2=(OQ)^2+z^2=x^2+y^2+z^2,$$

as required.



1.4 ⋆ Since $\mathbf{b}\cdot\mathbf{c}=bc\cos\theta$, it follows that

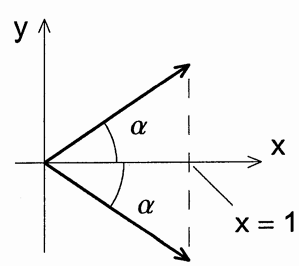
$$\theta=\arccos\left(\frac{\mathbf{b}\cdot\mathbf{c}}{bc}\right)=\arccos\left(\frac{4+4+4}{\sqrt{21}\cdot\sqrt{21}}\right)=55.2^\circ$$

1.5 ⋆ Let P be the corner at $(1,1,1)$ and Q the corner below P at $(1,1,0)$. Then OP is a body diagonal and OQ is a face diagonal, so the angle between OP and OQ is the required angle θ. Thus

$$\vec{OP}\cdot\vec{OQ}=\begin{cases}=|\vec{OP}|\,|\vec{OQ}|\cos\theta=\sqrt{3}\sqrt{2}\cos\theta\\ =(1,1,1)\cdot(1,1,0)=2\end{cases}$$

Equating these two, we find $\cos\theta=\sqrt{2/3}$ and hence $\theta=\arccos\sqrt{2/3}=0.615$ rad or 35.3° .

1.6 ⋆ $\mathbf{b}\cdot\mathbf{c}=1-s^2$, which is zero if and only if $s=\pm1$. The vectors $\mathbf{b}$ and $\mathbf{c}$ make equal angles, α, above and below (or below and above) the x axis. The angle between them can be 90° only if $\alpha=45^\circ$.



1.7 ⋆ If we choose our x axis in the direction of $\mathbf{r}$, then $\mathbf{r}=(r,0,0)$, whereas $\mathbf{s}=(s_x,s_y,s_z)$. As usual $s_x=s\cos\theta$, where θ is the angle between $\mathbf{s}$ and the x axis; with our choice of axes, this means that θ is the angle between $\mathbf{s}$ and $\mathbf{r}$. Thus, according to definition (1.7)

$$\mathbf{r}\cdot\mathbf{s}\text{ [definition (1.7)]}=\sum r_i s_i=rs_x+0+0=rs\cos\theta=\mathbf{r}\cdot\mathbf{s}\text{ [definition (1.6)]}$$

1.8 ⋆ **(a)** Starting from the definition (1.7), we see that

$$\mathbf{r}\cdot(\mathbf{u}+\mathbf{v}) = \sum r_i(u_i+v_i) = \sum(r_iu_i + r_iv_i) = \sum r_iu_i + \sum r_iv_i = \mathbf{r}\cdot\mathbf{u} + \mathbf{r}\cdot\mathbf{v}$$

where the second equality follows from the distributive property of ordinary numbers. The third equality is just a rearrangement of the six terms of the sum, and the last is just the definition (1.7) of the two scalar products.

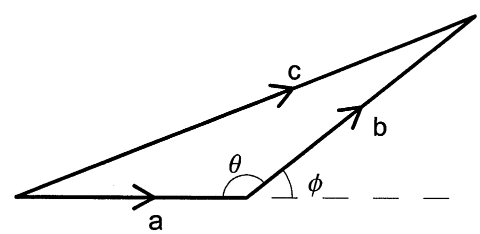
(b) Starting again from (1.7), we find

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \frac{d}{dt}\sum r_is_i = \sum\left(r_i\frac{ds_i}{dt} + \frac{dr_i}{dt}s_i\right) = \mathbf{r}\cdot\frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{s}.$$

1.9 ⋆ Consider the three vectors **a**, **b**, and $\mathbf{c} = \mathbf{a} + \mathbf{b}$ defined in the figure. The angle ϕ is the angle between **a** and **b**, and the angle θ of the triangle is $\theta = \pi - \phi$. By the given identity,

$$\begin{aligned} c^2 &= (\mathbf{a}+\mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} \\ &= a^2 + b^2 + 2ab\cos\phi = a^2 + b^2 - 2ab\cos\theta, \end{aligned}$$

since $\theta = \pi - \phi$ and hence $\cos\phi = -\cos\theta$.



1.10 ⋆ The particle's polar angle is $\phi = \omega t$, so $x = R\cos(\omega t)$ and $y = R\sin(\omega t)$ or

$$\mathbf{r} = \hat{\mathbf{x}}\,R\cos(\omega t) + \hat{\mathbf{y}}\,R\sin(\omega t).$$

Differentiating, we find that $\dot{\mathbf{r}} = -\hat{\mathbf{x}}\,\omega R\sin(\omega t) + \hat{\mathbf{y}}\,\omega R\cos(\omega t)$ and then

$$\ddot{\mathbf{r}} = -\hat{\mathbf{x}}\,\omega^2R\cos(\omega t) - \hat{\mathbf{y}}\,\omega^2R\sin(\omega t) = -\omega^2\mathbf{r} = -\omega^2R\hat{\mathbf{r}}.$$

That is, the acceleration is antiparallel to the radius vector and has magnitude $a = \omega^2R = v^2/R$, the well known centripetal acceleration.

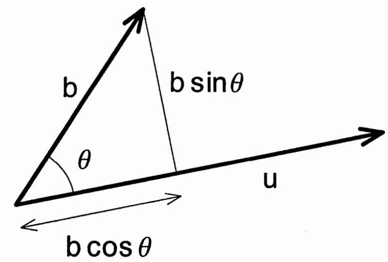
1.11 ⋆ The particle's coordinates are $x(t) = b\cos(\omega t), y(t) = c\sin(\omega t)$, and $z(t) = 0$. It remains in the plane $z = 0$ at all times, and, since

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = \cos^2(\omega t) + \sin^2(\omega t) = 1$$

it moves in an ellipse with semimajor and semiminor axes b and c in the xy plane. It is easy to see that it moves counterclockwise and returns to its starting point in a period $\tau = 2\pi/\omega$.

1.12 ⋆ Note first that the z coordinate increases at the constant rate v_o. Next consider the projection of **r** onto the xy plane, namely, the point $(x, y) = (b\cos\omega t, \sin\omega t)$. By inspection this satisfies $x^2/b^2 + y^2/c^2 = 1$, the equation of an ellipse centered on the origin. It is easy to see that (x, y) moves around this ellipse in a counter-clockwise direction and returns to its starting point in a time $\tau = 2\pi/\omega$. Meanwhile, z is increasing steadily, so the point **r** is moving up the surface of an elliptical cylinder centered on the z axis.

1.13 ⋆ If θ is the angle between $\mathbf{b}$ and the unit vector $\mathbf{u}$, then $\mathbf{u}\cdot\mathbf{b} = b\cos\theta$ is the component of $\mathbf{b}$ in the direction of $\mathbf{u}$. Similarly, $|\mathbf{u}\times\mathbf{b}| = b\sin\theta$ is the component of $\mathbf{b}$ perpendicular to $\mathbf{u}$. Thus the result $b^2 = (\mathbf{u}\cdot\mathbf{b})^2 + (\mathbf{u}\times\mathbf{b})^2$ is simply a statement of Pythagoras' theorem.



1.14 ⋆ $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})^2 = a^2+b^2+2\mathbf{a}\cdot\mathbf{b} = a^2+b^2+2ab\cos\theta \le a^2+b^2+2ab = (a+b)^2.$

The three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}+\mathbf{b}$ form the three sides of a triangle, and the inequality says that the length of any one side is less than the sum of the other two sides.

1.15 ⋆ With the proposed choice of axes, $\mathbf{r} = (r,0,0)$ and $\mathbf{s} = (s_x, s_y, 0)$ where $s_y = s\sin\theta$. (To be definite we can choose the y axis so that $\mathbf{s}$ lies in the upper half of the xy plane, so that $s_y \ge 0$.) Now

$$\mathbf{r}\times\mathbf{s} = (r,0,0)\times(s_x,s_y,0) = (0,0,rs_y) = (rs\sin\theta)\hat{\mathbf{z}}.$$

This says that $\mathbf{r}\times\mathbf{s}$ is perpendicular to both of $\mathbf{r}$ and $\mathbf{s}$, has magnitude $rs\sin\theta$, and has direction given by the right-hand rule.

1.16 ⋆⋆ **(a)** Using definition (1.7)

$$\mathbf{r}\cdot\mathbf{r} = \sum r_i r_i = x^2+y^2+z^2 = r^2$$

where the last equality is just the three-dimensional version of Pythagoras' theorem. It immediately follows that $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$.

(b) Using the result of part (a) and the distributive property of the scalar product [Problem 1.8(a)],

$$|\mathbf{r}+\mathbf{s}|^2 = (\mathbf{r}+\mathbf{s})\cdot(\mathbf{r}+\mathbf{s}) = \mathbf{r}\cdot\mathbf{r} + 2\mathbf{r}\cdot\mathbf{s} + \mathbf{s}\cdot\mathbf{s} = r^2 + 2\mathbf{r}\cdot\mathbf{s} + s^2,$$

which we can rewrite as

$$\mathbf{r}\cdot\mathbf{s} = \tfrac{1}{2}\left(|\mathbf{r}+\mathbf{s}|^2 - r^2 - s^2\right).$$

Each of the three terms on the right is the square of the length of a vector and is therefore independent of our choice of axes. This means that the whole right side is similarly independent, and so therefore is the left side, $\mathbf{r}\cdot\mathbf{s}$.

1.17 ⋆⋆ **(a)** Let us start with the x component of $\mathbf{r} \times (\mathbf{u} + \mathbf{v})$. From the definition (1.9), we see that

$$[\mathbf{r} \times (\mathbf{u} + \mathbf{v})]_{\mathbf{x}} = r_y(u_z + v_z) - r_z(u_y + v_y) = (r_y u_z - r_z u_y) + (r_y v_z - r_z v_y) = (\mathbf{r} \times \mathbf{u})_x + (\mathbf{r} \times \mathbf{v})_x .$$

Since the y and z components follow in the same way, we conclude that $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = \mathbf{r} \times \mathbf{u} + \mathbf{r} \times \mathbf{v}$.

(b) Starting again from (1.9), we find for the x component

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s})_x = \frac{d}{dt}(r_y s_z - r_z s_y) = \left(r_y \frac{ds_z}{dt} - r_z \frac{ds_y}{dt}\right) + \left(\frac{dr_y}{dt} s_z - \frac{dr_z}{dt} s_y\right) = \left(\mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}\right)_x .$$

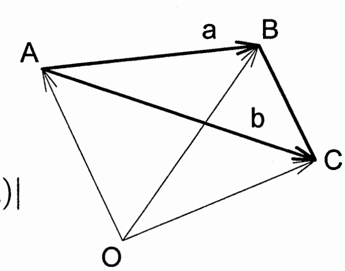
This is the x component of the desired identity. Since the y and z components follow in exactly the same way, our proof is complete.

1.18 ⋆⋆ **(a)** $|\mathbf{a} \times \mathbf{b}| = ab \sin\gamma = bh$, where $h = a \sin\gamma$ is the height of the triangle ABC. Therefore $|\mathbf{a} \times \mathbf{b}| = 2$(area of triangle), which is the required first result. The other two follow in the same way.

(b) By part (a), $|\mathbf{c} \times \mathbf{a}| = |\mathbf{b} \times \mathbf{c}|$ or $ca \sin\beta = bc \sin\alpha$, whence $a/\sin\alpha = b/\sin\beta$, as required. The third expression follows in the same way.

1.19 ⋆⋆ $\displaystyle \frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \frac{d\mathbf{a}}{dt} \cdot (\mathbf{v} \times \mathbf{r}) + \mathbf{a} \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{r}) = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}) + \mathbf{a} \cdot (\dot{\mathbf{v}} \times \mathbf{r} + \mathbf{v} \times \dot{\mathbf{r}})$.
The final term $\mathbf{a} \cdot (\mathbf{v} \times \dot{\mathbf{r}})$ is zero because $\dot{\mathbf{r}} = \mathbf{v}$ and $\mathbf{v} \times \mathbf{v} = 0$. The second to last term is $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{r}) = \mathbf{0}$, because $\mathbf{a} \times \mathbf{r}$ is perpendicular to $\mathbf{a}$, so their scalar product is zero. This leaves us with the requested identity.

1.20 ⋆⋆ Let two of the sides of the triangle be $\mathbf{a} = \mathbf{B} - \mathbf{A}$ and $\mathbf{b} = \mathbf{C} - \mathbf{A}$ as shown. Then, according to Problem 1.18



$$\begin{aligned} 2(\text{area}) = |\mathbf{a} \times \mathbf{b}| &= |(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})| \\ &= |(\mathbf{B} \times \mathbf{C}) - (\mathbf{B} \times \mathbf{A}) - (\mathbf{A} \times \mathbf{C}) + (\mathbf{A} \times \mathbf{A})| \\ &= |(\mathbf{B} \times \mathbf{C}) + (\mathbf{C} \times \mathbf{A}) + (\mathbf{A} \times \mathbf{B})| \end{aligned}$$

1.21 ⋆⋆ The base of the parallelepiped is a parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$. By Problem 1.18, the area of this base is $|\mathbf{b} \times \mathbf{c}|$. The vector $\mathbf{b} \times \mathbf{c}$ is normal to the base, so if θ is the angle between $\mathbf{a}$ and this normal,

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = a|\mathbf{b} \times \mathbf{c}|\,|\cos\theta| = (\text{area of base})\,|a \cos\theta| .$$

Now, $|a \cos\theta|$ is the height of the parallelepiped. Therefore $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is just "(area of base) $\times$ height," which is the volume, as claimed.

1.22 ⋆⋆ **(a)** According to definition (1.6), $\mathbf{a}\cdot\mathbf{b} = ab\cos(\alpha-\beta)$, but according to (1.7)

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y = ab(\cos\alpha\cos\beta + \sin\alpha\sin\beta).$$

Comparing these two expressions we get the desired result.

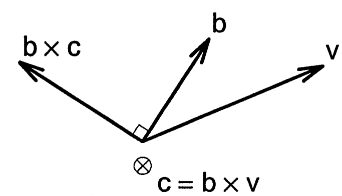
(b) According to the right-hand-rule definition, $\mathbf{a}\times\mathbf{b} = -ab\sin(\alpha-\beta)\hat{\mathbf{z}}$. (The minus sign is easy to check; for example, if $\alpha > \beta$, then the right-hand rule puts $\mathbf{a}\times\mathbf{b}$ in the negative z direction.) On the other hand, the definition (1.9) gives the z component as

$$(\mathbf{a}\times\mathbf{b})_z = a_x b_y - a_y b_x = ab(\cos\alpha\sin\beta - \sin\alpha\cos\beta).$$

Comparing these two expressions we again get the desired result.

1.23 ⋆⋆ The picture shows the plane of $\mathbf{b}$ and $\mathbf{v}$. Because $\mathbf{c} = \mathbf{b}\times\mathbf{v}$, $\mathbf{c}$ is perpendicular to this plane, as indicated. Therefore $\mathbf{b}\times\mathbf{c}$ lies in the plane and is perpendicular to $\mathbf{b}$. This means that $\mathbf{v}$ can be expressed in terms of the two mutually perpendicular vectors $\mathbf{b}$ and $\mathbf{b}\times\mathbf{c}$:

$$\mathbf{v} = \alpha\,\mathbf{b} + \beta\,\mathbf{b}\times\mathbf{c}$$



If we form the dot product of this equation with $\mathbf{b}$ we find that $\mathbf{b}\cdot\mathbf{v} = \alpha\mathbf{b}\cdot\mathbf{b}$. Therefore $\alpha = \mathbf{b}\cdot\mathbf{v}/b^2 = \lambda/b^2$. Similarly, if we form the cross product of the same equation with $\mathbf{b}$, we find that $\mathbf{b}\times\mathbf{v} = \beta\mathbf{b}\times(\mathbf{b}\times\mathbf{c})$. Since $\mathbf{b}\times\mathbf{v} = \mathbf{c}$ and (as you can easily check) $\mathbf{b}\times(\mathbf{b}\times\mathbf{c}) = -b^2\mathbf{c}$, this last implies that $\beta = -1/b^2$, and we conclude that $\mathbf{v} = (\lambda\mathbf{b} - \mathbf{b}\times\mathbf{c})/b^2$.

1.24 ⋆ Integrating the equation $df/f = dt$, we find that $\ln f = t + k$, or $f = Ae^t$, with one arbitrary constant, $A = e^k$.

1.25 ⋆ Integrating the equation $df/f = -3dt$, we find that $\ln f = -3t + k$, or $f = Ae^{-3t}$, with one arbitrary constant, $A = e^k$.

1.26 ⋆⋆ **(a)** Since the puck is frictionless, the net force on it is zero, and, as seen from frame $\mathcal{S}$ the puck heads due north with constant speed v_o, so $x = 0$ and $y = v_o t$, where v_o is the speed with which I kicked the puck.

(b) We must now find how to translate the coordinates (x, y) of any point P, as seen in frame $\mathcal{S}$, to the coordinates (x', y') of the same point P as seen in $\mathcal{S}'$. At time t, the origin of $\mathcal{S}'$ is at $(vt, 0)$ (as seen in $\mathcal{S}$), where v is the speed of $\mathcal{S}'$ relative to $\mathcal{S}$. Therefore $(x, y) = (vt, 0) + (x', y')$; whence $x' = x - vt$ and $y' = y$. This is for any point (x, y). If we substitute the coordinates of the puck from part (a), we find

$$x' = -vt \quad \text{and} \quad y' = v_o t.$$

As seen from frame $\mathcal{S}'$, the puck moves in a straight line toward the north-west quadrant with velocity $(-v, v_o)$. Since the velocity is constant, this motion is consistent with Newton's first law and frame $\mathcal{S}'$ is apparently inertial.

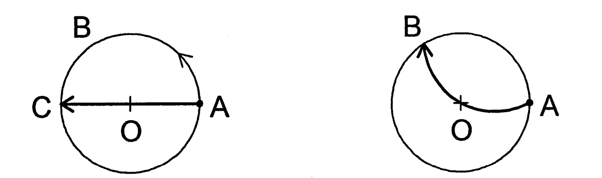
(c) At time t, the origin of frame $\mathcal{S}''$ is at the point $(\frac{1}{2}at^2, 0)$ as seen in $\mathcal{S}$. Therefore, the coordinates of any point (x, y) as seen in $\mathcal{S}$ are $x'' = x - \frac{1}{2}at^2$ and $y'' = y$ as seen in $\mathcal{S}''$. Substituting the coordinates of the puck from part (a), we find

$$x'' = -\tfrac{1}{2}at^2 \qquad \text{and} \qquad y' = v_o t.$$

Therefore, as seen from $\mathcal{S}''$, the puck moves toward the NW quadrant in a parabola, with v''_x steadily increasing (in the negative direction). Since the velocity is not constant, Newton's first law is not valid, and $\mathcal{S}''$ is not an inertial frame.

1.27 ⋆⋆ Since the puck is frictionless, the net force on it is zero, and, as seen from the ground, it travels in a straight line through the center O, as shown in the left picture. It starts from the point A at $t = 0$, travels "due west" with constant speed v_o, and falls onto the ground at point C after a time $T = 2R/v_o$ (where R is the radius of the turntable).

Now imagine an observer sitting on the turntable near A. As seen from the ground, he is traveling north with speed ωR. Therefore, as seen by the observer, the puck's initial velocity has a sideways (southerly) component ωR, in addition to the westerly component v_o; that is, the puck moves initially west *and* south, as shown in the right picture. (The magnitude of the southerly component depends on the table's rate of rotation ω.) As the puck moves in to a smaller radius r, the sideways component ωr gets less, so the puck's path curves to the right. Continuing to curve, its passes through O and eventually reaches the edge of the turntable at point B. The left picture shows the point B of the table at time $t = 0$. The position of B is determined by the following consideration: In the time $T = 2R/v_o$ for the puck to cross the table, point B of the table must move around to point C where we know the puck falls to the ground. Thus the angle BOC is equal to ωT. The faster the table rotates, the larger the angle BOC and the more sharply the puck's path (as seen from the table) is curved.



seen from ground   seen from turntable

1.28 ⋆ When we write out Equations (1.25) and (1.26) for the three particles, we get three equations:

$$\begin{aligned}
\dot{\mathbf{p}}_1 &= (\text{net force on particle 1}) = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_1^{\text{ext}} \\
\dot{\mathbf{p}}_2 &= (\text{net force on particle 2}) = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_2^{\text{ext}} \\
\dot{\mathbf{p}}_3 &= (\text{net force on particle 3}) = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_3^{\text{ext}}\,.
\end{aligned}$$

Adding these three equations, we find for $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 + \dot{\mathbf{p}}_3$,

$$\dot{\mathbf{P}} = (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{31} + \mathbf{F}_{32}) + \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}}. \qquad \text{(i)}$$

This corresponds to Equation (1.27). The first six terms on the right can be rearranged to give

$$\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{31} + \mathbf{F}_{32} = (\mathbf{F}_{12} + \mathbf{F}_{21}) + (\mathbf{F}_{13} + \mathbf{F}_{31}) + (\mathbf{F}_{23} + \mathbf{F}_{32}) = 0$$

since each of the three pairs on the right is zero by Newton's third law. Thus Equation (i) for $\dot{\mathbf{P}}$ reduces to

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} = \mathbf{F}^{\text{ext}}$$

which is the required Equation (1.29).

1.29 ⋆ When we write out Equations (1.25) and (1.26) for the four particles, we get four equations:

$$\begin{aligned}
\dot{\mathbf{p}}_1 &= (\text{net force on particle 1}) = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}} \\
\dot{\mathbf{p}}_2 &= (\text{net force on particle 2}) = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}} \\
\dot{\mathbf{p}}_3 &= (\text{net force on particle 3}) = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}} \\
\dot{\mathbf{p}}_4 &= (\text{net force on particle 4}) = \mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}}.
\end{aligned}$$

Adding these four equations, we find for $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 + \dot{\mathbf{p}}_3 + \dot{\mathbf{p}}_4$,

$$\begin{aligned}
\dot{\mathbf{P}} = {} & (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14}) + (\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24}) + (\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34}) + (\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43}) \\
& + (\mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}}).
\end{aligned} \tag{ii}$$

This corresponds to Equation (1.27). The twelve terms on the first line of the right side can be rearranged to give

$$(\mathbf{F}_{12} + \mathbf{F}_{21}) + (\mathbf{F}_{13} + \mathbf{F}_{31}) + (\mathbf{F}_{14} + \mathbf{F}_{41}) + (\mathbf{F}_{23} + \mathbf{F}_{32}) + (\mathbf{F}_{24} + \mathbf{F}_{42}) + (\mathbf{F}_{34} + \mathbf{F}_{43}) = 0$$

since each of the six pairs is zero by Newton's third law. Thus Equation (ii) for $\dot{\mathbf{P}}$ reduces to

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}} = \mathbf{F}^{\text{ext}}$$

which is the required Equation (1.29).

1.30 ⋆ Since mass 2 is at rest, the initial total momentum is just $\mathbf{P}_{\text{in}} = m_1\mathbf{v}$. The final total momentum is $\mathbf{P}_{\text{fin}} = (m_1 + m_2)\mathbf{v}'$. Equating these two and solving for $\mathbf{v}'$, we find that $\mathbf{v}' = \mathbf{v}m_1/(m_1 + m_2)$.

1.31 ⋆ We have to prove for any pair of particles (call them 1 and 2) that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. To do this, suppose that all forces on 1 and 2, except $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, have been switched off. For example, we could move all bodies with which 1 and 2 interact to a great distance. If the law of conservation of momentum holds, then $\mathbf{p}_1 + \mathbf{p}_2$ is consant, which implies that $\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0$. Since all other forces have been switched off, the only force on particle 1 is $\mathbf{F}_{12}$ and the only force on 2 is $\mathbf{F}_{21}$. Therefore this last equation implies that $\mathbf{F}_{12} + \mathbf{F}_{21} = 0$, which is the required result. (There is a subtle point here: We have assumed that switching off the external forces did not change the internal forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$. One could imagine a world where the presence or absence of external forces affected the internal forces. However, this seems not to be the case in our world.)

1.32 ⋆⋆ The fields produced by q_2 at the position of q_1 are as given in the problem. Thus the electric and magnetic forces on q_1 are

$$\mathbf{F}_{12}^{\text{el}} = q_1\mathbf{E}(\mathbf{r}_1) = \frac{1}{4\pi\epsilon_o}\frac{q_1q_2}{s^2}\hat{\mathbf{s}} \quad \text{and} \quad \mathbf{F}_{12}^{\text{mag}} = q_1\mathbf{v}_1 \times \mathbf{B}(\mathbf{r}_1) = \frac{\mu_o}{4\pi}\frac{q_1q_2}{s^2}\mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{s}}).$$

The ratio of these is

$$\frac{F_{12}^{\text{mag}}}{F_{12}^{\text{el}}} = \mu_o\epsilon_o v_1 v_2 \sin\alpha\sin\beta = \frac{v_1v_2}{c^2}\sin\alpha\sin\beta \le \frac{v_1v_2}{c^2}$$

where α and β are the angles involved in the cross products and I have used the fact that $\mu_o\epsilon_o = 1/c^2$.

1.33 ⋆⋆⋆ According to the Biot-Savart law, the total force on loop 1 due to loop 2 is

$$\mathbf{F}_{12} = \frac{\mu_o}{4\pi}I_1I_2\oint\oint\frac{d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \mathbf{s})}{s^3} = \frac{\mu_o}{4\pi}I_1I_2\left\{\oint\oint d\mathbf{r}_2\frac{d\mathbf{r}_1\cdot\mathbf{s}}{s^3} - \oint\oint\frac{\mathbf{s}}{s^3}d\mathbf{r}_1\cdot d\mathbf{r}_2\right\}.$$

Here the two integrals run around the two loops, and in writing the second equality I have used the "$BAC = CAB$" rule, that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$. In the first integral on the right, $d\mathbf{r}_1\cdot\mathbf{s}/s^3 = -d(1/s)$, so when we integrate around loop 1, this gives zero, and we are left with

$$\mathbf{F}_{12} = \frac{\mu_o}{4\pi}I_1I_2\oint\oint\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}d\mathbf{r}_1\cdot d\mathbf{r}_2.$$

Evidently, $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

1.34 ⋆⋆⋆ Let us introduce the notation $\boldsymbol{\ell}_\alpha$ for the angular momentum of the particle α, that is, $\boldsymbol{\ell}_\alpha = \mathbf{r}_\alpha \times \mathbf{p}_\alpha$. Our first task is to find the time derivative of $\boldsymbol{\ell}_\alpha$ (just as in the case of linear momentum our first task was to find $\dot{\mathbf{p}}_\alpha$). This is

$$\dot{\boldsymbol{\ell}}_\alpha = \frac{d}{dt}(\mathbf{r}_\alpha \times \mathbf{p}_\alpha) = \dot{\mathbf{r}}_\alpha \times \mathbf{p}_\alpha + \mathbf{r}_\alpha \times \dot{\mathbf{p}}_\alpha$$

where I have used the product rule to differentiate the cross product. The first term on the right is zero, since $\mathbf{p}_\alpha = m_\alpha\dot{\mathbf{r}}_\alpha$ and the cross product of two parallel vectors is zero. In the second term, we can use Newton's second law to replace $\dot{\mathbf{p}}_\alpha$ with $\mathbf{F}_\alpha$, the net force on particle α, to give

$$\dot{\boldsymbol{\ell}}_\alpha = \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \boldsymbol{\Gamma}_\alpha \tag{iii}$$

where I have introduced $\boldsymbol{\Gamma}_\alpha$ to denote $\mathbf{r}_\alpha \times \mathbf{F}_\alpha$, the net torque on particle α. The total angular momentum is $\mathbf{L} = \sum\boldsymbol{\ell}_\alpha$. By summing Equation (iii) over all particles we see immediately that

$$\dot{\mathbf{L}} = \sum \mathbf{\Gamma}_\alpha = \mathbf{\Gamma}. \tag{iv}$$

That is, the rate of change of $\mathbf{L}$ is the total torque on the system. Unfortunately, this isn't yet enough, since we need to separate the effects of the internal and external forces in accordance with Equation (1.19). This gives

$$\mathbf{\Gamma} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \sum_\alpha \sum_{\beta \neq \alpha} (\mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta}) + \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}}.$$

Here the final sum is just the net *external* torque $\mathbf{\Gamma}^{\text{ext}}$. In the double sum, we can pair off terms, grouping each $\mathbf{F}_{\alpha\beta}$ with $\mathbf{F}_{\beta\alpha} = -\mathbf{F}_{\alpha\beta}$, to give

$$\mathbf{\Gamma} = \sum_\alpha \sum_{\beta > \alpha} (\mathbf{r}_\alpha - \mathbf{r}_\beta) \times \mathbf{F}_{\alpha\beta} + \mathbf{\Gamma}^{\text{ext}}.$$

Now, the vector $(\mathbf{r}_\alpha - \mathbf{r}_\beta)$ points from particle β to particle α. (This is illustrated in Figure 3.7.) Thus, *provided* the internal forces $\mathbf{F}_{\alpha\beta}$ are *central*, these two vectors are collinear, and their cross product is zero, so that the whole double sum is zero. Therefore, $\mathbf{\Gamma} = \mathbf{\Gamma}^{\text{ext}}$ and, according to Equation (iv), $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$. In particular, if there are no external forces, $\mathbf{L}$ is constant.

1.35 ⋆ In the absence of air resistance, the net force on the ball is $\mathbf{F} = m\mathbf{g}$, and with the given choice of axes, $\mathbf{g} = (0, 0, -g)$. Thus Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, implies that $\ddot{\mathbf{r}} = \mathbf{g}$, or

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \text{and} \qquad \ddot{z} = -g.$$

The initial velocity has components $v_{ox} = v_o \cos\theta$, $v_{oy} = 0$, and $v_{oz} = v_o \sin\theta$, and we can choose the initial position to be the origin. The first of the above equations can be integrated once to give $\dot{x} = v_{ox}$, and again to give $x(t) = v_{ox}t$. In the same way, the y equation gives $y(t) = 0$, and the z equation gives $z(t) = v_{oz}t - \frac{1}{2}gt^2$. The ball returns to the ground when $z(t) = 0$ which gives $t = 2v_{oz}/g$. Substituting this time into the expression for $x(t)$ gives the range, range $= 2v_{oz}v_{oz}/g$.

1.36 ⋆ **(a)** During the flight the only force on the bundle is its weight, and Newton's second law reads $m\ddot{\mathbf{r}} = \mathbf{F} = m\mathbf{g}$, or $\ddot{\mathbf{r}} = \mathbf{g}$. If we choose the origin at sea level directly below the plane at the moment of launch and measure x in the direction of flight and y vertically up, then the solution is $x = v_o t$, $y = h - \frac{1}{2}gt^2$, and $z = 0$.

(b) The time for the bundle to drop to sea level ($y = 0$) is $t = \sqrt{2h/g}$ and the horizontal distance traveled in this time is $x = v_o t = v_o\sqrt{2h/g}$. With the given numbers this is about 220 m.

(c) If the drop is delayed by a time Δt, the bundle will overshoot by a distance $\Delta x = v_o \Delta t$, so $\Delta t = \Delta x / v_o = 0.2$ sec.

1.37 ⋆ **(a)** The two forces on the puck are its weight $m\mathbf{g}$ and the normal force $\mathbf{N}$ of the incline. If we choose axes with x measured up the slope, y along the outward normal, and z horizontally across the slope, then $\mathbf{N} = (0, N, 0)$ and $\mathbf{g} = (-g\sin\theta, g\cos\theta, 0)$. Thus Newton's second law reads

$$m\ddot{\mathbf{r}} = \mathbf{N} + m\mathbf{g} \quad \text{or} \quad \begin{cases} m\ddot{x} = -mg\sin\theta \\ m\ddot{y} = N - mg\cos\theta \\ m\ddot{z} = 0 \end{cases}$$

Since $\dot{z} = 0$ initially, it remains so and hence $z = 0$ for all t. The normal force adjusts itself so that $\ddot{y} = 0$, and $y = 0$ for all t. Finally, $\ddot{x} = -g\sin\theta$, which can be integrated twice to give $x = v_o t - \frac{1}{2}gt^2\sin\theta$.

(b) Solving for the times when $x = 0$, we find that $t = 0$ (at launch) or $t = 2v_o/(g\sin\theta)$ (the answer of interest).

1.38 ⋆ The two forces on the puck are its weight $m\mathbf{g}$ and the normal force $\mathbf{N}$ of the incline. With the suggested choice of axes, $\mathbf{N} = (0, 0, N)$ and $\mathbf{g} = (0, -g\sin\theta, -g\cos\theta)$. Thus Newton's second law reads

$$m\ddot{\mathbf{r}} = \mathbf{N} + m\mathbf{g} \quad \text{or} \quad \begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = -mg\sin\theta \\ m\ddot{z} = N - mg\cos\theta \end{cases}$$

By integrating the y equation twice, we find that $y = v_{oy}t - \frac{1}{2}gt^2\sin\theta$. Thus the time to return to the line $y = 0$ is $t = 2v_{oy}/(g\sin\theta)$ and the distance from O at that time is $x = v_{ox}t = 2v_{ox}v_{oy}/(g\sin\theta)$.

1.39 ⋆⋆ $x = v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi$, $y = v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi$, $z = 0$. When the ball returns to the plane, y is 0, which implies that $t = 2v_o\sin\theta/(g\cos\phi)$. Substituting this time into x and using a couple of trig identities yields the claimed answer for the range R. To find the maximum range, differentiate R with respect to θ and set the derivative equal to zero. This gives $\theta = (\pi - 2\phi)/4$, and substitution into R (plus another trig identity) yields the claimed value of R_{max}.

1.40 ⋆⋆⋆ **(a)** $x = (v_o\cos\theta)t$, $y = (v_o\sin\theta)t - \frac{1}{2}gt^2$, and $z = 0$

(b) $r^2 = x^2 + y^2 = \frac{1}{4}g^2t^4 - (v_o g\sin\theta)t^3 + v_o^2t^2$, so $d(r^2)/dt = g^2t^3 - 3(v_o g\sin\theta)t^2 + 2v_o^2t$. When t is sufficiently small, r certainly increases with time. It's derivative vanishes if and only if

$$t = \frac{v_o}{2g}\left(3\sin\theta \pm \sqrt{9\sin^2\theta - 8}\right).$$

If θ is small, the argument of the square root is negative, and r always increases. As θ increases, the first value of θ for which the derivative does vanish is given by $\sin\theta = \sqrt{8/9}$; that is, $\theta_{\text{max}} = 70.5°$. Thus for $0 \leq \theta < \theta_{\text{max}}$, r always increases. For $\theta_{\text{max}} < \theta \leq 90°$, r

increases initially, but then decreases for a while. (This last is particularly clear for the case that $\theta = 90°$.)

1.41 ⋆ The r and ϕ components of Newton's second law are $F_r = m(\ddot{r} - r\dot{\phi}^2)$ and $F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$. Since $r = R$ is constant $\dot{r} = \ddot{r} = 0$, and since $\dot{\phi} = \omega$ is constant $\ddot{\phi} = 0$. Finally the only force is the inward tension of the string, so $F_r = -T$. Thus the r equation becomes $-T = -mR\omega^2$, so $T = m\omega^2 R$, which is just the familiar "centripetal force" mv^2/R, since $\omega = v/R$.

1.42 ⋆ Elementary trigonometry applied to the triangle of Fig.1.10 shows that $x = r\cos\phi$ and $y = r\sin\phi$, and by Pythagoras' theorem $r = \sqrt{x^2 + y^2}$. Clearly $\tan\phi = y/x$, so it is tempting to say that $\phi = \arctan(y/x)$. Unfortunately this isn't quite satisfactory. The difficulty is that the two distinct vectors $\mathbf{r}$ and $-\mathbf{r}$ have the same ratio y/x [since $(-y)/(-x) = y/x$] but their polar angles should differ by π. The simple-minded claim that $\phi = \arctan(y/x)$ can't distinguish these two cases. One way out is to define a function $\arctan(x, y)$ which puts the angle in the right quadrant. For example, if arctan is defined to lie between $-\pi/2$ and $\pi/2$, then we could define

$$\arctan(x, y) = \begin{cases} \arctan(y/x) & \text{if } x > 0 \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & \text{if } x < 0 \end{cases}$$

If we define $\phi = \arctan(x, y)$, then every nonzero vector has a well defined polar angle ϕ, any two vectors in different directions have different polar angles in the range $-\pi/2 \le \phi < 3\pi/2$, and (most important) ϕ is always in the correct quadrant.

1.43 ⋆ **(a)** From Figure 1.11(b), you can see that the x and y components of $\hat{\mathbf{r}}$ are $\cos\phi$ and $\sin\phi$. (Remember that $|\hat{\mathbf{r}}| = 1$.) Therefore, $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$. Similarly, the x and y components of $\hat{\boldsymbol{\phi}}$ are $-\sin\phi$ and $\cos\phi$, so that $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$.

(b) Differentiating these two results with respect to t and using the chain rule, we find (Remember that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constant.)

$$\frac{d}{dt}\hat{\mathbf{r}} = \dot{\phi}(-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi) = \dot{\phi}\hat{\boldsymbol{\phi}} \quad \text{and} \quad \frac{d}{dt}\hat{\boldsymbol{\phi}} = \dot{\phi}(-\hat{\mathbf{x}}\cos\phi - \hat{\mathbf{y}}\sin\phi) = -\dot{\phi}\,\hat{\mathbf{r}}.$$

1.44 ⋆ If $\phi = A\sin(\omega t) + B\cos(\omega t)$, then

$$\dot{\phi} = \omega A\cos(\omega t) - \omega B\sin(\omega t) \quad \text{and} \quad \ddot{\phi} = -\omega^2 A\sin(\omega t) - \omega^2 B\cos(\omega t) = -\omega^2\phi$$

and ϕ does indeed satisfy $\ddot{\phi} = -\omega^2\phi$.

1.45 ⋆⋆ Since the magnitude of $\mathbf{v}(t)$ is the same as $\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$, the magnitude is constant if and only if $\mathbf{v}(t)\cdot\mathbf{v}(t)$ is. Since

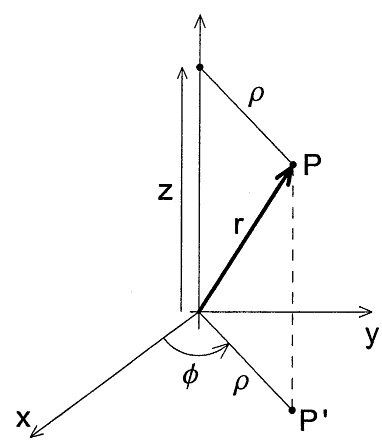
$$\frac{d}{dt}[\mathbf{v}(t)\cdot\mathbf{v}(t)] = 2\mathbf{v}(t)\cdot\dot{\mathbf{v}}(t),$$

this implies that the magnitude of $\mathbf{v}(t)$ is constant if and only if $\mathbf{v}(t)\cdot\dot{\mathbf{v}}(t) = 0$; that is, $\mathbf{v}(t)$ is orthogonal to $\dot{\mathbf{v}}(t)$

1.46 ⋆⋆ **(a)** As seen in the inertial frame $\mathcal{S}$ the puck moves in a straight line with $\phi = 0$ and $r = R - v_o t$

(b) As seen in $\mathcal{S}'$, $r' = r = R - v_o t$ and $\phi' = \phi - \omega t = -\omega t$. This path is sketched in the answer to Problem 1.27. Initially, the puck moves inward with speed v_o but also downward with speed ωR. It curves to its right, passing through the center and continuing to curve to the right until it slides off the turntable.

1.47 ⋆⋆ **(a)** $\rho = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$ (chosen to lie in the right quadrant), and z is the same as in Cartesians. The coordinate ρ is the perpendicular distance from P to the z axis. If we use r for the coordinate ρ, then r is not the same thing as $|\mathbf{r}|$ and $\hat{\mathbf{r}}$ is not the unit vector in the direction of $\mathbf{r}$ [see part (b)].



(b) The unit vector $\hat{\boldsymbol{\rho}}$ points in the direction of increasing ρ (with ϕ and z fixed), that is, directly away from the z axis; $\hat{\boldsymbol{\phi}}$ is tangent to a horizontal circle through P centered on the z axis (counter-clockwise, seen from above); $\hat{\mathbf{z}}$ is parallel to the z axis.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}.$$

(c) Differentiating this equation we find (Remember that $\hat{\mathbf{z}}$ is constant.)

$$\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

since $d\hat{\boldsymbol{\rho}}/dt = \dot{\phi}\hat{\boldsymbol{\phi}}$ [see (1.42)]. Differentiating again, we find similarly that

$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\mathbf{r}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}.$$

1.48 ⋆⋆ The unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are the same as in two dimensions (except that what was called r is now called ρ),

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \qquad \text{and} \qquad \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

and $\hat{\mathbf{z}}$ is the same as in Cartesians. Differentiating with respect to t, we find that

$$\frac{d}{dt}\hat{\boldsymbol{\rho}} = -\hat{\mathbf{x}}\,\dot{\phi}\sin\phi + \hat{\mathbf{y}}\,\dot{\phi}\cos\phi = \dot{\phi}\,\hat{\boldsymbol{\phi}} \quad \text{and} \quad \frac{d}{dt}\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\,\dot{\phi}\cos\phi - \hat{\mathbf{y}}\,\dot{\phi}\sin\phi = -\dot{\phi}\hat{\boldsymbol{\rho}}$$

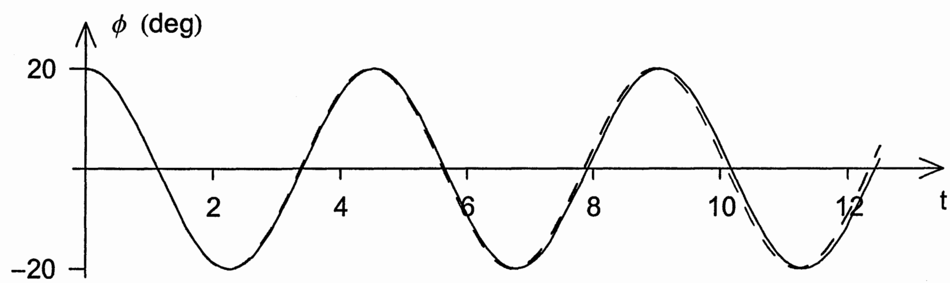
and, of course, $d\hat{\mathbf{z}}/dt = 0$.

1.49 ★★ There are two forces on the puck, the net normal force of the two cylinders and the force of gravity. So $\mathbf{F} = N\hat{\boldsymbol{\rho}} - mg\hat{\mathbf{z}}$. Since the puck is confined between the cylinders, $\rho = R$, a constant. The three components of $\mathbf{F} = m\mathbf{a}$ are:

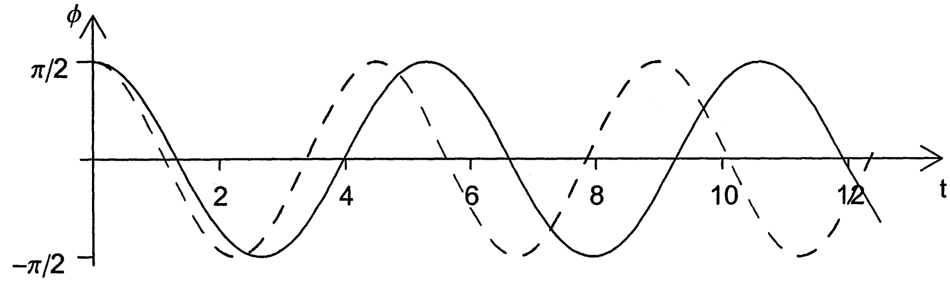
$$\begin{array}{lll} F_\rho = m(\ddot{\rho} - \rho\dot{\phi}^2) & \text{or} & N = -m\rho\dot{\phi}^2 \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) & \text{or} & 0 = mR\ddot{\phi} \\ F_z = m\ddot{z} & \text{or} & -mg = m\ddot{z}. \end{array}$$

The ρ equation tells us the magnitude and direction (inward) of the normal force. The ϕ equation tells us that $\dot{\phi}$ is constant. (This is actually conservation of angular momentum.) Thus $\dot{\phi} = \omega$, a constant, and hence $\phi = \phi_o + \omega t$. The puck moves around the cylinder at a constant rate ω. The z equation tells us that $\dot{z} = v_{oz} - gt$ and hence that $z = z_o + v_{oz}t - \frac{1}{2}gt^2$. That is, the vertical motion is precisely that of a body in free fall. The resulting path is a helix of downward increasing pitch.

1.50 ★★★ In the picture, the solid curve is a numerical solution of the differential equation, found with Mathematica's NDSolve. The dashed curve is the small-oscillation approximation (1.57) with the same initial conditions ($\phi_o = 20°$ and $\dot{\phi}_o = 0$). Given that 20° is certainly not a very small angle, the small-angle approximation does remarkably well, though you can just see that the approximate solution oscillates a little too fast; as one would expect. (For larger amplitudes, the true period is a little longer.)



1.51 ★★★ In the picture, the solid curve is a numerical solution of the differential equation, found with Mathematica's NDSolve. The dashed curve is the small-oscillation approximation (1.57) with the same initial condition ($\phi_o = \pi/2$). Considering how large the initial angle is, the small-angle approximation does remarkably well. The only obvious discrepancy is that the approximation oscillates somewhat too fast, as one would expect. (For large amplitudes, the true period is a little longer.) After two complete cycles, the approximate solution is nearly half a cycle ahead. Also, if you look closely you can maybe see that the actual motion is not perfectly sinusoidal at the extremes — the crests are a little wider and flatter.


ϕ
$\pi/2$
$-\pi/2$
2
4
6
8
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12
t

Chapter 2

Projectiles and Charged Particles

I covered this chapter in 3.5 fifty-minute lectures.

The topics of Chapter 2 are two relatively straightforward applications of Newton's laws of motion — the motion of projectiles subject to air resistance and of a charged particle in a magnetic field. Both of these are interesting topics that your students may not have studied in detail before. In particular, while air resistance is often mentioned in most freshman physics courses, it is usually mentioned only to be neglected. The inclusion of air resistance gives us a chance to discuss the solution of several non-trivial differential equations and (in the case of quadratic resistance) is our first chance to confront differential equations that can only be solved numerically. (My policy with respect to numerical solution of differential equations is to endorse no specific software, but rather to urge the use of whatever software — Mathematica, Maple, Matlab — is available. Some of the simpler problems can be done with a spreadsheet such as Excel, as can many of the harder problems if you can help your students with the numerical methods needed.) The motion of a charged particle in a magnetic field is another interesting problem and it gives me the opportunity to introduce the use of complex numbers in solving real problems.

These two topics are a natural context to introduce several other mathematical techniques besides differential equations and complex numbers. For example, in Section 2.3 Taylor's series emerges as a natural way to account for weak air resistance and Section 2.4 introduces the hyperbolic functions sinh, cosh, and tanh (of which most of my students claimed complete ignorance). As usual, the end-of-chapter problems offer several opportunities for the students to get up to speed on any math that they need help with. For instance, Problems 2.33 and 2.34 provide a crash course in the hyperbolic functions, and Problems 2.45 through 2.51 should refresh memories about complex numbers. There are several problems requiring the use of a computer to find orbits and ranges of projectiles subject to air resistance.

Solutions to Problems for Chapter 2

2.1 ⋆⋆ According to (2.7), $f_{\text{quad}}/f_{\text{lin}} = vD\gamma/\beta$, so this equals 1 when $v = 1/(D\gamma/\beta)$, which is roughly 1 cm/s if $D = 7$ cm. Since baseballs generally travel much faster than 1 cm/s, it is usually a very good approximation to ignore f_{lin}. For a beachball with $D = 70$ cm, the corresponding speed is about 1 mm/s.

2.2 ⋆ According to Stokes's law $f_{\text{lin}} = (3\pi\eta D)v$, which has precisely the form $f_{\text{lin}} = bv$ if we define $b = \beta D$ and $\beta = 3\pi\eta = 3\pi(1.7 \times 10^{-5}\ \text{N·s/m}^2) = 1.6 \times 10^{-4}\ \text{N·s/m}^2$.

2.3 ⋆ **(a)** From (2.84) and (2.82), $f_{\text{quad}}/f_{\text{lin}} = (\kappa\varrho Av^2)/(3\pi\eta Dv)$. With $\kappa = 1/4$ and $A = \pi D^2/4$, this becomes $\varrho Dv/(48\eta)$ or $R/48$, with R given by (2.83).

(b) With the given numbers, $R = 1.1 \times 10^{-2}$ and it is very safe to neglect the quadratic drag.

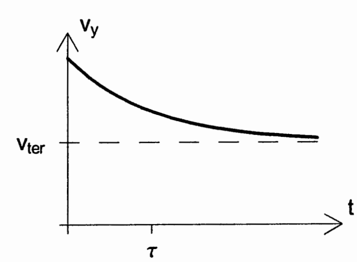
2.4 ⋆⋆ **(a)** In a short time dt the projectile moves a distance vdt, and the front sweeps out a cylinder of volume $Avdt$. Therefore the mass of fluid encountered is $\varrho Avdt$, and the rate at which mass is swept up is ϱAv.

(b) If a mass $\varrho Avdt$ is accelerated from 0 to v in time dt, the rate of change of its momentum is ϱAv^2. This is, therefore, the forward force on the fluid and, hence, the backward force on the projectile.

(c) Since $A \propto D^2$, it follows that $f_{\text{quad}} = \kappa\varrho Av^2 = cv^2$, where $c = \kappa\varrho A \propto D^2$. For a sphere in air, $\kappa = 1/4$, $A = \pi D^2/4$, and $\varrho = 1.29\ \text{kg/m}^3$, so $f_{\text{quad}} = (\kappa\varrho\pi D^2/4)v^2 = cv^2$, where $c = \gamma D^2$ and

$$\gamma = \kappa\varrho\pi/4 = \tfrac{1}{4} \times (1.29\ \text{kg/m}^3) \times \pi/4 = 0.25\ \text{N} \cdot \text{s}^2/\text{m}^4.$$

2.5 ⋆ With $v_y > v_{\text{ter}}$, the drag force is greater than the weight, and the net force is upward. Thus the projectile slows down, with v_y approaching v_{ter} as $t \to \infty$. This is clear from Eq.(2.30), as shown in the plot.



2.6 ⋆ **(a)** If we insert the Taylor series for $e^{-t/\tau}$ into (2.33), we get

$$v_y(t) = v_{\text{ter}}\left[1 - e^{-t/\tau}\right] = v_{\text{ter}}\left[1 - \left(1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} - \cdots\right)\right].$$

The first two terms on the right cancel, and, if t is sufficiently small, we can neglect terms in t^2 and higher. This leaves us with

$$v_y(t) \approx v_{\text{ter}}t/\tau = gt$$

where to get the second equality I replaced v_{ter} by $g\tau$ as in (2.34).

(b) Putting $v_{yo} = 0$ into (2.35) and then inserting the Taylor series for the exponential, we find:

$$y(t) = v_{\text{ter}}t - v_{\text{ter}}\tau\left[1 - e^{-t/\tau}\right] = v_{\text{ter}}t - v_{\text{ter}}\tau\left[1 - \left(1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} - \cdots\right)\right].$$

On the right side, the second and third terms cancel, as do the first and fourth. If we neglect all terms beyond t^2, this leaves us with $y(t) \approx v_{\text{ter}}t^2/(2\tau) = \frac{1}{2}gt^2$, since $v_{\text{ter}} = g\tau$.

2.7 ⋆ With $F = F(v)$, Newton's second law, $F = m\,dv/dt$, can be rewritten as $dt = m\,dv/F(v)$, which can be integrated to give the advertised result. If $F = F_o$, a constant, the integral gives $t = (m/F_o)(v - v_o)$ or $v = v_o + at$, where $a = F_o/m$. This is the well known kinematic formula for v when a is constant.

2.8 ⋆ $t = m\displaystyle\int_{v_o}^{v} \frac{dv'}{-cv'^{3/2}} = \frac{2m}{c}\left[v'^{-1/2}\right]_{v_o}^{v} = \frac{2m}{c}\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{v_o}}\right)$

or, solving for v, $v = v_o/(1 + ct\sqrt{v_o}/2m)^2$. Clearly, $v = 0$ only when $t \to \infty$.

2.9 ⋆ If we integrate the given equation, with the velocity running from v_{yo} to v_y and time from 0 to t, we find $m\ln[(v_y - v_{\text{ter}})/(v_{yo} - v_{\text{ter}})] = -bt$, or, solving for v_y,

$$v_y = v_{\text{ter}} + (v_{yo} - v_{\text{ter}})e^{-bt/m},$$

which is exactly (2.30), since $m/b = \tau$.

2.10 ⋆⋆ **(a)** The characteristic time is $\tau = m/b = m/(3\pi\eta D) = 1.4 \times 10^{-4}$ s. The buoyant force on the ball bearing is $F_b = \varrho_g V g$, where ϱ_g is the density of glycerin and V is the volume of the ball bearing. This can be rewritten as $F_b = (\varrho_g/\varrho_s)mg$, where ϱ_s is the density of steel and m is the mass of the ball bearing. This upward force must be added (with a minus sign) to the right side of the equation of motion (2.25). Therefore the terminal speed changes from $v_{\text{ter}} = mg/b$ to

$$v_{\text{ter}} = \frac{mg - F_b}{b} = \frac{mg(1 - \varrho_g/\varrho_s)}{b} = \tau g(1 - \varrho_g/\varrho_s) = 1.2 \text{ mm/s}.$$

The time to reach 95% of v_{ter} is $3\tau = 4.3 \times 10^{-4}$ s. [As you can easily check, the buoyant force changes the terminal speed but not the exponent $-t/\tau$ in Eq.(2.33).]

(b)
$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{\kappa\varrho_g A v^2}{3\pi\eta D v} = \frac{\frac{1}{4}\varrho_g(\frac{1}{4}\pi D^2)v}{3\pi\eta D} = \frac{\varrho_g D v}{48\eta} = 5 \times 10^{-6},$$

so it was an excellent approximation to neglect f_{quad}.

2.11 ⋆⋆ **(a)** Since we are now measuring y upward, the answers can be found from (2.30) and (2.35) by replacing $v_{\rm ter}$ with $-v_{\rm ter}$:

$$v_y(t) = -v_{\rm ter} + (v_{\rm o} + v_{\rm ter})e^{-t/\tau} \quad \text{and} \quad y(t) = -v_{\rm ter}t + (v_{\rm o} + v_{\rm ter})\tau(1 - e^{-t/\tau}).$$

(b) Setting $v_y = 0$ and solving for t, we find $t_{\rm top} = \tau \ln(1 + v_{\rm o}/v_{\rm ter})$. Sustituting this time into $y(t)$ we find $y_{\rm max} = [v_{\rm o} - v_{\rm ter} \ln(1 + v_{\rm o}/v_{\rm ter})]\tau$.

(c) In the vacuum $v_{\rm ter} = \infty$. Letting $v_{\rm ter} \to \infty$ in $y_{\rm max}$ and using the suggested approximation for the log term, we find

$$y_{\rm max} \to \left\{ v_{\rm o} - v_{\rm ter} \left[\frac{v_{\rm o}}{v_{\rm ter}} - \frac{1}{2}\left(\frac{v_{\rm o}}{v_{\rm ter}}\right)^2 \right] \right\} \tau = \frac{v_{\rm o}^2}{2g}$$

since the first two terms in the middle expression cancel each other and $v_{\rm ter} = g\tau$.

2.12 ⋆⋆ By the chain rule,

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}.$$

This lets us rewrite the second law, $m\dot{v} = F$, as

$$\frac{d}{dx}(v^2) = \frac{2}{m}F(x),$$

which can be integrated to give

$$v^2 - v_{\rm o}^2 = \frac{2}{m}\int_{x_{\rm o}}^{x} F(x')dx'$$

as claimed. If F is constant, this reduces to the well-known kinematic result $v^2 - v_{\rm o}^2 = 2a\,\Delta x$, where $a = F/m$ is the constant acceleration and $\Delta x = x - x_{\rm o}$.

2.13 ⋆⋆ With $F = -kx$ and $v_{\rm o} = 0$, Eq.(2.85) becomes

$$v^2 = -\frac{2k}{m}\int_{x_{\rm o}}^{x} x'dx' = \omega^2(x_{\rm o}^2 - x^2) \quad \text{or} \quad v = -\omega\sqrt{x_{\rm o}^2 - x^2} \tag{i}$$

where I have introduced the shorthand $\omega^2 = k/m$. [The second result is the square root of the first. Getting the right sign for the square root takes a little thought. Initially the velocity is clearly negative, and this is the phase of the motion I shall consider. After a while, the sign of v changes and the minus sign in (i) must be changed to a plus. Quite surprisingly, the final result is the same either way.]

Writing $v = dx/dt$ in (i), rearranging, and integrating, we find that

$$\omega t = -\int_{x_{\rm o}}^{x} dx'/\sqrt{x_{\rm o}^2 - x^2} = \arccos(x/x_{\rm o}) \quad \text{or} \quad x = x_{\rm o}\cos(\omega t),$$

which is simple harmonic motion. (To do the integral, I used the substitutions $x/x_{\rm o} = u$ and then $u = \cos\theta$.)

2.14 ⋆⋆⋆ **(a)** With $F = -F_o e^{v/V}$, the second law ($mdv/dt = F$) separates to become $me^{-v/V}dv = -F_o dt$. This is easily integrated (from time 0 to t or v_o to v) and the result solved to give

$$v = -V \ln\left(\frac{F_o t}{mV} + e^{-v_o/V}\right).$$

(b) This is zero when the argument of the log is 1, so that $t = (1 - e^{-v_o/V})mV/F_o$.

(c) Using the result that $\int \ln(x)dx = x\ln(x) - x$, we can evaluate the integral $x = \int v\,dt$ to give

$$x(t) = Vt - \frac{mV^2}{F_o}\left[\left(\frac{F_o t}{mV} + e^{-v_o/V}\right)\ln\left(\frac{F_o t}{mV} + e^{-v_o/V}\right) + \frac{v_o}{V}e^{-v_o/V}\right]$$

and, substituting the time from part (b),

$$x_{\text{max}} = \frac{mV^2}{F_o}\left[1 - e^{-v_o/V}\left(1 + \frac{v_o}{V}\right)\right].$$

2.15 ⋆ Since the only force is the projectile's weight $m\mathbf{g}$, Newton's second law implies that $\ddot{\mathbf{r}} = \mathbf{g}$ and its two components can be integrated twice to give the well-known results $x = v_{xo}t$ and $y = v_{yo}t - \frac{1}{2}gt^2$ (if we take $x_o = y_o = 0$). At landing, $y = 0$, which gives the time of flight as $t = 2v_{yo}/g$. The range is just the value of x at this time, namely $x = v_{xo}t = 2v_{xo}v_{yo}/g$.

2.16 ⋆ As usual, $x = (v_o\cos\theta)t$ and $y = (v_o\sin\theta)t - \frac{1}{2}gt^2$. The time to reach the plane of the wall ($x = d$) is $t = d/(v_o\cos\theta)$ and the ball's height at that time is $y = d\tan\theta - \frac{1}{2}gd^2/(v_o\cos\theta)^2$. Notice that this height decreases monotonically as v_o decreases. Thus there is indeed a minimum speed $v_o(\text{min})$ for which the ball clears the wall. Putting $y = h$ and solving for v_o we find that

$$v_o(\text{min}) = \sqrt{\frac{gd^2}{2(d\tan\theta - h)\cos^2\theta}}.$$

If $\tan\theta < h/d$, the argument of the square root is negative and there is no real $v_o(\text{min})$; physically, the ball's initial velocity is aimed below the top of the wall, so the ball cannot possibly clear the wall whatever its speed. With the given numbers, $v_o(\text{min}) = 26.4$ m/s or roughly 50 mi/hr.

2.17 ⋆ From the first of Equations (2.36) we find that $1 - e^{-t/\tau} = x(t)/(v_{xo}\tau)$ and hence $t = -\tau\ln(1 - x/v_{xo}\tau)$. Substituting these into the second of (2.36), we find

$$y = \frac{v_{yo} + v_{\text{ter}}}{v_{xo}}x + v_{\text{ter}}\tau\ln\left(1 - \frac{x}{v_{xo}\tau}\right)$$

which is (2.37).

2.18 $\star$ **(a)** If $f(x) = \ln(x)$, then, as you can easily check, $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, and $f^{(n)}(1) = (-1)^{n-1}(n-1)!$, so

$$\ln(1+\delta) = \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} - \frac{\delta^4}{4} + \cdots .$$

(b) If $f(x) = \cos(x)$, then $f(0) = 1$, $f'(0) = -\sin(0) = 0$, $f''(0) = -\cos(0) = -1$, $f'''(0) = \sin(0) = 0$, and so on. Thus

$$\cos(\delta) = 1 - \frac{\delta^2}{2!} + \frac{\delta^4}{4!} + \cdots .$$

(c) Similarly, $\sin(\delta) = \delta - \frac{\delta^3}{3!} + \frac{\delta^5}{5!} + \cdots$ and **(d)** $e^\delta = 1 + \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \cdots$.

2.19 $\star$ **(a)** In the absence of air resistance, we know that $x = v_{xo}t$ and $y = v_{yo}t - \frac{1}{2}gt^2$. If we solve the first of these to give $t = x/v_{xo}$ and then substitute into the second, we find

$$y = \frac{v_{yo}}{v_{xo}}x - \tfrac{1}{2}g\left(\frac{x}{v_{xo}}\right)^2,$$

which is the equation of a parabola.

(b) As air resistance is switched off, $\tau \to \infty$, and the second term inside the log term of (2.37) becomes small. Thus we can use the Taylor series (2.40) for the log,

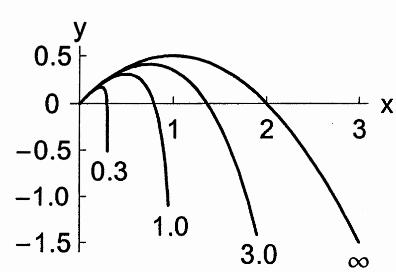
$$\ln\left(1 - \frac{x}{v_{xo}\tau}\right) = -\frac{x}{v_{xo}\tau} - \frac{1}{2}\left(\frac{x}{v_{xo}\tau}\right)^2 - \cdots,$$

in (2.37). For τ sufficiently large, we can neglect all remaining terms in this series and (2.37) becomes

$$y \approx \frac{v_{yo} + v_{\text{ter}}}{v_{xo}}x - v_{\text{ter}}\tau\left(\frac{x}{v_{xo}\tau} + \frac{1}{2}\frac{x^2}{v_{xo}^2\tau^2}\right).$$

The second and third terms on the right cancel, and, if we replace v_{ter} by $g\tau$, the two remaining terms give precisely the answer to part (a).

2.20 $\star\star$ The figure shows the trajectory for four different values of drag, given by $\tau = 0.3$, $1.0, 3.0$ and ∞ (the last meaning no resistance at all) all for the same time interval, $0 \le t \le 3$.



2.21 ⋆⋆⋆ The problem is axially symmetric about the z axis, so we can focus on any one direction ϕ. To be definite, let's take $\phi = 0$. If the gun is fired at an angle θ above the horizontal, then its trajectory is

$$z = v_o t \sin\theta - \tfrac{1}{2}gt^2 \qquad \text{and} \qquad \rho = v_o t \cos\theta.$$

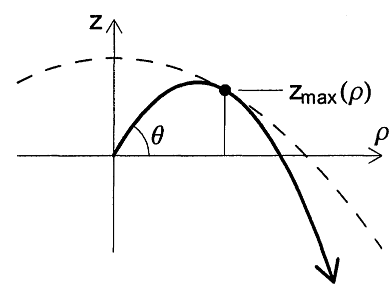
Solving the second equation for t and substituting into the first, we get the equation of the trajectory:

$$z = \rho \tan\theta - \frac{g\rho^2}{2v_o^2}\sec^2\theta.$$

This is the solid curve in the figure. For any distance ρ from the gun, the highest the shell can go, z_{max}, is found by differentiating z with respect to θ and finding where $\partial z/\partial\theta = 0$. This gives $\tan\theta = v_o^2/g\rho$. (You can easily check that $\partial^2 z/\partial\theta^2 < 0$, so this is a maximum of z.) Substituting into the expression for z, we find

$$z_{\text{max}}(\rho) = \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2.$$

This is the highest the shell can reach at the distance ρ. The locus of z_{max} is the dashed parabola shown. By changing the angle θ we can get the shell to pass through any point lower than z_{max}; therefore, all points below the dashed curve are accessible to the gun. When the parabola is rotated about the z axis we get a paraboloid of revolution. Its vertex is at $z = v_o^2/2g$ (well known as the maximum height of a projectile launched vertically up) and its radius in the plane $z = 0$ is $\rho = v_o^2/g$ (well known to be the maximum horizontal range of a gun with muzzle speed v_o).



2.22 ⋆⋆⋆ **(a)** The range in vacuum is $R = 2v_{xo}v_{xo}/g = v_o^2\sin(2\theta)/g$. This is maximum at $\theta = \pi/4$ and $R_{\text{max}} = v_o^2/g$, or $R_{\text{max}} = 1$ in the units suggested.

(b) Solving (2.39) for R with $\theta = 0.75$, you should find $R = 0.50$, about half the value in a vacuum.

(c)

θ	0.4	0.5	0.6	0.7	0.8
R	0.463	0.499	0.514	0.508	0.487

(d)

θ	0.58	0.59	0.60	0.61	0.62	0.63	0.64
R	0.51244	0.51313	0.51362	0.51393	0.51404	0.51397	0.51372

The maximum range occurs at $\theta = 0.62$ rad (about 36°, as compared with 45° in the vacuum) with $R_{\text{max}} = 0.514$ (as compared to $R_{\text{max}} = 1$ in the vacuum).

2.23 ⋆ According to Eq.(2.59), $v_{\rm ter} = \sqrt{mg/\gamma D^2}$. Since $m = \frac{4}{3}\pi R^3\varrho = \frac{1}{6}\pi D^3\varrho$, we can eliminate either m or D to give

$$v_{\rm ter} = \sqrt{\frac{\pi D\varrho g}{6\gamma}} = \left(\frac{\pi\varrho}{6m}\right)^{1/3}\sqrt{\frac{mg}{\gamma}}. \tag{ii}$$

In all three cases, $g = 9.8$ m/s^2 and $\gamma = 0.25$ kg/m^3.

(a) With $D = 3$ mm and $\varrho = 8$ g/cm^3, the second expression in Eq.(ii) gives $v_{\rm ter} = 22$ m/s.

(b) With $m = 16 \times 0.454 = 7.26$ kg and $\varrho = 8$ g/cm^3, the third expression in Eq.(ii) gives $v_{\rm ter} = 140$ m/s.

(c) With $m = 200 \times 0.454 = 90.8$ kg and $\varrho = 1$ g/cm^3, the third expression in Eq.(ii) gives $v_{\rm ter} = 107$ m/s.

2.24 ⋆ **(a)** For a sphere the drag force is $f_{\rm quad} = \frac{1}{4}\varrho_{\rm air}Av^2$, and at the terminal speed this must equal mg. Therefore

$$v_{\rm ter} = \sqrt{\frac{mg}{\frac{1}{4}\varrho_{\rm air}A}} = \sqrt{\frac{(\frac{1}{6}\pi D^3)\varrho_{\rm sph}g}{\frac{1}{4}\varrho_{\rm air}(\frac{1}{4}\pi D^2)}} = \sqrt{\frac{8}{3}Dg\frac{\varrho_{\rm sph}}{\varrho_{\rm air}}}$$

(b) For two spheres with $D_1 = D_2$ but $\varrho_1 > \varrho_2$, we see that $v_{\rm ter1} > v_{\rm ter2}$.

(c) If $\varrho_1 = \varrho_2$ but $D_1 > D_2$, then $v_{\rm ter1} > v_{\rm ter2}$.

2.25 ⋆ The steps of the derivations are already given in almost complete detail in the text between (2.47) and (2.51). Since the drag force is cv^2, the dimensions of c are $[c] = [F/v^2] = MLT^{-2}/(L/T)^2 = ML^{-1}$. Therefore

$$[\tau] = \left[\frac{m}{cv_{\rm o}}\right] = \frac{M}{(ML^{-1})(L/T)} = T.$$

2.26 ⋆ Putting in the numbers for the characteristic time, we find

$$\tau = m/cv_{\rm o} = 80/(0.20 \times 20) = 20 \text{ s}.$$

This tells us that, coasting from an initial 20 m/s, he will slow to half his initial speed (namely, to 10 m/s) in 20 seconds. From (2.45) we find that the time to slow to any speed v is

$$t = \frac{m}{c}\left(\frac{1}{v} - \frac{1}{v_{\rm o}}\right),$$

which gives

v (m/s)	15	10	5
t (sec)	6.7	20.0	60.0

Notice that in this approximation the time to come to rest ($v = 0$) is infinite; that is, the cyclist never comes to rest. This underscores that to ignore ordinary friction at very slow speeds is a very bad approximation.

2.27 ⋆ If we choose our x axis pointing straight up the slope, then for the upward journey the x component of the second law reads

$$m\dot{v} = -cv^2 - mg\sin\theta = -c(v^2 + v_{\text{ter}}^2)$$

where v denotes the x component of the velocity and I have introduced the terminal speed for the puck on the incline, defined so that $v_{\text{ter}}^2 = (mg\sin\theta)/c$. If we write this in the separated form $m\,dv/(v^2 + v_{\text{ter}}^2) = -c\,dt$, we can integrate both sides (the left side from v_o to v and the right from 0 to t) to give

$$\frac{m}{v_{\text{ter}}}[\arctan(v/v_{\text{ter}}) - \arctan(v_o/v_{\text{ter}})] = -ct \qquad \text{(iii)}$$

which can be solved to give

$$v = v_{\text{ter}}\tan\big(\arctan(v_o/v_{\text{ter}}) - cv_{\text{ter}}t/m\big).$$

Putting $v = 0$ in Eq.(iii), we find that the time to reach the top is $t = (m/cv_{\text{ter}})\arctan(v_o/v_{\text{ter}})$.

2.28 ⋆ The equation of motion in the suggested form reads $-mv^{-1/2}dv = c\,dx$. Integrating this (from 0 to x on the right and v_o to v on the left), we find that $x = (2m/c)(\sqrt{v_o} - \sqrt{v})$. The puck comes to rest when $v = 0$ and $x = (2m/c)\sqrt{v_o}$.

2.29 ⋆ According to Eq.(2.57), the actual speed is $v = v_{\text{ter}}\tanh(gt/v_{\text{ter}})$, whereas the speed in a vacuum would be just $v_{\text{vac}} = gt$. Putting $v_{\text{ter}} = 50$ m/s and $g = 9.8$ m/s^2, we get the following results:

time (s)	0	1	5	10	20	30
actual speed (m/s)	0	9.7	38	48	50	50
speed in vacuum (m/s)	0	9.8	49	98	196	294

2.30 ⋆ According to Eq.(2.59), $v_{\text{ter}} = \sqrt{mg/\gamma D^2}$, so

$$D = \frac{1}{v_{\text{ter}}}\sqrt{\frac{mg}{\gamma}} \approx \frac{1}{50 \text{ m/s}}\sqrt{\frac{(70 \text{ kg}) \times (9.8 \text{ m/s}^2)}{0.25 \text{ kg/m}^3}} \approx 1 \text{ m}$$

which seems about right.

2.31 ⋆⋆ **(a)** Using (2.59), we find the basketball's terminal speed is

$$v_{\text{ter}} = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{(0.60 \text{ kg}) \times (9.8 \text{ m/s}^2)}{(0.25 \text{ N·s}^2/\text{m}^4) \times (0.24 \text{ m})^2}} = 20.2 \text{ m/s}.$$

(b) Solving (2.58), we find that

$$t = (v_{\text{ter}}/g)\,\text{arccosh}[\exp(yg/v_{\text{ter}}^2)] = 2.78 \text{ s}$$

where y is the height of the tower ($y = 30$ m). This is to be compared with $t = \sqrt{2y/g} =$ 2.47 s in vacuum. According to (2.57), the speed at landing is

$$v = v_{\text{ter}} \tanh(gt/v_{\text{ter}}) = 17.7 \text{ m/s},$$

less than, though fairly close to, the terminal speed. In a vacuum the corresponding speed is $v = \sqrt{2gy} = 24.2$ m/s.

2.32 ⋆⋆ **(a)** The terminal speed is defined as the speed at which the drag of air resistance is equal to the body's weight. If $v \ll v_{\text{ter}}$, then the drag should be much less than the weight and hence not very important.

(b) From Eqs.(2.26) and (2.53) you can easily find the ratios of drag to weight to be

$$\frac{f_{\text{lin}}}{mg} = \frac{v}{v_{\text{ter}}} \qquad \text{whereas} \qquad \frac{f_{\text{quad}}}{mg} = \left(\frac{v}{v_{\text{ter}}}\right)^2.$$

As expected, both ratios are equal to 1 when $v = v_{\text{ter}}$, but if we reduce v below v_{ter}, the ratio for the linear case drops like v whereas that for the quadratic case drops like v^2. That is, the quadratic drag diminishes more quickly. (For instance, if $v = 0.1v_{\text{ter}}$, the linear ratio is $f_{\text{lin}}/mg = 0.1$, but the quadratic ratio is $f_{\text{quad}}/mg = 0.01$.)

2.33 ⋆⋆ **(a)** Note that when z is large and positive,

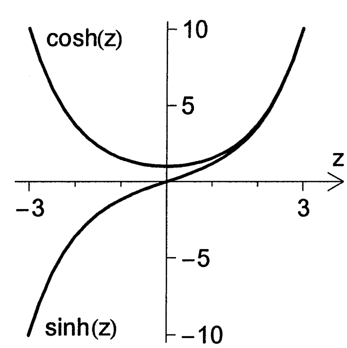
$\cosh z \approx \sinh z \approx e^z/2.$

Similarly, when z is large and negative,

$\cosh z \approx -\sinh z \approx e^{-z}/2.$

Also

$\cosh(0) = 1 \quad \text{and} \quad \sinh(0) = 0.$



(b) $\cos(iz) = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2} = \dfrac{e^{-z} + e^{z}}{2} = \cosh(z).$

Similarly, $\sinh(z) = -i\sin(iz)$.

(c) $\dfrac{d}{dz}\cosh(z) = \dfrac{d}{dz}\dfrac{e^z + e^{-z}}{2} = \dfrac{e^z - e^{-z}}{2} = \sinh(z)$, and likewise $\dfrac{d}{dz}\sinh(z) = \cosh(z)$.

Integrating these two results, we find that

$$\int \sinh(z)dz = \cosh(z) \qquad \text{and} \qquad \int \cosh(z)dz = \sinh(z)$$

(d) $\cosh^2(z) - \sinh^2(z) = [\cos(iz)]^2 - [-i\sin(iz)]^2 = [\cos(iz)]^2 + [\sin(iz)]^2 = 1$

(e) If we make the substitution $x = \sinh(z)$, then

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh z\, dz}{\sqrt{1+\sinh^2 z}} = \int dz = z = \operatorname{arcsinh}(x).$$

2.34 ⋆⋆ **(a)** $\tanh z = \dfrac{\sinh z}{\cosh z} = \dfrac{-i\sin(iz)}{\cos(iz)} = -i\tan(iz)$

(b) $\dfrac{d}{dz}\tanh z = \dfrac{d}{dz}\left(\dfrac{\sinh z}{\cosh z}\right) = \dfrac{\cosh z}{\cosh z} - \sinh z\,\dfrac{\sinh z}{\cosh^2 z} = \dfrac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \dfrac{1}{\cosh^2 z}$.

This is the same as $\text{sech}^2 z$ if we define $\text{sech} z = 1/(\cosh z)$.

(c) $\displaystyle\int \tanh(z)dz = \int \frac{\sinh(z)\,dz}{\cosh(z)} = \int \frac{d(\cosh z)}{\cosh z} = \ln\cosh z.$

(d) $1 - \tanh^2 z = 1 - \dfrac{\sinh^2 z}{\cosh^2 z} = \dfrac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \dfrac{1}{\cosh^2 z} = \text{sech}^2 z.$

(e) The substitution $x = \tanh z$ gives

$$\int \frac{dx}{1-x^2} = \int \frac{\text{sech}^2 z\,dz}{1-\tanh^2 z} = \int dz = z = \text{arctanh}x.$$

2.35 ⋆⋆ **(a)** Using (2.53) we can replace c by $c = mg/v_{\text{ter}}^2$ in (2.52) to give (2.54) and thence (2.55)

$$\frac{dv}{1-(v/v_{\text{ter}})^2} = g\,dt$$

which we can now integrate from time 0 to t to give

$$v_{\text{ter}}\int_0^{v/v_{\text{ter}}} \frac{du}{1-u^2} = gt.$$

(Here I made the substitution $v/v_{\text{ter}} = u$, so that $dv = v_{\text{ter}}\,du$.) The integral on the left can be done several ways. One route is to substitute $u = \tanh w$, so that $du = \text{sech}^2 w\,dw$ and $1 - u^2 = \text{sech}^2 w$, and the integral becomes $v_{\text{ter}}\int dw$. Thus

$$v_{\text{ter}}\,\text{arctanh}(v/v_{\text{ter}}) = gt$$

which can be solved at once for v to give (2.57).

To find the position $y(t)$ we have only to integrate (2.57) for $v(t)$. Substitute $u = gt/v_{\text{ter}}$, and all you have to do is integrate $\tanh u$. The simple way to do this is to write $\tanh u = \sinh u/\cosh u$ and then $w = \cosh u$, so that $(\tanh u)\,du = dw/w$. The required integral is now just $\int dw/w$, and you get (2.58) directly.

(b) If $\tau = v_{\text{ter}}/g$, we can rewrite the two equations as

$$v(t) = v_{\text{ter}}\tanh(t/\tau) \qquad \text{and} \qquad y(t) = v_{\text{ter}}\tau\ln[\cosh(t/\tau)].$$

The first of these gives the following values for $v(t)$

time t	τ	2τ	3τ
velocity $v(t)$	$0.76\,v_{\text{ter}}$	$0.96\,v_{\text{ter}}$	$0.995\,v_{\text{ter}}$
percent	76%	96%	99.5%

(c) If $x \gg 1$, then $\cosh x \approx \frac{1}{2}e^x$. Thus if $t \gg \tau$, then

$$y(t) \approx v_{\rm ter}\tau \ln\left(\tfrac{1}{2}e^{t/\tau}\right) = v_{\rm ter}\tau\,(t/\tau - \ln 2) = v_{\rm ter}t + \text{const}.$$

That is, the ball eventually falls with constant velocity.

(d) If $x \ll 1$, then $\cosh x \approx 1 + \frac{1}{2}x^2$. Thus if $t \ll \tau$, the expression of part (b) for $y(t)$ becomes

$$y(t) \approx v_{\rm ter}\tau \ln[1 + \tfrac{1}{2}(t/\tau)^2] \approx v_{\rm ter}\tau[\tfrac{1}{2}(t/\tau)^2] = \tfrac{1}{2}gt^2$$

where in the last step I replaced $v_{\rm ter}$ by $g\tau$. That is, at first, air resistance is negligible, and the ball falls freely.

2.36 ⋆⋆ **(a)** Using Eq.(2.59) you can check that $v_{\rm ter} = (\pi\varrho/6m)^{1/3}\sqrt{mg/\gamma}$. Putting in the numbers, with $m_1 = 100$ and $m_2 = 1$ lb (mass), we find that $v_{\rm ter1} = 191$ and $v_{\rm ter2} = 130$ m/s.

(b) Solving Eq.(2.58) for t we find for the time to drop through a height h,

$$t = (v_{\rm ter}/g)\operatorname{arccosh}(e^{hg/v_{\rm ter}^2}).$$

Putting in the numbers for ball 1, we find the time for it to travel the distance $h = 100$ cubits $= 60.96$ m is $t_1 = 3.537$ s. The distance y_2 traveled by ball 2 in this time as given by Eq.(2.58) is $y_2 = 60.58$ m. Thus when ball 1 lands, ball 2 still has a distance $h - y_2 = 38$ cm to go — a little more than a foot.

2.37 ⋆⋆ Starting from (2.55) and substituting $u = v/v_{\rm ter}$, we get $du/(1-u^2) = g\,dt/v_{\rm ter}$. Separating into partial fractions as suggested and integrating from time 0 to t, we find

$$\frac{1}{2}\int_0^u \left(\frac{1}{1+u'} + \frac{1}{1-u'}\right) du' = \frac{gt}{v_{\rm ter}} = T,$$

say. Thus $\ln(1+u)/(1-u) = 2T$ and hence $(1+u)/(1-u) = e^{2T}$. Solving for u, we find

$$u = \frac{e^{2T}-1}{e^{2T}+1} = \frac{e^T - e^{-T}}{e^T + e^{-T}} = \frac{\sinh T}{\cosh T} = \tanh T.$$

Since $u = v/v_{\rm ter}$ and $T = gt/v_{\rm ter}$, this is the same as (2.57).

2.38 ⋆⋆ **(a)** The equation of motion for the upward journey (with v measured upward) is

$$m\dot{v} = -mg - cv^2 \quad \text{or} \quad \dot{v} = -g(1 + v^2/v_{\rm ter}^2)$$

since $cv_{\rm ter}^2 = mg$. If we make the substitution $u = v/v_{\rm ter}$, this equation separates as $du/(1+u^2) = -(g/v_{\rm ter})dt$, which can be integrated to give

$$\arctan\left(\frac{v_o}{v_{\rm ter}}\right) - \arctan\left(\frac{v}{v_{\rm ter}}\right) = \frac{g}{v_{\rm ter}}t. \qquad \text{(iv)}$$

Therefore

$$v = v_{\text{ter}} \tan\left[\arctan\left(\frac{v_o}{v_{\text{ter}}}\right) - \frac{g}{v_{\text{ter}}}t\right].$$

(b) From Eq.(iv), $v = 0$ at the time $t_{\text{top}} = (v_{\text{ter}}/g)\arctan(v_o/v_{\text{ter}})$.

(c) With $v_{\text{ter}} = 35$ m/s and the given values of v_o, the times given by part (b) and the corresponding vacuum times ($t_{\text{vac}} = v_o/g$) are

v_o (m/s)	1	10	20	30	40
actual time (s)	0.102	0.99	1.85	2.53	3.04
time in vacuum (s)	0.102	1.02	2.04	3.06	4.08

2.39 ⋆⋆ **(a)** The equation of motion is $m\dot{v} = -f_{\text{fr}} - cv^2$, which separates to give

$$-m\frac{dv}{f_{\text{fr}} + cv^2} = dt.$$

This can be integrated from time 0 to t (and velocity from v_o to v). The integral over v gives an arctan function. (Make the substitutions $cv^2/f_{\text{fr}} = u^2$ and then $u = \tan w$.) The result is

$$t = \frac{m}{\sqrt{f_{\text{fr}}c}}\left(\arctan\sqrt{\frac{c}{f_{\text{fr}}}}v_o - \arctan\sqrt{\frac{c}{f_{\text{fr}}}}v\right).$$

(b) Putting in the numbers, with $v_o = 20$ m/s and the four given final velocities $v = 15, 10, 5,$ and 0 m/s, we find the following corresponding times:

v (m/s)	15	10	5	0
t (s)	6.3	18.4	48.3	142

The corresponding times if we neglect friction are (from Problem 2.26) 6.7, 20.0, 60.0, and ∞. To neglect friction, compared to the quadratic air resistance, is quite good at higher speeds, but terrible at very low speeds.

2.40 ⋆⋆ Newton's second law reads $m\dot{v} = -bv - cv^2$. This separates as $m\,dv/(bv + cv^2) = -dt$, which can be integrated from 0 to t to give

$$\frac{m}{b}\ln\left(\frac{v}{b + cv}\right) = -t + \text{const}.$$

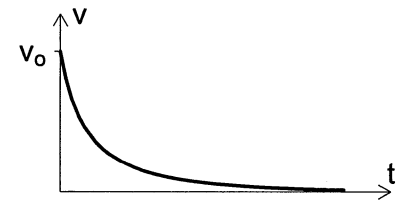
This can be solved to give $v/(b + cv) = Ae^{-bt/m}$, where A is another constant equal to $A = v_o/(b + cv_o)$, and thence

$$v = \frac{bAe^{-bt/m}}{1 - cAe^{-bt/m}}.$$

When $t \to \infty$, the second term in the denominator can be neglected and $v \approx (\text{const})\,e^{-bt/m}$. That is, v approaches zero exponentially, just as it would do if the drag were purely linear.

[Compare Equation (2.20).] This is because, as $v \to 0$, the quadratic force becomes negligible compared to the linear.

Although it is hard to see it, this graph decays exponentially for large t (corresponding to the dominant linear drag), but decays more steeply for small t (corresponding to the dominant quadratic drag).



2.41 ⋆⋆ On the upward journey the velocity v (measured upward) satisfies $m\dot{v} = -mg - cv^2$. Since the terminal speed satisfies $cv_{\text{ter}}^2 = mg$, this can be rewritten as $\dot{v} = -g(1 + v^2/v_{\text{ter}}^2)$, or, using the "$vdv/dx$" rule, $d(v^2)/(1 + v^2/v_{\text{ter}}^2) = 2\,g\,dy$. Integrating this, from v_o to v on the left and 0 to y on the right, we find that

$$v_{\text{ter}}^2 \left[\ln(v^2 + v_{\text{ter}}^2) - \ln(v_o^2 + v_{\text{ter}}^2)\right] = 2gy, \tag{v}$$

whence $v = \sqrt{(v_o^2 + v_{\text{ter}}^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2}$. Putting $v = 0$ in (v) gives y_{max} exactly as claimed in Eq.(2.89). With the given numbers this yields $y_{\text{max}} = 17.7$ m, compared with the vacuum value, $y_{\text{max}} = v_o^2/2g = 20.4$ m.

2.42 ⋆⋆ If we continue to measure y upward as in Problem 2.41, the equation of motion for the downward journey is $m\dot{v} = -mg + cv^2 = -mg(1 - v^2/v_{\text{ter}}^2)$. Using the "$vdv/dx$" rule (again as in Problem 2.41), this becomes $d(v^2)/(1 - v^2/v_{\text{ter}}^2) = -2\,g\,dy$. If we integrate this from the top ($v = 0$ and $y = y_{\text{max}}$) to an arbitrary position (velocity v and height y), we get

$$-v_{\text{ter}}^2 \ln(1 - v^2/v_{\text{ter}}^2) = -2g(y - y_{\text{max}})$$

whence

$$v = -v_{\text{ter}}\sqrt{1 - e^{2g(y - y_{\text{max}})/v_{\text{ter}}^2}}.$$

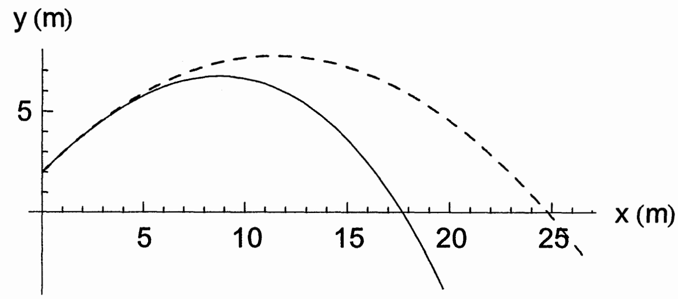
(Note that I took the negative square root since the velocity is downward.) Here y_{max} is given by Eq.(2.89) and, when the ball hits the ground, $y = 0$. Thus the velocity just before it hits the ground is $v = -v_{\text{ter}}\sqrt{1 - v_{\text{ter}}^2/(v_{\text{ter}}^2 + v_o^2)} = -v_{\text{ter}}v_o/\sqrt{(v_{\text{ter}}^2 + v_o^2)}$, as claimed. Putting in the numbers, we find that $v = -17.4$ m/s at landing. In a vacuum the speed at landing would be the same as that at launch, so v would be -20 m/s.

2.43 ⋆⋆⋆ **(a)** We have to solve the differential equations (2.61),

$$m\ddot{x} = -c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x}$$
$$m\ddot{y} = -mg - c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y}$$

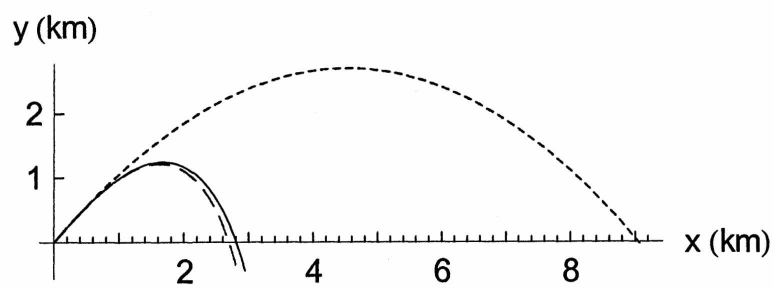
with the initial conditions $x(0) = 0$, $y(0) = 2$, $\dot{x}(0) = 15\cos(45°)$, and $\dot{y}(0) = 15\sin(45°)$. (I've chosen my origin on the ground directly below the launch point. The constant c is

$c = \gamma D^2$, as usual, with $\gamma = 0.25$ and $D = 0.24$ m.) Ignoring air resistance, I estimated that the ball would land after about 2.5 sec, so I solved the equations (using NDSolve in Mathematica) for $0 \leq t \leq 2.5$ s and plotted the solution as the continuous curve shown. The vacuum solution is $x = \dot{x}(0)t$ and $y = y(0) + \dot{y}(0)t - \frac{1}{2}gt^2$, which I have plotted as the dashed curve.



(b) From the plots, you can read off the true range as 17.7 m and the range in vacuum as 24.8 m.

2.44 ⋆⋆⋆ **(a)** The two equations to be solved are the same as for Problem 2.43, except that the constant c has to be replaced by $c(y) = \gamma D^2 \exp(-y/\lambda)$ and $D = 0.15$ m. In the absence of air resistance, the cannonball would land after a little less than 50 seconds, so I solved the equations (using NDSolve in Mathematica) for $0 \leq t \leq 50$ s. However, when I plotted the trajectory, I found that (with air resistance) the ball lands after only about 33 seconds, so I replotted the trajectory as the solid curve for $0 \leq t \leq 35$ s. From the graph we see that the horizontal range is about 2.8 km.



(b) If we wish to ignore the variation of atmospheric density, we have only to replace $c(y)$ by $c(0)$ in the equation of motion. This gives the long-dashed curve, with a range of about 2.7 km. Notice that this range is a little shorter than we found allowing for variable resistance. That it is shorter was to be expected, since the variation of density with height *reduces* the drag at high altitudes and so *increases* the range. The variation of resistance has only a small effect, since the ball only rises to about 1300 m, where the resistance hasn't dropped very much.

The vacuum trajectory is given by the well-known expressions $x = v_{ox}t$ and $y = v_{oy}t - \frac{1}{2}gt^2$ and is shown with short dashes. The range in this case turns out to be about 9.0 km — much more than either value with air resistance, confirming that air resistance is very important for an object traveling this fast.

2.45 ⋆ **(a)** Let the polar coordinates of $\mathbf{r} = (x, y)$ be r and θ, so that $x = r\cos\theta$ and $y = r\sin\theta$. Then

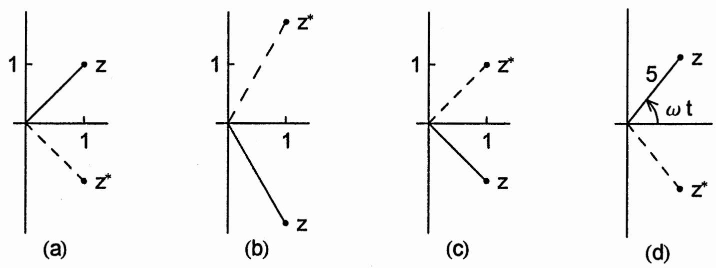
$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

(b) With $x = 3$ and $y = 4$, $r = 5$ and $\theta = 0.927$ (or about 53°).

(c) $z = 2e^{-i\pi/3} = 1 - i\sqrt{3}$.

2.46 ⋆

	z	Re(z)	Im(z)	$\lvert z\rvert$	θ	z^*
(a)	$1+i$	1	1	$\sqrt{2}$	$\pi/4$	$1-i$
(b)	$1-i\sqrt{3}$	1	$-\sqrt{3}$	2	$-\pi/3$	$1+i\sqrt{3}$
(c)	$\sqrt{2}e^{-i\pi/4}$	1	1	$\sqrt{2}$	$-\pi/4$	$\sqrt{2}e^{i\pi/4}$
(d)	$5e^{i\omega t}$	$5\cos(\omega t)$	$5\sin(\omega t)$	5	ωt	$5e^{-i\omega t}$



Note that the fourth picture is drawn to a different scale than the first three.

2.47 ⋆ **(a)** $z = 6 + 8i = 10e^{i\theta}$ and $w = 3 - 4i = 5e^{-i\theta}$, where $\theta = 0.927$ rad. (Note that the phase angles of z and w are exactly opposite — same θ in both expresions.) Therefore

$$z + w = 9 + 4i \quad \text{and} \quad z - w = 3 + 12i,$$

$$zw = (10e^{i\theta})(5e^{-i\theta}) = 50,$$

and

$$\frac{z}{w} = \frac{10e^{i\theta}}{5e^{-i\theta}} = 2e^{2i\theta} = 2\cos(2\theta) + 2i\sin(2\theta) = -0.56 + 1.92i,$$

or

$$\frac{z}{w} = \frac{zw^*}{ww^*} = \frac{(6+8i)(3+4i)}{(3-4i)(3+4i)} = \frac{-14+48i}{25} = -0.56 + 1.92i.$$

(b) $z = 8e^{i\pi/3} = 4 + 4\sqrt{3}i$ and $w = 4e^{i\pi/6} = 2\sqrt{3} + 2i$. Therefore,

$$z + w = (4 + 2\sqrt{3}) + (4\sqrt{3} + 2)i \quad \text{and} \quad z - w = (4 - 2\sqrt{3}) + (4\sqrt{3} - 2)i,$$

$$zw = (8e^{i\pi/3})(4e^{i\pi/6}) = 32e^{i\pi/2} = 32i \quad \text{and} \quad \frac{z}{w} = \frac{8e^{i\pi/3}}{4e^{i\pi/6}} = 2e^{i\pi/6} = \sqrt{3} + i.$$

2.48 ⋆ If $z = x + iy$, then $z^*z = (x - iy)(x + iy) = x^2 + y^2 = |z|^2$. Therefore, $|z| = \sqrt{z^*z}$.

2.49 ⋆ **(a)** On the one hand, $z^2 = e^{2i\theta} = \cos 2\theta + i \sin 2\theta$. On the other hand, $z^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$. Equating these two expressions, we find

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \text{and} \qquad \sin 2\theta = 2\cos\theta\sin\theta.$$

(b) In exactly the same way, we can show that

$$\cos 3\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta) \qquad \text{and} \qquad \sin 3\theta = \sin\theta(3\cos^2\theta - \sin^2\theta).$$

2.50 ⋆ $\dfrac{d}{dz}e^z = \dfrac{d}{dz}\left(1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!} + \cdots\right) = 0 + 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots = e^z.$

2.51 ⋆⋆

$$\begin{aligned}
e^z e^w &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right)\\
&= 1 + (z+w) + \frac{1}{2!}(z^2 + 2zw + w^2) + \frac{1}{3!}(z^3 + 3z^2w + 3zw^2 + w^3) + \cdots\\
&= 1 + (z+w) + \frac{1}{2!}(z+w)^2 + \frac{1}{3!}(z+w)^3 + \cdots\\
&= e^{z+w}.
\end{aligned}$$

2.52 ⋆ According to (2.77) $v_x + iv_y = ae^{i(\delta-\omega t)} = a\cos(\delta - \omega t) + i\sin(\delta - \omega t)$. Therefore,

$$v_x = a\cos(\omega t - \delta) \qquad \text{and} \qquad v_y = -a\sin(\omega t - \delta).$$

The two components of the transverse velocity $\mathbf{v}_\perp$ oscillate at the same frequency ω, exactly 90° out of phase. Since $v_x^2 + v_y^2 = a^2$, a constant, $\mathbf{v}_\perp$ rotates steadily clockwise with constant magnitude.

2.53 ⋆ The components of the force are $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = q(v_yB, -v_xB, E)$, so the three components of $m\mathbf{a} = \mathbf{F}$ are

$$m\dot{v}_x = qBv_y, \qquad m\dot{v}_y = -qBv_x, \qquad m\dot{v}_z = qE.$$

The first two of these are exactly the same as (2.64) and (2.65) for the case of no electric field, and the motion of x and y is therefore the same as in Figure 2.15: The transverse position (x, y) moves clockwise around a circle at constant angular velocity $\omega = qB/m$. The equation for v_z shows that there is a constant acceleration in the z direction, $a_z = qE/m$, so that $z = z_o + v_{zo}t + \frac{1}{2}a_zt^2$. The particle moves in a helix or spiral of constant radius around a line parallel to the z axis, with an increasing pitch as the motion in the z direction accelerates.

2.54 ⋆⋆ **(a)** If we differentiate the first of Equations (2.68), we find

$$\ddot{v}_x = \omega \dot{v}_y = -\omega^2 v_x$$

whose general solution we know to be $v_x = A\cos\omega t + B\sin\omega t$. From (2.68), $v_y = \dot{v}_x/\omega$, so the general solution according to this approach is

$$\left.\begin{aligned} v_x &= A\cos\omega t + B\sin\omega t \\ v_y &= -A\sin\omega t + B\cos\omega t \end{aligned}\right\} \qquad \text{(vi)}$$

(b) From (2.77), as disentangled in Problem 2.52, we have the general solution

$$\left.\begin{aligned} v_x &= a\cos(\delta - \omega t) = a\cos\delta\cos\omega t + a\sin\delta\sin\omega t \\ v_y &= a\sin(\delta - \omega t) = -a\cos\delta\sin\omega t + a\sin\delta\cos\omega t \end{aligned}\right\} \qquad \text{(vii)}$$

If we choose $A = a\cos\delta$ and $B = a\sin\delta$ in (vi), then solution (vi) takes exactly the form of the solution (vii). Conversely, in the solution (vii) we can take a and δ to be the hypotenuse and lower angle of a right triangle of base A and height B. With this choice, $a\cos\delta = A$ and $a\sin\delta = B$, so the solution (vii) takes exactly the form of the solution (vi).

2.55 ⋆⋆⋆ **(a)** The given fields are $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, B)$, so the force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = q(v_y B, E - v_x B, 0).$$

If we define $\omega = qB/m$ as in (2.67), the equation of motion, $\dot{\mathbf{v}} = \mathbf{F}/m$ can be written in components as

$$\dot{v}_x = \omega v_y\,, \qquad \dot{v}_y = -\omega(v_x - E/B)\,, \qquad \text{and} \qquad \dot{v}_z = 0\,. \qquad \text{(viii)}$$

Since $\dot{v}_z = 0$ and $v_{zo} = 0$ it is clear that $v_z = 0$ for all times, and the motion remains in the plane $z = 0$.

(b) The particle is undeflected if and only if both $\dot{v}_x$ and $\dot{v}_y$ remain zero for all times. From the second of Equations (viii) this requires that $v_x = E/B = v_{\text{dr}}$, say. If this condition is satisfied, then v_y remains zero, and the first of Equations (viii) implies that v_x remains constant, $v_x = v_{\text{dr}}$. The condition that $v = E/B$ is easily understood: With this velocity, the electric force qE in the $+y$ direction exactly balances the magnetic force qvB in the $-y$ direction.

(c) If we make the suggested change of variables,

$$u_x = v_x - v_{\text{dr}} \qquad \text{and} \qquad u_y = v_y$$

with $v_{\text{dr}} = E/B$, then the first two of Equations (viii) become

$$\dot{u}_x = \omega u_y \qquad \text{and} \qquad \dot{u}_y = -\omega u_x.$$

These two equations have exactly the form of Equations (2.68), and their solution is therefore $u_x + iu_y = Ae^{i\omega t}$, where, setting $t = 0$, we see that $A = v_{xo} - v_{\text{dr}}$. Rewritten in terms of $\mathbf{v}$, this solution is

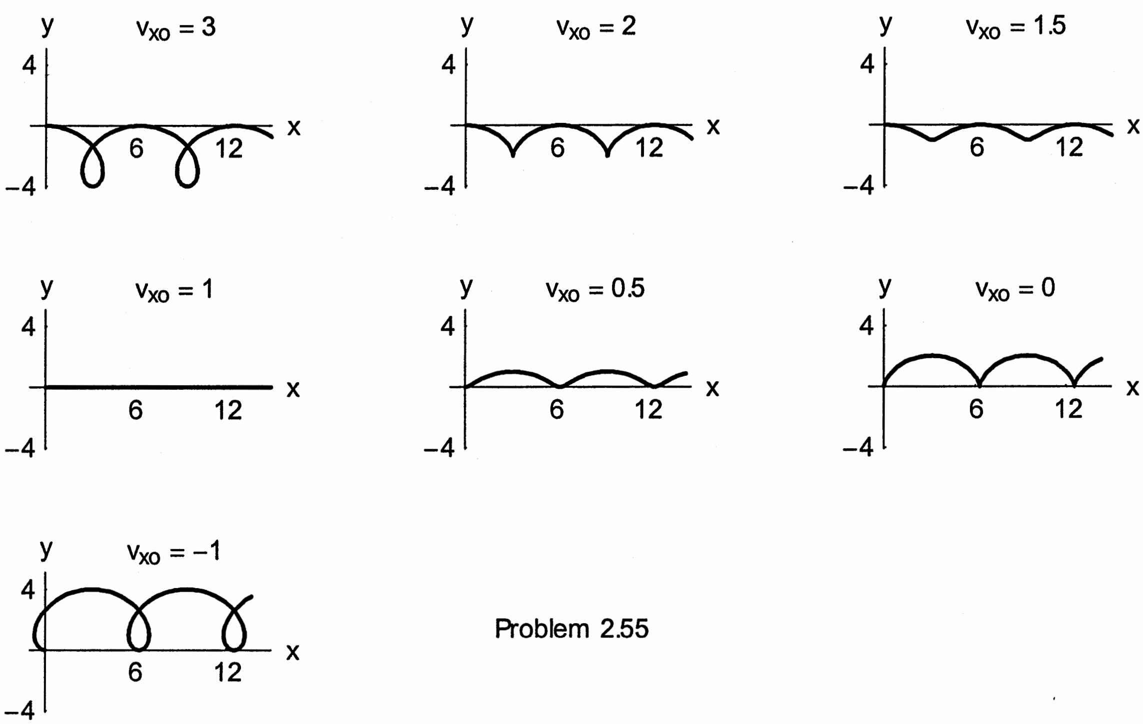
$$v_x = v_{\text{dr}} + (v_{xo} - v_{\text{dr}})\cos\omega t \qquad \text{and} \qquad v_y = -(v_{xo} - v_{\text{dr}})\sin\omega t \qquad \text{(ix)}$$

and, of course, $v_z = 0$. The transverse velocity (v_x, v_y) goes steadily around a circle of radius $(v_{xo} - v_{\rm dr})$, with a constant drift $v_{\rm dr}$ in the x direction superposed.

(d) To find the position as a function of time, we have only to integrate the equations (ix) for the velocity. Before we do this, it is helpful to define the length $R = (v_{xo} - v_{\rm dr})/\omega$, in terms of which we get

$$x = v_{\rm dr}t + R\sin\omega t \qquad \text{and} \qquad y = R(\cos\omega t - 1).$$

The point $(x, y) = R(\sin\omega t, -\cos\omega t)$ describes a circle of radius R about the origin. Thus the charge q describes a circle about the point $(v_{\rm dr}t, -R)$, which moves steadily to the right with speed $v_{\rm dr}$. The resulting curve is a cycloid, whose precise appearance depends on the initial velocity v_{xo}, as illustrated below for seven different values of v_{xo}. Notice, in particular, that if $v_{xo} = v_{\rm dr}$ then $R = 0$ and the charge drifts straight through the fields, as we already knew. (The values of v_{xo} are shown as multiples of $v_{\rm dr}$.)



Problem 2.55

Chapter 3

Momentum and Angular Momentum

I covered this chapter in 2 fifty-minute lectures.

This chapter, on momentum and angular momentum, is the shortest chapter (22 pages) in the book, in contrast with Chapter 4 on energy which is one of the longest (56 pages). I toyed with moving part of Chapter 4 into Chapter 3, but decided that the logical division belonged where it is and that there is really nothing wrong with having chapters of very different lengths. The main reason for the difference is that energy is a much more complicated notion than either kind of momentum. Another is that conservation of momentum has already been derived from Newton's third law in Chapter 1. Nevertheless, there are still some things to say about linear momentum. In Section 3.2, I use the relation $\mathbf{F}^{\text{ext}} = \dot{\mathbf{P}}$ to obtain the equation of motion of a rocket. And in Section 3.3, I introduce the center of mass, to allow me to derive the important relation $\mathbf{P} = M\dot{\mathbf{R}}$.

Section 3.4 defines the angular momentum of a single particle and derives Kepler's second law, (emphasizing that, unlike the other two Kepler laws, the second requires only that the gravitational force is central — not necessarily inverse square). And Section 3.5 proves the conservation of angular momentum for an isolated multiparticle system (emphasizing that this result requires not only the validity of Newton's third law, but also that the inter-particle forces are central).

There are several demonstration experiments related to conservation of linear and angular momentum and to the motion of rockets. You might consider using one or two to liven up the class.

Solutions to Problems for Chapter 3

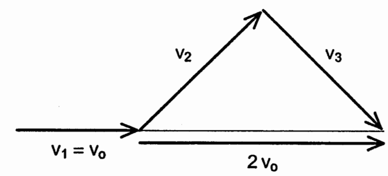
3.1 $\star$ Let v_{s} and v_{g} denote the speeds of the shell and gun relative to the ground. Then conservation of momentum implies that $mv_{\text{s}} = Mv_{\text{g}}$. (The shell and gun certainly move in opposite directions.) Their relative speed is $v = v_{\text{s}} + v_{\text{g}}$. Eliminating v_{g} between these two equations, we find that $v_{\text{s}} = Mv/(m + M)$, which is the desired result.

3.2 ⋆ Let's choose our x axis to point due north and y vertically up, and let $\mathbf{v}$ be the velocity of the second fragment just after the explosion. Then conservation of momentum implies that $\frac{1}{2}mv_x = mv_o$ and $\frac{1}{2}mv_y = -\frac{1}{2}mv_o$. Therefore, $\mathbf{v} = (2v_o, -v_o, 0)$ and the requested velocity is $\sqrt{5}v_o$ north at an angle $\theta = \arctan(1/2) = 26.6°$ below the horizontal.

3.3 ⋆ Let the original mass of the shell be $3m$ (so that each of the fragments has mass m). Conservation of momentum implies that $m\mathbf{v}_1 + m\mathbf{v}_2 + m\mathbf{v}_3 = 3m\mathbf{v}_o$. Since $\mathbf{v}_1 = \mathbf{v}_o$, this simplifies to $\mathbf{v}_2 + \mathbf{v}_3 = 2\mathbf{v}_o$. Squaring both sides and recalling that $\mathbf{v}_2$ is orthogonal to $\mathbf{v}_3$, we find that $v_2^2 + v_3^2 = 4v_o^2$, or, since $v_2 = v_3$,

$$v_2 = v_3 = \sqrt{2}\,v_o\,.$$

Notice that $\mathbf{v}_2$ and $\mathbf{v}_3$ are at 45° on either side of the initial direction.



3.4 ⋆⋆ **(a)** Let v be the speed of recoil of the flatcar, so that $u - v$ is the speed of either hobo relative to the ground just after they jump. Conservation of momentum implies that $2m_h(u - v) = m_{fc}v$, from which we find

$$v = \frac{2m_h}{2m_h + m_{fc}}u. \tag{i}$$

(b) Let v' be the recoil speed of the flatcar after the first hobo jumps and v'' that after the second. Conservation of momentum in the first jump works just as in part (a) (except that only one hobo jumps) and we conclude that

$$v' = \frac{m_h}{2m_h + m_{fc}}u\,. \tag{ii}$$

The second jump is more complicated because the flatcar is already moving with speed v'. In this case, conservation of momentum implies that

$$m_h(u - v'') - m_{fc}v'' = -(m_h + m_{fc})v'$$

or

$$v'' = \frac{m_h u + (m_h + m_{fc})v'}{m_h + m_{fc}} = \frac{3m_h + 2m_{fc}}{2m_h + 2m_{fc}}\,\frac{2m_h}{2m_h + m_{fc}}u = \frac{3m_h + 2m_{fc}}{2m_h + 2m_{fc}}v$$

where the second equality results from substitution of Eq.(ii) (plus some algebra) and the last one from use of Eq.(i). Clearly $v'' > v$, so the second procedure gives the larger final recoil velocity.

3.5 ⋆⋆ By conservation of momentum, $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$. Since $m_1 = m_2$ and $\mathbf{v}_2 = 0$, this reduces to

$$\mathbf{v}_1 = \mathbf{v}'_1 + \mathbf{v}'_2. \tag{iii}$$

Similarly, that the collision is elastic implies that $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$, whence

$$v_1^2 = v_1'^2 + v_2'^2. \tag{iv}$$

Squaring Equation (iii), we find that

$$v_1^2 = v_1'^2 + v_2'^2 + 2\mathbf{v}_1'\cdot\mathbf{v}_2',$$

and comparing this with Equation (iv), we conclude that $\mathbf{v}_1' \cdot \mathbf{v}_2' = 0$. That is $\mathbf{v}_1'$ is perpendicular to $\mathbf{v}_2'$ (except when one of them is zero, in which case the angle between them is undefined).

3.6 ⋆ Thrust $= -\dot{m}v_{\text{ex}} = (15{,}000 \text{ kg/s}) \times (2500 \text{ m/s}) = 3.75 \times 10^7 \text{ N} \approx 4200$ tons — more than, though not vastly more than, the vehicle's weight.

3.7 ⋆ Since $v_o = 0$, the final velocity as given by (3.8) is

$$v = v_{\text{ex}} \ln(m_o/m) = 3000 \ln(2) \approx 2100 \text{ m/s}.$$

The thrust is given by (3.7) as

$$\text{thrust} = -\dot{m}v_{\text{ex}} = \frac{m_o - m}{t} v_{\text{ex}} \approx 2.5 \times 10^7 \text{ N}.$$

This is just a little bigger than the initial weight, $m_o g \approx 2.0 \times 10^7$ N.

3.8 ⋆ **(a)** The condition for the rocket to hover is that $-\dot{m}v_{\text{ex}} = mg$. This requires that $-dm/m = g\,dt/v_{\text{ex}}$, which integrates to give $-\ln(m/m_o) = gt/v_{\text{ex}}$. The maximum time occurs when $m = (1-\lambda)m_o$, so $t_{\text{max}} = -\ln(1-\lambda)v_{\text{ex}}/g$.

(b) If $\lambda = 0.1$ and $v_{\text{ex}} = 3000$ m/s, this gives $t_{\text{max}} = 32$ seconds.

3.9 ⋆ From the data of Problem 3.7, $m_o = 2 \times 10^6$ kg, and $\dot{m} \approx -8.33 \times 10^3$ kg/s. The minimum exhaust speed is determined by the condition: thrust $= -\dot{m}v_{\text{ex}} = m_o g$, which gives $v_{\text{ex}} = -m_o g/\dot{m} \approx 2350$ m/s, compared with the actual value of about 3000 m/s.

3.10 ⋆ According to Eq.(3.8), $v = v_{\text{ex}} \ln(m_o/m)$. Therefore, $p = mv = mv_{\text{ex}} \ln(m_o/m)$ and

$$\dot{p} = v_{\text{ex}}[\dot{m} \ln(m_o/m) - m(\dot{m}/m)] = -\dot{m}v_{\text{ex}}[1 - \ln(m_o/m)].$$

Bearing in mind that $\dot{m}$ is negative, we see that $\dot{p}$ is initially positive but drops to 0 when $\ln(m_o/m) = 1$ and is negative thereafter. Therefore, p is maximum when $\ln(m_o/m) = 1$ or $m = m_o/e$.

3.11 ★★ **(a)** From (3.4) we know that the change in momentum (of rocket plus ejected fuel) in time dt is $dP = m\,dv + dm\,v_{\text{ex}}$. On the other hand we know on general grounds that $\dot{P} = F^{\text{ext}}$, so that $dP = F^{\text{ext}}dt$. Equating these two expressions for dP, we conclude that

$$m\,dv + dm\,v_{\text{ex}} = F^{\text{ext}}dt$$

or, dividing by dt and rearranging,

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}.$$

(b) With $-\dot{m} = k$ (a positive constant) and $F^{\text{ext}} = -mg$, the equation of motion becomes

$$m\dot{v} = kv_{\text{ex}} - mg$$

and since $\dot{m} = -k$ it follows that $m = m_{\text{o}} - kt$. The differential equation separates to give

$$dv = \left(\frac{kv_{\text{ex}}}{m_{\text{o}} - kt} - g\right) dt$$

which can be integrated from time 0 to t (and velocity 0 to v) to give

$$v \;=\; v_{\text{ex}} \ln\left(\frac{m_{\text{o}}}{m_{\text{o}} - kt}\right) - gt \;=\; v_{\text{ex}} \ln\left(\frac{m_{\text{o}}}{m}\right) - gt.$$

(c) Putting in the numbers ($v_{\text{ex}} = 3000$ m/s, $m_{\text{o}}/m = 2$, $g = 9.8$ m/s^2, and $t = 120$ s), we find $v = 900$ m/s. With $g = 0$, the corresponding number is 2100 m/s, so gravity reduces the speed acquired in the first two minutes to a little less than half its weight-free value.

(d) If the thrust kv_{ex} is less than the weight mg, the rocket will just sit on the ground until it has shed enough mass that the thrust *can* overcome the weight. Not a good design!

3.12 ★★ **(a)** If it uses all the fuel in a single burn, then according to (3.8) the final speed is

$$v = v_{\text{ex}} \ln\left(\frac{m_{\text{o}}}{0.4m_{\text{o}}}\right) = v_{\text{ex}} \ln(2.5) = 0.92v_{\text{ex}}.$$

(b) After the first stage the speed is

$$v_1 = v_{\text{ex}} \ln\left(\frac{m_{\text{o}}}{0.7m_{\text{o}}}\right) = v_{\text{ex}} \ln\left(\frac{1}{0.7}\right)$$

and after the second stage it is

$$v_2 = v_{\text{ex}} \ln\left(\frac{0.6m_{\text{o}}}{0.3m_{\text{o}}}\right) + v_1 = v_{\text{ex}} \ln\left(\frac{0.6}{0.3} \times \frac{1}{0.7}\right) = v_{\text{ex}} \ln(2.86) = 1.05v_{\text{ex}}.$$

3.13 ⋆⋆ We can find the height $y(t)$ by integrating $v(t)$ as found in Problem 3.11:

$$y(t) = \int_0^t v(t')dt' = v_{\text{ex}} \int_0^t [\ln m_\text{o} - \ln m(t')]dt' - \int_0^t gt'dt'$$
$$= v_{\text{ex}} t \ln m_\text{o} - v_{\text{ex}} \int_0^t \ln m(t')dt' - \tfrac{1}{2}gt^2. \qquad \text{(v)}$$

To do the remaining integral, notice that $m(t') = m_\text{o} - kt'$, so that $dt' = -dm'/k$, where m' is short for $m(t')$. Thus the remaining integral in (v) is

$$\int_0^t \ln m' dt' = -\frac{1}{k}\int_{m_\text{o}}^m \ln m'\, dm' = -\frac{1}{k}\Big[m' \ln m' - m'\Big]_{m_\text{o}}^m = \frac{1}{k}(m_\text{o} \ln m_\text{o} - m \ln m) - t,$$

where in the last expression I used the fact that $m_\text{o} - m = kt$. Substituting into (v), we get

$$y(t) = v_{\text{ex}} t - \tfrac{1}{2}gt^2 + \frac{v_{\text{ex}}}{k}(kt \ln m_\text{o} - m_\text{o} \ln m_\text{o} + m \ln m) = v_{\text{ex}} t - \tfrac{1}{2}gt^2 - \frac{m v_{\text{ex}}}{k} \ln\left(\frac{m_\text{o}}{m}\right)$$

where I have again used that $kt = m_\text{o} - m$. Putting in the numbers from Problem 3.11 and $t = 2$ min, we find that $y \approx 40$ km.

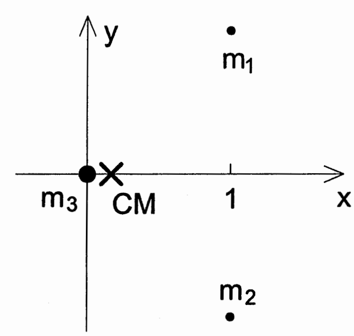
3.14 ⋆⋆ The equation of motion (3.29) reads $m\dot{v} = kv_{\text{ex}} - bv$, or $m\, dv/(kv_{\text{ex}} - bv) = dt$. Since $dm/dt = -k$ we can replace dt by $-dm/k$, and the equation of motion becomes $k\, dv/(kv_{\text{ex}} - bv) = -dm/m$. This integrates to give

$$\frac{k}{b}\ln\left(\frac{kv_{\text{ex}} - bv}{kv_{\text{ex}}}\right) = \ln\left(\frac{m}{m_\text{o}}\right) \quad \text{or} \quad v = \frac{kv_{\text{ex}}}{b}\left[1 - \left(\frac{m}{m_\text{o}}\right)^{b/k}\right].$$

3.15 ⋆

$$X = \frac{\sum m_\alpha x_\alpha}{M} = \frac{1+1+0}{12} = \frac{1}{6}$$
$$Y = \frac{\sum m_\alpha y_\alpha}{M} = \frac{1-1+0}{12} = 0$$
$$Z = \frac{\sum m_\alpha z_\alpha}{M} = \frac{0+0+0}{12} = 0$$

Because m_3 is much bigger than m_1 and m_2, the CM is much closer to m_3 than to the other two.



3.16 ⋆ If we put the origin at the sun's center and the x axis through the earth's center, then the CM lies on the x axis at

$$X = \frac{m_\text{s} x_\text{s} + m_\text{e} x_\text{e}}{m_\text{s} + m_\text{e}} = \frac{0 + m_\text{e} x_\text{e}}{m_\text{s} + m_\text{e}} = \frac{6.0 \times 10^{24}}{2.0 \times 10^{30}} \times (1.5 \times 10^8 \text{ km}) = 450 \text{ km},$$

which is very, very close to the center of the sun.

3.17 ⋆ If we put the origin at the earth's center and the x axis through the moon's center, then the CM lies on the x axis at

$$X = \frac{m_e x_e + m_m x_m}{m_e + m_m} = \frac{0 + m_m x_m}{m_e + m_m} = \frac{7.4 \times 10^{22}}{(600 + 7.4) \times 10^{22}} \times (3.8 \times 10^5 \text{ km}) = 4600 \text{ km}.$$

Since the earth's radius is 6400 km, the earth-moon CM is comfortably inside the earth

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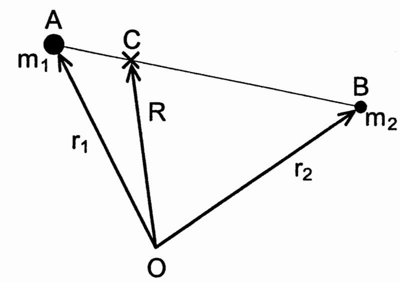
3.18 ⋆⋆ **(a)** The vector pointing from m_1 the CM is



$$\overrightarrow{AC} = \mathbf{R} - \mathbf{r}_1 = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} - \mathbf{r}_1$$
$$= \frac{m_2}{m_1 + m_2}(\mathbf{r}_2 - \mathbf{r}_1) = \frac{m_2}{M}\overrightarrow{AB}$$

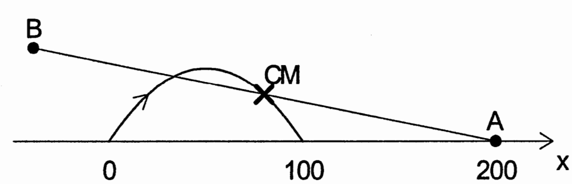
Since the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are in the same direction, the three points A, B, and C are collinear. Since $AC < AB$, the CM lies between A and B on the line joining them.

(b) By the same argument, $\overrightarrow{BC} = (m_1/M)\overrightarrow{BA}$. Thus the ratio of the two lengths is $AC/BC = m_2/m_1$, as claimed. If $m_1 \gg m_2$, then $AC \ll BC$ and C is very close to A.

3.19 ⋆⋆ **(a)** As long as all pieces are still in the air (so that the only force on them is gravity), (3.11) implies that the CM follows the path of a single particle of mass M, namely, the same parabola.

(b) At the moment when both pieces land ($y = 0$), the CM is also at $y = 0$ and is, therefore, at the target ($x = 100$). Since the CM is half way between the two pieces and the first piece is 100 m beyond the target, the second piece has to be 100 m short of the target, namely back at the launch site ($x = 0$).

(c) Let's call the two halves A and B, and let's imagine the ground removed, so that the pieces can continue to move like projectiles even after they pass the level $y = 0$. At all times, the CM is at the midpoint of the line AB. Suppose that A lands first, at $x_A = 200$. At the same time, B and hence the CM are still above ground, which means that the CM has $X < 100$ (since we know the CM *will* land at $X = 100$). This implies that at the moment when A lands, $x_B < 0$. In other words, B has already passed back over the launch point and will land still further behind it. A similar argument shows that if A lands after B, then B will land at a point $x_B > 0$.



3.20 ⋆⋆ Notice first that the definition (3.9) of the CM can be written as $M\mathbf{R} = \sum m_\alpha \mathbf{r}_\alpha$. Now suppose that we have two bodies made up as follows:

(body 1) = N_1 particles with masses $m_{1\alpha}$ at positions $\mathbf{r}_{1\alpha}$, with $\alpha = 1, \cdots, N_1$,

and

(body 2) = N_2 particles with masses $m_{2\beta}$ at positions $\mathbf{r}_{2\beta}$, with $\beta = 1, \cdots, N_2$.

The total mass of the two-body system is $M = M_1 + M_2 = \sum_\alpha m_{1\alpha} + \sum_\beta m_{2\beta}$ and the CM postion $\mathbf{R}$ of the whole system satisfies

$$M\mathbf{R} = \sum_\alpha m_{1\alpha}\mathbf{r}_{1\alpha} + \sum_\beta m_{2\beta}\mathbf{r}_{1\beta} = M_1\mathbf{R}_1 + M_2\mathbf{R}_2\,,$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the CM positions of the two separate bodies and the second equality follows from our rewritten definition of the CM applied to each body separately. This is the required result.

3.21 ⋆⋆ Let the disk's mass be M and its mass density (mass/area) be $\sigma = M/A$, where $A = \pi R^2/2$ is its area. The CM position is $\mathbf{R} = \int \sigma\mathbf{r}\, dA/M = \int \mathbf{r}\, dA/A$ where the integral runs over the area of the disk in the plane $z = 0$. Clearly $Z = 0$, and, by symmetry, $X = 0$. Finally

$$Y = \frac{1}{A}\int y\, dA = \frac{2}{\pi R^2}\int_0^R r\, dr \int_0^\pi d\phi(r\sin\phi) = \frac{4}{3\pi}R.$$

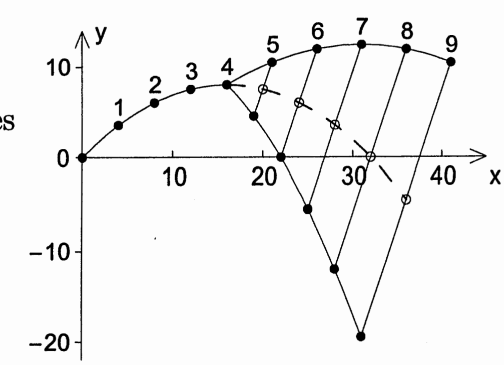
3.22 ⋆⋆ Let the hemisphere's mass be M and its density be $\varrho = M/V$, where $V = 2\pi R^3/3$ is its volume. The CM position is $\mathbf{R} = \int \varrho\mathbf{r}\, dV/M = \int \mathbf{r}\, dV/V$ where the integral runs over the volume of the hemisphere. By symmetry $X = Y = 0$, while

$$Z = \frac{1}{V}\int z\, dV = \frac{3}{2\pi R^3}\int_0^R r^2 dr \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi\, r\cos\theta = \frac{3}{2\pi R^3}\cdot\frac{R^4}{4}\cdot\frac{1}{2}\cdot 2\pi = \frac{3}{8}R$$

3.23 ⋆⋆⋆ **(a)** The orbit is $\mathbf{r}(t) = \mathbf{v}_o t + \frac{1}{2}\mathbf{g}t^2$.

(b) Because the two pieces have equal masses, $m_1 = m_2 = M/2$, conservation of momentum implies that $\mathbf{v}_1 + \mathbf{v}_2 = 2\mathbf{v}$, so that $\mathbf{v}_2 = 2\mathbf{v} - \mathbf{v}_1 = \mathbf{v} - \Delta\mathbf{v}$.

(c) The CM (hollow circles) is at the midpoint of the line joining the two pieces and clearly continues along the original parabola.



3.24 ⋆ If we orient the triangle so that $\mathbf{a}$ is the base, then the height is $h = b\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Therefore

$$\tfrac{1}{2}|\mathbf{a}\times\mathbf{b}| = \tfrac{1}{2}ab\sin\theta = \tfrac{1}{2}ah = \text{(area of triangle)}.$$

3.25 ⋆ The only force on the particle is the tension of the string, which is necessarily directed toward the hole in the table at O. Therefore the angular momentum $\boldsymbol{\ell}$ about O is constant. When the particle is travelling in a circle of radius r, the vertical component of $\boldsymbol{\ell} = \mathbf{r}\times\mathbf{p}$ is $\ell_z = rp = rmv = rm(r\omega) = mr^2\omega$. Therefore, the quantity $r^2\omega$ is constant and $r^2\omega = r_o^2\omega_o$; whence $\omega = (r_o/r)^2\omega_o$.

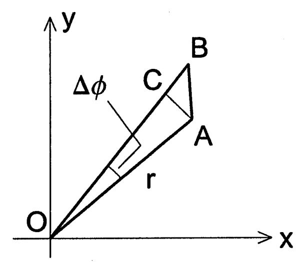
3.26 ⋆ **(a)** Since the force is central it has the form $\mathbf{F} = f(\mathbf{r})\hat{\mathbf{r}}$ and the torque on the particle is $\boldsymbol{\Gamma} = \mathbf{r}\times\mathbf{F} = 0$. Therefore the angular momentum $\boldsymbol{\ell}$ is constant.

(b) If $\boldsymbol{\ell}_o$ is the particle's initial angular momentum, then conservation of angular momentum implies that $\mathbf{r}\times\mathbf{p} = \boldsymbol{\ell}_o$ at any time and hence that $\mathbf{r}$ is always perpendicular to $\boldsymbol{\ell}_o$. In other words, the position vector $\mathbf{r}$ always lies in the plane through O perpendicular to $\boldsymbol{\ell}_o$.

3.27 ⋆⋆ **(a)** Since $\mathbf{r} = r\hat{\mathbf{r}}$ and $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$, it follows that

$$\boldsymbol{\ell} = \mathbf{r}\times m\dot{\mathbf{r}} = mr^2\dot{\phi}\,\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = mr^2\omega\hat{\mathbf{z}}$$

Therefore, $\ell = mr^2\omega$ as claimed. (Strictly speaking we should put absolute value signs in here if we are to insist that ℓ be positive.)



(b) Suppose that in a time Δt the planet moves from A to B and swings through an angle $\Delta\phi$. The area swept out is the area of the triangle OAB. In the limit of small Δt, this triangle is well approximated by the triangle OAC, with $OC = OA$. This has height r and base $r\Delta\phi$. Therefore the area swept out is $\Delta A \approx \tfrac{1}{2}r^2\Delta\phi$. Dividing both sides by Δt and letting $\Delta t \to 0$, we conclude that $\dot{A} = \tfrac{1}{2}r^2\dot{\phi} = \tfrac{1}{2}r^2\omega$, as claimed. Comparing this with the result of part (a), we see that $\dot{A} = \ell/2m$ and hence that the conservation of ℓ implies that $\dot{A}$ is constant.

3.28 ⋆ With three particles, Eq.(3.20) reads $\mathbf{L} = (\mathbf{r}_1\times\mathbf{p}_1) + (\mathbf{r}_2\times\mathbf{p}_2) + (\mathbf{r}_3\times\mathbf{p}_3)$, so

$$\dot{\mathbf{L}} = (\dot{\mathbf{r}}_1\times\mathbf{p}_1 + \mathbf{r}_1\times\dot{\mathbf{p}}_1) + (\dot{\mathbf{r}}_2\times\mathbf{p}_2 + \mathbf{r}_2\times\dot{\mathbf{p}}_2) + (\dot{\mathbf{r}}_3\times\mathbf{p}_3 + \mathbf{r}_3\times\dot{\mathbf{p}}_3).$$

The first term in each parenthesis is zero (because each $\dot{\mathbf{r}}$ is parallel to its corresponding $\mathbf{p}$). We can replace $\dot{\mathbf{p}}_1$ by the corresponding net force $\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_1^{\text{ext}}$ and so on to give

$$\begin{aligned}\dot{\mathbf{L}} &= (\mathbf{r}_1\times\mathbf{F}_1) + (\mathbf{r}_2\times\mathbf{F}_2) + (\mathbf{r}_3\times\mathbf{F}_3)\\ &= \left(\mathbf{r}_1\times[\mathbf{F}_{12}+\mathbf{F}_{13}+\mathbf{F}_1^{\text{ext}}]\right) + \left(\mathbf{r}_2\times[\mathbf{F}_{23}+\mathbf{F}_{21}+\mathbf{F}_2^{\text{ext}}]\right) + \left(\mathbf{r}_3\times[\mathbf{F}_{31}+\mathbf{F}_{32}+\mathbf{F}_3^{\text{ext}}]\right)\end{aligned}$$

Remembering that $\mathbf{F}_{12} = -\mathbf{F}_{21}$ and so on, we can regroup to give

$$\dot{\mathbf{L}} = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} + (\mathbf{r}_2 - \mathbf{r}_3) \times \mathbf{F}_{23} + (\mathbf{r}_3 - \mathbf{r}_1) \times \mathbf{F}_{31} + \mathbf{r}_1 \times \mathbf{F}_1^{\text{ext}} + \mathbf{r}_2 \times \mathbf{F}_2^{\text{ext}} + \mathbf{r}_3 \times \mathbf{F}_3^{\text{ext}}.$$

Provided all internal forces are central, each of the first three products here is zero (as discussed in connection with Fig.3.8) and we are left with

$$\dot{\mathbf{L}} = \mathbf{r}_1 \times \mathbf{F}_1^{\text{ext}} + \mathbf{r}_2 \times \mathbf{F}_2^{\text{ext}} + \mathbf{r}_3 \times \mathbf{F}_3^{\text{ext}} = \mathbf{\Gamma}_1^{\text{ext}} + \mathbf{\Gamma}_2^{\text{ext}} + \mathbf{\Gamma}_3^{\text{ext}} = \mathbf{\Gamma}^{\text{ext}}.$$

3.29 ⋆ We are told that the matter accreted by the asteroid is initially at rest. Therefore its initial angular momentum is zero and, by conservation of angular momentum, $I_o\omega_o = I\omega$, where I_o and I are the initial and final moments of inertia of the asteroid. Now,

$$I = \frac{2}{5}MR^2 = \frac{2}{5}\left(\frac{4}{3}\pi\varrho R^3\right)R^2 = \frac{8}{15}\pi\varrho R^5,$$

so, given that ϱ is constant, conservation of angular momentum implies that $R_o^5\omega_o = R^5\omega$, and $\omega = \omega_o(R_o/R)^5$. If $R = 2R_o$, then $\omega = \omega_o/32$.

3.30 ⋆⋆ **(a)** If a particle is a distance ρ from the axis of rotation and the body turns through an angle $d\phi$, then the particle moves a distance $\rho d\phi$ in the tangential (ϕ) direction. Dividing by dt we conclude that the particle's speed is $v = \rho d\phi/dt = \rho\omega$ in the ϕ direction. That is, $\mathbf{v} = \rho\omega\hat{\boldsymbol{\phi}}$.

(b) The particle's position is $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, so its angular momentum is $\boldsymbol{\ell} = \mathbf{r}\times\mathbf{p} = (\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}) \times m\rho\omega\hat{\boldsymbol{\phi}} = m\rho^2\omega\hat{\mathbf{z}} - mz\rho\omega\hat{\boldsymbol{\phi}}$. Therefore its z component is $\ell_z = m\rho^2\omega$.

(c) The total angular momentum has

$$L_z = \sum_{\alpha=1}^{N} \ell_{\alpha z} = \sum_{\alpha=1}^{N} m_\alpha\rho_\alpha^2\omega = I\omega \quad \text{where} \quad I = \sum_{\alpha=1}^{N} m_\alpha\rho_\alpha^2.$$

3.31 ⋆⋆ If we place the disk in the plane $z = 0$ centered on the origin, the sum (3.31) can be written as $I = \sum m_\alpha r_\alpha^2$ (because in the plane $z = 0$ the coordinate ρ is the same as what we usually call r). If the density (mass/area) of the disk is $\sigma = M/\pi R^2$, then, replacing the sum by the appropriate integral, we find that

$$I = \int \sigma r^2 dA = \frac{M}{\pi R^2}\int_0^R r dr \int_0^{2\pi} d\phi\, r^2 = \frac{M}{\pi R^2}\cdot\frac{R^4}{4}\cdot 2\pi = \tfrac{1}{2}MR^2$$

3.32 ⋆⋆ The sum (3.31) becomes the integral $I = \int \varrho\, dV \rho^2$, where $\varrho = M/V = 3M/(4\pi R^3)$ is the density and $\rho = r\sin\theta$ is the distance of a point from the z axis. Therefore

$$I = \frac{3M}{4\pi R^3}\int_0^R r^4 dr \int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} d\phi = \frac{3M}{4\pi R^3}\cdot\frac{R^5}{5}\cdot\frac{4}{3}\cdot 2\pi = \frac{2}{5}MR^2.$$

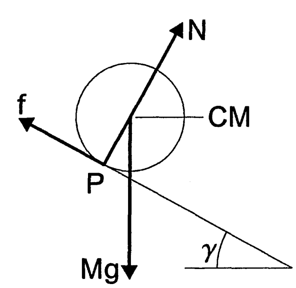
3.33 ⋆⋆ If we place the square in the plane $z = 0$, centered on the origin with its sides parallel to the x and y axes, the sum (3.31) takes the form $I = \sum m_\alpha \rho_\alpha^2 = \sum m_\alpha(x_\alpha^2 + y_\alpha^2) = 2\sum m_\alpha x_\alpha^2$, where the last expression holds because, by symmetry, the two terms of the previous expression are equal. If we denote the density (mass/area) of the square by $\sigma = M/A = M/(4b^2)$, then we can replace the sum by an integral:

$$I = 2\sigma \int x^2 dA = 2\sigma \int_{-b}^{b} x^2 dx \int_{-b}^{b} dy = 2\frac{M}{4b^2} \cdot \frac{2b^3}{3} \cdot 2b = \frac{2}{3}Mb^2$$

3.34 ⋆⋆ The CM moves just like a point mass M, so its height is $Y = v_o t - \frac{1}{2}gt^2$, and the time to return to $Y = 0$ is $t = 2v_o/g$. Since there is no torque about the CM, the angular momentum $L = I\omega$ is constant. Therefore $\omega = \omega_o$ is also constant and the number of complete revolutions in the time t is $n = \omega_o t/2\pi = \omega_o v_o/\pi g$. Therefore, he must arrange that $v_o = n\pi g/\omega_o$ where n is an integer.

3.35 ⋆⋆

(a)

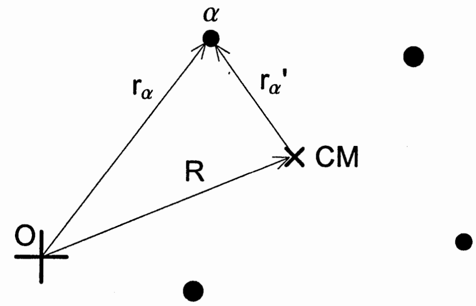


(b) The condition $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ applied about P becomes $I_P\,\dot{\omega} = MgR\sin\gamma$, whence $\dot{v} = \frac{2}{3}g\sin\gamma$.

(c) The same condition applied about the CM gives $I_{\text{cm}}\dot{\omega} = fR$. To eliminate the unknown frictional force f, we must use Newton's second law, $M\dot{v} = Mg\sin\gamma - f$. Eliminating f, we get the same answer as before.

3.36 ⋆⋆ The motion of the CM is the same as in Example 3.4; that is, the momentum of the CM just after the impulse is $\mathbf{P} = \mathbf{F}\Delta t$, so, since $\mathbf{P} = M\dot{\mathbf{R}}$, the initial CM velocity is $\dot{\mathbf{R}} = \mathbf{F}\Delta t/2m$ and, since $\mathbf{P}$ is constant, this velocity remains unchanged. That is, the CM moves with constant speed $F\Delta t/2m$ in the direction of the applied force. The applied torque is $\Gamma^{\text{ext}} = Fb\sin\alpha$, thus the initial angular momentum is $L = Fb\Delta t\sin\alpha$, which also remains constant. Thus the dumbell rotates with constant angular velocity $\omega = L/I = (F\Delta t\sin\alpha)/(2mb)$ about the CM. If $\alpha = 90°$, these results are the same as in the example. If $\alpha = 0$, the CM moves with the same speed but along the x axis, and there is no rotation ($\omega = 0$), just as you would expect.

3.37 ⋆⋆⋆ **(a)**



(b) $\sum m_\alpha \mathbf{r}'_\alpha = \sum m_\alpha(\mathbf{r}_\alpha - \mathbf{R})$
$= \sum m_\alpha \mathbf{r}_\alpha - (\sum m_\alpha)\mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0.$
The sum $(1/M)\sum m_\alpha \mathbf{r}'_\alpha$ defines the position of the CM *relative to the CM*, which is fairly obviously zero.

(c) The angular momentum about the CM is $\mathbf{L}(\text{about CM}) = \sum \mathbf{r}'_\alpha \times m_\alpha \dot{\mathbf{r}}'_\alpha$. Therefore

$$\begin{aligned}
\dot{\mathbf{L}}(\text{about CM}) &= \sum \dot{\mathbf{r}}'_\alpha \times m_\alpha \dot{\mathbf{r}}'_\alpha + \sum \mathbf{r}'_\alpha \times m_\alpha \ddot{\mathbf{r}}'_\alpha \\
&= 0 + \sum \mathbf{r}'_\alpha \times m_\alpha(\ddot{\mathbf{r}}_\alpha - \ddot{\mathbf{R}}) \\
&= \sum \mathbf{r}'_\alpha \times F_\alpha - \sum m_\alpha \mathbf{r}'_\alpha \times \ddot{\mathbf{R}} \\
&= \boldsymbol{\Gamma}(\text{about CM}) + 0 = \boldsymbol{\Gamma}^{\text{ext}}(\text{about CM}).
\end{aligned}$$

The first sum on the right of the top line was zero because the cross product of any two parallel vectors is zero. The second sum on the third line was zero by the result of part (b), and, in the last line, all internal torques canceled for the same reason as in (3.25).

Chapter 4

Energy

I covered this chapter in 6 fifty-minute lectures.

This is a long and important chapter. Its length reflects that energy is a complicated notion and also that I felt required me to prove several results that are simply swept under the rug in many traditional treatments. [For instance, the proof in Section 4.9 of the work-KE theorem for two interacting particles, Eq.(4.86), is often omitted but seems to me crucial to the use of the energy concept for multiparticle systems.]

All of my students were quite familiar with the concepts of kinetic and potential energy. Officially they had all met the gradient and the curl, but very few of them were at all confident using them. Thus I thought that the time spent in Section 4.3 convincing them that $\mathbf{F} = -\nabla U$ (both that this is true and what it means) was time well spent, and similarly with the amazing result of Section 4.4 that the condition $\nabla \times \mathbf{F} = \mathbf{0}$ guarantees the path-independence of the work done by $\mathbf{F}$. Similarly, the idea of a time-dependent potential energy is sometimes thrown into the ring without much explanation, and it seemed to me that a fair bit of explanation is called for.

It is rather usual to introduce PE and its relation to force in one dimension, and later to generalize to three dimensions. I don't care for this approach because the 1-D case is misleadingly simpler than the 3-D. Therefore, I decided to treat three dimensions first and then, in Section 4.6, show how much simpler life is in 1 D. The discussion of central forces in Section 4.8 is a good opportunity to introduce (or review, as the case may be) spherical polar coordinates. Sections 4.9 and 4.10 are about the energy of a multiparticle system. Some texts just ignore the complications of having several particles, but I felt that results like (4.96), that the force on particle α is $\mathbf{F}_\alpha = -\nabla_\alpha U$ are important and not particularly obvious. Hence the two sections on the subject. You could if you wish simply state the relevant results and save some time.

Problems 4.10 through 4.16 are exercises for students who need to brush up on the gradient, and 4.20 through 4.25 do the same for the curl.

Solutions to Problems for Chapter 4

4.1 ⋆ Same as Problem 1.8(b).

4.2 ⋆⋆ **(a)**

$$W = (\int_O^Q + \int_Q^P)\mathbf{F}\cdot d\mathbf{r} = \int_0^1 F_x(x,0)\,dx + \int_0^1 F_y(1,0)\,dy = \int_0^1 x^2\,dx + \int_0^1 2y\,dy = \frac{1}{3} + 1 = \frac{4}{3}.$$

(b) On this path $y = x^2$, so

$$W = \int_O^P F_x\,dx + \int_O^P F_y\,dy = \int_0^1 x^2\,dx + \int_0^1 (2x^3)(2x\,dx) = \frac{1}{3} + \frac{4}{5} = \frac{17}{15}.$$

(c) On this path $x = t^3$ and $y = t^2$, so

$$W = \int_O^P F_x\,dx + \int_O^P F_y\,dy = \int_0^1 t^6(3t^2\,dt) + \int_0^1 (2t^5)(2t\,dt) = 3 \times \frac{1}{9} + 4 \times \frac{1}{7} = \frac{19}{21}.$$

4.3 ⋆⋆ **(a)**

$$W = \int_P^Q \mathbf{F}\cdot d\mathbf{r} = \int_P^O F_x dx + \int_O^Q F_y dy = 0 + 0 = 0.$$

(b) On path (b), $y = 1 - x$, so $dy = -dx$ and

$$W = \int_P^Q (F_x dx + F_y dy) = \int_1^0 (F_x - F_y)dx = \int_0^1 (y + x)dx = \int_0^1 1dx = 1.$$

(c) On path (c), $\mathbf{r} = (\cos\phi, \sin\phi)$, so $\mathbf{F} = (-\sin\phi, \cos\phi)$ and $d\mathbf{r} = (-\sin\phi, \cos\phi)d\phi$. Therefore

$$W = \int_P^Q \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (\sin^2\phi + \cos^2\phi)d\phi = \pi/2.$$

4.4 ⋆⋆ **(a)** As in Problem 3.25, conservation of angular momentum implies that $mr^2\omega = mr_o^2\omega_o$, so $\omega = (r_o/r)^2\omega_o$.

(b) The tension force, which I must supply, is what keeps the particle in its circular path with centripetal acceleration $a_r = -\omega^2 r$. (This is where we must assume that I pull the string slowly — otherwise, $a_r = \ddot{r} - \omega^2 r$.) Thus the force which I exert is

$$F(r) = m\omega^2 r = m\left[\left(\frac{r_o}{r}\right)^2 \omega_o\right]^2 r = m\omega_o^2 r_o^4 \frac{1}{r^3}$$

where I used the result of part (a) for the second equality. The work I do is (Remember the distance I pull the string in any small displacement is $-dr$.)

$$W = \int_{r_o}^{r} F(r')(-dr') = -m\omega_o^2 r_o^4 \int_{r_o}^{r} \frac{dr'}{r'^3} = \frac{1}{2} m\omega_o^2 r_o^4 \left(\frac{1}{r^2} - \frac{1}{r_o^2}\right).$$

(c) The particle's KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$. Thus, with the result of part (a) for ω,

$$\Delta T = \frac{1}{2} m(r^2\omega^2 - r_o^2\omega_o^2) = \frac{1}{2} m\omega_o^2 \left(\frac{r_o^4}{r^2} - r_o^2\right)$$

which is the same as the work done in part (b), as it has to be.

4.5 ⋆ **(a)** If we choose axes with y measured vertically up (and x and z horizontal), then $\mathbf{F}_{\text{grav}} = (0, mg, 0)$ and

$$W_{\text{grav}}(1 \to 2) = \int_1^2 \mathbf{F}_{\text{grav}} \cdot d\mathbf{r} = -\int_1^2 mg\, dy = -mg(y_2 - y_1) = -mgh.$$

Since this is visibly independent of the path followed, $\mathbf{F}_{\text{grav}}$ is conservative.

(b) $U_{\text{grav}}(\mathbf{r}) = -W_{\text{grav}}(0 \to \mathbf{r}) = mgy.$

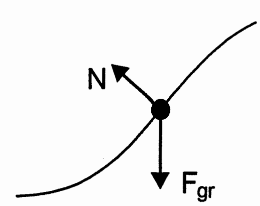
4.6 ⋆ $U = \sum_\alpha U_\alpha = \sum m_\alpha g y_\alpha = gMY$, where, in the last equality, I used the definition (3.9) of the CM, according to which $\sum m_\alpha y_\alpha = MY$.

4.7 ⋆ **(a)** Since $\mathbf{F}_{\text{gr}} = -m\gamma y^2 \hat{\mathbf{y}}$,

$$W(\mathbf{r}_1 \to \mathbf{r}_2) = \int_1^2 \mathbf{F}_{\text{gr}} \cdot d\mathbf{r} = -m\gamma \int_{y_1}^{y_2} y^2 dy = -\frac{m\gamma}{3}(y_2^3 - y_1^3).$$

Because this work is independent of path (and $\mathbf{F}_{\text{gr}}$ depends on only $\mathbf{r}$) $\mathbf{F}_{\text{gr}}$ is conservative, and

$$U(\mathbf{r}) = -W(0 \to \mathbf{r}) = \frac{m\gamma}{3} y^3.$$



(b) The forces are $\mathbf{F}_{\text{gr}}$, which is conservative, and the normal force $\mathbf{N}$, which is not. (For instance, $\mathbf{N}$ can depend on the bead's speed.)

(c) By conservation of energy, $T + U = \text{const}$, so $\frac{1}{2}mv_{\text{fin}}^2 = \frac{1}{3}m\gamma h^3$ and hence

$$v_{\text{fin}} = \sqrt{2\gamma h^3/3}.$$

4.8 ⋆⋆ We'll measure the puck's position by the angle θ it subtends at the sphere's center O (measured down from the top). The puck's PE (defined as zero at the level of O) is $U(\theta) = mgR\cos\theta$, and its total energy is $E = U(0) = mgR$. By conservation of energy,

$$T = \tfrac{1}{2}mv^2 = E - U = mgR(1 - \cos\theta). \qquad \text{(i)}$$

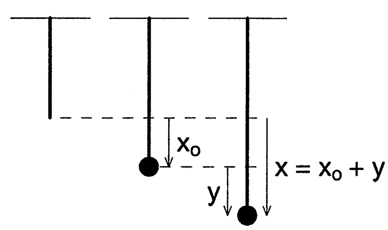
As long as the puck remains in contact with the sphere, the radial component of Newton's second law reads $N - mg\cos\theta = -mv^2/R$, where N denotes the normal force of the sphere on the puck. Substituting from Eq (i) for mv^2 we find

$$N = mg(3\cos\theta - 2).$$

As long as N is positive the puck remains on the sphere. Since the sphere cannot exert a negative normal force, once the predicted value of N becomes negative, the puck must have left the sphere. Therefore it leaves the sphere when $N = 0$ or $\theta = \arccos(2/3) = 48.2°$.

4.9 ⋆⋆ **(a)** $U = -\int_0^x F(x')dx' = k\int_0^x x'dx' = \tfrac{1}{2}kx^2$

(b) The two forces on m are gravity, mg down, and the spring, kx up, and at the new equilibrium x_o they must balance, so $x_o = mg/k$. The total PE is



$$\begin{aligned} U &= \tfrac{1}{2}kx^2 - mgx \\ &= \tfrac{1}{2}k(x_o + y)^2 - mg(x_o + y) \\ &= \tfrac{1}{2}ky^2 + y(kx_o - mg) + \text{const.} \end{aligned}$$

In the last line the second term is zero (the condition for equilibrium) and we can drop the constant, to give $U = \tfrac{1}{2}ky^2$.

4.10 ⋆

function, f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
$ax^2 + bxy + cy^2$	$2ax + by$	$bx + 2cy$	0
$\sin(axyz^2)$	$ayz^2\cos(axyz^2)$	$axz^2\cos(axyz^2)$	$2axyz\cos(axyz^2)$
ae^{xy/z^2}	$(ay/z^2)e^{xy/z^2}$	$(ax/z^2)e^{xy/z^2}$	$(-2axy/z^3)e^{xy/z^2}$

4.11 ⋆

function, f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
$ay^2 + 2byz + cz^2$	0	$2ay + 2bz$	$2by + 2cz$
$\cos(axy^2z^3)$	$-ay^2z^3\sin(axy^2z^3)$	$-2axyz^3\sin(axy^2z^3)$	$-3axy^2z^2\sin(axy^2z^3)$
$ar = a\sqrt{x^2 + y^2 + z^2}$	ax/r	ay/r	az/r

4.12 ⋆ **(a)** $\nabla f = \hat{\mathbf{x}}\, 2x + \hat{\mathbf{z}}\, 3z^2$. **(b)** $\nabla f = k\,\hat{\mathbf{y}}$. **(c)** $\nabla f = \hat{\mathbf{r}}$.

4.13 ⋆

function, f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$	∇f
$\ln(r)$	x/r^2	y/r^2	z/r^2	$\hat{\mathbf{r}}/r$
r^n	nxr^{n-2}	nyr^{n-2}	nzr^{n-2}	$n\,r^{n-1}\,\hat{\mathbf{r}}$
$g(r)$	$g'(r)x/r$	$g'(r)y/r$	$g'(r)z/r$	$g'(r)\,\hat{\mathbf{r}}$

4.14 ⋆ Consider the x component

$$[\nabla(fg)]_x = \frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x} = f(\nabla g)_x + g(\nabla f)_x\,.$$

Since the other two components work the same way, we conclude that $\nabla(fg) = f\nabla g + g\nabla f$.

4.15 ⋆ If $f(\mathbf{r}) = x^2 + 2y^2 + 3z^2$, the gradient is $\nabla f = (2x, 4y, 6z)$, so Equation (4.35) gives the approximation

$$\begin{aligned} df \approx \nabla f\cdot d\mathbf{r} &= (2x, 4y, 6z)\cdot(dx, dy, dz) \\ &= (2, 4, 6)\cdot(0.01, 0.03, 0.05) = 0.02 + 0.12 + 0.30 = 0.4400. \end{aligned}$$

This compares favorably with the exact result

$$df = f(1.01, 1.03, 1.05) - f(1, 1, 1) = 6.4494 - 6.0000 = 0.4494.$$

4.16 ⋆ If $U = k(x^2 + y^2 + z^2)$, then $\mathbf{F} = -\nabla U = -2k(x, y, z) = -2k\mathbf{r}$.

4.17 ⋆ **(a)** The force $\mathbf{F} = q\mathbf{E}_o$ certainly doesn't depend on anything but $\mathbf{r}$ (and doesn't even depend on $\mathbf{r}$ since $\mathbf{E}_o$ is constant). The work integral is $W(1 \to 2) = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = q\mathbf{E}_o\cdot\int_1^2 d\mathbf{r} = q\mathbf{E}_o\cdot(\mathbf{r}_2 - \mathbf{r}_1)$, which is independent of path. Therefore, the force is conservative, and the PE is $U(\mathbf{r}) = -W(0 \to \mathbf{r}) = -q\mathbf{E}_o\cdot\mathbf{r}$.

(b) Since $U = -q(E_{ox}x + E_{oy}y + E_{oz}z)$, $-\nabla U = q(E_{ox}, E_{oy}, E_{oz}z) = q\mathbf{E}_o = \mathbf{F}$.

4.18 ⋆⋆ **(a)** According to (4.35), the change in $f(\mathbf{r})$ resulting from any small displacement $d\mathbf{r}$ is $df = \nabla f\cdot d\mathbf{r}$. If, in particular, we consider any infinitesimal displacement $d\mathbf{r}$ in a surface of constant f, then df will be zero. This implies that $\nabla f\cdot d\mathbf{r} = \mathbf{0}$, that is, ∇f is perpendicular to the surface of constant f.

(b) Consider a displacement $d\mathbf{r} = \epsilon\mathbf{u}$ with fixed magnitude ϵ but variable direction $\mathbf{u}$. Our job is to find the direction of $\mathbf{u}$ for which the corresponding change df is largest. Since $df = \nabla f\cdot d\mathbf{r} = \epsilon\nabla f\cdot\mathbf{u} = \epsilon|\nabla f|\cos\theta$, where θ is the angle between ∇f and $\mathbf{u}$, we see that df is maximum if $\theta = 0$, or ∇f and $\mathbf{u}$ are parallel. That is, the direction of ∇f is the direction in which f increases most rapidly.

4.19 ⋆⋆ **(a)** The equation $x^2 + 4y^2 = K$ defines an ellipse in the xy plane, centered on $x = y = 0$. Since the value of $x^2 + 4y^2$ is unchanged by any variation of z, the surface is an elliptical cylinder, centered on the z axis, with semi-major and semi-minor axes equal to $\sqrt{K}$ and $\sqrt{K}/2$.

(b) We know that the vector ∇f is perpendicular to the surface f = const. In the present case $\nabla f = (2x, 8y, 0) = (2, 8, 0)$ at the point $(1, 1, 1)$. Therefore the unit normal is $\mathbf{n} = (1, 4, 0)/\sqrt{17}$ or $-\mathbf{n}$. The direction of maximum increase is $\mathbf{n}$.

4.20 ⋆

$\mathbf{F} =$	(kx, ky, kz)	(Ax, By^2, Cz^3)	(Ay^2, Bx, Cz)
$\nabla \times \mathbf{F} =$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, B - 2Ay)$

4.21 ⋆ This is exactly the same as Example 4.5 except that $\gamma = kqQ$ is replaced by $\gamma = -GMm$. Thus, if we choose $U = 0$ at $r = \infty$, then $U(r) = \gamma/r = -GMm/r$.

4.22 ⋆ Since $\mathbf{F} = \gamma\hat{\mathbf{r}}/r^2$, the spherical polar components of $\mathbf{F}$ are $F_\theta = F_\phi = 0$ and $F_r = \gamma/r^2$. Since F_r is independent of θ and ϕ, its two derivatives $\partial F_r/\partial\theta$ and $\partial F_r/\partial\phi$ are both zero. Inspection of the expression for $\nabla \times \mathbf{F}$ in spherical polars should convince you that each of the six terms is zero, so $\nabla \times \mathbf{F} = 0$. Since $\mathbf{F}$ certainly depends on $\mathbf{r}$ only, $\mathbf{F}$ is conservative.

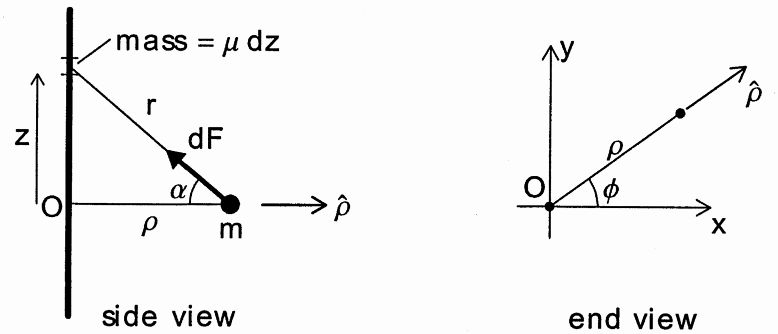
4.23 ⋆⋆ All three given forces depend only on $\mathbf{r}$; that is, all satisfy the first condition. It remains to checks their curls.

(a) With $\mathbf{F} = k(x, 2y, 3z)$, we find $\nabla \times \mathbf{F} = (0, 0, 0)$, so $\mathbf{F}$ is conservative. The corresponding PE is $U = -\int \mathbf{F}\cdot d\mathbf{r}' = -k\int(x'dx' + 2y'dy' + 3z'dz') = -k(\frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2)$. Clearly $-\nabla U = k(x, 2y, 3z) = \mathbf{F}$.

(b) With $\mathbf{F} = k(y, x, 0)$, we find $\nabla\times\mathbf{F} = (0, 0, 1-1) = (0, 0, 0)$, so $\mathbf{F}$ is conservative. The corresponding PE is $U = -\int_0^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r}' = -k\int_0^{\mathbf{r}}(y'dx' + x'dy')$. This integral can be evaluated along any path. One simple choice is to go from the origin out to x on the x axis (this contributes 0 to the integral) and then parallel to the y axis to y (which contributes $-kxy$). Thus $U = -kxy$. Clearly $-\nabla U = k(y, x, 0) = \mathbf{F}$.

(c) With $\mathbf{F} = k(-y, x, 0)$, we find $\nabla\times\mathbf{F} = (0, 0, 1+1) = (0, 0, 2)$, so $\mathbf{F}$ is not conservative.

4.24 ⋆⋆⋆ **(a)** Consider first the force on m due to a short segment dz of the rod at a height z above m. This force has magnitude $dF = Gm\mu\, dz/r^2$ in the direction shown in the left picture, where r is the distance from the element dz to m. To find the total force we must integrate this from $z = -\infty$ to ∞. When we do this, the z components F_z from points z and $-z$ will cancel. Since the component into the page is clearly zero, we have only to worry



about the component in the direction of $\hat{\boldsymbol{\rho}}$ (the unit vector in the ρ direction, pointing away from the z axis):

$$dF_\rho = -Gm\mu\cos\alpha\frac{dz}{r^2} = -Gm\mu\rho\frac{dz}{r^3}$$

where the last expression follows because $\cos\alpha = \rho/r$. Thus the net force has ρ component

$$F_\rho = -Gm\mu\rho\int_{-\infty}^{\infty}\frac{dz}{r^3} = -Gm\mu\rho\int_{-\infty}^{\infty}\frac{dz}{(z^2+\rho^2)^{3/2}} = -\frac{Gm\mu}{\rho}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta$$

where the last form results from the substitution $z/\rho = \tan\theta$. The final integral is just 2, and we conclude that

$$\mathbf{F} = -\frac{2Gm\mu}{\rho}\hat{\boldsymbol{\rho}}\,. \tag{ii}$$

(b) The unit vector $\hat{\boldsymbol{\rho}}$ lies in the xy plane. If we denote its polar angle by ϕ as in the right picture, then

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$$

where $\cos\phi = x/\rho$ and $\sin\phi = y/\rho$. Substituting into Eq. (ii), we find

$$\mathbf{F} = -\frac{2Gm\mu}{\rho^2}(\hat{\mathbf{x}}\,x + \hat{\mathbf{y}}\,y + \hat{\mathbf{z}}\,0)$$

where $\rho = \sqrt{x^2+y^2}$. It is now a straightforward matter to evaluate the components of $\boldsymbol{\nabla}\times\mathbf{F}$. For instance, $(\boldsymbol{\nabla}\times\mathbf{F})_x = \partial_y F_z - \partial_z F_y$ where I have introduced the abbreviation ∂_x for $\partial/\partial x$ and so on. Since $F_z = 0$ and F_y is independent of z, it follows that $(\boldsymbol{\nabla}\times\mathbf{F})_x = 0$. The y component works in exactly the same way, and

$$(\boldsymbol{\nabla}\times\mathbf{F})_z = \partial_x F_y - \partial_y F_x = -2Gm\mu\left(y\,\partial_x\rho^{-2} - x\,\partial_y\rho^{-2}\right). \tag{iii}$$

Now, it is a simple matter to check that $\partial_x\rho^{-2} = -2x\rho^{-4}$ and likewise $\partial_y\rho^{-2} = -2y\rho^{-4}$, so the two terms on the right of Eq.(iii) cancel exactly. Thus all three components of $\boldsymbol{\nabla}\times\mathbf{F}$ are zero, and $\mathbf{F}$ is conservative.

(c) From Eq.(ii), we see that $\mathbf{F}$ is especially simple in cylindrical polar coordinates. Specifically $F_\rho = -2Gm\mu/\rho$, which is independent of ϕ and z, while the other two components are zero, $F_\phi = F_z = 0$. Sustituting into the expression inside the back cover for $\boldsymbol{\nabla}\times\mathbf{F}$ in cylindrical polars, we see immediately the $\boldsymbol{\nabla}\times\mathbf{F} = 0$.

(d) The potential energy $U(\mathbf{r})$ is given by the integral $-\int \mathbf{F}\cdot d\mathbf{r}$ taken from any chosen reference point $\mathbf{r}_o$ to the point of interest $\mathbf{r}$. Since the integral is independent of path, we can choose any convenient path. One such choice is given in cylindrical polar coordinates as follows: Let the reference point $\mathbf{r}_o$ be given by coordinates (ρ_o, ϕ_o, z_o) and $\mathbf{r}$ by (ρ, ϕ, z). Now define the path in three stages:

1. Go from $\mathbf{r}_o$ parallel to the z axis until you reach the desired final value z.
2. Next move in a circle of constant ρ and z until you reach the desired final value of ϕ.
3. Finally go radially out in the direction of $\hat{\boldsymbol{\rho}}$ to the final value of ρ.

In the first two legs of this journey the force does no work. In the final leg, $\mathbf{F}$ and $d\mathbf{r}$ point in the $\hat{\boldsymbol{\rho}}$ direction, and the work integral is easily written as an integral over ρ to give

$$U(\mathbf{r}) = -\int_{\rho_o}^{\rho}\left(-\frac{2Gm\mu}{\rho'}\right)d\rho' = 2Gm\mu\ln(\rho/\rho_o).$$

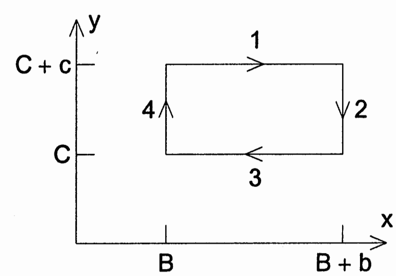
4.25 ★★★ **(a)** Let 1 and 2 denote any two points and Γ_a and Γ_b be any two paths leading from point 1 to point 2. Next let Γ be the closed path that starts at point 1, goes to point 2 via Γ_a, and then returns to point 1 tracing the path Γ_b backwards. Obviously

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_{\Gamma_a}\mathbf{F}\cdot d\mathbf{r} - \int_{\Gamma_b}\mathbf{F}\cdot d\mathbf{r}$$

The work integral is path-independent if and only if the right side is zero for any two paths joining any two points 1 and 2, and the left side is zero if and only if the work integral is zero around any closed path. Therefore the two statements are equivalent.

(b) If we accept Stokes's theorem, $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA$, then obviously $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ if $\nabla\times\mathbf{F} = 0$ everywhere.

(c) The integral going around the closed path Γ can be divided into four integrals, each along one of the straight paths labelled 1, 2, 3, and 4 in the picture.



$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \left(\int_1 + \int_2 + \int_3 + \int_4\right)\mathbf{F}\cdot d\mathbf{r}$$

Now

$$\int_1 + \int_3 = \int_B^{B+b} F_x(x, C+c, z)dx - \int_B^{B+b} F_x(x, C, z)dx$$

and

$$F_x(x, C+c, z)dx - F_x(x, C, z) = \int_C^{C+c} dy\,\frac{\partial F_x(x,y,z)}{\partial y}.$$

Combining the last two results, we find that

$$\int_1 + \int_3 = \int_B^{B+b} dx \int_C^{C+c} dy\,\frac{\partial F_x(x,y,z)}{\partial y} = \int \frac{\partial F_x}{\partial y}dA$$

where the final integral is a two-dimensional integral over the whole rectangle. There is a similar expression (with a minus sign) for $\int_2 + \int_4$, and, combining these two, we conclude that

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right) dA = \int (\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\, dA.$$

4.26 ★ If $U = mgy$, then, whether or not g depends on t, $-\boldsymbol{\nabla}U = -mg\boldsymbol{\nabla}y = -mg(0,1,0) = \mathbf{F}$. But

$$\frac{dE}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}m\mathbf{v}^2 + mgy\right) = m\dot{\mathbf{v}}\cdot\mathbf{v} + (mg\dot{y} + my\dot{g}) = -mgv_y + mgv_y + my\dot{g} = my\dot{g} \neq 0.$$

4.27 ★★ Stokes's theorem guarantees that the integral (4.48) is path-independent and hence that it defines a unique function $U(\mathbf{r}, t)$. To show that $-\boldsymbol{\nabla}U = \mathbf{F}$, consider the difference between the values of U at two neighboring points $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$:

$$\begin{aligned} dU &= U(\mathbf{r}+d\mathbf{r}, t) - U(\mathbf{r}, t) && \text{(iv)} \\ &= -\int_{\mathbf{r}}^{\mathbf{r}+d\mathbf{r}} \mathbf{F}(\mathbf{r}', t)\cdot d\mathbf{r}' = -\mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{r}. && \text{(v)} \end{aligned}$$

But $dU = (\boldsymbol{\nabla}U)\cdot d\mathbf{r}$, and since both of these expressions for dU are valid for all (small) $d\mathbf{r}$, it follows that $\mathbf{F}(\mathbf{r}, t) = -\boldsymbol{\nabla}U(\mathbf{r}, t)$. The argument leading to Eq.(4.19) requires that we look at the change in $E = T + U$ as we follow the particle moving from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ as the time advances from t to $t + dt$. The change in T is $dT = \mathbf{F}\cdot d\mathbf{r}$. The change in U is

$$dU = U(\mathbf{r}+d\mathbf{r}, t+dt) - U(\mathbf{r}, t) \qquad \text{(vi)}$$

and here's the difficulty. This dU is not the same as the dU in Eq.(iv). Here we are concerned with the change dU as the particle moves from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ *and* the time changes from t to $t + dt$. In Eq.(iv) dU is the difference between the values at $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ *at the same time* t. Thus the difference (vi) is not the same as $-\mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{r}$ as in (v); it does not cancel the change in KE when we evaluate $dE = dT + dU$, and mechanical energy is not conserved.

4.28 ★★ **(a)** Since $E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$, it follows that $\dot{x}(x) = \sqrt{2/m}\sqrt{E - \frac{1}{2}kx^2}$.

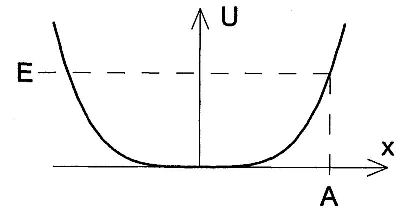
(b) At the end point $x = A$, we know that $T = 0$, so $E = \frac{1}{2}kA^2$. Substituting into the result of part (a), we find $\dot{x}(x) = \omega\sqrt{A^2 - x^2}$, where I have defined $\omega = \sqrt{k/m}$. From (4.58) we find

$$t = \int_0^x \frac{dx'}{\dot{x}(x')} = \frac{1}{\omega}\int_0^x \frac{dx'}{\sqrt{A^2 - x'^2}}.$$

The integral can be evaluated with the substitution $x' = A\sin\theta$ and gives $\arcsin(x/A)$. So $t = (1/\omega)\arcsin(x/A)$.

(c) Solving for x we find $x(t) = A\sin\omega t$. This shows that x is a sinusoidal function of t, which is the definition of simple harmonic motion. In particular, $x(t)$ repeats itself after a time t such that $\omega t = 2\pi$, or $t = 2\pi/\omega = 2\pi\sqrt{m/k}$.

4.29 ⋆⋆ **(a)** The mass moves to the right to the turning point at $x = A$, where $E = U = kA^4$ so that T and hence v are zero. It speeds up as it moves back to O and then moves out to the left until $x = -A$, where $E = U$ again. It then moves back to O, and repeats the whole cycle indefinitely (in the absence of any friction). Notice how this quartic well is much flatter at the bottom than the parabolic harmonic well.



(b) Since $T = E - U = k(A^4 - x^4)$, the velocity is $\dot{x} = \sqrt{2k/m}\sqrt{A^4 - x^4}$ (moving to the right). According to (4.58), the time to move out to $x = A$ is

$$t(0 \to A) = \int_0^A \frac{dx}{\dot{x}} = \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{\sqrt{A^4 - x^4}}.$$

(c) The period is four times this time. Thus, if we change variables to $u = x/A$,

$$\tau = \frac{1}{A}\sqrt{\frac{8m}{k}} \int_0^1 \frac{du}{\sqrt{1 - u^4}}$$

which is inversely proportional to A as claimed.

(d) Using any suitable software, the integral can be found to be 1.31, and setting $m = k = A = 1$ we find $\tau = 3.71$.

4.30 ⋆ **(a)** As the toy tips, the hemisphere rolls and its center O remains at a fixed height. On the other hand the height of the CM above O changes from $h - R$ to $(h - R)\cos\theta$. Therefore, the PE of the toy is now $U(\theta) = mg[R + (h - R)\cos\theta]$.

(b) Since $dU/d\theta = -mg(h - R)\sin\theta$, which vanishes at $\theta = 0$, we see that the upright position is an equilibrium, as expected. Next, $d^2U/d\theta^2 = -mg(h - R)\cos\theta = mg(R - h)$ at $\theta = 0$. Thus the equilibrium is stable if and only if $R > h$. [If $R = h$, then $U(\theta) = mgR =$ const, and the equilibrium is neutral.]

4.31 ⋆ **(a)** Because the string is inextensible, the height of m_2 below the wheel is $y = k - x$, where k is a constant, and its velocity is $\dot{y} = -\dot{x}$. Thus the total energy is $E = T + U = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 - m_1gx - m_2g(k - x)$ or $E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2)gx$, if we drop the uninteresting constant $-m_2gk$.

(b) The equation $dE/dt = 0$ yields $(m_1 + m_2)\dot{x}\ddot{x} - (m_1 - m_2)g\dot{x} = 0$, or

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g. \qquad \text{(vii)}$$

Applying the second law to each separate mass, we find

$$m_1\ddot{x} = m_1g - F_T \qquad \text{and} \qquad m_2\ddot{x} = F_T - m_2g$$

where F_T is the tension in the string. Adding these two equations, we can eliminate the tension and we get precisely Eq.(vii).

4.32 ⋆⋆ **(a)** Consider an infinitesimal displacement $d\mathbf{r} = (dx, dy, dz)$ occurring in a time dt. The bead's velocity is $\mathbf{v} = (dx/dt, dy/dt, dz/dt)$, so the speed is

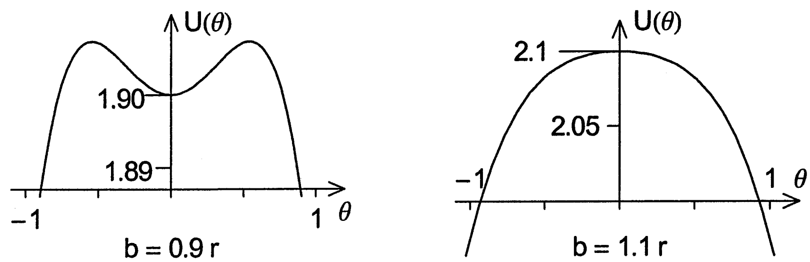
$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \frac{ds}{dt} = \dot{s}$$

(b) Differentiating the equation $v^2 = \mathbf{v}^2$, we get $2v\dot{v} = 2\mathbf{v}\cdot\dot{\mathbf{v}} = 2v\hat{\mathbf{v}}\cdot\mathbf{F}/m$. Cancelling the factor of $2v$ and recognising from part (a) that $\dot{v} = \ddot{s}$, we find that $m\ddot{s} = \hat{\mathbf{v}}\cdot\mathbf{F} = F_{\text{tang}}$

(c) The net force on the particle is $\mathbf{F} = \mathbf{N} - \nabla U$. Therefore, $F_{\text{tang}} = -\hat{\mathbf{v}}\cdot\nabla U$ (since $\mathbf{N}$ is perpendicular to $\hat{\mathbf{v}}$). Now, if we imagine a small displacement ds along the wire, $d\mathbf{r} = ds\,\hat{\mathbf{v}}$ (since $\hat{\mathbf{v}}$ is a unit vector tangent to the wire), so $dU = d\mathbf{r}\cdot\nabla U = ds\,\hat{\mathbf{v}}\cdot\nabla U$. Therefore, $F_{\text{tang}} = -\hat{\mathbf{v}}\cdot\nabla U = -dU/ds$.

4.33 ⋆⋆ **(a)** The derivation of $U(\theta)$ is given above Eq.(4.59).

(b)



(c) These two plots do bear out the finding of Example 4.7 that for $b < r$ the equilibrium at $\theta = 0$ is stable, whereas for $b > r$ it is unstable. In addition, they show that for $b < r$ there can be two further equilibrium points (symmetrically placed on either side of $\theta = 0$), both of which are unstable

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4.34 ⋆⋆ **(a)** The distance of the mass m below the support is $l\cos\phi$. Therefore, its height measured up from the equilibrium position is $l - l\cos\phi = l(1-\cos\phi)$ and its PE is $U = mgl(1-\cos\phi)$. The total energy is $E = \frac{1}{2}ml^2\dot{\phi}^2 + mgl(1-\cos\phi)$.

(b) The equation $dE/dt = 0$ reads $ml^2\dot{\phi}\ddot{\phi} + mgl\dot{\phi}\sin\phi = 0$ or $ml^2\ddot{\phi} = -mgl\sin\phi$. That is, $I\alpha = \Gamma$.

(c) Provided ϕ remains small, the equation of motion is well-approximated by $l\ddot{\phi} = -g\phi$, whose solution is $\phi = A\cos(\omega t) + B\sin(\omega t)$, where $\omega = \sqrt{g/l}$. This has period $\tau_o = 2\pi\sqrt{l/g}$.

4.35 ⋆⋆ **(a)** Because the string is inextensible, the height of m_2 below the wheel is $y = k - x$, where k is a constant, and its velocity is $\dot{y} = -\dot{x}$. The angular velocity of the wheel is $\omega = \dot{x}/R$. Thus the total energy is

$$\begin{aligned} E = T + U &= \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2\dot{x}^2 + \tfrac{1}{2}I\omega^2 - m_1 gx - m_2 g(k-x) \\ &= \tfrac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2 - (m_1 - m_2)gx, \end{aligned}$$

if we drop the uninteresting constant $-m_2gk$.

(b) The equation $dE/dt = 0$ yields $(m_1 + m_2 + I/R^2)\dot{x}\ddot{x} - (m_1 - m_2)g\dot{x} = 0$, or

$$(m_1 + m_2 + I/R^2)\ddot{x} = (m_1 - m_2)g. \tag{viii}$$

Applying the second law to each separate mass and to the rotating wheel, we find

$$m_1\ddot{x} = m_1 g - F_{\rm TR}, \quad m_2\ddot{x} = F_{\rm TL} - m_2 g, \quad \text{and} \quad I\dot{\omega} = (F_{\rm TR} - F_{\rm TL})R$$

where $F_{\rm TR}$ and $F_{\rm TL}$ are the tensions in the right and left lengths of string. Eliminating the two tensions and we get precisely Eq.(viii).

4.36 ⋆⋆ **(a)** It is easy to see that $h = b/\tan\theta$ and $H = l - b/\sin\theta$. Thus

$$U = -mgh - MgH = gb\left(\frac{m}{\sin\theta} - \frac{M}{\tan\theta}\right) = \frac{gb}{\sin\theta}(m - M\cos\theta)$$

where, in the third expression, I dropped an uninteresting constant.

(b) As you can check, the derivative of U is $dU/d\theta = gb(M - m\cos\theta)/\sin^2\theta$. If $M > m$, this never vanishes and there are no equilibrium points. If $M = m$, it vanishes at $\theta = 0$ which is impossible (unless the string is infinitely long). If $M < m$, there is an equilibrium point at $\theta_{\rm o} = \arccos(M/m)$. Since $\cos\theta$ decreases as θ increases, the factor $(M - m\cos\theta)$ is negative for $\theta < \theta_{\rm o}$ and positive for $\theta > \theta_{\rm o}$. Therefore, $U(\theta)$ has a minimum at $\theta_{\rm o}$ and the equilibrium is stable.

4.37 ⋆⋆⋆ **(a)** Let us take the zero of PE when $\phi = 0$. As the wheel turns through angle ϕ, the mass M rises by $R(1 - \cos\phi)$ and m descends by $R\phi$. Therefore the total PE is

$$U(\phi) = MgR(1 - \cos\phi) - mgR\phi.$$

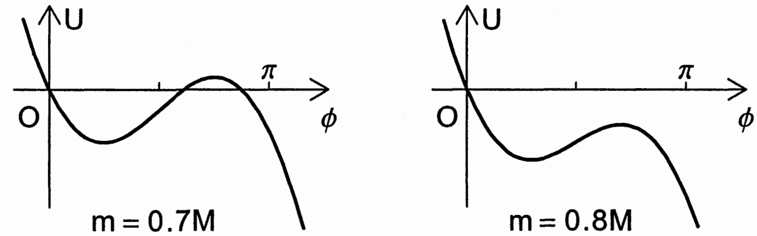
(b) The condition for equilibrium is that $dU/d\phi = 0$, that is,

$$MgR\sin\phi = mgR \qquad \text{or} \qquad \sin\phi = m/M. \tag{ix}$$

The equation $\sin\phi = m/M$ has no solutions if $m > M$. If $m = M$, there is one solution, at $\phi = \pi/2$. If $m < M$, there are two equilibrium angles, given by $\sin\phi = m/M$, one with M below the axle ($\phi < \pi/2$) and the other symmetrically located with M above the axle ($\phi > \pi/2$). All of these results are easy to understand in terms of torques. Equation (ix) is just the condition that the clockwise torque of M balance the counterclockwise torque of m. If $m > M$, they cannot balance; if $m < M$, they can balance in one position below the axle and one above.

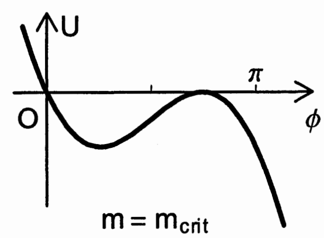
The stability depends on the second derivative, $d^2U/d\phi^2 = MgR\cos\phi$. This is positive (stable equilibrium) for the equilibrium with $\phi < \pi/2$, but negative (unstable) for $\phi > \pi/2$. This also is easy to understand. If M is at the equilibrium below the axle and ϕ increases, the torque of M increases and returns M to its equilibrium. If M is at the equilibrium above the axle, an increase of ϕ reduces the torque of M, and M continues to move away from equilibrium. Both the stable and unstable equilibrium points are clearly visible as a valley and a hill in each of the pictures below.

(c)



The behavior of the wheel when released from $\phi = 0$ is quite different in the two cases $m = 0.7M$ or $0.8M$. When $m = 0.7M$, the hill that represents the unstable equilibrium is higher than zero; when the wheel is released from $\phi = 0$ it will rotate past the stable valley but will stop before it reaches the unstable hill, and swing back again. In this case the wheel oscillates indefinitely between the two turning points. When $m = 0.8M$, the unstable hill is lower than zero, and, when released from $\phi = 0$, the wheel will rotate past the stable valley *and on past the hill.* Once past the hill, ϕ will increase, and the wheel rotate counterclockwise indefinitely (or until the string holding m runs out).

(d) As the mass m increases, the tendency for our system to turn in the direction of increasing ϕ increases, and the graph of $U(\phi)$ against ϕ steadily tilts down to the right. (This is clearly visible in the two pictures for $m = 0.7M$ and $0.8M$.) Somewhere between these two cases there must be a critical value m_{crit} for which the unstable hill just touches the horizontal axis, as shown. To find when this happens notice that at this critical value, U vanishes at the same point as $dU/d\phi$. This gives two simultaneous equations:



$$M(1 - \cos\phi) = m\phi \qquad \text{[that is, } U = 0\text{]}$$

and

$$M\sin\phi = m \qquad \text{[that is, } dU/d\phi = 0\text{]}$$

By dividing one of these by the other, we can eliminate m and M to give an equation for ϕ, namely $1 - \cos\phi = \phi\sin\phi$. This equation cannot be solved analytically, but is easly solved with a numerical equation solver (such as FindRoot in Mathematica) to give $\phi = 2.33$ rad. From the second of the equations above, it follows that the critical value of the mass is $m = M\sin\phi = 0.725M$. The graph shown was drawn for this value of the mass and confirms that this is indeed the critical value. If m is any less, the wheel released from O will remain trapped and oscillate; if m is any more, the wheel will escape and rotate for ever.

4.38 ⋆⋆⋆ **(a)** Using the trig identity $(1 - \cos\phi) = 2\sin^2\phi/2$, we can write the PE as $U(\phi) = 2mgl\sin^2\phi/2$. The KE is $\frac{1}{2}ml^2\dot{\phi}^2$ and the total energy is just $E = U(\Phi)$, since $T = 0$ at the endpoint $\phi = \Phi$. Thus we can solve the equation $T = U(\Phi) - U(\phi)$ to give

$$\dot{\phi} = \pm 2\sqrt{g/l}\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)} = \pm\frac{4\pi}{\tau_o}\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)}.$$

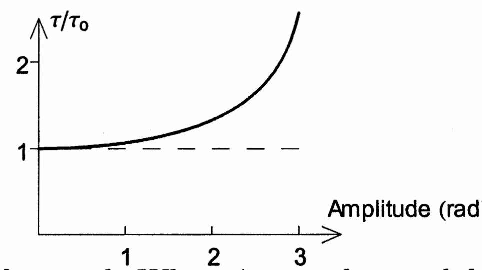
(To get the last expression I used $\sqrt{l/g} = \tau_o/2\pi$.) The time to swing from $\phi = 0$ to Φ is given by the integral $\int d\phi/\dot{\phi}$ taken from 0 to Φ (using the positive square root for $\dot{\phi}$). Since the period is four times this amount, we conclude that

$$\tau = 4\int_0^{\Phi} \frac{d\phi}{\dot{\phi}} = \frac{\tau_o}{\pi}\int_0^{\Phi} \frac{d\phi}{\sqrt{\sin^2\Phi/2 - \sin^2\phi/2}}.$$

If we make the recommended substitutions, $\sin\Phi/2 = A$ and $\sin\phi/2 = Au$, the square root in the denominator becomes $A\sqrt{1-u^2}$ and $d\phi = 2A\,du/\cos\phi/2 = 2A\,du/\sqrt{1-A^2u^2}$. Putting all this together, we find

$$\tau = \tau_o\frac{2}{\pi}\int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-A^2u^2}} = \tau_o\frac{2}{\pi}K(A^2). \tag{x}$$

(b) When the amplitude is small (less than about 0.5 rad), τ is very close to τ_o, as we would expect. When the amplitude is 3 rad (about 170 degrees), τ is about 2.5 times longer than its small-amplitude value. The graph suggests that as the amplitude approaches π, the period may approach infinity, a conclusion that is confirmed by putting $A = 1$ in the integral (x). This is easy to understand: When $\phi = \pi$, the pendulum is actually in unstable equilibrium. Thus the closer it gets to π, the more slowly it moves, causing the period to approach infinity.



4.39 ⋆⋆⋆ **(a)** Same as 4.38(a). **(b)** Ignoring the second square root completely, we get

$$\tau = \tau_o\frac{2}{\pi}\int_0^1 \frac{du}{\sqrt{1-u^2}} = \tau_o$$

as expected. (To show the integral is just $\pi/2$, use the substitution $u = \sin\alpha$.)

(c) If we make the binomial approximation suggested, we get

$$\tau = \tau_o\frac{2}{\pi}\left(\int_0^1 \frac{du}{\sqrt{1-u^2}} + \tfrac{1}{2}A^2\int_0^1 \frac{u^2du}{\sqrt{1-u^2}}\right).$$

The first integral is $\pi/2$ as before, while the same substitution shows the second integral to be $\pi/4$. Recalling that $A = \sin\Phi/2$, we find that $\tau = \tau_o[1 + \frac{1}{4}\sin^2(\Phi/2)]$, as stated. If $\Phi = 45°$, this gives $\tau = 1.037\tau_o$, which represents a 3.7% correction to the small-amplitude approximation (τ_o), and is itself within 0.3% of the exact answer ($1.040\tau_o$).

4.40 ⋆ **(a)** Referring to Fig.4.16, notice that the side OQ has length $OQ = r\sin\theta$, thus

$$x = OQ\cos\phi = r\sin\theta\cos\phi, \quad y = OQ\sin\phi = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

(b) By Pythagoras' theorem, $r = \sqrt{x^2 + y^2 + z^2}$. By elementary trig,

$$\theta = \arccos(z/r) = \arccos(z/\sqrt{x^2 + y^2 + z^2}) \text{ and } \phi = \arctan(y/x).$$

(But note that this leaves an ambiguity of π in ϕ. See the solution to Problem 1.42.)

4.41 ⋆ Since the mass moves in a circle, the radial component of its accelerations is just $-v^2/r$, the centripetal acceleration. By Newton's second law, $m(-v^2/r) = F_r = -dU/dr = -nkr^{n-1}$, from which we see that $mv^2 = nkr^n = nU$, and hence $T = \frac{1}{2}mv^2 = \frac{1}{2}nU$.

4.42 ⋆ Assuming that the spring doesn't bend, its extension is $r - r_o$ and the force which it exerts is $\mathbf{F} = -k(r - r_o)\hat{\mathbf{r}}$. Because this is central and spherically symmetric, it is automatically conservative.

4.43 ⋆⋆ **(a)** Since $\hat{\mathbf{r}} = \mathbf{r}/r$ we can write $\mathbf{F} = f(r)\mathbf{r}/r = g(r)\mathbf{r}$, say, where $g(r)$ is simply a convenient new name for $f(r)/r$. Now consider the x component of $\nabla \times \mathbf{F}$:

$$\begin{aligned}(\nabla \times \mathbf{F})_x &= \partial_y F_z - \partial_z F_y \\ &= \partial_y[g(r)z] - \partial_z[g(r)y] = z\,\partial_y g(r) - y\,\partial_z g(r).\end{aligned}$$

Now by the chain rule, $\partial_y g(r) = g'(r)\partial r/\partial y = g'(r)y/r$, and similarly for $\partial_z g(r)$. Therefore

$$(\nabla \times \mathbf{F})_x = zg'(r)y/r - yg'(r)z/r = 0.$$

Since the other two components work the same way, we conclude that $\nabla \times \mathbf{F} = 0$ and $\mathbf{F}(\mathbf{r})$ is conservative.

(b) If $\mathbf{F}(\mathbf{r})$ is central and spherically symmetric, its spherical polar components are $F_r = f(r)$ and $F_\theta = F_\phi = 0$. When we insert these in the given expression for $\nabla \times \mathbf{F}$ in spherical polars, only the two terms involving F_r survive since $F_\theta = F_\phi = 0$, but both of these remaining terms involve derivatives of F_r with respect to θ or ϕ, which are also zero. Therefore $\nabla \times \mathbf{F} = 0$.

4.44 ⋆⋆ Since $\mathbf{F} = f(r)\hat{\mathbf{r}}$, the work done going radially out from A to C is $W_{AC} = \int_A^C \mathbf{F}{\cdot}d\mathbf{r} = \int_{r_A}^{r_B} f(r)dr$. The same argument applies to W_{DB}, so $W_{AC} = W_{DB}$. On the other hand, on the paths CB and AD, $\mathbf{F}$ is perpendicular to $d\mathbf{r}$, so $W_{CB} = W_{AD} = 0$. Therefore

$$W_{ACB} = W_{AC} + W_{CB} = W_{AD} + W_{DB} = W_{ADB}$$

4.45 ⋆⋆ That $\mathbf{F}$ is conservative tells us that the amounts of work done along the two paths ACB and ADB are equal, $W_{ACB} = W_{ADB}$. That $\mathbf{F}$ is central implies that no work is done along CB and AD, that is, $W_{CB} = W_{AD} = 0$. Therefore $W_{AC} = W_{DB}$. Since $\mathbf{F} = f(\mathbf{r})\hat{\mathbf{r}}$, $W_{AC} = f(\mathbf{r}_A)dr$ and $W_{DB} = f(\mathbf{r}_D)dr$, and since these two are equal, it follows that $f(\mathbf{r}_A) = f(\mathbf{r}_D)$. Finally, since A and D are any two points at the same distance from O, this says that $f(\mathbf{r})$ depends only on $|\mathbf{r}|$. That is, $f(\mathbf{r}) = f(r)$.

4.46 ⋆ In an elastic collision KE is conserved, so

$$m_1 v_1^2 = m_1 {v'_1}^2 + m_2 {v'_2}^2. \tag{xi}$$

Since momentum is also conserved, $m_1\mathbf{v}_1 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$, or, squaring both sides,

$$m_1^2 v_1^2 = m_1^2 {v'_1}^2 + m_2^2 {v'_2}^2 + 2m_1 m_2 \mathbf{v}'_1 \cdot \mathbf{v}'_2. \tag{xii}$$

If we multiply both sides of (xi) by m_1 and subtract from (xii), four terms cancel and we're left with

$$0 = m_2(m_2 - m_1){v'_2}^2 + 2m_1 m_2 \mathbf{v}'_1 \cdot \mathbf{v}'_2.$$

Therefore

$$\mathbf{v}'_1 \cdot \mathbf{v}'_2 = \frac{m_1 - m_2}{2m_1} {v'_2}^2.$$

If $m_1 > m_2$, $\cos\theta$ has to be positive, so $\theta < \pi/2$. Whereas if $m_1 < m_2$, $\cos\theta$ has to be negative, so $\theta > \pi/2$.

4.47 ⋆ Since the problem is one-dimensional, we can use v to denote velocity (strictly speaking the x component of velocity). Then conservation of KE (a little rearranged) says

$$m_1(v_1^2 - {v'_1}^2) = m_2({v'_2}^2 - v_2^2)$$

and, similarly, conservation of momentum

$$m_1(v_1 - v'_1) = m_2(v'_2 - v_2).$$

Dividing the first of these by the second, we find

$$v_1 + v'_1 = v'_2 + v_2 \quad \text{or} \quad v_1 - v_2 = v'_2 - v'_1\,.$$

4.48 ⋆ Let the initial speed of particle 1 be v_1 and the final speed of the composite be v'. Then, conservation of momentum says that $m_1 v_1 = (m_1 + m_2)v'$. Therefore the initial and final KEs are $T = \frac{1}{2}m_1 v_1^2$ and $T' = \frac{1}{2}(m_1 + m_2)v'^2 = \frac{1}{2}m_1^2 v_1^2/(m_1 + m_2)$, and the fractional loss of KE is

$$\frac{T - T'}{T} = \frac{m_1(m_1 + m_2) - m_1^2}{m_1(m_1 + m_2)} = \frac{m_2}{m_1 + m_2}.$$

If $m_1 \ll m_2$, almost all the initial KE is lost; if $m_2 \ll m_1$, almost none of the initial KE is lost.

4.49 ⋆⋆ Let's define $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, so that $\mathbf{r}$ is a vector pointing from particle 2 to particle 1. The PE is $U = \gamma/r$, the force on particle 1 due to particle 2 is $\mathbf{F}_{12} = (\gamma/r^2)\hat{\mathbf{r}}$, and that on particle 2 due to particle 1 is $\mathbf{F}_{21} = -(\gamma/r^2)\hat{\mathbf{r}}$. We'll start with the x component of $-\nabla_1 U$:

$$(-\nabla_1 U)_x = -\frac{\partial U}{\partial x_1} = \frac{\gamma}{r^2}\frac{\partial r}{\partial x_1} = \frac{\gamma}{r^3}(x_1 - x_2) \tag{xiii}$$

where for the second equality I used the chain rule and for the third I used the following:

$$\frac{\partial r}{\partial x_1} = \frac{\partial}{\partial x_1}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2} = \frac{x_1-x_2}{\sqrt{\cdots}} = \frac{x_1-x_2}{r}. \tag{xiv}$$

Combining (xiii) with the corresponding results for the y and z components, we see that

$$-\nabla_1 U = \frac{\gamma}{r^3}(\mathbf{r}_1-\mathbf{r}_2) = \frac{\gamma}{r^3}\mathbf{r} = \frac{\gamma}{r^2}\hat{\mathbf{r}} = \mathbf{F}_{12}. \tag{xv}$$

If we had evaluated $-\nabla_2 U$, the only difference would have been that in the equivalent of (xiv), we would have had an extra minus sign because $\partial(x_1-x_2)^2/\partial x_2 = -2(x_1-x_2)$. Thus in place of (xv), we would have found $-\nabla_2 U = -(\gamma/r^2)\hat{\mathbf{r}} = \mathbf{F}_{21}$.

4.50 ⋆⋆ By the chain rule

$$\frac{\partial}{\partial x_1}f(x_1-x_2) = f'(x_1-x_2)\frac{\partial}{\partial x_1}(x_1-x_2) = f'(x_1-x_2)$$

where f' denotes the derivative of f with respect to its argument. On the other hand,

$$\frac{\partial}{\partial x_2}f(x_1-x_2) = f'(x_1-x_2)\frac{\partial}{\partial x_2}(x_1-x_2) = -f'(x_1-x_2).$$

This establishes the required result for the one-dimensional function $f(x_1-x_2)$. We can prove the corresponding three-dimensional result, one component at a time. For example, $(\nabla_1 U)_x = \partial U/\partial x_1$, whereas $(\nabla_2 U)_x = \partial U/\partial x_2$, and by the one-dimensional result, the latter is minus the former. (The four extra variables y_1, z_1, y_2, z_2 act as constants and do not affect the result.) Since the y and z components work the same way, this establishes the required three-dimensional result.

4.51 ⋆⋆

$$\begin{aligned} U(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = U_{12}(\mathbf{r}_1-\mathbf{r}_2) &+ U_{13}(\mathbf{r}_1-\mathbf{r}_3) + U_{14}(\mathbf{r}_1-\mathbf{r}_4) \\ &+ U_{23}(\mathbf{r}_2-\mathbf{r}_3) + U_{24}(\mathbf{r}_2-\mathbf{r}_4) + U_{34}(\mathbf{r}_3-\mathbf{r}_4) \\ &+ U_1^{\text{ext}}(\mathbf{r}_1) + U_2^{\text{ext}}(\mathbf{r}_2) + U_3^{\text{ext}}(\mathbf{r}_3) + U_4^{\text{ext}}(\mathbf{r}_4) \end{aligned}$$

so

$$\begin{aligned} -\nabla_3 U = 0 &+ \mathbf{F}_{31} + 0 \\ &+ \mathbf{F}_{32} + 0 + \mathbf{F}_{34} \\ &+ 0 + 0 + \mathbf{F}_3^{\text{ext}} + 0 = \mathbf{F}_3^{\text{net}}. \end{aligned}$$

4.52 ⋆⋆ **(a)** By the work-KE theorem for each of the four particles, $dT_1 = W_1, dT_2 = W_2, dT_3 = W_3$, and $dT_4 = W_4$. Adding these four equations, we conclude that $dT = W_{\text{tot}}$.

(b) The work done on the four separate particles has the form

$$
\begin{aligned}
W_1 &= (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}})\cdot d\mathbf{r}_1 \\
W_2 &= (\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}})\cdot d\mathbf{r}_2 \\
W_3 &= (\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}})\cdot d\mathbf{r}_3 \\
W_4 &= (\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}})\cdot d\mathbf{r}_4
\end{aligned}
$$

When we add these, we get 12 terms involving internal forces which we can group in pairs ($\mathbf{F}_{12}$ with $\mathbf{F}_{21} = -\mathbf{F}_{12}$ and so on), to give

$$
\begin{aligned}
W = \mathbf{F}_{12}\cdot(d\mathbf{r}_1 - d\mathbf{r}_2) &+ \mathbf{F}_{13}\cdot(d\mathbf{r}_1 - d\mathbf{r}_3) + \mathbf{F}_{14}\cdot(d\mathbf{r}_1 - d\mathbf{r}_4) \\
&+ \mathbf{F}_{23}\cdot(d\mathbf{r}_2 - d\mathbf{r}_3) + \mathbf{F}_{24}\cdot(d\mathbf{r}_2 - d\mathbf{r}_4) + \mathbf{F}_{34}\cdot(d\mathbf{r}_3 - d\mathbf{r}_4) \\
&+ \mathbf{F}_1^{\text{ext}}\cdot d\mathbf{r}_1 + \mathbf{F}_2^{\text{ext}}\cdot d\mathbf{r}_2 + \mathbf{F}_3^{\text{ext}}\cdot d\mathbf{r}_3 + \mathbf{F}_4^{\text{ext}}\cdot d\mathbf{r}_4.
\end{aligned}
$$

The first term on the right is equal to $-dU_{12}$ [exactly as in Equations (4.85) and (4.86)], and the next five terms behave similarly. The first term on the third line is just $-dU_1^{\text{ext}}$, and so on. Thus,

$$W = (-dU_{12} - dU_{13} - \cdots - dU_{34}) + (-dU_1^{\text{ext}} - \cdots - dU_4^{\text{ext}}) = -dU,$$

as required. Combining these two results, we have $dT = -dU$ or $E = T + U = \text{const}$.

4.53 ⋆⋆ **(a)** The centripetal acceleration is supplied by the Coulomb force, so $mv^2/r = ke^2/r^2$ and thence $T = \frac{1}{2}mv^2 = \frac{1}{2}ke^2/r = -\frac{1}{2}U$.

(b) $E = T_1 + T_2 + U_1 + U_2 + U_{12} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - ke^2\left(\dfrac{1}{r_1} + \dfrac{1}{r_2} - \dfrac{1}{r_{12}}\right)$.

(c) Long before: $E = T_1 + T_2 + U_1 + 0 + 0 = T_2 - \dfrac{ke^2}{2r}$.

Long after: $E' = T_1' + T_2' + 0 + U_2' + 0 = T_1' - \dfrac{ke^2}{2r'}$.

Therefore, by conservation of energy, $T_1' = T_2 + \frac{1}{2}ke^2\left(\dfrac{1}{r'} - \dfrac{1}{r}\right)$.

Chapter 5

Oscillations

I covered this chapter in 4.5 *fifty-minute lectures.*

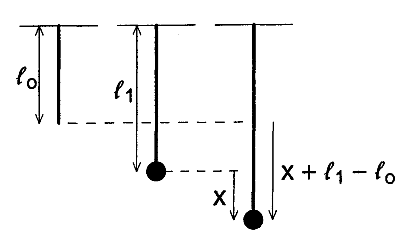
The first two sections of this chapter deal with the one-dimensional simple harmonic oscillator, something almost all our students are familiar with. On the other hand, I found many students distressingly weak at the trigonometry behind the various ways to write oscillations — hence the rather ponderous enumeration of the four equivalent forms for an oscillatory solutions in Section 5.2. Section 5.3 is about the two-dimensional oscillator, which is obviously important and comes up from time to time later in the book but plays no role in the rest of the chapter; so you could omit this section.

Section 5.4 is about the damped oscillator (back in one dimension) and 5.5 and 5.6 treat the driven damped oscillator and resonance — topics that every physics major should surely understand and that are essential if you're going to cover Chapter 12 on the chaos of non-linear oscillators. Finally, Sections 5.7–5.9 use Fourier series to solve for the motion of an oscillator that is driven by an arbitrary periodic force. While these three sections could certainly be omitted, my class voted overwhelmingly to include them.

There are several demonstration experiments you can show on driven damped oscillations, including the very simple on described in footnote 14.

Solutions to Problems for Chapter 5

5.1 ⋆ **(a)** When the spring is at the new equilibrium position it is extended by an amount $l_1 - l_o$, so its tension is $k(l_1 - l_o)$. This must balance the weight mg. Therefore, $k(l_1 - l_o) = mg$. When it is stretched a further x, its total extension is $x + l_1 - l_o$ and the tension is $k(x + l_1 - l_o)$ upward. Thus the total downward force on the mass is

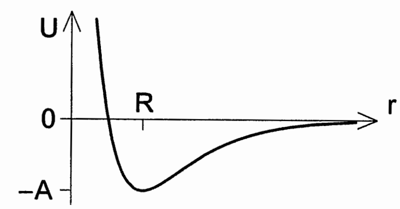


$$F = -k(x + l_1 - l_o) + mg = -kx.$$

(b) The total PE is $U = U_{\text{sp}} + U_{\text{gr}}$ or

$$U = \tfrac{1}{2}k(x + l_1 - l_o)^2 - mgx = \tfrac{1}{2}kx^2 + k(l_1 - l_o)x - mgx + \tfrac{1}{2}k(l_1 - l_o)^2 = \tfrac{1}{2}kx^2 + \text{const}$$

5.2 ⋆ When $r = 0$, $U = A[(e^{R/S} - 1)^2 - 1]$, which is large and positive since $R \gg S$. When $r \to \infty$, U is negative and approaches 0. The smallest possible value of U is when $r = R$ and $U = -A$; that is, the equilibrium separation is $r_o = R$. If we set $r = R + x$ and make a Taylor expansion of the exponential term in U, then

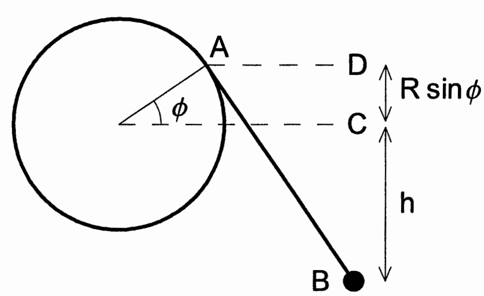


$$U = A\left[\left(\left\{1 - \frac{x}{S} + \cdots\right\} - 1\right)^2 - 1\right] \approx -A + A\left(\frac{x}{S}\right)^2 = \text{const} + \tfrac{1}{2}kx^2$$

where $k = 2A/S^2$.

5.3 ⋆ The height of the mass below the pivot is $l\cos\phi$. Therefore the height above the bottom is $l(1 - \cos\phi)$ and the PE is $U = mgl(1 - \cos\phi)$. If ϕ is small, $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ and $U \approx \frac{1}{2}mgl\phi^2 = \frac{1}{2}k\phi^2$, where $k = mgl$.

5.4 ⋆⋆ The PE is $U = -mgh$ where h is the height of the mass, measured down from the level of the cylinder's center. To find h, note first that as the pendulum swings from equilibrium to angle ϕ, a length $R\phi$ of string unwinds from the cylinder. Thus the length of string away from the cylinder is $AB = (l_o + R\phi)$, and the height BD is $BD = (l_o + R\phi)\cos\phi$. Since the height $CD = R\sin\phi$, we find by subtraction that $h = BD - CD = l_o\cos\phi + R(\phi\cos\phi - \sin\phi)$. Therefore



$$U = -mgh = -mg[l_o\cos\phi + R(\phi\cos\phi - \sin\phi)].$$

If ϕ remains small we can write $\cos\phi \approx 1 - \phi^2/2$ and $\sin\phi \approx \phi$, to give

$$U \approx -mg\left\{l_o - \tfrac{1}{2}l_o\phi^2 + R\left[\phi(1 - \tfrac{1}{2}\phi^2) - \phi\right]\right\} \approx -mgl_o + \tfrac{1}{2}mgl_o\phi^2 = \text{const} + \tfrac{1}{2}k\phi^2$$

where in the third expression I dropped the term in ϕ^3. The constant $k = mgl_o$, which is the same as for a simple pendulum of length l_o. Evidently, wrapping the string around a cylinder makes no difference for small oscillations.

5.5 ⋆ **(a)** In the form (I), $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$, we can replace the exponentials using Euler's formula $e^{\pm i\omega t} = \cos\omega t \pm i \sin\omega t$ to give

$$x(t) = B_1 \cos\omega t + B_2 \sin\omega t \qquad \text{(II)}$$

where $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$.

(b) To get form (III) from form (II), define A and δ to be the hypotenuse and lower angle of a right triangle with base B_1 and height B_2 as in Figure 5.4. Then (II) can be rewritten as in Equation (5.11) to give the form (III).

(c) Since $\cos\theta = \operatorname{Re} e^{i\theta}$, we can rewrite the form (III) as

$$x(t) = \operatorname{Re} Ae^{i(\omega t - \delta)} = \operatorname{Re} Ce^{i\omega t} \qquad \text{(IV)}$$

where $C = Ae^{-i\delta}$.

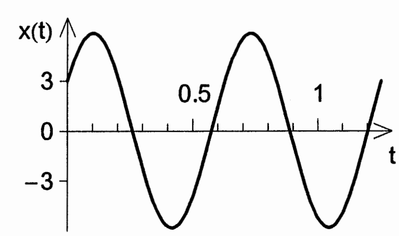
(d) Finally, since $\operatorname{Re} z = \frac{1}{2}(z + z^*)$, we can rewrite (IV) as

$$x(t) = \tfrac{1}{2}(Ce^{i\omega t} + C^* e^{-i\omega t}) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \qquad \text{(I)}$$

where $C_1 = C/2$ and $C_2 = C^*/2$.

5.6 ⋆ The position can be written as $x(t) = A\cos(\omega t - \delta)$, where $A = 2x_o$ and we know that $x(0) = x_o$ and $v(0) < 0$. Putting $t = 0$, we see that $\cos(-\delta) = 0.5$ and hence that $\delta = \pm\pi/3$. The velocity is $v(t) = -A\omega \sin(\omega t - \delta)$, so the condition that $v(0) < 0$ implies that $\sin(\delta) < 0$. Therefore $\delta = -\pi/3$ and $x(t) = 2x_o \cos(\omega t + \pi/3)$.

5.7 ⋆ **(a)** Since $x(t) = B_1 \cos\omega t + B_2 \sin\omega t$, $x(0) = B_1$ and $v(0) = \omega B_2$. Therefore, $B_1 = x_o$ and $B_2 = v_o/\omega$.



(b) If $k = 50$ N/m and $m = 0.5$ kg, then $\omega = \sqrt{k/m} = 10\ \text{s}^{-1}$. If $x_o = 3.0$ m and $v_o = 50$ m/s, then $B_1 = 3$ m, $B_2 = 5$ m.

(c) If we rewrite $x(t)$ in the form $x(t) = A\cos(\omega t - \delta)$, with $\delta = \arctan(B_2/B_1) = 1.03$ rad, then x first vanishes when $(\omega t - \delta) = \pi/2$ or $t = 0.26$ s, and v first vanishes when $(\omega t - \delta) = 0$ or $t = \delta/\omega = 0.10$ s.

5.8 ⋆ **(a)** $\omega = \sqrt{k/m} = \sqrt{80/0.2} = 20\ \text{s}^{-1}$, $f = \omega/2\pi = 3.2$ Hz, and $\tau = 2\pi/\omega = 0.31$ s.

(b) Since $x_o = 0$, $A\cos(-\delta) = 0$, so $\delta = \pm\pi/2$. Since $v_o = \omega A \sin\delta = 40$ m/s, δ must be positive, $\delta = +\pi/2$, and therefore $A = v_o/\omega = 2$ m.

5.9 ⋆ We are given that $A = 0.2$ m and $v_o = 1.2$ m/s. We know that $E = \frac{1}{2}kA^2 = \frac{1}{2}mv_o^2$. Therefore $m/k = A^2/v_o^2$ and $\tau = 2\pi\sqrt{m/k} = 2\pi A/v_o = 1.05$ s.

5.10 ⋆ If $F = -F_o \sinh \alpha x$, then $U = -\int F\,dx = (F_o/\alpha)\cosh \alpha x$. The only equilibrium position is at $x = 0$ and, for points close to this, Taylor's series gives

$$U(x) \approx (F_o/\alpha)(1 + \tfrac{1}{2}\alpha^2 x^2) = \tfrac{1}{2}kx^2 + \text{const},$$

where $k = \alpha F_o$. The angular frequency of oscillations is $\omega = \sqrt{k/m} = \sqrt{\alpha F_o/m}$.

5.11 ⋆ The given information gives two expressions of the total energy E

$$E = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 \qquad \text{and} \qquad E = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2. \tag{i}$$

Equating these two, we find $m(v_1^2 - v_2^2) = k(x_2^2 - x_1^2)$. This implies that

$$\omega^2 = \frac{k}{m} = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}.$$

We know also that $E = \frac{1}{2}kA^2$, and inserting this in the first of Eqs.(i) we conclude that

$$A^2 = \frac{m}{k}v_1^2 + x_1^2 = \frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}v_1^2 + x_1^2 = \frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}.$$

5.12 ⋆⋆ Because $\sin^2(\omega t - \delta)$ oscillates symmetrically between 0 and 1, its average over a cycle is fairly obviously $\frac{1}{2}$. To prove it, write $\sin^2\theta = \frac{1}{2}[1 - \cos 2\theta]$, so that

$$\langle \sin^2(\omega t - \delta) \rangle = \frac{1}{2\tau}\int_0^\tau [1 - \cos 2(\omega t - \delta)]dt = \frac{1}{2} - \frac{1}{2\tau}\big[\sin 2(\omega t - \delta)\big]_0^\tau = \frac{1}{2}$$

where the final square bracket is zero because the sine function is τ-periodic. The corresponding result with a cosine follows in exactly the same way.

We know from (5.16) that $E = \frac{1}{2}kA^2$, and, since $T = \frac{1}{2}kA^2\sin^2(\omega t - \delta)$, it follows that $\langle T \rangle = \frac{1}{4}kA^2 = \frac{1}{2}E$, and similarly for $\langle U \rangle$.

5.13 ⋆⋆ Since $U(r) = U_o(r/R + \lambda^2 R/r)$, its derivative is $dU/dr = U_o(1/R - \lambda^2 R/r^2)$, which vanishes at $r = \lambda R$ and nowhere else. Clearly $U(r) \to +\infty$ when $r \to 0$ or ∞, so $U(r)$ has a minimum at $r = r_o = \lambda R$. If we write $r = r_o + x$ then

$$U = \lambda U_o\left(\frac{r_o + x}{r_o} + \frac{r_o}{r_o + x}\right) = \lambda U_o\left(1 + \frac{x}{r_o} + \left[1 + \frac{x}{r_o}\right]^{-1}\right)$$
$$\approx \lambda U_o\left(1 + \frac{x}{r_o} + 1 - \frac{x}{r_o} + \frac{x^2}{r_o^2}\right) = \lambda U_o\left(2 + \frac{x^2}{\lambda^2 R^2}\right).$$

where, in the second line, I dropped all terms in $(x/r_o)^3$ and higher. This has the expected form $U = \frac{1}{2}kx^2 + \text{const}$, where $k = 2U_o/(\lambda R^2)$. The angular frequency is $\omega = \sqrt{k/m} = \sqrt{2U_o/(m\lambda R^2)}$.

5.14 ⋆ Given that $\mathbf{F} = -(k_x x, k_y y)$, the PE is

$$U(\mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\cdot d\mathbf{r}' = \int_0^{\mathbf{r}} (k_x x' dx' + k_y y' dy') = k_x \int_0^x x' dx' + k_y \int_0^y y' dy' = \tfrac{1}{2}(k_x x^2 + k_y y^2).$$

5.15 ⋆ If we replace the variable t by $t' = t + t_o$, Eq. (5.19) becomes

$$x = A_x \cos(\omega t' - \omega t_o - \delta_x) \qquad \text{and} \qquad y = A_y \cos(\omega t' - \omega t_o - \delta_y)$$

and if we then choose t_o such that $\omega t_o = -\delta_x$ these become

$$x = A_x \cos(\omega t') \qquad \text{and} \qquad y = A_y \cos(\omega t' - [\delta_y - \delta_x]).$$

If we rename t' as t and set $\delta_y - \delta_x = \delta$, this is the desired form.

5.16 ⋆ With $\delta = \pi/2$ Eq.(5.20) reads

$$x = A_x \cos(\omega t) \qquad \text{and} \qquad y = A_y \cos(\omega t - \pi/2) = A_y \sin(\omega t)$$

from which it follows that $x^2/A_x^2 + y^2/A_y^2 = \cos^2(\omega t) + \sin^2(\omega t) = 1$, the equation of an ellipse with semi-major and semi-minor axes A_x and A_y.

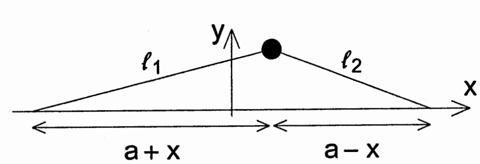
5.17 ⋆⋆ **(a)** Suppose that the ratio of frequencies is rational, that is $\omega_x/\omega_y = p/q$, where p and q are integers. Then let $\tau = 2\pi p/\omega_x = 2\pi q/\omega_y$. Now consider the following

$$x(t+\tau) = A_x \cos[\omega_x(t+\tau)]) = A_x \cos[\omega_x t + 2\pi p] = A_x \cos[\omega_x t] = x(t)$$

where in the second equality I used our definition of τ and in the second the fact that if p is an integer then $\cos(\theta + 2\pi p) = \cos(\theta)$. This shows that $x(t)$ is periodic with period τ. By exactly the same argument, $y(t)$ is also periodic with the same period τ, and we've proved that the whole motion is likewise. What we usually call *the* period of the motion is the value of $\tau = 2\pi p/\omega_x$ with p and q the *smallest* integers for which $\omega_x/\omega_y = p/q$.

(b) Suppose the motion is periodic. Then there is a τ such that $x(t+\tau) = x(t)$ and $y(t+\tau) = y(t)$. Running the previous argument backward, we see that $\omega_x\tau$ must be an integer multiple of 2π, that is $\omega_x\tau = 2\pi p$ for some integer p. Similarly $\omega_y\tau = 2\pi q$ for some integer q. Dividing these two conclusions, we see that $\omega_x/\omega_y = p/q$ and the ratio of frequencies is rational. Therefore, if the ratio is irrational, the motion cannot be periodic.

5.18 ⋆⋆⋆ When the mass is at position (x, y), the lengths of the two springs are l_1 and l_2, where



$$l_1 = \sqrt{(a+x)^2 + y^2} = a\left(1 + \frac{2x}{a} + \frac{x^2+y^2}{a^2}\right)^{1/2}$$

$$\approx a\left[1 + \frac{1}{2}\left(\frac{2x}{a} + \frac{x^2+y^2}{a^2}\right) - \frac{1}{8}\left(\frac{2x}{a}\right)^2\right] = a + x + \frac{y^2}{2a}.$$

Here, in passing from the first to the second line, I have used the Taylor expansion $(1+\epsilon)^{1/2} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \cdots$, dropping all terms of third degree in x or y, but being careful to keep all terms of second degree. The PE of spring 1 is therefore

$$\begin{aligned} U_1 = \tfrac{1}{2}k(l_1 - l_o)^2 &= \tfrac{1}{2}k[(a - l_o) + x + y^2/2a]^2 \\ &\approx \tfrac{1}{2}k[(a - l_o)^2 + 2(a - l_o)x + x^2 + (1 - l_o/a)y^2] \end{aligned}$$

where, again, I have dropped terms of degree three in x or y. To find U_2, we have only to replace x by $-x$, and for the total PE we just add U_1 and U_2. When we do this, the terms linear in x cancel, leaving

$$U = U_1 + U_2 = k[x^2 + (1 - l_o/a)y^2] + \text{const}$$

which has the form (5.104), apart from the unimportant constant.

If $a < l_o$, the coefficient of y^2 is negative and the equilibrium at the origin O is unstable. This is because, with $a < l_o$, the springs are in compression at O. When the mass moves a little from O along the y axis, the compression in the springs forces it further away, causing the instability.

5.19 ⋆⋆⋆ The simplest way to find the total PE of all four springs is treat them two at a time. The two springs anchored on the x axis constitute the system of Problem 5.18, for which we found that

$$U_1 + U_2 = k[x^2 + (1 - l_o/a)y^2].$$

(See the solution to that problem. I've dropped an uninteresting constant here.) In exactly the same way, the PE of the two springs anchored on the y axis is

$$U_3 + U_4 = k[y^2 + (1 - l_o/a)x^2].$$

Adding these, we find for the total PE of all four springs

$$U = U_1 + U_2 + U_3 + U_4 = k[x^2 + y^2 + (1 - l_o/a)(x^2 + y^2)] = k\frac{2a - l_o}{a}r^2,$$

where I've used the fact that $x^2 + y^2 = r^2$. This has the advertised form $U = \frac{1}{2}k'r^2$ with an effective spring constant $k' = 2k(2a - l_o)/a$. The corresponding force is $\mathbf{F} = -\boldsymbol{\nabla}U = -k'\mathbf{r}$.

5.20 ⋆ The derivative of the decay parameter, $\beta - \sqrt{\beta^2 - \omega_o^2}$, is $1 - \beta/\sqrt{\beta^2 - \omega_o^2}$. With $\beta > \omega_o$ the second term is clearly greater than 1, so the derivative is negative, and the decay parameter *decreases* as β increases. The graph is shown in Fig.5.13.

5.21 ⋆ If $x = te^{-\beta t}$, then

$$\dot{x} = (1 - \beta t)e^{-\beta t} \qquad \text{and} \qquad \ddot{x} = (\beta^2 t - 2\beta)e^{-\beta t}.$$

Therefore

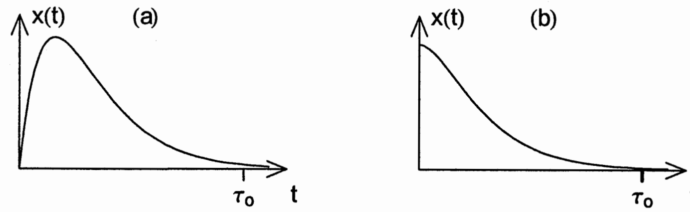
$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = \left[(\beta^2 t - 2\beta) + 2\beta(1 - \beta t) + \omega_o^2 t\right] e^{-\beta t} = [\omega_o^2 - \beta^2]e^{-\beta t} = 0$$

if (and only if) $\beta = \omega_o$. That is, $x = te^{-\beta t}$ *does* satisfy Eq.(5.28) if the damping is critical.

5.22 ⋆ **(a)** The general solution for a critically damped oscillator ($\beta = \omega_o$) is given in (5.44) as $x(t) = e^{-\omega_o t}(C_1 + C_2 t)$. Thus

$$x_o = x(0) = C_1 \qquad \text{and} \qquad v_o = \dot{x}(0) = C_2 - \omega_o C_1 \,. \tag{ii}$$

Here $x_o = 0$, so $C_1 = 0$ and $C_2 = v_o$. Therefore, $x(t) = v_o t e^{-\omega_o t}$.



(b) In this case $v_o = 0$ and Eqs.(ii) imply that $C_1 = x_o$ and $C_2 = \omega_o x_o$. Therefore $x(t) = x_o e^{-\omega_o t}(1 + \omega_o t)$. When $t = \tau_o$, the natural period, $x = x_o e^{-2\pi}(1 + 2\pi) = 0.0136 x_o$. The motion is almost 99% damped out.

5.23 ⋆ Because $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$,

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = \dot{x}(-b\dot{x}) = vF_{\text{dmp}}$$

where, for the third equality, I used the differential equation (5.24). Since vF_{dmp} is the rate at which F_{dmp} does work on the oscillator, this is the requested result.

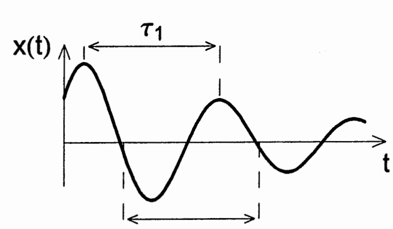
5.24 ⋆ As long as $\beta < \omega_o$, we can define $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ and we can use as two independent solutions

$$x_1(t) = e^{-\beta t}\cos(\omega_1 t) \qquad \text{and} \qquad x_2(t) = e^{-\beta t}\sin(\omega_1 t).$$

If $\beta \to \omega_o$, then $\omega_1 \to 0$ and so $x_1(t) \to e^{-\beta t}$. That is, as $\beta \to \omega_o$, our first solution for the case $\beta < \omega_o$ becomes the first solution for the case $\beta = \omega_o$.

Unfortunately, in the same limit, $x_2(t) \to 0$, and our second solution evaporates. However, for our second solution we could equally have used $\tilde{x}_2(t) = e^{-\beta t}\sin(\omega_1 t)/\omega_1$. Since $\sin(kt)/k \to t$ as $k \to 0$, this new second solution does not vanish as $\omega_1 \to 0$; instead, as $\beta \to \omega_o$, the new solution satisfies $\tilde{x}_2(t) \to te^{-\beta t}$ and we obtain the second solution for the case $\beta = \omega_o$.

5.25 ⋆⋆ **(a)** Because $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$, its derivative is $dx/dt = -Ae^{-\beta t}[\beta\cos(\cdots) + \omega_1\sin(\cdots)]$. The maxima and minima of $x(t)$ occur when this derivative vanishes, that is, when $\tan(\omega_1 t - \delta) = -\beta/\omega_1$. Because $\tan\theta$ is π-periodic, the zeroes of dx/dt are equally spaced, with separation π/ω_1. The zeroes of the derivatives correspond alternately to maxima and minima, so the maxima are separated by a time $\tau_1 = 2\pi/\omega_1$.



(b) The zeroes of $x(t)$ occur when $\cos(\omega_1 t - \delta) = 0$. Thus they are regularly spaced with separation of π/ω_1, which equals $\tau_1/2$.

(c) With $\beta = \omega_o/2$, the amplitude shrinks by a factor

$$e^{-\beta\tau_1} = e^{-2\pi\beta/\sqrt{\omega_o^2-\beta^2}} = e^{-2\pi/\sqrt{3}} = 0.027$$

(This is much more shrinkage than in the picture, for which β was chosen to be $\omega_o/10$ and the shrinkage factor is about 0.53).

5.26 ⋆⋆ The damping changes the frequency to $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$, which we can solve to give

$$\beta = \omega_o\sqrt{1 - \frac{\omega_1^2}{\omega_o^2}} = \omega_o\sqrt{1 - \frac{\tau_o^2}{\tau_1^2}} = \omega_o\sqrt{1 - 0.998} = 0.0447\omega_o = 0.281 \text{ s}^{-1}$$

After a time $t = 10\tau_1 \approx 10\tau_o$, the amplitude will have changed by a factor of

$$e^{-\beta t} \approx e^{-10\beta\tau_o} = e^{-20\pi\beta/\omega_o} = e^{-20\pi(0.0447)} = 0.060\,.$$

In other words, the amplitude will have diminished by a factor of $1/0.060 = 17$. Clearly the change of amplitude of by a factor of 17 is far more noticeable than the change of period by 0.1%.

5.27 ⋆⋆ The question is: How many times can the function $x(t)$ vanish? If the oscillator is weakly damped ($\beta < \omega_o$), then according to Eq.(5.38) $x(t)$ contains a factor $\cos(\omega t - \delta)$, which vanishes infinitely many times as it oscillates.

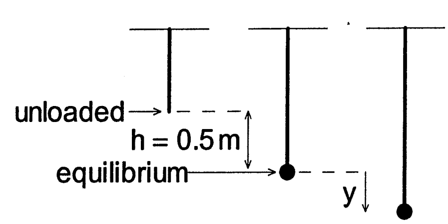
(a) If the oscillator is critically damped ($\beta = \omega_o$), then, according to (5.44), $x(t) = e^{-\beta t}(C_1 + C_2 t)$. This vanishes if and only if $t = -C_1/C_2$; therefore, $x(t)$ vanishes at most once. (It may never vanish — for example, if the motion starts at $t = 0$ and $-C_1/C_2 < 0$.)

(b) If the oscillator is overdamped ($\beta > \omega_o$), then according to (5.40),

$$x(t) = e^{-\beta t}(C_1 e^{\lambda t} + C_2 e^{-\lambda t}) = e^{-(\beta-\lambda)t}(C_1 + C_2 e^{-2\lambda t})$$

where $\lambda = \sqrt{\beta^2 - \omega_o^2}$. This vanishes if and only if $e^{-2\lambda t} = -C_1/C_2$, and, since $e^{-2\lambda t}$ is a monotonic function (always decreasing), this happens at most once.

5.28 ⋆⋆ The final resting position is the equilibrium position, at a height $h = 0.5$ m below the unloaded position. This height is determined by the condition $kh = mg$, from which we see that $\omega_o^2 = k/m = g/h$.



In discussing the oscillations, it is simplest to measure the mass's position $y(t)$ from its equilibrium position. The initial conditions are $y(0) = -h$ and $\dot{y}(0) = 0$. Since the damping is critical, $\beta = \omega_o$ and, according to Equation (5.44),

$$y(t) = (C_1 + C_2 t)e^{-\omega_o t} = -h(1 + \omega_o t)e^{-\omega_o t},$$

because the initial conditions require that $C_1 = -h$ and that $-\omega_o C_1 + C_2 = 0$. Putting in $h = 0.5$ m and $t = 1$ s, we get $\omega_o = \sqrt{g/h} = \sqrt{19.6} = 4.43\ \mathrm{s}^{-1}$ and then

$$y(t) = -(0.5\ \mathrm{m})(1 + 4.43)e^{-4.43} = -0.032\ \mathrm{m} = -3.2\ \mathrm{cm}.$$

That is, it is just 3.2 cm above its final resting position.

5.29 ★★ After one period, the amplitude has shrunk by a factor $e^{-\beta\tau_1} = 0.5$. Therefore $\beta\tau_1 = \ln 2$. Meanwhile, because $\tau_1 = 2\pi/\omega_1 = 2\pi/\sqrt{\omega_o^2 - \beta^2}$, it follows that $\beta\tau_1 = 2\pi/\sqrt{(\omega_o/\beta)^2 - 1}$. Equating these two expressions for $\beta\tau_1$ and solving, we find $\beta/\omega_o = 1/\sqrt{1 + (2\pi/\ln 2)^2} = 0.1097$. That is, $\beta = 0.11\omega_o$.

$$\frac{\tau_1}{\tau_o} = \frac{\omega_o}{\omega_1} = \frac{\omega_o}{\sqrt{\omega_o^2 - \beta^2}} = \frac{1}{\sqrt{1 - \beta^2/\omega_o^2}} \approx 1 + \frac{1}{2}\frac{\beta^2}{\omega_o^2} = 1.006.$$

Therefore $\tau_1 = 1.006$ s.

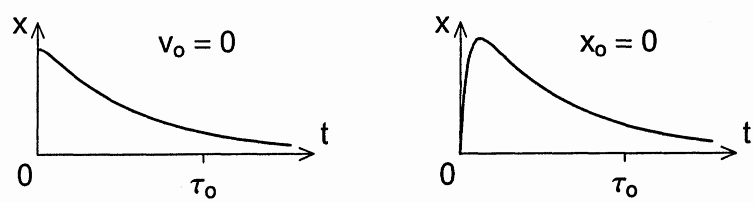
5.30 ★★ **(a)** From Eq.(5.40) we know that $x = e^{-\beta t}(C_1 e^{\lambda t} + C_2 e^{-\lambda t})$, where $\lambda = \sqrt{\beta^2 - \omega_o^2}$. We can differentiate this to get the velocity v and then set $t = 0$ to give the two equations

$$x_o = C_1 + C_2 \qquad \text{and} \qquad v_o = \lambda(C_1 - C_2) - \beta(C_1 + C_2)$$

which we can solve to give

$$C_1 = \frac{1}{2\lambda}[x_o(\lambda + \beta) + v_o] \qquad \text{and} \qquad C_2 = \frac{1}{2\lambda}[x_o(\lambda - \beta) - v_o]$$

(b)



(c) If we let $\beta \to 0$, then $\lambda \to i\omega_o$ and the coefficients C_1 and C_2 become

$$C_1 = \frac{x_o}{2} + \frac{v_o}{2i\omega_o} \qquad \text{and} \qquad C_2 = \frac{x_o}{2} - \frac{v_o}{2i\omega_o}$$

and our solution becomes

$$x = \frac{x_o}{2}\left(e^{i\omega_o t} + e^{-i\omega_o t}\right) + \frac{v_o}{2i\omega_o}\left(e^{i\omega_o t} - e^{-i\omega_o t}\right) = x_o\cos(\omega_o t) + \frac{v_o}{\omega_o}\sin(\omega_o t)$$

which you should recognize as the general solution for undamped oscillations.

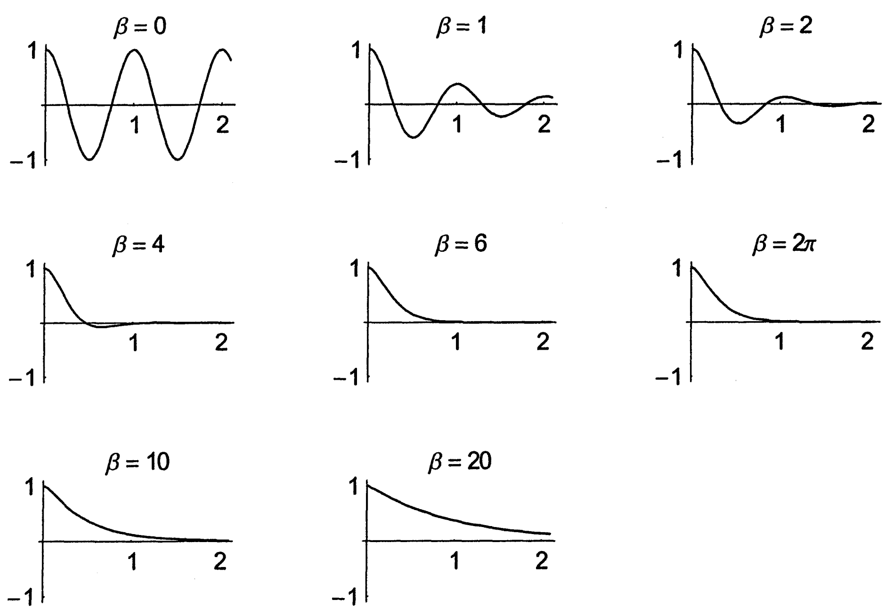
5.31 ★★ Starting with $\beta < \omega_o$, we know that $x(t)$ has the form (5.37), which we can rewrite as $x(t) = e^{-\beta t}(A\cos\omega_1 t + B\sin\omega_1 t)$, where, as usual, $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$. Given that $x(0) = 1$ and $\dot{x}(0) = 0$, we can solve for A and B and find

$$x(t) = e^{-\beta t}\left(\cos(\omega_1 t) + \frac{\beta}{\omega_1}\sin(\omega_1 t)\right). \tag{iii}$$

Given that $\omega_o = 2\pi$, we can use this to plot $x(t)$ for any value of $\beta < \omega_o$. You can also use it for $\beta > \omega_o$. The only complication is that ω_1 is now imaginary, $\omega_1 = i\sqrt{\beta^2 - \omega_o^2} = i\tilde{\omega}_1$, say. If your plotting software is happy to work with complex numbers, you can continue to use exactly the form (iii), but it may be safer to rewrite it in terms of real functions as

$$x(t) = e^{-\beta t}\left(\cosh(\tilde{\omega}_1 t) + \frac{\beta}{\tilde{\omega}_1}\sinh(\tilde{\omega}_1 t)\right). \tag{iv}$$

Finally for $\beta = \omega_o$, you almost certainly need to use (5.44) which, for the given initial conditions has the form $x(t) = e^{-\beta t}(1 + \beta t)$. The results are shown below.



5.32 ★★ **(a)** If we write $x(t)$ in the suggested form, it is easy to see that $x(0) = B_1$ and $\dot{x}(0) = \omega_1 B_2 - \beta B_1$. Therefore, $B_1 = x_o$ and $B_2 = \beta x_o/\omega_1$, so that

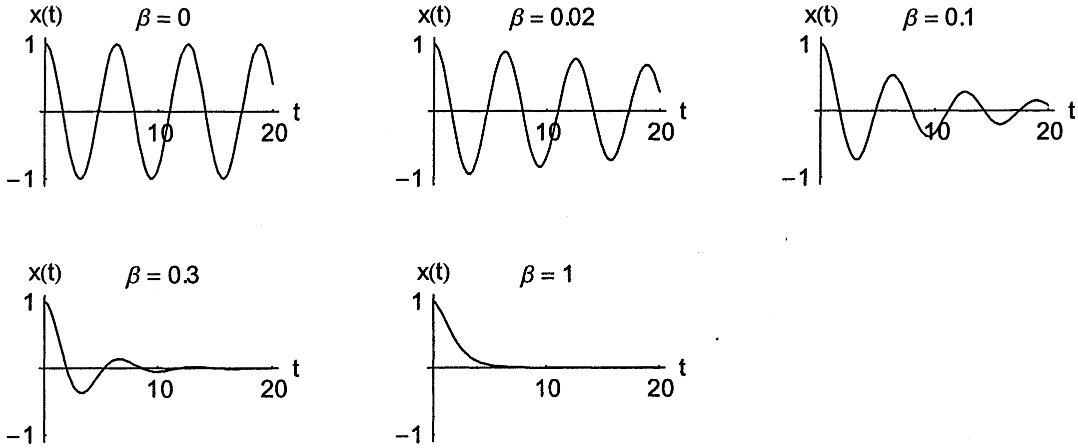
$$x(t) = x_o e^{-\beta t}\left(\cos\omega_1 t + \frac{\beta}{\omega_1}\sin\omega_1 t\right).$$

(b) When $\beta \to \omega_o$, the denominator $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ approaches zero, but so does the numerator $\sin\omega_1 t$, and $(\sin\omega_1 t)/\omega_1 \to t$. Therefore, the solution of part (a) goes over to

$$x(t) = x_o e^{-\beta t}(1 + \beta t),$$

which is precisely the critical solution (5.44) satisfying the given initial conditions.

(c)



5.33 ⋆ According to Eq.(5.69), $x(t) = A\cos(\omega t - \delta) + e^{-\beta t}[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)]$. Setting $t = 0$, we obtain

$$x_o = x(0) = A\cos\delta + B_1 \quad \Longrightarrow \quad B_1 = x_o - A\cos\delta.$$

Similarly, differentiating $x(t)$ and then setting $t = 0$, we find

$$v_o = \dot{x}(0) = A\omega\sin\delta - \beta B_1 + \omega_1 B_2 \quad \Longrightarrow \quad B_2 = \frac{1}{\omega_1}(v_o - A\omega\sin\delta + \beta B_1)$$

as in Eq.(5.70).

5.34 ⋆ We are given that both x_p and x satisfy the same inhomogeneous equation, $Dx_p = f$ and $Dx = f$. Therefore, since D is linear, $D(x - x_p) = Dx - Dx_p = f - f = 0$. That is, the difference $x - x_p = x_h$ is a solution of the homogeneous equation $Dx_h = 0$. Therefore x can always be written as $x = x_p + x_h$ as claimed.

5.35 ⋆⋆ **(a)** $z = x + iy = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

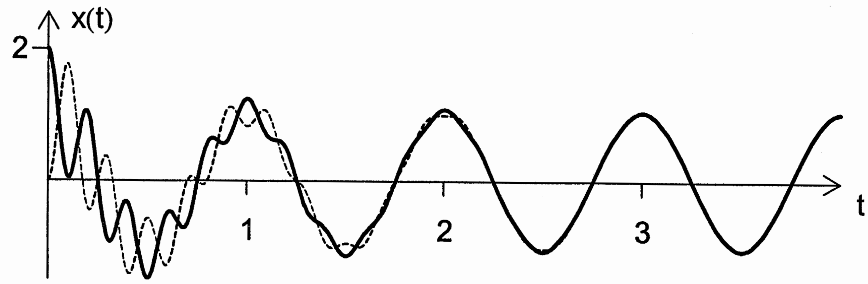
(b) $zz^* = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$.

(c) $z^* = x - iy = r\cos\theta - i(r\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)] = re^{-i\theta}$

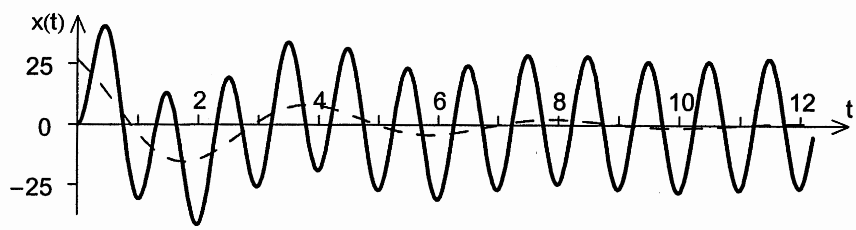
(d) If $z = re^{i\theta}$ and $w = se^{i\phi}$, then $zw = rse^{i(\theta+\phi)}$. Therefore, $(zw)^* = rse^{-i(\theta+\phi)} = (re^{-i\theta})(se^{-i\phi}) = z^*w^*$. If $z = re^{i\theta}$, then $1/z = 1/(re^{i\theta}) = (1/r)e^{-i\theta}$, so $(1/z)^* = (1/r)e^{i\theta}$. Finally, $1/z^* = 1/(re^{-i\theta}) = (1/r)e^{i\theta} = (1/z)^*$.

(e) If $z = \dfrac{a}{b + ic}$, then $|z|^2 = zz^* = \dfrac{a}{b + ic}\left(\dfrac{a}{b + ic}\right)^* = \dfrac{a}{b + ic}\cdot\dfrac{a}{b - ic} = \dfrac{a^2}{b^2 + c^2}$.

5.36 ⋆⋆ From Eqs.(5.64), (5.65), and (5.70) we can calculate the various constants. A and δ are the same as in Example 5.3 (changing the initial condition doesn't affect them), but $B_1 = 0.945$ and $B_2 = 0.0429$. The resulting graph is the solid curve shown. The dashed curve is that of Fig.5.15(b). With different initial conditions, the two curves differ at first, but after a couple of cycles the transients have pretty well died out and the two graphs are indistinguishable.

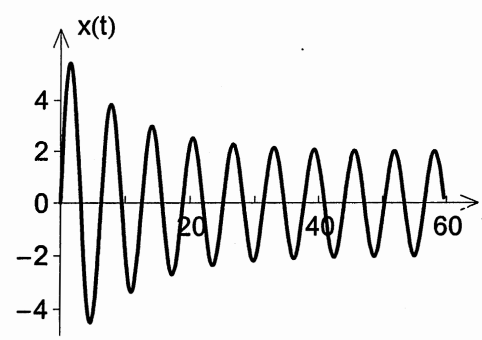


5.37 ⋆⋆ With the help of Eqs.(5.64), (5.65), and (5.70) we find that $A = 26.9$, $\delta = 3.04$, $B_1 = 26.7$, and $B_2 = -6.18$. The corresponding motion is shown by the solid curve, while the dashed curve is the transient alone.



The most obvious difference between the present case and that of Example 5.3 is that here $\omega_o < \omega$, so that the transient imposes a slow rise and fall on the driven oscillations. In Fig.5.15 of Example 5.3, where $\omega_o > \omega$, the transient superposes a rapid oscillation on the driven motion. A subtler difference is that in Fig.5.15 where $\omega_o > \omega$ the phase shift δ is close to zero and the motion is in step with the driving force. Here, with $\omega_o < \omega$, $\delta \approx \pi$ and the motion is nearly 180° out of phse with the driving force. [The driving force has maxima at integer values of t whereas $x(t)$ has minima.]

5.38 ⋆⋆ With the given parameters, the constants in (5.69) are easily found. Given that $\omega = \omega_o$, (5.65) tells us that $\delta = \pi/2$ and (5.64) that $A = f_o/2\beta\omega = 2$. Thus (5.70) implies that $B_1 = 0$ and $B_2 = (v_o - \omega A)/\omega_1$, or, since $\omega_1 = \sqrt{\omega_o^2 - \beta^2} = 0.995$, $B_2 = 4.02$. The graph of (5.69) with these values of the constants is as shown. Initially the oscillations have a large amplitude (somewhere about 5) but after a few cycles, the transients have died out and the motion is indistinguishable from the "attractor" with amplitude $A = 2$.



5.39 ⋆⋆ See Figure 5.15.

5.40 $\star$ We can save ourselves a little trouble if we note that A is maximum if and only is A^2 is maximum, which occurs if and only if $1/A^2$ is minimum. In fact, let's consider $(f_o/A)^2 = (\Omega - \omega_o^2)^2 + 4\beta^2\Omega$, where I'll use the variable $\Omega = \omega^2$. To find the minimum of this quantity we have only to differentiate and set the derivative equal to zero:

$$\frac{d}{d\Omega}\left(\frac{f_o^2}{A^2}\right) = 2(\Omega - \omega_o^2) + 4\beta^2,$$

which vanishes when $\Omega = \omega_o^2 - 2\beta^2$, that is, $\omega = \sqrt{\omega_o^2 - 2\beta^2}$. It is easy to check that the second derivative is positive. Therefore $(f_o/A)^2$ is minimum and A is maximum as expected.

5.41 $\star$ Provided β is significantly less that ω_o, the maximum of A^2 comes when $\omega \approx \omega_o$ and, at this point, the denominator of Eq.(5.71) is approximately $4\beta^2\omega_o^2$. Thus A^2 is equal to half its maximum when the denominator is equal to $8\beta^2\omega_o^2$, or when $(\omega^2 - \omega_o^2)^2 = 4\beta^2\omega_o^2$. This simplifies to $(\omega - \omega_o)(\omega + \omega_o) = \pm 2\beta\omega_o$. Since $(\omega + \omega_o) \approx 2\omega_o$, this says that the half maximum occurs at $\omega = \omega_o \pm \beta$.

5.42 $\star$ The period of the pendulum is $\tau = 2\pi\sqrt{l/g} = 10.99$ s. Therefore the quality factor is $Q = \pi(\text{decay time})/\tau = \pi \times (8\text{ h})/(10.99\text{ s}) \approx 8,000$.

5.43 $\star\star$ **(a)** Assuming that the weight of the four men is evenly distributed among the four springs, we can substitute $m = 80$ kg and $x = 2$ cm into the equation $mg = kx$ for any one spring. This gives $k = mg/x = 80 \times 9.8/0.02 \approx 4 \times 10^4$ N/m.

(b) Each axle assembly is attached to two springs, so the effective spring constant of its support is $2k$ and its natural frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2 \times (4 \times 10^4)}{50}} \approx 6\text{ Hz}.$$

(c) If the distance between bumps is d, then $v = fd = 6 \times 0.8 \approx 5$ m/s or roughly 10 mi/h.

5.44 $\star\star$ **(a)** Since $x = A\cos(\omega t - \delta)$, the total energy is

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2A^2\cos^2(\omega t - \delta) + \tfrac{1}{2}kA^2\sin^2(\omega t - \delta).$$

Because $\omega \approx \omega_o$, we can replace $k = m\omega_o^2$ by $m\omega^2$, and then, since $\cos^2\theta + \sin^2\theta = 1$, we get $E = \frac{1}{2}m\omega^2A^2$, as claimed.

(b) The rate at which the damping force dissipates energy is $F_{\text{dmp}}v = bv^2 = 2m\beta v^2$. Therefore the energy dissipated in one period is

$$\Delta E_{\text{dis}} = \int_0^\tau 2m\beta v^2 dt = 2m\beta\omega^2A^2\int_0^\tau \sin^2(\omega t - \delta)dt.$$

The remaining integral is just π/ω. (To see this use the trig identity $\sin^2\theta = \frac{1}{2}(1-\sin 2\theta)$ and note that the integral of the sine term over a period is zero.) Therefore, $\Delta E_{\text{dis}} = 2\pi m\beta\omega A^2$.

(c) Combining the results of parts (a) and (b), we find that

$$\frac{E}{\Delta E_{\text{dis}}} = \frac{\frac{1}{2}m\omega^2 A^2}{2\pi m\beta\omega A^2} = \frac{\omega_o}{4\pi\beta} = \frac{Q}{2\pi}$$

where I have again used the fact that $\omega = \omega_o$. That is, the ratio of the total energy to the energy lost per cycle is $2\pi Q$.

5.45 ⋆⋆⋆ The key relationships we need are

$$F(t) = F_o\cos(\omega t) \;\; (5.56), \qquad x(t) = A\cos(\omega t - \delta) \;\; (5.66)$$

and

$$A = (F_o/m)/\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2} \;\; (5.64), \qquad \tan\delta = 2\beta\omega/(\omega_o^2 - \omega^2) \;\; (5.65).$$

(a) The rate at which the driving force F does work is

$$P = F\dot{x} = -F_o\omega A\cos(\omega t)\sin(\omega t - \delta) = \tfrac{1}{2}\omega A F_o[\sin\delta - \sin(2\omega t - \delta)] \qquad \text{(v)}$$

where, in the last equality I used the trig identity for $\cos\theta\sin\phi$. The average power over any number of complete cycles is just $\langle P\rangle = (1/\tau)\int_0^\tau P dt$. When we integrate the square bracket of (v), the first term gives us $\tau\sin\delta$ and the second gives zero. Therefore, $\langle P\rangle = \frac{1}{2}\omega A F_o\sin\delta$. Finally, if you'll look at Fig.5.14, you'll see that $\sin\delta = 2\beta\omega/\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}$. So

$$\langle P\rangle = \tfrac{1}{2}\omega A F_o\sin\delta = \tfrac{1}{2}\omega A F_o\frac{2\beta\omega}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}} = m\beta\omega^2 A^2 \qquad \text{(vi)}$$

where, in the last equality, I used (5.64) to replace F_o and the large square root by mA.

(b) The damping force F_{dmp} has magnitude $2m\beta v$ in the opposite direction to the velocity, so the rate of loss of energy is $P_{\text{loss}} = F_{\text{dmp}}v = 2m\beta v^2 = 2m\beta\omega^2 A^2\sin^2(\omega t - \delta)$. When we average this over a cycle, the integration of the $\sin^2$ term gives $\tau/2$ [use the appropriate trig identity to write $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$], and we find that $\langle P_{\text{loss}}\rangle = m\beta\omega^2 A^2$, which is equal to the average power delivered, as found in part (a).

(c) Using (vi) and (5.64), we can write $\langle P\rangle$ in the form $\langle P\rangle = K\Omega/[(\Omega - \omega_o^2)^2 + 4\beta^2\Omega]$, where I have introduced the temporary shorthand $\Omega = \omega^2$. Now, to maximize $\langle P\rangle$ is the same as to minimize $1/\langle P\rangle = K'[4\beta^2 + (\Omega - \omega_o^2)^2/\Omega]$. The derivative of the latter is proportional to $(\Omega - \omega_o^2)$ (times a positive function). Thus the derivataive vanishes when $\Omega = \omega_o^2$. As you can check, the second derivative is positive, so $1/\langle P\rangle$ has a minimum, and $\langle P\rangle$ the expected maximum, when $\omega = \omega_o$.

5.46 ⋆ If we insert the Fourier series (5.82) into the definitiion of the average, we find that

$$\langle f\rangle = \frac{1}{\tau}\int_0^\tau f(t)dt = \frac{1}{\tau}\left[\int_0^\tau a_0\,dt + \sum_{n=1}^{\infty}\left(a_n\int_0^\tau \cos(n\omega t)dt + b_n\int_0^\tau \sin(n\omega t)dt\right)\right].$$

The first term here gives just a_0, while every one of the remaining terms is the integral of $\cos(n\omega t)$ [or $\sin(n\omega t)$] over one or more periods and is therefore zero. Thus $\langle f\rangle = a_0$.

5.47 ⋆⋆ To treat the integrals involving two cosines, we need the trig identity $\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta+\phi)+\cos(\theta-\phi)]$. For the case $n = m \neq 0$, this gives

$$\int_{-\tau/2}^{\tau/2}\cos^2(n\omega t)dt = \tfrac{1}{2}\int_{-\tau/2}^{\tau/2}[\cos(2n\omega t)+1]dt = \tfrac{1}{2}\tau$$

because the first integral is zero [because $\sin(2n\omega t)$ is τ-periodic] and the second gives just τ. For the case $m \neq n$,, we find

$$\begin{aligned}\int_{-\tau/2}^{\tau/2}\cos(n\omega t)\cos(m\omega t)dt &= \tfrac{1}{2}\int_{-\tau/2}^{\tau/2}[\cos(n+m)\omega t + \cos(n-m)\omega t]dt \\ &= \tfrac{1}{2}\left[\frac{\sin(n+m)\omega t}{n+m} + \frac{\sin(n-m)\omega t}{n-m}\right]_{-\tau/2}^{\tau/2} = 0\end{aligned}$$

because both sine functions are τ periodic. Using the corresponding trig identities for $\sin\theta\sin\phi$ and $\cos\theta\sin\phi$, you can prove the corresponding results for two sine functions and for one sine and one cosine.

5.48 ⋆⋆ If we multiply the Fourier series (5.82) by $\cos(m\omega t)$ ($m = 1, 2, 3, \cdots$) and integrate over a period, we find

$$\int \cos(m\omega t)f(t)dt = \sum_{n=0}^{\infty} a_n\int\cos(m\omega t)\cos(n\omega t)dt + \sum_{n=1}^{\infty} b_n\int\cos(m\omega t)\sin(n\omega t)dt$$

where all integrals run from $-\tau/2$ to $\tau/2$. By (5.106) every integral in the second sum is zero. By (5.105) every integral in the first sum is zero except the one with $n = m$, which is equal to $\tau/2$. Thus the whole right side collapses to a single term and

$$\int\cos(m\omega t)f(t)dt = a_m\tau/2$$

which establishes (5.83) for a_m. To establish (5.84) for b_m we do exactly the same thing except that we multiply by $\sin(m\omega t)$.

Finally, to find a_0 we just integrate the Fourier series (5.82) over a period to give

$$\int f(t)dt = \sum_{n=0}^{\infty} a_n\int\cos(n\omega t)dt + \sum_{n=0}^{\infty} b_n\int\sin(n\omega t)dt$$

where all integrals run from $-\tau/2$ to $\tau/2$. The integrals of the cosines or sines give sines or cosines, and are zero because both sines and cosines are periodic. The only exception is the integral of the cosine with $n = 0$. Since $\cos(0) = 1$, this integral is just τ, and we conclude that $\int f(t)dt = a_0\tau$, which establishes (5.85).

5.49 ⋆⋆⋆ The given function is even, $f(-t) = +f(t)$. Therefore, $\sin(m\omega t)f(t)$ is odd and all of the integrals (5.84) for the coefficients b_m are zero. Since $\cos(m\omega t)f(t)$ is even, the coefficients a_m are not necessarily zero. Bearing in mind that $\tau = 2$, so $\omega = \pi$, we find that

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)dt = \frac{f_{\max}}{2}$$

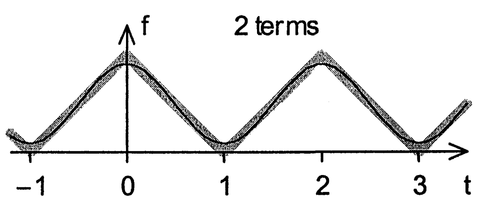
while for $m \geq 1$.

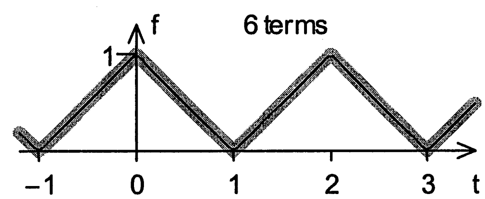
$$a_m = \frac{4}{\tau}\int_0^{\tau/2} \cos(m\omega t)f(t)dt = 2f_{\max}\int_0^1 \cos(m\pi t)(1-t)dt.$$

This integral can be evaluated (using integration by parts), and we find that

$$a_m = -\frac{2f_{\max}}{(m\pi)^2}\Big[\cos(m\pi t)\Big]_0^1 = \begin{cases} 0 & [m \text{ even}] \\ 4f_{\max}/(m\pi)^2 & [m \text{ odd}] \end{cases}$$

The left picture shows the sum of the first two terms (constant term plus first cosine) and the sawtooth function itself in gray. The right picture shows the first six terms; these follow the sawtooth so closely that it is hard to tell them apart except at the corners.

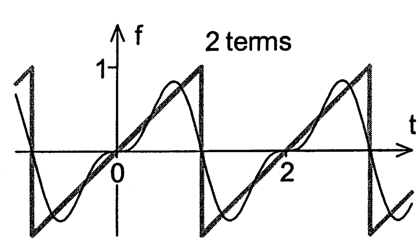


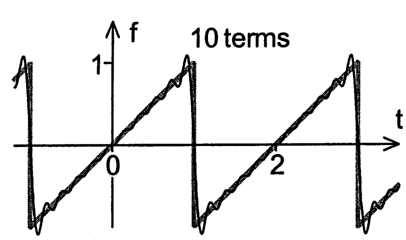


5.50 ⋆⋆⋆ The given function is odd, $f(-t) = -f(t)$. Therefore, $\cos(m\omega t)f(t)$ is odd and all of the integrals (5.83) and (5.85) for the coefficients a_m are zero. Since $\sin(m\omega t)f(t)$ is even, the coefficients b_m are not necessarily zero, and

$$b_m = \frac{4}{\tau}\int_0^{\tau/2} \sin(m\omega t)f(t)dt = 2f_{\max}\int_0^1 \sin(m\pi t)tdt.$$

(Remember that $\tau = 2$, so $\omega = \pi$.) This integral is easily evaluated by parts, and we find that $b_m = -2f_{\max}(-1)^m/(m\pi)$. The sums of the first 2 and first 10 terms of the Fourier series are as shown, with $f(t)$ itself shown in gray.





5.51 ★★ If the real force is $f(t) = \sum_0^\infty f_n \cos(n\omega t)$, and we define the complex function $g(t) = \sum_0^\infty f_n e^{in\omega t}$, then it is obviously true that $f(t) = \mathrm{Re}[g(t)]$. (A major advantage of using complex functions is that by allowing the coefficients in the series for $g(t)$ to be complex, we can include the cosines *and sines* of the general Fourier series, but we'll continue to discuss a force that contains only cosine terms.) We can now try to solve the complex equation

$$Dz = \ddot{z} + 2\beta\dot{z} + \omega_o^2 z = g(t). \tag{vii}$$

Before we solve this, notice that, if we succeed, then it is obvious that $x(t) = \mathrm{Re}[z(t)]$ satisfies $Dx = f$. That is, the real part of $z(t)$ satisfies the actual equation of motion for the actual physical problem. To solve the complex equation (vii), we can try a function of the form $z(t) = \sum_0^\infty C_n e^{in\omega t}$. If we substitute this function into Eq.(vii), we obtain

$$Dz = \sum_0^\infty C_n(-n^2\omega^2 + 2i\beta n\omega + \omega_o^2)e^{in\omega t} = \sum_0^\infty f_n e^{in\omega t}.$$

This equation is satisfied if and only if the separate coefficients on the left are equal to the corresponding coefficients on the right, that is,

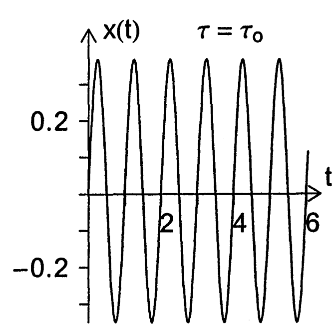
$$C_n = \frac{f_n}{\omega_o^2 - n^2\omega^2 + 2i\beta n\omega}.$$

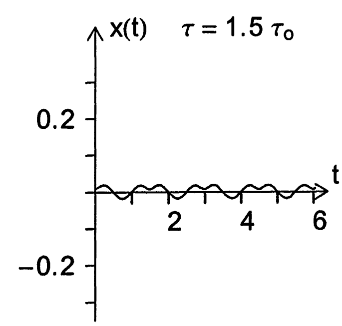
By choosing the coefficients according to this equation, we obtain a complex solution, z, whose real part, $x = \mathrm{Re}(z)$, is a real solution of the actual real problem. Needless to say, this is the same solution we got before, just in a slightly tidier form.

5.52 ★★★ The first six Fourier coefficients, all multiplied by 10^4, are

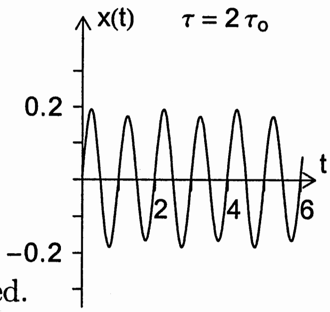
	A_0	A_1	A_2	A_3	A_4	A_5
$\tau = 1.0\,\tau_o$	63	3582	27	5	0	1
$\tau = 1.5\,\tau_o$	42	145	90	18	6	2
$\tau = 2.0\,\tau_o$	32	82	1791	40	13	6
$\tau = 2.5\,\tau_o$	25	59	131	98	25	11

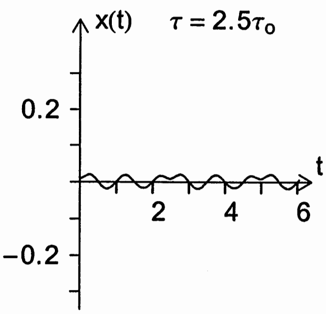
The width parameter β is half as big as in Example 5.5. This means that close to resonance, the amplitude should be twice as big, but away from resonance it should be almost the same. This is born out by the coefficients shown. When $\tau = \tau_o$ the coefficient A_1 is twice as big as in Table 5.1, and





the same applies to A_2 when $\tau = 2\tau_o$, but all the other coefficients are virtually unchanged. The same shows up in the graphs. Those for $\tau = \tau_o$ and $\tau = 2\tau_o$ are twice as high as in Fig.5.25, whereas those for $\tau = 1.5\tau_o$ and $\tau = 2.5\tau_o$ are unchanged.





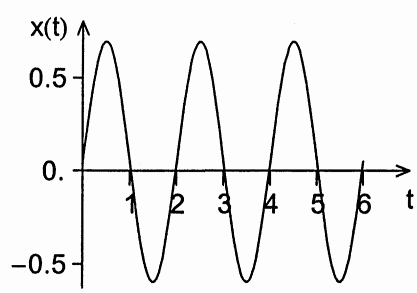
5.53 ★★★ From Problem 5.49, we know that the Fourier series for $f(t)$ contains only cosine terms and that $f_0 = \frac{1}{2}$, while $f_n = 4/(n\pi)^2$ for n odd, but zero for n even. It follows that

$$x(t) = A_0 + \sum_{n\,\text{odd}} A_n \cos(n\omega t - \delta_n)$$

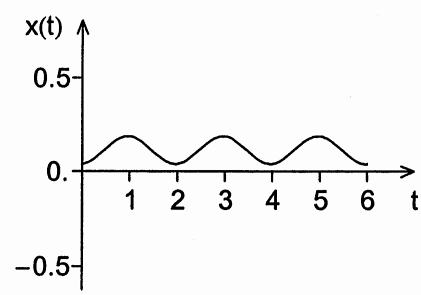
where, according to (5.92) and (5.93), $A_0 = 1/2\omega_o^2$, $\delta_0 = 0$, while for $n \geq 1$,

$$A_n = \frac{4}{n^2\pi^2\sqrt{(\omega_o^2 - n^2\omega^2)^2 + (2\beta n\omega)^2}} \quad \text{and} \quad \delta_n = \arctan\left(\frac{2\beta n\pi}{\omega_o^2 - n^2\pi^2}\right).$$

(a) With $\tau_o = 2$ and $\omega_o = \pi$, the first four coefficients A_n $(n = 0, 1, 2, 3)$ are 0.0507, 0.6450, 0, and 0.0006. (Note the large value of A_1, because this term is on resonance.)



(a)



(b)

(b) With $\tau_o = 3$ and $\omega_o = 2\pi/3$, the first four coefficients A_n $(n = 0, 1, 2, 3)$ are 0.1140, 0.0734, 0, and 0.0005. (Notice that here the constant term A_0 is certainly not negligible compared to the others and it throws the oscillations off center.)

5.54 ★ Consider the average of $f(t)$ over an interval smaller than one period (for instance, the interval from 0 to $\tau/2$). The values of $f(t)$ in this subset could all be larger than the average value over the whole period; then the average over the interval would be larger than that over the whole period. For example, consider the function $f(t) = \sin t$, which is periodic with period 2π; its average over a whole period is 0, but, in the interval between 0 and π, $f(t)$ is everywhere greater than 0 and the same is therefore true of its average (average $= 2/\pi$).

Consider now a long interval from 0 to T. We can write $T = n\tau + \delta$ where n is a large integer and $0 \le \delta < \tau$. Then

$$\begin{aligned}(\text{average from 0 to } T) = \frac{1}{T}\int_0^T f(t)dt &= \frac{1}{n\tau+\delta}\left(\int_0^{n\tau} + \int_{n\tau}^{n\tau+\delta}\right) f(t)dt \\ &= \frac{n}{n\tau+\delta}\int_0^{\tau} f(t)dt + \frac{1}{n\tau+\delta}\int_0^{\delta} f(t)dt \\ &\to \frac{1}{\tau}\int_0^{\tau} f(t)dt, \qquad \text{as } n\to\infty\end{aligned}$$

where in moving to the second line I used the periodicity of $f(t)$ to simplify both integrals. The final expression is, of course, the average over one period.

5.55 ⋆⋆ Using the trig identity for $\cos\theta\cos\phi$ (inside front cover), we find for the integrand of Eq.(5.99)

$$\cos(n\omega t-\delta_n)\cos(m\omega t-\delta_m) = \tfrac{1}{2}\cos\big[(n+m)\omega t-(\delta_n+\delta_m)\big] + \tfrac{1}{2}\cos\big[(n-m)\omega t-(\delta_n-\delta_m)\big]$$

Let us denote by I_{nm} the integral of Eq.(5.99). If $n = m = 0$, this becomes [note that according to (5.93) $\delta_0 = 0$]

$$I_{00} = \int_{-\tau/2}^{\tau/2} 1dt = \tau.$$

If $n = m \neq 0$, then

$$I_{nn} = \tfrac{1}{2}\int_{-\tau/2}^{\tau/2}[\cos(2n\omega t-2\delta_n)+1]dt = \tau/2$$

because the first integral is zero. Finally if $n \neq m$,

$$I_{nm} = \tfrac{1}{2}\left[\frac{\sin(\cdots)}{n+m} + \frac{\sin(\cdots)}{n-m}\right]_{-\tau/2}^{\tau/2} = 0$$

because both sines are τ-periodic. This completes the proof of (5.99).

With a little rearrangement, (5.98) becomes

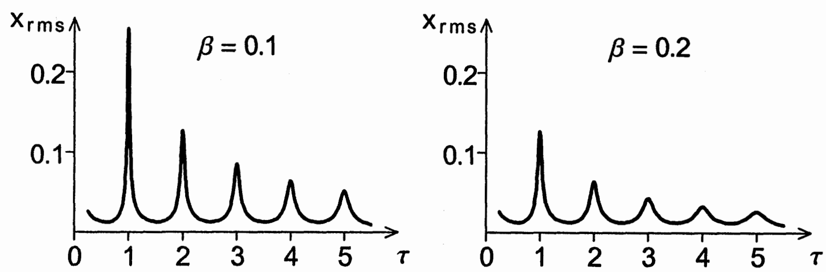
$$\langle x^2\rangle = \sum_n\sum_m A_nA_m\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\cos(n\omega t-\delta_n)\cos(m\omega t-\delta_m)\,dt.$$

By the previous result, the integral here is zero unless $m = n$. Thus the double sum reduces to a single sum with $m = n$. The remaining integrals are τ (for $n = 0$) and $\tau/2$ (for $n > 0$), so we're left with $\langle x^2\rangle = A_0^2 + \frac{1}{2}\sum_1^\infty A_n^2$, which is the Parseval relation.

5.56 ⋆⋆ If $x(t)$ has a Fourier series of the type suggested, then $\langle x^2\rangle$ will look like the double series of (5.98) except that there will also be terms involving products of two sine functions and others involving a sine and a cosine. To evaluate this sum, we need the "orthogonality relations" (5.99) and their equivalents for products of sines and of a sine and a cosine. Using the trig identity for $\sin\theta\sin\phi$ it is easy to prove the $\int \sin(n\omega t - \delta_n)\sin(m\omega t - \delta_m)dt$ is also given by (5.99) (except that when $n = m = 0$ the integral is zero). Similarly, using the trig identity for $\cos\theta\sin\phi$ you can show that all integrals of the form $\int \sin(n\omega t - \delta_n)\cos(m\omega t - \delta_m)dt$ are zero. And using these results, you can easily show that the Parseval relation is

$$\langle x^2\rangle = A_0^2 + \tfrac{1}{2}\sum_1^{\infty}\left(A_n^2 + B_n^2\right).$$

5.57 ⋆⋆



The left picture shows the data for this problem; the right shows the data of Figure 5.26 (though drawn here to a slightly different scale). Notice that the resonances with $\beta = 0.1$ are twice as high and half as wide as those for $\beta = 0.2$.

Chapter 6

Calculus of Variations

I covered this chapter in 2 fifty-minute lectures.

This chapter is one of the two shortest chapters of the book, and it is the only chapter devoted exclusively to a mathematical prerequisite. I had always believed that it should be possible to derive Lagrange's equation without getting involved with Hamilton's principle, but when I tried to do this I learned that Hamilton's principle is *by far* the best way to prove that Lagrange's equations hold in any coordinate system. This is a pity in a way, because Hamilton's principle is a large intellectual leap for the student and is one without any simple and immediate applications (other than the proof of Lagrange's equations). Nevertheless, it seems to be the best way to arrive at Lagrange's equations and is, of course, centrally important in modern theoretical physics. Therefore, I make essential use of the principle when I set up the Lagrangian formalism in Chapter 7. Because one can't discuss Hamilton's principle without a knowledge of the calculus of variations, and because our students don't seem to learn about the calculus of variations anywhere else, Chapter 6 is an introduction to the calculus of variations.

My treatment is pretty standard. The central problem of the calculus of variations is, of course, to minimize (or, more generally, make stationary) some line integral. I start in Section 6.1 with two examples of the kinds of integrals that one might want to minimize — the distance between two points, and Fermat's integral for the transit time for light traveling between two points. Then in Section 6.2, I derive the Euler-Lagrange equation for the desired stationary path. Section 6.3 contains a couple of examples, including the well worn, but always fascinating, brachistochrone problem. Section 6.4 generalizes to more than two dimensions, a generalization that is obviously essential for applications in mechanics.

As always, it is crucial that the student try some of the problems at the end of the chapter. These include a few theoretical developments and a few problems that fill in details omitted from the text, but most are just applications of the Euler-Lagrange equation. Many of these latter are rather straightforward, but a few are quite challenging and, I thought, fascinating (for example, Problem 6.23 on the best flight path for a plane in a shearing wind).

Solutions to Problems for Chapter 6

6.1 ⋆ Imagine first an infinitesimal section of path on the sphere, in which θ increases by $d\theta$ and ϕ by $d\phi$. This carries us a distance $R\,d\theta$ to the "south," and $R\sin\theta\,d\phi$ to the "east." The distance ds along the path is therefore

$$ds = \sqrt{(R\,d\theta)^2 + (R\sin\theta\,d\phi)^2} = R\sqrt{1+\sin^2\theta\,\phi'(\theta)^2}\,d\theta.$$

Therefore, the total path length is $R\int_{\theta_1}^{\theta_2}\sqrt{1+\sin^2\theta\,\phi'(\theta)^2}\,d\theta$, as claimed.

6.2 ⋆ Imagine first an infinitesimal section of path on the cylinder, in which ϕ increases by $d\phi$ and z by dz. This carries us a distance $R\,d\phi$ around the cylinder and dz up it. The distance ds along the path is therefore

$$ds = \sqrt{(R\,d\phi)^2 + (dz)^2} = \sqrt{R^2\phi'(z)^2+1}\,dz.$$

Therefore, the total path length is $R\int_{z_1}^{z_2}\sqrt{R^2\phi'(z)^2+1}\,dz$.

6.3 ⋆⋆ We already know that the actual path is a straight line within one medium. Therefore the segments from P_1 to Q and from Q to P_2 are straight and the corresponding distances are $P_1Q = \sqrt{x^2+y_1^2+z^2}$ and $QP_2 = \sqrt{(x-x_1)^2+y_2^2+z^2}$. Therefore the total time for the journey P_1QP_2 is

$$T = \left(\sqrt{x^2+y_1^2+z^2} + \sqrt{(x-x_1)^2+y_2^2+z^2}\right)/c.$$

To find the position of $Q = (x, 0, z)$ for which this is minimum we must differentiate with respect to z and x and set the derivatives equal to zero:

$$\frac{\partial T}{\partial z} = \frac{z}{c\sqrt{\cdots}} + \frac{z}{c\sqrt{\cdots}} = 0 \quad\Longrightarrow\quad z = 0,$$

which says that Q must lie in the same vertical plane as P_1 and P_2, and

$$\frac{\partial T}{\partial x} = \frac{x}{c\sqrt{\cdots}} + \frac{x-x_1}{c\sqrt{\cdots}} = 0 \quad\Longrightarrow\quad \sin\theta_1 = \sin\theta_2 \quad\text{or}\quad \theta_1 = \theta_2.$$

6.4 ⋆⋆ The lengths of the paths P_1Q and QP_2 are

$$P_1Q = \sqrt{x^2+h_1^2+z^2} \qquad\text{and}\qquad QP_2 = \sqrt{(x_2-x)^2+h_2^2+z^2}$$

The time for light to traverse each path is the path length divided by the speed of light, $v = c/n$. Thus the total time is

$$t = \frac{1}{c}\left(n_1\sqrt{x^2+h_1^2+z^2} + n_2\sqrt{(x_2-x)^2+h_2^2+z^2}\right).$$

To find where this is minimum we must set $\partial t/\partial z$ and $\partial t/\partial x$ equal to zero:

$$\frac{\partial t}{\partial z} = \frac{1}{c}\left(\frac{n_1 z}{\sqrt{x^2+h_1^2+z^2}} + \frac{n_2 z}{\sqrt{(x_2-x)^2+h_2^2+z^2}}\right),$$

which is zero if and only if $z = 0$. That is, Fermat's principle requires that Q lie in the plane containing P_1 and P_2 and normal to the interface. Also

$$\frac{\partial t}{\partial x} = \frac{1}{c}\left(\frac{n_1 x}{\sqrt{x^2+h_1^2+z^2}} - \frac{n_2(x_2-x)}{\sqrt{(x_2-x)^2+h_2^2+z^2}}\right) = \frac{1}{c}(n_1\sin\theta_1 - n_2\sin\theta_2)$$

which is zero if and only if $n_1\sin\theta_1 = n_2\sin\theta_2$, and this is Snell's law.

6.5 ⋆⋆ The distance from A to P is $AP = 2R\sin[(90° - \theta)/2]$, where R is the radius of the mirror. (To see this, drop a perpendicular from the center of the mirror to the line AP.) Similarly $PB = 2R\sin[(90° + \theta)/2]$. Thus the total distance APB is

$$APB = AP + PB = 2R\left(\sin\frac{90° - \theta}{2} + \sin\frac{90° + \theta}{2}\right) = 4\sin 45° \cos\frac{\theta}{2}$$

which is *maximum* when $\theta = 0$.

6.6 ⋆⋆ For curves in a plane:

curve	$y = y(x)$	$x = x(y)$	$r = r(\phi)$	$\phi = \phi(r)$
$ds =$	$\sqrt{1+y'^2}\,dx$	$\sqrt{1+x'^2}\,dy$	$\sqrt{r^2+r'^2}\,d\phi$	$\sqrt{1+r^2\phi'^2}\,dr$

For curves on a cylinder (first two) or sphere (last two):

curve	$\phi = \phi(z)$	$z = z(\phi)$	$\theta = \theta(\phi)$	$\phi = \phi(\theta)$
$ds =$	$\sqrt{1+R^2\phi'^2}\,dz$	$\sqrt{R^2+z'^2}\,d\phi$	$R\sqrt{\sin^2\theta+\theta'^2}\,d\phi$	$R\sqrt{1+\sin^2\theta\phi'^2}\,d\theta$

6.7 ⋆ The length of a small arc on the surface of the cylinder is $ds = \sqrt{R^2 d\phi^2 + dz^2} = \sqrt{R^2\phi'^2+1}\,dz$, where the last expression results from thinking of $\phi = \phi(z)$ as a function of z. Thus the integrand of the integral $\int ds = \int f dz$ is $f = \sqrt{R^2\phi'^2+1}$, and the Euler-Lagrange equation

$$\frac{\partial f}{\partial \phi} = \frac{d}{dz}\frac{\partial f}{\partial \phi'} \quad \text{becomes} \quad \frac{\partial f}{\partial \phi'} = \frac{\phi'}{\sqrt{R^2\phi'^2+1}} = \text{const.}$$

If you solve this last for ϕ' you will see that $\phi' = \text{const}$ (a different constant), from which we deduce that $\phi = az+b$, with a and b chosen so that the path passes through the given points. This equation defines a path which spirals around the cylinder from (ϕ_1, z_1) to (ϕ_2, z_2), with the angle ϕ changing linearly with z.

The geodesic connecting points 1 and 2 is certainly not unique. First, we can spiral from 1 to 2 going either way (clockwise or counter-clockwise) around the cylinder, making less than one complete revolution. Further, we could spiral less steeply, making one or more complete revolutions before arriving at point 2. Generally there is a unique *shortest* path, namely the spiral on which ϕ changes by less than π, but if ϕ_1 and ϕ_2 differ by exactly π, there are two shortest paths with equal lengths. If you unwrap and flatten the cylinder, the spiral paths become straight lines, which are well known to be the shortest paths on a flat surface.

6.8 ⋆ The PE is $U = -mgy$ and the initial KE and PE are both 0. Therefore, by conservation of energy, $\frac{1}{2}mv^2 - mgy = 0$ and $v = \sqrt{2gy}$.

6.9 ⋆ The integrand is $f = y'^2 + yy' + y^2$, so its derivatives are $\partial f/\partial y = y' + 2y$ and $\partial f/\partial y' = 2y' + y$ and the Euler-Lagrange equation (6.13) is

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'} \quad \Longrightarrow \quad y' + 2y = 2y'' + y' \quad \Longrightarrow \quad y'' = y,$$

the general solution of which is $y(x) = A\sinh(x) + B\cosh(x)$. Since $y(0) = 0$ and $y(1) = 1$, we see that $B = 0$ and $A = 1/\sinh(1)$, so the solution is $y(x) = \sinh(x)/\sinh(1)$.

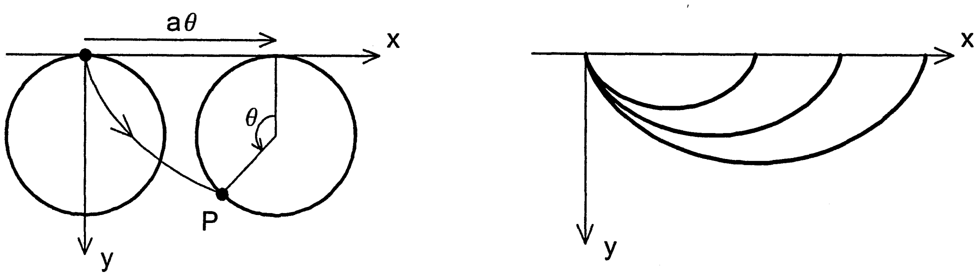
6.10 ⋆ If $\partial f/\partial y = 0$, then the Euler-Lagrange equation (6.13) reduces to $\dfrac{d}{dx}\dfrac{\partial f}{\partial y'} = 0$, which implies that $\partial f/\partial y' = \text{const}$.

6.11 ⋆⋆ The integrand is $f(y, y', x) = \sqrt{x}\sqrt{1 + y'^2}$. Since this is independent of y, $\partial f/\partial y = 0$ and the Euler-Lagrange equation (6.13) implies simply that $\partial f/\partial y'$ is a constant; that is, $\sqrt{x}\,y'/\sqrt{1 + y'^2} = k$. This can be solved for y' to give $y' = k/\sqrt{x - k^2}$, which integrates to give $y = 2k\sqrt{x - k^2} + D$, where D is a constant of integration. Rearranging we find that $x = k^2 + (y - D)^2/4k^2$, which is a parabola with it's axis along the line $y = D$.

6.12 ⋆⋆ The integrand is $f(y, y', x) = x\sqrt{1 - y'^2}$. Since this is independent of y, $\partial f/\partial y = 0$ and the Euler-Lagrange equation (6.13) implies simply that $\partial f/\partial y'$ is a constant; that is, $xy'/\sqrt{1 - y'^2} = k$. This can be solved for y' to give $y' = k/\sqrt{x^2 + k^2}$, which integrates to give $y = k\sinh^{-1}(x/k) + c$, where c is a constant of integration. (Make the substitution $x/k = \sinh u$.) Rearranging we find that $x = k\sinh[(y - c)/k]$.

6.13 ⋆⋆ If we write the path as $\phi = \phi(r)$, the distance from O to P is $\int_O^P ds = \int_0^R f\,dr$, where $f = [2/(1 - r^2)]\sqrt{1 + r^2\phi'^2}$. Since $\partial f/\partial\phi = 0$, the Euler-Lagrange equation (6.13) implies simply that $\partial f/\partial\phi'$ is a constant; that is, $[2/(1 - r^2)]r^2\phi'/\sqrt{1 + r^2\phi'^2} = k$. Because the path passes through the origin, $r = 0$, the constant k must in fact be zero, and we find that $\phi' = 0$. This defines a straight line through the origin.

6.14 ⋆⋆ **(a)** The left picture shows two positions of a wheel of radius a rolling on the underside of the x axis. The point which started in contact with the x axis has moved to the point P while the wheel has rotated through an angle θ and moved a distance $a\theta$ to the right. The coordinates of the point P are easily seen to be $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, in exact agreement with Eq.(6.26). That is, the curve traced by the point P on the wheel (the cycloid) is the same as the brachistochrone curve of (6.26).



(b) The right picture shows three cycloids with three successively larger values of a. It is clear from the picture (or from the equations) that as we increase a from 0 to ∞, the loop steadily expands, eventually sweeping exactly once across any point in the positive quadrant.

(c) If the point 2 has coordinates $x_2 = \pi b$ and $y_2 = 2b$, it is easy to see that this corresponds to $a = b$ and $\theta = \pi$; that is, point 2 is at the exact bottom of the cycloid. The time for the journey is

$$t = \int_1^2 \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_0^\pi \frac{\sqrt{x'^2 + y'^2}\, d\theta}{\sqrt{y}} = \sqrt{\frac{b}{2g}} \int_0^\pi \frac{\sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta}{\sqrt{1-\cos\theta}} = \pi\sqrt{\frac{b}{g}}.$$

If point 2 has coordinates $x_2 = 2\pi b$ and $y_2 = 0$, then again $a = b$ but now $\theta = 2\pi$. That is point 2 is at the top of the cycloid, with point P back on the x axis. We can easily calculate the time to reach point 2, but there is no need, because it is obviously twice that to reach the bottom. That is, in this case, $t = 2\pi\sqrt{b/g}$.

6.15 ⋆⋆ The analysis for the case that the car is launched from point 1 with fixed speed v_o is very similar to the case of Example 6.2 with $v_o = 0$, except that, by conservation of energy, the car's speed at height y is $v = \sqrt{2gy + v_o^2}$ instead of $v = \sqrt{2gy}$. If we make the change of variables from y to $\tilde{y} = y + v_o^2/2g$, the analysis goes through exactly as before and we get the same answer as in Eq.(6.26) except that it is now $\tilde{y}$ that is equal to $a(1 - \cos\theta)$. Therefore $y = a(1 - \cos\theta) - v_o^2/2g$ (with x exactly the same as before), and the curve is the same cycloid except that it is shifted up by a height $h = v_o^2/2g$.

6.16 ⋆⋆ By Problem 6.1, the path length is $L = \int f\, d\theta$, with $f = f(\phi, \phi', \theta) = \sqrt{1 + \sin^2\theta\, \phi'^2}$. Since $\partial f/\partial\phi = 0$, the Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta\, \phi'}{\sqrt{1 + \sin^2\theta\, \phi'^2}} = c.$$

If we choose our polar axis to go through the point 1, then $\theta_1 = 0$ and the constant c has to be zero. Thus the Euler-Lagrange equation implies that $\phi' = 0$ and hence that ϕ is constant. The curves of constant ϕ are the lines of longitude and are great circles. Therefore, the geodesics are great circles.

6.17 ⋆⋆ The length of an element of path in cylindrical coordinates is

$$ds = \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2} = \sqrt{(1+\lambda^2)d\rho^2 + \rho^2 d\phi^2} = \sqrt{(1+\lambda^2) + \rho^2\phi'^2}\, d\rho$$

where in the second equality I used the fact that $dz = \lambda\, d\rho$ on the cone, and in the last I assumed that the path was written as $\phi = \phi(\rho)$. Thus the path length has the standard form $L = \int f d\rho$, with $f = f(\phi, \phi', \rho) = \sqrt{1+\lambda^2+\rho^2\phi'^2}$. Because $\partial f/\partial\phi = 0$, the Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial \phi'} = \frac{\rho^2\phi'}{\sqrt{1+\lambda^2+\rho^2\phi'^2}} = K \quad \text{whence} \quad \phi' = \frac{K\sqrt{1+\lambda^2}}{\rho\sqrt{\rho^2-K^2}}.$$

This can be integrated (make the subtitution $K/\rho = \cos u$) to give

$$\phi - \phi_o = \sqrt{1+\lambda^2}\arccos(K/\rho) \quad \text{or} \quad \rho = \frac{K}{\cos\left[(\phi-\phi_o)/\sqrt{1+\lambda^2}\right]}.$$

This is not easily recognized as any simple curve, but in the limit that $\lambda \to 0$ the cone approaches the plane $z = 0$ and the geodesic approaches the curve $\rho = K/\cos(\phi - \phi_o)$, which you can show is a straight line perpendicular to the direction $\phi = \phi_o$, a distance K from the origin.

6.18 ⋆⋆ If we use polar coordinates (r, ϕ) and write the path in the form $\phi = \phi(r)$, then the path length takes the form $L = \int f dr$ with $f = f(\phi, \phi', r) = \sqrt{1+r^2\phi'^2}$. Because $\partial f/\partial\phi = 0$, the Euler-Lagrange equation is

$$\frac{\partial f}{\partial \phi'} = \frac{r^2\phi'}{\sqrt{1+r^2\phi'^2}} = \text{const} \quad \text{whence} \quad \phi' = \frac{K}{r\sqrt{r^2-K^2}}.$$

This can be integrated (make the subtitution $K/r = \cos u$) to give

$$\phi - \phi_o = \arccos(K/r) \quad \text{or} \quad r = \frac{K}{\cos(\phi-\phi_o)}.$$

This is the equation of a straight line perpendicular to the direction $\phi = \phi_o$, a distance K from the origin. [To see this, note that $r\cos(\phi - \phi_o)$ is the component of $\mathbf{r}$ in the direction $\phi = \phi_o$; that this equals the constant K says that $\mathbf{r}$ lies on the line indicated.]

6.19 ⋆⋆ The area of the surface of revolution is $A = \int 2\pi y\, ds = 2\pi \int_{y_1}^{y_2} y\sqrt{1+x'^2}dy$, which we can write as $A = 2\pi \int f dy$ where $f = y\sqrt{1+x'^2}$. Because $\partial f/\partial x = 0$, the Euler-Lagrange equation reads

$$\frac{\partial f}{\partial x'} = \frac{yx'}{\sqrt{1+x'^2}} = y_o \quad \text{whence} \quad x' = \frac{y_o}{\sqrt{y^2 - y_o^2}},$$

where y_o is a constant. Using the substitution $y/y_o = \cosh u$, you can integrate this to give $x - x_o = y_o \text{arccosh}(y/y_o)$ (where x_o is another constant) or $y = y_o \cosh[(x - x_o)/y_o]$.

6.20 ⋆⋆ If $f = f(y, y')$, then, by the standard result of two-variable calculus, $df = (\partial f/\partial y)dy + (\partial f/\partial y')dy'$. Dividing both sides by dx, we find

$$\frac{df}{dx} = \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' = \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right)y' + \frac{\partial f}{\partial y'}y'' = \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)$$

where, for the second equality, I used the Euler-Lagrange equation and, in the last, I used the product rule. Moving the right side across to the left, we see that $f - y'\partial f/\partial y'$ is constant.

6.21 ⋆⋆ The time for the car to go from 1 to 2 is $t = \int_1^2 ds/v$. As usual, $ds = \sqrt{1+y'^2}\,dx$, and, as before, $v = \sqrt{2gy}$. Therefore, $t = (1/\sqrt{2g})\int f\, dx$, where $f = \sqrt{(1+y'^2)/y}$. Because f does not depend explicitly on x (that is, $\partial f/\partial x = 0$), we can use the first integral (6.43) from Problem 6.20 and obtain

$$f - y'\frac{\partial f}{\partial y'} = \frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\frac{y'}{\sqrt{y}\sqrt{1+y'^2}} = \frac{1}{\sqrt{y}\sqrt{1+y'^2}} = k.$$

Solving for y' we find

$$\frac{dy}{dx} = \sqrt{\frac{1}{k^2 y} - 1} = \sqrt{\frac{2a - y}{y}}$$

if we rename $k^2 = 1/(2a)$. Separating the variables x and y and integrating gives $x = \int \sqrt{y/(2a-y)}\,dy$, which is precisely the result (6.23). From here on, the derivation of the brachistochrone is exactly the same as in Example 6.2.

6.22 ⋆⋆⋆ The area between the string and the x axis is $A = \int y\, dx$. The length of a small element of string satisfies $ds^2 = dx^2 + dy^2$, so $dx = \sqrt{ds^2 - dy^2} = \sqrt{1 - y'^2}\, ds$, if we regard y as a function of s and $y' = dy/ds$. Therefore, the area A can be written as $A = \int_0^l f\, ds$ where $f(y, y', s) = y\sqrt{1-y'^2}$. Because f does not depend on s explicitly, we can use the first intergral (6.43):

$$f - y'\frac{\partial f}{\partial y'} = y\sqrt{1-y'^2} + .y'\frac{yy'}{\sqrt{1-y'^2}} = \frac{y}{\sqrt{1-y'^2}} = R,$$

where R is some constant. This last equation implies that $y' = \sqrt{1 - y^2/R^2}$ or equivalently $dy/\sqrt{1-y^2/R^2} = ds$. Integrating both sides we conclude that $\arcsin(y/R) = s/R$. (The constant of integration is zero because $y = 0$ when $s = 0$.) Therefore, $y = R\sin(s/R)$. Since $y = 0$ when $s = l$, we see that $l/R = \pi$. (It is fairly easy to see that the other solutions, $l/R = 2\pi, 3\pi, \cdots$ yield a smaller area.) Finally, we saw that $dx = \sqrt{1 - y'^2}\, ds$, so $x = \int \sqrt{1 - y'^2}\, ds = R - R\cos(s/R)$. Combining these results for x and y, we see that $(x - R)^2 + y^2 = R^2$, so the string must lie on the semicircle with radius R centered on the point $(R, 0)$.

6.23 ⋆⋆⋆ **(a)** If the plane is at position (x, y) and is heading at an angle ϕ north of east, its velocity relative to the air is $(v_o \cos\phi, v_o \sin\phi)$, while the wind's velocity is $(Vy, 0)$. The plane's velocity relative to the ground is the sum of these two vectors, so its ground speed is

$$v = \sqrt{(v_o \cos\phi + Vy)^2 + (v_o \sin\phi)^2} = \sqrt{v_o^2 + 2v_o Vy\cos\phi + V^2y^2} \approx v_o + Vy = v_o(1 + ky)$$

where I have made the approximation $\cos\phi \approx 1$, assuming that ϕ is always small, and where $k = V/v_o$.

(b) The time of flight is

$$t = \int_O^P \frac{ds}{v} = \int_0^D \frac{\sqrt{1 + y'^2}\, dx}{v} \approx \int_0^D \frac{1 + \frac{1}{2}y'^2}{v_o(1 + ky)} dx = \frac{1}{v_o} \int_0^D f\, dx \qquad \text{(i)}$$

where I used the binomial approximation in the numerator (assuming y' is always small) and where $f = (1 + \frac{1}{2}y'^2)/(1 + ky)$.

(c) With a little algebra, you can show that the Euler-Lagrange equation is

$$y''(1 + ky) - \tfrac{1}{2}ky'^2 + k = 0.$$

If you substitute the proposed guess, $y = \lambda x(D - x)$, this becomes (after a little more algebra) $kD^2\lambda^2 + 4\lambda - 2k = 0$, of which the relevant solution is $\lambda = (\sqrt{4 + 2k^2D^2} - 2)/(kD^2)$. (The second solution is negative which makes no sense.)

The maximum displacement to the north is $y_{\max} = \lambda D^2/4$, which, if we put in the given numbers, is $y_{\max} = 366$ miles. If we put the same numbers in Eq.(i) and do the integral numerically, we find for the time of flight $t = 3.556$ hours. The time along the direct path would be 4 hours, so the saving is 0.444 hours or 27 minutes.

6.24 ⋆⋆⋆ As in Section 6.1, the time for light to travel between any two points is an integral of the form $t = \int ds/v$, where $v = c/n = cr^2/a$. In two-dimensional polars, $ds = \sqrt{dr^2 + r^2 d\phi^2} = \sqrt{1 + r^2\phi'^2}\, dr$. Thus the integral that has to be made stationary has the standard form $\int f(\phi, \phi', r)\, dr$ where

$$f(\phi, \phi', r) = \sqrt{1 + r^2\phi'^2}/r^2$$

and I have dropped a couple of uninteresting constant factors. Since f does not involve ϕ, the Euler-Lagrange equation implies simply that $\partial f/\partial\phi'$ is constant. That is, $\phi'/\sqrt{1 + r^2\phi'^2} = \text{const} = k^2$, say. We can solve this to give

$$\phi' = k/\sqrt{1-k^2r^2}$$

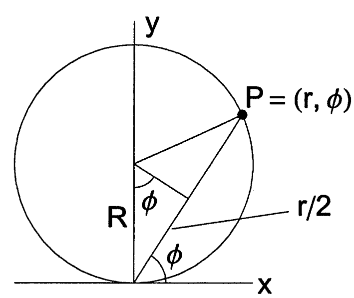
which we can integrate to give $\phi = \phi_o + \arcsin kr$ (where ϕ_o is just the constant of integration) or

$$r = 2R\sin(\phi - \phi_o)$$

(where I have introduced the new constant $R = 1/2k$). This is the equation of the path followed by the light. To identify this path is actually quite hard, but we are given a generous hint, that it is a circle through the origin. To verify this claim, let us first simplify the equation by redefining the direction $\phi = 0$ to get rid of the constant ϕ_o, so that the equation is

$$r = 2R\sin\phi. \tag{ii}$$

Notice that $r = 0$ when $\phi = 0$, and when ϕ is small the curve is close to the x axis. (When ϕ is small, $y = r\sin\phi$ is much smaller than $x = r\cos\phi$.) Thus the curve is tangent to the x axis at the origin and the claimed circle has to be as shown. Consider, then, a point P with polar coordinates (r, ϕ) on the circle shown, with radius R. From the indicated geometry, it is clear that $r/2 = R\sin\phi$, which is precisely the equation (ii) of the path of the light. That is, the path of the light *is* the circle shown.



The closer the light is to the origin, O, the slower it travels, and it never actually reaches the origin. Rather, as $t \to \infty$, the light approaches O from the right or left, depending on its direction of travel around the circle.

6.25 ⋆⋆⋆ The parametric equation (6.25) of the cycloid is $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, from which we find the derivatives $x' = a(1-\cos\theta)$ and $y' = a\sin\theta$. Therefore, the element of path length is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'^2 + y'^2}\,d\theta = a\sqrt{(1-\cos\theta)^2 + \sin^2\theta}\,d\theta = a\sqrt{2(1-\cos\theta)}\,d\theta.$$

The speed of the cart is given by conservation of energy as

$$v = \sqrt{2g(y_o - y)} = \sqrt{2ga(\cos\theta_o - \cos\theta)}.$$

Therefore the required time is

$$t = \int_{P_o}^{P} \frac{ds}{v} = \int_{\theta_o}^{\pi} \frac{a\sqrt{2(1-\cos\theta)}}{\sqrt{2ga(\cos\theta_o - \cos\theta)}}\,d\theta.$$

If we make the substitution $\theta = \pi - 2\alpha$ and use a couple of trig identities, this becomes

$$t = 2\sqrt{\frac{a}{g}}\int_0^{\alpha_o} \frac{\cos\alpha}{\sqrt{\sin^2\alpha_o - \sin^2\alpha}}\,d\alpha.$$

Finally, the substitutions $\sin\alpha = u$ and then $u/u_o = v$ reduce this to

$$t = 2\sqrt{\frac{a}{g}}\int_0^{u_o}\frac{du}{\sqrt{u_o^2-u^2}} = 2\sqrt{\frac{a}{g}}\int_0^1\frac{dv}{\sqrt{1-v^2}} = \pi\sqrt{\frac{a}{g}}$$

which *is* independent of θ_o.

The higher the starting point P_o, the further the car has to go, but the steeper the initial slope and the faster the car goes. On a cycloid, these two effects perfectly cancel, so that the time to reach P is independent of the position of P_o.

6.26 ⋆⋆ Let $S(\alpha,\beta)$ be the integral taken along the "wrong" path (6.32),

$$S(\alpha,\beta) = \int_{u_1}^{u_2} f(x+\alpha\xi, y+\beta\eta, x'+\alpha\xi', y'+\beta\eta', u)du.$$

This has to be stationary when $\alpha=\beta=0$, for any choice of the functions $\xi(u)$ and $\eta(u)$ (as long as they vanish at the endpoints u_1 and u_2). Thus we can differentiate with respect to α and β and set the derivatives equal to zero:

$$\frac{\partial S}{\partial\alpha} = \int_{u_1}^{u_2}\left(\frac{\partial f}{\partial x}\xi + \frac{\partial f}{\partial x'}\xi'\right)du = \int_{u_1}^{u_2}\xi(u)\left(\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'}\right)du = 0$$

where for the second equality I used integration by parts. This has to be true for any choice of the function $\xi(u)$, and this implies that the factor in parentheses must be zero,

$$\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'},$$

which is the first of the Euler-Lagrange equations (6.34). The second one follows in exactly the same way when we set $\partial S/\partial\beta = 0$.

6.27 ⋆⋆ The element of path length is $ds = \sqrt{dx^2+dy^2+dz^2} = \sqrt{x'^2+y'^2+z'^2}\,du$. Thus the total path length is $L = \int f\,du$ where $f = \sqrt{x'^2+y'^2+z'^2}$. There are three Euler-Lagrange equations, which involve the following six derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0,$$

and

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2+y'^2+z'^2}},\quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2+y'^2+z'^2}},\quad \frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{x'^2+y'^2+z'^2}}.$$

Since the first three derivatives are zero, the Euler-Lagrange equations imply simply that each of the last three is constant. This means that the ratios, $x' : y' : z'$ are constant, which implies in turn that as we move along the curve the ratios $dx : dy : dz$ are constant. In other words, the curve is a straight line.

Chapter 7

Lagrange's Equations

I covered this chapter in 6 *fifty-minute lectures.*

Chapter 7 is probably the most important chapter of the book. For most readers, much of the material of Chapters 1 through 5 should already have been familiar. By contrast the material of Chapter 7 is probably completely new. (Chapter 6 was also probably new, being the mathematical prerequisite for Chapter 7.) It's not just that your students likely haven't met Lagrangian mechanics before. The Lagrangian method is an entirely new way of approaching mechanics, and they will need to be broken in gently, to be lead through several worked examples, and work lots of examples themselves.

There are, of course, several other important topics that most of our students haven't met before — Hamiltonian mechanics, normal modes, rotation of rigid bodies, non-inertial frames, and several more. But if we were forced to choose just one such topic, I believe most of us would choose Lagrangian mechanics, both because it is such a powerful tool itself and because it is an essential preliminary to most of these other important topics. Thus, this chapter is in many ways the pivotal chapter of the book. If your students are well prepared you may choose to skip quickly through several of the first five chapters, and you may choose to omit several of the later chapters. But you can't omit Chapter 7.

I have tried hard to make completely clear that Lagrangian mechanics can be derived from Newtonian (and vice versa). The logic is easily sketched

$$\text{Newton} \iff \text{Lagrange's equations} \iff \text{Hamilton's principle.} \tag{i}$$

In Section 7.1, I first prove this logical chain for an unconstrained single particle (subject to conservative forces) using Cartesian coordinates. The next crucial step is to observe that (unlike Newton's laws) Hamilton's principle is invariant under almost arbitrary changes of coordinates and, hence, that the same is true of Lagrange's equations. This establishes Lagrange's equations in all their wonderful generality. The generalization to unconstrained multiparticle systems is then easily sketched. The generalization to constrained systems is, of course, harder. In Section 7.3, I give reasonably careful definitions of generalized coordinates, degrees of freedom, and (briefly) holonomic systems. Then Section 7.4 has the main proof, followed in Section 7.5 by lots of worked examples. Section 7.6 has a brief discussion of ignorable coordinates, and 7.7 is a summary of the chapter so far.

One of the controversial aspects of Lagrangian mechanics concerns what to do next. My own view is that, having swallowed this huge and unfamiliar pill, our students should be given a chance to digest it by studying its many applications in well known, interesting, and reasonably straightforward problems — planetary motion, rigid-body rotation, normal modes of coupled oscillators, and so on — and this is what I do in Chapters 8 through 11. However, there are also many theoretical developments that call out to be pursued. Three of these, the relation between invariance principles and conservation laws, the Lagrangian for a charge in a magnetic field, and the method of Lagrange multipliers, I decided to treat as optional sections at the end of this chapter. If you feel that any of these are pressingly important, you could include them now, though my preference would be to come back to them later. The most important and obvious theoretical sequel to Lagrangian mechanics is Hamiltonian mechanics, and several colleagues argued that Hamiltonian mechanics should be the subject of Chapter 8. Obviously I did not agree. When our students are still reeling from the challenge of mastering Lagrange is, I believe, the worst moment to hit them with Hamilton, and I postponed introduction of the latter to Chapter 13. Nevertheless, Hamiltonian mechanics is a tremendously important part of modern physics[1] and I designed Chapter 13 to be read at any time after Chapter 7. If your students are doing well and feeling strong, you could even jump to Chapter 13 immediately after Chapter 7.

This chapter offers several opportunities to bring out demonstration experiments to give your students a break from theory. Few students have seen an Atwood machine (Example 7.3) and even fewer know what it was designed for (to measure g). You can quite easily make demonstrations of Examples 7.5 and 7.6, and of several of the problems at the end of the chapter (for example, the "yoyo" of Problem 7.14). Even more than usual, it's important that your students do lots of problems for themselves. Many of the "$\star$" problems are intentionally very simple and are solvable by Newtonian mechanics. I think you should assign several of these just to get your students thinking in Lagrangian mode.

Solutions to Problems for Chapter 7

7.1 $\star$ The kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad \text{and} \qquad U = mgz.$$

Therefore

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

The three Lagrange equations (7.7) are

[1] Though I can't agree with the view that a good understanding of classical Hamiltonian mechanics is a prerequisite for quantum mechanics.

$$0 = m\ddot{x}, \qquad 0 = m\ddot{y}, \qquad \text{and} \qquad -mg = m\ddot{z},$$

which are just the three components of the equation $\mathbf{F} = m\mathbf{a}$ for a projectile with $\mathbf{F} = m\mathbf{g}$.

7.2 $\star$ With $F = -kx$, the PE is $U = \frac{1}{2}kx^2$ and the Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$. The Lagrange equation is $-kx = m\ddot{x}$ and its solution is $x = A\cos(\omega t - \delta)$ where $\omega = \sqrt{k/m}$ and A and δ are arbitrary constants.

7.3 $\star$ $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2)$ and the two Lagrange equations are $-kx = m\ddot{x}$ and $-ky = m\ddot{y}$. In the general solution, x and y oscillate with the same angular frequency $\omega = \sqrt{k/m}$ and the point (x, y) moves around an ellipse.

7.4 $\star$ The kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad \text{and} \qquad U = mgh = -mgy\sin\alpha.$$

Therefore

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy\sin\alpha$$

The two Lagrange equations (7.7) are

$$0 = m\ddot{x} \qquad \text{and} \qquad mg\sin\alpha = m\ddot{y}$$

which imply that the acceleration across the slope is zero while that down the slope is $g\sin\alpha$, as expected.

7.5 $\star$ If we make a small displacement from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$, we know from (4.35) that the change in $f(\mathbf{r})$ is $df = \nabla f \cdot d\mathbf{r}$. In two-dimensional polars, $d\mathbf{r} = (dr, r\,d\theta)$. Therefore,

$$df = (\nabla f)_r\,dr + (\nabla f)_\theta\,r\,d\theta.$$

On the other hand, we know from two-variable calculus that

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta.$$

Both of these are valid for arbitrary (small) values of dr and $d\theta$, so we can compare coefficients to give

$$(\nabla f)_r = \frac{\partial f}{\partial r} \qquad \text{and} \qquad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}.$$

7.6 ⋆ **(a)** Newton's second law gives the two vector equations (or six scalar equations)

$$\mathbf{F}_1 = -\boldsymbol{\nabla}_1 U = m_1\ddot{\mathbf{r}}_1 \qquad \text{and} \qquad \mathbf{F}_2 = -\boldsymbol{\nabla}_2 U = m_2\ddot{\mathbf{r}}_2\,. \tag{ii}$$

(b) The Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - U(x_1, \cdots, z_2).$$

There are six Lagrange equations, one for each of the coordinates x_1, y_1, z_1, x_2, y_2, and z_2:

$$\frac{\partial\mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_1} \qquad \Longrightarrow \qquad -\partial U/\partial x_1 = m_1\ddot{x}_1$$

$$\cdots\cdots\cdots$$

$$\frac{\partial\mathcal{L}}{\partial z_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{z}_2} \qquad \Longrightarrow \qquad -\partial U/\partial z_2 = m_2\ddot{z}_2$$

which you will recognize as precisely the six components of Newton's second law as in Eq.(ii).

7.7 ⋆ **(a)** Newton's second law gives the N vector equations (or $3N$ scalar equations)

$$\mathbf{F}_\alpha = -\boldsymbol{\nabla}_\alpha U = m_\alpha\ddot{\mathbf{r}}_\alpha \quad [\alpha = 1, \cdots, N]. \tag{iii}$$

(b) The Lagrangian is

$$\mathcal{L}(\mathbf{r}_1, \cdots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \cdots, \dot{\mathbf{r}}_N) = T - U = \sum_1^N \tfrac{1}{2}m_\alpha\dot{\mathbf{r}}_\alpha^2 - U(\mathbf{r}_1, \cdots, \mathbf{r}_N)$$

and the $3N$ Lagrange equations are

$$\frac{\partial\mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_1} \qquad \Longrightarrow \qquad -\partial U/\partial x_1 = m_1\ddot{x}_1$$

and so on. These are just the $3N$ components of the N Newtonian equations (iii).

7.8 ⋆⋆ **(a)** $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}kx^2$.

(b) Solving for x_1 and x_2 in terms of X and x, we find

$$x_1 = X + \tfrac{1}{2}x + \tfrac{1}{2}l \qquad \text{and} \qquad x_2 = X - \tfrac{1}{2}x - \tfrac{1}{2}l.$$

Differentiating these, and substituting into $\mathcal{L}$, we find

$$\mathcal{L} = \tfrac{1}{2}m[(\dot{X} + \tfrac{1}{2}\dot{x})^2 + (\dot{X} - \tfrac{1}{2}\dot{x})^2] - \tfrac{1}{2}kx^2 = m\dot{X}^2 + \tfrac{1}{4}m\dot{x}^2 - \tfrac{1}{2}kx^2.$$

The two Lagrange equations are

$$\text{X eqn:} \quad \frac{\partial\mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} \qquad \text{or} \qquad 0 = 2m\ddot{X}$$

and

$$\text{x eqn:} \quad \frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \qquad \text{or} \qquad -kx = \tfrac{1}{2}m\ddot{x}.$$

(c) The X equation implies that $\dot{X}(t) = \text{const} = V_o$ and hence that $X(t) = V_o t + X_o$; that is, the CM moves like a free particle, which we could have anticipated, since there are no external forces. The x equation has the general solution $x(t) = A\cos(\omega t - \delta)$; that is, the two masses oscillate in and out, relative to each other, with frequency $\omega = \sqrt{2k/m}$. The factor of $2k$ inside the square root, can be understood in several ways; for example, the spring is compressed (or stretched) by twice the amount that either separate mass moves. Thus the force on either mass is as if the spring had force constant $2k$.

7.9 ⋆ The requested equations are just the standard equations for two-dimensional polar coordinates, with the radius fixed at that of the hoop, $r = R$, namely $x = R\cos\phi$, $y = R\sin\phi$ and, in the other direction, $\phi = \arctan(y/x)$ with ϕ chosen to lie in the right quadrant.

7.10 ⋆ $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = \rho/\tan\alpha$, and, in the other direction, $\rho = \sqrt{x^2+y^2}$ (or $\rho = z\tan\alpha$) and $\phi = \arctan(y/x)$ with ϕ chosen to lie in the right quadrant.

7.11 ⋆ $x = x_s + l\sin\phi = A\cos(\omega t) + l\sin\phi$, and $y = y_s + l\cos\phi = l\cos\phi$. In the other direction, $\phi = \arctan[(x - A\cos\omega t)/y]$.

7.12 ⋆ If we define $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - U(x)$, then $\partial\mathcal{L}/\partial x = -\partial U/\partial x = F$ and $(d/dt)(\partial\mathcal{L}/\partial\dot{x}) = m\ddot{x}$. Substituting into Newton's second law $F + F_{\text{fric}} = m\ddot{x}$, we find that

$$\frac{\partial\mathcal{L}}{\partial x} + F_{\text{fric}} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}\,.$$

7.13 ⋆⋆ The Lagrangian is $\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(\mathbf{r}_1, \mathbf{r}_2, t)$. Suppose that in the actual motion the two particles follow the "right" path $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, and we'll compare the action along this path with that on the "wrong" one $\mathbf{R}_1(t) = \mathbf{r}_1(t) + \epsilon_1(t)$ and $\mathbf{R}_2(t) = \mathbf{r}_2(t) + \epsilon_2(t)$. The difference between the Lagrangians on these two paths is

$$\begin{aligned}\delta\mathcal{L} &= \mathcal{L}(\mathbf{R}_1, \mathbf{R}_2, \dot{\mathbf{R}}_1, \dot{\mathbf{R}}_2, t) - \mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, t)\\ &= \tfrac{1}{2}m_1[(\dot{\mathbf{r}}_1 + \dot{\epsilon}_1)^2 - \dot{\mathbf{r}}_1^2] + \tfrac{1}{2}m_2[(\dot{\mathbf{r}}_2 + \dot{\epsilon}_2)^2 - \dot{\mathbf{r}}_2^2] - [U(\mathbf{r}_1 + \epsilon_1, \mathbf{r}_2 + \epsilon_2, t) - U(\mathbf{r}_1, \mathbf{r}_2, t)]\\ &\approx m_1\dot{\mathbf{r}}_1\cdot\dot{\epsilon}_1 + m_2\dot{\mathbf{r}}_2\cdot\dot{\epsilon}_2 - \epsilon_1\cdot\nabla_1 U - \epsilon_2\cdot\nabla_2 U.\end{aligned}$$

Therefore, the difference between the two action integrals is

$$\begin{aligned}\delta S = \textstyle\int \delta\mathcal{L}\,dt &= -\textstyle\int[\epsilon_1\cdot(m_1\ddot{\mathbf{r}}_1 + \nabla_1 U) + \epsilon_2\cdot(m_2\ddot{\mathbf{r}}_2 + \nabla_2 U)]dt\\ &= -\textstyle\int[\epsilon_1\cdot\mathbf{F}_1^{\text{cstr}} + \epsilon_2\cdot\mathbf{F}_2^{\text{cstr}}]dt.\end{aligned}$$

(In the first line I used integration by parts to move a time derivative from each $\dot{\epsilon}$ to the corresponding $\dot{\mathbf{r}}$, and in the second I used Newton's second law.) The integral on the second line is the work done by the constraint forces in the displacement from $\mathbf{r}_1$ to $\mathbf{r}_1 + \boldsymbol{\epsilon}_1$ and from $\mathbf{r}_2$ to $\mathbf{r}_2 + \boldsymbol{\epsilon}_2$. Provided this displacement is consistent with the constraints, this work is zero. Thus we have proved that the action integral is stationary for any displacement of path consistent with the constraints. If we now introduce generalized coordinates $q_1, \cdots, q_n$, then *any* variation of $q_1, \cdots, q_n$ is consistent with the constraints. Therefore the action integral $S = \int \mathcal{L}(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)dt$ is stationary for any variations of $q_1, \cdots, q_n$, and the correct path must satisfy the n Euler-Lagrange equations (7.52).

7.14 ⋆ Recalling the $I = \frac{1}{2}mR^2$ and that $\omega = \dot{x}/R$, we see that the kinetic energy is $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{3}{4}m\dot{x}^2$. Therefore, the Lagrangian is $\mathcal{L} = \frac{3}{4}m\dot{x}^2 + mgx$, the Lagrange equation is $mg = 3m\ddot{x}/2$, and $\ddot{x} = 2g/3$.

7.15 ⋆ Since both masses have the same speed $\dot{x}$, the total KE is $T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$, whereas the PE is due to the second mass alone, $U = -m_2gx$. Therefore,

$$\mathcal{L} = T - U = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2gx.$$

The Lagrange equation is $m_2g = (m_1 + m_2)\ddot{x}$, from which it follows $\ddot{x} = gm_2/(m_1 + m_2)$.

7.16 ⋆ Since $\omega = v/R$, the cylinder's KE is $T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m + I/R^2)\dot{x}^2$. The PE is $U = -mgx\sin\alpha$, so the Lagrangian is $\mathcal{L} = \frac{1}{2}(m + I/R^2)\dot{x}^2 + mgx\sin\alpha$ and the Lagrange equation is $mg\sin\alpha = (m + I/R^2)\ddot{x}$. Therefore $\ddot{x} = (mg\sin\alpha)/(m + I/R^2)$.

7.17 ⋆ The KE of rotation of the pulley is $\frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{x}^2/R^2$ since $\omega = \dot{x}/R$. Therefore, the total KE is $T = \frac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2$, while the PE is $U = -(m_1 - m_2)gx$ as before. Thus the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2 + (m_1 - m_2)gx$$

and the Lagrange equation is

$$(m_1 - m_2)g = (m_1 + m_2 + I/R^2)\ddot{x}.$$

That is, $\ddot{x} = (m_1 - m_2)g/(m_1 + m_2 + I/R^2)$.

7.18 ⋆ We'll use as our generalized coordinate the height x of m below the axis of the cylinder. The angular velocity of the cylinder is $\omega = \dot{x}/R$, so the total KE is $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m+I/R^2)\dot{x}^2$, while the PE is $U = -mgx$. Therefore, $\mathcal{L} = T - U = \frac{1}{2}(m+I/R^2)\dot{x}^2 + mgx$ and the Lagrange equation is $mg = (m+I/R^2)\ddot{x}$, which implies that $\ddot{x} = mg/(m+I/R^2)$.

7.19 ⋆ With its mass $M = 0$, the presence of the wedge should be irrelevant; that is, the block should fall straight down with acceleration g, pushing the massless wedge aside as it goes. To verify this, note that the position of the block relative to the table is $\mathbf{r} = (q_2 + q_1 \cos\alpha, q_1 \sin\alpha)$ (with axes chosen as in Figure 7.8). Therefore the acceleration is

$$\mathbf{a} = (\ddot{q}_2 + \ddot{q}_1 \cos\alpha,\ \ddot{q}_1 \sin\alpha) = \ddot{q}_1 \left(\left[1 - \frac{m}{M+m}\right] \cos\alpha,\ \sin\alpha \right) \tag{iv}$$

where in the last expression I have replaced $\ddot{q}_2$ using Equation (7.66). As $M \to 0$, the x component approaches zero, as expected. To see what happens to the y component, we use (7.67) for $\ddot{q}_1$, which, in the limit that $M = 0$, gives

$$\ddot{q}_1 = \frac{g \sin\alpha}{1 - \cos^2\alpha} = \frac{g}{\sin\alpha}.$$

Inserting this into Eq.(iv), we find that, when $M = 0$, the y component of $\mathbf{a}$ is just g, as anticipated.

7.20 ⋆ The helix satisfies $\rho = R$ and $z = \lambda\phi$. Thus the bead's velocity is $\mathbf{v} = (\dot{\rho}, \rho\dot{\phi}, \dot{z}) = (0, R\dot{\phi}, \dot{z}) = \dot{z}(0, R/\lambda, 1)$ and its KE is $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{z}^2(1 + R^2/\lambda^2)$. The PE is $U = mgz$, so the Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m\dot{z}^2(1 + R^2/\lambda^2) - mgz$. The Lagrange equation is $-g = (1 + R^2/\lambda^2)\ddot{z}$ (after canceling a factor of m), and $\ddot{z} = -g/(1 + R^2/\lambda^2)$. When $R \to 0$ this answer reduces to $\ddot{z} = -g$, which is correct because in this limit the helix reduces to a vertical frictionless wire, on which the acceleration is just g vertically down.

7.21 ⋆ If we use two-dimensional polar coordinates, the bead's velocity is $\mathbf{v} = (\dot{r}, r\dot{\phi}) = (\dot{r}, r\omega)$, where ω is the fixed angular velocity with which the rod is forced to rotate. Thus $\mathcal{L} = T - U = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$. ($U$ is a constant, which we may as well take to be zero.) The Lagrange equation is $\ddot{r} = \omega^2 r$, the general solution of which is $r(t) = Ae^{\omega t} + Be^{-\omega t}$. If $r(0) = \dot{r}(0) = 0$, then $A = B = 0$ and the bead stays put; that is, $r = 0$ is an equilibrium point (though unstable, as we'll see). If $r(0) = r_o \neq 0$, but $\dot{r}(0) = 0$, then $A = B = r_o/2$ and

$$r(t) = \tfrac{1}{2}r_o(e^{\omega t} + e^{-\omega t}) \to \tfrac{1}{2}r_o e^{\omega t}$$

as $t \to \infty$. As seen in the rotating frame of the rod, there is an ouward "centrifugal force" $m\omega^2 r$. This causes the bead to accelerate outward, and as r increases, the acceleration increases in proportion — hence the exponential growth of r.

7.22 ⋆ We must first write down the Lagrangian in an inertial frame, for which the natural choice is a frame fixed to the earth, relative to which the elevator is accelerating upward. The point of support in the elevator's ceiling has velocity $\mathbf{V} = (0, at)$ (if we measure x horizontally and y vertically up) and position $(0, \frac{1}{2}at^2)$. The bob's velocity relative to the elevator is $\mathbf{v}_{\rm rel} = (l\dot{\phi}\cos\phi, l\dot{\phi}\sin\phi)$. Thus its velocity relative to the ground is $\mathbf{v} = \mathbf{V} + \mathbf{v}_{\rm rel} =$

$(l\dot{\phi}\cos\phi, at + l\dot{\phi}\sin\phi)$. The bob's height above the ground is $y = \frac{1}{2}at^2 - l\cos\phi$. You can now write down the KE and PE and (after a little algebra) the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m\left(a^2t^2 + 2atl\dot{\phi}\sin\phi + l^2\dot{\phi}^2\right) - mg\left(\tfrac{1}{2}at^2 - l\cos\phi\right).$$

The Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \implies matl\dot{\phi}\cos\phi - mgl\sin\phi = \frac{d}{dt}(ml^2\dot{\phi} + matl\sin\phi)$$
$$= ml^2\ddot{\phi} + matl\dot{\phi}\cos\phi + mal\sin\phi.$$

Making a couple of cancelations and rearranging, we arrive at the equation $l\ddot{\phi} = -(g+a)\sin\phi$, which is the equation for a normal (non-accelerating) pendulum, except that g has been replaced by $(g + a)$.

7.23 ⋆ The small cart's KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x} + \dot{X})^2 = \frac{1}{2}m(\dot{x} - A\omega\sin\omega t)^2$, and $U = \frac{1}{2}kx^2$. Thus $\partial\mathcal{L}/\partial\dot{x} = m(\dot{x} - A\omega\sin\omega t)$ and Lagrange's equation reads

$$-kx = m\ddot{x} - mA\omega^2\cos\omega t \qquad \text{or} \qquad \ddot{x} + \omega_o^2 x = B\cos\omega t$$

where I have replaced k/m by ω_o^2 and renamed $A\omega^2$ as B.

7.24 ⋆ The Lagrangian for the Atwood machine is given by Eq.(7.54) as $\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx$. Therefore

$$(\text{generalized force}) = \frac{\partial \mathcal{L}}{\partial x} = (m_1 - m_2)g \;\;\&\;\; (\text{generalized momentum}) = \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_1 + m_2)\dot{x}.$$

The "effective weight" is $(m_1 - m_2)g$ and the "effective mass" is $(m_1 + m_2)$.

7.25 ⋆ If $\mathbf{F} = kr^n\hat{\mathbf{r}}$, then $U(\mathbf{r}) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\cdot d\mathbf{r}' = -\int_{r_o}^{r} kr'^n dr' = -kr^{n+1}/(n+1)$ (plus a constant that we can choose to be zero).

7.26 ⋆ From Eq.(7.79), $\Omega'^2 = \omega^2\sin^2\theta_o = \omega^2(1-\cos^2\theta_o)$, and, from (7.76), $\cos\theta_o = g/(\omega^2 R)$. Combining these, we find that $\Omega' = \sqrt{\omega^2 - g^2/(\omega R)^2}$ as claimed.

7.27 ⋆⋆ Let x be the distance from the top pulley down to the lower one and y that from the lower pulley to the mass $3m$. Then

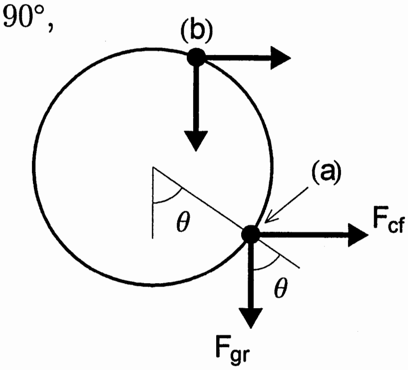
$$\mathcal{L} = \tfrac{1}{2}m\left[4\dot{x}^2 + (\dot{x} - \dot{y})^2 + 3(\dot{x} + \dot{y})^2\right] - mg[4x + (y - x) - 3(x + y)]$$
$$= 2m(2\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2) + 2mgy.$$

The x equation is $0 = 4\ddot{x} + \ddot{y}$ and the y equation is $g = \ddot{x} + 2\ddot{y}$, from which we find $\ddot{x} = -g/7$. That is, the mass $4m$ accelerates down at $g/7$.

If the top pulley were stationary, then the same would be true of the mass $4m$. Thus the tension in the top string would be $4mg$. This would mean that the net force on the system consisting of the lower pulley and its two masses would be zero, and hence that its

CM could not be accelerating. But with the upper string stationary, the lower string would behave just like a single Atwood machine and the CM of the two masses m and $3m$ would clearly accelerate down. This is a contradiction, so the top pulley has to move (and, in fact, accelerate).

7.28 ⋆⋆ **(a)** Consider the equilibrium point with $0 < \theta < 90°$, labeled (a) in the picture. As seen in the rotating frame, the bead is subject to three forces, the normal force of the hoop (not shown in the picture), the force of gravity, $\mathbf{F}_{\rm gr} = m\mathbf{g}$, and the centrifugal force $\mathbf{F}_{\rm cf} = m\omega^2\rho$, radially out from the axis of rotation, where $\rho = R\sin\theta$ is the distance of the bead from the axis. The bead will be in equilibrium if and only if the tangential component of the net force is zero. Since the tangential component of the normal force is zero, this condition is,



$$F_{\rm tang} = -(mg)\sin\theta + (m\omega^2 R\sin\theta)\cos\theta = m(\omega^2 R\cos\theta - g)\sin\theta = 0.$$

This condition is satisfied if and only if $\cos\theta = g/\omega^2 R$, which is precisely the condition (7.71).

(b) Suppose the bead has moved a little away from the equilibrium at the top of the hoop, as indicated by (b) in the picture. At this position the tangential components of $\mathbf{F}_{\rm gr}$ and $\mathbf{F}_{\rm cf}$ are both pulling the bead away from equilibrium. Therefore the equilibrium at the top is definitely unstable.

(c) Consider the equilibrium with θ negative [across from (a) in the picture] and suppose the bead moves a little up from the equilibrium (θ more negative). This makes $\cos\theta$ smaller, and the first parenthesis on the right of Eq.(7.73) becomes negative. Since $\sin\theta$ is also negative, $\ddot\theta$ is positive, and the bead accelerates back toward equilibrium. Similarly, if the bead moves down from equilibrium, $\ddot\theta$ becomes negative and, again, the bead accelerates back toward equilibrium. Therefore, the equilibrium is stable.

7.29 ⋆⋆ Because the angle between the line OP and the horizontal is ωt, the position of P is $(R\cos\omega t, R\sin\omega t)$. Therefore the position of the pendulum's bob is

$$\mathbf{r} = (x, y) = (R\cos\omega t + l\sin\phi, R\sin\omega t - l\cos\phi)$$

and its velocity is

$$\mathbf{v} = (\dot x, \dot y) = (-\omega R\sin\omega t + \dot\phi l\cos\phi, \omega R\cos\omega t + \dot\phi l\sin\phi).$$

Therefore the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m[\omega^2R^2 + \dot\phi^2 l^2 + 2\omega R\dot\phi l\sin(\phi - \omega t)] - mg(R\sin\omega t - l\cos\phi)$$

where I have used a couple of trig identities to combine various terms. The two derivatives of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial \phi} = m\omega R\dot{\phi}l\cos(\phi - \omega t) - mgl\sin\phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m[\dot{\phi}l^2 + \omega Rl\sin(\phi - \omega t)]$$

and the Lagrange equation, after a couple of cancellations, is

$$l\ddot{\phi} = -g\sin\phi + \omega^2\cos(\phi - \omega t).$$

As $\omega \to 0$, this becomes $l\ddot{\phi} = -g\sin\phi$, the equation for an ordinary simple pendulum.

7.30 ⋆⋆ **(a)** As in Eq.(7.39) the position of the bob (relative to the ground-based frame) is

$$\mathbf{r} = (l\sin\phi + \tfrac{1}{2}at^2, l\cos\phi) \quad \text{and hence} \quad \mathbf{v} = (l\dot{\phi}\cos\phi + at, -l\dot{\phi}\sin\phi).$$

Therefore, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}mv^2 + mgy = \tfrac{1}{2}m(l^2\dot{\phi}^2 + 2atl\dot{\phi}\cos\phi + a^2t^2) + mgl\cos\phi$$

and the two derivatives are

$$\frac{\partial \mathcal{L}}{\partial \phi} = -matl\dot{\phi}\sin\phi - mgl\sin\phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ml^2\dot{\phi} + matl\cos\phi.$$

The Lagrange equation is (after a couple of cancellations)

$$l\ddot{\phi} = -g\sin\phi - a\cos\phi.$$

Finally, imagine a right triangle with base g, height a, base angle β, and hypotenuse $\sqrt{g^2 + a^2}$. Multiplying top and bottom of this equation by $\sqrt{g^2 + a^2}$, we find that

$$l\ddot{\phi} = -\sqrt{g^2 + a^2}\,(\cos\beta\sin\phi + \sin\beta\cos\phi) = -\sqrt{g^2 + a^2}\,\sin(\phi + \beta).$$

(b) The condition for equilibrium is that $\ddot{\phi} = 0$, which implies that $\phi = -\beta$. That is, in equilibrium the pendulum hangs at an angle $\beta = \arctan(a/g)$ behind the vertical. If ϕ moves a little to the left of $-\beta$, then $\phi + \beta$ is negative, so $\sin(\phi + \beta)$ is negative, so $\ddot{\phi}$ is positive, and the pendulum accelerates to the right, back toward equilibrium. Similarly if ϕ moves a little to the right. Therefore the equilibrium is stable.

If $\phi = -\beta + \epsilon$, (with ϵ small) then the equation of motion becomes $l\ddot{\epsilon} = -\sqrt{g^2 + a^2}\,\sin\epsilon \approx -\sqrt{g^2 + a^2}\,\epsilon$. Therefore, the frequency of small oscillations is $\omega = \sqrt{\sqrt{g^2 + a^2}/l}$.

7.31 ⋆⋆ **(a)** The position and hence velocity of the mass M are

$$\mathbf{r}_M = (x + L\sin\phi, L\cos\phi) \quad \text{and} \quad \mathbf{v}_M = (\dot{x}_M, \dot{y}_M) = (\dot{x} + L\dot{\phi}\cos\phi, -L\dot{\phi}\sin\phi).$$

Therefore the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}(mv_m^2 + Mv_M^2) + Mgy_M - \tfrac{1}{2}kx^2 \\ &= \tfrac{1}{2}(m + M)\dot{x}^2 + \tfrac{1}{2}M(L^2\dot{\phi}^2 + 2\dot{x}L\dot{\phi}\cos\phi) + MgL\cos\phi - \tfrac{1}{2}kx^2\end{aligned}$$

and the two Lagrange equations are (after a little tidying up)

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \implies (m+M)\ddot{x} + ML\ddot{\phi}\cos\phi - ML\dot{\phi}^2\sin\phi = -kx$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \implies L\ddot{\phi} + \ddot{x}\cos\phi = -g\sin\phi$$

(b) If ϕ remains small, we can write $\cos\phi \approx 1$ and $\sin\phi \approx \phi$ and ignore powers of ϕ or $\dot{\phi}$ higher than second to give

$$(m+M)\ddot{x} + ML\ddot{\phi} = -kx \qquad \text{and} \qquad L\ddot{\phi} + \ddot{x} = -g\phi.$$

7.32 ⋆⋆ The position of the CM of the block (labeled C in Fig.4.14) is

$$\mathbf{r} = (x, y) = \big([r+b]\sin\theta + r\theta\cos\theta, [r+b]\cos\theta + r\theta\sin\theta\big).$$

So its velocity is

$$\mathbf{v} = (\dot{x}, \dot{y}) = \big(b\cos\theta + r\theta\sin\theta, -b\sin\theta + r\theta\cos\theta\big)\dot{\theta}.$$

and $v^2 = (b^2 + r^2\theta^2)\dot{\theta}^2$. Therefore

$$\mathcal{L} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 - mgy = m\big(\tfrac{1}{2}(b^2 + r^2\theta^2) + \tfrac{1}{3}b^2\big)\dot{\theta}^2 - mg\big([r+b]\cos\theta + r\theta\sin\theta\big).$$

This simplifies a bit, and even more if we make the small angle approximation ($\cos\theta \approx 1$ and $\sin\theta \approx \theta$), ignoring all products of more than two small quantities:

$$\mathcal{L} \approx \tfrac{5}{6}mb^2\dot{\theta}^2 - \tfrac{1}{2}mg(r-b)\theta^2 + \text{const.}$$

With this approximation, the Lagrange equation is $\frac{5}{3}mb^2\ddot{\theta} = -mg(r-b)\theta$, and, provided $r > b$, the block oscillates with angular frequency $\omega = \sqrt{3g(r-b)/(5b^2)}$.

7.33 ⋆⋆ The soap's height above the table is $y = x\sin(\omega t)$, and the velocity has component $\dot{x}$ up the slope and $x\omega$ normal to the slope. Therefore,

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m(\dot{x}^2 + x^2\omega^2) - mgx\sin(\omega t)$$

and the Lagrange equation is

$$\ddot{x} = \omega^2 x - g\sin(\omega t).$$

The homogeneous equation $\ddot{x} = \omega^2 x$ has as its general solution $x_\text{h} = C\cosh(\omega t) + D\sinh(\omega t)$. If we substitute the suggested particular solution $x = A\sin(\omega t)$ into the equation of motion, we find that the equation is satisfied if $A = g/2\omega^2$. Therefore the general solution is $x(t) = (g/2\omega^2)\sin(\omega t) + C\cosh(\omega t) + D\sinh(\omega t)$. Using the initial conditions that $x(0) = x_\text{o}$ and $\dot{x}(0) = 0$, we see that $C = x_\text{o}$ and $D = g/2\omega^2$, so, the solution is

$$x(t) = x_\text{o}\cosh(\omega t) + (g/2\omega^2)[\sin(\omega t) - \sinh(\omega t)].$$

7.34 ⋆⋆ **(a)** Let the unstretched length of the spring be l and consider a short segment of spring a distance ξ from the fixed end and of length $d\xi$. Since the spring is uniform, the mass of this segment is $Md\xi/l$ and since the spring stretches uniformly its velocity (when the cart has velocity $\dot{x}$) is $\dot{x}\xi/l$. Therefore the KE of this segment is $\frac{1}{2}M\dot{x}^2\xi^2 d\xi/l^3$, and the total KE of the whole spring is

$$T_{\text{spr}} = \frac{1}{2}\frac{M\dot{x}^2}{l^3}\int_0^l \xi^2 d\xi = \frac{1}{6}M\dot{x}^2$$

Therefore the Lagrangian for the system of spring and cart is $\mathcal{L} = \frac{1}{2}(m + M/3)\dot{x}^2 - \frac{1}{2}kx^2$.

(b) The Lagrange equation is $-kx = (m + M/3)\ddot{x}$, which is the same as for the usual massless spring except that m, the mass of the cart, has been replaced by $m + M/3$. In particular, the angular frequency of the oscillations is $\omega = \sqrt{k/(m + M/3)}$.

7.35 ⋆⋆ Let O be the center of the hoop. The line AO is turning with angular velocity ω, so the speed of O relative to A is $v_o = R\omega$. The line from O to the bead is turning with angular velocity $\dot{\phi} + \omega$, so the speed of the bead relative to O is $v_b = R(\dot{\phi} + \omega)$. The velocity of the bead relative to A is $\mathbf{v} = \mathbf{v}_b + \mathbf{v}_o$, and

$$T = \tfrac{1}{2}m\mathbf{v}^2 = \tfrac{1}{2}m(\mathbf{v}_b^2 + \mathbf{v}_o^2 + 2\mathbf{v}_b\cdot\mathbf{v}_o) = \tfrac{1}{2}mR^2[(\dot{\phi} + \omega)^2 + \omega^2 + 2(\dot{\phi} + \omega)\omega\cos\phi]$$

since ϕ is the angle between $\mathbf{v}_b$ and $\mathbf{v}_o$. Because the hoop is horizontal, the potential energy is constant, and we may as well take it to be $U = 0$, so that $\mathcal{L} = T$. The Lagrange equation is

$$\frac{\partial T}{\partial \phi} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} \quad \text{or} \quad -(\dot{\phi} + \omega)\omega\sin\phi = \frac{d}{dt}[(\dot{\phi} + \omega) + \omega\cos\phi] = (\ddot{\phi} - \dot{\phi}\omega\sin\phi),$$

where I have cancelled a common factor of mR^2. The first term on the left cancels the second on the right, and we're left with $\ddot{\phi} = -\omega^2\sin\phi$, which is exactly the equation of a simple pendulum with g/L replaced by ω^2, oscillating about the point $\phi = 0$ (that is, B). The frequency of small oscillations is evidently just ω.

7.36 ⋆⋆⋆ **(a)** The kinetic energy is $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$. The gravitational PE (relative to O) is $-mgr\cos\phi$, and the spring PE is $\frac{1}{2}k(r - L_o)^2$. Therefore

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mgr\cos\phi - \tfrac{1}{2}k(r - L_o)^2.$$

(b) The two Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad \text{or} \quad mr\dot{\phi}^2 + mg\cos\phi - k(r - L_o) = m\ddot{r} \tag{v}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad \text{or} \quad -mgr\sin\phi = \frac{d}{dt}(mr^2\dot{\phi}). \tag{vi}$$

If you move the term $mr\dot{\phi}^2$ to the right side, you will recognize (v) as the r component of $\mathbf{F} = m\mathbf{a}$. Equation (vi) is just the equation "torque = rate of change of angular momentum." Alternatively, if you divide through by r, it becomes the ϕ component of $\mathbf{F} = m\mathbf{a}$.

(c) The equilibrium length with the mass in place is given by $k(L-L_o) = mg$. If we write $r = L + \epsilon$ and then ignore all terms that are quadratic in ϵ or ϕ or their derivatives, then (v) reduces to (the first term drops out entirely, and, in the second, $\cos\phi$ can be replaced by 1)

$$mg - k\epsilon - k(L - L_o) = m\ddot{\epsilon}.$$

The first and third terms on the left cancel exactly, and we're left with $m\ddot{\epsilon} = -k\epsilon$. Thus, the mass oscillates in the radial direction with the usual spring frequency $\sqrt{k/m}$. When we substitute $r = L + \epsilon$ in (vi), all terms involving ϵ drop out (since they are already multiplied by $\sin\phi$ or $\dot{\phi}$) and we are left with $-g\phi = L\ddot{\phi}$. Thus the mass oscillates in the ϕ direction with the usual pendulum frequency $\sqrt{g/L}$.

7.37 ⋆⋆⋆ **(a)** The hanging mass is a distance $L - r$ below the table. Thus it's KE is $\frac{1}{2}m\dot{r}^2$ and its PE is $-mg(L-r)$, or just mgr, if we drop an uninteresting constant. The mass on the table has KE $= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$ and PE which is constant and we may as well take to be zero. Thus

$$\mathcal{L} = m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 - mgr.$$

(b) The two Lagrange equations are

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} \qquad \text{or} \qquad mr\dot{\phi}^2 - mg = 2m\ddot{r}$$

and

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \qquad \text{or} \qquad 0 = \frac{d}{dt}(mr^2\dot{\phi}).$$

The ϕ equation says simply that the angular momentum $\ell = mr^2\dot{\phi}$ is constant.

(c) Clearly $\dot{\phi} = \ell/mr^2$, and the r equation can be rewritten as

$$2m\ddot{r} = \frac{\ell^2}{mr^3} - mg. \tag{vii}$$

The length r can remain constant if and only if $\ddot{r} = 0$. This requires that $\ell^2/mr_o^3 = mg$. (This condition says that the centripetal force needed to keep the upper mass in a circular path must equal the tension needed to hold the lower mass at a fixed height.) Therefore, $r_o = [\ell^2/(m^2g)]^{1/3}$

(d) If $r = r_o + \epsilon$, then (vii) becomes

$$2m\ddot{\epsilon} = \frac{\ell^2}{m(r_o+\epsilon)^3} - mg = \frac{\ell^2}{mr_o^3}\left(1 + \frac{\epsilon}{r_o}\right)^{-3} - mg \approx \frac{\ell^2}{mr_o^3}\left(1 - 3\frac{\epsilon}{r_o}\right) - mg$$

where, to get the final expression, I used the binomial approximation. The first and last terms in the final expression cancel, and we are left with $2m\ddot{\epsilon} = -3(\ell^2/mr_o^4)\epsilon$, which implies that ϵ oscillates sinusoidally with frequency $\sqrt{3/2}\,\ell/mr_o^2$. In particular, since the displacement ϵ oscillates, the equilibrium at $r = r_o$ is stable.

7.38 ⋆⋆⋆ **(a)** In spherical polar coordinates, the angle θ is fixed at $\theta = \alpha$, so the KE is just $T = \frac{1}{2}m(\dot{r}^2 + r^2 \sin^2\!\alpha\, \dot{\phi}^2)$. Since the PE is $U = mgz = mgr\cos\alpha$,

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2 \sin^2\!\alpha\, \dot{\phi}^2) - mgr\cos\alpha$$

(b) Since $\mathcal{L}$ does not depend on ϕ, the ϕ equation says simply that $\partial\mathcal{L}/\partial\dot{\phi} = mr^2\sin^2\!\alpha\,\dot{\phi}$ is constant. That is, ℓ_z is conserved.

The r equation is

$$m\ddot{r} = mr\sin^2\!\alpha\,\dot{\phi}^2 - mg\cos\alpha \quad \text{or} \quad \ddot{r} = \frac{\ell_z^2}{m^2 r^3 \sin^2\alpha} - g\cos\alpha \tag{viii}$$

where, in the second version, I have canceled an m and replaced $\dot{\phi}$ by $\ell_z/(mr^2\sin^2\!\alpha)$. If $\ell_z = 0$, the particle simply slides in the radial direction with the well-known acceleration $g\cos\alpha$. (Remember α is the angle of the incline with the *vertical.*) The condition that the particle can remain in a horizontal circle is that $\ddot{r} = 0$ and hence that $r = r_o = [\ell_z^2/(mg\sin^2\!\alpha\cos\alpha)]^{1/3}$.

(c) If we write $r = r_o + \epsilon$, Eq.(viii) becomes

$$\ddot{\epsilon} = \frac{\ell_z^2}{m^2 r_o^3 \sin^2\alpha}\left(1 + \frac{\epsilon}{r_o}\right)^{-3} - g\cos\alpha \;\approx\; \frac{\ell_z^2}{m^2 r_o^3 \sin^2\alpha}\left(1 - 3\frac{\epsilon}{r_o}\right) - g\cos\alpha = \frac{-3\ell_z^2}{m^2 r_o^4 \sin^2\alpha}\epsilon$$

where the first and last terms in the penultimate expression cancel because r_o is the equilibrium radius. This is the equation of simple harmonic oscillations, so the circular path is stable and the particle oscillates about this path with frequency $\omega = \sqrt{3}\ell_z/(mr_o^2)$.

7.39 ⋆⋆⋆ **(a)** The particle's velocity has spherical polar components

$$\mathbf{v} = (\dot{r}, r\dot{\theta}, r\sin\theta\,\dot{\phi}) \tag{ix}$$

Therefore

$$\mathcal{L} = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\!\theta\,\dot{\phi}^2\right) - U(r)$$

(b) The r, θ, and ϕ equations are

$$m\ddot{r} = mr\left(\dot{\theta}^2 + \sin^2\!\theta\,\dot{\phi}^2\right) - \partial U/\partial r \tag{x}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\sin\theta\cos\theta\,\dot{\phi}^2 \tag{xi}$$

$$\frac{d}{dt}\left(mr^2\sin^2\!\theta\,\dot{\phi}\right) = 0 \tag{xii}$$

The first term on the right of the r equation (x) can be recognized as $mv_{\rm tg}^2/r$, where $v_{\rm tg}$ is the tangential velocity [obtained from (ix) by dropping the first component]. This is just the centrifugal force $F_{\rm cf}$, so the radial equation is just $m\ddot{r} = F_{\rm cf} + F_r$.

The term in parentheses in the ϕ equation (xii) can be recognized as ℓ_z. Therefore the ϕ equation states simply that ℓ_z is constant, as expected.

The term in parentheses in the θ equation (xi) can be recognized as ℓ_ϕ. (To see this, just evaluate $\boldsymbol{\ell} = m\mathbf{r}\times\mathbf{v}$ in polar coordinates.) Thus the θ equation gives the rate of change of ℓ_ϕ. This result is surprising at first, since we know that the vector $\boldsymbol{\ell}$ is constant. However, as the particle moves, the vector $\hat{\boldsymbol{\phi}}$ changes; thus, ℓ_ϕ is the component of a fixed vector in a *variable* direction and does in fact change, as in (xi).

(c) If initially $\theta = \pi/2$ and $\dot\theta = 0$, then the θ equation (xi) shows that the product $mr^2\dot\theta$ remains constant. Thus $\dot\theta$ remains equal to zero and θ remains equal to $\pi/2$. That is, the motion remains in the equatorial plane $\theta = \pi/2$, consistent with our knowledge that the motion is confined to a plane.

(d) If initially $\dot\phi = 0$, then the ϕ equation (xii) shows that the product $mr^2\sin^2\theta\,\dot\phi$ remains zero. Thus $\dot\phi$ remains zero and ϕ is constant. Therefore, the motion is confined to the longitudinal plane $\phi = \phi_o$.

7.40 ⋆⋆⋆ **(a)** The bob's velocity is $\mathbf{v} = (0, R\dot\theta, R\dot\phi\sin\theta)$ and its height below the support is $z = R\cos\theta$. Therefore

$$\mathcal{L} = T - U = \tfrac{1}{2}mR^2(\dot\theta^2 + \dot\phi^2\sin^2\theta) + mgR\cos\theta$$

The θ and ϕ equations are (after a little tidying)

$$R\dot\phi^2\sin\theta\cos\theta - g\sin\theta = R\ddot\theta \qquad \text{and} \qquad mR^2\dot\phi\sin^2\theta = \text{const.}$$

(b) The ϕ equation tells us that the z component of angular momentum, $\ell_z = mR^2\dot\phi\sin^2\theta$, is constant.

(c) If ϕ is constant, the θ equation reduces to $-g\sin\theta = R\ddot\theta$, which is the equation for a simple pendulum. That is, in this case, the pendulum swings in a single vertical plane, $\phi = \phi_o$, just like a simple pendulum.

(d) If we replace $\dot\phi$ by $\ell_z/(mR^2\sin^2\theta)$, the θ equation becomes

$$R\ddot\theta = k\frac{\cos\theta}{\sin^3\theta} - g\sin\theta, \tag{xiii}$$

where k denotes the positive constant $k = \ell_z^2/m^2R^3$. Now θ can remain constant if and only if $\ddot\theta = 0$, which requires that θ satisfy $k\cos\theta = g\sin^4\theta$. This equation can only be satisfied if $0 < \theta < \pi/2$. (If $\pi/2 < \theta \le \pi$, the left side is negative while the right is positive.) If we vary θ from 0 to $\pi/2$, the left side decreases steadily from k to 0 while the right side increases steadily from 0 to g. Therefore there is exactly one value $\theta = \theta_o$ at which the angle θ can remain constant. In this motion the string of the pendulum traces out a vertical cone of half angle θ_o.

(e) We can rewrite the θ equation (xiii) as $R\ddot\theta = f(\theta)$, where $f(\theta) = k(\cos\theta/\sin^3\theta) - g\sin\theta$. Now, at the "equilibrium" value $\theta = \theta_o$ we know that $f(\theta_o)$ is zero. Thus if θ is close to the equilibrium value, $\theta = \theta_o + \epsilon$, we can write $R\ddot\epsilon \approx f'(\theta_o)\epsilon$, and by differentiating you can check that $f'(\theta)$ is the sum of three terms, all negative in the range $0 < \theta < \pi/2$. Therefore the equation of motion has the form $\ddot\epsilon =$ (negative constant)ϵ, and θ executes simple harmonic motion about θ_o. The motion of the bob is uniform motion in a horzontal circle with a superposed small sinusoidal motion in the $\hat{\boldsymbol{\theta}}$ direction.

7.41 ⋆⋆⋆ The potential energy is just $U = mgz = mgk\rho^2$. The kinetic energy is

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\Big(\dot\rho^2 + (\rho\dot\phi)^2 + \dot z^2\Big) = \tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\omega^2 + 4k^2\rho^2\dot\rho^2\big)\,.$$

Therefore the Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\omega^2 + 4k^2\rho^2\dot\rho^2\big) - mgk\rho^2,$$

and the equation of motion is (after a little algebra)

$$(1 + 4k^2\rho^2)\ddot\rho + 8k^2\rho\dot\rho^2 = (\omega^2 - 2gk)\rho. \tag{xiv}$$

A position ρ_o is an equilibrium point if placing the bead at ρ_o with $\dot\rho = 0$ ensures that $\ddot\rho = 0$ (guaranteeing that $\dot\rho$ remains zero and ρ constant). According to Eq.(xiv) this will be true if and only if

$$(\omega^2 - 2gk)\rho = 0. \tag{xv}$$

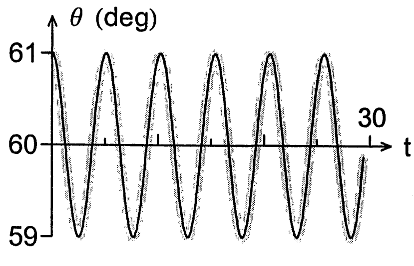
This can be satisfied in two ways: First the system is in equilibrium if $\rho = 0$, that is, if the bead is exactly at the bottom of the wire. To decide whether this equilibrium is stable, we have only to imagine pulling the bead a small distance to one side. With ρ and $\dot\rho$ very small Eq.(xiv) implies that

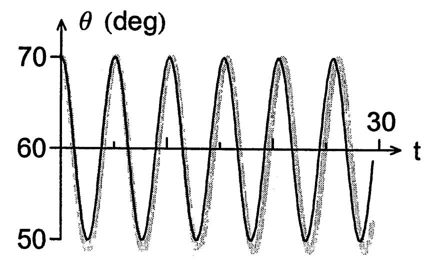
$$\ddot\rho \approx (\omega^2 - 2gk)\rho.$$

If $\omega^2 < 2gk$, the term in parentheses is negative, and the bead accelerates back to the bottom, so the equilibrium is stable. If $\omega^2 > 2gk$, the term in parentheses is positive, and the bead accelerates away from the bottom, so the equilibrium is unstable.

If $\omega^2 = 2gk$, the condition (xv) is satisfied for *any* value of ρ. That is, with $\omega = \sqrt{2gk}$, the bead will be in equilibrium anywhere on the wire. To investigate the stability of this equilibrium, we imagine giving the bead a small nudge. From Eq.(xiv), with the right side equal to zero, we see that $\ddot\rho$ is always negative for any positive ρ. Thus if we nudge the bead outward, its negative acceleration will slow it down, though not actually stop it. If we nudge it inward it will actually speed up. Either way it will not return to its original position.

7.42 ⋆⋆⋆ With $g = R = 1$ and $\omega^2 = 2$, the equilibrium angle is $\theta_o = \arccos(g/\omega^2 R) = 60°$. The frequency of the approximate (small-angle) solution is $\Omega' = \sqrt{\omega^2 - (g/\omega R)^2} = \sqrt{3/2}$. The left picture (overleaf) shows the oscillations when the bead is released at $\theta_o = 61°$; the exact motion (thick, gray curve) and the approximation (thin, black curve) are indistinguishable at this scale. In the right picture, the bead was released at $\theta_o = 70°$; the approximation is still remarkably good, but the two can be told apart — for example, the exact curve is a bit unsymmetrical, with maxima at 70° (10° above center), but minima at about 49° (a little more than 11° below).



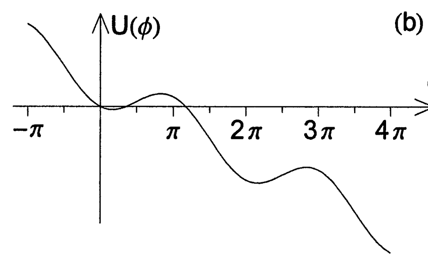


7.43 ★★★ **(a)** $\mathcal{L} = \frac{1}{2}(M+m)R^2\dot{\phi}^2 - MgR(1-\cos\phi) + mgR\phi$, and the equation of motion is

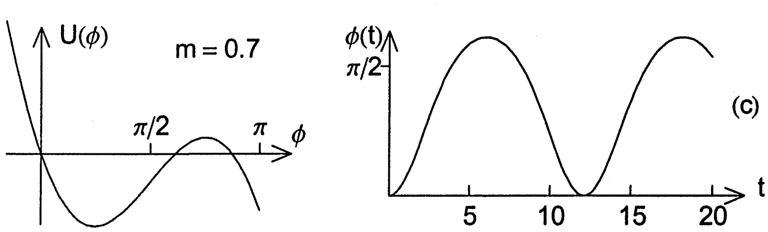
$$(M+m)R\ddot{\phi} = -Mg\sin\phi + mg. \tag{xvi}$$

Equilibrium positions require that $\ddot{\phi} = 0$ or $\sin\phi = m/M$. Provided $m < M$, this has two solutions, $\phi = \arcsin(m/M)$, one with $0 < \phi < \pi/2$ and one with $\pi/2 < \phi < \pi$. If the wheel turns a little away from the first of these, it is easy to see from (xvi) that it accelerates back and the equilibrium is stable. In the same way, you can see that the second is unstable.

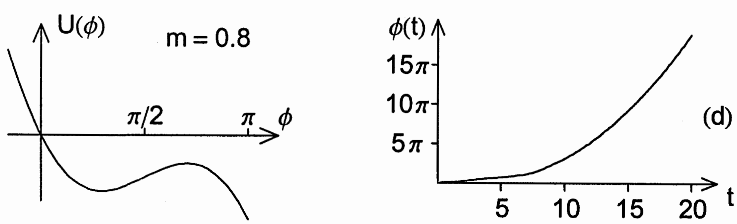
(b) The stable equilibrium is the minimum at about $\pi/6$; the unstable is the maximum at about $5\pi/6$. Notice that there is actually a whole sequence of equilibriums, spaced at intervals of 2π (that is, differing by one or more complete revolutions).



(c) Based on the left-hand plot of $U(\phi)$ against ϕ, it is clear that when released from $\phi = 0$, the mass M will swing out to an angle a bit bigger than $\pi/2$, swing back to $\phi = 0$, and continue to oscillate indefinitely, as confirmed by the right-hand picture of $\phi(t)$ against t.

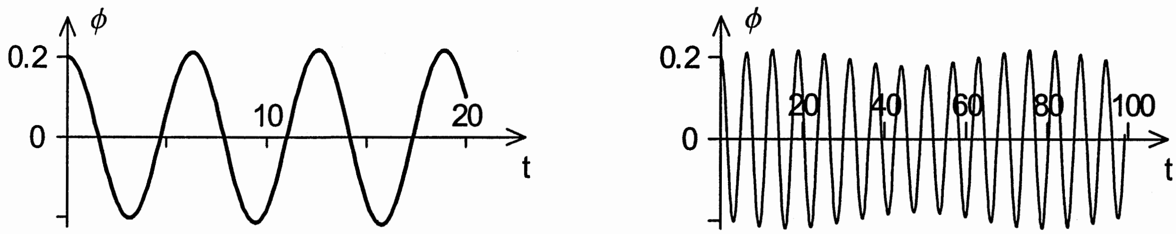


(d) With $m = 0.8$, the maximum in $U(\phi)$ just beyond $\phi = \pi/2$ is not high enough to stop the wheel, which continues to roll indefinitely. You can see in the right-hand picture that, by the time $t = 20$, it has made about 8 complete revolutions. (Note the very different vertical scale.)



7.44 ⋆⋆⋆ **(a)** In the left figure, which shows the range $0 \le t \le 20$, the pendulum is clearly oscillating pretty much sinusoidally and with a period of about 6 — certainly close to its natural period of 2π.

(b) In the right figure, which shows the range $0 \le t \le 100$, you can see that the amplitude is itself oscillating slightly with a period of about 60 — close to the period of the wheel.



7.45 ⋆⋆ **(a)** If you look at the definition (7.95) of A_{jk}, you will see that A_{jk} differs from A_{kj} only in the order of the two terms in the scalar product. Since the scalar product is commutative, these two expressions are equal.

(b) If we consider first the case of just two variables, the sum in question is

$$\begin{aligned} S = \sum_{j,k} A_{jk} v_j v_k &= A_{11} v_1^2 + A_{12} v_1 v_2 + A_{21} v_2 v_1 + A_{22} v_2^2 \\ &= A_{11} v_1^2 + 2A_{12} v_1 v_2 + A_{22} v_2^2 \end{aligned}$$

where, in the second line, I used the fact that $A_{12} = A_{21}$. Differentiating with respect to v_1, we find that $\partial S/\partial v_1 = 2A_{11}v_1 + 2A_{12}v_2 = \sum_j A_{1j}v_j$, which is the claimed result for $i = 1$. The case $i = 2$ works in the same way.

If there are n variables, then, before differentiating with respect to v_i, it helps to separate out the terms that depend on v_i from those that do not:

$$\begin{aligned} S = \sum_{j,k} A_{jk} v_j v_k &= A_{ii} v_i^2 + \sum_{k \ne i} A_{ik} v_i v_k + \sum_{j \ne i} A_{ji} v_j v_i + \text{terms not involving } v_i \\ &= A_{ii} v_i^2 + 2 \sum_{j \ne i} A_{ij} v_i v_j + \text{terms not involving } v_i. \end{aligned}$$

Here, in passing to the second line, I replaced the dummy index k by j in the sum $\sum_{k \ne i}$, and used the fact that $A_{ji} = A_{ij}$ in the sum $\sum_{j \ne i}$. Differentiating with respect to v_i we find that

$$\frac{\partial S}{\partial v_i} = 2A_{ii}v_i + 2\sum_{j\neq i} A_{ij}v_j = 2\sum_j A_{ij}v_j\,.$$

7.46 ⋆⋆ **(a)** A rotation through angle ϵ about the z axis changes the coordinates of particle α thus: $(r_\alpha, \theta_\alpha, \phi_\alpha) \to (r_\alpha, \theta_\alpha, \phi_\alpha + \epsilon)$. Therefore, the invariance of $\mathcal{L}$ when the whole system undergoes this rotation means that

$$\mathcal{L}(r_1, \theta_1, \phi_1 + \epsilon, \cdots, r_N, \theta_N, \phi_N + \epsilon) = \mathcal{L}(r_1, \theta_1, \phi_1, \cdots, r_N, \theta_N, \phi_N).$$

By the definition of partial derivatives, the difference between the two sides of this equation is

$$\text{difference} = \sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\epsilon = 0 \quad\Longrightarrow\quad \sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0. \tag{xvii}$$

(b) Lagrange's equations tell us that $\partial\mathcal{L}/\partial\phi_\alpha = (d/dt)(\partial\mathcal{L}/\partial\dot\phi_\alpha) = d\ell_{\alpha z}/dt$. (Recall that $\partial\mathcal{L}/\partial\dot\phi_\alpha = \ell_{\alpha z}$, the z component of the angular momentum of particle α.) Therefore the result (xvii) implies that $(d/dt)\sum \ell_{\alpha z} = 0$; that is, the z component of the total angular momentum is constant, $L_z = \sum \ell_{\alpha z} = \text{const}$.

7.47 ⋆⋆⋆ **(a)** The KE is $T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2$ and, because $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q)$, it follows that $\dot{\mathbf{r}}_\alpha = \dot q(d\mathbf{r}_\alpha/dq)$. Therefore

$$T = \tfrac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2 = \tfrac{1}{2}\dot q^2 \sum_\alpha m_\alpha \frac{d\mathbf{r}_\alpha}{dq}\cdot\frac{d\mathbf{r}_\alpha}{dq} = \tfrac{1}{2}A(q)\dot q^2 \quad \text{where} \quad A(q) = \sum_\alpha m_\alpha \frac{d\mathbf{r}_\alpha}{dq}\cdot\frac{d\mathbf{r}_\alpha}{dq} > 0$$

Because $\mathcal{L} = T - U = \frac{1}{2}A(q)\dot q^2 - U(q)$, the two derivatives of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial q} = \tfrac{1}{2}A'(q)\dot q^2 - U'(q) \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot q} = A(q)\dot q$$

and the Lagrange equation is (after a little algebra) $A(q)\ddot q = -U'(q) - \frac{1}{2}A'(q)\dot q^2$.

(b) If q_o is an equilibrium point and if the system is placed at q_o with $\dot q = 0$, then the system stays put, so $\ddot q = 0$. By the Lagrange equation this implies that $U'(q_o)$ must be 0. Conversely, if $U'(q_o) = 0$, then when the system is placed at q_o with $\dot q = 0$, $\ddot q = 0$ and the system stays put.

(c) Suppose that q_o is stable and the system is placed at rest close to q_o at $q_o + \epsilon$ with $\epsilon > 0$. Because q_o is stable, the system accelerates back toward q_o, so $\ddot q$ must be negative. Inspecting the Lagrange equation, we see that U' must be positive, which implies that U is at a minimum at q_o. Running the argument backward, we see that if U is minimum, then the equiibrium is stable.

(d) In Problem 7.30, the relation between the position $\mathbf{r}$ of the pendulum and the generalized coordinate ϕ is time-*dependent*. The PE is minimum at $\phi = 0$, whereas the equilibrium occurs at $\phi = -\beta \neq 0$.

7.48 ⋆⋆ If $F = F(q_1, \cdots, q_n)$, then $dF/dt = \sum_j \dot{q}_j \partial F/\partial q_j$. Therefore, if $\mathcal{L}' = \mathcal{L} + dF/dt$, its derivatives of $\mathcal{L}'$ are

$$\frac{\partial \mathcal{L}'}{\partial q_i} = \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial}{\partial q_i}\frac{dF}{dt} = \frac{\partial \mathcal{L}}{\partial q_i} + \sum_j \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j \tag{xviii}$$

and

$$\frac{\partial \mathcal{L}'}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial F}{\partial q_i}$$

so

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{d}{dt}\frac{\partial F}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_j \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j\,. \tag{xix}$$

If you compare the two equations (xviii) and (xix), you will see that the two last terms are identical. Thus if $\mathcal{L}$ satisfies Lagrange's equation, so does $\mathcal{L}'$, and vice versa.

7.49 ⋆⋆ **(a)** If $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r} = \frac{1}{2}(B_y z - B_z y, B_z x - B_x z, B_x y - B_y x)$, then

$$(\boldsymbol{\nabla}\times\mathbf{A})_x = \partial_y A_z - \partial_z A_y = B_x.$$

(Remember that $\mathbf{B}$ is uniform and constant.) Since the y and z components work the same way, we conclude that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. In polar coordinates, $\mathbf{B} = B\hat{\mathbf{z}}$ and $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, so

$$\mathbf{A} = \tfrac{1}{2}\mathbf{B}\times\mathbf{r} = \tfrac{1}{2}B\hat{\mathbf{z}}\times(\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}) = \tfrac{1}{2}B\rho\hat{\boldsymbol{\phi}}$$

since $\hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}$.

(b) Since there is no electric field, $V = 0$, and since $\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$,

$$\mathcal{L} = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}}\cdot\mathbf{A} = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \tfrac{1}{2}qB\rho^2\dot{\phi}.$$

The three Lagrange equations are

$$m\ddot{\rho} = m\rho\dot{\phi}^2 + qB\rho\dot{\phi}, \qquad \frac{d}{dt}\left(m\rho^2\dot{\phi} + \tfrac{1}{2}qB\rho^2\right) = 0, \qquad \text{and} \qquad m\ddot{z} = 0.$$

(c) In any case, the solution of the z equation is $z = z_o + v_{zo}t$; that is, the particle moves uniformly in the direction of $\mathbf{B}$. If $\rho =$ constant, the ρ equation reduces to $m\dot{\phi}^2 + qB\dot{\phi} = 0$. Therefore, either $\dot{\phi} = 0$ (in which case the particle moves straight along a field line) or $\dot{\phi} = -qB/m$. In this second case, the particle moves clockwise around the z axis (assuming q is positive) at the same time it moves in the z direction with constant velocity; this results in the helical motion described in Section 2.7, with angular velocity equal to the cyclotron freqency $\omega = qB/m$.

7.50 ⋆ The constraint equation is

$$f(x, y) = x + y = \text{const.}$$

The Lagrangian is $\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_2 g y$, and the two modified Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \Longrightarrow \qquad 0 + \lambda = m_1\ddot{x}$$

and

$$\frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \qquad \Longrightarrow \qquad m_2 g + \lambda = m_1 \ddot{y}.$$

These three equations are easily solved for the three unknowns, $\ddot{x}$, $\ddot{y}$, and λ, to give $\ddot{y} = -\ddot{x} = gm_2/(m_1+m_2)$, and $\lambda = -gm_1m_2/(m_1+m_2)$. The constraint force on m_2 (for example) is $F^{\text{cstr}} = \lambda \partial f/\partial y = -gm_1m_2/(m_1+m_2)$, where the minus sign is because the tension in the string acts upward on m_2, whereas we're measuring y downward. If we wrote down the constraint equation and Newton's second law for the two masses, we would get the same three equations (with λ replaced by minus the tension), so we would naturally get the same solutions.

7.51 ⋆ $\mathcal{L}(x,y) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy$.

(a) With the constraint $f(x,y) = \sqrt{x^2+y^2} = l$, the two modified Lagrange equations are

$$\lambda \frac{x}{l} = m\ddot{x} \qquad \text{and} \qquad mg + \lambda\frac{y}{l} = m\ddot{y}.$$

If F^{t} denotes the tension in the rod, the two Newtonian equations are:

$$-F^{\text{t}}\sin\phi = -F^{\text{t}}\frac{x}{l} = m\ddot{x} \qquad \text{and} \qquad mg - F^{\text{t}}\cos\phi = mg - F^{\text{t}}\frac{y}{l} = m\ddot{y}. \tag{xx}$$

Comparing corresponding equations we see that $\lambda = -F^{\text{t}}$. Perhaps more important, (7.122) is satisfied for both x and y components:

$$\lambda\frac{\partial f}{\partial x} = \lambda\frac{x}{l} = -F^{\text{t}}\sin\phi = F^{\text{t}}_x$$

and likewise for the y component.

(b) If instead we use the constraint equation $f'(x,y) = x^2+y^2 = l^2$, the two modified Lagrange equations are

$$\lambda' 2x = m\ddot{x} \qquad \text{and} \qquad mg + \lambda' 2y = m\ddot{y}.$$

Comparing with the two Newtonian equations (xx), we find that in this case $\lambda' = -F^{\text{t}}/2l$, but again (7.122) is satisfied:

$$\lambda'\frac{\partial f'}{\partial x} = \lambda'(2x) = -\frac{F^{\text{t}}}{2l}(2l\cos\phi) = F^{\text{t}}_x$$

and likewise for the y component.

7.52 ⋆ As the string unwinds, it is clear that $x = R\phi$, so the constraint equation is

$$f = x - R\phi = 0. \tag{xxi}$$

The Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx$ and the two modified Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \Longrightarrow \qquad mg + \lambda = m_1\ddot{x} \tag{xxii}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} + \lambda \frac{\partial f}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad \Longrightarrow \quad 0 - \lambda R = I\ddot{\phi}. \tag{xxiii}$$

Solving these three equations we find that $\ddot{x} = gm/(m + I/R^2)$ and $\ddot{\phi} = \ddot{x}/R$. If you write down Newton's second law as applied to the mass and the wheel, you should get two equations with exactly the form of Eqs.(xxii) and Eqs.(xxiii) except that λ is replaced by $-F^{\mathrm{t}}$, (minus the tension in the string). Naturally these give the same answer for $\ddot{x}$ and $\ddot{\phi}$. The simplest way to identify λ is to compare the Lagrange equation (xxii) with the Newtonian equation to give $\lambda = -F^{\mathrm{t}}$. Since the constraint function is $f = x - R\phi$, we see that $\lambda \partial f/\partial x = -F^{\mathrm{t}}$, as it should. On the other hand, $\lambda \partial f/\partial \phi = F^{\mathrm{t}} R$, which is the torque on the wheel, as one might have anticipated.

Chapter 8

Two-Body Central-Force Problems

I covered this chapter in 3 *fifty-minute lectures.*

This relatively short chapter offers a fairly standard treatment of the two-body problem, introducing the concepts of relative motion and reduced mass and proving Kepler's first and third laws. (The second was covered in Chapter 3.) I tried to emphasize how — as with most problems — both Newton and Lagrange can both add insights. I want our students to see that it's good to have both approaches ready to use at any time. I also tried to make clear the wonderful simplifications that are possible with this problem: How the 6-dimensional problem reduces to 3 dimensions when one separates out the relative motion, then to 2 when one invokes conservation of angular momentum, and finally 1 with the introduction of the effective potential energy that governs the radial motion. The final and optional Section 8.8 is an elementary introduction to the complicated business of changing between orbits.

Solutions to Problems for Chapter 8

8.1 ⋆ You can regard Eqs.(8.4) and (8.7) as two simultaneous equations to be solved for $\mathbf{r}_1$ and $\mathbf{r}_2$. If you multiply (8.4) by m_2/M and add to (8.7), this gives $\mathbf{r}_1 = \mathbf{R} + (m_2/M)\mathbf{r}$. Similarly, if you multiply (8.4) by m_1/M and subtract from (8.7), this gives $\mathbf{r}_2 = \mathbf{R} - (m_1/M)\mathbf{r}$. If you differentiate these results with respect to t, the required expression for T follows exactly as in (8.10).

8.2 ⋆⋆ **(a)** The Lagrangian is $\mathcal{L} = T - U$ or

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \tfrac{1}{2}m_2\dot{\mathbf{r}}_2^2 - \left[m_1 g z_1 + m_2 g z_2 + U(r)\right] \\ &= \left[\tfrac{1}{2}M\dot{\mathbf{R}}^2 - MgZ\right] + \left[\tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(\mathbf{r})\right] = \mathcal{L}_{\rm cm} + \mathcal{L}_{\rm rel},\end{aligned}$$

where I have chosen rectangular coordinates with z measured vertically up and Z is the z coordinate of the CM position, $Z = (m_1 z_1 + m_2 z_2)/M$.

(b) The Lagrange equations for the three components of $\mathbf{R}$ are

$$M\ddot{X} = 0, \quad M\ddot{Y} = 0, \quad M\ddot{Z} = -g,$$

so the CM moves just like a projectile of mass M. The Lagrange equations for the relative coordinates are

$$\mu\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}} U(r)$$

where $\nabla_{\mathbf{r}}$ denotes the gradient with respect to the relative coordinates. This last equation is precisely Newton's second law for the motion of a single particle of mass μ, position $\mathbf{r}$, and potential energy $U(r)$.

8.3 ⋆⋆ The motion is obviously confined to a vertical line, so we can treat it as a one-dimensional problem with coordinates y_1 and y_2 measured vertically up from the table. The Lagrangian is $\mathcal{L} = \frac{1}{2}M\dot{Y}^2 - MgY + \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}k(y-L)^2$, where M and μ are the total and reduced masses, and Y and y are the CM and relative positions. The two Lagrange equations are

$$\ddot{Y} = -g \quad \text{and} \quad \mu\ddot{y} = -k(y-L),$$

with solutions $Y = Y_o + \dot{Y}_o t - \frac{1}{2}gt^2$ (where $Y_o = Lm_1/M$ and $\dot{Y}_o = v_o m_1/M$) and $y = L + A\sin\omega t$ (where $\omega = \sqrt{k/\mu}$ and $A = v_o/\omega$). That is, the CM moves up then down like a body in free fall, while the relative position oscillates in SHM. Using Eq.(8.9), we can find the individual positions

$$y_1 = L + \frac{m_1}{M}v_o t - \tfrac{1}{2}gt^2 + \frac{m_2 v_o}{M\omega}\sin\omega t \quad \text{and} \quad y_2 = \frac{m_1}{M}v_o t - \tfrac{1}{2}gt^2 - \frac{m_1 v_o}{M\omega}\sin\omega t.$$

8.4 ⋆ The x equation is

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \quad \text{or} \quad -\frac{\partial U}{\partial x} = \mu\ddot{x}$$

with corresponding equations for y and z. These are precisely Newton's second law, $\mathbf{F} = \mu\ddot{\mathbf{r}}$, for a single particle of mass μ and position $\mathbf{r} = (x, y, z)$, subject to the force $\mathbf{F} = -\nabla U$.

8.5 ⋆ Because $\mathcal{L} = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$, we find $p_x = \partial\mathcal{L}/\partial\dot{x} = \mu\dot{x}$. Combining this with the corresponding results for the y and z components, we find $\mathbf{p} = \mu\dot{\mathbf{r}}$.

Since $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the last result implies that

$$\mathbf{p} = \frac{m_1 m_2}{m_1 + m_2}(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2) = \frac{1}{m_1 + m_2}(m_2\mathbf{p}_1 - m_1\mathbf{p}_2).$$

This last result is true in any frame, but in the CM frame, $\mathbf{p}_2 = -\mathbf{p}_1$, so the result becomes $\mathbf{p} = \mathbf{p}_1$. In the CM frame, the relative momentum is the same as $\mathbf{p}_1$.

8.6 ⋆ In the CM frame, with the origin fixed at the CM, we know that $\mathbf{p}_2 = -\mathbf{p}_1$ and that $\mathbf{r}_2 = -(m_1/m_2)\mathbf{r}_1$. Therefore

$$\mathbf{L} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = \left(1 + \frac{m_1}{m_2}\right)\mathbf{r}_1 \times \mathbf{p}_1 = \frac{M}{m_2}\boldsymbol{\ell}_1$$

and $\boldsymbol{\ell}_1 = (m_2/M)\mathbf{L}$. Similary with $\boldsymbol{\ell}_2$.

8.7 ⋆⋆ **(a)** Newton's second law in the form $m\mathbf{a} = \mathbf{F}$, with $\mathbf{a}$ equal to the centripetal acceleration (magnitude $\omega^2 r$), implies that $m_1\omega^2 r = Gm_1m_2/r^2$. Substituting $\omega = 2\pi/\tau$ and solving for the period τ, we find $\tau = 2\pi r^{3/2}/\sqrt{Gm_2}$. (Note how the mass m_1 cancels out.)

(b) The equation of motion, $\mu\ddot{\mathbf{r}} = \mathbf{F}$, for the relative motion (with μ equal to the *reduced* mass m_1m_2/M) implies that $\mu\omega^2 r = Gm_1m_2/r^2$. This is the same as before, except that the mass on the left is now the reduced mass μ instead of m_1. This no longer cancels the factor m_1 on the right, so the final answer contains m_1m_2/μ instead of m_2:

$$\tau = \frac{2\pi r^{3/2}}{\sqrt{Gm_1m_2/\mu}} = \frac{2\pi r^{3/2}}{\sqrt{GM}}. \tag{i}$$

If $m_2 \gg m_1$, then $M \approx m_2$ and (i) approaches the answer of part (a).

(c) If $m_1 = m_2$, then $M = 2m_2$, and the period (i) is $1/\sqrt{2}$ times that in part (a), or, for the case at hand, 0.71 years.

8.8 ⋆⋆ The Lagrangian is $\mathcal{L} = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}kr^2$. The Lagrange equation for $\mathbf{R}$ is $\ddot{\mathbf{R}} = 0$, so the CM moves with constant velocity. The equation for $\mathbf{r}$ is $\mu\ddot{\mathbf{r}} = -k\mathbf{r}$, which implies that the relative position moves like a three-dimensional isotropic oscillator with angular frequency $\omega = \sqrt{k/\mu}$.

8.9 ⋆⋆ **(a)**

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}m(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2) - \tfrac{1}{2}k\big(|\mathbf{r}_1 - \mathbf{r}_2| - L\big)^2 \\ &= \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - \tfrac{1}{2}k(r - L)^2 \\ &= \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \tfrac{1}{2}k(r - L)^2\end{aligned}$$

(b) The equations are $M\ddot{X} = 0$ and $M\ddot{Y} = 0$, with solutions $\mathbf{R} = \mathbf{R}_\text{o} + \dot{\mathbf{R}}_\text{o}t$. That is, the CM moves like a free particle.

(c) The r and ϕ equations are

$$\mu\ddot{r} = \mu r\dot{\phi}^2 - k(r - L) \quad \text{and} \quad \mu r^2\dot{\phi} = \text{const.}$$

If $r = $ const, the ϕ equation tells us that $\dot{\phi} = \text{const} = \omega$, say, and the r equation becomes $k(r - L) = \mu r\omega^2$, which says that the spring force supplies the centripetal acceleration $r\omega^2$. The two particles circle each other at fixed radius.

If ϕ is constant, the r equation becomes $\mu\ddot{r} = -k(r - L)$, which implies that $r = L + A\cos(\omega t - \delta)$, where $\omega = \sqrt{k/\mu} = \sqrt{2k/m_1}$. The arrangement doesn't rotate, but undergoes radial oscillations with angular frequency $\sqrt{2k/m_1}$.

8.10 ⋆⋆ **(a)** The KE is given by (8.12), and the PE is just $U_1 + U_2 + U_{12}$. Using (8.9) you can check that

$$U = U_1 + U_2 + U_{12} = \tfrac{1}{2}k(r_1^2 + r_2^2) + \tfrac{1}{2}\alpha k r^2 = k\mathbf{R}^2 + \tfrac{1}{2}k(\alpha + \tfrac{1}{2})\mathbf{r}^2.$$

(In deriving the last expression, remember that $m_1 = m_2$ so that $\mathbf{r}_1$ and $\mathbf{r}_2$ are just $\mathbf{R} \pm \mathbf{r}/2$. Also, $M = 2m_1$ and the reduced mass is $\mu = \frac{1}{2}m_1$.) Therefore

$$\mathcal{L} = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - k\mathbf{R}^2 - \tfrac{1}{2}k(\alpha + \tfrac{1}{2})\mathbf{r}^2$$

where $\mathbf{r}^2 = x^2 + y^2$, and so on.

(b) There are four Lagrange equations. That for the CM coordinate X reads

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \qquad \text{or} \qquad -2kX = M\ddot{X}$$

with exactly the same equation for Y. Thus both components of the CM position oscillate with the same frequency $\sqrt{2k/M}$, and the CM moves around an elliptical path, as described in Section 5.3.

The equation for the relative coordinate x reads

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \text{or} \qquad -k(\alpha + \tfrac{1}{2})x = \mu\ddot{x}$$

with exactly the same equation for y. Therefore both components of the relative position $\mathbf{r}$ oscillate with the same frequency $\sqrt{k(\alpha + \frac{1}{2})/\mu}$, and $\mathbf{r}$ also moves around an ellipse.

8.11 ⋆⋆ From the Lagrangian $\mathcal{L} = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}kr^2$ we find the equation of motion $\mu\ddot{\mathbf{r}} = -k\mathbf{r}$. Therefore

$$x = A\cos\omega t + B\sin\omega t \qquad \text{and} \qquad y = C\cos\omega t + D\sin\omega t.$$

From these there follows

$$Cx - Ay = (BC - DA)\sin\omega t \qquad \text{and} \qquad Dx - By = -(BC - DA)\cos\omega t.$$

Squaring both of these and adding, we find that $ax^2 + 2bxy + cy^2 = k$, where

$$a = C^2 + D^2, \qquad b = -(CA + DB), \qquad \text{and} \qquad c = A^2 + B^2$$

and $k = (BC - DA)^2 > 0$. We are promised that this equation defines an ellipse provided a and c are positive (which they certainly are) and $ac > b^2$. So let us examine the quantity

$$ac - b^2 = (C^2 + D^2)(A^2 + B^2) - (CA + DB)^2 = (BC - DA)^2 > 0. \qquad \text{(ii)}$$

Therefore $\mathbf{r}$ moves around an ellipse. In the CM frame, $\mathbf{r}_1 = (m_2/M)\mathbf{r}$ and $\mathbf{r}_2 = -(m_1/M)\mathbf{r}$, so both particles move around similar ellipses at opposite ends of a line through the CM. [In Eq.(ii), $ac - b^2$ can actually be zero. In this case, the ellipse collapses to a straight line.]

8.12 ⋆⋆ **(a)** According to Eq.(8.29), $\mu\ddot{r} = -dU_{\text{eff}}/dr$. Therefore, the planet can orbit at a fixed radius if and only if $dU_{\text{eff}}/dr = 0$. Since $U_{\text{eff}} = -\gamma/r + \ell^2/2\mu r^2$, it follows that $dU_{\text{eff}}/dr = \gamma/r^2 - \ell^2/\mu r^3$, which is zero when $r = r_o = \ell^2/\gamma\mu$.

(b) The "equilibrium" radius r_o is stable if and only if U_{eff} is minimum at r_o; that is, its second derivative must be positive. This derivative is

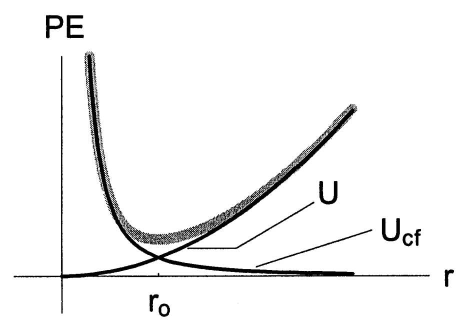
$$\left[\frac{d^2U_{\text{eff}}}{dr^2}\right]_{r=r_o} = \left[\frac{-2\gamma}{r^3} + \frac{3\ell^2}{\mu r^4}\right]_{r=r_o} = \frac{\gamma}{r_o^{\,3}}$$

where, in the second equality, I used the result of part (a) to write $\ell^2 = \gamma\mu r_o$. Since this second derivative is positive, the equilibrium is stable. Near the minimum, the effective PE has the approximate form $U_{\text{eff}} \approx \text{const} + \frac{1}{2}(\gamma/r_o^{\,3})(r - r_o)^2$. Substituting this into the equation of motion, we get $\mu\ddot{r} = -dU_{\text{eff}}/dr = -(\gamma/r_o^{\,3})(r - r_o)$, which shows that r oscillates about r_o with angular frequency $\omega_{\text{osc}} = \sqrt{\gamma/\mu r_o^{\,3}}$, which is exactly the same as the angular velocity of the planet in its circular orbit. (To see this, set the centripetal acceleration $\omega^2 r$ equal to F_{grav}/m.) Therefore, the period of oscillation is equal to the orbital period.

8.13 ⋆⋆⋆ **(a)** Since $U = \frac{1}{2}kr^2$ and $U_{\text{cf}} = \ell^2/2\mu r^2$,

$$U_{\text{eff}} = U + U_{\text{cf}} = \tfrac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2},$$

shown as the thick, gray curve.



(b) We know from (8.29) that $\mu\ddot{r} = -U'_{\text{eff}}(r)$ (where the prime denotes differentiation with respect to r). Thus r can remain constant if and only if this derivative is zero. That is,

$$U'_{\text{eff}}(r_o) = kr_o - \frac{\ell^2}{\mu r_o^{\,3}} = 0$$

Therefore the "equilibrium" radius is $r_o = (\ell^2/k\mu)^{1/4}$.

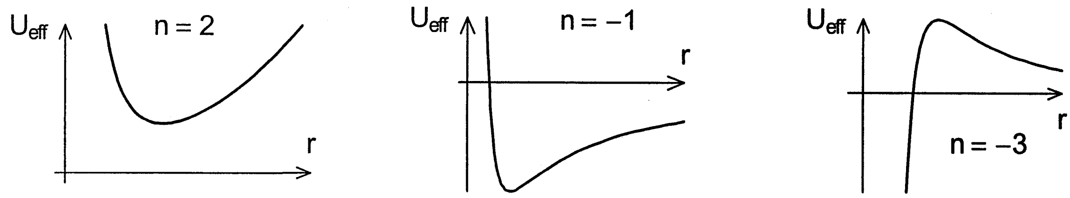
(c) The Taylor expansion of $U(r)$ about r_o gives

$$U_{\text{eff}}(r) = U_{\text{eff}}(r_o) + U'_{\text{eff}}(r_o)(r - r_o) + \tfrac{1}{2}U''_{\text{eff}}(r_o)(r - r_o)^2$$

if I ignore terms higher than quadratic. The first term on the right is constant, the second is zero [by part (b)], and the derivative in the third is easily evaluated as $U''_{\text{eff}}(r_o) = 4k$. If we write $r - r_o = \epsilon$, then the equation of motion (8.29) becomes $\mu\ddot{\epsilon} = -4k\epsilon$, and ϵ oscillates about zero with angular frequency $\omega = \sqrt{4k/\mu}$.

8.14 ⋆⋆⋆ **(a)** If $U = kr^n$, the force is $F = -dU/dr = -knr^{n-1}$. That $kn > 0$ means simply that the force is attractive (inward, toward the origin).

(b) An orbit of fixed radius r_o occurs if the derivative of U_{eff} is zero at $r = r_o$. The relevant functions are



$$U_{\text{eff}} = kr^n + \frac{\ell^2}{2\mu r^2}, \quad U'_{\text{eff}} = knr^{n-1} - \frac{\ell^2}{\mu r^3}, \quad \text{and} \quad U''_{\text{eff}} = kn(n-1)r^{n-2} + 3\frac{\ell^2}{\mu r^4}.$$

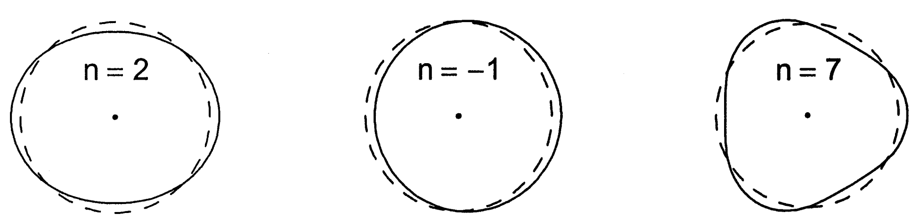
The derivative vanishes at radius $r_o = (\ell^2/\mu kn)^{1/(n+2)}$. The corresponding circular orbit is stable if the second derivative is positive at $r = r_o$. As you can check, after a little algebra, $U''_{\text{eff}}(r_o) = (n+2)\ell^2/\mu r_o^{\,4}$. Therefore the circular orbit is stable if $n > -2$. This agrees with the graphs where you can see that U_{eff} has a minimum for $n = 2$ and $n = -1$, but a maximum for $n = -3$.

(c) When $r = r_o + \epsilon$, with ϵ small, the effective PE is approximately

$$U_{\text{eff}}(r) = \text{const} + \tfrac{1}{2}U''_{\text{eff}}(r_o)\epsilon^2 = \text{const} + \frac{1}{2}\frac{(n+2)\ell^2}{\mu r_o^{\,4}}\epsilon^2$$

and, provided $n > 2$, the equation of motion, $\mu\ddot{\epsilon} = -U'_{\text{eff}}$, implies radial oscillations of angular frequency $\omega_{\text{osc}} = \sqrt{n+2}\,\ell/\mu r_o^{\,2} = \sqrt{n+2}\,\omega$, where $\omega = \ell/\mu r_o^{\,2}$ is the angular velocity of the circular orbit. That is, $\tau_{\text{osc}} = \tau/\sqrt{n+2}$, as claimed.

If $\sqrt{n+2}$ is rational, $\sqrt{n+2} = p/q$ where p and q are integers, then after a time $t = p\tau_{\text{osc}} = q\tau$ both the orbital motion and the radial oscillations will be back where they started; that is, the whole motion will be about to repeat itself. In the pictures, the dashed circles show the circular orbits and the solid curves the motion with small radial oscillations.



8.15 ⋆ Equation (8.54) (which is exact) states that $\tau^2 = ka^3$, where $k = 4\pi^2\mu/\gamma$. Because $\mu = M_s m_p/(M_s + m_p)$ and $\gamma = GM_s m_p$, the "constant" k is exactly $k = 4\pi^2/G(M_s + m_p)$. Thus k has its minimum value for Jupiter, $k_{\text{min}} = 4\pi^2/G(M_s + m_j)$, while the maximum value (for a planet of zero mass) is $k_{\text{max}} = 4\pi^2/GM_s$. The fractional difference is

$$(\text{fractional difference}) = \frac{k_{\text{max}} - k_{\text{min}}}{k_{\text{max}}} = \frac{m_j}{M_s + m_j} \approx \frac{m_j}{M_s} = 10^{-3} = 0.1\%.$$

8.16 ★★ Multiplying both sides of the given equation by $(1+\epsilon\cos\theta)$ gives $r+\epsilon x = c$ (since $r\cos\theta = x$) or $r = c - \epsilon x$. Squaring both sides, setting $r^2 = x^2+y^2$, and rearranging, we find $(1-\epsilon^2)x^2 + 2c\epsilon x + y^2 = c^2$. If we divide both sides by $(1-\epsilon^2)$ and define $d = c\epsilon/(1-\epsilon^2)$, this gives

$$(x^2+2dx) + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}.$$

Next we can add d^2 to both sides to "complete the square" on the left, to give

$$(x+d)^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2} + d^2 = \frac{c^2}{1-\epsilon^2}\left(1+\frac{\epsilon^2}{1-\epsilon^2}\right) = \frac{c^2}{(1-\epsilon^2)^2} = a^2$$

if we define $a = c/(1-\epsilon^2)$. Finally, dividing through by a^2, we arrive at

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{a^2(1-\epsilon^2)} = \frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where in the second expression I have introduced the definition $b = a\sqrt{1-\epsilon^2}$. Collecting our definitions of a, b, and d, we see that

$$a = \frac{c}{1-\epsilon^2}, \quad b = \frac{c}{\sqrt{1-\epsilon^2}} \quad \text{and} \quad d = a\epsilon$$

exactly as in (8.52).

8.17 ★★ **(a)** If $G = \mathbf{r}\cdot\mathbf{p}$, then $dG/dt = \dot{\mathbf{r}}\cdot\mathbf{p} + \dot{\mathbf{p}}\cdot\mathbf{r} = mv^2 + \mathbf{F}\cdot\mathbf{r}$. If we integrate this from 0 to t, we get $G(t) - G(0) = \int_0^t (2T + \mathbf{F}\cdot\mathbf{r})dt$, or, dividing both sides by t,

$$\frac{G(t)-G(0)}{t} = 2\langle T\rangle + \langle \mathbf{F}\cdot\mathbf{r}\rangle. \tag{iii}$$

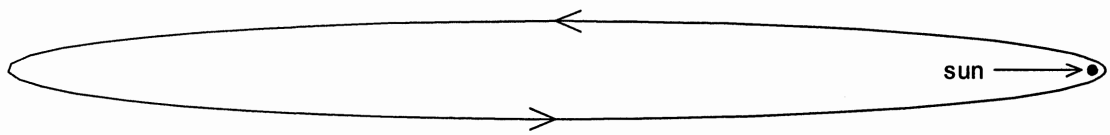
(b) If the motion is periodic and if K denotes the maximum value of $|G|$ during any one cycle, then the numerator of the left side of this relation can never exceed $2K$. Therefore, as we let $t \to \infty$, the left side approaches zero.

(c) If $U = kr^n$, then $\mathbf{F} = -\nabla U = nkr^{n-1}\hat{\mathbf{r}}$, so $\mathbf{F}\cdot\mathbf{r} = nkr^n = nU$. Inserting this in Eq.(iii) and letting $t\to\infty$, we find $0 = 2\langle T\rangle + \langle nU\rangle$, if we now understand the two angle brackets $\langle\,\rangle$ to denote the long-term average of whatever is between them. Therefore, $\langle T\rangle = -n\langle U\rangle/2$.

8.18 ★★ We are given the satellite's height $h_{\min} = 250$ km and speed $v_{\max} = 8500$ m/s at perigee. The distance from the earth's center is then $r_{\min} = R_e + h_{\min} = 6650$ km. For any known satellite, we can certainly ignore the difference between the mass m and the reduced mass $\mu \approx m$. Thus the angular momentum is $\ell = mv_{\max}r_{\min}$ and the parameter c of Eq.(8.48) is $c = \ell^2/\gamma\mu = (v_{\max}r_{\min})^2/GM_e$. (Recall that $\gamma = GM_e m$.) Putting in the given numbers, we get $c = 7960$ km. The rest is easy: From Eq.(8.50), $r_{\min} = c/(1+\epsilon)$, so $\epsilon = (c - r_{\min})/r_{\min} = 0.197$. Similarly, from Eq.(8.50) $r_{\max} = c/(1-\epsilon) = 9910$ km, so $h_{\max} = r_{\max} - R_e = 3510$ km.

8.19 ⋆⋆ We are given the satellite's heights $h_{\rm min} = 300$ km at perigee and $h_{\rm max} = 3000$ km at apogee. The corresponding distances from the earth's center are $r_{\rm min} = R_{\rm e} + r_{\rm min}$ and $r_{\rm max} = R_{\rm e} + r_{\rm max}$. We know from (8.50) that $r_{\rm min} = c/(1+\epsilon)$ and $r_{\rm max} = c/(1-\epsilon)$, from which it follows that $\epsilon = (r_{\rm max} - r_{\rm min})/(r_{\rm max} + r_{\rm min}) = 0.17$. From this we can find $c = (1+\epsilon)r_{\rm min} = 7824$ km. This is the distance from the earth's center when the satellite crosses the y axis, so the corresponding height is $h = c - R_{\rm e} \approx 1400$ km.

8.20 ⋆⋆ From Eq.(8.48), $c = \ell^2/\gamma\mu$. Therefore, as $\ell \to 0$, the length $c \to 0$. From (8.50), $r_{\rm max} = c/(1-\epsilon)$, so, if we want to keep $r_{\rm max}$ fixed, ϵ must approach 1. That is, the ellipse becomes long and thin, like a cigar. Also from (8.50), $r_{\rm min} = c/(1+\epsilon) \to 0$, so the perihelion gets very close to the sun, as shown. Because $2a = r_{\rm min} + r_{\rm max}$, in the limit $a \to r_{\rm max}/2$.

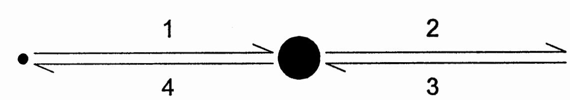


8.21 ⋆⋆⋆ **(b)** For the narrow elliptical orbit, with $\ell \to 0$ but $\ell \neq 0$, we've seen that $a = r_{\rm max}/2$. Thus, by Kepler's third law, (8.55), $\tau_{(\ell\to 0)} = \pi(r_{\rm max})^{3/2}/\sqrt{2GM_{\rm s}}$.

(c) With $\ell = 0$, the problem is one-dimensional. When the comet is a distance r from the sun, its energy is $\frac{1}{2}mv^2 - \gamma/r = E = -\gamma/r_{\rm max}$, where, as usual $\gamma = GM_{\rm s}m$. Therefore, as it falls toward the sun (segment 1 in the figure) its radial velocity is $v_r = -\sqrt{2GM_{\rm s}}\sqrt{1/r - 1/r_{\rm max}}$. (The minus sign is because it is moving inward.) The time to fall from $r_{\rm max}$ to the center of the sun is

$$t = \int_{r_{\rm max}}^{0} \frac{dr}{v_r} = \frac{1}{\sqrt{2GM_{\rm s}}} \int_0^{r_{\rm max}} \frac{dr}{\sqrt{1/r - 1/r_{\rm max}}} = \frac{\pi(r_{\rm max})^{3/2}}{2\sqrt{2GM_{\rm s}}}.$$

(To do the integral, make the sustitution $r/r_{\rm max} = u$.)

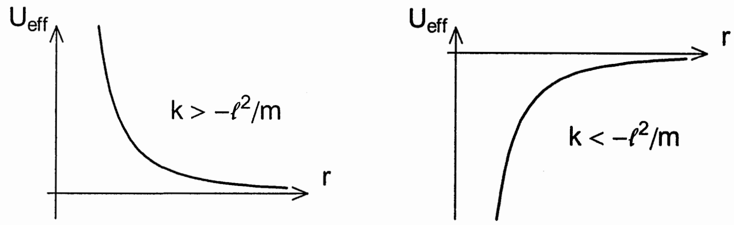


(d) When the comet passes the center of the sun it heads straight out the other side (segment 2 in the picture) and moves out to the same distance $r_{\rm max}$. Next it falls back to the center (segment 3), and finally it passes through and moves back to its starting point (segment 4). The time to do all this is 4 times the time t of part (c), so the period is $\tau_{(\ell=0)} = 2\pi(r_{\rm max})^{3/2}/\sqrt{2GM_{\rm s}}$.

(e) The period $\tau_{(\ell=0)}$ is twice as long as $\tau_{(\ell\to 0)}$. This is because if the comet has ℓ small but nonzero, then when it reaches the sun it swings around the sun and effectively jumps from segment 1 to segment 4. If it has ℓ exactly zero, it doesn't know which side to swing around and it effectively doubles its journey by going straight through and out on segments 2 and 3.

8.22 ⋆⋆⋆ **(a)** If $F = k/r^3$, then $U = k/2r^2$, and the effective potential energy is

$$U_{\text{eff}} = U + \frac{\ell^2}{2mr^2} = \frac{k + \ell^2/m}{2r^2}.$$



If $k > -\ell^2/m$, the effective PE is positive (left picture) and the particle can come in from afar but must eventually move out to infinity. If $k < -\ell^2/m$, the effective PE is negative (right picture); if $E > 0$ the particle can come in from afar and then move out again, but if $E < 0$ it is trapped in a bounded orbit.

(b) The transformed equation reads $u'' = -(1 + km/\ell^2)u$. If $k > -\ell^2/m$, the number in parentheses is positive (call it κ^2) and the general solution is $u(\phi) = A\cos(\kappa\phi - \delta)$. By conservation of angular momentum, the angle ϕ always changes in one direction (always increasing or always decreasing). Therefore, the factor $\cos(\kappa\phi - \delta)$ must eventually vanish, so that $u \to 0$ and hence $r \to \infty$; that is, the particle eventually moves off to infinity, as predicted.

If $k < -\ell^2/m$, the transformed equation has the form $u'' = \lambda^2 u$, with the general solution $u(\phi) = Ae^{\lambda\phi} + Be^{-\lambda\phi}$. This solution may or may not vanish, depending on the values of A and B. If it vanishes, then r moves off to infinity at some value of ϕ. (In this case $E \geq 0$.) If $u(\phi)$ remains bounded away from zero ($u \geq u_o$ for some $u_o > 0$), then r remains bounded and the particle stays within some $r_{\max}$ at all times. (This is the case that $E < 0$.)

8.23 ⋆⋆⋆ **(a) & (b)** Since $F = -ku^2 + \lambda u^3$, the transformed radial equation is

$$u''(\phi) = -u - \frac{mF}{\ell^2 u^2} = -u + \frac{mk}{\ell^2} - \frac{m\lambda}{\ell^2}u = -\beta^2 u + \frac{mk}{\ell^2} = -\beta^2(u - K),$$

where for the third equality I have defined $\beta = \sqrt{1 + m\lambda/\ell^2}$ and, for the fourth, $K = mk/(\ell\beta)^2$. The general solution is $u = K + A\cos(\beta\phi - \delta)$, but by redefining the direction $\phi = 0$ we can make $\delta = 0$. If we define $\epsilon = A/K$, then $u = K(1 + \epsilon\cos\beta\phi)$ and we can write $r = 1/u = c/(1 + \epsilon\cos\beta\phi)$, where $c = 1/K = \ell^2\beta^2/mk$.

The denominator of r is periodic in ϕ with period $2\pi/\beta < 2\pi$; if $0 < \epsilon < 1$, the denominator never vanishes, so r oscillates between minimum and maximum values, starting out at a minimum when $\phi = 0$. Thus, the particle "tries" to follow an ellipse, but r returns to its minimum before ϕ has made a complete revolution.

(c) If β is a rational number, $\beta = p/q$ (where p and q are integers), then the orbit will close after q revolutions. If β is irrational, the orbit will never close. If $\lambda \to 0$, then $\beta \to 1$ and the orbit becomes a Kepler ellipse.

8.24 ★★★ With $\lambda < 0$ and $\ell^2 < -\lambda m$, the transformed equation for $u = 1/r$ can be written as

$$u''(\phi) = \left(\frac{m\lambda}{\ell^2} - 1\right)u + \frac{mk}{\ell^2} = \kappa^2(u + K)$$

with the general solution $u(\phi) = -K + Ae^{\kappa\phi} + Be^{-\kappa\phi}$. It is easy to see that this function can vanish no more than twice. Thus there are really just two cases to consider: (Remember that, because angular momentum is conserved, ϕ always increases or always decreases. To be definite, let's assume ϕ always increases. Remember also that by definition r and $u = 1/r$ are positive.)

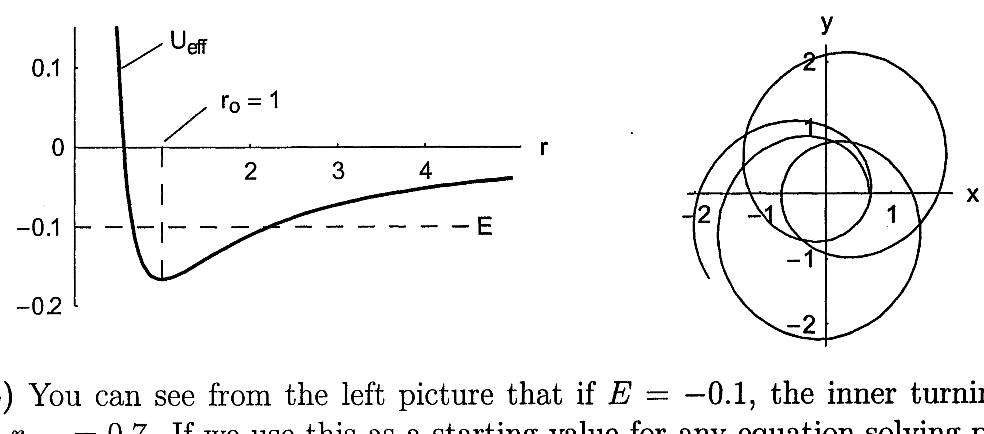
(1) In the range $\phi_{(t=0)} \le \phi \le \infty$, the function $u(\phi)$ never vanishes, so that $u(\phi) \ge u_{\min} > 0$. In this case $r \le r_{\max}$; that is, r is bounded. As $\phi \to \infty$, the function $u(\phi)$ approaches $Ae^{\kappa\phi}$ which approaches infinity. That is, $r \to 0$ and the particle eventually spirals in toward the origin. (As you can check, A cannot be zero in this case.)

(2) In the range $\phi_{(t=0)} \le \phi \le \infty$, the function u vanishes at least once, and the first time it does so is at $\phi = \phi_o$ (where ϕ_o may be infinity). In this case, as $\phi \to \phi_o$, $u(\phi)$ decreases toward 0 and $r = 1/u$ increases toward infinity; that is, the particle spirals out toward $r = \infty$.

8.25 ★★★ **(a)** The effective potential energy $U_{\text{eff}}(r)$ is

$$U_{\text{eff}} = U_{\text{cf}} + U = \frac{\ell^2}{2mr^2} - \frac{2k}{3r^{3/2}} = \frac{1}{2r^2} - \frac{2}{3r^{3/2}}$$

where the last expression results from the choice of units with $m = \ell = k = 1$. This function is plotted in the left picture below. The derivative is $U'_{\text{eff}}(r) = -r^{-3} + r^{-5/2}$, and this vanishes at $r = 1$. That is, the effective PE is minimum at $r_o = 1$ as indicated.



(b) You can see from the left picture that if $E = -0.1$, the inner turning point is at about $r_{\min} = 0.7$. If we use this as a starting value for any equation-solving program (such as Mathematica's FindRoot), we find that this root of the equation $U_{\text{eff}}(r) = -0.1$ is actually $r_{\min} = 0.6671$.

(c) The transformed radial equation (8.41) reads $u'' = -u + u^{1/2}$, with the initial conditions $u(0) = 1/r_{\min}$ and $u'(0) = 0$. This has to be solved numerically and produces the orbit shown on the right. Obviously this orbit has not closed after 3.5 revolutions, and we can see that it won't close for a very long time: If you look closely, you can see that r has returned to r_{min} at about $\theta = 2.9\pi$ (actually 2.8989π to 5 significant figures). Clearly, therefore, it cannot return to r_{min} at an integer multiple of 2π for at least 20,000 revolutions. (In fact, it never does, but this is harder to prove.)

8.26 ⋆⋆⋆ We have seen that Kepler's second law ("equal areas in equal times") is equivalent to conservation of angular momentum, which in turn implies that the force is central. Since the force is central and conservative, the variable $u = 1/r$ satisfies the "transformed radial equation" (8.41), which we can rewrite as

$$F = -[u''(\phi) + u(\phi)]\ell^2 u(\phi)^2/\mu.$$

Next, Kepler's first law states that the path of any body orbiting the sun is an ellipse with the sun at one focus, and we have seen that the equation for such an ellipse has the form (8.49), namely, $u(\phi) = (1 + \epsilon\cos\phi)/c$, where ϵ and c are positive constants for any given ellipse. When this form is substituted into the transformed radial equation, we find that

$$F = -\frac{\ell^2 u^2}{c\mu} = -\frac{\ell^2/(c\mu)}{r^2}.$$

Finally, because the force is conservative, it cannot depend on the angular momentum of the body, and the constant c must be proportional to ℓ^2, and we're left with $F = -\gamma/r^2$ where γ is a positive constant.

8.27 ⋆⋆⋆ We are given the initial speed and radius, v_o and r_o, and the angle α between $\mathbf{v}_o$ and the line from the comet to the sun. (And we know G and the sun's mass M.) If we denote the comet's unknown mass by m, then, in terms of m and the givens, we can calculate the following:

$$T_o = \tfrac{1}{2}mv_o^2 \quad \text{and} \quad U_o = -GMm/r_o$$

and hence

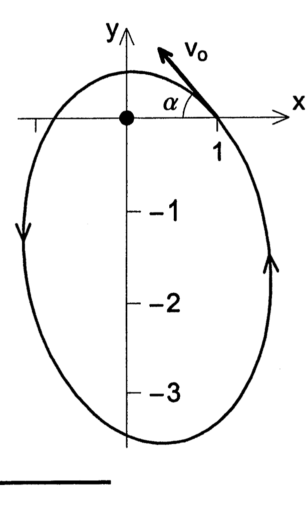
$$E = T_o + U_o, \quad \ell = v_o r_o \sin\alpha, \quad \text{and} \quad c = \ell^2/(GMm^2).$$

From Eq.(8.58),

$$\epsilon = \sqrt{1 + 2E\ell^2/(G^2M^2m^3)}$$

and since $r = c/[1 + \epsilon\cos(\phi - \delta)]$

$$\delta = \arccos[(c/r_o - 1)/\epsilon].$$



If you look carefully, you'll see that the unknown mass m cancels from the three parameters c, ϵ, and δ. (So, in practice, we may as well set $m = 1$.) Thus we can find the numerical values of these parameters, and we've got the orbit. With the given numbers, we get

$$c = 8.87 \times 10^{10}\ \text{m}, \quad \epsilon = 0.753, \quad \text{and} \quad \delta = 1.72.$$

The figure shows distances in units of 10^{11} m.

8.28 ⋆ For any Kepler orbit we can write $r = c/(1 + \epsilon\cos\phi)$, where $c = \ell^2/\gamma\mu$. For a circular orbit, $\epsilon = 0$ and $r = c$. For a parabolic orbit, $\epsilon = 1$ and $r_{\text{min}} = c/(1+\epsilon) = c/2$.

8.29 ⋆⋆ When the mass of the sun is suddenly halved, the earth's PE is immediately halved, $U = \frac{1}{2}U_o$. On the other hand, the KE is unchanged, $T = T_o$. Therefore, the total energy becomes $E = T + U = T_o + \frac{1}{2}U_o = 0$, because $T_o = -\frac{1}{2}U_o$ in a circular orbit (virial theorem). Since the final orbit has $E = 0$, it is a parabola, and the earth would eventually escape from the sun.

8.30 ⋆⋆ If we multiply both sides of Eq.(8.49), $r = c/(1 + \epsilon\cos\phi)$ by $(1 + \epsilon\cos\phi)$, replace $r\cos\phi$ by x, and rearrange, we find that $r = c - \epsilon x$. Squaring both sides gives $x^2 + y^2 = c^2 - 2c\epsilon x + \epsilon^2 x^2$. We now have two cases to consider. **(a)** If $\epsilon = 1$, the terms in x^2 cancel and we're left with $y^2 = c^2 - 2cx$, a parabola. **(b)** If $\epsilon > 1$, we find $(\epsilon^2 - 1)x^2 - 2c\epsilon x - y^2 = -c^2$. Completing the square for x gives $(\epsilon^2-1)(x-\delta)^2 - y^2 = -c^2 + \epsilon^2c^2/(\epsilon^2 - 1) = c^2/(\epsilon^2-1)$. Finally, multiplying both sides by $(\epsilon^2-1)/c^2$, we get

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1 \quad \text{where} \quad \alpha = \frac{c}{\epsilon^2 - 1} \quad \text{and} \quad \beta = \frac{c}{\sqrt{\epsilon^2-1}}$$

which is the equation of a hyperbola.

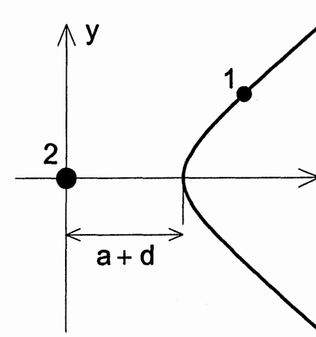
8.31 ⋆⋆⋆ If we write $F_r = \gamma/r^2$ (with γ positive), the $E = T + U = T + \gamma/r$, which is never negative. The transformed radial equation (8.41) reads $u'' = -u - \gamma\mu/\ell^2$. [Compare (8.45).] This has the solution

$$r = \frac{1}{u} = \frac{c}{\epsilon\cos\phi - 1}, \tag{iv}$$

where $c = \ell^2/\gamma\mu > 0$ and ϵ is a positive constant. (In general the cosine has the form $\cos(\phi - \delta)$, but by redefining the zero of ϕ we can make $\delta = 0$.] Before showing that this is a hyperbola, we need to note that, as you can easily check, ϵ is related to E as in (8.57), $E = (\gamma^2\mu/2\ell^2)(\epsilon^2 - 1)$. Since $E \geq 0$, it follows that $\epsilon \geq 1$. If $\epsilon = 1$, it is easy to check that (iv) defines a parabola. If $\epsilon > 1$, then multiply (iv) by $\epsilon\cos\phi - 1$ to give $\epsilon x - r = c$, which becomes (after some algebra)

$$\frac{(x-d)^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a = c/(\epsilon^2 - 1)$, $b = a\sqrt{\epsilon^2 - 1}$, and $d = a\epsilon$. This is the equation of a hyperbola, as in the sketch, which shows the path of the relative position $\mathbf{r} = (x, y)$, that is, the orbit of particle 1 as seen from particle 2.



8.32 ⋆ For a circular orbit the centripetal acceleration is v^2/r and is supplied by the gravitational force $F = \gamma/r^2$. Thus, by the second law, $mv^2/r = \gamma/r^2$, whence $v = \sqrt{\gamma/(mr)}$.

8.33 ⋆⋆ The boost here occurs at P' which is the apogee, so the continuity of the orbit implies that $c_2/(1-\epsilon_2) = c_3/(1-\epsilon_3) = c_3$, since $\epsilon_3 = 0$. If the boost factor at P' is λ', then $\ell_3 = \lambda'\ell_2$, which implies that $c_3 = \lambda'^2 c_2$. Combining these two results, we get $\lambda'^2 = 1/(1-\epsilon_2)$. But we already know that $\epsilon_2 = \lambda^2 - 1$, and that $\lambda^2 = 2R_3/(R_1+R_3)$. Combining these, we conclude that $\lambda'^2 = 1/(2-\lambda^2) = (R_1+R_3)/2R_1$, as claimed.

8.34 ⋆⋆ If we use the notation of Example 8.6, $R_1 = 1$ AU and $R_3 = 30$ AU. Therefore the semi-major axis of the transfer orbit is $a_2 = (R_1+R_3)/2 = 15.5$ AU and the period of the transfer orbit is $\tau_2 = \tau_e(a_2/a_e)^{3/2} = \tau_e(15.5)^{3/2}$. The time for the transfer is half of this period, namely $\frac{1}{2}\tau_2 = \frac{1}{2}\tau_e(15.5)^{3/2} = 30.5$ years.

8.35 ⋆⋆⋆ The simplest way to do this problem is to imagine Example 8.6 run backwards and to invert the thrust factors found there. The first thrust comes at the point P' of Figure 8.13, and its thrust factor is [compare (8.74)] $\lambda' = \sqrt{2R_1/(R_1+R_3)} = \sqrt{2/5} = 0.63$, where R_3 denotes the larger radius and R_1 the smaller, as in Figure 8.13. Similarly, the second thrust comes at P and its thrust factor is [compare (8.73)] $\lambda = \sqrt{(R_1+R_3)/2R_3} = \sqrt{5/8} = 0.79$. The speed changes during the transfer orbit, and, by conservation of angular momentum, $v(\text{at } P) = v(\text{at } P')R_3/R_1$. Therefore

$$v_{\text{fin}} = \lambda \cdot \frac{R_3}{R_1} \cdot \lambda' \cdot v_{\text{in}} = \sqrt{\frac{R_1+R_3}{2R_3}} \cdot \frac{R_3}{R_1} \cdot \sqrt{\frac{2R_1}{(R_1+R_3}} \cdot v_{\text{in}} = \sqrt{\frac{R_3}{R_1}} \cdot v_{\text{in}} = 2v_{\text{in}}$$

Chapter 9

Mechanics in Noninertial Frames

I covered this chapter in 4.5 *fifty-minute lectures.*

In this chapter you will probably have to struggle with a prejudice your students have learned in their introductory physics course. For good reason, most teachers of freshman physics push the view that "fictitious" or "inertial" forces — most especially centrifugal forces — are an abomination to be avoided by all right-thinking physicists. It's easy to see where this view came from: Most students struggling for the first time with Newton's laws get hopelessly confused by the centrifugal force, which is almost certainly best left out of the introductory syllabus. Nevertheless, non-inertial frames are a fact of life, and we now have to persuade our students that it really is alright (and often desirable) to work in noninertial frames and live with the accompanying "fictitious" forces.

Section 9.1 introduces the inertial force $-m\mathbf{A}$ experienced by a mass m in a frame accelerating (relative to an inertial frame) with linear acceleration $\mathbf{A}$, and Section 9.2 describes the theory of tides based on this force.

The remainder of the chapter is about the centrifugal and Coriolis forces associated with rotating frames — of which the most important example is the earth itself. Section 9.3 introduces the angular velocity vector. Section 9.4 treats the tricky relation between time derivatives of a given vector with respect to two frames in relative rotation, and 9.5 establishes the equation of motion of a particle, as seen from a rotating frame, including the centrifugal and Coriolis forces. (If you don't care for the tricky analysis of Section 9.4, note that Problem 9.11 gives an alternative derivation of the equation of motion using the Lagrangian formalism.) Sections 9.6 and 9.7 discuss the centrifugal and Coriolis forces one at a time, then 9.8 and 9.9 discuss their effects on a projectile and on the Foucault pendulum. Finally, Section 9.10 relates the centrifugal and Coriolis forces encountered in rotating frames to the centripetal and Coriolis accelerations encountered if you use polar coordinates in a nonrotating frame.

There are a few good demonstrations possible with this chapter. If your students do Problem 9.14, about the parabolic shape of the surface of water in a spinning bucket, they'll enjoy a demonstration of this neat effect. Half fill a large beaker with water and start it spinning on a turntable. If you keep it spinning and wait patiently, it will form a beautiful parabola, which they can easily see if you let them crowd around your table. Tell them this is how large parabolic mirrors used to be made from molten glass.

Solutions to Problems for Chapter 9

9.1 ⋆ The bob of a pendulum at rest in an inertial frame is subject to two forces, the tension $\mathbf{T}$ in the string and the force of gravity $m\mathbf{g}$. Therefore, in equilibrium $\mathbf{T} + m\mathbf{g} = 0$ or $\mathbf{T} = -m\mathbf{g}$; that is, the tension force must be vertically up. Since the direction of the tension force in a string is always the direction of the string, the string must point vertically up from the bob. That is, the pendulum must hang vertically down.

We know that a pendulum in a car that is accelerating forward can be treated in the same way, provided we replace $\mathbf{g}$ with $\mathbf{g}_{\text{eff}} = \mathbf{g} - \mathbf{A}$, which points down and backward. By the same argument, this means the string must point up and forward from the bob, and the pendulum as a whole tilts backward.

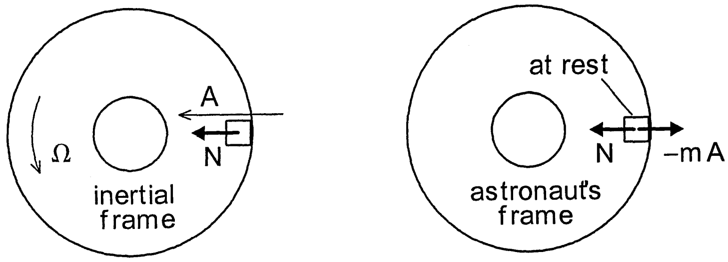
With a pendulum it is legitimate to ignore the buoyant force $\mathbf{F}_\text{b}$ of the air on the bob. With a helium balloon it is certainly not, since $\mathbf{F}_\text{b}$ is actually greater than the weight. Thus the condition for equilibrium of a helium balloon in a non-accelerating car is

$$\mathbf{T} = -(m\mathbf{g} + \mathbf{F}_\text{b}). \tag{i}$$

The direction of $\mathbf{F}_\text{b}$ is opposite to the pressure gradient, which is in the direction of $\mathbf{g}$. ($\mathbf{F}_\text{b}$ pushes toward lower pressures.) Thus $\mathbf{F}_\text{b}$ is vertically upward and greater in magnitude than $m\mathbf{g}$. Therefore the right side of Eq.(i) is downward, and the tension must be pulling down on the balloon. That is, the balloon floats vertically upward.

If the car is accelerating forward we have only to replace $\mathbf{g}$ in (i) with $\mathbf{g}_{\text{eff}}$, which points down and back. Now the buoyant force is opposite to $\mathbf{g}_{\text{eff}}$, so $\mathbf{F}_\text{b}$ points up and forward. Since $\mathbf{F}_\text{b}$ is greater in magnitude than $m\mathbf{g}$, Eq.(i) implies that $\mathbf{T}$ must point down and back, which means the balloon has to be floating up and forward. The direction of the string is that of $\mathbf{g}_{\text{eff}}$, so the angle of tilt is $\arctan(A/g)$.

9.2 ⋆ **(a)** As seen by inertial observers outside the station, the (square) astronaut has a centripetal acceleration $A = \omega^2 R$ which is supplied by the normal force $\mathbf{N}$.



(b) As seen by the crew inside the station, the astronaut is at rest under the action of two forces, the normal force $\mathbf{N}$ and the inertial force $-m\mathbf{A}$. To simulate normal gravity, we must have $A = \omega^2 R = g$ or $\omega = \sqrt{g/R} = 0.5$ rad/s $= 4.8$ rpm.

(c) The apparent gravity $g_{\text{app}} = \omega^2 R$ is proportional to R. Thus if we decrease R from 40 m to 38 m, the fractional change in g_{app} is $\delta g_{\text{app}}/g_{\text{app}} = \delta R/R = -5\%$.

9.3 ⋆⋆ **(a)** Referring to Fig.9.3, we see that, as suggested, the distance d is $d = d_o - R_e = d_o(1 - R_e/d_o)$ at the point P, and clearly $\hat{\mathbf{d}} = \hat{\mathbf{d}}_o = \hat{\mathbf{x}}$. Therefore, by Eq.(9.12)

$$\mathbf{F}_{\text{tid}} = -\frac{GM_m m}{d_o^2}\left(\frac{1}{(1 - R_e/d_o)^2} - 1\right)\hat{\mathbf{x}}. \tag{ii}$$

Obviously this has the direction of $-\hat{\mathbf{x}}$ as claimed. If we use the binomial approximation to write $(1 - R_e/d_o)^{-2} \approx 1 + 2R_e/d_o$, this gives

$$\mathbf{F}_{\text{tid}} = -2\frac{GM_m m R_e}{d_o^3}\hat{\mathbf{x}}. \tag{iii}$$

Since the gravitational force of the earth has magnitude $mg = GM_e m/R_e^2$, their ratio is

$$\frac{F_{\text{tid}}}{mg} = 2\frac{M_m}{M_e}\left(\frac{R_e}{d_o}\right)^3 = 2 \times \frac{7.35 \times 10^{22}}{5.98 \times 10^{24}} \times \left(\frac{6.37 \times 10^6}{3.84 \times 10^8}\right)^3 = 1.1 \times 10^{-7}.$$

(b) At the point R of Fig.9.4, $d = d_o + R_e$. Therefore $\mathbf{F}_{\text{tid}}$ is given by Eq.(ii) except that the denominator has a plus sign instead of a minus. Clearly $\mathbf{F}_{\text{tid}}$ is now in the direction of $+\hat{\mathbf{x}}$. If we make the binomial approximation, then we get the same answer as in (iii) except that the minus sign has become a plus. That is, the tidal force at R is equal and opposite to that at P.

9.4 ⋆⋆ If we put the mass m of Fig.9.3 at the point Q of Fig.9.4, then $d^2 = d_o^2 + R_e^2$. Thus Eq.(9.12) for the tidal force becomes

$$\mathbf{F}_{\text{tid}} = -GM_m m\left(\frac{\mathbf{d}}{d^3} - \frac{\mathbf{d}_o}{d_o^3}\right) = \frac{-GM_m m}{d_o^3}\left(\frac{\mathbf{d}}{[1 + (R_e/d_o)^2]^{3/2}} - \mathbf{d}_o\right).$$

Next we write $\mathbf{d} = \mathbf{d}_o + R_e\hat{\mathbf{y}}$ and use the binomial approximation to replace $[1+(R_e/d_o)^2]^{-3/2}$ by $1 - \frac{3}{2}(R_e/d_o)^2$. This leaves five terms inside the large parentheses, of which the two largest (both linear in $\mathbf{d}_o$) exactly cancel. The next largest term is $R_e\hat{\mathbf{y}}$ and the remaining two terms are smaller by a factor of order R_e/d_o or less. Therefore, to an excellent approximation, $\mathbf{F}_{\text{tid}} = -GM_m m R_e\hat{\mathbf{y}}/d_o^3$ at Q. This points in the direction shown in Fig.9.4. Comparing with the gravitational force $mg = GM_e m/R_e^2$, we find $F_{\text{tid}}/mg = 0.56 \times 10^{-7}$, which is half its value at P.

9.5 ⋆⋆ At the point P, $d = d_o - R_e$ and $x = -R_e$. Therefore

$$U_{\text{tid}} = -GM_m m\left(\frac{1}{d} + \frac{x}{d_o^2}\right) = -GM_m m\left(\frac{1}{d_o - R_e} - \frac{R_e}{d_o^2}\right)$$
$$= -\frac{GM_m m}{d_o}\left(\frac{1}{1 - R_e/d_o} - \frac{R_e}{d_o}\right) \approx -\frac{GM_m m}{d_o}\left(1 + \frac{R_e^2}{d_o^2}\right)$$

where in the last equality I used the binomial series to write $(1 - \epsilon)^{-1} \approx 1 + \epsilon + \epsilon^2$.

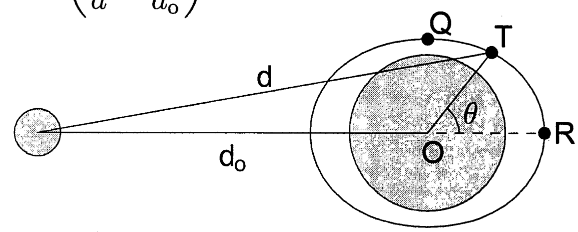
9.6 ⋆⋆⋆ Just as in (9.14), the requirement that the ocean's surface be an equipotential implies that

$$U_{\text{eg}}(T) - U_{\text{eg}}(Q) = U_{\text{tid}}(Q) - U_{\text{tid}}(T), \tag{iv}$$

and the left hand side is just $mgh(\theta)$. The term $U_{\text{tid}}(T)$ on the right is given by (9.13)

$$U_{\text{tid}}(T) = -GM_{\text{m}}m\left(\frac{1}{d} + \frac{x}{d_{\text{o}}}\right) \tag{v}$$

where we must find the values of x and d for the point T. Obviously $x = R_{\text{e}}\cos\theta$, but d requires more care. By the law of cosines, $d^2 = d_{\text{o}}^2 + 2d_{\text{o}}R_{\text{e}}\cos\theta + R_{\text{e}}^2$. Thus



$$\frac{1}{d} = \frac{1}{\sqrt{d_{\text{o}}^2 + 2d_{\text{o}}R_{\text{e}}\cos\theta + R_{\text{e}}^2}} = \frac{1}{d_{\text{o}}}\left(1 + 2\frac{R_{\text{e}}}{d_{\text{o}}}\cos\theta + \frac{R_{\text{e}}^2}{d_{\text{o}}^2}\right)^{-1/2}.$$

Since $R_{\text{e}} \ll d_{\text{o}}$, we can approximate the term in parenthesis using the binomial series, $(1+\epsilon)^{-1/2} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \cdots$. Although we need to keep the term ϵ^2, we can drop from it anything higher than $(R_{\text{e}}/d_{\text{o}})^2$, so we find

$$\begin{aligned}\frac{1}{d} &= \frac{1}{d_{\text{o}}}\left[1 - \frac{1}{2}\left(2\frac{R_{\text{e}}}{d_{\text{o}}}\cos\theta + \frac{R_{\text{e}}^2}{d_{\text{o}}^2}\right) + \frac{3}{8}\left(2\frac{R_{\text{e}}}{d_{\text{o}}}\cos\theta\right)^2\right] \\ &= \frac{1}{d_{\text{o}}}\left[1 - \frac{R_{\text{e}}}{d_{\text{o}}}\cos\theta + \frac{1}{2}\frac{R_{\text{e}}^2}{d_{\text{o}}^2}(3\cos^2\theta - 1)\right].\end{aligned} \tag{vi}$$

When this is substituted into (v) the term which is linear in $\cos\theta$ exactly cancels the second term on the right of (v) leaving

$$U_{\text{tid}}(T) = -\frac{GM_{\text{m}}m}{d_{\text{o}}}\left[1 + \frac{1}{2}\frac{R_{\text{e}}^2}{d_{\text{o}}^2}(3\cos^2\theta - 1)\right].$$

The value of $U_{\text{tid}}(Q)$ is found by putting $\theta = \pi/2$ (and hence $\cos\theta = 0$), and the difference on the right of (iv) is

$$U_{\text{tid}}(Q) - U_{\text{tid}}(T) = \frac{3GM_{\text{m}}mR_{\text{e}}^2}{2d_{\text{o}}^3}\cos^2\theta.$$

Since the left side of (iv) is $mgh(\theta)$, we conclude that $h(\theta) = h_{\text{o}}\cos^2\theta$, where

$$h_{\text{o}} = \frac{3GM_{\text{m}}mR_{\text{e}}^2}{2d_{\text{o}}^3 mg} = \frac{3M_{\text{m}}R_{\text{e}}^4}{2M_{\text{e}}d_{\text{o}}^3}$$

since $g = GM_{\text{e}}/R_{\text{e}}^2$.

The height $h(\theta) = h_{\text{o}}\cos^2\theta$ is zero at the point Q where $\theta = \pi/2$ — as it had to be, since it was defined as the height measured up from sea level at Q. It is positive for all other values of θ and symmetrical about $\theta = \pi/2$, rising to a maximum at $\theta = 0$ and π. This produces the oval shape shown in the picture.

9.7 ⋆ **(a)** If the vector $\mathbf{Q}$ is fixed in the frame $\mathcal{S}$, then $(d\mathbf{Q}/dt)_{\mathcal{S}} = 0$. Thus the relation (9.30)

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_o} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \mathbf{\Omega} \times \mathbf{Q}$$

reduces to

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_o} = \mathbf{\Omega} \times \mathbf{Q}.$$

This is just the "useful relation" (9.22) for the rate of change, as seen in $\mathcal{S}_o$ of a vector $\mathbf{Q}$ fixed in a body rotating with angular velocity $\mathbf{\Omega}$.

(b) If $\mathbf{Q}$ is fixed in $\mathcal{S}_o$, then $(d\mathbf{Q}/dt)_{\mathcal{S}_o} = 0$, and the relation (9.30) reduces to

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} = -\mathbf{\Omega} \times \mathbf{Q}.$$

Again this is just the "useful relation" (9.22), except that $\mathbf{Q}$ is fixed in $\mathcal{S}_o$ and the angular velocity of $\mathcal{S}_o$ relative to $\mathcal{S}$ is $-\mathbf{\Omega}$.

9.8 ⋆

For a person moving	$\mathbf{F}_{\text{cf}}$ is	$\mathbf{F}_{\text{cor}}$ is
(a) south near the north pole	south (and a little up)	west
(b) east on the equator	vertically up	vertically up
(c) south on the equator	vertically up	zero

9.9 ⋆ $\mathbf{F}_{\text{cor}} = 2m\mathbf{v} \times \mathbf{\Omega} = 2mv_o\Omega\cos\theta$ due east, and

$$\frac{F_{\text{cor}}}{mg} = \frac{2v_o\Omega\cos\theta}{g} = \frac{2 \times (1000\text{ m/s}) \times (7.3 \times 10^{-5}\text{ rad/s}) \times (\cos 40°)}{9.8\text{ m/s}^2} = 0.0114.$$

9.10 ⋆⋆ From Eq.(9.31) to (9.32) the derivation is exactly the same whether $\mathbf{\Omega}$ varies or not. If $\mathbf{\Omega}$ varies, then the first time derivative on the right of (9.32) picks up an extra term involving $\dot{\mathbf{\Omega}}$. Specifically, in place of (9.33) we now have $(d^2\mathbf{r}/dt^2)_{\mathcal{S}_o} = \ddot{\mathbf{r}} + 2\mathbf{\Omega} \times \dot{\mathbf{r}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + \dot{\mathbf{\Omega}} \times \mathbf{r}$. If we multiply both sides by m, the left side becomes $\mathbf{F}$, the net "real" force, and we get the equation of motion $m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega} + m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega} + m\mathbf{r} \times \dot{\mathbf{\Omega}}$.

9.11 ⋆⋆⋆ **(a)** The KE evaluated in the inertial frame $\mathcal{S}_o$ is $T = \frac{1}{2}m\mathbf{v}_o^2 = \frac{1}{2}m(\mathbf{v} + \mathbf{\Omega} \times \mathbf{r})^2$, so the Lagrangian is $\mathcal{L} = \frac{1}{2}m(\dot{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{r})^2 - U$.

(b) The derivatives of $\mathcal{L}$ are as follows:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= m(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r})\cdot\frac{\partial}{\partial x}(\boldsymbol{\Omega} \times \mathbf{r}) - \frac{\partial U}{\partial x}\\ &= m(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r})\cdot(0, \Omega_z, -\Omega_y) - \frac{\partial U}{\partial x}\\ &= m[(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}]_x - \frac{\partial U}{\partial x}.\end{aligned}$$

We can combine this with the two corresponding y and z equations, into a single vector equation

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}} = m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} + \mathbf{F}$$

Similarly

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = m(\ddot{\mathbf{r}} + \boldsymbol{\Omega} \times \dot{\mathbf{r}})$$

So the three Lagrange equations are $m\ddot{\mathbf{r}} = \mathbf{F} + 2m(\dot{\mathbf{r}} \times \boldsymbol{\Omega}) + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$, in agreement with Eq.(9.34).

9.12 ⋆ **(a)** All the rules of statics (including those concerned with total torques being zero) are derivable from the requirement that the net force $\mathbf{F}$ on every particle of the system must be zero, $\mathbf{F} = 0$. If we wish a structure to remain static in a rotating frame, then we must use the equation of motion (9.34) for each particle in the rotating frame. Since all of the particles are to be stationary (in the rotating frame), this reduces to $0 = \mathbf{F} + \mathbf{F}_{\text{cf}}$. This leads to all of the usual conditions except that where we usually use the net force $\mathbf{F}$ we must include the centrifugal force and use $\mathbf{F} + \mathbf{F}_{\text{cf}}$.

(b) For the puck on the rotating horizontal turntable, there are four forces, its weight $m\mathbf{g}$, the normal force $\mathbf{N}$ of the table, the force of friction $\mathbf{f}$, and the centrifugal force. If the puck is not to move on the table these must sum to zero, $m\mathbf{g} + \mathbf{N} + \mathbf{f} + \mathbf{F}_{\text{cf}} = 0$. The two vertical forces must balance, so $N = mg$, and the two horizontal forces must also balance, so $F_{\text{cf}} = m\Omega^2 r = f \le \mu N = \mu mg$. Therefore $r \le \mu g/\Omega^2$.

9.13 ⋆ The radial and tangential components of $\mathbf{g}$ are given by (9.45) and (9.47) as

$$g_{\text{rad}} = g_{\text{o}} - \Omega^2 R \sin\theta \approx g_{\text{o}} \approx g \qquad \text{and} \qquad g_{\text{tang}} = \Omega^2 R \sin\theta\cos\theta = \tfrac{1}{2}\Omega^2 R \sin(2\theta)$$

where I use "radial" to mean the direction of the center of the earth. From Figure 9.11, we see that

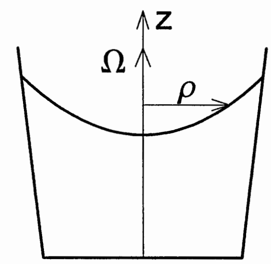
$$\tan\alpha = \frac{g_{\text{tang}}}{g_{\text{rad}}} = \frac{\Omega^2 R}{2g}\sin(2\theta).$$

Thus $|\alpha|$ is minimum at the poles and the equator, where it is zero, and maximum at latitude 45°, where it is $\Omega^2 R/2g \approx 0.1°$ as in (9.48).

9.14 ⋆⋆ In the rotating frame of the bucket, the water is in equilibrium and its surface is an equipotential surface for the combined gravitational force (PE $= mgz$) and centrifugal force (force $= m\Omega^2\rho$ and hence PE $= -m\Omega^2\rho^2/2$). Therefore, the surface is given by $mgz - m\Omega^2\rho^2/2 = \text{const}$, or

$$z = \frac{\Omega^2\rho^2}{2g} + \text{const},$$

which is a parabola, as claimed.

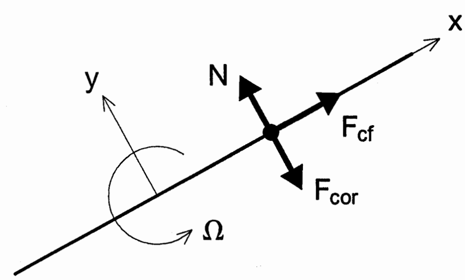


9.15 ⋆⋆ At the equator, we know that $g = g_o - \Omega^2 R$, but we are told this is λg_o. Therefore $\Omega^2 R = (1-\lambda)g_o$. Now, according to (9.44), at colatitude θ,

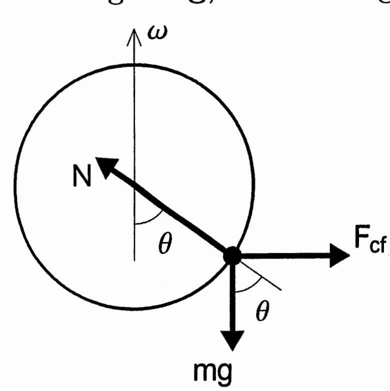
$$\mathbf{g}(\theta) = -g_o\hat{\mathbf{r}} + \Omega^2 R\sin\theta\hat{\boldsymbol{\rho}} = -g_o[\hat{\mathbf{r}} + (\lambda-1)\sin\theta\hat{\boldsymbol{\rho}}] = -g_o[\cos\theta\hat{\mathbf{z}} + \lambda\sin\theta\hat{\boldsymbol{\rho}}]$$

since $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\boldsymbol{\rho}}$. Therefore, $g(\theta) = \sqrt{\cos^2\theta + \lambda^2\sin^2\theta}$.

9.16 ⋆ With axes fixed on the rotating rod as shown, the bead stays on the x axis and its velocity is $\mathbf{v} = \dot{x}\hat{\mathbf{x}}$. The three forces on the bead are the normal force $\mathbf{N} = N\hat{\mathbf{y}}$, the centrifugal force $\mathbf{F}_{\text{cf}} = m\Omega^2 x\hat{\mathbf{x}}$, and the Coriolis force $\mathbf{F}_{\text{cor}} = -2m\dot{x}\hat{\mathbf{y}}$. The two components of the equation of motion are $m\ddot{x} = F_{\text{cf}} = m\Omega^2 x$ and $N = F_{\text{cor}}$. The solution is $x(t) = Ae^{\Omega t} + Be^{-\Omega t}$. The centrifugal force drives the bead out along the rod. The normal and Coriolis forces just balance out.



9.17 ⋆ As seen in a frame rotating with the hoop, there are five forces on the bead. The first three, all of which act in the plane of the hoop, are the bead's weight $m\mathbf{g}$, the centrifugal force $\mathbf{F}_{\text{cf}} = m\omega^2 R\sin\theta\hat{\boldsymbol{\rho}}$, and the normal force $\mathbf{N}$ (actually the component of the normal force in the plane of the hoop). The other two are the Coriolis force $\mathbf{F}_{\text{cor}}$ and the component of the normal force normal to the hoop (neither of which is shown in the picture). Since these last two both act normal to the hoop, they cancel one another and need not concern us further. The bead can move only in the tangential direction, and its equation of motion is $ma_{\text{tang}} = F_{\text{tang}}$ or



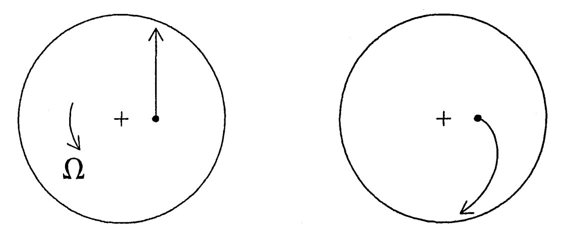
$$mR\ddot{\theta} = F_{\text{cf}}\cos\theta - mg\sin\theta = (m\omega^2 R\sin\theta)\cos\theta - mg\sin\theta,$$

whence $\ddot{\theta} = (\omega^2\cos\theta - g/R)\sin\theta$, in agreement with Eq.(7.69).

If the bead is in equilibrium, the tangential components of the centrifugal force and the weight must balance. That is, $F_{cf} \cos\theta + mg \sin\theta$, or $\cos\theta - g/(\omega^2 R)$, since $F_{cf} = m\omega^2 R \sin\theta$. This is the same as Eq.(7.71).

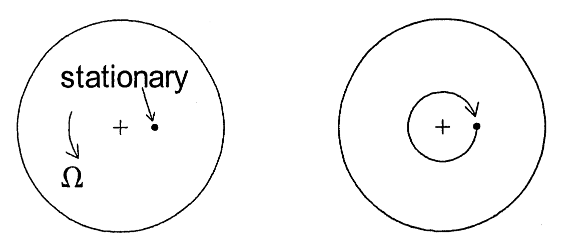
9.18 ⋆⋆ As seen in a frame rotating with the system, there are four forces on the mass: its weight $-mg\hat{\mathbf{y}}$, the centrifugal force $m\Omega^2 x\hat{\mathbf{x}}$, the normal force $\mathbf{N}$ of the confining plane, and the Coriolis force $\mathbf{F}_{\text{cor}}$. The last two both act in the z direction (normal to the confining plane) and must cancel each other, because there is no motion in this direction. The equations of motion in the remaining two directions are $\ddot{y} = -g$ with solution $y = y_o + v_{yo}t - \frac{1}{2}gt^2$, and $\ddot{x} = \Omega^2 x$ with solution $x = Ae^{\Omega t} + Be^{-\Omega t}$. The vertical motion is the same as that of a body in free fall. Except in the special case that $A = 0$, the x motion may be inward or outward initially, but eventually the particle moves outward at an exponentially increasing rate, caused by the centrifugal force. In the case that $A = 0$, the particle moves inward, slowing down because of the centrifugal force, and approaches the y axis as $t \to \infty$.

9.19 ⋆⋆ **(a)** As seen by a ground-based observer, the puck has initial velocity ΩR in the tangential direction. Since it is subject to zero net force, it travels in a straight line at constant speed (left picture). As seen from the merry-go-round, the puck is subject to the two inertial forces (centrifugal and Coriolis). It is initially at rest, so the Coriolis force is initially zero, and the puck is accelerated outward by the centrifugal force. As it speeds up, the Coriolis force becomes increasingly important and the puck curves to the right, spiralling outward.



View from ground from merry-go-round

(b) As seen from the ground, the puck is initially at rest. Since it is subject to zero net force, it remains at rest indefinitely. This means that, as seen from the merry-go-round, the puck describes a clockwise circle with angular velocity Ω and speed ΩR. This is quite a subtle result in the rotating frame. The centrifugal force is $m\Omega^2 R$ outward, and the Coriolis force is $2m\Omega^2 R$ inward; thus the net force is $m\Omega^2 R$ inward (as seen by observers on the merry-go-round), and this is just the required centripetal force to hold it in the circular orbit!



View from ground from merry-go-round

9.20 ⋆⋆ **(a)** The net "real" force is zero, so we have to consider only the centrifugal and Coriolis forces. Therefore

$$m\ddot{\mathbf{r}} = \mathbf{F}_{\text{cf}} + \mathbf{F}_{\text{cor}} = m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}\,.$$

Given that $\mathbf{r} = (x, y, 0)$ and $\boldsymbol{\Omega} = (0, 0, \omega)$, the various vector products are easily found, and the equation of motion becomes

$$\ddot{\mathbf{r}} = \Omega^2(x, y, 0) + 2\Omega(\dot{y}, -\dot{x}, 0)$$

or, in terms of components,

$$\ddot{x} = \Omega^2 x + 2\Omega\dot{y} \qquad \text{and} \qquad \ddot{y} = \Omega^2 y - 2\Omega\dot{x}\,.$$

(b) If we multiply the equation for $\ddot{y}$ by i and add it to that for $\ddot{x}$, we find (with $\eta = x + iy$)

$$\ddot{\eta} = \Omega^2\eta - 2i\Omega\dot{\eta}. \tag{vii}$$

If we guess that there could be a solution of the form $\eta = e^{-i\alpha t}$ and substitute into this equation, we find that this guess *is* a solution if and only if α satisfies the auxiliary equation

$$-\alpha^2 = \Omega^2 - 2\Omega\alpha$$

or $(\alpha - \Omega)^2 = 0$. This has just the one solution $\alpha = \Omega$, and gives us one solution of the equation of motion, namely, $\eta = e^{-i\Omega t}$. As in Section 5.4, the second solution is just t times the first; that is, as you can easily check, $te^{-i\Omega t}$ satisfies the equation of motion (vii), and the general solution is

$$\eta(t) = e^{-i\Omega t}(C_1 + C_2 t).$$

(c) The given initial conditions imply that $\eta(0) = x_o$ and $\dot{\eta}(0) = v_{xo} + iv_{yo}$, while from part (b) we see that $\eta(0) = C_1$ and $\dot{\eta}(0) = C_2 - i\Omega C_1$. This gives two equations for C_1 and C_2, which are easily solved to give

$$\eta(t) = e^{-i\Omega t}[x_o + v_{xo}t + i(v_{yo} + \Omega x_o)t].$$

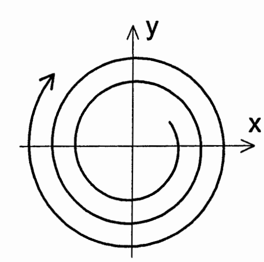
Taking real and imaginary parts, we obtain $x(t)$ and $y(t)$ exactly as in Equation (9.72).

(d) If we exclude the exceptional case mentioned, then for t sufficiently large we can neglect the terms that do not contain a factor of t [that is, the first term on the right in each line of (9.72)]. In this case (9.72) becomes

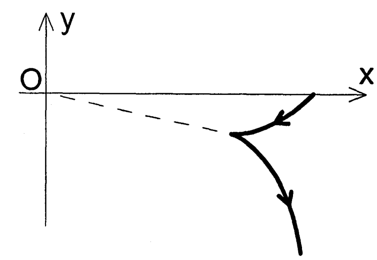
$$x(t) = t(B_1 \cos\Omega t + B_2 \sin\Omega t) \qquad \text{and} \qquad y(t) = t(-B_1 \sin\Omega t + B_2 \cos\Omega t)$$

with $B_1 = v_{xo}$ and $B_2 = v_{yo} + x_o$. If we define $A = \sqrt{B_1^2 + B_2^2}$ as in (5.10), then these become, as in (5.11),

$$x(t) = tA\cos(\Omega t - \delta) \qquad \text{and} \qquad y(t) = -tA\sin(\Omega t - \delta).$$



Without the factor of t, this point would move clockwise round a circle of radius A. The factor of t makes this circle grow at a constant rate, and the puck actually describes an expanding spiral, as shown. Remember that this is the behavior for large t; when t is small, the terms I have neglected can produce a quite different behavior, as in Problem 9.24. Nevertheless, the motion eventually approaches the simple spiral shown.



9.21 ⋆⋆ Suppose that the puck comes instantaneously to rest at a time t_o. As the puck comes to rest, the Coriolis force becomes less and less important, and the centrifugal force is what actually stops it. Since the centrifugal force is radially outward, the puck can only stop when it is headed inward, toward the axis of rotation O. Once it has stopped, the centrifugal force accelerates it radially outward again, and the orbit has the cusp shape shown. This is what happens in Problem 9.24(d).

9.22 ⋆⋆ Let $\mathcal{S}_o$ be the inertial frame in which the charge q orbits Q in a weak magnetic field $\mathbf{B}$. In this frame the equation of motion is

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\mathcal{S}_o} = -\frac{kqQ}{r^2}\hat{\mathbf{r}} + q\left(\frac{d\mathbf{r}}{dt}\right)_{\mathcal{S}_o} \times \mathbf{B} \tag{viii}$$

where the first term on the right is the Coulomb attraction of Q and the second is the magnetic force $q\mathbf{v} \times \mathbf{B}$. Let us now move to a frame $\mathcal{S}$ rotating with angular velocity $\mathbf{\Omega}$ relative to $\mathcal{S}_o$. We can rewrite the two derivatives of Eq.(viii) in terms of the corresponding derivatives in $\mathcal{S}$, as in Section 9.5. (I'll call these latter derivatives $\ddot{\mathbf{r}}$ and $\dot{\mathbf{r}}$ as before.) In $\mathcal{S}$ Eq.(viii) becomes

$$m\ddot{\mathbf{r}} - 2m\dot{\mathbf{r}} \times \mathbf{\Omega} - m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega} = -\frac{kqQ}{r^2}\hat{\mathbf{r}} + q(\dot{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{r}) \times \mathbf{B}.$$

If we choose the angular velocity so that $\mathbf{\Omega} = -q\mathbf{B}/(2m)$, then the terms involving $\dot{\mathbf{r}}$ on either side cancel exactly. The terms involving double cross products don't quite cancel, and we're left with

$$m\ddot{\mathbf{r}} = -\frac{kqQ}{r^2}\hat{\mathbf{r}} - \frac{q^2}{4m}(\mathbf{B} \times \mathbf{r}) \times \mathbf{B}.$$

If the B field is sufficiently weak, we can drop the second term on the right, and we're left with the equation for a body orbiting in an inverse square force (the Kepler problem). Therefore, in the rotating frame $\mathcal{S}$ the charge q moves in an ellipse (or hyperbola), and in the original frame $\mathcal{S}_o$ (relative to which $\mathcal{S}$ is rotating slowly), the elliptical orbit precesses slowly.

9.23 ⋆⋆ Let $\mathcal{S}_o$ be the original inertial frame and choose the plane in which the motion occurs to be the $x_o y_o$ plane. Now consider a frame $\mathcal{S}$ with the same origin and z axis as $\mathcal{S}_o$ and with $\mathcal{S}$ rotating about the z_o axis with angular velocity Ω. For any object confined to the xy plane, the centrifugal force is $m\Omega^2\mathbf{r}$ and the equation of motion (in frame $\mathcal{S}$) is

$$m\ddot{\mathbf{r}} = -k\mathbf{r} + \mathbf{F}_{\text{cf}} + \mathbf{F}_{\text{cor}} = (-k + m\Omega^2)\mathbf{r} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega}.$$

If we choose $\Omega = \sqrt{k/m}$ (the natural frequency of the mass on the spring), the centrifugal term exactly cancels the spring term and we're left with $m\ddot{\mathbf{r}} = 2m\dot{\mathbf{r}} \times \mathbf{\Omega}$, which is exactly the equation for a charge $q = 2m$ in a uniform magnetic field $\mathbf{B} = \mathbf{\Omega}$. In Section 2.7, we saw that the general solution has the form $x + iy = ce^{-i\omega t} + d$, where c and d are any two complex constants and $\omega = qB/m = 2\Omega$. [See the equation between (2.79) and (2.80).]

To translate this back to frame $\mathcal{S}_\text{o}$, notice that since $\mathcal{S}$ is rotating counterclockwise with angular velocity Ω, the complex coordinates satisfy $x_\text{o} + iy_\text{o} = (x + iy)e^{i\Omega t}$. Therefore

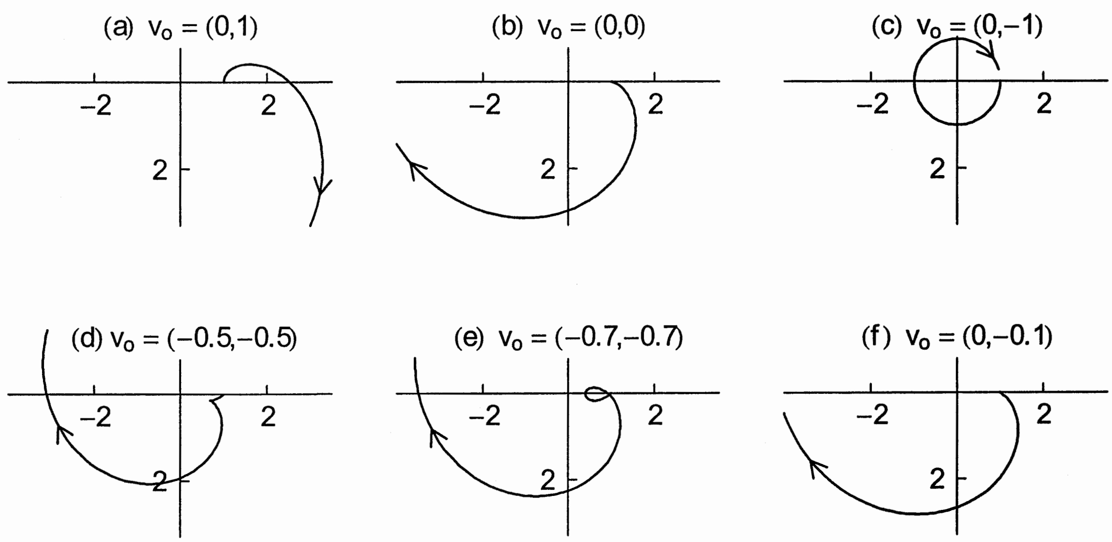
$$x_\text{o} + iy_\text{o} = (ce^{-2i\Omega t} + d)e^{i\Omega t} = ce^{-i\Omega t} + de^{i\Omega t} = a\cos\Omega t + b\sin\Omega t$$

where a and b are two other complex constants. Taking real and imaginary parts we conclude that

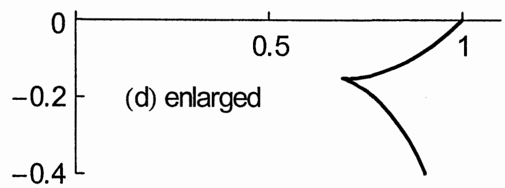
$$x_\text{o} = A\cos\Omega t + B\sin\Omega t \qquad \text{and} \qquad y_\text{o} = C\cos\Omega t + D\sin\Omega t$$

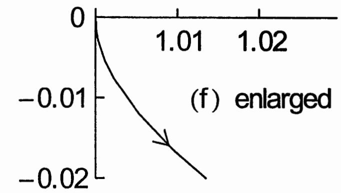
where A, B, C, D are four arbitrary real constants. This is the same solution as was found in the solution to Problem 8.11, and is an ellipse for the reasons given there.

9.24 ⋆⋆⋆ In (a) you can see clearly that the puck starts out at the right place $(1, 0)$ and



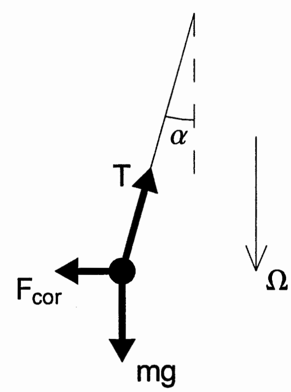
in the right direction $(0, 1)$. It curves to the right and spirals outward. Cases (b) and (c) are the two cases of Problem 9.19. In (d) the puck appears to come to rest, as you can see more clearly in the enlargement below. As it slows down, the Coriolis force becomes less and less important, and it is the centrifugal force that actually stops it. Since the centrifugal force is always outward, the puck can only stop when it is headed inward, toward the axis of rotation. In (e) the puck makes a little loop-the-loop. In the unenlarged version of (f), it is hard to tell that it starts out headed the right way. In the enlargement, we see that it does. Because it is going so slowly, the centrifugal force dominates and the puck actually curves to the left at first. In every case, except (c), the puck eventually approaches an outward, clockwise spiral, as found in Problem 9.20.





9.25 ⋆ At the south pole the centrifugal force is zero and the Coriolis force is horizontal and to the left of the train, with magnitude $F_{\rm cor} = 2mv\Omega$. There are three forces on the mass at the end of the plumb line: $\mathbf{F}_{\rm cor}$, the tension $\mathbf{T}$ in the line, and mass's weight $m\mathbf{g}$. (See the picture, which shows the plumb line as seen from the rear of the train. The train's velocity is into the page and the $\mathbf{\Omega}$ is vertically down.) The condition for equilibrium is

$$\mathbf{F}_{\rm cor} + \mathbf{T} + m\mathbf{g} = 0.$$



For the plumb line in the stationary hut, $\mathbf{F}_{\rm cor} = 0$, so $\mathbf{T} = -m\mathbf{g}$ and the line hangs vertically. On the train the angle with the vertical satisfies $\tan\alpha = F_{\rm cor}/mg = 2v\Omega/g$, and the angle between the two lines is $\alpha = \arctan(2v\Omega/g) = 0.13°$ with the line on the train hanging toward the left.

9.26 ⋆⋆ The equations of motion are given by (9.53). To zeroth order in Ω these reduce to

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \text{and} \quad \ddot{z} = -g,$$

with the familiar solutions

$$x = v_{xo}t, \quad y = v_{yo}t, \quad \text{and} \quad z = v_{zo}t - \tfrac{1}{2}gt^2.$$

If you substitute these into those terms of (9.53) that already contain a factor of Ω (and hence are only small corrections), you will find the equations

$$\left.\begin{aligned} \ddot{x} &= 2\Omega(v_{yo}\cos\theta - v_{zo}\sin\theta) + 2\Omega gt\sin\theta \\ \ddot{y} &= -2\Omega v_{xo}\cos\theta \\ \ddot{z} &= -g + 2\Omega v_{xo}\sin\theta. \end{aligned}\right\}$$

These three equations can be integrated twice to give precisely the requested equations (9.73).

9.27 $\star\star$ **(a)** The dominant effect as the object falls is that it acquires an increasing velocity vertically downward. Therefore the Coriolis force $2m\mathbf{v} \times \boldsymbol{\Omega}$ (with $\boldsymbol{\Omega}$ pointing due north) points due east.The left picture shows the experiment as seen by someone standing north of the equator and looking south.

(a) View from earth (b) View from space

(b) As seen from space, the earth and the ladder on which the object initially rests are both rotating with angular velocity $\dot{\phi} = \Omega$ toward the east. When the object is released its angular momentum $\ell = mr^2\dot{\phi}$ must remain constant. Thus, as r gets smaller (slightly), $\dot{\phi}$ gets bigger, and the object moves slightly ahead of the ladder from which it was released. Therefore the object lands slightly to the east of the ladder's new position.

9.28 $\star\star$ **(a)** If we ignore Ω entirely and set $v_{yo} = 0$, Eqs.(9.73) become $x = v_{xo}t$, $y = 0$, and $z = v_{zo} - \frac{1}{2}gt^2$. Thus the time of flight (time until $z = 0$ again) is $t = 2v_{zo}/g$ and the range R (value of x at landing) is $R = 2v_{xo}v_{zo}/g = 2v_o^2\cos(\alpha)\sin(\alpha)/g$. If $v_o = 500$ m/s and $\alpha = 20°$, these become $t = 34.9$s and $R = 16.4$ km.

(b) According to the second of Eqs. (9.73) (with $v_{yo} = 0$), $y = -\Omega v_o \cos(\alpha)\cos(\theta)t^2$. At latitude 50° north, $\theta = 40°$ and

$$y = -(7.3 \times 10^{-5}\ \mathrm{s}^{-1}) \times (500\ \mathrm{m/s}) \times \cos(20°) \times \cos(40°) \times (34.9\ \mathrm{s})^2 = -32\mathrm{m};$$

that is, the shell lands 32 m to the south of the target. At latitude 50° south, the factor $\cos\theta$ has the opposite sign, and the shell lands 32 m to the north.

9.29 $\star\star$ **(a)** With $v_{xo} = v_{yo} = 0$ and $v_{zo} = v_o$, the trajectory given by (9.73) is

$$x = -\Omega\sin\theta(v_o - \tfrac{1}{3}gt)t^2, \quad y = 0, \quad \text{and} \quad z = v_o t - \tfrac{1}{2}gt^2.$$

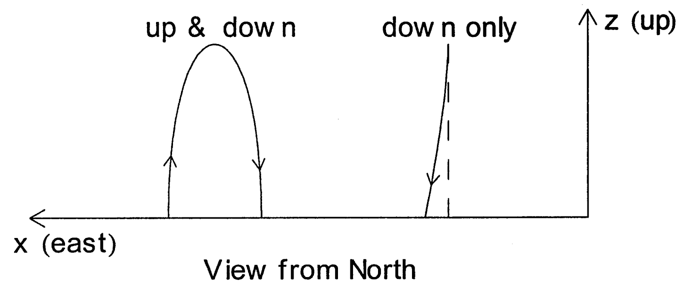
We see that the ball does not stray in the north-south (y) direction (at least to first order in Ω), but it does move in the east-west (x) direction. The time for the ball to return to the ground is found from the z equation to be $t_{gr} = 2v_o/g$. Substituting this value into the x equation, we find $x_{gr} = -(4\Omega v_o^3 \sin\theta)/(3g^2)$. Since this is negative, we conclude that the Coriolis effect causes the ball to land to the west of its point of departure.

(b) The westerly displacement is maximum at the equator, where $\sin\theta = 1$, and the displacement is

$$\frac{4\,\Omega\, v_o^3}{3\,g^2} = \frac{4 \times (7.3 \times 10^{-5}) \times (40)^3}{3 \times (9.8)^2} \approx 0.065\ \mathrm{m}$$

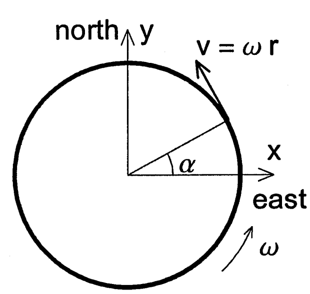
or about 6 cm — not an effect that would be easy to detect.

(c) On the upward journey, the Coriolis force accelerates the ball to the west and on the downward journey to the east. Thus v_x starts from zero, increases (to the west) as the ball climbs, and decreases back to zero by the time the ball returns to the ground. Throughout the trip v_x is to the west, and the ball lands to the west of its starting position. The dropped ball starts with $v_x = 0$ at the top and its whole journey is downward, so that the Coriolis force accelerates it to the east throughout. Thus v_x is to the east at all times, and the ball lands to the east of its initial position. In the pictures, the Coriolis effect is much exaggerated.



9.30 ★★★ I'll choose axes as usual, with x east, y north, and z vertically up. The picture shows the hoop as seen from above. Consider first a small segment of hoop subtending an angle $d\alpha$ with polar angle α. The mass of this segment is $dm = m d\alpha/2\pi$, and the Coriolis force on it is

$$d\mathbf{F}_{\text{cor}} = 2\, dm\, (\mathbf{v} \times \mathbf{\Omega})$$



where

$$\mathbf{v} = \omega r(-\sin\alpha, \cos\alpha, 0) \quad \text{and} \quad \mathbf{\Omega} = \Omega(0, \sin\theta, \cos\theta).$$

The segment's position vector is $\mathbf{r} = r(\cos\alpha, \sin\alpha, 0)$ and the torque on it is

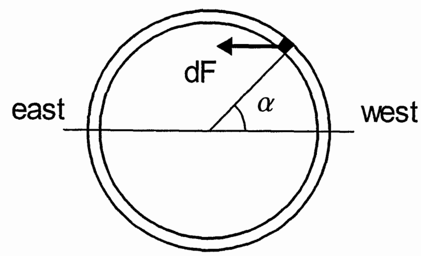
$$\begin{aligned} d\mathbf{\Gamma}_{\text{cor}} = \mathbf{r} \times d\mathbf{F}_{\text{cor}} = 2\, dm\, \mathbf{r} \times (\mathbf{v} \times \mathbf{\Omega}) &= 2\, dm[\mathbf{v}(\mathbf{r}\cdot\mathbf{\Omega}) - \mathbf{\Omega}(\mathbf{r}\cdot\mathbf{v})] \\ &= 2\, dm\, \omega r^2 \Omega(-\sin^2\alpha, \sin\alpha\cos\alpha, 0)\sin\theta. \end{aligned}$$

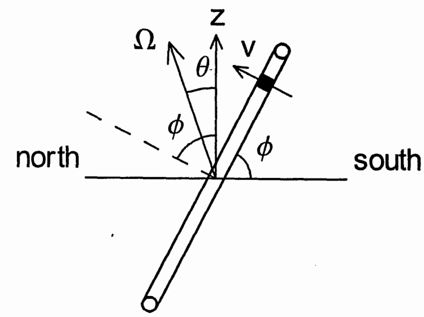
To find the total torque, we must replace dm by $m\, d\alpha/2\pi$ and integrate over α from 0 to 2π. The integral of $\sin^2\alpha$ gives π, while that of $\sin\alpha\cos\alpha$ is zero. Thus, the total torque on the hoop is

$$\mathbf{\Gamma}_{\text{cor}} = -(m\omega r^2 \Omega \sin\theta)\hat{\mathbf{x}},$$

which points west with magnitude $m\omega r^2\Omega\sin\theta$.

9.31 ★★★ The left picture shows the ring face-on, viewed from the north and above. The shaded segment of water has mass $dm = m\, d\alpha/2\pi$, where m is the total mass of water, and it experiences a Coriolis force $d\mathbf{F}$ to the east as it is swung toward the viewer. The right picture shows the ring side on, viewed from the west. The velocity of the shaded segment of water is $v = \dot{\phi} R \sin\alpha$ normal to the ring.





The force $d\mathbf{F}$ is

$$d\mathbf{F} = 2\,dm\,\mathbf{v}\times\mathbf{\Omega} = 2\,dm\,(\dot{\phi}R\sin\alpha)\Omega\sin(\phi-\theta)\text{ east.}$$

This produces a torque of magnitude

$$d\Gamma = |\mathbf{r}\times d\mathbf{F}| = R\,dF\sin\alpha = 2\,dm\,R^2\dot{\phi}\Omega\sin(\phi-\theta)\sin^2\alpha$$

tending to push the water counterclockwise as seen in the left picture. To find the total torque, we replace dm by $m\,d\alpha/2\pi$ and integrate from $\alpha=0$ to 2π. This gives

$$\Gamma = \int d\Gamma = mR^2\dot{\phi}\Omega\sin(\phi-\theta).$$

The total angular momentum given to the water is

$$L = \int \Gamma dt = mR^2\Omega\int\sin(\phi-\theta)\dot{\phi}dt = mR^2\Omega\int_0^\pi\sin(\phi-\theta)d\phi = 2mR^2\Omega\cos\theta.$$

If V is the final speed of the water, then $L = mRV$, and equating these two expressions, we see that $V = 2R\Omega\cos\theta$. With the given numbers,

$$V = 2\times(1\text{ m})\times(7.3\times10^{-5}\text{ rad/s})\times(\cos 40°) = 0.11\text{ mm/s}.$$

9.32 ★★★ The enemy ship is due east of the gun, which is aimed in that direction. That is, $v_{yo}=0$, and Eqs.(9.73) simplify to

$$\left.\begin{aligned} x &= v_{xo}t - (\Omega v_{zo}\sin\theta)t^2 + \tfrac{1}{3}(\Omega g\sin\theta)t^3 \\ y &= -(\Omega v_{xo}\cos\theta)t^2 \\ z &= v_{zo}t - \tfrac{1}{2}gt^2 + (\Omega v_{xo}\sin\theta)t^2. \end{aligned}\right\}\qquad\text{(ix)}$$

(a) If we ignore Ω entirely, we get the same answers as in part (a) of Problem 9.28. In particular, the range is $R_o = 2v_{xo}v_{zo}/g$. (I've called it R_o to emphasize that it's the range ignoring Ω.)

(b) We now wish to work to first order in Ω, and we must first use the third of Eqs.(ix) to find the time at which the shell lands. Solving that equation for t when $z=0$, we find

$$v_{zo} = \tfrac{1}{2}g\left(1-\frac{2\Omega v_{xo}\sin\theta}{g}\right)t \quad\text{whence}\quad t\approx\frac{2v_{zo}}{g}\left(1+\frac{2\Omega v_{xo}\sin\theta}{g}\right)$$

to first order in Ω. (I used the binomial approximation in solving for t.) This gives t as the sum of two terms. The first is the answer of part (a) (ignoring Ω entirely) and the second is the first order correction to t. To find by how much the shell misses the target, we must substitute this corrected time into the expressions for x and y in Eqs.(ix). The expression for y already contains a factor of Ω, so, to first order, we can just use the zeroth order time, to give

$$y = -(\Omega v_{xo}\cos\theta)\left(\frac{2v_{zo}}{g}\right)^2 = 32 \text{ m}.$$

This is the same answer as in Problem 9.28. The east-west position x requires more care. The first term in the expression for x in Eqs.(ix) does not involve Ω at all. Thus to get x correct to first order in Ω we must include the first order correction to t in this term. (In the other two terms we don't need to do this, because they already contain one factor of Ω.) Thus from (ix) we get

$$\begin{aligned} x &= v_{xo}\frac{2v_{zo}}{g}\left(1+\frac{2\Omega v_{xo}\sin\theta}{g}\right) - (\Omega v_{zo}\sin\theta)\left(\frac{2v_{zo}}{g}\right)^2 + \tfrac{1}{3}(\Omega g\sin\theta)\left(\frac{2v_{zo}}{g}\right)^3 \\ &= R_o + \frac{4\Omega v_{zo}\sin\theta}{g^2}\left(v_{xo}^2 - v_{zo}^2 + \frac{2}{3}v_{zo}^2\right) = R_o + \frac{4\Omega v_o^3\sin\alpha\sin\theta}{g^2}\left(\cos^2\alpha - \frac{1}{3}\sin^2\alpha\right). \end{aligned}$$

Because R_o is the actual distance of the target, the second term is the distance (east-west) by which the shell misses. Putting in the given numbers this gives +71 m. That is, the shell overshoots by 71 m to the east (in addition to being 32 m to the north).

9.33 ⋆⋆ According to Eq.(9.66), $\eta(t) = x + iy = e^{-i\Omega_z t}(C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t})$. The initial conditions are $\eta(0) = A$ and $\dot\eta(0) = 0$, which give us two equations for the two coefficients C_1 and C_2

$$C_1 + C_2 = A \quad\text{and}\quad -i\Omega_z(C_1 + C_2) + i\omega_o(C_1 - C_2) = 0.$$

These are easily solved to give

$$C_1 = \frac{A}{2}\left(1+\frac{\Omega_z}{\omega_o}\right) \approx \frac{A}{2} \quad\text{and}\quad C_2 = \frac{A}{2}\left(1-\frac{\Omega_z}{\omega_o}\right) \approx \frac{A}{2},$$

where the approximations follow because $\Omega_z/\omega_o \lesssim 10^{-4}$. (The period of a Foucault pendulum is of order 10 s, while that of the earth's spinning is 1 day $\sim 10^5$ s.)

9.34 ⋆⋆⋆ As suggested, I'll write the puck's position as $\mathbf{R} + \mathbf{r}$, where $\mathbf{R}$ points from the earth's center to P and $\mathbf{r}$ from P to the puck. Notice that $\mathbf{R}$ and $\mathbf{r}$ are perpendicular and it is certainly true that $r \ll R$. The equation of motion is

$$\ddot{\mathbf{r}} = \mathbf{g}_o(\mathbf{r}) + 2\dot{\mathbf{r}}\times\boldsymbol{\Omega} + [\boldsymbol{\Omega}\times(\mathbf{r}+\mathbf{R})]\times\boldsymbol{\Omega} \qquad \text{(x)}$$

where $\mathbf{g}_o(\mathbf{r})$ is the "true" acceleration of gravity at the position of the puck,

$$\mathbf{g}_o(\mathbf{r}) = -GM\frac{\mathbf{R}+\mathbf{r}}{|\mathbf{R}+\mathbf{r}|^3} = -GM\frac{\mathbf{R}+\mathbf{r}}{R^3}(1+r^2/R^2)^{-3/2} \approx -GM\frac{\mathbf{R}+\mathbf{r}}{R^3} = \mathbf{g}_o(0) + g_o(0)\frac{\mathbf{r}}{R},$$

where in the approximation I dropped terms of order $(r/R)^2$. Returning to the equation of motion, Eq.(x), we note that the last term (the centrifugal term) consists of two terms. The one involving $\mathbf{r}$ can be ignored (I'll justify this later) and the one involving $\mathbf{R}$ combines with $\mathbf{g}_o(0)$ to give $\mathbf{g}(0)$, the observed free-fall acceleration at P. Therefore

$$\ddot{\mathbf{r}} = \mathbf{g}(0) - g\mathbf{r}/R + 2\dot{\mathbf{r}} \times \mathbf{\Omega}.$$

[In the second term on the right, I have replaced $g_o(0)$ by $g = g(0)$, because we can ignore their tiny difference in this term, which is already small.] Bearing in mind that $\mathbf{r}$ lies in the xy plane and that $\mathbf{g}(0)$ is perpendicular to that plane, we can write down the x and y components of this equation as [the components of $\dot{\mathbf{r}} \times \mathbf{\Omega}$ are given in Eq.(9.52) if you don't want to work them out]

$$\ddot{x} = -gx/R + 2\dot{y}\Omega\cos\theta \qquad \text{and} \qquad \ddot{y} = -gy/R - 2\dot{x}\Omega\cos\theta$$

These two equations have exactly the form of the Foucault equation (9.61) except that the length of the pendulum L has been replaced by the radius of the earth.

The frequency of the puck's oscillations is $\omega_o = \sqrt{g/R} = 1.24 \times 10^{-3}\ \mathrm{s}^{-1}$, giving a period of $\tau_o = 2\pi/\omega_o \approx 5000$ s or an hour and a bit. This frequency is at least an order of magnitude greater than the frequency of precession, $\Omega_z = 7.3 \times 10^{-5}\ \mathrm{s}^{-1}$, so it makes sense to say that the puck oscillates with frequency ω_o and precesses with frequency Ω_z.

If the amplitude of oscillations is A, then the puck's speed v is of order $v \sim A\omega_o$. The three forces to be compared are

$$\text{gravitational restoring force} = mgr/R \sim mgA/R = mA\omega_o^2$$
$$\text{Coriolis force} = 2m|\mathbf{v} \times \mathbf{\Omega}| \sim 2mA\omega_o\Omega$$
$$|m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega}| \sim mA\Omega^2.$$

Since $\omega_o \gg \Omega$, this confirms that the gravitational restoring force is much bigger than the Coriolis force and that the term $m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega}$ in the centrifugal force can, indeed, be neglected.

Chapter 10

Rotational Motion of Rigid Bodies

I covered this chapter in 5 *fifty-minute lectures.*

This chapter is important because the rotational motion of rigid bodies is important. But it is also important because it is an excellent opportunity to give your students practice at handling matrices and to introduce them to the concept of a tensor, in the shape of the inertia tensor. I take for granted that they can add and multiply matrices and calculate determinants, and that they know the equation $\mathbf{Ax} = 0$ (where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{x}$ is an $n \times 1$ column) has nontrivial solutions only if $\det \mathbf{A} = 0$. Concerning tensors, I don't yet go much beyond the idea of a tensor as a 3×3 matrix. (In Chapter 15, I discuss the transformation properties of tensors under rotations of space and Lorentz transformations of space-time. I resisted the temptation to discuss tensors of rank greater than 2 anywhere.)

Section 10.1 is a final look at the separation of the motion of the CM and the rotational motion relative to the CM. The remaining sections are all devoted to rotation. Section 10.2 treats rotation about a fixed axis and introduces the moment and products of inertia. Section 10.3 treats rotation about an arbitrary axis and introduces the inertia tensor as the 3×3 matrix $\mathbf{I}$ in the relation $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$. Here I introduce the idea that a three-dimensional vector $\mathbf{L} = (L_x, L_y, L_z)$ is sometimes best thought of as a 3×1 column matrix comprising the same three elements. (Concerning notation, I urge our students to be a bit relaxed and, when there is no danger of confusion, to use the same symbol $\mathbf{L}$ for a vector and for the 3×1 matrix that represents it. In class, I sometimes used an underscore to distinguish the matrix $\underline{\mathbf{L}}$ from the vector $\mathbf{L}$ or $\vec{L}$ — a notation that is recommended in Problem 10.38.)

Section 10.4 introduces the notion of principal axes and 10.5 describes how to find them using the corresponding eigenvalue equation. Finally in Section 10.6, we're ready to discuss some applications, starting with an approximate, but hopefully instructive, discussion of the precession of a top. (This is a nice opportunity to entertain the class with a demonstration.) Sections 10.7 and 10.8 are about Euler's equations, ending with a discussion of the subtle business of the free precession of an axially symmetric body and the different rates of this precession as seen from the body and space frames. The stability or otherwise of a book thrown into the air (page 398) is a demonstration that you shouldn't miss here.

Sections 10.9 and 10.10 introduce the Euler angles (θ, ϕ, ψ). My own prejudice is that, unless your students are very well prepared, Euler angles are a subject best left to graduate school, but the majority of professors whom we surveyed were emphatic that any self-respecting text at this level should include the subject. Certainly, the topic is an important one and several students seemed to enjoy it very much. Anyway, these last two sections are classified as "omittable," and you can cover them or not as you see fit.

As always, the end-of-chapter problems are crucial. Problems 10.3 through 10.8 are about finding the center of mass of various bodies. It may seem a bit late to be worrying about this exercise, but the fact is that we haven't had much need for this skill before now, and we need it now as a prelude to calculating moments of inertia.

Solutions to Problems for Chapter 10

10.1 ⋆ From (10.4), $\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$, so

$$\sum m_\alpha \mathbf{r}'_\alpha = \sum m_\alpha \mathbf{r}_\alpha - \sum m_\alpha \mathbf{R}.$$

Now, by the definition (10.1) of the CM position, the first sum on the right is just $M\mathbf{R}$, and, by factoring the $\mathbf{R}$ from the second sum, you can see that the second term is the same. Therefore, the two terms on the right cancel, and the sum on the left is zero.

10.2 ⋆ Let us denote by T_1 the KE of the motion of the CM plus the rotational KE about the CM:

$$T_1 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}(\tfrac{1}{2}MR^2)(v/R)^2 = \tfrac{3}{4}Mv^2$$

where, for the second equality, I used $I_{\text{cm}} = \frac{1}{2}MR^2$ and $\omega = v/R$. On the other hand, the rotational KE about the instantaneous point of contact P is (Note that ω is the same either way.)

$$T_2 = \tfrac{1}{2}I_P\omega^2 = \tfrac{1}{2}(\tfrac{3}{2}MR^2)(v/R)^2 = \tfrac{3}{4}Mv^2.$$

These two are clearly equal.

10.3 ⋆ If we choose the x axis parallel to one of the base's edges, the positions of the five masses are

$$\tfrac{1}{2}(L, L, 0),\ \ \tfrac{1}{2}(L, -L, 0),\ \ \tfrac{1}{2}(-L, L, 0),\ \ \tfrac{1}{2}(-L, -L, 0),\ \ \text{and}\ \ (0, 0, H)$$

Therefore

$$X = \frac{1}{M}\sum m x_\alpha = \frac{m}{M}\sum x_\alpha = 0, \quad Y = \frac{1}{M}\sum m y_\alpha = \frac{m}{M}\sum y_\alpha = 0$$

and

$$Z = \frac{1}{M}\sum m z_\alpha = \frac{m}{M}\sum z_\alpha = \frac{1}{5}H.$$

10.4 ⋆⋆ The suggested small volume is approximately rectangular and has edges $dr, rd\theta$, and $(r\sin\theta)d\phi$. Therefore, its volume is $dV = r^2 dr(\sin\theta)d\theta d\phi$, and the claimed result follows. If the integral runs through all space, the limits are $0 \le r < \infty$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$.

10.5 ⋆⋆ The x and y coordinates are easy: For instance, $X = (1/M)\int \varrho x dV$. Since every point (x, y, z) in the hemisphere can be paired with a point $(-x, y, z)$ and the contributions from these two points exactly cancel, we conclude that $X = 0$. Similarly $Y = 0$. Finally

$$\begin{aligned} Z = \frac{1}{M}\int \varrho z dV &= \frac{\varrho}{M}\int_0^R r^2 dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi\, r\cos\theta \\ &= \frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi \;= \frac{3}{8}R \end{aligned}$$

where, in passing to the second line, I used the fact that $V = \frac{2}{3}\pi R^3$.

10.6 ⋆⋆ **(a)** I'll choose my origin at the center of the hemisphere with the hemisphere in the region $z > 0$. For the same reasons as in Problem 10.5, $X = Y = 0$, and

$$\begin{aligned} Z = \frac{1}{M}\int \varrho z dV &= \frac{\varrho}{M}\int_a^b r^2 dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi\, r\cos\theta \\ &= \frac{1}{V}\int_a^b r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi \;= \frac{3(b^4 - a^4)}{8(b^3 - a^3)} \end{aligned}$$

since $V = \frac{2}{3}\pi(b^3 - a^3)$.

(b) If $a = 0$, the answer reduces to $Z = 3b/8$, which is the correct CM position for a solid hemisphere of radius b. (See Problem 10.5.)

(c) If $b \to a$, we can write $b = a + \epsilon$ and the binomial expansion reduces the answer to $Z = (3 \times 4a^3\epsilon)/(8 \times 3a^2\epsilon) = \frac{1}{2}a$. This is the CM position of a thin hemispherical shell of radius a.

10.7 ⋆⋆ **(a)** The "rounded cone" is like an icecream cone that has been licked down until the top surface is nearly flat, but is actually part of a sphere centered on the bottom (apex) of the cone. Its volume is

$$V = \int dV = \int_0^R r^2 dr \int_0^{\theta_o} \sin\theta d\theta \int_0^{2\pi} d\phi = \tfrac{2}{3}\pi R^3(1 - \cos\theta_o)$$

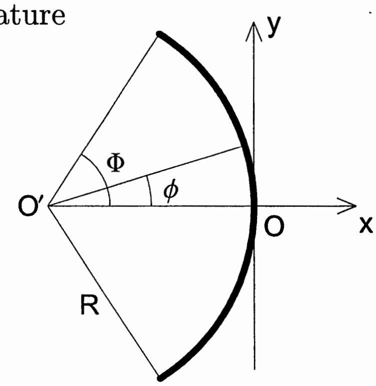
If $\theta_o = \pi$, the "cone" is a sphere and our answer becomes the familiar $V = \frac{4}{3}\pi R^3$. When $\theta_o \to 0$ the rounded top of the cone becomes flat and, with $\cos\theta_o \approx 1 - \theta_o^2/2$, our answer becomes $V = \frac{1}{3}\pi R^3\theta_o^2 = \frac{1}{3}\pi r^2 R$ where $r = R\theta_o$ is the radius of the top of the cone; this is the correct expression for the volume of an ordinary cone of radius r and height R.

(b) As in Problems 10.5 and 10.6, $X = Y = 0$ and

$$Z = \frac{1}{M}\int \varrho z dV = \frac{\varrho}{M}\int_0^R r^2 dr \int_0^{\theta_o} \sin\theta d\theta \int_0^{2\pi} d\phi\, r\cos\theta$$
$$= \frac{1}{V}\int_0^R r^3 dr \int_0^{\theta_o} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi \;=\; \frac{3R}{16}\frac{1-\cos 2\theta_o}{1-\cos\theta_o}$$

If $\theta_o = \pi$ the "cone" is a sphere and our answer becomes $Z = 0$; that is, the CM is at the origin, as expected. As $\theta_o \to 0$, the answer becomes $Z \to \frac{3}{4}R$, which is the correct distance from the apex of an ordinary cone to its CM.

10.8 ⋆⋆ I'll use a temporary origin O' at the center of curvature of the bent wire, as shown. Let's consider a short segment of wire at angle ϕ, where ϕ runs from $-\Phi$ to Φ and $2R\Phi = L$. If the segment subtends an angle $d\phi$ at O', its mass is $dm = Md\phi/2\Phi = MRd\phi/L$ and its x' coordinate (relative to the origin O') is $x' = R\cos\phi$. By symmetry, the CM has $Y = Z = 0$, and



$$X' = \frac{1}{M}\int x' dm = \frac{R^2}{L}\int_{-\Phi}^{\Phi} \cos\phi d\phi$$
$$= \frac{2R^2}{L}\sin\Phi = \frac{2R^2}{L}\sin(L/2R).$$

Thus the CM position relative to the origin O is $X = X' - R = R\left[\frac{2R}{L}\sin\left(\frac{L}{2R}\right) - 1\right]$.

If $R \to \infty$, the wire returns to its straight configuration and $L/2R \to 0$. Now, when t is small, $(1/t)\sin t \approx 1 - t^2/6$, so $X \approx R(L/2R)^2/6 \to 0$. This correctly reflects that the CM returns to the origin O.

If $2\pi R = L$, $L/2R = \pi$ and our answer reduces to $X = -R$. This is correct because the wire is now a single complete circle and the CM is at its center at $X = -R$.

10.9 ⋆ Let us put the cylinder with its axis on the z axis and use cylindrical polar coordinates. The density is $\varrho = M/V$ where $V = \pi R^2 h$, and the element of volume is $dV = \rho\, d\rho\, d\phi\, dz$. Thus

$$I_{zz} = \int \varrho\rho^2 dV = \frac{M}{V}\int_0^R \rho^3 d\rho \int_0^{2\pi} d\phi \int_0^h dz = \frac{M}{\pi R^2 h}\cdot\frac{R^4}{4}\cdot 2\pi\cdot h = \tfrac{1}{2}MR^2.$$

The products of inertia I_{xz} and I_{yz} are zero because the cylinder is axially symmetric about the z axis.

10.10 ⋆ (a) Since $\mu = M/L$, $I = \int_0^L \mu x^2 dx = \frac{M}{L}\left[x^3/3\right]_0^L = \frac{1}{3}ML^2.$

(b) $I = \int_{-L/2}^{L/2} \mu x^2 dx = \frac{M}{L}\left[x^3/3\right]_{-L/2}^{L/2} = \frac{1}{12}ML^2.$

10.11 ⋆⋆ **(a)** In terms of spherical polar coordinates, the distance from the axis out to a point (r, θ, ϕ) is $\rho = r\sin\theta$. Therefore, for a sphere of radius R,

$$I = \varrho \int \rho^2 dV = \varrho \int_0^R r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi = \frac{8\pi}{15}\varrho R^5 = \frac{2}{5}MR^2$$

since $\varrho = M/(\frac{4}{3}\pi R^3)$.

(b) If $I(b, a)$ denotes the moment of inertia of a hollow sphere of outer and inner radii b and a, then $I(b, 0)$ is that of a solid sphere of radius b and obviously $I(b, 0) = I(b, a) + I(a, 0)$. Therefore

$$I(b, a) = I(b, 0) - I(a, 0) = \frac{8\pi}{15}\varrho(b^5 - a^5) = \frac{2}{5}M\frac{b^5 - a^5}{b^3 - a^3}$$

since $\varrho = M/V = M/\left[\frac{4}{3}\pi(b^3 - a^3)\right]$.

10.12 ⋆⋆ The area of the triangular ends is $A = \sqrt{3}a^2$ and

$$I_{zz} = \frac{M}{V}\int (x^2 + y^2)dV = \frac{M}{A}\int x^2 dx \int dy + \frac{M}{A}\int dx \int y^2 dy. \qquad \text{(i)}$$

These two integrals take a little care (You need to draw a picture and decide on the limits of integration.) The result is that they are equal and

$$\int x^2 dx \int dy = \int dx \int y^2 dy = \sqrt{3}a^4/6.$$

Substituting into Eq.(i), we get $I_{zz} = \frac{1}{3}Ma^2$. The two products are zero, $I_{xz} = I_{yz} = 0$, because the prism has reflection symmetry in the xy plane.

10.13 ⋆⋆ **(a)** The equation $\dot{L}_z = \Gamma_z$ implies that $I\ddot{\phi} = -mga\sin\phi \approx -mga\phi$. This implies SHM with angular frequency $\omega = \sqrt{mga/I}$ and hence period $\tau = 2\pi\sqrt{I/(mga)}$.

(b) The period of a simple pendulum is $\omega = \sqrt{g/L}$ and these two periods are equal if $L = I/(ma)$.

10.14 ⋆⋆ **(a)** The moment of inertia of the wheel is $I_\text{w} = \frac{1}{2}M_\text{w}R_\text{w}^2 = 0.05$ kg·m^2, and, from Problem 10.11, that of the ship is $I_\text{s} = 2M_\text{s}(b^5 - a^5)/5(b^3 - a^3) = 1.23 \times 10^5$ kg·m^2. By conservation of angular momentum, $\omega_\text{s} = \omega_\text{w}I_\text{w}/I_\text{s} = 4.08 \times 10^{-4}$ rpm, and the time to turn through $\Delta\theta = 10°$ is $\Delta\theta/\omega_\text{s} = 68$ minutes.

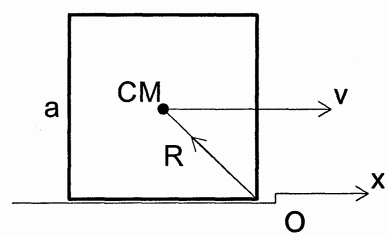
(b) The KE given to the flywheel is $T_\text{w} = \frac{1}{2}I_\text{w}\omega_\text{w}^2 = 274$ J, and, as you can easily check, that of the ship is totally negligible. If the designers of the ship were very conscientious, they could have arranged that this energy could be recouped when the turn is complete (by using it to recharge a battery, for example), but, assuming it goes to waste, the energy needed for the whole maneuver is 274 J.

10.15 ⋆⋆ **(a)** $I = \frac{2}{3}Ma^2$. [See Example 10.2, Eq.(10.47)].

(b) The initial energy of the cube is pure PE and, since the CM is a height $a/\sqrt{2}$ above the table, $E_{\text{in}} = Mga/\sqrt{2}$. The KE just before impact is $\frac{1}{2}I\omega^2$ and the PE is $Mga/2$. Therefore, by conservation of energy

$$Mg\frac{a}{\sqrt{2}} = \frac{1}{3}Ma^2\omega^2 + Mg\frac{a}{2} \quad \text{whence} \quad \omega = \sqrt{\frac{3g}{2a}(\sqrt{2}-1)}.$$

10.16 ⋆⋆ **(a)** The moment of inertia of the cube about any edge is worked out in Example 10.2 and is given by (10.47) as $\frac{2}{3}Ma^2$. During the collision, kinetic energy will be lost (the collision is inevitably inelastic), but the angular momentum L_y about the edge of the step is conserved. (I take the x direction as that of the incident velocity, as shown, and y into the page.) Just before the collision the angular momentum is $\sum m_\alpha \mathbf{r}_\alpha \times \mathbf{v} = M\mathbf{R} \times \mathbf{v}$, so that $L_y = Mav/2$. Just after the collision, the cube is rotating about the edge of the step and $L_y = I_{yy}\omega_o = \frac{2}{3}Ma^2\omega_o$. Equating these two expressions for L_y, we find that $\omega_o = 3v/(4a)$.



(b) If the initial speed is small, the cube's rotational motion about O will stop before the CM has passed the step, and the cube will fall backward. If v is big enough, the CM will pass the step and the cube will roll forward. At the critical speed that divides these possibilities, the CM will just come to rest vertically above O. Since mechanical energy is conserved in the rotational phase of motion, this critical speed is determined by the condition

$$\tfrac{1}{2}I_{yy}{\omega_o}^2 + Mga/2 = Mga/\sqrt{2}.$$

(The height of the CM above O is $a/2$ initially, and $a/\sqrt{2}$ when the CM is vertically above O.) Substituting for ω_o from part (a), we can solve for v and find $v_{\text{crit}} = [8(\sqrt{2}-1)ga/3]^{1/2}$.

10.17 ⋆⋆ The moment of inertia for rotation about the z axis is $I_{zz} = \varrho \int dx \int dy \int dz\,(x^2 + y^2)$, where the integral runs through the interior of the given ellipsoid. If we make the suggested change of variables, then $dx = a\,d\xi$ and so on, and the integral becomes

$$\begin{aligned} I_{zz} &= \varrho abc \int d\xi \int d\eta \int d\zeta\,(a^2\xi^2 + b^2\eta^2) \\ &= \varrho abc\left(a^2 \int d\xi \int d\eta \int d\zeta\,\xi^2 + b^2 \int d\xi \int d\eta \int d\zeta\,\eta^2\right) \qquad \text{(ii)} \end{aligned}$$

where the boundary for these new integrals is the unit sphere $\xi^2 + \eta^2 + \zeta^2 = 1$. Each of the two integrals in the second line can be easily evaluated. For example, in $\int d\xi \int d\eta \int d\zeta\,\xi^2$ the integral $\int d\eta \int d\zeta$ gives the area of a disk of radius $\sqrt{1-\xi^2}$, so that

$$\int d\xi \int d\eta \int d\zeta\,\xi^2 = \pi \int_{-1}^{1} d\xi (1-\xi^2)\xi^2 = \frac{4\pi}{15}.$$

The second integral in (ii) is the same, and we conclude that

$$I_{zz} = \frac{4}{15}\varrho\pi abc(a^2 + b^2) = \frac{1}{5}M(a^2 + b^2)$$

where I got the second equality by substituting $\varrho = M/(\frac{4}{3}\pi abc)$. (If you don't remember that the volume of an ellipsoid is $\frac{4}{3}\pi abc$, you can find it by doing the integral $\int dV$ using the same changes of variables.) Notice that for a sphere of radius R this reduces to the familiar $I_{zz} = \frac{2}{5}MR^2$.

10.18 ⋆⋆⋆ **(a)** The angular momentum (about the pivot) just after impact is $L = \Gamma dt = Fbdt = \xi b$. This is the same as $I\omega$, so $\omega = \xi b/I$. Therefore the CM velocity is $v_{\rm cm} = a\omega = a\xi b/I$, and the total momentum is $P = Mv_{\rm cm} = Ma\xi b/I$.

(b) The total impulse delivered to the rod is $\xi + \eta$ and this is equal to the total momentum $P = Ma\xi b/I$. Therefore, $\eta = P - \xi = (Mab/I - 1)\xi$.

(c) The impulse at the pivot is zero if and only if $Mab/I = 1$, so the sweet spot is at $b_{\rm o} = I/Ma$.

10.19 ⋆ **(a)** Because

$$\boldsymbol{\omega} = (\omega_x, \omega_y\omega_z) \qquad \text{and} \qquad \mathbf{r} = (x, y, z)$$

it follows that

$$\boldsymbol{\omega} \times \mathbf{r} = \big(\omega_y z - \omega_z y,\ \omega_z x - \omega_x z,\ \omega_x y - \omega_y x\big)$$

and hence

$$[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_x = y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z) = (y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z. \qquad \text{(iii)}$$

This is exactly the x component of Eq.(10.35), and, since the other two components work the same way, we've proved (10.35).

(b) By the $BAC - CAB$ rule, $\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}(\mathbf{r}\cdot\mathbf{r}) - \mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega})$, so

$$[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_x = \omega_x(x^2 + y^2 + z^2) - x(x\omega_x + y\omega_y + z\omega_z).$$

On the right side, the first and fourth terms cancel and the remaining four reproduce the right side of Eq.(iii). Once again the other two components work the same way, and we've proved (10.35).

10.20 ⋆ **(a)** If we use A to denote the set of all points of the first body, then A is the union $A = B \cup C$ and, from the definition (10.37),

$$I^A_{xx} = \sum_{\alpha\in A} m_\alpha(y_\alpha^2 + z_\alpha^2) = \sum_{\alpha\in B} m_\alpha(y_\alpha^2 + z_\alpha^2) + \sum_{\alpha\in C} m_\alpha(y_\alpha^2 + z_\alpha^2) = I^B_{xx} + I^C_{xx}.$$

The other two diagonal elements of $\mathbf{I}$ work in the same way, as do the six off-diagonal elements as defined by Eq.(10.38). Therefore, $\mathbf{I}^A = \mathbf{I}^B + \mathbf{I}^C$, as claimed.

(b) If A is the result of removing C from B, then $B = A \cup C$ and, by part (a), $\mathbf{I}^B = \mathbf{I}^A + \mathbf{I}^C$, whence $\mathbf{I}^A = \mathbf{I}^B - \mathbf{I}^C$.

10.21 ⋆⋆ Let us start from the given expression and show that it's elements *are* equal to the usual definitions (10.37) and (10.38). First

$$I_{xx} = \int \varrho(r^2 - xx)dV = \int \varrho(x^2 + y^2)dV$$

which is exactly the definition (10.37) (in integral form). The two other diagonal elements work the same way. Meanwhile,

$$I_{xy} = \int \varrho(0 - xy)dV = -\int \varrho(xy)dV$$

which is exactly the definition (10.38). The other off-diagonal elements all work the same way, and we've shown that the proposed definition agrees with the original one.

10.22 ⋆⋆ **(a)** From (10.37), $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2) = m\left(\sum y_\alpha^2 + \sum z_\alpha^2\right)$. In the first sum in the last expression, four of the points lie in the plane $y = 0$, while the other four have $y_\alpha = a$; thus this first sum is $4a^2$. The same applies to the second sum, and we conclude that $I_{xx} = 8ma^2$. The other two diagonal elements are clearly the same. Similarly, from (10.38), $I_{xy} = -m\sum x_\alpha y_\alpha$. In this sum, four of the points lie in the plane $x = 0$, and of the remaining four points, two lie in the plane $y = 0$. This leaves two points, both with $x_\alpha = y_\alpha = a$. Thus $I_{xy} = -2ma^2$. All the remaining off-diagonal elements are the same, and the inertia tensor is as shown on the left below.

$$\mathbf{I}(\text{part a}) = ma^2\begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{I}(\text{part b}) = ma^2\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) As in part (a), $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2) = m\left(\sum y_\alpha^2 + \sum z_\alpha^2\right)$, but now all eight terms in both sums are the same and equal to $(a/2)^2$. Therefore $I_{xx} = 4ma^2 = I_{yy} = I_{zz}$. Because the body has reflection symmetry in all three coordinate planes, all of the off-diagonal elements are zero, and the inertia tensor is as shown above right.

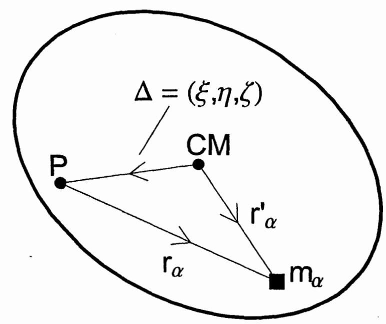
10.23 ⋆⋆ Since the whole body lies in the plane $z = 0$, the four products of inertia involving z are all zero, $I_{xz} = I_{yz} = I_{zx} = I_{zy} = 0$. For example,

$$I_{xz} = -\sum m_\alpha x_\alpha z_\alpha = 0.$$

For the same reason,

$$I_{xx} + I_{yy} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2) + \sum m_\alpha(z_\alpha^2 + x_\alpha^2) = \sum m_\alpha(x_\alpha^2 + y_\alpha^2) = I_{zz}.$$

10.24 ⋆⋆ **(a)** For rotation about P, the moment of inertia $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2)$. From the picture, you can see that $\mathbf{r}_\alpha = \mathbf{r}'_\alpha - \boldsymbol{\Delta}$, so that $x_\alpha = x'_\alpha - \xi$ and so on. Therefore



$$
\begin{aligned}
I_{xx} &= \sum m_\alpha[(y'_\alpha - \eta)^2 + (z'_\alpha - \zeta)^2] \\
&= \sum m_\alpha({y'_\alpha}^2 + {z'_\alpha}^2) + \sum m_\alpha(\eta^2 + \zeta^2) \\
&\quad - 2\eta \sum m_\alpha y'_\alpha - 2\zeta \sum m_\alpha z'_\alpha .
\end{aligned}
$$

The first sum on the second line is just $I^{\rm cm}_{xx}$. The second is $M(\eta^2 + \zeta^2)$, and the last two are zero by (10.7). Thus

$$I_{xx} = I^{\rm cm}_{xx} + M(\eta^2 + \zeta^2) \tag{iv}$$

as claimed. The other two diagonal elements work the same way, as do the six off-diagonal terms; for instance,

$$I_{yz} = I^{\rm cm}_{yz} - M\eta\zeta. \tag{v}$$

(b) In Example 10.2(b) we found $\mathbf{I}^{\rm cm}$ for a cube in (10.52), which gives

$$I^{\rm cm}_{xx} = \tfrac{1}{6}Ma^2 \qquad \text{and} \qquad I^{\rm cm}_{yz} = 0.$$

In part (a) of the same example, we found $\mathbf{I}$ for for the same cube rotating about a corner, which is displaced from the CM by $\boldsymbol{\Delta} = (-a/2, -a/2, -a/2)$. There we found in (10.49)

$$I_{xx} = \tfrac{2}{3}Ma^2 = \tfrac{1}{6}Ma^2 + 2M(-a/2)^2 \qquad \text{and} \qquad I_{yz} = -\tfrac{1}{4}Ma^2 = 0 - M(-a/2)(-a/2).$$

As you can easily see these are precisely the relations (iv) and (v) with $\eta = \zeta = -a/2$.

10.25 ⋆⋆ **(a)** As in (10.45)

$$I_{xx} = \int_{-a}^{a} dx \int_{-b}^{b} dy \int_{-c}^{c} dz\, \varrho\,(y^2 + z^2).$$

This is the sum of two terms, each of which is easily evaluated. For example, the first term is

$$\varrho \int_{-a}^{a} dx \int_{-b}^{b} y^2\, dy \int_{-c}^{c} dz = \frac{8}{3}\varrho a b^3 c = \frac{1}{3}Mb^2.$$

The second term works the same way, and we conclude that $I_{xx} = \frac{1}{3}M(b^2 + c^2)$, with corresponding results for I_{yy} and I_{zz}. Since the plane $z = 0$ is a plane of reflection symmetry, it follows that the four off-diagonal elements $I_{xz}, \cdots$ that involve z are all zero. Since the other two planes are likewise symmetry planes, we see that all six off-diagonal elements are zero. Thus the the whole inertia tensor (which I'll rename $\mathbf{I}^{\rm cm}$ since it is for rotation about the CM) is

$$\mathbf{I}^{\rm cm} = \frac{1}{3}M \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$$

(b) The "generalized parallel axis theorem" gives the inertia tensor $\mathbf{I}$ with respect to axes through an arbitrary pivot, in terms of the tensor $\mathbf{I}^{\text{cm}}$ referred to parallel axes through the CM. The displacement $\boldsymbol{\Delta} = (\xi, \eta, \zeta)$ from the CM to the corner A is just (a, b, c). Thus, $I_{xx} = I_{xx}^{\text{cm}} + M(b^2 + c^2) = \frac{4}{3}M(b^2 + c^2)$, with similar results for the other two diagonal elements. The off-diagonal elements of $\mathbf{I}^{\text{cm}}$ are all zero, so $I_{yz} = -Mbc$, and so on. Thus the whole inertia tensor with respect to A is

$$\mathbf{I} = \frac{1}{3}M \begin{bmatrix} 4(b^2+c^2) & -3ab & -3ac \\ -3ba & 4(c^2+a^2) & -3bc \\ -3ca & -3cb & 4(a^2+b^2) \end{bmatrix}.$$

(c) If $\boldsymbol{\omega}$ is parallel to the x axis, then $\boldsymbol{\omega}$ has components ω, 0, and 0 and, carrying out the indicated matrix multiplication, we find that $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ has components $\frac{4}{3}M(b^2 + c^2)\omega$, $-Mab\omega$, and $-Mac\omega$.

10.26 $\star\star$ **(a)** Consider a small volume element between ρ and $\rho + d\rho$, between ϕ and $\phi + d\phi$, and between z and $z + dz$. As long as it is small in all three directions, it is approximately rectangular and has volume $dV = (d\rho)\cdot(\rho d\phi)\cdot(dz)$. Thus $\int dV\, f = \int \rho\, d\rho \int d\phi \int dz\, f$. Note that the order in which we do the three integrals doesn't matter, as long as we're careful with the limits.

(b)

$$I_{zz} = \varrho \int \rho^2\, dV = \frac{M}{V} \int_0^h dz \int_0^{2\pi} d\phi \int_0^{Rz/h} \rho^3 d\rho. \tag{vi}$$

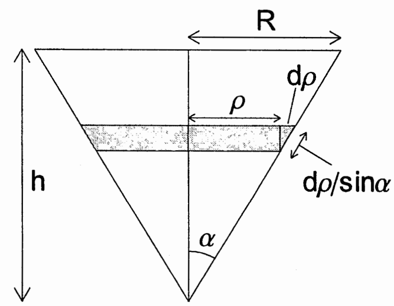
The limits on these three integrals can be understood with reference to Figure 10.6: If we do the z integration last, then clearly z runs from 0 to h; in any case, ϕ runs from 0 to 2π; and with z and ϕ fixed ρ runs from the z axis ($\rho = 0$) out to a radius $r = Rz/h$. If we now replace V by $V = \frac{1}{3}\pi R^2 h$, and do the ϕ integral ($= 2\pi$) and the ρ integral ($= \frac{1}{4}R^4 z^4/h^4$), we get

$$I_{zz} = \frac{3MR^2}{2h^5} \int_0^h z^4 dz = \frac{3MR^2}{10}.$$

(c) $I_{xx} = \varrho \int (y^2 + z^2) dV = \varrho \int y^2 dV + \varrho \int z^2 dV$. If you compare the first term here with the middle expression of Eq.(vi), you will see that the former is just half the latter. (Remember $\rho^2 = x^2 + y^2$.) Thus

$$I_{xx} = \tfrac{1}{2} I_{zz} + \frac{M}{V} \int_0^h z^2 dz \int_0^{2\pi} d\phi \int_0^{Rz/h} \rho\, d\rho = \frac{3MR^2}{20} + \frac{3Mh^2}{5}.$$

10.27 ⋆⋆⋆ Let the mass density of the cone be σ (mass/area) and let us use the cylindrical coordinates ρ and ϕ to specify positions on the cone. (We could equally use z and ϕ, but ρ and ϕ are marginally more convenient.) If we imagine dividing the surface into strips as shown, and the strips into small increments of angle $d\phi$, then the element of area is $dA = (d\rho/\sin\alpha)(\rho\, d\phi)$. The moment about the z axis is then



$$I_{zz} = \int \sigma(x^2+y^2)\, dA = \sigma \int_0^R \int_0^{2\pi} \rho^2 \frac{\rho\, d\rho\, d\phi}{\sin\alpha} = \frac{\sigma\pi R^4}{2\sin\alpha} \tag{vii}$$

since the ϕ integral is just 2π and the ρ integral is $R^4/4$. The area of the cone is $A = \pi R^2/\sin\alpha$. (You can check this by doing the integral $A = \int dA$ if you want.) Thus $\sigma\pi R^2/\sin\alpha = M$, the total mass. Therefore, $I_{zz} = MR^2/2$.

The other two moments, I_{xx} and I_{yy}, are equal by rotational symmetry, and

$$I_{xx} = \int \sigma(y^2+z^2)\, dA.$$

The first term here is the same as the second term in (vii), where the two terms are equal (by rotational symmetry again). Thus the first term of I_{xx} is just half of I_{zz}. In the second term of I_{xx} we can replace z by $\rho h/R$ (by similar triangles), and we see that the second term is h^2/R^2 times I_{zz}. Putting these together, we find that

$$I_{xx} = I_{yy} = (\tfrac{1}{2} + h^2/R^2)I_{zz} = \tfrac{1}{4}M(R^2+2h^2).$$

Finally, all of the off-diagonal terms are zero by rotational symmetry about the z axis. Therefore

$$\mathbf{I} = \frac{1}{4}M\begin{bmatrix} (R^2+2h^2) & 0 & 0 \\ 0 & (R^2+2h^2) & 0 \\ 0 & 0 & 2R^2 \end{bmatrix}.$$

10.28 ⋆⋆⋆ We know from Problem 10.12 (see that solution) the $I_{zz} = \frac{1}{3}Ma^2$. The reflection symmetry in the xy plane ensures that $I_{xz} = I_{yz} = 0$ and the reflection symmetry in the yz plane that $I_{xy} = 0$. Next

$$I_{xx} = \varrho \int (y^2+z^2)dV = \frac{M}{V}\int y^2 dV + \frac{M}{V}\int z^2 dV. \tag{viii}$$

Comparing with the solution to Problem 10.12, you can see that the first term here is equal to $\frac{1}{2}I_{zz} = \frac{1}{6}Ma^2$. The integral in the second term is easily evalated to give $Ah^3/12 = Vh^2/12$. Therefore $I_{xx} = \frac{1}{6}Ma^2 + \frac{1}{12}Mh^2$. To find I_{yy} we simply replace y by x in Eq,(viii), and we find that $I_{yy} = I_{xx}$. Thus

$$\mathbf{I} = \frac{1}{12}M\begin{bmatrix} 4a^2 & 0 & 0 \\ 0 & 2a^2h^2 & 0 \\ 0 & 0 & 2a^2h^2 \end{bmatrix}.$$

10.29 ⋆ That Ox is a principal axis means that if $\boldsymbol{\omega}$ is along Ox then $\mathbf{L}$ is also along Ox. If we think in terms of matrices, this says that if $\boldsymbol{\omega}$ is a column with entries $\omega, 0, 0$, then $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ is a column whose second and third entries are also zero. This requires that $I_{yx} = I_{zx} = 0$. Similarly, that Oy and Oz are principal axes requires that $I_{xy} = I_{zy} = 0$ and $I_{xz} = I_{yz} = 0$. This leaves $\mathbf{I}$ with I_{xx}, I_{yy}, and I_{zz} down the diagonal and zeroes everywhere else.

10.30 ⋆ Choose the xy plane to contain the lamina. Then, according to Problem 10.23,

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

If $\boldsymbol{\omega}$ points along the z axis, then $\boldsymbol{\omega}$ has components $(0, 0, \omega)$, and $\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = (0, 0, I_{zz}\omega)$ (both to be thought of as column vectors), which also points along the z axis. Therefore the z axis is a principal axis.

10.31 ⋆⋆ **(a)** If the z axis is an axis of rotational symmetry, then we saw in Example 10.1(c) that $I_{xz} = I_{yz} = 0$. Therefore, if $\boldsymbol{\omega} = (0, 0, \omega)$, then $\mathbf{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega) = (0, 0, I_{zz}\omega)$ and $\hat{\mathbf{z}}$ is a principal axis.

(b) Let us choose any two perpendicular directions in the plane $z = 0$ as our x and y axes. Because $\hat{\mathbf{z}}$ is an axis of rotational symmetry, the xz plane is a plane of reflection symmetry, and this implies that $I_{xy} = 0$ [Example 10.1(b)]. Therefore all off-diagonal elements of $\mathbf{I}$ are zero, and the x and y axes are principal axes.

(c) $I_{xx} = \int \varrho(y^2 + z^2)dV$ and $I_{yy} = \int \varrho(x^2 + z^2)dV$. Now consider a rotation through 90 degrees about the z axis. On the one hand this interchanges y and x (but leaves z unchanged), but, because of the rotational symmetry, it doesn't change the body at all. Therefore, the two integrals are equal and $I_{xx} = I_{yy}$; that is, $\lambda_1 = \lambda_2$.

10.32 ⋆⋆ **(a)** Let us choose the principal axes as our coordinate directions. Then

$$\lambda_1 = \int \varrho(y^2 + z^2)dV \quad \text{and} \quad \lambda_2 = \int \varrho(x^2 + z^2)dV.$$

Adding these two equations, we get

$$\lambda_1 + \lambda_2 = \int \varrho(x^2 + y^2)dV + 2\int \varrho z^2 dV \geq \int \varrho(x^2 + y^2)dV = \lambda_3.$$

(b) The "$\geq$" in the above relations is an "$=$" if and only if all parts of the body have $z = 0$, that is, the body is a lamina lying in the plane $z = 0$.

10.33 ⋆⋆⋆ **(a)** First

$$T = \tfrac{1}{2}\sum m_\alpha v_\alpha^2 = \tfrac{1}{2}\sum m_\alpha(\boldsymbol{\omega}\times\mathbf{r}_\alpha)^2. \qquad \text{(ix)}$$

Now, for any two vectors, $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$ and $|\mathbf{a}\times\mathbf{b}| = ab\sin\theta$, so $(\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{a}\times\mathbf{b})^2 = a^2b^2$, which is the required identity. Substituting into (ix), we get the required result

$$T = \tfrac{1}{2}\sum m_\alpha[\omega^2 r_\alpha^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)^2]. \qquad \text{(x)}$$

(b)

$$\begin{aligned}\mathbf{L} &= \sum m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha = \sum m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega}\times\mathbf{r}_\alpha)\\ &= \sum m_\alpha[\boldsymbol{\omega}(\mathbf{r}_\alpha\cdot\mathbf{r}_\alpha) - \mathbf{r}_\alpha(\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)]. \qquad \text{(xi)}\end{aligned}$$

(c) If we dot Equation (xi) with $\boldsymbol{\omega}$ we find

$$\boldsymbol{\omega}\cdot\mathbf{L} = \sum m_\alpha[\omega^2 r_\alpha^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)^2] = 2T,$$

as given by (x). Therefore,

$$T = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}. \qquad \text{(xii)}$$

The second identity concerns the translation between vector and matrix notations. It may help to have a temporary notation that distinguishes between vectors and matrices. Let us agree to identify matrices with an underscore. Thus if $\mathbf{r} = (x, y, z)$, I shall write

$$\underline{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Similarly, I shall write the inertia tensor as $\underline{\mathbf{I}}$ and the angular momentum (column vector) as $\underline{\mathbf{L}} = \underline{\mathbf{I}}\,\underline{\boldsymbol{\omega}}$. To see what the vector dot product looks like in matrix notation, consider the following (as usual, a tilde denotes the transpose):

$$\underline{\tilde{\mathbf{a}}}\,\underline{\mathbf{b}} = [a_x\ a_y\ a_z]\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z = \mathbf{a}\cdot\mathbf{b}.$$

In other words, the vector dot product $\mathbf{a}\cdot\mathbf{b}$ is the same as the matrix product $\underline{\tilde{\mathbf{a}}}\,\underline{\mathbf{b}}$ of the row representing $\mathbf{a}$ and the column representing $\mathbf{b}$. Substituting into Eq.(xii), we see that $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\underline{\tilde{\boldsymbol{\omega}}}\,\underline{\mathbf{I}}\,\underline{\boldsymbol{\omega}}$, as claimed.

(d) With respect to the principal axes $\mathbf{I}$ is a diagonal matrix with the principal moments λ_1, λ_2, and λ_3 on the diagonal. Therefore $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ is the column with elements $\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3$ and $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$.

10.34 ⋆ The matrix $\mathbf{I}$ is given in Eq.(10.72) and $\mathbf{I} - \lambda\mathbf{1}$ in the next equation. Thus

$$\det(\mathbf{I} - \lambda\mathbf{1}) = \begin{vmatrix} 8\mu-\lambda & -3\mu & -3\mu \\ -3\mu & 8\mu-\lambda & -3\mu \\ -3\mu & -3\mu & 8\mu-\lambda \end{vmatrix} = (2\mu-\lambda)\begin{vmatrix} 1 & -3\mu & -3\mu \\ 1 & 8\mu-\lambda & -3\mu \\ 1 & -3\mu & 8\mu-\lambda \end{vmatrix}$$

$$= (2\mu - \lambda) \begin{vmatrix} 1 & -3\mu & -3\mu \\ 0 & 11\mu - \lambda & 0 \\ 0 & 0 & 11\mu - \lambda \end{vmatrix} = (2\mu - \lambda)(11\mu - \lambda)^2$$

where, for the second equality, I added the second and third columns to the first and factored out the resulting $(2\mu - \lambda)$, and, in moving to the second line, I subtracted the first row from both the second and third rows.

10.35 ⋆⋆ **(a)**

$$I_{xx} = \sum m_\alpha (y_\alpha^2 + z_\alpha^2) = 0 + 2m(a^2 + a^2) + 3m(a^2 + a^2) = 10ma^2$$

$$I_{yy} = \sum m_\alpha (z_\alpha^2 + x_\alpha^2) = m(a^2) + 2m(a^2) + 3m(a^2) = 6ma^2 = I_{zz}$$

$$I_{yz} = -\sum m_\alpha (y_\alpha z_\alpha) = 0 - 2m(a^2) - 3m(-a^2) = ma^2$$

$$I_{zx} = -\sum m_\alpha (z_\alpha x_\alpha) = 0 + 0 + 0 = 0 = I_{xy}$$

Therefore

$$\mathbf{I} = ma^2 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

(b) For convenience write the eigenvalues of $\mathbf{I}$ as $\lambda = ma^2\lambda'$. Then

$$(\mathbf{I} - \lambda\mathbf{1}) = ma^2 \begin{bmatrix} 10 - \lambda' & 0 & 0 \\ 0 & 6 - \lambda' & 1 \\ 0 & 1 & 6 - \lambda' \end{bmatrix}. \tag{xiii}$$

This has determinant $\det(\mathbf{I} - \lambda\mathbf{1}) = (ma^2)^3(10 - \lambda')(7 - \lambda')(5 - \lambda')$. Therefore the three eigenvalues (that is, the three principal moments) are

$$\lambda_1 = 10ma^2, \quad \lambda_2 = 7ma^2, \quad \text{and} \quad \lambda_3 = 5ma^2$$

To find the corresponding principal axes, we must substitute λ_1, λ_2, and λ_3 in turn into (xiii) and solve the equation $(\mathbf{I} - \lambda\mathbf{1})\mathbf{a} = 0$. For the first eigenvalue, this gives three equations, $0 = 0$, $-4a_2 + a_3 = 0$, and $a_2 - 4a_3 = 0$. Therefore, $a_2 = a_3 = 0$ and $\mathbf{a} = (a_1, 0, 0)$. Thus for a unit vector in the direction of the first principal axis we can take $\mathbf{e}_1 = (1, 0, 0)$; that is, the first principal axis is the x axis. The other two principal axis are found in the same way to be

$$\mathbf{e}_2 = \tfrac{1}{\sqrt{2}}(0, 1, 1) \quad \text{and} \quad \mathbf{e}_3 = \tfrac{1}{\sqrt{2}}(0, 1, -1).$$

In this case, the second axis points toward the mass $2m$, and the third toward the mass $3m$.

10.36 ⋆⋆ **(a)** The three masses are equal, $m_1 = m_2 = m_3 = m$ and their positions are

$$\mathbf{r}_1 = a(1,0,0), \quad \mathbf{r}_2 = a(0,1,2), \quad \text{and} \quad \mathbf{r}_3 = a(0,2,1).$$

Therefore

$$\left.\begin{aligned}
I_{xx} &= \textstyle\sum m_\alpha(y_\alpha^2 + z_\alpha^2) = ma^2(0+5+5) = 10ma^2 \\
I_{yy} &= \textstyle\sum m_\alpha(x_\alpha^2 + z_\alpha^2) = ma^2(1+4+1) = 6ma^2 \\
I_{zz} &= \textstyle\sum m_\alpha(x_\alpha^2 + y_\alpha^2) = ma^2(1+1+4) = 6ma^2 \\
I_{xy} &= -\textstyle\sum m_\alpha x_\alpha y_\alpha = -ma^2(0+0+0) = 0 \\
I_{xz} &= -\textstyle\sum m_\alpha x_\alpha z_\alpha = -ma^2(0+0+0) = 0 \\
I_{yz} &= -\textstyle\sum m_\alpha y_\alpha z_\alpha = -ma^2(0+2+2) = -4ma^2
\end{aligned}\right\} \quad \text{or } \mathbf{I} = 2ma^2 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

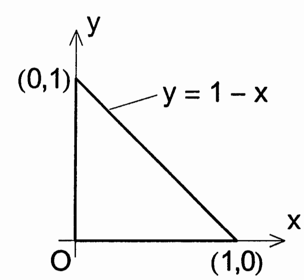
(b) As you can check, the characteristic equation is

$$\det(\mathbf{I} - \lambda\mathbf{1}) = (10ma^2 - \lambda)^2(2ma^2 - \lambda) = 0$$

Therefore, the principal moments are $\lambda_1 = \lambda_2 = 10ma^2$ and $\lambda_3 = 2ma^2$. If we set $\lambda = 10ma^2$, the equation $(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$ yields three equations, $0 = 0$, $\omega_2 + \omega_3 = 0$, and $\omega_2 + \omega_3 = 0$, of which only one is independent. Thus there are two independent eigenvectors with $\lambda = 10ma^2$, which we can take to be $\mathbf{e}_1 = (1,0,0)$ and $\mathbf{e}_2 = (0,1,-1)/\sqrt{2}$ or any other two perpendicular directions in the plane of these two. If we set $\lambda = 2ma^2$, the equation $(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$ yields three equations, $\omega_1 = 0$, $\omega_2 - \omega_3 = 0$, and $-\omega_2 + \omega_3 = 0$. There is just one independent eigenvector with $\lambda = 2ma^2$, which we can take to be $\mathbf{e}_3 = (0,1,1)/\sqrt{2}$.

10.37 ⋆⋆⋆ **(a)** Since all mass is confined to the plane $z = 0$,

$$I_{xx} = \int \sigma y^2 dA = \sigma \int_0^1 dx \int_0^{1-x} y^2 dy = \sigma/12 = 2.$$



I_{yy} is the same, and from Problem 10.23, we know that

$$I_{zz} = I_{xx} + I_{yy} = 4.$$

I_{xz} and I_{yz} are both zero, and

$$I_{xy} = -\int \sigma xy\, dA = -\sigma \int_0^1 x\, dx \int_0^{1-x} y\, dy = -1.$$

Therefore

$$\mathbf{I} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) The characteristic equation is $\det(\mathbf{I} - \lambda\mathbf{1}) = (1-\lambda)(3-\lambda)(4-\lambda) = 0$, so the principal moments are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 4$. If we set $\lambda = \lambda_1 = 1$, the equation $(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$ implies the three equations $\omega_1 - \omega_2 = 0$, $\omega_1 - \omega_2 = 0$, and $\omega_3 = 0$; thus, the corresponding principal direction is $\mathbf{e}_1 = (1,1,0)/\sqrt{2}$. Setting $\lambda = \lambda_2$ and $\lambda = \lambda_3$ in turn, we can similarly find the other two principal directions to be $\mathbf{e}_2 = (1,-1,0)/\sqrt{2}$ and $\mathbf{e}_3 = (0,0,1)$.

10.38 ★★★ **(a)** I shall use the notation suggested, with matrices indicated with an underline. In particular, the (3×1) column representing a vector $\mathbf{a}$ is $\underline{\mathbf{a}}$, and the scalar product $\mathbf{a}\cdot\mathbf{b}$ of two vectors is the matrix product $\tilde{\underline{\mathbf{a}}}\,\underline{\mathbf{b}}$. Now consider the following: Since $\underline{\mathbf{I}}\,\underline{\mathbf{e}}_i = \lambda_i\,\underline{\mathbf{e}}_i$, it follows that

$$\tilde{\underline{\mathbf{e}}}_j\,\underline{\mathbf{I}}\,\underline{\mathbf{e}}_i = \tilde{\underline{\mathbf{e}}}_j\,\lambda_i\,\underline{\mathbf{e}}_i = \lambda_i \mathbf{e}_j\cdot\mathbf{e}_i\,. \tag{xiv}$$

On the other hand, since the left side of Eq.(xiv) is a number, it is equal to its own transpose, and, since $(\underline{\mathbf{A}}\,\underline{\mathbf{B}}\,\underline{\mathbf{C}})^{\sim} = \tilde{\underline{\mathbf{C}}}\,\tilde{\underline{\mathbf{B}}}\,\tilde{\underline{\mathbf{A}}}$,

$$\tilde{\underline{\mathbf{e}}}_j\,\underline{\mathbf{I}}\,\underline{\mathbf{e}}_i = \tilde{\underline{\mathbf{e}}}_i\,\tilde{\underline{\mathbf{I}}}\,\underline{\mathbf{e}}_j = \lambda_j \mathbf{e}_i\cdot\mathbf{e}_j \tag{xv}$$

where in the last step I have used the symmetry of $\underline{\mathbf{I}}$ (that is, $\tilde{\underline{\mathbf{I}}} = \underline{\mathbf{I}}$) and that $\underline{\mathbf{e}}_j$ is an eigenvector of $\underline{\mathbf{I}}$ with eigenvalue λ_j. Comparing (xiv) and (xv), we see that

$$(\lambda_i - \lambda_j)\mathbf{e}_i\cdot\mathbf{e}_j = 0.$$

Thus if $\lambda_i \neq \lambda_j$, it follows that $\mathbf{e}_i\cdot\mathbf{e}_j = 0$.

(b) If all three eigenvalues are different, then it follows the all three eigenvectors are orthogonal. Now suppose that $\mathbf{e}_1'$ was a *different* eigenvector with the same eigenvalue λ_1. By part (a), both $\mathbf{e}_1$ and $\mathbf{e}_1'$ have to be orthogonal to $\mathbf{e}_2$ and $\mathbf{e}_3$. Therefore $\mathbf{e}_1$ and $\mathbf{e}_1'$ must point along the same direction.

(c) If $\lambda_1 = \lambda_2 = \lambda$, say, the proof of orthogonality in part (a) breaks down. In this case,

$$\underline{\mathbf{I}}\,\underline{\mathbf{e}}_1 = \lambda\,\underline{\mathbf{e}}_1 \quad \text{and} \quad \underline{\mathbf{I}}\,\underline{\mathbf{e}}_2 = \lambda\,\underline{\mathbf{e}}_2$$

with the *same* eigenvalue λ. If $\mathbf{a}$ is any vector in the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$, then $\mathbf{a} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2$ and likewise for the corresponding (3×1) matrices. Therefore

$$\underline{\mathbf{I}}\,\underline{\mathbf{a}} = \underline{\mathbf{I}}\,(\alpha\,\underline{\mathbf{e}}_1 + \beta\,\underline{\mathbf{e}}_2) = \alpha\,\underline{\mathbf{I}}\,\underline{\mathbf{e}}_1 + \beta\,\underline{\mathbf{I}}\,\underline{\mathbf{e}}_2 = \alpha\lambda\,\underline{\mathbf{e}}_1 + \beta\lambda\,\underline{\mathbf{e}}_2 = \lambda\,\underline{\mathbf{a}}.$$

We conclude that any vector in the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$ is an eigenvector of $\underline{\mathbf{I}}$ with the same eigenvalue λ. That is, any direction in the plane is a principal axis with same principal moment. Thus, if $\mathbf{e}_1$ and $\mathbf{e}_2$ are not orthogonal, we are free to use any other two vectors in the plane that are.

(d) *Any* vector $\mathbf{a}$ can be written as a linear combination of the three independent vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. If $\lambda_1 = \lambda_2 = \lambda_3$, then the same argument just given shows that $\mathbf{a}$ is an eigenvector with the same eigenvalue. That is, any direction is a principal axis with the same principal moment.

10.39 ★ According to Eq.(10.83), the rate of precession is $\Omega = MgR/(\lambda_3\omega)$, where R is the distance from the tip to the CM of the cone, $R = \frac{3}{4}h$, and λ_3 is the moment of inertia about the cone's axis, $\lambda_3 = \frac{3}{10}Mr^2$ [Eq.(10.59)]. Therefore $\Omega = 5gh/(2r^2\omega) = 21$ rad/s ≈ 200 rpm.

10.40 ★★ **(a)** Multipying the first of Equations (10.86) by $\lambda_1\omega_1$, the left side becomes $\lambda_1^2\omega_1\dot{\omega}_1$, which is the same as $\frac{1}{2}d(\lambda_1^2\omega_1^2)/dt$. Therefore

$$\frac{d}{dt}\left(\lambda_1^2\omega_1^2\right) = 2\lambda_1(\lambda_2 - \lambda_3)\omega_1\omega_2\omega_3.$$

Similarly, the second and third equations give

$$\frac{d}{dt}\left(\lambda_2^2\omega_2^2\right) = 2\lambda_2(\lambda_3 - \lambda_1)\omega_1\omega_2\omega_3 \qquad \text{and} \qquad \frac{d}{dt}\left(\lambda_3^2\omega_3^2\right) = 2\lambda_3(\lambda_1 - \lambda_2)\omega_1\omega_2\omega_3.$$

Adding these three equations and remembering that $\mathbf{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$, we find that $d\mathbf{L}^2/dt = 0$.

(b) If, instead, we multiply the first of Equations (10.88) by ω_1, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\lambda_1\omega_1^2\right) = (\lambda_2 - \lambda_3)\omega_1\omega_2\omega_3.$$

Adding this to the corresponding two equations for the second and third components, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right) = \frac{d}{dt}T_{\text{rot}} = 0.$$

10.41 ⋆⋆ Choose axes 1, 2, and 3 to be the principal axes through O, with axis 3 perpendicular to the lamina. (By Problem 10.30, this *is* a principal axis.) By Problem 10.23, the corresponding principal moments satisfy $\lambda_1 + \lambda_2 = \lambda_3$. The component of $\boldsymbol{\omega}$ in the plane of the lamina has magnitude squared $\omega_1^2 + \omega_2^2$, whose time derivative is

$$\frac{d}{dt}(\omega_1^2 + \omega_2^2) = \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2. \tag{xvi}$$

Now, by Euler's equations,

$$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3 = -\lambda_1\omega_2\omega_3$$

where the second equality follows from the result of Problem 10.23. Thus,

$$\dot{\omega}_1 = -\omega_2\omega_3 \qquad \text{and similarly} \qquad \dot{\omega}_2 = +\omega_3\omega_1.$$

Substituting into Eq.(xvi), we conclude that

$$\frac{d}{dt}(\omega_1^2 + \omega_2^2) = -\omega_1\omega_2\omega_3 + \omega_2\omega_3\omega_1 = 0.$$

10.42 ⋆ The inertia tensor for the book (sides $a = 30$, $b = 20$, and $c = 3$, all in cm) can be evaluated as in Example 10.2. With the origin at the CM, all off-diagonal elements are zero, and the diagonal elements (which are the principal moments) are $\lambda_1 = M(b^2 + c^2)/12$, and so on. If the book's spin axis is close to the shortest symmetry axis (the z axis), then according to (10.91) the frequency of wobble is given by

$$\begin{aligned}\Omega^2 = \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1\lambda_2}\omega_3^2 &= \frac{(a^2 + b^2 - b^2 - c^2)(a^2 + b^2 - c^2 - a^2)}{(b^2 + c^2)(c^2 + a^2)}\omega_3^2 \\ &= \frac{(a^2 - c^2)(b^2 - c^2)}{(a^2 + c^2)(b^2 + c^2)}\omega_3^2\end{aligned} \tag{xvii}$$

Putting in the given numbers, we find $\Omega = 0.968\omega_3 = 174$ rpm. If the book is spinning about the longest (x) axis we have only to swap λ_1 and λ_3, and we find $\Omega = 0.614\omega_3 = 111$ rpm.

10.43 ⋆⋆ **(a)** From (10.94) we see that

$$\omega^2 = \omega_o^2 \cos^2(\Omega t) + \omega_o^2 \sin^2(\Omega t) + \omega_3^2 = \omega_o^2 + \omega_3^2 = \text{const},$$

since both ω_o and ω_3 are constant.

(b) From (10.94) and (10.95), we see that $\boldsymbol{\omega}$, $\mathbf{L}$ and $\mathbf{e}_3$ lie in a plane and, as seen in the body frame, $\boldsymbol{\omega}$ and $\mathbf{L}$ precess around $\mathbf{e}_3$ with angular frequency (10.93)

$$\Omega_b = \frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3 = \omega\cos\alpha$$

since (from Problem 10.23) $\lambda_3 = 2\lambda_1$ and $\omega_3 = \omega\cos\alpha$. As seen by me (in the "space" frame), the vector $\mathbf{L}$ is fixed and $\boldsymbol{\omega}$ precesses around $\mathbf{L}$ with frequency Ω_s given by Eq.(10.118) in Problem 10.46

$$\Omega_s = \omega\,\frac{\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}}{\lambda_1} = \omega\sqrt{4 - 3\sin^2\alpha}\,.$$

10.44 ⋆⋆ Because $\lambda_1 = \lambda_2$, Euler's equations (10.88) simplify. In particular, the third equation reads $\lambda_3\dot{\omega}_3 = \Gamma$, which is easily solved to give $\omega_3 = \omega_{30}(1 + 2\beta t)$, where ω_{30} is the initial value of ω_3 and the constant $\beta = \Gamma/(2\lambda_3\omega_{30})$. That is, the spin about the symmetry axis accelerates linearly.

The first two Euler equations (10.88) now become (remember $\lambda_1 = \lambda_2$)

$$\dot{\omega}_1 = -\frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3\omega_2 = -\Omega(1 + 2\beta t)\omega_2 \qquad \text{and} \qquad \dot{\omega}_2 = +\frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3\omega_1 = +\Omega(1 + 2\beta t)\omega_1$$

where the constant $\Omega = \omega_{30}(\lambda_3 - \lambda_1)/\lambda_1$. Setting $\omega_1 + i\omega_2 = \eta$, we can combine these two equations as $\dot{\eta} = i\Omega(1 + 2\beta t)\eta$, which can be solved by separation of variables to give $\eta = \omega_{10}e^{i\Omega(t+\beta t^2)}$, or

$$\omega_1 = \omega_{10}\cos\Omega(t + \beta t^2) \qquad \text{and} \qquad \omega_2 = \omega_{10}\sin\Omega(t + \beta t^2).$$

Thus, while ω_3 accelerates, the component of $\boldsymbol{\omega}$ in the xy plane precesses with a fixed magnitude ω_{10} but at an increasing rate.

10.45 ⋆⋆ **(a)** From Equation (10.93) the rate of precession of $\boldsymbol{\omega}$ about the earth's axis $\mathbf{e}_3$ is $\Omega_b = \omega_3(\lambda_1 - \lambda_3)/\lambda_1 = 0.00327\omega_3$. The period of this precession is

$$\tau_b = \frac{2\pi}{\Omega_b} = \frac{1}{0.00327}\frac{2\pi}{\omega_3} = 306 \text{ days}$$

because $2\pi/\omega_3 = 1$ day. This is very nearly, but not quite, the claimed 305 days. The discrepancy is because $2\pi/\omega_3$ is actually 1 sidereal day (the time for one rotation of the earth relative to the stars), and a sidereal day is less than a solar day (what we normally consider to be a day) by about one part in 365. Thus 306 sidereal days are equal to about 305 solar days.

(b) From Fig.10.9 we see that $\tan\alpha = \omega_o/\omega_3$. Since α is tiny ($\alpha = 0.2$ arcseconds $\approx 10^{-6}$ rad), this means that $\omega_o \ll \omega_3$ and hence $\omega = |\boldsymbol{\omega}| \approx \omega_3$. Similarly from (10.95), $L = |\mathbf{L}| \approx L_3 = \lambda_3\omega_3$. Therefore the rate of precession of $\boldsymbol{\omega}$ in the space frame, as given by Eq.(10.96), is $\Omega_s = L/\lambda_1 \approx \lambda_3\omega_3/\lambda_1 \approx \omega_3$, and the corresponding period is $\tau_s = 2\pi/\Omega_s \approx 2\pi/\omega_3 = 1$ day.

10.46 ⋆⋆⋆ **(a)** Referring to Figure 10.9, you can see that

$$\boldsymbol{\Omega}_\mathrm{b} = (\text{angular velocity of the vector } \boldsymbol{\omega} \text{ relative to the body})$$

and

$$\boldsymbol{\omega} = (\text{angular velocity of the body relative to the space frame}).$$

Therefore

$$\boldsymbol{\Omega}_\mathrm{s} = (\text{angular velocity of the vector } \boldsymbol{\omega} \text{ relative to the space frame}) = \boldsymbol{\Omega}_\mathrm{b} + \boldsymbol{\omega}.$$

(b) Let us take the component of the last equation perpendicular to $\mathbf{e}_3$. Since $\boldsymbol{\Omega}_\mathrm{b}$ is parallel to $\mathbf{e}_3$, this tells us that

$$(\boldsymbol{\Omega}_\mathrm{s})_\perp = \omega_\perp. \tag{xviii}$$

Again referring to Fig.10.9, you can see that this is the same thing as $\Omega_\mathrm{s} \sin\theta = \omega \sin\alpha$, which is the requested result.

(c) The result of part (b) immediately implies that $\Omega_\mathrm{s} = (\omega\sin\alpha)/\sin\theta$, which is the first requested expression. Alternatively, note that $L_\perp = L\sin\theta$, but also, from (10.95), $L_\perp = \lambda_1\omega_\mathrm{o}$. Therefore $\omega_\perp = \omega_\mathrm{o} = (L\sin\theta)/\lambda_1$, and Eq.(xviii) can be rewritten as $\Omega_\mathrm{s}\sin\theta = (L\sin\theta)/\lambda_1$. This gives us $\Omega_\mathrm{s} = L/\lambda_1$, which is the second requested expression.

Finally, from Eq.(10.95),

$$L = \sqrt{\lambda_1^2\omega_\mathrm{o}^2 + \lambda_3^2\omega_3^2} = \omega\sqrt{\lambda_1^2\sin^2\alpha + \lambda_3^2\cos^2\alpha} = \omega\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}.$$

Substituting this into the previoius result, $\Omega_\mathrm{s} = L/\lambda_1$, gives the third requested expression.

10.47 ⋆⋆⋆ Once the mountain has been added, the earth has only one axis of symmetry $\mathbf{e}_3$, which goes through the mountain. In accordance with (10.93), the angular velocity $\boldsymbol{\omega}$ now precesses about the mountain at a rate Ω_b (as seen by us on earth), where (if M denotes the mass of the earth and $m = 10^{-8}M$ that of the mountain)

$$\Omega_\mathrm{b} = \frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3 = \frac{mR^2}{\frac{2}{5}MR^2}\omega_3 = \frac{5m}{2M}\omega\cos\alpha.$$

The distance $d = 100$ mi moved by the pole in time t is $d = (R\sin\alpha)\Omega_\mathrm{b}t$. Therefore,

$$t = \frac{d}{(R\sin\alpha)\Omega_\mathrm{b}} = \frac{2Md}{5mR\omega\sin\alpha\cos\alpha}$$

or, putting in the numbers (including $\omega = 2\pi/\text{day}$), $t = 3.68 \times 10^5$ days ≈ 1000 years.

10.48 ⋆⋆ **(a)** Starting from Eq.(10.97), we get (see Fig.10.10 to check the expressions for the various unit vectors)

$$\begin{aligned}\boldsymbol{\omega} &= \dot\phi\hat{\mathbf{z}} + \dot\theta\mathbf{e}_2' + \dot\psi\mathbf{e}_3 \\ &= \dot\phi\hat{\mathbf{z}} + \dot\theta(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}) + \dot\psi[\cos\theta\hat{\mathbf{z}} + \sin\theta(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}})] \\ &= (-\dot\theta\sin\phi + \dot\psi\sin\theta\cos\phi)\hat{\mathbf{x}} + (\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi)\hat{\mathbf{y}} + (\dot\phi + \dot\psi\cos\theta)\hat{\mathbf{z}}.\end{aligned}$$

(b) Starting from Eq.(10.99), we get

$$\begin{aligned}\boldsymbol{\omega} &= (-\dot{\phi}\sin\theta)\mathbf{e}'_1 + \dot{\theta}\mathbf{e}'_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3 \\ &= (-\dot{\phi}\sin\theta)(\cos\psi\mathbf{e}_1 - \sin\psi\mathbf{e}_2) + \dot{\theta}(\sin\psi\mathbf{e}_1 + \cos\psi\mathbf{e}_2) + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3 \\ &= (-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\mathbf{e}_1 + (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\mathbf{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3\end{aligned}$$

10.49 ⋆⋆ Equation (10.100) gives the angular momentum $\mathbf{L}$ of the top in terms of the basis vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}_3$. To find $L_z = \mathbf{L}\cdot\hat{\mathbf{z}}$ we need to know the scalar product of each of these three vectors with $\hat{\mathbf{z}}$. These are easily read off from Fig.10.10 as

$$\mathbf{e}'_1\cdot\hat{\mathbf{z}} = -\sin\theta, \quad \mathbf{e}'_2\cdot\hat{\mathbf{z}} = 0, \quad \text{and} \quad \mathbf{e}_3\cdot\hat{\mathbf{z}} = \cos\theta.$$

According to (10.100)

$$\mathbf{L} = (-\lambda_1\dot{\phi}\sin\theta)\mathbf{e}'_1 + \lambda_1\dot{\theta}\mathbf{e}'_2 + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3.$$

Taking the dot product of this with $\hat{\mathbf{z}}$, we get

$$L_z = \mathbf{L}\cdot\hat{\mathbf{z}} = \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$$

which is Eq.(10.102). Finally, using (10.101) we can replace $(\dot{\psi} + \dot{\phi}\cos\theta)$ with L_3/λ_3 to give $L_z = \lambda_1\dot{\phi}\sin^2\theta + L_3\cos\theta$, which is (10.103).

10.50 ⋆⋆ Equation (10.99) gives $\boldsymbol{\omega}$ in terms of the basis vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}_3$,

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{e}'_1 + \dot{\theta}\,\mathbf{e}'_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3. \tag{xix}$$

Unfortunately, if $\lambda_1 \neq \lambda_2$, we need to rewrite this in terms of the principal axes $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. From Fig.10.10, you can check that

$$\mathbf{e}'_1 = \mathbf{e}_1\cos\psi - \mathbf{e}_2\sin\psi \quad \text{and} \quad \mathbf{e}'_2 = \mathbf{e}_1\sin\psi + \mathbf{e}_2\cos\psi.$$

When these are substituted into (xix), we get

$$\boldsymbol{\omega} = -(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\mathbf{e}_1 + (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\mathbf{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3$$

and hence

$$\begin{aligned}T &= \tfrac{1}{2}\lambda_1\omega_1^2 + \tfrac{1}{2}\lambda_2\omega_2^2 + \tfrac{1}{2}\lambda_3\omega_3^2 \\ &= \tfrac{1}{2}\lambda_1(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 + \tfrac{1}{2}\lambda_2(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 + \tfrac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2.\end{aligned}$$

10.51 ⋆ From (10.105) we find

$$\begin{aligned}E = T + U &= \tfrac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + MgR\cos\theta \\ &= \tfrac{1}{2}\lambda_1\dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta = \tfrac{1}{2}\lambda_1\dot{\theta}^2 + U_{\text{eff}}(\theta)\end{aligned}$$

where, in moving to the second line, I used (10.104) to replace $\dot{\phi}$ by $(L_3 - L_z\cos\theta)/(\lambda_1\sin^2\theta)$ and (10.101) to replace $(\dot{\psi} + \dot{\phi}\cos\theta)$ by L_3/λ_3. The value of U_{eff} in the last expression is easily read off from the previous one and agrees exactly with Eq.(10.114).

10.52 ⋆⋆ **(a)** If θ is constant ($\dot\theta = 0$), the expression (10.100) for $\mathbf{L}$ simplifies to

$$\mathbf{L} = -\lambda_1 \dot\phi \sin\theta\, \mathbf{e}_1' + L_3\, \mathbf{e}_3$$

where I have used (10.101) to replace $\lambda_3(\dot\psi + \dot\phi\cos\theta)$ by L_3. Next, using Fig.10.10, I shall read off the horizontal component L_{hor} of $\mathbf{L}$ and then replace $\dot\phi$ by the value found in (10.112) for the fast steady precession:

$$L_{\text{hor}} = -\lambda_1\dot\phi\sin\theta\cos\theta + L_3\sin\theta = -\lambda_1\frac{L_3}{\lambda_1\cos\theta}\sin\theta\cos\theta + L_3\sin\theta = 0$$

That is, $\mathbf{L}$ is in the vertical direction. (Since the value used for $\dot\phi$ is only approximate, the same is true of this result — $\mathbf{L}$ is close to the vertical.)

(b) Because $\mathbf{L}$ is vertical, θ is the angle between $\mathbf{L}$ and $\mathbf{e}_3$, so $L_3 = L\cos\theta$, and (10.112) becomes $\Omega = L_3/(\lambda_1\cos\theta) = L/\lambda_1$, which is the same as the rate Ω_s found in (10.96).

10.53 ⋆⋆ **(a)** We can write Eq.(10.110) for Ω as $a\Omega^2 + b\Omega + c = 0$ if we define $a = \lambda_1\cos\theta$, $b = -\lambda_3\omega_3$, and $c = MgR$. The two solutions for Ω are, of course,

$$\Omega = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

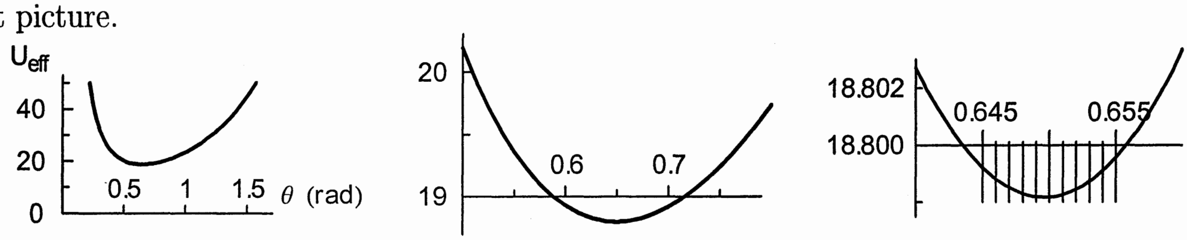
If $b^2 \gg 4ac$, the solution with the minus sign is approximately $\Omega \approx (-b-b)/2a = -b/a$. The solution with a plus sign requires a little more care. With $b^2 \gg 4ac$, we have $\sqrt{b^2-4ac} = b(1-4ac/b^2)^{1/2} \approx b - 2ac/b$, so the second solution is $\Omega \approx (-b+b-2ac/b)/2a = -c/b$. The condition for the validity of these two approximate solutions is just that $b^2 \gg 4ac$.

(b) If we put back the values of a, b, and c, the two solutions become $\Omega = -b/a = \lambda_3\omega_3/(\lambda_1\cos\theta)$ in agreement with (10.112) and $\Omega = -c/b = MgR/(\lambda_3\omega_3)$ in agreement with (10.111). The condition $b^2 \gg 4ac$ for these to be good approximations translates to $\lambda_3^2\omega_3^2 \gg 4\lambda_1 MgR\cos\theta$; if we assume λ_1 and λ_3 are not too different, this condition is roughly $\frac{1}{2}\lambda_3\omega_3^2 \gg MgR\cos\theta$ (I've dropped a factor of 2) or, even more roughly, that the KE of rotation is much greater than the gravitational PE.

10.54 ⋆⋆⋆ **(a)** The function we have to plot is (if we omit the constant term)

$$U_{\text{eff}}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + MgR\cos\theta = \frac{(10-8\cos\theta)^2}{2\sin^2\theta} + \cos\theta$$

if we put in all the suggested numbers. This function is plotted from $\theta = 0$ to 1.5 rad in the left picture.



(b) The top can precess steadily with θ = const, only at the angle θ_o for which $U_{\text{eff}}(\theta)$ is minimum. From the left picture it is clear that θ_o is somewhere near 0.6 or 0.7 rad. The middle picture zooms in on the range from 0.5 to 0.8 rad, and, from this one, it is clear θ_o is near 0.65 rad. The right picture zooms in again and from this one it's clear that θ_o is between 0.649 and 0.650m, probably closer to the latter. Thus, to three significant figures $\theta_o = 0.650$ rad or 37.2°.

(c) If we put all these numbers into (10.115) we get for the rate of precession $\Omega = 9.92$ rad/s. The approximate formula (10.112) gives $\Omega \approx 10.0$.

10.55 ⋆⋆⋆ **(a)** With $\hat{\mathbf{z}}$ chosen in the direction of $\mathbf{L}$ and given by (10.98),

$$\mathbf{L} = L\hat{\mathbf{z}} = -L\sin\theta\,\mathbf{e}_1' + L\cos\theta\,\mathbf{e}_3. \tag{xx}$$

(b) Comparing this with Eq.(10.100), $\mathbf{L} = (-\lambda_1\dot{\phi}\sin\theta)\mathbf{e}_1' + \lambda_1\dot{\theta}\mathbf{e}_2' + \lambda_3(\dot{\psi}+\dot{\phi}\cos\theta)\mathbf{e}_3$, we find the following three equations:

$$\lambda_1\dot{\phi}\sin\theta = L\sin\theta, \quad \dot{\theta} = 0, \quad \text{and} \quad \lambda_3(\dot{\psi}+\dot{\phi}\cos\theta) = L\cos\theta. \tag{xxi}$$

(c) The second of Eqs.(xxi) implies that θ is constant. Cancelling the $\sin\theta$ from the first equation, we see that $\dot{\phi}$ is constant, $\dot{\phi} = L/\lambda_1$. Since $\dot{\phi}$ is the rate at which the body's axis precesses about the space axis $\hat{\mathbf{z}}$, this agrees with the prediction (10.96), $\Omega_s = L/\lambda_1$, for that same rate.

(d) According to (10.99)

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\mathbf{e}_1' + (\dot{\psi}+\dot{\phi}\cos\theta)\mathbf{e}_3. \tag{xxii}$$

We aleady know that the coefficient of $\mathbf{e}_1'$ is constant, and, by the third of Eqs.(xxii) the coefficient of $\mathbf{e}_3$ is also constant. Therefore, both ω_3 and $|\boldsymbol{\omega}|$ are constant, and so, therefore, is the angle between $\boldsymbol{\omega}$ and $\mathbf{e}_3$.

From Eqs.(xx) and (xxii) it is clear that both $\mathbf{L}$ and $\boldsymbol{\omega}$ lie in the plane defined by $\mathbf{e}_1'$ and $\mathbf{e}_3$. Therefore, $\mathbf{L}$, $\boldsymbol{\omega}$, and $\mathbf{e}_3$ are coplanar at all times.

10.56 ⋆⋆⋆ **(a)** Because $E = \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{\text{eff}}(\theta)$, it is clear that at no time can $U_{\text{eff}}(\theta)$ exceed E. Now, if you look carefully at (10.114), you will see that, because of the factor of $\sin^2\theta$ in the denominator, $U_{\text{eff}}(\theta) \to \infty$ as $\theta \to 0$, *unless* $L_3 = L_z$. Therefore, unless $L_3 = L_z$ the top cannot pass through the position $\theta = 0$, and, conversely, if the top does visit $\theta = 0$, L_3 and L_z must be equal.

(b) Setting $L_3 = L_z = \lambda_3\omega_3$ in (10.114) we find that

$$\begin{aligned}U_{\text{eff}}(\theta) &= \frac{(\lambda_3\omega_3)^2(1-\cos\theta)^2}{2\lambda_1\sin^2\theta} + MgR\cos\theta \\ &\approx \frac{(\lambda_3\omega_3)^2(\theta^2/2)^2}{2\lambda_1\theta^2} + MgR\frac{\theta^2}{2} + \text{const} = \frac{1}{8}\left(\frac{(\lambda_3\omega_3)^2}{\lambda_1} - 4MgR\right)\theta^2 + \text{const}\end{aligned}$$

where, in passing to the second line, I expanded everything through order θ^2.

(c) Oscillations of θ about $\theta = 0$ will be stable if and only if the coefficient of θ^2 in $U_{\text{eff}}(\theta)$ is positive. Thus if $\omega_3 > 2\sqrt{MgR\lambda_1/\lambda_3^2} = \omega_{\text{min}}$ the equilibrium at $\theta = 0$ will be stable. If $\omega_3 < \omega_{\text{min}}$, it is unstable.

10.57 ⋆⋆⋆ **(a)** Writing the KE as the sum of that of the CM plus that of the rotation about the CM, we get

$$\mathcal{L} = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}\lambda_1^{\text{cm}}(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}\lambda_3^{\text{cm}}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

where λ_1^{cm} and λ_3^{cm} are the principal moments for rotation about the CM. [Compare Eq.(10.106).] By the parallel axis theorem (Problem 10.24) the moments about the CM are related to those about the tip as follows:

$$\lambda_3^{\text{cm}} = \lambda_3^{\text{tip}} \qquad \text{and} \qquad \lambda_1^{\text{cm}} = \lambda_1^{\text{tip}} - MR^2. \tag{xxiii}$$

(b) Clearly the motion of X and Y is independent of that of θ, ϕ, and ψ, and the position (X, Y) moves like a free particle with $\dot{X}$ and $\dot{Y}$ constant.

(c) The rate of the fast precession is given by (10.112) as $\Omega_{\text{fast}} = \lambda_3\omega_3/(\lambda_1\cos\theta)$. In view of the relations (xxiii), you can see that $\Omega_{\text{fast}}^{\text{cm}} > \Omega_{\text{fast}}^{\text{tip}}$ (for given θ and ω_3). That is, the fast precession is even faster when the tip is free to move.

The rate of the slow precession is given by (10.111) as $\Omega_{\text{slow}} \approx MgR/(\lambda_3\omega_3)$, and, since $\lambda_3^{\text{cm}} = \lambda_3$, this is the same whether the tip is fixed or not. However, (10.111) is only an approximation. If we keep the next term in that approximation, we find

$$\Omega_{\text{slow}} \approx \frac{MgR}{\lambda_3\omega_3}\left[1 + \frac{\lambda_1 MgR\cos\theta}{(\lambda_3\omega_3)^2}\right].$$

(To check this, do Problem 10.53 and keep one extra term.) Thus the slow precession is a little slower when the tip is free to move.

Chapter 11

Coupled Oscillators and Normal Modes

I covered this chapter in 4 fifty-minute lectures.

Like its predecessor (Chapter 10 on rigid bodies) this chapter is an excellent opportunity for our students to learn some interesting physics and to hone their skills with matrices. Classical coupled oscillators are conceptually simple, but also quite fascinating (though I did have one student complain that he was getting bored with pendulums and masses on springs). There are also several opportunities to bring experiments into class to tickle the students interest. A pair of simple pendulums coupled by a weak spring can illustrate beautifully the behavior of weakly coupled oscillators, as described in Section 11.3, and brought applause from one of my classes.

There is a brief mention of normal coordinates at the end of Section 11.2 and a much more detailed discussion in Section 11.7. The latter is a bit more sophisicated mathematically and could — though it saddens me to suggest this — be omitted on a first reading.

Solutions to Problems for Chapter 11

11.1 $\star$ **(a)** When the two carts are in equilibrium, the tensions in the three springs must be equal, so

$$k_1(L_1 - \ell_1) = k_2(L_2 - \ell_2) = k_3(L_3 - \ell_3).$$

(b) The net force on cart 1 is

$$F(\text{on cart 1}) = -k_1(x_1 + L_1 - \ell_1) + k_2(x_2 - x_1 + L_2 - \ell_2) = -k_1x_1 + k_2(x_2 - x_1)$$

because the term involving $L_1 - \ell_1$ cancels that involving $L_2 - \ell_2$. This is exactly the expression (11.1), derived on the assumption that the springs were unstretched in equilibrium. The force on cart 2 works similarly.

11.2 ⋆⋆ Let x_1 and x_2 be the extensions of the two springs from their unstretched lengths, and x_{10} and x_{20} their values at equilibrium. The displacements from equilibrium are

$$y_1 = x_1 - x_{10} \qquad \text{and} \qquad y_2 = x_2 - x_{20}. \tag{i}$$

The net downward forces on the two masses are

$$F_1 = m_1 g - k_1 x_1 + k_2(x_2 - x_1) \qquad \text{and} \qquad F_2 = m_2 g - k_2(x_2 - x_1), \tag{ii}$$

and the conditions for equilibrium are

$$m_1 g = k_1 x_{10} - k_2(x_{20} - x_{10}) \qquad \text{and} \qquad m_2 g = k_2(x_{20} - x_{10}).$$

Using (i) to eliminate x_1 and x_2 from (ii), we find that

$$\begin{aligned} m_1\ddot{y}_1 = F_1 &= m_1 g - k_1(y_1 + x_{10}) + k_2[y_2 - y_1 + (x_{20} - x_{10})] \\ &= -k_1 y_1 + k_2(y_2 - y_1) \end{aligned}$$

where, in the second line I used the equilibrium consition to cancel several terms. Similarly

$$\begin{aligned} m_2\ddot{y}_2 = F_2 &= m_2 g - k_2(x_2 - x_1) \\ &= -k_2(y_2 - y_1). \end{aligned}$$

The last two results combine to give the matrix equation $\mathbf{M}\ddot{\mathbf{y}} = -\mathbf{K}\mathbf{y}$ where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

11.3 ⋆ Using the matrices $\mathbf{M}$ and $\mathbf{K}$ given in Eq.(11.5), we find

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2\omega^2 \end{bmatrix}$$

[Compare Eq.(11.14).] Hence

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = m_1 m_2 \omega^4 - [m_1(k_2 + k_3) + m_2(k_1 + k_2)]\omega^2 + (k_1k_2 + k_1k_3 + k_2k_3)$$

This is zero when

$$\omega^2 = \frac{1}{2m_1m_2}\Big\{ m_1(k_2 + k_3) + m_2(k_1 + k_2) \\ \pm\sqrt{m_1^2(k_2+k_3)^2 + m_2^2(k_1+k_2)^2 + 2m_1m_2(k_2^2 - k_1k_2 - k_1k_3 - k_2k_3)}\ \Big\}$$

Therefore, these are the two normal frequencies (squared).

If $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$, these reduce to $\omega^2 = (k/m)(2 \pm 1)$ in agreement with Eq.(11.15).

11.4 ⋆⋆ **(a)** Putting $m_1 = m_2$ and $k_1 = k_3$ in (11.5), we find that the mass and spring-constant matrices are

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}$$

and hence

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} k_1 + k_2 - m\omega^2 & -k_2 \\ -k_2 & k_1 + k_2 - m\omega^2 \end{bmatrix}.$$

The determinant of the last matrix is $\det(\mathbf{K} - \omega^2\mathbf{M}) = (m\omega^2 - k_1)(m\omega^2 - k_1 - 2k_2)$. The two normal frequencies are the roots of the equation $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ and are $\sqrt{k_1/m}$ and $\sqrt{(k_1 + 2k_2)/m}$. If I set $k_1 = k_2 = k$, these reduce to the results (10.15) for all three springs equal.

(b) The motion in each normal mode is determined by the vector $\mathbf{a}$ satisfying the eigenvector equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. For $\omega = \omega_1$ this is easily seen to be exactly the same as for the equal-spring case; in particular, the motion is as given by (11.18) and as shown in Figure 11.2. The two carts oscillate in phase with equal amplitudes, so that the middle spring is undisturbed. This means its strength is irrelevant and we get the same motion with the same frequency whatever the value of k_2. For the second mode, with $\omega = \omega_2$, the motion is again the same as for the corresponding mode of the equal-spring case, namely (11.20) and Figure 11.4. This is a little subtle: In this mode, the middle spring does change length, and the frequency does depend on the value of k_2. Nevertheless, the motion is independent of k_2 since the symmetric arrangement, with the outside springs equally stretched and the middle one compressed, or vice versa, leads to equal (but opposite) forces on the two equal-mass carts and allows them to oscillate with equal amplitudes exactly out of phase.

11.5 ⋆⋆ **(a)** The quickest way to find the equation of motion for the system of Fig.11.15 is to set $k_3 = 0$ in Fig.11.1. With $m_1 = m_2$ and $k_1 = k_2$ as well, the mass and spring-constant matrices are given by Eq.(11.5) as

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}.$$

If we define $\omega_o = \sqrt{k/m}$ and hence $k = m\omega_o^2$, the characteristic equation becomes

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2(\omega^4 - 3\omega_o^2\omega^2 + \omega_o^4) = 0,$$

so the normal frequencies are given by $\omega^2 = \omega_o^2(3 \pm \sqrt{5})/2$.

(b) If we substitute $\omega = \omega_1 = \omega_o\sqrt{(3 - \sqrt{5})/2}$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ yields $a_2 = a_1(1 + \sqrt{5})/2 = 1.62a_1$. Thus the first normal mode has the form

$$x_1 = A\cos(\omega_1 t - \delta) \qquad \text{and} \qquad x_2 = 1.62A\cos(\omega_1 t - \delta).$$

where A and δ are arbitrary constants. In the first mode the two carts oscillate in phase, the second one with the larger amplitude. Similarly, in the second mode, we find $a_2 = a_1(1-\sqrt{5})/2 = -0.62a_1$ and so

$$x_1 = A\cos(\omega_2 t - \delta) \qquad \text{and} \qquad x_2 = -0.62A\cos(\omega_2 t - \delta).$$

where (in general) A and δ are different constants. In this mode the two carts move 180° out of phase, and cart 2 has the smaller amplitude.

11.6 ⋆⋆ **(a)** In this case

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix}$$

with determinant $\det(\mathbf{K} - \omega^2\mathbf{M}) = (m\omega^2 - k)(m\omega^2 - 6k)$. Thus the two normal frequencies are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{6k/m}$.

(b) The motion in each normal mode is determined by the vector $\mathbf{a}$ satisfying the eigenvector equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. For $\omega = \omega_1$ this gives $a_2 = 2a_1$, so the two carts oscillate in phase, with the second cart's amplitude equal to twice that of the first. If $\omega = \omega_2$ then $a_2 = -a_1/2$, so the two carts oscillate exactly out of phase, with the second cart's amplitude equal to half that of the first.

11.7 ⋆⋆ **(a)** Since $A\cos(\omega t - \delta) = A\cos\omega t\cos\delta + A\sin\omega t\sin\delta = B\cos\omega t + C\sin\omega t$, we can immediately rewrite (11.21) in the given form.

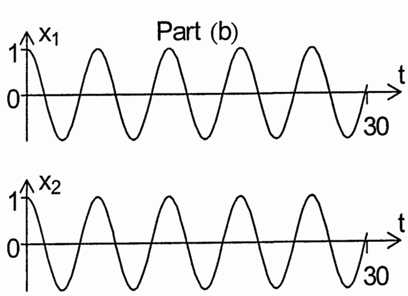
(b) From the given form, you can see that

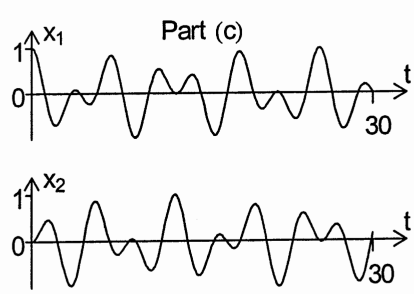
$$\mathbf{x}(0) = \begin{bmatrix} B_1 + B_2 \\ B_1 - B_2 \end{bmatrix} \qquad \text{and} \qquad \dot{\mathbf{x}}(0) = \begin{bmatrix} \omega_1 C_1 + \omega_2 C_2 \\ \omega_1 C_1 - \omega_2 C_2 \end{bmatrix}, \tag{iii}$$

but we are given that $x_1(0) = x_2(0) = A$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$. Solving for the coefficients in (iii), we find that

$$B_1 = A \qquad \text{and} \qquad B_2 = C_1 = C_2 = 0.$$

Substituting into the given form, you can see that this solution happens to be the first normal mode, a result we could have anticipated since the system started out with $x_1 = x_2$ and $\dot{x}_1 = \dot{x}_2$.





(c) In this case the initial conditions are $x_1(0) = A$ and $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$. Putting these into the left side of (iii), you will find that

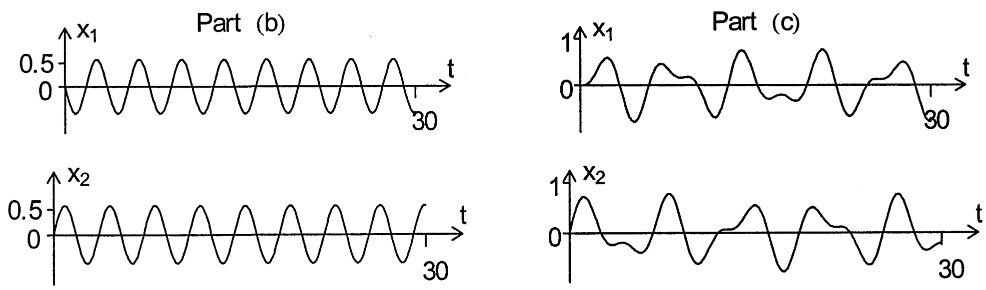
$$B_1 = B_2 = A/2 \qquad \text{and} \qquad C_1 = C_2 = 0.$$

This is an equal mixture of the two normal modes and produces the graphs shown in the right-hand figures above.

11.8 ⋆⋆ **(b)** The initial positions and velocities are given by Eq.(iii) of the solution to Problem 11.7. Given that $x_1(0) = x_2(0) = 0$ and $\dot{x}_2(0) = -\dot{x}_1(0) = v_o$, the coefficients in (iii) are

$$B_1 = B_2 = C_1 = 0 \qquad \text{and} \qquad C_2 = -v_o/\omega_2.$$

This solution is the second normal mode, as you probably foresaw.



(c) In this case the initial conditions are $x_1(0) = x_2(0) = \dot{x}_1(0) = 0$ and $\dot{x}_2(0) = v_o$. Putting these into the left sides of Eqs.(iii), you will find that

$$B_1 = B_2 = 0, \qquad C_1 = \frac{v_o}{2\omega_1}, \qquad \text{and} \qquad C_2 = \frac{v_o}{2\sqrt{3}\omega_1}.$$

This is a mixture of the two normal modes and produces the graphs shown on the right above.

11.9 ⋆⋆ **(a)** With identical masses and springs, the two equations of motion (11.2) are

$$m\ddot{x}_1 = -2kx_1 + kx_2 \qquad \text{and} \qquad m\ddot{x}_2 = kx_1 - 2kx_2\,.$$

If we add these two equations and define $\xi_1 = \frac{1}{2}(x_1+x_2)$, we find that $\ddot{\xi}_1 = -k\xi_1$. Similarly, if we subtract the second equation from the first and define $\xi_2 = \frac{1}{2}(x_1-x_2)$, we get $\ddot{\xi}_2 = -3k\xi_2$. These equations for ξ_1 and ξ_2 are uncoupled, as claimed.

(b) The general solutions of the equations of motion for ξ_1 and ξ_2 are $\xi_1 = A_1\cos(\omega_1 t - \delta_1)$ and $\xi_1 = A_2\cos(\omega_2 t - \delta_2)$, where A_1, A_2, δ_1 and δ_2 are arbitrary constants, $\omega_1 = \sqrt{k/m}$, and $\omega_2 = \sqrt{3k/m}$. Therefore

$$x_1 = \xi_1 + \xi_2 = A_1\cos(\omega_1 t - \delta_1) + A_2\cos(\omega_2 t - \delta_2)$$

and

$$x_2 = \xi_1 - \xi_2 = A_1\cos(\omega_1 t - \delta_1) - A_2\cos(\omega_2 t - \delta_2)$$

in agreement with Eq.(11.21).

11.10 ⋆⋆⋆ **(a)** As in Section 5.4, let us define $\beta = b/2m$ and $\omega_o^2 = k/m$. Then the equations of motion are

$$\ddot{x}_1 = -2\beta\dot{x}_1 - 2\omega_o^2 x_1 + \omega_o^2 x_2$$
$$\ddot{x}_2 = -2\beta\dot{x}_2 + \omega_o^2 x_1 - 2\omega_o^2 x_2.$$

(b) If you take first the sum and then the difference of these two equations, you should get

$$\ddot{\xi}_1 = -2\beta\dot{\xi}_1 - \omega_o^2\xi_1 \quad \text{and} \quad \ddot{\xi}_2 = -2\beta\dot{\xi}_2 - 3\omega_o^2\xi_2 .$$

These two equations for ξ_1 and ξ_2 are uncoupled, as advertised.

(c) The equation for ξ_1 is exactly the equation (5.28) that we found for a single damped oscillator and has the solution (5.37), which we can rewrite as

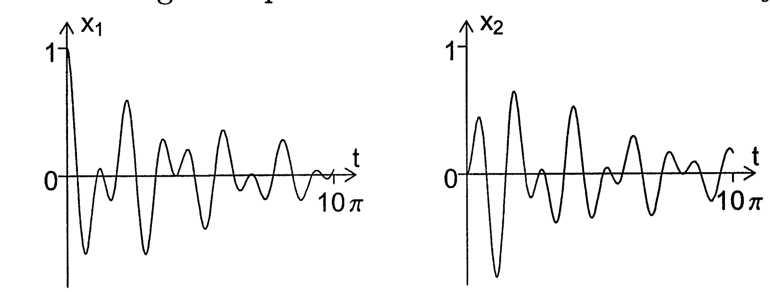
$$\xi_1(t) = e^{-\beta t}(B_1 \cos\omega_1 t + C_1 \sin\omega_1 t)$$

where $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$. Similarly,

$$\xi_2(t) = e^{-\beta t}(B_2 \cos\omega_2 t + C_2 \sin\omega_2 t)$$

where $\omega_2 = \sqrt{3\omega_o^2 - \beta^2}$. The expressions for $x_1(t)$ and $x_2(t)$ follow at once by adding and subtracting these expressions for $\xi_1(t)$ and $\xi_2(t)$.

(d) The given initial conditions imply that $\xi_1(0) = \xi_2(0) = A/2$, with both derivatives zero. Therefore, $B_1 = B_2 = A/2$, $C_1 = \beta A/2\omega_1$ and $C_2 = \beta A/2\omega_2$, from which you can write down $\xi_1(t)$ and $\xi_2(t)$, and thence $x_1(t)$ and $x_2(t)$ as shown. These plots of $x_1(t)$ and $x_2(t)$ are complicated because each is a combination of both normal frequencies, but you can clearly see the effects of the damping. Plots of $\xi_1(t)$ and $\xi_2(t)$ would be much simpler — looking just like Figure 5.11 for a single damped oscillator — since each contains just one frequency.



11.11 ⋆⋆⋆ **(a)** As in Section 5.4, let us define $\beta = b/2m$ and $\omega_o^2 = k/m$. Then the equations of motion are

$$\begin{aligned}\ddot{x}_1 &= -2\beta\dot{x}_1 - 2\omega_o^2 x_1 + \omega_o^2 x_2 + (F_o/m)\cos\omega t \\ \ddot{x}_2 &= -2\beta\dot{x}_2 + \omega_o^2 x_1 - 2\omega_o^2 x_2.\end{aligned} \tag{iv}$$

(b) If you take first the sum and then the difference of these two equations, you should get

$$\ddot{\xi}_1 + 2\beta\dot{\xi}_1 + \omega_o^2\xi_1 = f_o\cos\omega t \quad \text{and} \quad \ddot{\xi}_2 + 2\beta\dot{\xi}_2 + 3\omega_o^2\xi_2 = f_o\cos\omega t, \tag{v}$$

where $f_o = F_o/(2m)$. These two equations for ξ_1 and ξ_2 are uncoupled, as advertised.

(c) The equation for ξ_1 is the same as Eq.(5.57) for a single driven oscillator and can be solved in the same way. We write $\xi_1(t) = \mathrm{Re}\,\zeta_1(t)$ and try a solution of the form $\zeta_1(t) =$

$C_1 e^{i\omega t}$. Substituting this guess into the first of Eqs.(v), we find that it is a solution provided $(-\omega^2 + 2i\beta\omega + \omega_o^2)C_1 = f_o$. Thus we have found a solution with $C_1 = A_1 e^{-i\delta}$ where A_1 is given by the resonance formula (5.64)

$$A_1^2 = \frac{f_o^2}{(\omega^2 - \omega_o^2)^2 + 4\beta^2\omega^2} \tag{vi}$$

and the phase shift δ_1 is given by (5.65). To this particular solution we can add any solution of the corresponding homogeneous equation, and the general solution has the form

$$\xi_1(t) = A_1 \cos(\omega t - \delta_1) + B_1 e^{-\beta t} \cos(\omega_1 t - \delta_1')$$

where B_1 and δ_1' are arbitrary constants, determined by the initial conditions, and the frequncy of the transient is $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$. The second of Eqs.(v) can be solved in exactly the same way giving an analogous expression for $\xi_2(t)$, the only important differences being that the amplitude of the particular solution is given by

$$A_2^2 = \frac{f_o^2}{(\omega^2 - 3\omega_o^2)^2 + 4\beta^2\omega^2} \tag{vii}$$

and the frequency of the transient is $\omega_2 = \sqrt{3\omega_o^2 - \beta^2}$.

(d) If $\beta \ll \omega_o$, it is evident from Eq.(vi) that A_1 has a sharp maximum close to $\omega^2 = \omega_o^2$ and, from (vii) A_1 has a sharp maximum close to $\omega^2 = 3\omega_o^2$

(e) If both carts are driven in phase with the same driving force, then both of the equations of motion (iv) contain the term $(F_o/m)\cos\omega t$. Therefore the equation in (v) for ξ_1 has the same form as before (except that $f_o = F_o/m$ now) but the equation for ξ_2, which comes from taking the difference of the two equations (iv), contains no driving force at all. Therefore, ξ_1 shows the same resonance as before (with twice the amplitude), but ξ_2 shows none.

The reason for this difference is easy to see. The mode represented by ξ_1 has the two carts oscillating exactly in phase, while in that of ξ_2 they oscillate 180° out of phase. By driving them in phase, we are driving mode 1 but not mode 2.

11.12 ⋆⋆⋆ **(a)** The force of viscous drag is $\beta m(\dot{x}_2 - \dot{x}_1)$, to the right on cart 1 and to the left on cart 2. The equation of motion for cart 1 is

$$m\ddot{x}_1 = -kx_1 + \beta m(\dot{x}_2 - \dot{x}_1) \quad \text{or} \quad \ddot{x}_1 + \omega_o^2 x_1 + \beta\dot{x}_1 - \beta\dot{x}_2 = 0$$

and that for cart 2 is

$$m\ddot{x}_2 = -kx_2 - \beta m(\dot{x}_2 - \dot{x}_1) \quad \text{or} \quad \ddot{x}_2 + \omega_o^2 x_2 - \beta\dot{x}_1 + \beta\dot{x}_2 = 0.$$

These two equations combine as a single matrix equation $\ddot{\mathbf{x}} + \omega_o^2\mathbf{x} + \beta\mathbf{D}\dot{\mathbf{x}} = 0$, where $\mathbf{D}$ and $\mathbf{x}$ are the matrices

$$\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) If we substitute the proposed complex solution $\mathbf{z}(t) = \mathbf{a}e^{rt}$ into the matrix equation of motion, we find the $\mathbf{a}$ must satisfy

$$[(r^2 + \omega_o^2)\mathbf{1} + \beta r\mathbf{D}]\mathbf{a} = 0. \tag{viii}$$

This has nontrivial solutions only if the deteminant of the matrix in brackets is zero. That is,

$$\det[(r^2 + \omega_o^2)\mathbf{1} + \beta r\mathbf{D}] = (r^2 + \omega_o^2)(r^2 + \omega_o^2 + 2\beta r) = 0$$

Thus the values of r that give a solution are $r = r_1 = i\omega_o$ and $r = r_2 = -\beta + i\sqrt{\omega_o^2 - \beta^2} = -\beta + i\omega_1$. [There are actually two more solutions with the opposite sign to the imaginary part, but these give the same actual motion $\mathbf{x}(t)$.] If we put $r = r_1$ in (viii) we find that $a_1 = a_2 = A$, say. Thus the first mode has $x_1(t) = x_2(t) = A\cos(\omega_o t - \delta)$, and the two carts move together with equal amplitudes. Because cart 2 is stationary with respect to cart 1, the drag force is zero and the motion is undamped. If we put $r = r_2$ in (viii) we find that $a_1 = -a_2 = A$, say, and the second mode has $x_1(t) = -x_2(t) = A\cos(\omega_1 t - \delta)e^{-\beta t}$. In this mode the two carts move in opposite directions and the drag force causes the motion to damp out.

11.13 ⋆⋆⋆ **(a)** The two equations of motion in the form (11.2) are

$$m\ddot{x}_1 = -b\dot{x}_1 - (k + k_2)x_1 + k_2x_2 \qquad \text{and} \qquad m\ddot{x}_2 = -b\dot{x}_2 + k_2x_1 - (k + k_2)x_2\,. \tag{ix}$$

If we add these two equations, we find that

$$m\ddot{\xi}_1 = -b\dot{\xi}_1 - k\xi_1 \quad \text{or} \quad \ddot{\xi}_1 + 2\beta\dot{\xi}_1 + \omega_o^2\xi_1 = 0$$

where I have introduced $\beta = b/2m$ as usual and $\omega_o^2 = k/m$. Similarly if we subtract the second of Eqs.(ix) from the first we get

$$\ddot{\xi}_2 + 2\beta\dot{\xi}_2 + \omega_o'^2\xi_2 = 0$$

where $\omega_o'^2 = (k + 2k_2)/m$. The equations for ξ_1 and ξ_2 are visibly uncoupled.

(b) The solutions of these two equations are given in (5.38), though they are slightly more convenient in the equivalent form

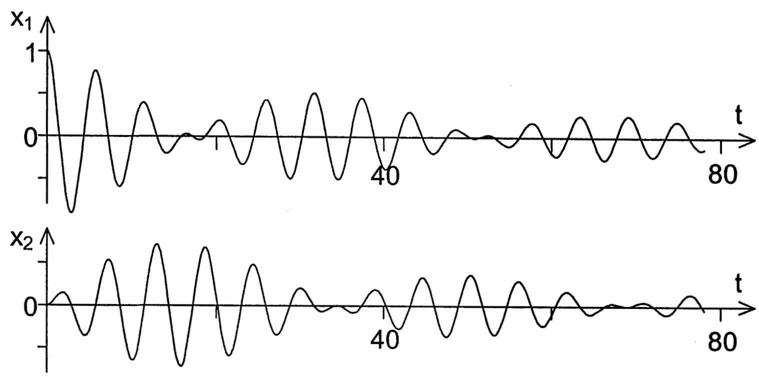
$$\xi_1 = e^{-\beta t}(B_1\cos\omega_1 t + C_1\sin\omega_1 t) \qquad \text{and} \qquad \xi_2 = e^{-\beta t}(B_2\cos\omega_2 t + C_2\sin\omega_2 t),$$

where $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ and $\omega_2 = \sqrt{\omega_o'^2 - \beta^2}$.

(c) The initial conditions imply that $\xi_1(0) = \xi_2(0) = A/2$, and hence that $B_1 = B_2 = A/2$. The initial velocities are a little more complicated since $\dot{\xi}_1(0)$ and $\dot{\xi}_2(0)$ both contain two terms, one proportional to β (from differentiating the factor $e^{-\beta t}$. However, to the extent that $\beta \ll \omega_o$, we can ignore these factors and the initial conditions tell us simply that $C_1 = C_2 = 0$. Thus

$$x_1 = \xi_1 + \xi_2 = (A/2)e^{-\beta t}(\cos\omega_1 t + \cos\omega_2 t) \text{ and } x_2 = \xi_1 - \xi_2 = (A/2)e^{-\beta t}(\cos\omega_1 t - \cos\omega_2 t).$$

Putting in the given numbers we get the two plots shown below, where you can see how the energy is transfer back and forth between the two carts and, at the same time, slowly dissipated.



11.14 ★★ **(a)** The kinetic energy is $T = \frac{1}{2}mL^2(\dot\phi_1^2 + \dot\phi_2^2)$. The gravitational potential energy of either pendulum has the form $mgL(1-\cos\phi) \approx \frac{1}{2}mgL\phi^2$, and the spring's PE is $\frac{1}{2}kx^2 \approx \frac{1}{2}kL^2(\phi_2-\phi_1)^2$. Putting these together,

$$\mathcal{L} = \tfrac{1}{2}mL^2(\dot\phi_1^2 + \dot\phi_2^2) - \tfrac{1}{2}mgL(\phi_1^2+\phi_2^2) - \tfrac{1}{2}kL^2(\phi_2-\phi_1)^2$$

from which we get the Lagrange equations:

$$\begin{aligned}\ddot\phi_1 &= -\omega_o{}^2\phi_1 + (k/m)(\phi_2-\phi_1)\\ \ddot\phi_2 &= -\omega_o{}^2\phi_2 - (k/m)(\phi_2-\phi_1)\end{aligned}$$

where I have divided through by mL^2 and introduced the natural frequency for either pendulum (without the spring) given by $\omega_o{}^2 = g/L$.

(b) From the equations of motion, you can write down the "mass matrix" $\mathbf{M}$ and "spring matrix" $\mathbf{K}$, and thence the matrix

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix}\omega_o{}^2 + k/m - \omega^2 & -k/m\\ -k/m & \omega_o{}^2 + k/m - \omega^2\end{bmatrix}.$$

The determinant of this matrix is $(\omega_o{}^2-\omega^2)(\omega_o{}^2+2k/m-\omega^2)$, and the two normal frequencies are

$$\omega_1 = \omega_o \qquad \text{and} \qquad \omega_2 = \sqrt{\omega_o{}^2 + 2k/m}.$$

The corresponding motions are found by solving the equation $(\mathbf{K}-\omega^2\mathbf{M})\mathbf{a} = 0$ with ω set equal to ω_1 and ω_2 in turn. For the first mode, this gives the eigenvector $\mathbf{a} = A(1,1)$ (actually a 2×1 column, of course). This means that in the first mode the two pendulums oscillate in unison (in phase with equal amplitudes). In this mode the spring is unstretched, its presence is irrelevant, and the frequency is just the natural frequency for a single pendulum.

For the second mode, $\mathbf{a} = A(1,-1)$, and the two pendulums oscillate with equal amplitudes but exactly out of phase. Notice that, in either mode, the two pendulums behave just like the two carts of Section 11.2.

11.15 ⋆⋆ Combining (11.38) and (11.37), we find

$$\mathcal{L} = \tfrac{1}{2}(m_1+m_2)L_1^2\dot\phi_1^2 + m_2L_1L_2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) + \tfrac{1}{2}m_2L_2^2\dot\phi_2^2 \\ - (m_1+m_2)gL_1(1-\cos\phi_1) - m_2gL_2(1-\cos\phi_2).$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot\phi_1} = (m_1+m_2)L_1^2\dot\phi_1 + m_2L_1L_2\dot\phi_2\cos(\phi_1-\phi_2)$$

the ϕ_1 equation is

$$(m_1+m_2)L_1^2\ddot\phi_1 + m_2L_1L_2\ddot\phi_2\cos(\phi_1-\phi_2) - m_2L_1L_2\dot\phi_2(\dot\phi_1-\dot\phi_2)\sin(\phi_1-\phi_2) \\ = -m_2L_1L_2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - (m_1+m_2)gL_1\sin\phi_1.$$

Since the first term on the right cancels the second to last term on the left, this simplifies slightly to

$$(m_1+m_2)L_1^2\ddot\phi_1 + m_2L_1L_2\ddot\phi_2\cos(\phi_1-\phi_2) + m_2L_1L_2\dot\phi_2^2\sin(\phi_1-\phi_2) \\ = -(m_1+m_2)gL_1\sin\phi_1.$$

In the same way the ϕ_2 equation is

$$m_2L_1L_2\ddot\phi_1\cos(\phi_1-\phi_2) + m_2L_2^2\ddot\phi_2 - m_2L_1L_2\dot\phi_1^2\sin(\phi_1-\phi_2) \\ = -m_2gL_2\sin\phi_2.$$

If both ϕ_1 and ϕ_2 are small, then $\cos(\phi_1-\phi_2)\approx 1$ and $\sin\phi\approx\phi$ (for either angle). Also the last term on the left of both equations is the product of three small quantities and can be neglected. With these approximations, the two equations here reduce to precisely Equations (11.41) and (11.42).

11.16 ⋆⋆ **(a)** The matrices **M** and **K** are given in Eq.(11.44) and the determinant of $\mathbf{K}-\omega^2\mathbf{M}$ is easily evaluated to give

$$\det(\mathbf{K}-\omega^2\mathbf{M}) = m_2L_1L_2[m_1L_1L_2\omega^4 - Mg(L_1+L_2)\omega^2 + Mg^2]$$

where $M = m_1+m_2$. The normal frequencies are found by setting this equal to zero and are

$$\omega^2 = \frac{Mg(L_1+L_2)\pm\sqrt{M^2g^2(L_1+L_2)^2 - 4m_1ML_1L_2g^2}}{2m_1L_1L_2} \qquad \text{(x)}$$

(b) Putting $m_1 = m_2$ and $L_1 = L_2 = L$, we find $\omega^2 = (2\pm\sqrt{2})g/L$ in agreement with (11.47).

(c) In the limit that $m_2\to 0$, Eq.(x) becomes

$$\omega^2 = \frac{g}{2L_1L_2}[(L_1+L_2)\pm(L_1-L_2)] = \frac{g}{L_2} \text{ or } \frac{g}{L_1}.$$

The first of these corresponds to the very light lower pendulum oscillating at its natural frequency while the upper remains unaffected and stationary. The second has the upper heavy pendulum swinging at its natural frequency, unaffected by the presence of the very light lower one.

11.17 ⋆⋆ **(a)** From (11.44) we find

$$\mathbf{M} = mL^2 \begin{bmatrix} 9 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = mL^2\omega_o^2 \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}.$$

whence

$$\mathbf{K} - \omega^2\mathbf{M} = mL^2 \begin{bmatrix} 9(\omega_o^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_o^2 - \omega^2) \end{bmatrix} \tag{xi}$$

and $\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2L^4(4\omega^2 - 3\omega_o^2)(2\omega^2 - 3\omega_o^2)$. Thus the normal frequencies are

$$\omega_1^2 = \tfrac{3}{4}\omega_o^2 \qquad \text{and} \qquad \omega_2^2 = \tfrac{3}{2}\omega_o^2.$$

If we put $\omega = \omega_1$ in Eq.(xi), the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ implies that $3a_1 = a_2$. If we put $\omega = \omega_2$, we find $3a_1 = -a_2$. Thus,

$$\text{for the first mode, } \mathbf{a} = A_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and for the second, } \mathbf{a} = A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

where A_1 and A_2 are arbitrary complex constants. In the first mode the pendulums oscillate in phase with the amplitude of the lower one three times that of the upper. In the second mode the pendulums oscillate exactly out of phase with the amplitude of the lower one three times that of the upper.

(b) The general solution is a sum of the two modes given above. The initial conditions that $\phi_1(0) = 0$ while $\phi_2(0) = \alpha$ and $\dot\phi_1 = \dot\phi_2 = 0$ require that

$$\mathrm{Re}\left\{A_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \qquad \text{and} \qquad \mathrm{Re}\left\{i\omega_1 A_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + i\omega_2 A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

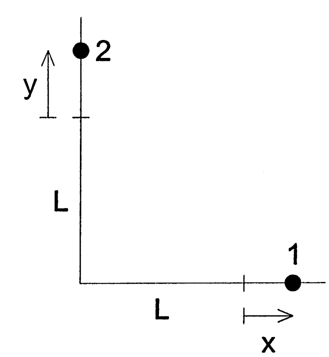
which imply that $A_1 = -A_2 = \alpha/6$. Putting in the time dependent exponentials and taking the real part, we conclude that

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{\alpha}{6}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cos\omega_1 t - \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cos\omega_2 t\right\}.$$

Because $\omega_2 = \sqrt{2}\omega_1$, this is not periodic.

11.18 ⋆⋆ Let the equilibrium length of the first two springs be L and that of the one that connects the two masses $\sqrt{2}L$. Let x and y be the displacements of the two masses from their equilibrium positions, as shown. The total KE is $T = \frac{1}{2}m(\dot x^2 + \dot y^2)$ and the total PE is

$$U = \tfrac{1}{2}k(x^2 + y^2) + \tfrac{1}{2}k'z^2,$$



where z is the extension of the diagonal spring. Given that we are interested in small displacements, the extension z can be calculated as follows:

$$z = \sqrt{(L+x)^2 + (L+y)^2} - \sqrt{2}L \approx \sqrt{2L^2 + 2L(x+y)} - \sqrt{2}L$$
$$= \sqrt{2}L\left(\sqrt{1+(x+y)/L} - 1\right) \approx \sqrt{2}L\cdot\tfrac{1}{2}(x+y)/L$$

where for the last expression on the first line I dropped terms higher than linear in x and y and for the final expression I used the binomial approximation for the square root. Substituting into U we get

$$U = \tfrac{1}{2}k(x^2+y^2) + \tfrac{1}{4}k'(x+y)^2 = \tfrac{1}{2}\left[(k+k'/2)x^2 + (k+k'/2)y^2 + k'xy\right]$$

The matrices $\mathbf{M}$ and $\mathbf{K}$ can be read off directly from the expressions for T and U. Thus, $\mathbf{M} = m\mathbf{1}$ and

$$\mathbf{K} = \begin{bmatrix} (k+k'/2) & k'/2 \\ k'/2 & (k+k'/2) \end{bmatrix} \text{ and } \mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} (k+k'/2) - m\omega^2 & k'/2 \\ k'/2 & (k+k'/2) - m\omega^2 \end{bmatrix}.$$

From this last we find $\det(\mathbf{K}-\omega^2\mathbf{M}) = (m\omega^2-k)(m\omega^2-k-k')$. Thus the normal frequencies are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{(k+k')/m}$.

The corresponding motions are determined by the vector $\mathbf{a}$ satisfying $(\mathbf{K}-\omega^2\mathbf{M})\mathbf{a} = 0$. For the first mode ($\omega = \omega_1$) this gives $a_1 = -a_2$. That is, in the first mode the two masses oscillate with equal amplitudes but out of step (as x increases y decreases and vice versa). For small oscillations, this means the length of the diagonal spring is constant, so its presence is irrelevant. In the second mode, we find $a_1 = a_2$ and the two masses oscillate in step (both x and y increase and then both decrease). In this mode the second spring changes its length, which increases the frequency compared to the first mode.

11.19 ⋆⋆⋆ **(a)** The PE (measured from the equilibrium positions) is

$$U = \tfrac{1}{2}kx^2 + MgL(1-\cos\phi) \approx \tfrac{1}{2}kx^2 + \tfrac{1}{2}MgL\phi^2.$$

The KE is a bit trickier because the velocity of the bob M is the vector sum of the velocity $\dot{x}$ of the support and $L\dot{\phi}$ of the bob relative to the support. In general these are in different directions, but as long as ϕ remains small they are essentially parallel, and the KE is

$$T \approx \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M(\dot{x}+L\dot{\phi})^2 = \tfrac{1}{2}(m+M)\dot{x}^2 + ML\dot{x}\dot{\phi} + \tfrac{1}{2}ML^2\dot{\phi}^2.$$

From these you can write down the Lagrangian $\mathcal{L}$ and the two Lagrange equations. The x equation is

$$(m+M)\ddot{x} + ML\ddot{\phi} = -kx$$

and the ϕ equation is

$$ML\ddot{x} + ML^2\ddot{\phi} = -MgL\phi.$$

We can write these as a single matrix equation, $\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}$, if we define

$$\mathbf{q} = \begin{bmatrix} x \\ \phi \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m+M & ML \\ ML & ML^2 \end{bmatrix}, \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k & 0 \\ 0 & MgL \end{bmatrix}.$$

(b) With the given values, the matrix $(\mathbf{K}-\omega^2\mathbf{M})$ is

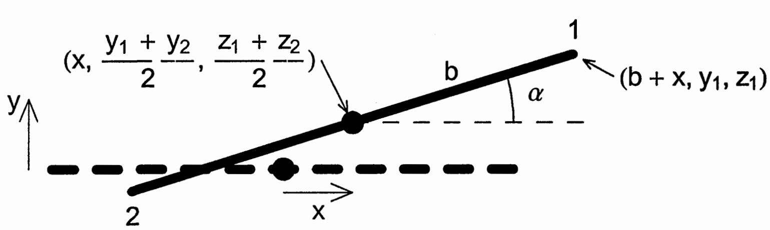
$$\mathbf{K}-\omega^2\mathbf{M}=\begin{bmatrix}2-2\omega^2 & -\omega^2\\ -\omega^2 & 1-\omega^2\end{bmatrix}$$

which has determinant $\det(\mathbf{K}-\omega^2\mathbf{M}) = 2(1-\omega^2)^2-\omega^4 = \omega^4-4\omega^2+2$. The normal frequencies are the zeros of this determinant and are

$$\omega_1=\sqrt{2-\sqrt{2}}=0.77 \qquad \text{and} \qquad \omega_2=\sqrt{2+\sqrt{2}}=1.85.$$

The motion in each corresponding mode is given by the equation $(\mathbf{K}-\omega^2\mathbf{M})\mathbf{a}=0$. For the first mode, this gives $a_2=\sqrt{2}a_1$, so the cart and bob oscillate in phase (both moving to the right and then both moving to the left), with the bob's amplitude (of motion relative to the cart) $\sqrt{2}$ times bigger than the cart's. For the second mode, $a_2=-\sqrt{2}a_1$, so the cart and bob oscillate exactly out of phase, again with the amplitude of the bob equal to $\sqrt{2}$ times that of the cart.

11.20 ⋆⋆⋆ The picture shows the rod as seen from vertically above, in its equilibrium position (dashed) and at an arbitrary small displacement (solid). The three coordinate directions are: x to the right, y toward the top of the page, and z vertically up, out of the page. The CM of the rod has coordinate x, while (y_1, z_1) and (y_2, z_2) refer to the two ends of the rod. The angle α measures the rotation of the rod. The positions of the CM and of the end 1 are as indicated in the picture (at least if the displacements are small).



The angle α is not an independent variable, because $\alpha=(y_1-y_2)/2b$. Similarly the coordinates z_1 and z_2 are not independent, being determined by the fixed length of the strings. For example, the length of string 1 is $\sqrt{x^2+y_1^2+(l-z_1)^2}$. Because this is also equal to l, we see that (for small displacements) $z_1=(x^2+y_1^2)/2l$, with a similar expression for z_2. Thus the PE is

$$U=mgz_{\rm cm}=mg\frac{z_1+z_2}{2}=\frac{mg}{4l}(2x^2+y_1^2+y_2^2)$$

while the KE is

$$T=\tfrac{1}{2}mv_{\rm cm}^2+\tfrac{1}{2}I\dot{\alpha}^2=\frac{m}{2}\left[\dot{x}^2+\left(\frac{\dot{y}_1+\dot{y}_2}{2}\right)^2\right]+\frac{1}{2}\left(\frac{1}{3}mb^2\right)\left(\frac{\dot{y}_1-\dot{y}_2}{2b}\right)^2$$

$$=\frac{m}{2}\left(\dot{x}^2+\frac{1}{3}\dot{y}_1^2+\frac{1}{3}\dot{y}_2^2+\frac{1}{3}\dot{y}_1\dot{y}_2\right)$$

and the Lagrangian is, as usual, $\mathcal{L} = T - U$. The three equations of motion are (after a little simplification and setting $g/l = \omega_o^2$)

$$\begin{aligned} \ddot{x} &= -\omega_o^2 x & &[x \text{ equation}] \\ 2\ddot{y}_1 + \ddot{y}_2 &= -3\omega_o^2 y_1 & &[y_1 \text{ equation}] \\ \ddot{y}_1 + 2\ddot{y}_2 &= -3\omega_o^2 y_2 & &[y_2 \text{ equation}] \end{aligned}$$

The x equation is completely uncoupled from the other two, and the first normal mode has the devise oscillating in the x direction only just like a simple pendulum, with frequency ω_o, as you probably anticipated. The equations for y_1 and y_2 are coupled, but if you add them you will find that $\ddot{y}_1 + \ddot{y}_2 = -\omega_o^2(y_1 + y_2)$. This says that the whole pendulum can oscillate in the y direction just like a simple pendulum, with frequency ω_o. Finally, if you subtract the y_2 equation from the y_1, you will find that $\ddot{y}_1 - \ddot{y}_2 = -3\omega_o^2(y_1 - y_2)$. This says that the devise can oscillate in a twisting motion (ends 1 and 2 moving oppositely) with frequency $\sqrt{3}\omega_o$.

11.21 $\star$ Let us focus our attention on a single coordinate q_i. The double sum that defines U contains n^2 terms because both j and k run from 1 to n. We can divide these n^2 terms into four groups: first, a single term with $j = k = i$; second, $(n-1)$ terms with $j = i$ but $k \neq i$; third, $(n-1)$ terms with $j \neq i$ but $k = i$; and finally $(n-1)^2$ terms with both j and k not equal to i. This gives

$$U = \tfrac{1}{2}K_{ii}q_i^2 + \tfrac{1}{2}\sum_{k\neq i} K_{ik}q_i q_k + \tfrac{1}{2}\sum_{j\neq i} K_{ji}q_j q_i + \tfrac{1}{2}\sum_{j\neq i}\sum_{k\neq i} K_{jk}q_j q_k \qquad \text{(xii)}$$

$$= \tfrac{1}{2}K_{ii}q_i^2 + q_i\sum_{j\neq i} K_{ij}q_j + \tfrac{1}{2}\sum_{j\neq i}\sum_{k\neq i} K_{jk}q_j q_k\,. \qquad \text{(xiii)}$$

Here, in passing from (xii) to (xiii), I replaced the dummy index k in the second term on the right of (xii) by j and used the fact that $K_{ji} = K_{ij}$ in the third term. We can now differentiate with respect to q_i to give [note that the whole third sum in (xiii) is independent of q_i, so its derivative is zero.]

$$U = K_{ii}q_i + \sum_{j\neq i} K_{ij}q_j = \sum_j K_{ij}q_j\,.$$

11.22 $\star$ The position of any one mass relative to its equilibrium position is

$$(x, y) = [L\sin\phi, L(1-\cos\phi)]$$

The gravitational PE of mass 1 is therefore $U_1^{\text{gr}} = mgL(1-\cos\phi_1)$, with corresponding expressions for the other two masses. The length of the first spring is

$$d_1 = \sqrt{(d_o + L\sin\phi_2 - L\sin\phi_1)^2 + (L\cos\phi_2 - L\cos\phi_1)^2} \qquad \text{(xiv)}$$

where d_o is the unstretched length of either spring. The PE of spring 1 is therefore

$$U_1^{\text{sp}} = \tfrac{1}{2}k(d_1 - d_o)^2 = \tfrac{1}{2}k\left(\sqrt{(d_o + L\sin\phi_2 - L\sin\phi_1)^2 + (L\cos\phi_2 - L\cos\phi_1)^2} - d_o\right)^2 \quad \text{(xv)}$$

with a corresponding expression for $U_2^{\rm sp}$ (just replace ϕ_2 and ϕ_1 by ϕ_3 and ϕ_2). The total PE is then

$$U = U_1^{\rm gr} + U_2^{\rm gr} + U_3^{\rm gr} + U_1^{\rm sp} + U_2^{\rm sp}. \tag{xvi}$$

If all three angles are small the gravitational PE's simplify as usual to $U_1^{\rm gr} \approx \frac{1}{2}mgL\phi_1^2$ and so on. The length (xiv) becomes $d_1 \approx d_{\rm o} + L\phi_2 - L\phi_1$ and the PE (xv) becomes $U_1^{\rm sp} \approx \frac{1}{2}kL^2(\phi_2 - \phi_1)^2$ with a corresponding expression for $U_2^{\rm sp}$. Putting all of these into (xvi) (and setting $m = L = 1$) we obtain the approximate expression (11.68).

11.23 ⋆⋆ Equation (11.73) asserts that $\omega_1^2 = g$, $\omega_2^2 = g + k$, and $\omega_3^2 = g + 3k$ in a system of units where $L = m = 1$, that is, a system in which the unit of length is L and the unit of mass is m. Taking the difference of the first two results of (11.73), we see that $\omega_2^2 - \omega_1^2 = k$ or

$$\frac{k}{\omega_2^2 - \omega_1^2} = 1 \tag{xvii}$$

in a system of units where the unit of mass is m. Now, as you can easily check, the dimension of k is $[k] = MT^{-2}$ while that of ω is $[\omega] = T^{-1}$. Therefore, $[k/(\omega_2^2 - \omega_1^2)] = M$, and (xvii) asserts that $k/(\omega_2^2 - \omega_1^2)$ is a mass whose value is 1 in our special units where $m = 1$. Therefore its value in any system is

$$\frac{k}{\omega_2^2 - \omega_1^2} = m$$

and $\omega_2^2 = \omega_1^2 + k/m = g/L + k/m$. Similarly $\omega_3^2 = g/L + 3k/m$.

11.24 ⋆⋆ If we let the strings have length $L_{\rm o}$ when both masses are in their equilibrium positions, then when the first mass is at position y_1 the length of the first string is $L = \sqrt{L_{\rm o}^2 + y_1^2} \approx L_{\rm o}(1 + \frac{1}{2}y_1^2/L_{\rm o}^2)$. The string is stretched by an amount $d = \frac{1}{2}y_1^2/L_{\rm o}$ and, with the tension T essentially constant, its PE is therefore $Td = \frac{1}{2}y_1^2T/L_{\rm o}$. Using the same argument for the other two strings, you can check that the total PE is

$$U = \tfrac{1}{2}[y_1^2 + (y_1 - y_2)^2 + y_2^2]T/L_{\rm o} = [y_1^2 - y_1y_2 + y_2^2]T/L_{\rm o}$$

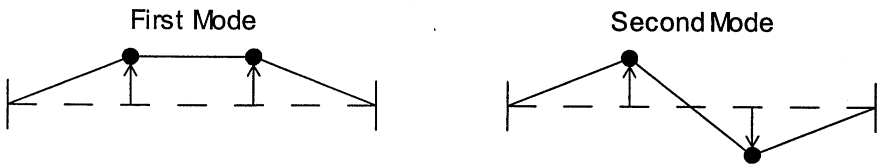
and the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2) - [y_1^2 - y_1y_2 + y_2^2]T/L_{\rm o}.$$

The matrix $(\mathbf{K} - \omega^2\mathbf{M})$ is

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} 2T/L_{\rm o} - \omega^2 m & -T/L_{\rm o} \\ -T/L_{\rm o} & 2T/L_{\rm o} - \omega^2 m \end{bmatrix}.$$

The eigenvalues are $\omega_1^2 = T/(mL_{\rm o})$ and $\omega_2^2 = 3T/(mL_{\rm o})$. For the first mode, the vector that describes the motion is $\mathbf{a} = A(1,1)$ (actually a 2×1 column), and the two masses oscillate in unison. For the second mode $\mathbf{a} = A(1,-1)$, and the two masses oscillate with equal amplitudes but exactly out of phase as shown on the next page.



11.25 ⋆⋆ The mass and spring constant matrices are

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

from which we find

$$\det(K - \omega^2 M) = (2\omega_o^2 - \omega^2)\left[(2-\sqrt{2})\omega_o^2 - \omega^2\right]\left[(2+\sqrt{2})\omega_o^2 - \omega^2\right]$$

where I have introduced the shorthand, $\omega_o = \sqrt{k/m}$. Thus the normal frequencies are

$$\omega_1 = \omega_o\sqrt{2-\sqrt{2}}, \qquad \omega_2 = \omega_o\sqrt{2}, \qquad \text{and} \qquad \omega_3 = \omega_o\sqrt{2+\sqrt{2}}$$

In the first mode, the eigenvector $\mathbf{a}$ has $a_1 = a_3 = a_2/\sqrt{2}$, so all three carts oscillate in phase, with the middle cart's amplitude $\sqrt{2}$ bigger than the outer two. In the second, $a_2 = 0$, while $a_1 = -a_3$, so the middle cart is stationary, while the first and third oscillate exactly out of phase. In the third mode, $a_1 = a_3 = -a_2/\sqrt{2}$, so the first and third carts oscillate in phase, while the middle one is exactly out of phase with amplitude $\sqrt{2}$ times bigger.

11.26 ⋆⋆ The moment of inertia of the hoop about its edge is $I = 2mR^2$, so its KE is $T_1 = \frac{1}{2}I\dot{\phi}_1^2 = mR^2\dot{\phi}_1^2$. The speed of the bead is (for small oscillations) just $v_2 = R(\dot{\phi}_1 + \dot{\phi}_2)$, so its KE is $T_2 = \frac{1}{2}mR^2(\dot{\phi}_1 + \dot{\phi}_2)^2$. Therefore

$$T = \tfrac{1}{2}mR^2(3\dot{\phi}_1^2 + 2\dot{\phi}_1\dot{\phi}_2 + \dot{\phi}_2^2).$$

The total PE is

$$U = U_1 + U_2 = mgR(1-\cos\phi_1) + mgR[(1-\cos\phi_1) + (1-\cos\phi_2)] \approx \tfrac{1}{2}mgR(2\phi_1^2 + \phi_2^2)$$

for small oscillations. Therefore the matrices $\mathbf{M}$ and $\mathbf{K}$ are

$$\mathbf{M} = mR^2\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = mR^2\omega_o^2\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\omega_o = \sqrt{g/R}$, the frequency of a pendulum of length R. From these you can check that $\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2R^4(2\omega^2 - \omega_o^2)(\omega^2 - 2\omega_o^2)$, so that the natural frequencies are given by $\omega_1^2 = \frac{1}{2}\omega_o^2$ and $\omega_2^2 = 2\omega_o^2$. If we substitute $\omega = \omega_1$ into the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find $a_1 = a_2$. Thus in the first mode the two angles oscillate in phase with equal amplitudes; that is, the bead and hoop oscillate together, with the bead stationary relative to the hoop. Substituting $\omega = \omega_2$, we find $a_2 = -2a_1$ so the bead and hoop oscillate 180° out of phase with the amplitude of ϕ_2 twice that of ϕ_1.

11.27 ⋆⋆ **(a)** The Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \tfrac{1}{2}k(x_1 - x_2)^2 = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \tfrac{1}{2}k(x_1^2 - 2x_1x_2 + x_2^2)$$

so the matrices $\mathbf{M}$ and $\mathbf{K}$ are

$$\mathbf{M} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = m\omega_o^2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(where $\omega_o^2 = k/m$) and $\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2\omega^2(\omega^2 - 2\omega_o^2)$. Therefore, the two normal frequencies are $\omega_1 = 0$ and $\omega_2 = \sqrt{2}\omega_o$.

(b) If we put $\omega = \omega_2$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ requires that $a_2 = -a_1$. That is, in mode 2, the two carts oscillate exactly out of phase while their CM remains stationary.

(c) If we put $\omega = \omega_1 = 0$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ reduces to $\mathbf{Ka} = 0$, which requires that $a_2 = a_1$; that is, $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (times any constant). If we try a solution of the form $\mathbf{x}(t) = \mathbf{a}f(t)$, then the equation of motion $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{Kx}$ becomes $\ddot{f} = 0$, so that $f(t) = x_o + v_o t$. In this mode the separation of the two carts is constant and their CM moves with constant velocity. The mode is possible because neither cart is attached to any fixed point, so there are no external forces; total momentum is conserved, and one possible motion is uniform motion of the CM with no internal motion.

11.28 ⋆⋆ **(a)** For small oscillations, the KE is

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M(\dot{x} + L\dot{\phi}^2) = \tfrac{1}{2}(m + M)\dot{x}^2 + ML\dot{x}\dot{\phi} + \tfrac{1}{2}ML^2\dot{\phi}^2$$

while the PE is $U = MgL(1 - \cos\phi) \approx \tfrac{1}{2}MgL\phi^2$. Thus if we take x and ϕ as our coordinates (in that order) the matrices $\mathbf{M}$ and $\mathbf{K}$ are

$$\mathbf{M} = \begin{bmatrix} m+M & ML \\ ML & ML^2 \end{bmatrix} = M\begin{bmatrix} 1+\lambda & L \\ L & L^2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 0 & 0 \\ 0 & MgL \end{bmatrix} = M\begin{bmatrix} 0 & 0 \\ 0 & L^2\omega_o^2 \end{bmatrix}$$

where I have introduced the mass ratio $\lambda = m/M$ and the frequency $\omega_o = \sqrt{g/L}$. Therefore

$$(\mathbf{K} - \omega^2\mathbf{M}) = -M\begin{bmatrix} \omega^2(1+\lambda) & \omega^2 L \\ \omega^2 L & (\omega^2 - \omega_o^2)L^2 \end{bmatrix}$$

and (as you can check) $\det(\mathbf{K} - \omega^2\mathbf{M}) = ML^2\omega^2[\lambda\omega^2 - (1+\lambda)\omega_o^2]$. The normal frequencies are $\omega_1 = 0$ and $\omega_2 = \omega_o\sqrt{(1+\lambda)/\lambda}$.

(b) If we set $\omega = \omega_1 = 0$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ reduces to $\mathbf{Ka} = 0$, which requires that $a_2 = 0$; that is, $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (times any constant). If we try a solution of the form $\mathbf{x}(t) = \mathbf{a}f(t)$, then the equation of motion $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{Kx}$ becomes $\ddot{f} = 0$, so that $f(t) = x_o + v_o t$. In this mode, the cart moves with constant velocity, while the pendulum is stationary relative to the cart.

If we set $\omega = \omega_2 = 0$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ requires that $(1+\lambda)a_1 = -La_2$, so $x = A\cos(\omega_2 t - \delta)$ and $\phi = -A\cos(\omega_2 t - \delta)(1+\lambda)/L$. In this mode, the cart and bob oscillate in opposite directions leaving their CM stationary.

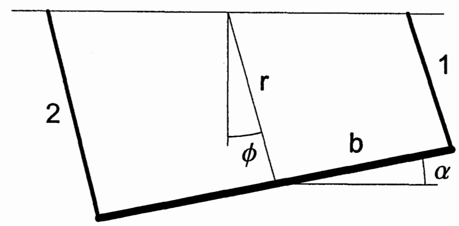
11.29 ⋆⋆⋆ I'll introduce the following notations:

L_o = unstretched length of either spring

r_o = distance of rod below ceiling in equilibrium

(r, ϕ) = position of CM of rod

α = angle between rod and horizontal



For small oscillations:

L_1 = (length of spring 1) $\approx r - b\alpha$

L_2 = (length of spring 2) $\approx r + b\alpha$

$r = r_o + \epsilon$, where ϵ is small.

In equilibrium, the total tension in the two springs must balance the weight of the rod, so

$$2k(r_o - L_o) = mg. \tag{xviii}$$

Spring 1 is stretched by the amount $L_1 - L_o$, so its PE is $U_1 = \frac{1}{2}k(L_1 - L_o)^2 = \frac{1}{2}k(r - L_o - b\alpha)^2$. Similarly, spring 2 has PE $U_2 = \frac{1}{2}k(L_2 - L_o)^2 = \frac{1}{2}k(r - L_o + b\alpha)^2$. Therefore, the total spring PE is

$$\begin{aligned} U_{\text{sp}} = U_1 + U_2 &= k(r - L_o)^2 + kb^2\alpha^2 \\ &= k(r_o - L_o + \epsilon)^2 + kb^2\alpha^2 \\ &= \text{const} + 2k(r_o - L_o)\epsilon + k\epsilon^2 + kb^2\alpha^2 \end{aligned} \tag{xix}$$

Meanwhile the gravitational PE is

$$U_{\text{gr}} = -mgr\cos\phi \approx -mg(r_o + \epsilon)(1 - \phi^2/2) \approx \text{const} - mg\epsilon + mgr_o\phi^2/2 \tag{xx}$$

where in the first approximation I set $\cos\phi \approx 1 - \phi^2/2$ and in the second I dropped all terms with products of three or more small quantities. The total PE is found by adding Eqs.(xix) and (xx). By Eq.(xviii), the terms linear in ϵ cancel, and we find

$$U = U_{\text{sp}} + U_{\text{gr}} = k\epsilon^2 + \tfrac{1}{2}mgr_o\phi^2 + kb^2\alpha^2.$$

The KE is

$$T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \tfrac{1}{2}I\dot{\alpha}^2 \approx \tfrac{1}{2}m\dot{\epsilon}^2 + \tfrac{1}{2}mr_o^2\dot{\phi}^2 + \tfrac{1}{6}mb^2\dot{\alpha}^2.$$

The Lagrangian is $\mathcal{L} = T - U$, as usual.

The r equation is $m\ddot{\epsilon} = -2k\epsilon$. This says the rod can oscillate vertically up and down, with angular frequency $\omega_1 = \sqrt{2k/m}$. The factor of 2 is because both springs are being stretched or compressed.

The ϕ equation is $mr_o^2\ddot{\phi} = -mgr_o\phi$. This says the rod can swing sideways like a simple pendulum with angular frequency $\omega_2 = \sqrt{g/r_o}$.

The α equation is $\frac{1}{3}mb^2\ddot{\alpha} = -2kb^2\alpha$. This says the rod can oscillate (up on the left and down on the right, then vice versa), with angular frequency $\omega_3 = \sqrt{6k/m}$. Notice that because $\mathcal{L}$ involves no cross terms (no terms involving more than one variable) the three Lagrange equations are uncoupled. That is, the three coordinates r, ϕ and α are already normal coordinates and oscillate independently.

11.30 ★★★ I'll use units such that m and k are both equal to 1. With these units the natural frequency $\omega_o = \sqrt{k/m}$ of a single cart on a single spring is just $\omega_o = 1$, and when we calculate the normal-mode frequencies they'll come out in units of ω_o. The matrices **M** and **K** are

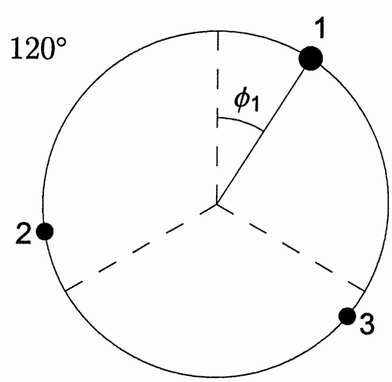
$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{1} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

The normal-mode frequencies are the solutions of the equation $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ and the corresponding motions are specified by the corresponding solutions **a** of the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. The results are

$$\begin{array}{c|cccc} \omega^2 & 0.382 & 1.382 & 2.618 & 3.618 \\ \hline \mathbf{a} & \begin{bmatrix} +1 \\ +1.62 \\ +1.62 \\ +1 \end{bmatrix} & \begin{bmatrix} -1 \\ -0.62 \\ +0.62 \\ +1 \end{bmatrix} & \begin{bmatrix} +1 \\ -0.62 \\ -0.62 \\ +1 \end{bmatrix} & \begin{bmatrix} -1 \\ +1.62 \\ -1.62 \\ +1 \end{bmatrix} \end{array}$$

where ω^2 is given in units of $\omega_o^2 = k/m$. In the first mode all four carts oscillate in phase. In the second, carts 1 and 2 are in phase with one another, but out of phase with carts 3 and 4. And so on.

11.31 ★★★ The three equilibrium positions are located at 120° to one another, as indicated by the dashed lines. The three masses are $m_1 = 2m$ and $m_2 = m_3 = m$, and their positions are their angles ϕ_1 and so on from their equilibrium positions. The total KE is



$$T = \tfrac{1}{2}mR^2(2\dot\phi_1^2 + \dot\phi_2^2 + \dot\phi_3^2)$$

and the PE is

$$U = \tfrac{1}{2}kR^2\left[(\phi_1 - \phi_2)^2 + (\phi_2 - \phi_3)^2 + (\phi_3 - \phi_1)^2\right]$$

To simplify the writing, I'll use units with $m = k = R = 1$. This means, in particular, that the unit of frequency is $\omega_o = \sqrt{k/m}$, the frequency of a single mass m on a single spring. The matrices **M** and **K** are

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

and

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 2(1-\omega^2) & -1 & -1 \\ -1 & 2-\omega^2 & -1 \\ -1 & -1 & 2-\omega^2 \end{bmatrix}.$$

Therefore, as you can check, $\det(\mathbf{K}-\omega^2\mathbf{M}) = -2\omega^2(\omega^2-2)(\omega^2-3)$ and the normal frequencies are $\omega_1 = 0$, $\omega_2 = \sqrt{2}$, and $\omega_3 = \sqrt{3}$. (In arbitrary units the last two would be $\omega_2 = \sqrt{2}\omega_o$, and $\omega_3 = \sqrt{3}\omega_o$.) If we put $\omega = 0$ in the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find that $a_1 = a_2 = a_3$. Thus, in the first mode the three masses move with constant speed around the hoop at their equilibrium separation. (Compare Problems 11.27 or 11.28.) If we put $\omega = \omega_2$, we find $a_1 = -a_2 = -a_3$; thus, in the second mode, mass 1 oscillates one way while the other two oscillate the other. Finally, with $\omega = \omega_3$, we find $a_1 = 0$ and $a_2 = -a_3$, and in the third mode, mass 1 is stationary, while 2 and 3 oscillate 180° out of phase.

11.32 ⋆⋆⋆ I'll introduce the notation $\lambda = M/m$ for the ratio of the two different masses, and I'll use units with $m = k = 1$. With this arrangement, any frequencies are measured in units of $\omega_o = \sqrt{k/m}$, the natural frequency of a mass m on a spring k. The total KE is $T = \frac{1}{2}(\dot{x}_1^2 + \lambda\dot{x}_2^2 + \dot{x}_3^2)$ and the PE is $U = \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}k(x_2 - x_3)^2$. The matrices $\mathbf{M}$ and $\mathbf{K}$ are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 1-\omega^2 & -1 & 0 \\ -1 & 2-\lambda\omega^2 & -1 \\ 0 & -1 & 1-\omega^2 \end{bmatrix}.$$

Therefore, as you can check, $\det(\mathbf{K} - \omega^2\mathbf{M}) = -\omega^2(\omega^2 - 1)(\lambda\omega^2 - 2 - \lambda)$ and the normal frequencies are $\omega_1 = 0$, $\omega_2 = 1$, and $\omega_3 = \sqrt{(2+\lambda)/\lambda}$. (In arbitrary units the last two would be $\omega_2 = \omega_o$, and $\omega_3 = \sqrt{(2+\lambda)/\lambda}\,\omega_o$.)

(b) If we put $\omega = \omega_2 = 1$ in the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find that $a_2 = 0$ and $a_1 = -a_3$. Thus, in the second mode the center atom is stationary, while the outer two oscillate with frequency ω_o and equal amplitudes but 180° out of phase. If we put $\omega = \omega_3$ in the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find that $a_1 = a_3 = -a_2(\lambda/2)$. Thus, in the third mode, atoms 1 and 3 oscillate in phase with the same amplitude, while the center atom oscillates 180° out of phase with amplitude $2/\lambda$ times that of the others.

(c) If we put $\omega = 0$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ becomes $\mathbf{K}\mathbf{a} = 0$, and we find that $a_1 = a_2 = a_3$. As in Problem 11.27, if we try a solution of the form $\mathbf{x} = \mathbf{a}f(t)$, the eqution of motion implies that $\ddot{f} = 0$, so all three atoms move with the same constant velocity separated by their equilibrium separation.

11.33 ⋆ From (11.78) $\mathbf{a}_{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{a}_{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $\mathbf{r}$ is any 2×1 column, $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, then by inspection

$$\mathbf{r} = \frac{x+y}{2}\mathbf{a}_{(1)} + \frac{x-y}{2}\mathbf{a}_{(2)}$$

11.34 ⋆⋆

$$\mathbf{a}_{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{a}_{(3)} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{xxi}$$

If $\mathbf{x}$ is an arbitrary 3×1 column (with elements x_1, x_2, x_3, or ϕ_1, ϕ_2, ϕ_3 for pendulums) then we can try expanding $\mathbf{x}$ in terms of the three eigenvectors (xxi),

$$\mathbf{x} = \xi_1 \mathbf{a}_{(1)} + \xi_2 \mathbf{a}_{(2)} + \xi_3 \mathbf{a}_{(3)}. \tag{xxii}$$

This gives three equations for the three coefficients ξ_1, ξ_2, ξ_3,

$$\xi_1 + \xi_2 + \xi_3 = x_1, \quad \xi_1 - 2\xi_3 = x_2 \quad \text{and} \quad \xi_1 - \xi_2 + \xi_3 = x_3.$$

These are easily solved to give

$$\xi_1 = \frac{x_1 + x_2 + x_3}{3}, \quad \xi_2 = \frac{x_1 - x_3}{2}, \quad \text{and} \quad \xi_3 = \frac{x_1 - 2x_2 + x_3}{6}.$$

If you substitute these values into the right side of Eq.(xxii), you can check that you do indeed get $\mathbf{x}$. Thus we have successfully expanded an arbitrary $\mathbf{x}$ in terms of the eigenvectors (xxi).

Normal coordinates are supposed to have the property that any one of them can oscillate while the other two remain zero and that this motion is one of the normal modes. For example, if ξ_1 oscillates while ξ_2 and ξ_3 remain zero, then the condition $\xi_2 = 0$ implies that $x_1 = x_3$ and then the condition $\xi_3 = 0$ implies that $x_1 = x_2 = x_3$. When $\xi_1 = (x_1 + x_2 + x_3)/3$ oscillates, this means that x_1, x_2, and x_3 all oscillate in phase with equal amplitudes, and this is indeed the first mode of Fig.11.14. You can check that the other two modes work similarly.

11.35 ⋆⋆ **(a)** The first normal mode has $\phi_1 = \phi_2$ and the second has $\phi_1 = -\phi_2$. Thus if we define $\xi_1 = \phi_1 + \phi_2$ and $\xi_2 = \phi_1 - \phi_2$, the first mode is an oscillation of ξ_1 with ξ_2 remaining zero, and the second mode is the other way round.

(b) The gravitational torque on either pendulum has the form $\Gamma_{\text{grav}} = -mgL\sin\phi \approx -mgL\phi$. The tension in the spring is approximately $kL(\phi_2 - \phi_1)$ and the velocity of either bob has the form $v = L\dot{\phi}$. The equations of motion in the form $I\ddot{\phi} = \Gamma$ read (after a little tidying up)

$$\ddot{\phi}_1 = -\left(\frac{g}{L} + \frac{k}{m}\right)\phi_1 + \frac{k}{m}\phi_2 - 2\beta\dot{\phi}_1 \quad \text{and} \quad \ddot{\phi}_2 = \frac{k}{m}\phi_1 - \left(\frac{g}{L} + \frac{k}{m}\right)\phi_2 - 2\beta\dot{\phi}_2$$

where $2\beta = b/m$. Adding these equations gives an equation for ξ_1 and subtracting them gives a separate equation for ξ_2:

$$\ddot{\xi}_1 = -\frac{g}{L}\xi_1 - 2\beta\dot{\xi}_1 \quad \text{and} \quad \ddot{\xi}_2 = -\left(\frac{g}{L} + \frac{2k}{m}\right)\xi_2 - 2\beta\dot{\xi}_2 .$$

(c) These equations can be solved as in Section 5.4 to give

$$\xi_1 = A_1 e^{-\beta t}\cos(\omega_1 t - \delta_1) \quad \text{and} \quad \xi_2 = A_2 e^{-\beta t}\cos(\omega_2 t - \delta_2)$$

where $\omega_1 = \sqrt{(g/L) - \beta^2}$ and $\omega_2 = \sqrt{(g/L) + (2k/m) - \beta^2}$, and $A_1, \delta_1, A_2, \delta_2$ are constants determined by the initial conditions. Both modes die out exponentially with the same decay constant β. In mode 1 ($\xi_2 = 0$) the two pendulums oscillate with frequency ω_1. In mode 2 ($\xi_1 = 0$), they oscillate out of phase with frequency ω_2.

Chapter 12

Nonlinear Mechanics and Chaos

This chapter is the first of the five chapters of Part II of the book. I tried to make all of these chapters mutually independent. Thus you can cover this chapter without covering any of the other four, and you can cover any of them without including this one. On the other hand, it was my plan that students should know most of the material of Part I (Chapters 1–11) before tackling any of Part II, though even this is not true in every case. In particular, you could cover Chapter 12 immediately after Chapter 5 on oscillations, if you're eager to get to some truly modern mechanics quickly (just as you could launch into Chapter 13 on Hamiltonian mechanics immediately after Chapter 7 on the Lagrangian approach).

Because the mathematics of chaos is distinctly harder than anything else in the book, this chapter is much more qualitative and descriptive than any of the others. For similar reasons, it is much more selective. In fact, the only mechanical system that I cover in any detail is the driven damped pendulum, and I entirely omit discussion of chaotic Hamiltonian systems. I know that this omission will offend some practitioners of chaos, but I felt it was much more important that the students' introduction to the subject center on systems that they could really understand.

It would be eminently possible to cover just some of this chapter. You could stop after Section 12.5 on sensitivity to initial condition, or after Section 12.6 on bifurcation diagrams. Or you could cover Sections 12.1–12.6, and then, omitting the sections on state-space orbits and Poincaré sections, jump to 12.9 on the logistic map.

As usual, there are some experiments you could bring into class. There are several cheap toys that show chaotic behavior. If you happen to have one of the beautiful chaotic pendulum made by Daedalon, that would be a sensation. You could also show several computer simulations.

I confess that I never covered any of the chapters of Part II in class. Instead, they were assigned as term projects, with one fifth of the class studying each of the five chapters. All of the chapters produced some excellent reports and solutions to end-of-chapter problems, though, as you might guess, the chapter on chaos really excited more of the students than did any other. In particular, several students produced some beautiful bifurcations diagrams. (But be warned that a couple of students put more energy into finding canned programs on the Web than into doing the problems themselves!)

Solutions for Problems for Chapter 12

12.1 ⋆ **(a)** The equation $\dot{x} = 2\sqrt{x-1}$ can be separated as $dx/(2\sqrt{x-1}) = dt$, which can be integrated to give $\sqrt{x-1} = t+k$. Solving for x we see that the function $x_1(t) = (t+k)^2+1$ is a solution of the original equation for any value of the constant k.

(b) Clearly the function $x_2(t) = 1$ satisfies the original equation and, equally clearly, it is not of the form $x_1(t)$ for any choice of the constant k.

(c) If $x_3 = Ax_1$, then $\dot{x}_3 = A\dot{x}_1 = 2A\sqrt{x_1 - 1} = 2A\sqrt{x_3/A - 1}$ which is not equal to $2\sqrt{x_3 - 1}$ unless $A = 1$. Similarly, if $x_4 = Bx_2 = B$, then $\dot{x}_4 = 0$, which is not equal to $2\sqrt{x_4 - 1}$ unless $B = 1$. Finally, if $x_5 = x_1 + x_2 = x_1 + 1$, then $\dot{x}_5 = \dot{x}_1 = 2\sqrt{x_1 - 1} = 2\sqrt{x_5}$, which is not equal to $2\sqrt{x_5 - 1}$.

12.2 ⋆ The equation $\dot{x} = 2\sqrt{x}$ can be separated as $dx/(2\sqrt{x}) = dt$, which can be integrated to give $\sqrt{x} = t + k$. Solving for x we see that the function $x = (t+k)^2$ is a solution of the original equation for any value of the constant k. The initial condition that $x(0) = 0$ implies that $k = 0$, and we have as one solution of the problem (differential equation plus initial condition) $x_1(t) = t^2$. On the other hand, it is clear by inspection that $x_2(t) = 0$ is another solution.

12.3 ⋆ **(a)** If x_1 and x_2 are solutions of (12.6), then

$$p\ddot{x}_1 + q\dot{x}_1 + rx_1 = 0 \qquad \text{and} \qquad p\ddot{x}_2 + q\dot{x}_2 + rx_2 = 0.$$

If we multiply the first of these by any constant a_1 and the second by a_2 and add, we find

$$p(a_1\ddot{x}_1 + a_2\ddot{x}_2) + q(a_1\dot{x}_1 + a_2\dot{x}_2) + r(a_1x_1 + a_2x_2) = 0.$$

That is, the function $a_1x_1 + a_2x_2$ is also a solution of the same differential equation.

(b) If x_1 and x_2 are solutions of the proposed nonlinear equation,

$$p\ddot{x} + q\dot{x} + r\sqrt{x} = 0 \tag{i}$$

then

$$p\ddot{x}_1 + q\dot{x}_1 + r\sqrt{x_1} = 0 \qquad \text{and} \qquad p\ddot{x}_2 + q\dot{x}_2 + r\sqrt{x_2} = 0. \tag{ii}$$

Let us consider whether the function $x = x_1 + x_2$ is also a solution of this same equation. To this end we try adding the two equations (ii) to give

$$p\ddot{x} + q\dot{x} + r(\sqrt{x_1} + \sqrt{x_2}) = 0. \tag{iii}$$

This equation is nearly, but not quite, the proposed equation (i). Unfortunately, unless either x_1 or x_2 is zero, $\sqrt{x_1} + \sqrt{x_2} \neq \sqrt{x_1 + x_2}$. Thus because $x = x_1 + x_2$ satisfies (iii), it does *not* satisfy (i), and the superposition principle does not hold for the nonlinear equation (i). [Another, slightly simpler, way to show the same thing would be to prove that ax_1 does not satisfy (i) unless $a = 0$ or 1.]

12.4 ⋆ **(a)** If both x and x_p satisfy the inhomogeneous linear equation (12.59), then by subtraction we find that $x - x_\mathrm{p}$ satisfies the homogeneous equation

$$p(\ddot{x} - \ddot{x}_\mathrm{p}) + q(\dot{x} - \dot{x}_\mathrm{p}) + r(x - x_\mathrm{p}) = 0.$$

Now, we know that any solution of this homogeneous linear equation can be written as a linear combination of any two independent solutions x_1 and x_2. Therefore, $x - x_\mathrm{p} = a_1x_1 + a_2x_2$, which is the required result (12.60).

(b) If both x and x_p satisfy the proposed nonlinear equation, then, proceeding as before, we find that

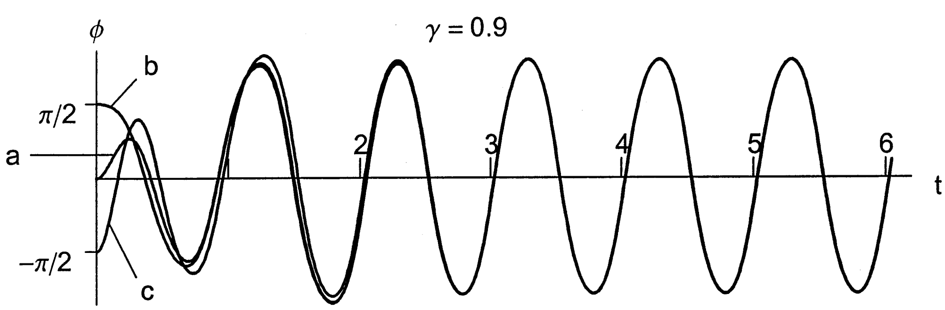
$$p(\ddot{x} - \ddot{x}_\mathrm{p}) + q(\dot{x} - \dot{x}_\mathrm{p}) + r(\sqrt{x} - \sqrt{x_\mathrm{p}}) = 0.$$

Because $\sqrt{x} - \sqrt{x_\mathrm{p}} \neq \sqrt{x - x_\mathrm{p}}$, this means that $x - x_\mathrm{p}$ does not satisfy corrresponding "homogeneous" equation, and we do not get a result corresponding to that of part (a).

12.5 ⋆ Because $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$,

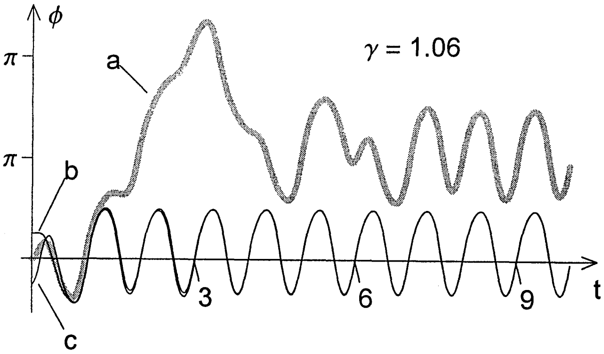
$$\begin{aligned}\cos^3 x &= \tfrac{1}{8}\left(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}\right)\\ &= \tfrac{1}{4}\left[\tfrac{1}{2}(e^{3ix} + e^{-3ix}) + \tfrac{3}{2}(e^{ix} + e^{-ix})\right] = \tfrac{1}{4}(\cos 3x + 3\cos x).\end{aligned}$$

12.6 ⋆⋆

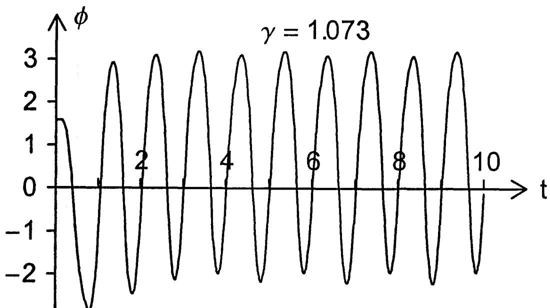


All of these three solutions certainly approach the same attractor.

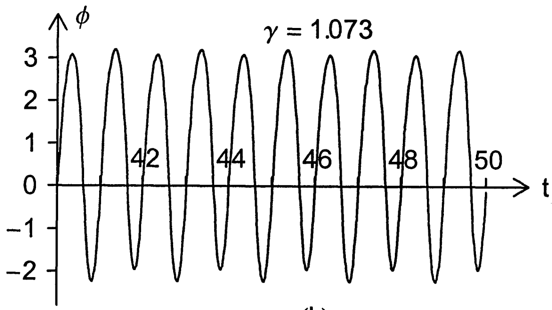
12.7 ⋆⋆ The solutions with the initial conditions (b) and (c) merge within about 3 cycles and oscillate nearly sinusoidally about $\theta = 0$. The solution (a) is much slower to settle down and eventually oscillates about $\theta = 2\pi$. To check that all three solutions do really have the same long- term behavior, you could plot $\theta(t) - 2\pi$ for solution (a); when this is done, all three agree perfectly (on the scale shown) after about $t = 8$.



12.8 ⋆⋆



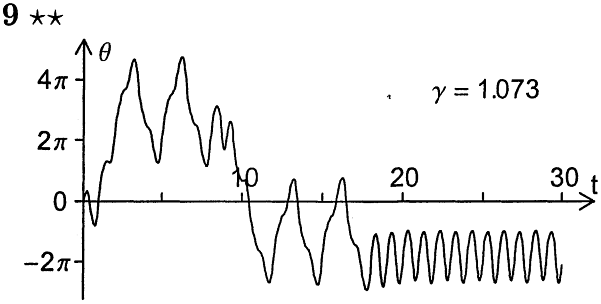
(a)

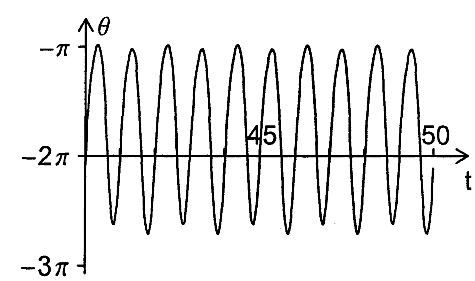


(b)

Having drawn part (b), we can see that, in fact, the motion has settled down after just three or four cycles. The long-term motion has period 2. (This is especially clear if you hold a horizontal ruler up to the plots.)

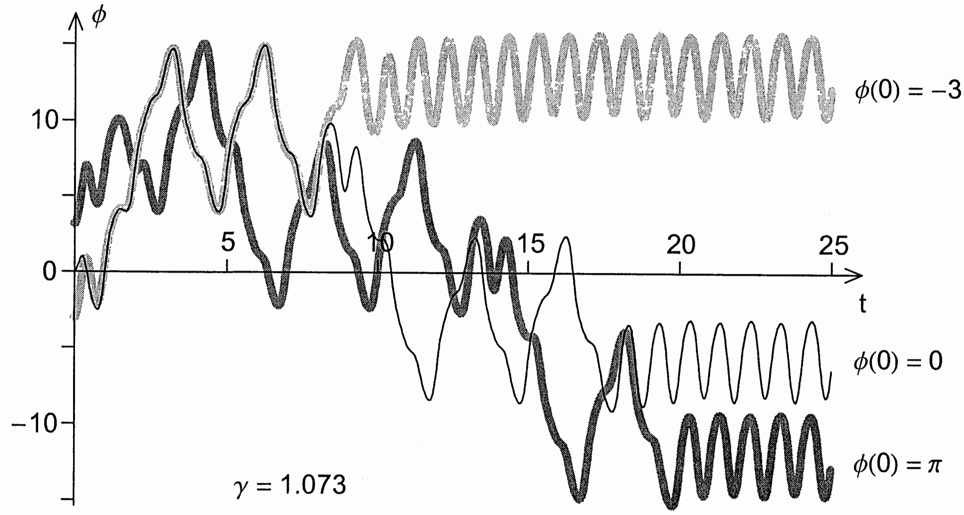
12.9 ⋆⋆





The left plot agrees well with Figure 12.5(a). From the right plot it is pretty clear that the period is 2. [Important note on precision: Although chaos has not set in, the solution is quite sensitive to initial conditions and hence also to rounding errors. I found the curves shown using Mathematica's NDSolve function, starting with the default precision of 15 significant figures. I then increased the precision to 16 (WorkingPrecision $\rightarrow$ 16), and again to 17, and on to 21 in integer steps. The curves for precision 15 and 16 looked quite like one another, but when I increased the precision to 17 the curve changed radically. From then on it made no visible change, indicating that the rounding errors had probably become negligible (for the range $0 \leq t \leq 50$)

12.10 ⋆⋆



I chose just three different initial conditions, all with $\dot{\phi}(0) = 0$ but with $\phi(0) = -3$, 0, and π, as indicated. The most obvious features are that the initial behaviors are very different, but that all three curves eventually become the same, within multiples of 2π. A remarkable feature is that the curves for $\phi(0) = -3$ and 0 quickly approach one another, so that from $t \approx 1$ to 8 it is hard to say if there is any difference, but the two curves then move rapidly apart, ending up three complete revolutions (that is, 6π) apart.

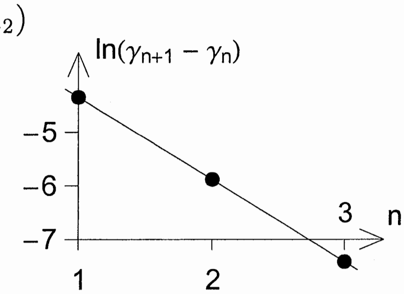
12.11 ⋆⋆ **(a)** $(\gamma_{n+1} - \gamma_n) = \dfrac{1}{\delta}(\gamma_n - \gamma_{n-1}) = \dfrac{1}{\delta^2}(\gamma_{n-1} - \gamma_{n-2})$

$$= \cdots\cdots = \frac{1}{\delta^{n-1}}(\gamma_2 - \gamma_1).$$

Therefore

$$\ln(\gamma_{n+1} - \gamma_n) = -(n-1)\ln\delta + \ln(\gamma_2 - \gamma_1).$$

(b) The least-squares line has slope -1.54, giving $\delta = e^{1.54} = 4.66$, compared with the correct value 4.67



12.12 ⋆⋆ **(a)** As in Problem 12.11, $(\gamma_{n+1} - \gamma_n) = (\gamma_2 - \gamma_1)/\delta^{n-1}$. Replacing n by $n-1$ and tidying up a bit, we can write this as $(\gamma_n - \gamma_{n-1}) = C/\delta^n = C\epsilon^n$, where C is a constant and $\epsilon = 1/\delta$. Applying this repeatedly, we find

$$\gamma_n = \gamma_{n-1} + C\epsilon^n = \gamma_{n-2} + C(\epsilon^{n-1} + \epsilon^n) = \cdots = \gamma_1 + C(\epsilon^2 + \epsilon^3 + \cdots \epsilon^n).$$

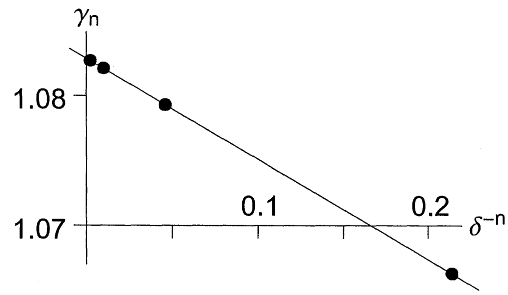
Since $|\epsilon| < 1$, this approaches a finite limit as $n \to \infty$,

$$\gamma_c = \gamma_1 + C(\epsilon^2 + \epsilon^3 + \cdots).$$

Taking the difference of the last two equations, we see that

$$\begin{aligned}\gamma_n &= \gamma_c - C(\epsilon^{n+1} + \epsilon^{n+2} + \cdots) \\ &= \gamma_c - C\epsilon^n(\epsilon + \epsilon^2 + \cdots) = \gamma_c - K\epsilon^n\end{aligned}$$

where $K = C(\epsilon + \epsilon^2 + \cdots)$. Since $\epsilon = 1/\delta$, this is the required relation.



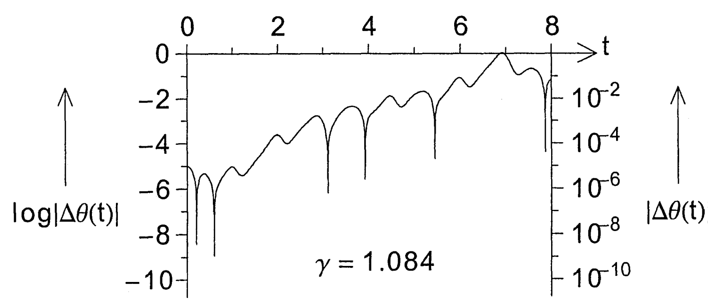
(b) The four data fit the straight line beautifully. The vertical intercept of the least-squarres line is at 1.08286, in excellent agreement with the value $\gamma_c = 1.0829$ claimed in (12.20).

12.13 ⋆ The crests in Fig.12.13 fall pretty close to a straight line with positive slope, confirming that the crests of $\Delta\phi$ grow exponentially. From the graph we see that $\Delta\phi(0) = 10^{-4}$ and $\Delta\phi(14.5) \approx 1$. According to Eq.(12.26), $\Delta\phi(t)/\Delta\phi(0) = e^{\lambda t}$. Therefore

$$\lambda = \frac{1}{t}\ln\left(\frac{\Delta\phi(t)}{\Delta\phi(0)}\right) = \frac{1}{14.5}\ln\left(\frac{1}{10^{-4}}\right) \approx 0.64\,.$$

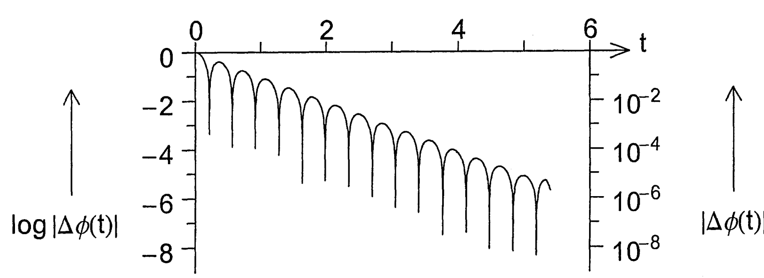
12.14 ⋆⋆

The crests in the graph of $\log|\Delta\phi(t)|$ increase linearly with t, indicating that $\Delta\phi(t)$ grows exponentially. This is the behavior that we expect when the motion is chaotic.



12.15 ⋆⋆

From $t = 0$ to 5, the crests of $\log|\Delta\phi(t)|$ decrease perfectly linearly, confirming that $\Delta\phi(t)$ decreases exponentially.

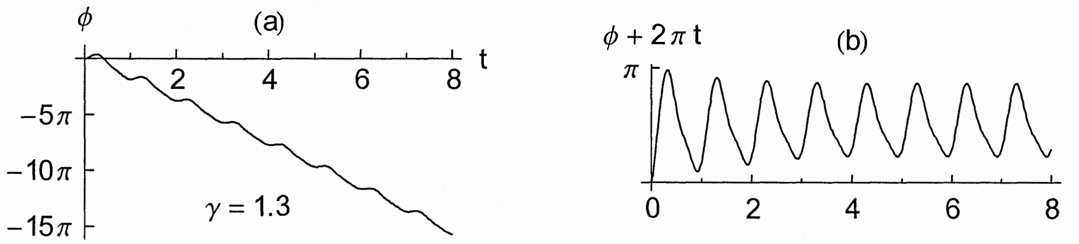


12.16 ⋆⋆ **(a)** The uncertainty $\Delta\phi(0)$ (the amount by which we recognize the true value of $\phi(0)$ may differ from our measured value) is 10^{-6} rad. According to (12.26), $\Delta\phi(t)$ grows exponentially, $\Delta\phi(t) = \Delta\phi(0)e^{\lambda t}$, and the maximum tolerable uncertainty is $\Delta\phi_{\text{max}} = 10^{-2}$. Therefore the time horizon t_{max} is determined by the condition $\Delta\phi_{\text{max}} = \Delta\phi(0)e^{\lambda t_{\text{max}}}$ or

$$t_{\text{max}} = \frac{1}{\lambda}\ln\left(\frac{\Delta\phi_{\text{max}}}{\Delta\phi(0)}\right) = 1 \times \ln(10^4) = 9.21\,. \tag{iv}$$

(b) If we reduce $\phi(0)$ from 10^{-6} to 10^{-9} rad, the argument of the natural log in Eq.(iv) increases by a factor of 10^3 and $t_{\text{max}} = 1 \times \ln(10^7) = 16.1$. The improvement by a factor of 1000 in the initial accuracy has increased t_{max} by a factor of only 1.75.

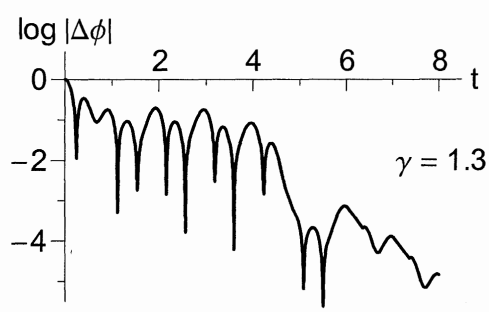
12.17 ⋆⋆ **(a)** The pendulum rolls through -2π each cycle, with a regular looking oscillation superposed.



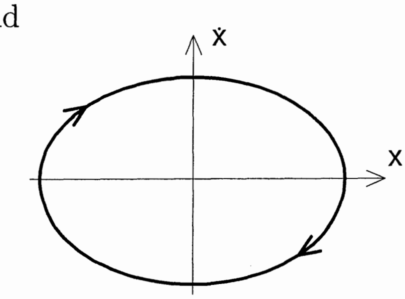
(b) In the plot of $\phi(t) + 2\pi t$, with the rolling motion subtracted out, we can see clearly that the oscillations become very regular after two or three cycles.

12.18 ⋆⋆

There is no disputing that $\log|\Delta\phi|$ decreases with time, and, while it does so somewhat erratically, one could probably say that it decreases approximately linearly. Therefore $\Delta\phi$ decreases approximately exponentially, as expected for nonchaotic motion.

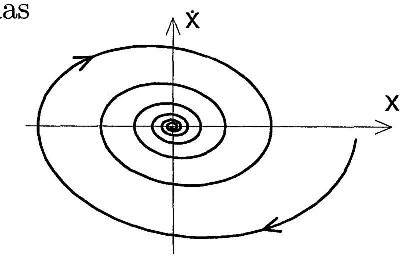


12.19 ⋆ **(a)** The general solution is $x = A\cos(\omega t - \delta)$ and hence $\dot{x} = -\omega A\sin(\omega t - \delta)$, where $\omega = \sqrt{k/m}$, and A and δ are arbitrary constants. These are the parametric equations of an ellipse in the plane of $(x, \dot{x})$. In the upper half plane ($\dot{x} > 0$), x is increasing and the orbit moves to the right; in the lower half plane ($\dot{x} < 0$), x is decreasing and the orbit moves to the left. Therefore, the orbit is traced clockwise.



(b) $E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = \text{const}$, which is the equation of an ellipse.

12.20 $\star$ **(a)** According to Eq.(5.38) the general motion has
$x = Ae^{-\beta t}\cos(\omega t - \delta)$ and $\dot{x} = -\omega Ae^{-\beta t}\sin(\omega t - \delta)$.
In state space (the two-dimensional space with coordinates x and $\dot{x}$), these are the parametric equations for an ellipse whose radius is shrinking exponentially. As time advances, the point representing the particle moves clockwise around this curve.



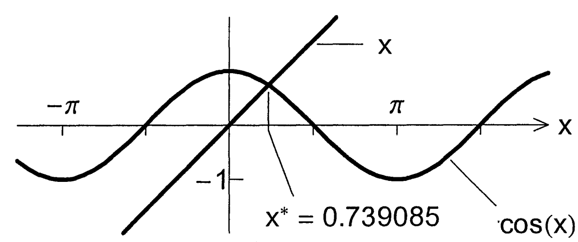
(b) As $t \to \infty$ both x and $\dot{x}$ approach zero. Thus the attractor is the origin of state space, $x = \dot{x} = 0$. In terms of energy, this is because the damping force dissipates the energy, so that $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \to 0$. This requires that both x and $\dot{x}$ tend to zero.

12.21 $\star\star$ **(a)** Here are the values of x_t for three different initial values, $x_0 = 0$, 3, and 100.

x_0	x_1	x_2	x_3	x_4	x_5	$\cdots$	x_{28}	x_{29}	x_{30}
0	1.00	0.54	0.86	0.65	0.793	$\cdots$	0.7391	0.7391	0.7391
3	−0.99	0.55	0.85	0.66	0.791	$\cdots$	0.7391	0.7391	0.7391
100	0.86	0.65	0.80	0.70	0.765	$\cdots$	0.7391	0.7391	0.7391

For all three initial values x_0, the sequence converges on the attractor $x = 0.7391$.

(b) The attractor must be a solution of the fixed-point equation $f(x^*) = x^*$. In the present case, this is $\cos(x^*) = x^*$, which has exactly one solution, $x^* = 0.739085$, and this is stable since $|f'(x^*)| < 1$.



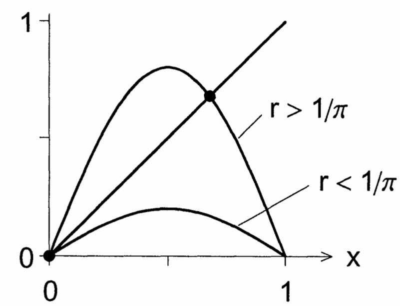
12.22 $\star\star$ **(a)** The map is $x_{t+1} = f(x_t)$ where $f(x) = x^2$. Thus the successive values are

$$x_0, x_0^2, x_0^4, x_0^8, \cdots$$

and $x_t = (x_0)^{2^n}$. A point x^* is fixed if and only if $f(x^*) = x^*$, that is, $x^{*2} = x^*$. Therefore there are just two fixed points, $x^* = 0$ and $x^* = 1$. To find whether a fixed point is stable, we have only to check whether $|f'(x^*)| < 1$. At the first fixed point, $x^* = 0$, we see that $f'(x^*) = 2x^* = 0$, so the fixed point at 0 is stable. At the second fixed point, $x^* = 1$, we see that $f'(x^*) = 2x^* = 2$, so the fixed point at 1 is unstable.

(b) Since $x_t = (x_0)^{2^n}$, we see that $x_t \to 0$ if and only if $|x_0| < 1$, that is, $-1 < x_0 < 1$.

(c) Similarly, $x_t \to \infty$ if and only if $|x_0| > 1$. These results show why the fixed point $x^* = 1$ is unstable. If x_0 is exactly 1, then, of course, all of the $x_t = 1$. But if we make x_0 a tiny bit bigger than 1, then x_t will eventually move off to infinity, and if we make x_0 a tiny bit smaller than 1, then x_t will eventually move off to zero.



12.23 ⋆⋆ **(a)** and **(b)** The fixed points x^* must satisfy $x^* = f(x^*)$; that is, they are determined by the intersections (if any) of the line $y = x$ and the curve $y = f(x)$, where, in this case, $f(x) = r \sin \pi x$. As you can see from the picture, there is always a fixed point at $x^* = 0$. Because the slope of the curve $y = f(x)$ at $x = 0$ is $f'(0) = r\pi$, we see that when $r < 1/\pi$, the only fixed point is at $x^* = 0$, but if we increase r until $r > 1/\pi$, then a second fixed point appears. The value of r at which this happens is evidently $r_0 = 1/\pi$.

The test for stability is that $|f(x^*)| < 1$. Thus the fixed point at $x^* = 0$ is stable provided $f'(0) = r\pi < 1$ and unstable if $r\pi > 1$. Therefore this first fixed point becomes unstable at the same value $r_0 = 1/\pi$ at which the second fixed point appears.

(c) The second fixed point at $x^* > 0$ is stable as long as $|f'(x^*)| < 1$. As we increase r, this fixed point becomes unstable at the value $r = r_1$ for which $f'(x^*) = -1$. This transition is determined by the two equations

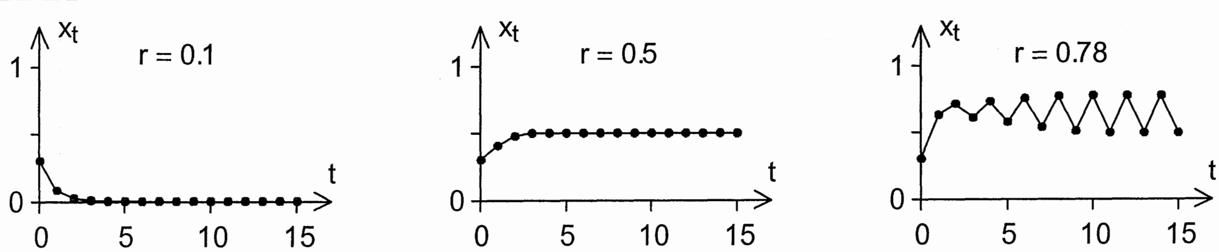
$$f(x^*) = x^* \quad \text{that is,} \quad r_1 \sin(\pi x^*) = x^*$$

and

$$f'(x^*) = -1 \quad \text{that is,} \quad r_1 \pi \cos(\pi x^*) = -1.$$

These two simultaneous equations for r_1 and x^* can be solved numerically in several ways. (One straightforward method is to use Mathematica's FindRoot.) The result is that $r_1 = 0.71996$.

12.24 ⋆⋆

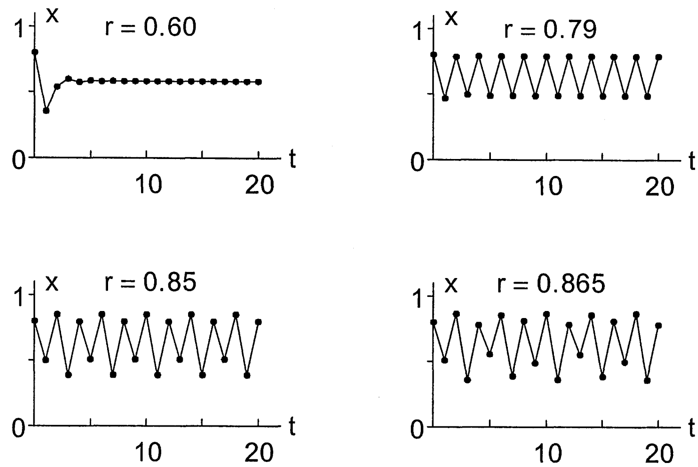


(a) When $r = 0.1$, x_t approaches 0, consistent with the result of Problem 12.23 that when $r < 1/\pi = 0.32$ the only stable attractor is $x^* = 0$.

(b) When $r = 0.5$, x_t approaches 0.5, consistent with the result of Problem 12.23 that when $0.32 < r < 0.72$ there is only one stable attractor given by $r \sin(\pi x^*) = x^*$ (which, with $r = 0.5$ happens to give $x^* = 0.5$).

(c) When $r = 0.78$, x_t does not approach any one limit. This is consistent with the result of Problem 12.23 that when $r > 0.72$ there is no stable attractor. Instead x_t oscillates between two fixed values.

12.25 ⋆⋆ When $r = 0.6$, x_t approaches a single attractor at $x^* = 0.58$. When $r = 0.79$, x_t is exhibiting period 2 oscillations, bouncing between 0.49 and 0.79. By the time $r = 0.85$ the period has doubled again to period 4, and when $r = 0.865$ it is period 8.



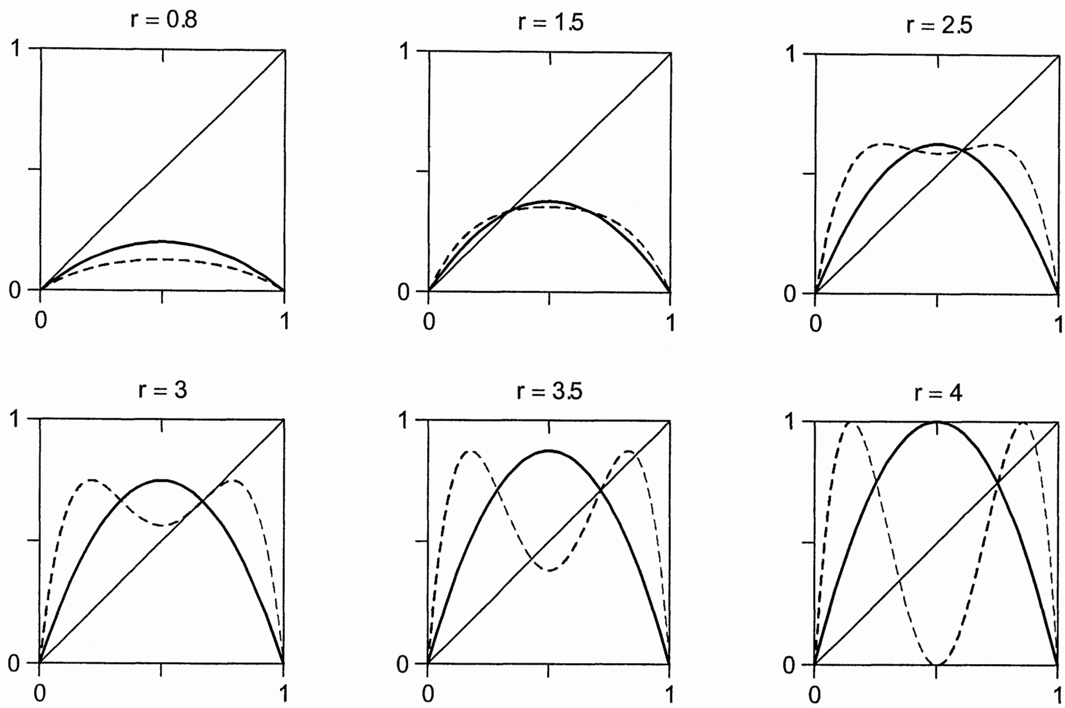
12.26 ⋆⋆ As suggested, I'll assume that $f(x) = r\phi(x)$. If our map is the logistic map, then $\phi(x) = x(1-x)$. For the sine map, $\phi(x) = \sin(\pi x)$. The same conclusions apply to any map for which $\phi(x)$ is a single arch, but to be definite I'll consider explicitly the familiar case of the logistic map. I'll first plot the functions $f(x)$ and $g(x) = f[f(x)]$ for six successive values of the growth parameter $r = 0.8$, 1.5, 2.5, 3, 3.5, and 4. Then I'll discuss why they behave the way they do. In the plots I have shown $f(x)$ as a solid curve and $g(x)$ with dashes. I have also shown the 45° line, whose intersections with $f(x)$ are the fixed points. Because both $f(x)$ and $g(x)$ are symmetric about the line $x = 0.5$, we actually need to analyze their behavior only for $0 \leq x \leq 0.5$.

As the growth parameter r increases, the arch of $f(x)$ steadily rises. For each value of r, if we let x move from 0 to 0.5, $f(x)$ increases monotonically from 0 to a maximum, $f_{\max}$ at $x = 0.5$. While $r < 1$ (first picture) the curve of $f(x)$ is entirely below the 45° line, and its only intersection with that line is at $x = 0$; that is, the only fixed point of the map is $x = 0$.

When $r > 1$ the initial slope of $f(x)$ (namely r) puts $f(x)$ above the 45° line. Thus $f(x)$ has to intersect that line a second time (as the curve bends back down to 0 at $x = 1$) and a second fixed point has materialized (second picture).

Until we reach $r = 2$, both of the curves $f(x)$ and $f[f(x)]$ continue to be simple arches. But once $r > 2$, the arch of $f[f(x)]$ developes a dip at its top (third picture). To see why this happens, note that once $r > 2$, the maximum $f_{\max} = r/4 > 0.5$. Thus as we take x from 0 to 0.5, $f(x)$ increases from 0 to 0.5 and then on to $f_{\max}$. Now, as $f(x)$ goes from 0 to 0.5, $f[f(x)]$ increases from 0 to $f_{\max}$, and then as $f(x)$ moves on from 0.5 to $f_{\max}$, $f[f(x)]$ *decreases* from $f_{\max}$ to $f[f_{\max}]$. Hence the dip at the top of the curve for $f[f(x)]$.

As we increase r from 2 to 4, the dip in $f[f(x)]$ gets steadily deeper. (As r increases, $f_{\max}$ increases beyond 0.5 and so $f[f_{\max}]$ gets smaller.) Eventually when $r = 4$, $f_{\max} = 1$ and $f[f_{\max}] = 0$, so the curve of $f[f(x)]$ dips all the way back to zero, as in the final picture.



Somewhere between $r = 2$ and $r = 4$, the curve of $f[f(x)]$ has to cross the 45° line in the downward direction, causing two new fixed points appear, as in either of the last two figures. This crossing occurs when the curve of $f[f(x)]$ is tangent to the 45° line at the upper fixed point x^*. Since $f[f(x)]$ is tangent to the 45° line its slope is 1 and by (12.57) $f(x^*) = -1$, which means that x^* is just about to become unstable. Therefore, x^* becomes unstable exactly when the two two-cycles materialize. (For the logistic map this occurs when $r = 3$ as in the picture on the bottom left.)

12.27 ⋆⋆ **(a)** Because $f(x) = rx(1-x)$

$$f[f(x)] = rf(x)[1 - f(x)] = r^2x(1-x)[1 - rx(1-x)] = -r^2x[rx^3 - 2rx^2 + (1+r)x - 1].$$

Therefore

$$\begin{aligned} x - f[f(x)] &= x[r^3x^3 - 2r^3x^2 + r^2(r+1)x + (1-r^2)] \\ &= x(rx - r + 1)[r^2x^2 - r(r+1)x + r + 1] \end{aligned}$$

in agreement with (12.61).

The fixed points of $f[f(x)]$ are the zeroes of this expression. The first two are $x = 0$ and $x = (r-1)/r$, which are just the two known fixed points of $f(x)$. The other two are the zeroes of the quadratic expression in square brackets and are

$$x_a, x_b = \frac{r(r+1) \pm \sqrt{r^2(r+1)^2 - 4r^2(r+1)}}{2r^2} = \frac{(r+1) \pm \sqrt{(r+1)(r-3)}}{2r}$$

in agreement with (12.62). Because these are fixed points of $f[f(x)]$ but not of $f(x)$, they have to be the two points of a two-cycle of $f(x)$.

(b) It is always true that $0 \le r \le 4$. If, in addition, $r < 3$, the product inside the square root is negative and both x_a and x_b are complex. Therefore there are no real two-cycles for $r < 3$.

(c) Putting $r = 3.2$ in our expressions for x_a and x_b, we get the two values 0.51 and 0.80, in apparent agreement with Fig.12.37.

12.28 ⋆⋆ **(a)** The two fixed points x_a and x_b of the two-cycle are given by (12.62). The derivative of the double map $g(x) = f[f(x)]$ at either fixed point is given by (12.57) as $g'(x_a) = g'(x_b) = f'(x_a)f'(x_b)$, and, because $f(x) = rx(1-x)$, its derivative is $f'(x) = r(1-2x)$. Putting all this together we find (skipping a bit of algebra)

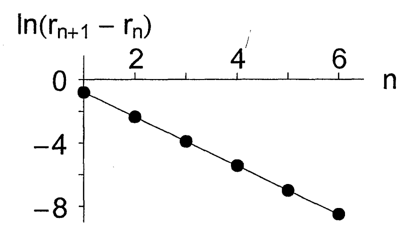
$$g'(x_a) = r^2(1-2x_a)(1-2x_b) = 4 + 2r - r^2$$

(b) The two-cycle is stable if $|g'(x_a)| < 1$. Thus the boundaries of stability are the values of r for which $g'(x_a) = 4 + 2r - r^2 = \pm 1$. These two equations have four roots, $r = 3$ or -2, and $1 \pm \sqrt{6}$, of which only two, $r_1 = 3$ and $r_2 = 1 + \sqrt{6} = 3.449$ lie in the relevant range, $0 \le r \le 4$. It is easy to show that $g'(x_a)$ moves from 1 to -1 as r moves from 3 to $1 + \sqrt{6}$. (Just sketch the graph of $4 + 2r - r^2$.) Thus the stable two-cycle comes into existence when $r = 3$ and becomes unstable when $r = 1 + \sqrt{6}$.

12.29 ⋆⋆ **(a)** See solution to Problem 12.11.

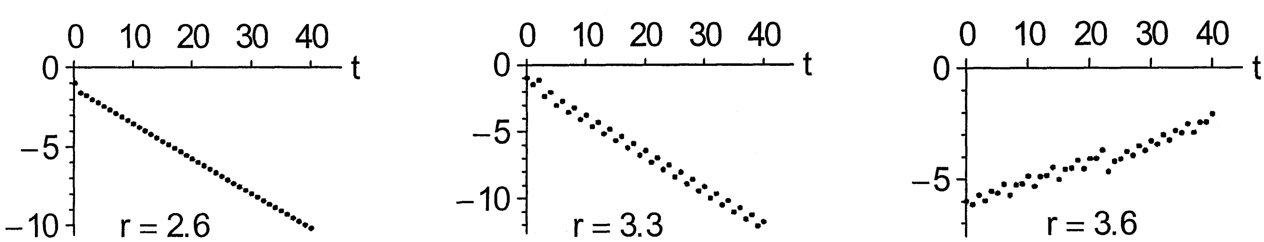
(b) From the values of r_n in Eq.(12.58) we can calculate the following:

n	1	2	3	4	5
$\ln(r_{n+1} - r_n)$	-0.800	-2.358	-3.897	-5.426	-7.013



A least-squares fit to the line $\ln(r_{n+1} - r_n) = \ln(K) - n\ln(\delta)$ yields $\ln(K) = 0.7497$ and $\ln(\delta) = 1.5495$, whence $\delta = 4.71$, compared with the accepted value $\delta = 4.67$. The data and the best-fit line are plotted above.

12.30 ⋆⋆



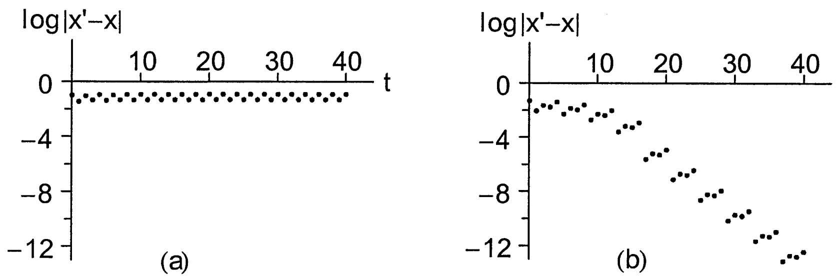
In all three graphs the vertical axis shows $\log|x'_t - x_t|$.

(a) With $r = 2.6$, $\log|x'_t - x_t|$ decreases linearly, so the difference $|x'_t - x_t|$ decreases exponentially.

(b) With $r = 3.3$, $\log|x'_t - x_t|$ decreases roughly linearly, so the difference $|x'_t - x_t|$ decreases roughly exponentially (with a small period-two oscillation superposed).

(c) With $r = 3.6$, $\log|x'_t - x_t|$ increases roughly linearly, so the difference $|x'_t - x_t|$ increases roughly exponentially.

12.31 ⋆⋆⋆



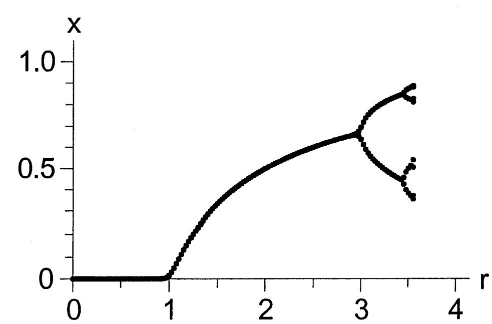
(a) With $x_0 = 0.4$ and $x'_0 = 0.5$, the difference $x'_t - x_t$ remains pretty constant with a small periodic oscillation.

(b) Here with $x_0 = 0.45$ and $x'_0 = 0.5$, the difference $|x'_t - x_t| \to 0$ exponentially.

To understand these different behaviors, consider the case of a period-2 orbit x_t with fixed points x_a and x_b. We could launch a second orbit x'_t so that it was exactly one period out of step with the first. (Just take $x'_1 = x_0$.) Then every time the first orbit is at x_a, the second is at x_b, and vice versa. Thus $|x'_t - x_t|$ will be a constant. On the other hand, if we launch the two orbits sufficiently close, they will eventually get into step and $|x'_t - x_t|$ will approach zero.

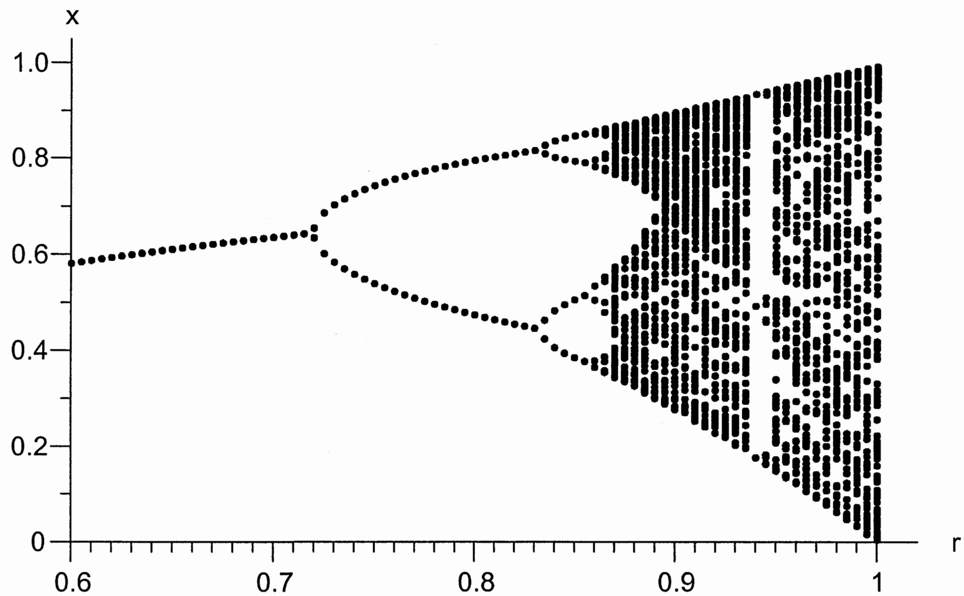
12.32 ⋆⋆⋆

For $0 \le r \le 1$, there is just one stable fixed point at $x = 0$; with r in this range, x_t always approaches 0. For $1 < r \le 3$, there is still only one stable fixed point, but it now increases with r. At $r = 3$ the second fixed point becomes unstable and a two-cycle takes over. At $r = 3.45$ the two-cycle becomes unstable and a four-cycle takes over. Near $r = 3.5$ you can just make out the four-cycle becoming an eight-cycle.



12.33 ⋆⋆⋆ See Fig.12.41.

12.34 ⋆⋆⋆



For $r < 0.715$ (about) the sine map has just one stable fixed point. Near to $r = 0.715$ a period doubling cascade begins, and chaos sets in somewhere about $r = 0.87$. The general appearance of this bifurcation diagram for the sine map is remarkably like Fig.12.41 for the logistic map. For instance, the sine map has a prominent window of period three near $r = 0.94$, very similar to the period-three window of the logistic map near $r = 3.84$.

Chapter 13

Hamiltonian Mechanics

A couple of colleagues tried to persuade me that Hamiltonian mechanics is the most important topic covered in this book and should be introduced sooner than Chapter 13. I certainly agree that Hamiltonian mechanics plays a crucial role in several areas of contemporary physics — plasma physics, accelerator design, chaos theory, and more.[1] But I would argue that Lagrangian mechanics is at least a strong contender for the rank of "most important topic" and, since it seems to be an unavoidable prerequisite for the Hamiltonian formalism, it certainly has to come first (in Chapter 7). Moreover, the Lagrangian approach (like the Hamiltonian) is a tremendous intellectual leap for our students, most of whom need lots of exposure to it before they feel truly comfortable and are ready for the next large leap. This is why I put Chapters 8 to 12, all but one of which make liberal use of Lagrangian mechanics, before this chapter. However, I did design this chapter so that it could be read immediately after Chapter 7. If your students were well prepared and seem on top of Lagrangian mechanics after studying Chapter 7, you could jump directly to Chapter 13 and introduce them to the third great formulation of classical mechanics right away.

A sad thing about teaching Hamiltonian mechanics at this level is that most of the advantages of the Hamiltonian formalism seem too advanced to be appreciated by most of our students. Nevertheless, I have tried to introduce several of the more advanced concepts, such as phase space and Liouville's theorem and to hint at some of the great versatility of the approach, such as the possibility of making canonical transformations. As I've already confessed, I didn't cover any of Chapters 12 through 16 in lecture, but the several students who chose to study this chapter for their term project seemed to enjoy it and to come away with some appreciation for the power of Hamilton's beautiful invention.

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[1] On the other hand, I do not agree with the oft heard view that classical Hamiltonian mechanics is a prerequisite for the study of quantum mechanis. Many of us learned quantum mechanics quite satisfactorily long before we knew about the classical Hamiltonian.

Solutions to Problems for Chapter 13

13.1 ⋆ $\mathcal{L} = \frac{1}{2}m\dot{x}^2$, $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2m$. The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/m \qquad \text{and} \qquad \dot{p} = -\partial\mathcal{H}/\partial x = 0,$$

with solutions $p = p_o =$ const, and $x = x_o + v_o t$, where $v_o = p_o/m$.

13.2 ⋆ The Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{x}^2 + mgx$, so $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2m - mgx$.The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/m \qquad \text{and} \qquad \dot{p} = -\partial\mathcal{H}/\partial x = mg.$$

Combining the two Hamilton equations, we find that $\ddot{x} = g$ as expected.

13.3 ⋆ The moment of inertia of a uniform disc is $I = \frac{1}{2}MR^2$, and its kinetic energy is $\frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{x}^2/R^2$. Therefore, $\mathcal{L} = \frac{1}{2}(m_1+m_2+\frac{1}{2}M)\dot{x}^2 + (m_1-m_2)gx$, $p = (m_1+m_2+\frac{1}{2}M)\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2(m_1+m_2+\frac{1}{2}M) - (m_1-m_2)gx$. The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/(m_1+m_2+\tfrac{1}{2}M) \qquad \text{and} \qquad \dot{p} = -\partial\mathcal{H}/\partial x = (m_1-m_2)g,$$

and the acceleration is $\ddot{x} = g(m_1-m_2)/(m_1+m_2+\frac{1}{2}M)$.

13.4 ⋆ The two original coordinates are x and y, and the constraint equation is $x + y + \pi R$ = const. Thus the equations for x and y in terms of the generalized coordinate x are $x = x$ (of course) and $y = -x +$ const, both of which are independent of time.

13.5 ⋆⋆ Since $\rho = R$ is fixed and $z = c\phi$, there is just one generalized coordinate, which we can choose to be ϕ. The bead's kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2) = \frac{1}{2}m(c^2+R^2)\dot{\phi}^2$, the potential energy is $U = mgz = mgc\phi$, and the generalized momentum is $p = \partial T/\partial\dot{\phi} = m(c^2+R^2)\dot{\phi}$. From these we find the Hamiltonian:

$$\mathcal{H} = T + U = \frac{p^2}{2m(c^2+R^2)} + mgc\phi,$$

and the two Hamilton equations:

$$\dot{\phi} = \frac{\partial\mathcal{H}}{\partial p} = \frac{p}{m(c^2+R^2)} \qquad \text{and} \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial\phi} = -mgc.$$

Combining the last two, we can find $\ddot{\phi}$ and thence

$$\ddot{z} = c\ddot{\phi} = -g\frac{c^2}{c^2+R^2} = -g\sin^2\alpha$$

where in the last expression I have introduced tha pitch α of the helix. (The bead rises a height $2\pi c$ in a horizontal run of $2\pi R$, so $\tan\alpha = c/R$ and $\sin\alpha = c/\sqrt{c^2+R^2}$.)

According to Newton, we could argue that the bead's tangential acceleration is $a_{\text{tang}} = g\sin\alpha$ (the well known acceleration down an incline) and its vertical component is $\ddot{z} = -a_{\text{tang}}\sin\alpha = -g\sin^2\alpha$.

If $R = 0$, then $\alpha = 90°$, and $\ddot{z} = -g$, as expected on a vertical wire.

13.6 ⋆⋆ Let the unstretched length of the spring be l and consider a short segment of spring a distance ξ from the fixed end and of length $d\xi$. Since the spring is uniform, the mass of this segment is $Md\xi/l$ and since the spring stretches uniformly its velocity (when the cart has velocity $\dot{x}$) is $\dot{x}\xi/l$. Therefore the KE of this segment is $\frac{1}{2}M\dot{x}^2\xi^2 d\xi/l^3$, and the KE of the whole spring is

$$T_{\text{spr}} = \frac{1}{2}\frac{M\dot{x}^2}{l^3}\int_0^l \xi^2 d\xi = \frac{1}{6}M\dot{x}^2.$$

The total KE of cart plus spring is $T = \frac{1}{2}m_{\text{eff}}\dot{x}^2$, where $m_{\text{eff}} = m + M/3$. Therefore the Lagrangian for the system is $\mathcal{L} = \frac{1}{2}m_{\text{eff}}\dot{x}^2 - \frac{1}{2}kx^2$, the generalized momentum is $p = \partial\mathcal{L}/\partial\dot{x} = m_{\text{eff}}\dot{x}$, and the Hamiltonian is $\mathcal{H} = p^2/(2m_{\text{eff}}) + \frac{1}{2}kx^2$. The two Hamilton equations are $\dot{x} = \partial\mathcal{H}/\partial p = p/m_{\text{eff}}$ and $\dot{p} = -\partial\mathcal{H}/\partial x = -kx$. Combining these last two, we find $\ddot{x} = -(k/m_{\text{eff}})x$, from which it follows that the cart oscillates with angular frequency $\omega = \sqrt{(k/m_{\text{eff}})}$, which is the claimed result.

13.7 ⋆⋆⋆ **(a)** The height of the track is $y = h(x)$. Therefore the distance traveled by the car in a small displacement is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + h'(x)^2}dx. \qquad \text{(i)}$$

It follows that the car's speed satisfies $v^2 = (1 + h'^2)\dot{x}^2$, so the Lagrangian and generalized momentum are

$$\mathcal{L} = \tfrac{1}{2}m\dot{x}^2[1 + h'(x)^2] - mgh(x) \quad \text{and} \quad p = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x}[1 + h'(x)^2]. \qquad \text{(ii)}$$

The second equation is easily solved for $\dot{x}$ and the Hamiltonian is

$$\mathcal{H} = \dot{x}p - \mathcal{L} = \frac{p^2}{2m(1 + h'^2)} + mgh$$

(b) Hamilton's equations are

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p} = \frac{p}{m(1 + h'^2)} \quad \text{and} \quad \dot{p} = -\frac{\partial\mathcal{H}}{\partial x} = \frac{p^2h'h''}{m(1 + h'^2)^2} - mgh' = m(\dot{x}^2h'h'' - gh') \quad \text{(iii)}$$

where in the last step I used Eq.(ii) to replace p by $p = m\dot{x}(1 + h'^2)$.

Before we do anything with this Hamiltonian result, let us look at the Newtonian prediction,

$$m\ddot{s} = F_{\text{tang}} = -dU/ds = -mgh'/\sqrt{1 + h'^2}, \qquad \text{(iv)}$$

where, in the last step, I wrote $U = mgh$ and I used Eq.(i) to replace ds by $dx\sqrt{1 + h'^2}$. To replace $\ddot{s}$ by $\ddot{x}$, note that

$$\ddot{s} = \frac{d}{dt}\frac{ds}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\sqrt{1 + h'^2}\right) = \ddot{x}\sqrt{1 + h'^2} + \frac{h'h''\dot{x}^2}{\sqrt{1 + h'^2}}.$$

Inserting this result into (iv) and solving for $\ddot{x}$ we find that, according to Newton,

$$\ddot{x} = -\frac{gh' + h'h''\dot{x}^2}{1 + h'^2}. \tag{v}$$

Let's now see that we get the same result from the Hamilton equations (iii). From the first of Eqs.(iii) we find

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}\frac{p}{m(1+h'^2)} = \frac{\dot{p}}{m(1+h'^2)} - \frac{p}{m}\frac{2h'h''\dot{x}}{(1+h'^2)^2}$$

If we use the second Hamilton equation (iii) to eliminate $\dot{p}$ and the second of Eqs.(ii) to eliminate p, this is easily seen to be exactly the same as the Newtonian result (v).

There is a much simpler way to accomplish the same result, though it may seem a cheat at first sight. Hamilton's equations, like Lagrange's from which we derived them, are true with respect to any choice of generalized coordinates. Therefore we can handle the same problem using as our generalized coordinate s, the distance measured along the track. If we do this, then the Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{s}^2 - U(s)$ and the generalized momentum is $p = \partial\mathcal{L}/\partial s = m\dot{s}$. Thus the Hamiltonian is $\mathcal{H} = p^2/(2m) + U(s)$ and the second Hamilton equation is $\dot{p} = -\partial\mathcal{H}/\partial s = -dU/ds$ or $m\ddot{s} = -dU/ds$ in agreement with the Newtonian result Eq.(iv).

13.8 $\star$ Since $U = 0$, $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Therefore, $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and similarly $p_y = m\dot{y}$ and $p_z = m\dot{z}$, and finally $\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} = \mathbf{p}^2/2m$. The six Hamilton equations are

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p_x} = \frac{p_x}{m} \quad \text{and} \quad \dot{p_x} = -\frac{\partial\mathcal{H}}{\partial x} = 0$$

with similar equations for the y and z components. We can combine these into two vector equations $\dot{\mathbf{r}} = \mathbf{p}/m$ and $\dot{\mathbf{p}} = 0$, with the expected solutions $\mathbf{p} = \text{const} = \mathbf{p}_\text{o}$ and $\mathbf{r} = \mathbf{r}_\text{o} + \mathbf{v}_\text{o}t$ where $\mathbf{v}_\text{o} = \mathbf{p}_\text{o}/m$.

13.9 $\star$ The Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$, and the generalized momentum has components $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$ and $p_y = \partial\mathcal{L}/\partial\dot{y} = m\dot{y}$. Therefore, the Hamiltonian is $\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} = (p_x^2 + p_y^2)/2m + mgy$. The Hamilton equations are

$$\dot{x} = p_x/m, \quad \dot{y} = p_y/m, \quad \dot{p_x} = 0, \quad \text{and} \quad \dot{p_y} = -mg.$$

The first two of these simply reproduce the known relations for $\mathbf{p}$ in terms of $\dot{\mathbf{r}}$. The third says that the x component of $\mathbf{p}$ is constant, and the last that p_y changes at the expected rate $-mg$.

13.10 $\star$ The KE is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. If we choose the PE to be zero at the origin, $U = -\int_0^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}kx^2 + Ky$. The generalized momenta are given by $\mathbf{p} = m\dot{\mathbf{r}}$ and the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + p_y^2) + \tfrac{1}{2}kx^2 + Ky.$$

The two Hamilton equations for x are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx$$

which combine to give $\ddot{x} = -(k/m)x$. Thus x oscillates in SHM, $x = A\cos(\omega t - \delta)$, with angular frequency $\omega = \sqrt{k/m}$. Meanwhile, the two y equations are

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} \quad \text{and} \quad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = -K$$

which combine to give $\ddot{y} = -K/m$. Thus y accelerates in the negative y direction, $y = -\frac{1}{2}(K/m)t^2 + v_{yo}t + y_o$, with constant acceleration $-K/m$.

13.11 ⋆ Let's measure x along the tracks (in the direction of travel), y crossways, and z vertically up, all three relative to the car. The ball's velocity relative to the ground is $(V + \dot{x}, \dot{y}, \dot{z})$, so the Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m[(V + \dot{x})^2 + \dot{y}^2 + \dot{z}^2] - mgz$. The generalized momentum has components

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m(V + \dot{x}), \qquad p_y = m\dot{y}, \qquad \text{and} \qquad p_z = m\dot{z}$$

(Notice that, perhaps unexpectedly, the generalized momentum is the momentum relative to the ground, not to the moving car.) We can solve for $\dot{x} = (p_x - mV)/m$ and then

$$\mathcal{H} = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \mathcal{L} = \frac{\mathbf{p}^2}{2m} - p_xV + mgz.$$

From this, you can derive the expected equation of motion (as you could check), but our point here is to note that $\mathcal{H}$ is not equal to the energy $T + U$ (neither relative to the car nor relative to the ground), because

$$\begin{aligned}(T + U)_{(\text{rel to car})} &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz \\ &= \frac{\mathbf{p}^2}{2m} - p_xV + \tfrac{1}{2}mV^2 + mgz \neq \mathcal{H}.\end{aligned}$$

and

$$(T + U)_{(\text{rel to ground})} = \frac{\mathbf{p}^2}{2m} + mgz \neq \mathcal{H}.$$

13.12 ⋆ As generalized coordinate I'll use the bead's position x relative to the axis of spin, as measured in the frame of the rod. The bead's PE is zero and its KE (relative to the ground) is $T = \frac{1}{2}m(\dot{x}^2 + x^2\omega^2)$, so $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + x^2\omega^2)$ and the generalized momentum is $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$. Therefore the Hamiltonian is

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{2m} - \tfrac{1}{2}mx^2\omega^2.$$

This is not equal to the energy $T+U$ (neither relative to the rod nor relative to the ground), because

$$(T+U)_{\text{(rel to rod)}} = \tfrac{1}{2}m\dot{x}^2 = \frac{p^2}{2m} \neq \mathcal{H}.$$

and

$$(T+U)_{\text{(rel to ground)}} = \tfrac{1}{2}m(\dot{x}^2 + x^2\omega^2) = \frac{p^2}{2m} + \tfrac{1}{2}mx^2\omega^2 \neq \mathcal{H}.$$

13.13 ⋆⋆ The KE is $T = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2)$, and the PE is $U = \frac{1}{2}kr^2 = \frac{1}{2}k(R^2 + z^2)$. Thus the two generalized momenta are

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi} \quad \text{and} \quad p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

and the Hamiltonian is

$$\mathcal{H} = T + U = \frac{1}{2m}\left(\frac{p_\phi^2}{R^2} + p_z^2\right) + \frac{1}{2}k(R^2 + z^2).$$

The two Hamilton equations for z are

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \quad \text{and} \quad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz.$$

These two combine to give $\ddot{z} = -(k/m)z$, which shows that the motion in the z direction is SHM with frequency $\omega = \sqrt{k/m}$. The two Hamilton equations for ϕ are

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mR^2} \quad \text{and} \quad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0.$$

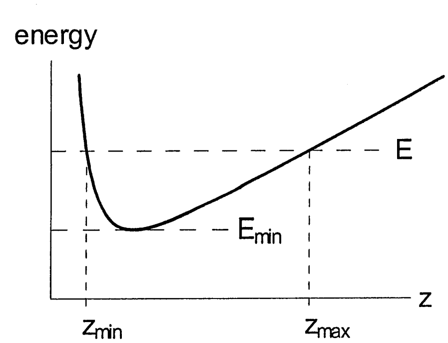
The first of these simply repeats the relation between $\dot{\phi}$ and p_ϕ. The second tells us that p_ϕ (namely, the z component of angular momentum) is constant and hence that the motion around the cylinder proceeds with constant $\dot{\phi}$.

13.14 ⋆⋆ According to Eq.(13.32), $\dot{z} = p_z/m(c^2+1)$, so $\dot{z}$ can vanish if and only if $p_z = 0$. According to (13.33)

$$\frac{1}{2m}\left[\frac{p_z^2}{(c^2+1)} + \frac{p_\phi^2}{c^2z^2}\right] + mgz = E.$$

Therefore, if $p_z = 0$ it must be that

$$\frac{p_\phi^2}{2mc^2z^2} + mgz = E.$$



To find how many values of z can satisfy this equation consider the graph of the LHS plotted against z. When $z \to 0$ or $z \to \infty$ it is clear that the LHS approaches $+\infty$. By differentiating it, you can check that its derivative vanishes exactly once, so the graph of the LHS has the simple cup shape shown. If $E < E_{\text{min}}$, no motion is possible. But if $E > E_{\min}$, the equation had exactly two solutions, shown as $z_{\min}$ and $z_{\max}$.

As the mass mass moves around the cone, $\dot{\phi} = p_\phi/(mc^2z^2)$ always keeps the same sign (always positive or always negative). Thus the mass always moves in the same direction around the cone. Meanwhile z moves down to $z_{\min}$ and up to $z_{\max}$, oscillating between the two.

13.15 ⋆⋆ According to Eq.(13.24)

$$\mathcal{H} = \mathcal{H}(\mathbf{q}, \mathbf{p}, t) = \sum_{j=1}^{n} p_j \dot{q}_j(\mathbf{q}, \mathbf{p}, t) - \mathcal{L}\big(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t), t\big). \tag{vi}$$

Differentiating with respect to q_i, we find

$$\frac{\partial \mathcal{H}}{\partial q_i} = \sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i} - \left[\frac{\partial \mathcal{L}}{\partial q_i} + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_i}\right] = -\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\dot{p}_i$$

where, in the second expression, I replaced $\partial\mathcal{L}/\partial\dot{q}_i$ in the second sum by p_i, so that the two sums cancelled, and for the third equality I used Lagrange's equations. Differentiating (vi) with respect to p_i we find

$$\frac{\partial \mathcal{H}}{\partial p_i} = \left[\dot{q}_i + \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i}\right] - \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i$$

where, in the second sum of the second expression, I replaced $\partial\mathcal{L}/\partial\dot{q}_j$ by p_j, so that the two sums cancel.

13.16 ⋆⋆ For a system with just one degree of freedom, (13.24) reads

$$\mathcal{H} = p\dot{q}(q, p, t) - \mathcal{L}\big(q, \dot{q}(q, p, t), t\big).$$

Therefore

$$\frac{\partial \mathcal{H}}{\partial t} = p\frac{\partial \dot{q}}{\partial t} - \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} + \frac{\partial \mathcal{L}}{\partial t}\right] = -\frac{\partial \mathcal{L}}{\partial t}$$

where, in the second term of the second expression, I replaced $\partial\mathcal{L}/\partial\dot{q}$ by p, so that the first two terms cancelled. (Remember that, in the Hamiltonian approach, the independent variables are q, p, and t; thus, in evaluating $\partial/\partial t$, we hold q and p fixed.)

The case with n degrees of freedom proceeds in exact parallel starting from (13.24):

$$\frac{\partial \mathcal{H}}{\partial t} = \sum_j p_j \frac{\partial \dot{q}_j}{\partial t} - \left[\sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial t} + \frac{\partial \mathcal{L}}{\partial t}\right] = -\frac{\partial \mathcal{L}}{\partial t}$$

13.17 ⋆⋆⋆ **(a)** From (13.32) we see that $\dot{z} = 0$ if and only if $p_z = 0$, and from (13.34) that this implies that

$$\frac{p_\phi^2}{mc^2 z_o^3} = mg \quad \text{or} \quad z_o = \left(\frac{p_\phi^2}{m^2c^2g}\right)^{1/3}.$$

(b) If we combine the two equation (13.34) to give $\ddot{z}$ and then put $z = z_o + \epsilon$, we find

$$\ddot{\epsilon} = \ddot{z} = \frac{\dot{p}_z}{m(c^2+1)} = \frac{1}{m(c^2+1)}\left[\frac{p_\phi^2}{mc^2z^3} - mg\right]$$
$$= \frac{1}{m(c^2+1)}\left[\frac{p_\phi^2}{mc^2z_o^3}\left(1 - 3\frac{\epsilon}{z_o}\right) - mg\right] = -\frac{3p_\phi^2\epsilon}{m^2c^2(c^2+1)z_o^4}$$

where in passing to the second line I used the binomial approximation for z^{-3} and where, in the first expression of the second line, the first and last terms cancelled because of the result of part (a). This equation implies that ϵ oscillates in SHM.

(c) The last equation of part (b) implies that the frequency of these oscillations is

$$\omega = \frac{p_\phi}{mcz_o^2}\sqrt{\frac{3}{c^2+1}} = \sqrt{3}\dot{\phi}_o\frac{c}{\sqrt{c^2+1}} = \sqrt{3}\dot{\phi}_o \sin\alpha$$

where for the second equality I used (13.32) to replace p_ϕ by $mc^2z_o^2\dot{\phi}_o$ and for the third I replaced $c/\sqrt{c^2+1}$ by $\sin\alpha$ where α is the half-angle of the cone.

(d) For ω to be equal to $\dot{\phi}_o$, it must be that $\sin\alpha = 1/\sqrt{3}$ or $\alpha = 35.3°$. If this is the case, the height z will return to its initial value in the time for one complete orbit around the cone. Therefore the orbit is closed and is (approximately at least) a circle, tilted at a small angle $\epsilon_{\text{max}}/cz_o$.

13.18 ⋆⋆⋆ **(a)** According to (7.103), $\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}}\cdot\mathbf{A})$. Therefore, the generalized momentum has $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x} + qA_x$, with similar expressions for p_y and p_z. Thus

$$\mathbf{p} = m\dot{\mathbf{r}} + q\mathbf{A} \quad \text{or} \quad \dot{\mathbf{r}} = \frac{\mathbf{p} - q\mathbf{A}}{m}. \tag{vii}$$

The Hamiltonian is

$$\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} = \mathbf{p}\cdot\frac{\mathbf{p} - q\mathbf{A}}{m} - \left[\frac{1}{2}m\left(\frac{\mathbf{p} - q\mathbf{A}}{m}\right)^2 - q\left(V - \frac{\mathbf{p} - q\mathbf{A}}{m}\cdot\mathbf{A}\right)\right]$$
$$= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV$$

(b) Hamilton's equations are

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p_x} = \frac{p_x - qA_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial\mathcal{H}}{\partial x} = q\left(\sum_{i=1}^{3}\dot{r}_i\frac{\partial A_i}{\partial x} - \frac{\partial V}{\partial x}\right)$$

with similar equations for y and z. Combining these two equations we find

$$m\ddot{x} = \dot{p}_x - q\frac{dA_x}{dt} = q\left(\sum_i \dot{r}_i\frac{\partial A_i}{\partial x} - \frac{\partial V}{\partial x}\right) - q\left(\sum_i \frac{\partial A_x}{\partial r_i}\dot{r}_i + \frac{\partial A_x}{\partial t}\right)$$
$$= q\left[-\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) + \dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

which you can recognize as the x component of the Lorentz-force equation $m\ddot{\mathbf{r}} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$. Since the other two components work in exactly the same way, we're home.

13.19 ⋆ The KE is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and the PE is $U = U(r) = U\left(\sqrt{x^2+y^2}\right)$. Thus the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + U\left(\sqrt{x^2+y^2}\right).$$

Clearly neither $\partial\mathcal{H}/\partial x$ nor $\partial\mathcal{H}/\partial y$ is zero (unless U is a constant); that is, neither x nor y is ignorable.

13.20 ⋆ **(a)** $U(\mathbf{r}) = -\int \mathbf{F}\cdot d\mathbf{r} = -\mathbf{F}\cdot\mathbf{r}$. Therefore

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} - \mathbf{F}\cdot\mathbf{r} = \frac{p_x^2 + p_y^2}{2m} - F_x x - F_y y.$$

(b) If we choose our x axis in the direction of $\mathbf{F}$, then $F_y = 0$ and the coordinate y is ignorable.

(c) If neither axis is parallel to $\mathbf{F}$, then neither F_x nor F_y is zero, and neither $\partial\mathcal{H}/\partial x$ nor $\partial\mathcal{H}/\partial y$ is zero, so neither of the coordinates is ignorable.

13.21 ⋆⋆ **(a)** The KE is T $= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)$, and the PE is $U = \frac{1}{2}k(r - l_o)^2$. The generalized momenta are $\mathbf{P} = M\dot{\mathbf{R}}$, $p_r = \mu\dot{r}$, and $p_\phi = \mu r^2\dot{\phi}$, so the Hamiltonian is

$$\mathcal{H} = \frac{1}{2M}(P_x^2 + P_y^2) + \frac{1}{2\mu}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + \tfrac{1}{2}k(r - l_o)^2.$$

The CM coordinates X and Y are ignorable, because there are no external forces, so that total momentum is conserved. The coordinate ϕ is ignorable because the force between the two masses is central, so their angular momentum is conserved. The coordinate r is not ignorable, because there is a radial force.

(b) The two Hamilton equations for X are

$$\dot{X} = \frac{\partial\mathcal{H}}{\partial P_x} = \frac{P_x}{M} \quad\text{and}\quad \dot{P}_x = -\frac{\partial\mathcal{H}}{\partial X} = 0$$

with corresponding equations for Y. These four equations say that the CM position $\mathbf{R}$ moves like a free particle, with constant velocity. The two equations for r are

$$\dot{r} = \frac{\partial\mathcal{H}}{\partial p_r} = \frac{p_r}{\mu} \quad\text{and}\quad \dot{p}_r = -\frac{\partial\mathcal{H}}{\partial r} = \frac{p_\phi^2}{\mu r^3} - k(r - l_o),$$

and the two ϕ equations are

$$\dot{\phi} = \frac{\partial\mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{\mu r^2} \quad\text{and}\quad \dot{p}_\phi = -\frac{\partial\mathcal{H}}{\partial\phi} = 0$$

which last says that p_ϕ is conserved as expected.

(c) The two r equations combine to give the familiar radial equation

$$\mu\ddot{r} = \dot{p}_r = \frac{p_\phi^2}{\mu r^3} - k(r - l_o). \qquad \text{(viii)}$$

In the special case that $p_\phi = 0$ this shows that r executes SHM about the equilibrium length $r = l_o$. The CM moves with constant velocity while the two masses oscillate toward and away from the CM.

(d) If $p_\phi \neq 0$, the radial equation (viii) is no longer linear and cannot be solved in terms of elementary functions. In this case, the two masses still oscillate in and out, but not in SHM, while they orbit around the CM with constant angular momentum.

13.22 ★★ **(a)** According to (13.14) $\mathcal{H}(q,p) = p\dot{q}(q,p) - \mathcal{L}\big(q, \dot{q}(q,p)\big)$, so

$$\frac{\partial\mathcal{H}}{\partial q} = p\frac{\partial\dot{q}}{\partial q} - \left[\frac{\partial\mathcal{L}}{\partial q} + \frac{\partial\mathcal{L}}{\partial\dot{q}}\frac{\partial\dot{q}}{\partial q}\right] = -\frac{\partial\mathcal{L}}{\partial q}$$

where the first and third terms in the middle expression cancel because $p = \partial\mathcal{L}/\partial\dot{q}$.

(b) According to (13.24),

$$\mathcal{H} = \mathcal{H}(\mathbf{q},\mathbf{p},t) = \sum_{j=1}^{n} p_j\dot{q}_j(\mathbf{q},\mathbf{p},t) - \mathcal{L}\big(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t), t\big).$$

so

$$\frac{\partial\mathcal{H}}{\partial q_i} = \sum_j p_j\frac{\partial\dot{q}_j}{\partial q_i} - \left[\frac{\partial\mathcal{L}}{\partial q_i} + \sum_j \frac{\partial\mathcal{L}}{\partial\dot{q}_j}\frac{\partial\dot{q}_j}{\partial q_i}\right] = -\frac{\partial\mathcal{L}}{\partial q_i}$$

where the two sums cancel because $p_j = \partial\mathcal{L}/\partial\dot{q}_j$.

13.23 ★★★ **(a)** The gravitational PE is $U_{\text{gr}} = Mgy - mgy - mg(x+y) + \text{const} = -mgx$ if we drop the uninteresting constant. The spring PE is harder. If we let l_o denote the spring's natural, unloaded length, then $k(l_e - l_o) = mg$ and if x' denotes the spring's true extension (from its unloaded length), then $l_o + x' = l_e + x$ so

$$x' = x + (l_e - l_o) = x + \frac{mg}{k}$$

Thus the spring PE is

$$U_{\text{spr}} = \tfrac{1}{2}kx'^2 = \tfrac{1}{2}k\left(x + \frac{mg}{k}\right)^2 = \tfrac{1}{2}kx^2 + mgx + \text{const.}$$

If we add this to $U_{\text{gr}} = -mgx$, the terms in mgx cancel and (dropping another uninteresting constant) we get $U = U_{\text{gr}} + U_{\text{spr}} = \tfrac{1}{2}kx^2$ as claimed.

(b) The KE is $T = \tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}m(\dot{x}+\dot{y})^2 = \tfrac{1}{2}m[3\dot{y}^2 + (\dot{x}+\dot{y})^2]$, from which we find the momenta,

$$p_x = \frac{\partial T}{\partial\dot{x}} = m(\dot{x}+\dot{y}) \quad\text{and}\quad p_y = \frac{\partial T}{\partial\dot{y}} = m(\dot{x}+4\dot{y})$$

whence

$$\dot{x}+\dot{y} = \frac{p_x}{m} \quad\text{and}\quad \dot{y} = \frac{1}{3m}(p_y - p_x).$$

From these we can calculate the Hamiltonian,

$$\mathcal{H} = T + U = \frac{1}{2m}\left[\frac{(p_x - p_y)^2}{3} + p_x^2\right] + \frac{1}{2}kx^2.$$

Because this doesn't depend on y, the coordinate y is ignorable. This is traceable to the fact that the total mass on each side is the same.

(c) The Hamilton equations for x are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{3m}(4p_x - p_y) \quad \text{and} \quad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx \tag{ix}$$

and those for y

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{1}{3m}(p_y - p_x) \quad \text{and} \quad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = 0. \tag{x}$$

The initial conditions are that $x(0) = x_o$, $y(0) = y_o$, and $\dot{x}(0) = \dot{y}(0) = 0$. These imply that $p_x(0) = p_y(0) = 0$, and, because p_y is constant, $p_y = 0$ for all time. Combining the two equations (ix) and setting $p_y = 0$, we find that $\ddot{x} = 4\dot{p}_x/3m = -4kx/3m$. Therefore $x = x_o \cos\omega t$, where $\omega = \sqrt{4k/3m}$. Next, from the first of Eqs.(ix) (with $p_y = 0$) we find that $p_x = \frac{3}{4}m\dot{x} = -\frac{3}{4}m\omega x_o \sin\omega t$ and finally, from the first of Eqs.(x), $\dot{y} = -p_x/3m = \frac{1}{4}\omega x_o \sin\omega t$, so $y = -\frac{1}{4}x_o \cos\omega t + \text{const} = y_o + \frac{1}{4}x_o(1 - \cos\omega t)$.

13.24 ⋆ The new variables are defined as $Q = p$ and $P = -q$. So

$$\dot{Q} = \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -\frac{\partial \mathcal{H}}{\partial(-P)} = \frac{\partial \mathcal{H}}{\partial P}$$

and

$$\dot{P} = -\dot{q} = -\frac{\partial \mathcal{H}}{\partial p} = -\frac{\partial \mathcal{H}}{\partial Q}.$$

These are precisely Hamilton's equations with respect to the new variables.

13.25 ⋆⋆⋆ **(a)** The new variables Q and P are defined so that

$$q = \sqrt{2P}\sin Q \quad \text{and} \quad p = \sqrt{2P}\cos Q \tag{xi}$$

and we are promised that the old variables q and p satisfy Hamilton's equations. So now consider this:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial Q} &= \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial Q} = -\dot{p}(\sqrt{2P}\cos Q) - \dot{q}(\sqrt{2P}\sin Q) \\ &= -(\dot{p}p + \dot{q}q) = -\frac{1}{2}\frac{d}{dt}(p^2 + q^2) = -\frac{d}{dt}(P\cos^2 Q + P\sin^2 Q) = -\dot{P}.\end{aligned}$$

This proves the Hamilton equation for $\dot{P}$. Next

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial P} &= \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial P} = -\dot{p}\left(\frac{1}{\sqrt{2P}}\sin Q\right) + \dot{q}\left(\frac{1}{\sqrt{2P}}\cos Q\right) \\ &= -\dot{p}\frac{q}{2P} + \dot{q}\frac{p}{2P} = \frac{p^2}{2P}\frac{d}{dt}\left(\frac{q}{p}\right) = \frac{p^2}{2P}\frac{d}{dt}\tan Q = \frac{p^2}{2P}\dot{Q}\sec^2 Q = \dot{Q}.\end{aligned}$$

Here in moving to the second line I used Eqs.(xi), and for the final equality I used the second of Eqs.(xi). This proves the Hamilton equation for $\dot{Q}$.

(b) $\mathcal{H} = (p^2/2m) + \frac{1}{2}kq^2 = \frac{1}{2}(p^2 + q^2)$ because $m = k = 1$.

(c) $\mathcal{H} = \frac{1}{2}(p^2 + q^2) = \frac{1}{2}(2P\cos^2 Q + 2P\sin^2 Q) = P$. We see that Q is ignorable and that the new Hamiltonian is just P (so that conservation of P is just conservation of energy).

(d) The Hamilton equation for Q reads $\dot{Q} = \partial\mathcal{H}/\partial P = 1$, which implies that $Q = (t - \delta)$ where δ is an arbitrary constant. Substituting this into the first of Eqs.(xi) and putting $P = E$, we find $q = \sqrt{2E}\sin(t - \delta)$, which correctly describes an SHO with energy E and frequency $\omega = \sqrt{k/m} = 1$.

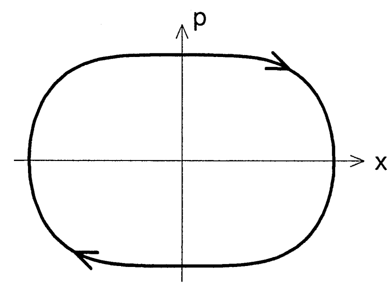
13.26 ⋆ The potential energy is

$$U = -\int F\,dx = \tfrac{1}{4}kx^4,$$

and the Hamiltonian is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{4}kx^4 = E.$$

In the two-dimensional phase space, with coordinates x and p, this defines the flattened ellipse shown.



13.27 ⋆⋆ The position and momentum of an object in free fall are $x = x_o + (p_o/m)t + \frac{1}{2}gt^2$ and $p = p_o + mgt$. Thus the positions in phase space of the eight points in Fig.13.6 are as follows:

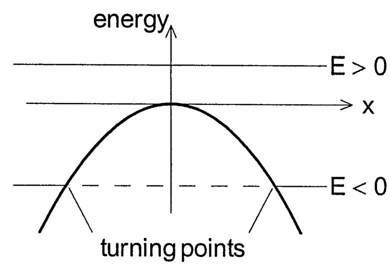
	time 0			time t	
	x_o	p_o		x	p
A_o	0	0	A	$\frac{1}{2}gt^2$	mgt
B_o	X	0	B	$X + \frac{1}{2}gt^2$	mgt
C_o	X	P	C	$X + (P/m)t + \frac{1}{2}gt^2$	$P + mgt$
D_o	0	P	D	$(P/m)t + \frac{1}{2}gt^2$	$P + mgt$

Inspecting the data in this table, you can see the heights of the two points D_o and C_o above the x axis are both equal to P, so the lines D_oC_o and A_oB_o are parallel. Since both of these lines have the same length (X), $A_oB_oC_oD_o$ is a parallelogram. (In fact it is also a rectangle,but this doesn't matter here.) In the same way, the lines AB and DC parallel and of equal length, so $ABCD$ is also a parallogram. Again from the data in the table you can check that both parallograms have bases equal to X and heights equal to P. Thus both have area $X \cdot P$. In particular, their areas are the same.

13.28 ⋆⋆ **(a)** The potential energy is

$$U = -\int F\,dx = -\tfrac{1}{2}kx^2,$$

which gives the inverted parabola shown. If $E > 0$, the mass can go anywhere. It can come in from the left, slowing down, and move out to the right, speeding up again, or vice versa.

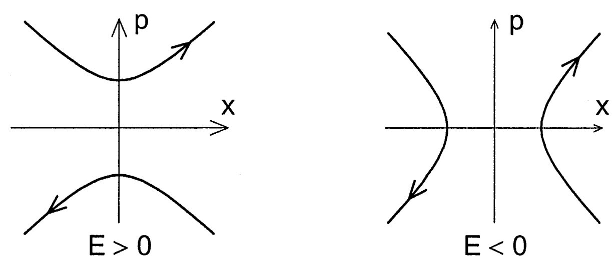


If $E < 0$, the mass is excluded from the interval between the two turning points. It can come in from the left, slowing to a halt at the turning point, and then reverse out; or similarly from the right.

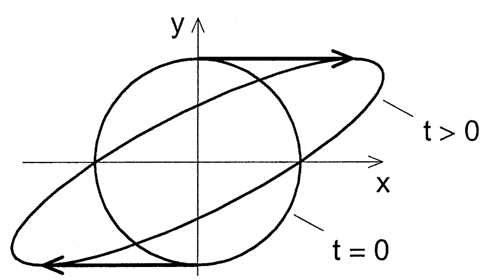
(b) The Hamiltonian is

$$\mathcal{H} = \frac{p^2}{2m} - \frac{1}{2}kx^2 = E.$$

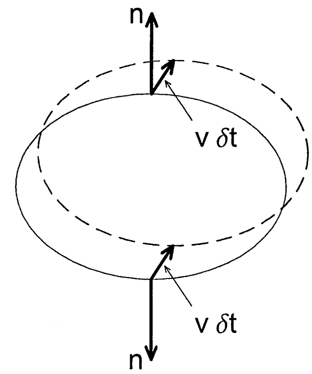
In the phase space, with coordinates x and p, this is the equation of a hyperbola. If $E > 0$ the hyperbola is as shown in the left picture. The upper curve shows a mass coming in from the left, passing through, and going out on the right. The lower curve shows a mass going from right to left. If $E < 0$, the hyperbola is as shown on the right. The left curve shows a mass coming in from the left and reversing out; the right curve shows a mass coming in from the right and reversing out.



13.29 $\star$



13.30 ⋆ The crucial point is that **n** is defined as the *outward* normal from the surface. Thus if the velocity **v** is pretty much in the same direction all over the surface, then $\mathbf{n}\cdot\mathbf{v}\delta t$ will have opposite signs on opposite sides of the surface.



13.31 ⋆

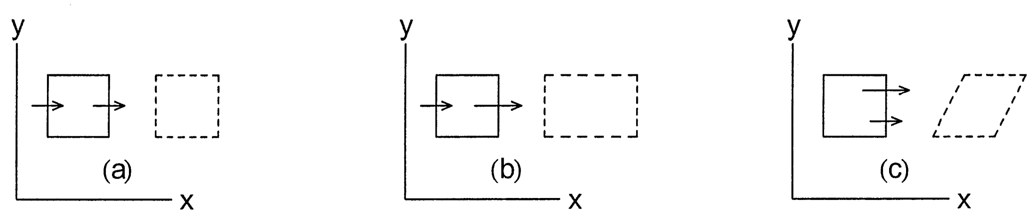
(a) If $\mathbf{v} = k\mathbf{r} = (kx, ky, kz)$, then $\nabla\cdot\mathbf{v} = k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3k$

(b) If $\mathbf{v} = k(z, x, y)$, then $\nabla\cdot\mathbf{v} = k\left(\frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z}\right) = 0$

(c) If $\mathbf{v} = k(z, y, x)$, then $\nabla\cdot\mathbf{v} = k\left(\frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z}\right) = k$

(d) If $\mathbf{v} = k(x, y, -2z)$, then $\nabla\cdot\mathbf{v} = k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} - 2\frac{\partial z}{\partial z}\right) = k(1 + 1 - 2) = 0$

13.32 ⋆⋆ **(a)** With $\mathbf{v} = (k, 0, 0)$, $\nabla\cdot\mathbf{v} = \partial k/\partial x + 0 + 0 = 0$. This flow is sketched in the left picture. Fluid is flowing into the square on its left, but out on its right, and these two effects cancel. To put it another way, the square as a whole is moving to the right, but its front and back are moving at the same speed, so its volume doesn't change.



(b) With $\mathbf{v} = (kx, 0, 0)$, $\nabla\cdot\mathbf{v} = k\partial x/\partial x = k$. This flow is shown in the middle picture. Fluid is flowing into the square on its left, and out on its right. Since the speed of flow is greater on the right, there is a net outflow. The square as a whole is stretching.

(c) With $\mathbf{v} = (ky, 0, 0)$, $\nabla\cdot\mathbf{v} = k\partial y/\partial x = 0$. This flow is shown in the right picture. Fluid is flowing into the square on its left, and out on its right. The speed of flow is greater near the top, but the net flows on the left and right cancel. The square as a whole is becoming a parallelogram, but its volume isn't changing.

13.33 ⋆⋆ The divergence theorem asserts that

$$\int_S \mathbf{v}\cdot\mathbf{n}\,dA = \int_V \nabla\cdot\mathbf{v}\,dV. \tag{xii}$$

(a) If S is a sphere of radius R centered on the origin, then the unit normal $\mathbf{n}$ points radially out from the origin, so $\mathbf{n} = \hat{\mathbf{r}}$, and the surface integral on the left side of Eq.(xii) is easily evaluated (with $\mathbf{v} = k\mathbf{r}$):

$$\text{LHS} = \int_S \mathbf{v}\cdot\mathbf{n}\,dA = \int_S k\mathbf{r}\cdot\hat{\mathbf{r}}dA = kR\int_S dA = 4\pi kR^3. \tag{xiii}$$

Because $\nabla\cdot\mathbf{v} = 3k$, the volume integral on the right side of Eq.(xii) is easily seen to be

$$\text{RHS} = \int_V \nabla\cdot\mathbf{v}\,dV = 3k\int_V dV = 3k\left(\frac{4}{3}\pi R^3\right) = 4\pi kR^3 \tag{xiv}$$

in agreement with (xiii). That is, this example confirms the divergence theorem.

(b) If S is not centered on the origin, then the normal $\mathbf{n}$ is not the same as $\hat{\mathbf{r}}$ and the integrand $\mathbf{v}\cdot\mathbf{n}$ in the LHS of Eq.(xii) is a somewhat complicated function of position on the sphere. This makes the surface integral harder to evaluate directly. On the other hand, the volume integral in the RHS can be evaluated exactly as berfore in (xiv). Therefore, the surface integral has the same value as before, $4\pi kR^3$.

13.34 ⋆⋆ **(a)** In rectangular coordinates, the vector $\mathbf{v}$ is

$$\mathbf{v} = \frac{k\hat{\mathbf{r}}}{r^2} = \frac{k\mathbf{r}}{r^3} = k\left(\frac{x}{r^3}, \frac{x}{r^3}, \frac{x}{r^3}\right).$$

The divergence $\nabla\cdot\mathbf{v}$ involves three derivatives, of which the first is

$$\frac{\partial v_x}{\partial x} = k\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = k\left(\frac{1}{r^3} - x\frac{3}{r^4}\frac{\partial r}{\partial x}\right) = k\left(\frac{1}{r^3} - x\frac{3}{r^4}\frac{x}{r}\right) = \frac{k}{r^5}(r^2 - 3x^2),$$

with corresponding expressions for the other two derivatives. Therefore

$$\begin{aligned}\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} &= \frac{k}{r^5}\left[(r^2 - 3x^2) + (r^2 - 3y^2) + (r^2 - 3z^2)\right] \\ &= \frac{3k}{r^5}\left[r^2 - (x^2 + y^2 + z^2)\right] = 0.\end{aligned}$$

(b) If we use spherical polar coordinates (r, θ, ϕ), then only the r component of $\mathbf{v}$ is different from zero; specifically, $v_r = k/r^2$, while $v_\theta = v_\phi = 0$. The expression inside the back cover for $\nabla\cdot\mathbf{v}$ has three terms, but, in the present case, two of them are zero, and we're left with

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(k) = 0.$$

13.35 ★★ The initial volume occupied by the beam is $V_o = (\pi R_o^2 L_o)[\pi(\Delta p_\perp)^2 2\Delta p_z]$. (This is the volume in phase space.) By Liouville's theorem, this volume can't change. Thus when R shrinks (with L_o and Δp_z fixed), $\Delta p_\perp$ has to grow. In the long run, this means that R will increase again.

13.36 ★★ The velocity in the $2n$-dimensional phase space is

$$\mathbf{v} = \dot{\mathbf{z}} = (\dot{q}_1, \cdots, \dot{q}_n, \dot{p}_1, \cdots, \dot{p}_n) = \left(\frac{\partial\mathcal{H}}{\partial p_1}, \cdots, \frac{\partial\mathcal{H}}{\partial p_n}, -\frac{\partial\mathcal{H}}{\partial q_1}, \cdots, -\frac{\partial\mathcal{H}}{\partial q_n}\right).$$

where the last equality follows from Hamilton's equations. The $2n$-dimensional divergence is $\boldsymbol{\nabla}\cdot\mathbf{v} = \sum_{i=1}^{2n} \partial v_i/\partial z_i$, and each of the first n terms in this sum exactly cancels the corresponding one of the final n terms, so that $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$. If now V denotes the volume of an arbitrary closed surface, then, according to (13.59), $dV/dt = 0$, which is Liouville's theorem.

13.37 ★★★ **(a)** As discussed, the surface integral on the left side of Eq.(13.63) has the form $I_1 + \cdots + I_6$, where $I_1 = \int_{S_1} \mathbf{n}\cdot\mathbf{v}\,dA$ and so on. Now, on the surface S_1 the outward normal is $\mathbf{n} = -\hat{\mathbf{x}}$, while on S_2, $\mathbf{n} = \hat{\mathbf{x}}$. Therefore,

$$\begin{aligned} I_1 + I_2 &= -\int_Y^{Y+B} dy \int_Z^{Z+C} dz\, v_x(X, y, z) + \int_Y^{Y+B} dy \int_Z^{Z+C} dz\, v_x(X + A, y, z) \\ &= \int_Y^{Y+B} dy \int_Z^{Z+C} dz \left[v_x(X + A, y, z) - v_x(X, y, z)\right]. \end{aligned} \tag{xv}$$

(b) By the fundamental theorem of calculus, $f(X + A) - f(X) = \displaystyle\int_X^{X+A} \frac{df}{dx} dx$, so the integrand in (xv) is $\displaystyle\int_X^{X+A} \frac{\partial v_x}{\partial x} dx$.

(c) Combining the results of parts (a) and (b), we see that

$$I_1 + I_2 = \int_X^{X+A} dx \int_Y^{Y+B} dy \int_Z^{Z+C} dz\, \frac{\partial v_x}{\partial x} = \int_V \frac{\partial v_x}{\partial x} dV.$$

Adding this to the corresponding results for the other two pairs of integrals, we conclude that the surface integral of (13.63) is

$$\int_S \mathbf{v}\cdot\mathbf{n}\, dA = I_1 + \cdots + I_6 = \int_V \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dV = \int_V \boldsymbol{\nabla}\cdot\mathbf{v}\, dV$$

which is the divergence theorem.

Chapter 14

Collision Theory

It is a little unusual to have a separate chapter on collision theory in a book of this type. My decision to do this reflects in part my interest in the subject, but mainly a conviction that if one's going to introduce the subject at all, one had better do it properly. Ideas like the differential cross section and the relation of CM to lab frames are just too subtle to be introduced in a couple of sections at the end of a chapter. Sections 14.1–14.3 introduce the basic ideas of the scattering angle, the impact parameter, and the various total cross sections (total elastic and inelastic cross sections, and so on). Sections 14.4–14.6 are about the differential cross section and how to calculate it, including hard-sphere and Rutherford scattering as examples. While all of Sections 14.1–14.6 treat the target as fixed, Sections 14.7 and 14.8 (both advertized as omittable) take up the different cross sections in different reference frames and derive the crucial relations (14.45) and (14.56) between the lab and CM cross sections.

Solutions to Problems for Chapter 14

14.1 ⋆ The area of a blueberry is $\sigma = \pi r^2 = \pi d^2/4 = \pi/4$ cm^2 $= 0.79$ cm^2. The density of targets (number/area) is $n_{\text{tar}} = N_{\text{tar}}/A = N_{\text{tar}}/(\pi D^2/4) = 0.034$ cm^{-2}. The probability of a hit in one try is $n_{\text{tar}}\sigma = 0.027$.

14.2 ⋆ **(a)** With $r = 5$ fm $= 5 \times 10^{-15}$ m, the cross section is $\sigma = \pi r^2 = 25\pi \times 10^{-30}$ m$^2 \approx 0.8$ barns.

(b) If $r = 0.1$ nm $= 10^{-10}$ m, the cross section is $\sigma = \pi r^2 = \pi \times 10^{-20}$ m$^2 = 3 \times 10^8$ barns.

14.3 ⋆ The density is $\varrho = 0.07$ g/cm^3, the length of the tank, $L = 50$ cm, and the mass of an H atom is $m_{\text{H}} = 1.66 \times 10^{-27}$ kg. The number density (number/volume) is ϱ/m_{H}, so the target density (number/area) is

$$n_{\text{tar}} = (\text{number density}) \times L = \frac{\varrho L}{m_{\text{H}}} = \frac{(0.07 \times 10^3\ \text{kg/m}^3) \times (0.5\ \text{m})}{1.66 \times 10^{-27}\ \text{kg}} \approx 2.1 \times 10^{28}\ \text{atoms/m}^3$$

14.4 ⋆⋆ As in Eq.(14.3), $n_{\text{tar}} = \varrho t/m$, so

$$N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \sigma = 10^9 \times \frac{(8.9 \times 10^3\ \text{kg/m}^3) \times (10^{-5}\ \text{m})}{63.5 \times 1.66 \times 10^{-27}\ \text{kg}} \times (2.0 \times 10^{-28}\ \text{m}^2) = 1.69 \times 10^5\ \text{particles.}$$

14.5 ⋆⋆ The target density is $n_{\text{tar}} =$ (number density) $\times$ thickness, where in an ideal gas of nitrogen the number density is $2 \times (6.02 \times 10^{23})/(22.4\ \text{liters}) = 5.38 \times 10^{25}$ particles/m^3. (Remember that a mole of gas occupies 22.4 liters and that each N_2 molecule has two atoms.) Thus $n_{\text{tar}} = 5.38 \times 10^{24}$ particles/m^2. Therefore

$$N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \sigma = 10^{11} \times (5.38 \times 10^{24}\ \text{m}^{-2}) \times (0.5 \times 10^{-28}\ \text{m}^2) = 2.7 \times 10^7\ \text{particles.}$$

14.6 ⋆⋆ The probability that any one target will be struck by a projectile is

$$\text{prob(any given target is struck)} = n_{\text{inc}} \sigma.$$

Therefore, with N_{tar} targets in all, the expected total number of strikes is

$$N_{\text{sc}} = N_{\text{tar}} \times \text{prob(any given target is struck)} = N_{\text{tar}} n_{\text{inc}} \sigma.$$

14.7 ⋆ A sphere of radius R at a large distance d subtends a solid angle $\Delta\Omega \approx A/d^2 = \pi R^2/d^2$. For the moon this gives $\Delta\Omega_{\text{moon}} \approx 6.45 \times 10^{-5}$ sr, and for the sun, $\Delta\Omega_{\text{sun}} \approx 6.76 \times 10^{-5}$ sr. Because $\Delta\Omega_{\text{moon}} \approx \Delta\Omega_{\text{sun}}$, the moon and sun *appear* to be about the same size.

14.8 ⋆ $\Delta\Omega = A/r^2 = (1\ \text{mm}^2)/(10\ \text{mm})^2 = 0.01$ sr.

14.9 ⋆ $\Omega = \int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} (\sin\theta) d\theta = 2\pi\big[-\cos\theta\big]_0^{\pi} = 4\pi.$

14.10 ⋆

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \sigma_o \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta (1 + 3\cos\theta + 3\cos^2\theta) = 2\pi\sigma_o \int_{-1}^{1} du (1 + 3u + 3u^2) = 8\pi\sigma_o.$$

14.11 ⋆⋆ The target density is

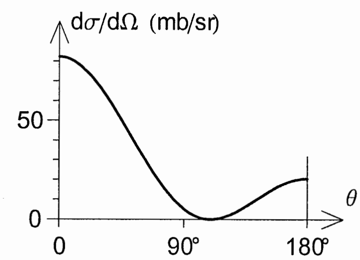
$$n_{\text{tar}} = \frac{\varrho t}{m_{\text{Ag}}} = \frac{(10.5 \times 10^3 \text{ kg/m}^3) \times (10^{-6} \text{ m})}{108 \times (1.66 \times 10^{-27} \text{ kg})} = 5.86 \times 10^{22} \text{ m}^{-2}$$

and the solid angle subtended by the counter is $\Delta\Omega = (0.1 \text{ mm}^2)/(10 \text{ mm})^2 = 10^{-3}$ sr. Therefore, the number of alphas scattered into $\Delta\Omega$ should be

$$N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \frac{d\sigma}{d\Omega} \Delta\Omega = (10^{10}) \times (5.86 \times 10^{22} \text{ m}^{-2}) \times (0.5 \times 10^{-28} \text{ m}^2/\text{sr}) \times (10^{-3} \text{ sr}) \approx 29.$$

14.12 ⋆⋆⋆ **(a)** Since $E = p^2/2m$ and $p = \sqrt{2mE}$,

$$f(\theta) = \frac{\hbar}{\sqrt{2mE}} \left(e^{i\delta_0} \sin \delta_0 + 3e^{i\delta_1} \sin \delta_1\right).$$



(b) With $\delta_0 = -30°$ and $\delta_1 = 150°$,

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\hbar^2}{8mE} \left(1 + 6\cos\theta + 9\cos^2\theta\right).$$

The figure shows the differential cross section in millibarns/steradian.

(c) $\sigma = \int |f(\theta)|^2 = \frac{\hbar^2}{8mE} \int_0^{2\pi} d\phi \int_{-1}^{1} (1 + 6u + 9u^2) du = \frac{\hbar^2}{8mE} \cdot 16\pi = 260$ mb,

where the second integral in the third expression results from the substitution $u = \cos\theta$.

14.13 ⋆⋆ In Fig.14.10, let us temporarilly rename as α' the second of the angles labelled α (the angle of reflection), and let the incoming and outgoing speeds be v and v'. That the collision is elastic implies that $v = v'$, and conservation of angular momentum implies that $mvR \sin\alpha = mv'R \sin\alpha'$. Together, these imply that $\sin\alpha = \sin\alpha'$ and hence that $\alpha = \alpha'$.

14.14 ⋆⋆ **(a)** If we consider Fig.14.1 to show a two-dimensional scattering event, then on the one hand the definition of the differential cross section (really differential width) is that the number of scatterings between θ and $\theta + d\theta$ is

$$N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \frac{d\sigma}{d\theta} d\theta,$$

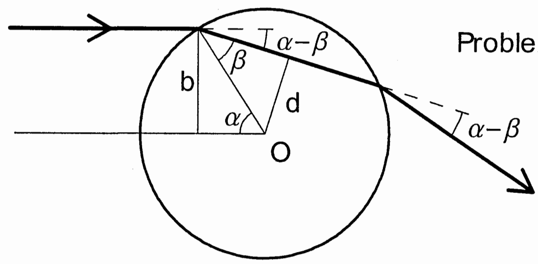
where n_{tar} is the density (number/width) of targets. [This is the analog of Eq.(14.17) for two-dimensional scattering.] On the other hand, by the familiar argument (modified for two dimensions) $N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} db$. Comparing these two equations, we see that $d\sigma/d\theta = |db/d\theta|$, where the absolute values signs are because $d\theta$ and db were both taken to be positive.

(b) From Fig.14.10 (considered as a two-dimensional picture) we see that $b = R\sin\alpha$ and that $\theta = \pi - 2\alpha$ or $\alpha = \frac{1}{2}(\pi - \theta)$. Therefore $b = R \sin \frac{1}{2}(\pi - \theta) = R\cos(\theta/2)$, and

$$\frac{d\sigma}{d\theta} = \left|\frac{db}{d\theta}\right| = \frac{R}{2} |\sin(\theta/2)|.$$

(c) $\sigma = \int_{-\pi}^{\pi} \frac{d\sigma}{d\theta} d\theta = 2\int_0^{\pi} (R/2)\sin(\theta/2)d\theta = 2R\int_0^{\pi/2} \sin(u)du = 2R.$

14.15 ⋆⋆⋆ **(a)**

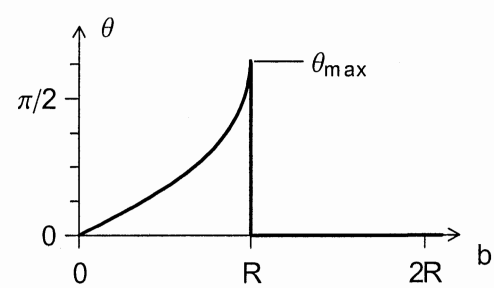


Problem 14.15(a)

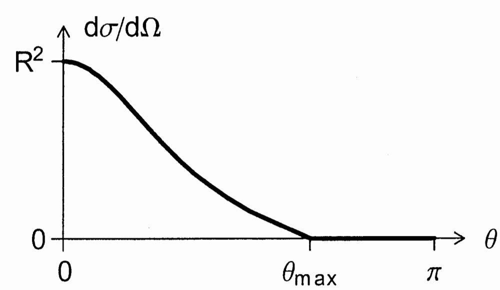
Note that the angles shown are $\alpha = \arcsin(b/R)$ and $\beta = \arcsin(d/R)$, and the angle of deflection on entering the sphere is $\alpha - \beta$. By symmetry the deflection on leaving the sphere is the same, so the total angle of deflection is $\theta = 2(\alpha - \beta)$.

(b) The projectile's initial energy is $\frac{1}{2}mv_o^2 = p_o^2/2m$, while its energy inside the sphere is $p^2/2m - U_o$. Equating these two, we find that $p = \sqrt{p_o^2 + 2mU_o}$. The initial angular momentum is bp_o and that when inside the sphere is dp. Equating these we find that $d = bp_o/p = \zeta b$.

(c) From part (a), it is clear that $\theta = 2(\alpha - \beta) = 2[\arcsin(b/R) - \arcsin(\zeta b/R)]$, as in Equation (14.60). This is zero when $b = 0$ (the projectile goes straight through, undeflected). As b increases from 0 to R, the scattering angle θ increases steadily to a maximum, $\theta_{\max} = \pi - 2\arcsin\zeta$ when $b = R$, then drops abruptly to 0 and remains zero for all $b > R$ because the projectile misses the target entirely.



Problem 14.15(c)



(d)

(d) Differentiating (14.60), we find

$$\frac{d\sigma}{d\Omega} = \frac{b}{(\sin\theta)d\theta/db} \quad \text{where} \quad \frac{d\theta}{db} = 2\left(\frac{1}{\sqrt{R^2 - b^2}} - \frac{\zeta}{\sqrt{R^2 - \zeta^2 b^2}}\right).$$

This is good for $0 \le b \le R$ (equivalently $0 \le \theta \le \theta_{\max}$); for $\theta > \theta_{\max}$, $d\sigma/d\Omega = 0$.

(e) $\sigma_{\text{tot}} = 2\pi\int_0^{\pi} \frac{d\sigma}{d\Omega}\sin\theta\, d\theta = 2\pi\int_0^{\pi} \frac{b}{(\sin\theta)d\theta/db}\sin\theta\, d\theta = 2\pi\int_0^R b\, db = \pi R^2.$

14.16 $\star$

Number of mica sheets:	0	1	2	3	4	5	6
Counts, N_{sc} (per min):	24.7	29	33.4	44	81	101	255
Speed, v (arb. units):	1.0	0.95	0.90	0.85	0.77	0.69	0.57
$N_{sc}v^4$:	24.7	23.6	21.9	23.0	28.5	22.9	26.9

The numbers $N_{sc}v^4$ are reasonably (though certainly not exactly) constant. Without knowing the experimental uncertainties, we can't really assess their constancy, but at the time the results were considered good confirmation that N_{sc} is inversely proportional to v^4.

14.17 $\star$

target:	Gold	Platinum	Tin	Silver	Copper	Aluminum
N_{sc}:	1319	1217	467	420	152	26
Z:	79	78	50	47	29	13
N_{sc}/Z^2 :	0.21	0.20	0.19	0.19	0.18	0.15

14.18 $\star\star$ The cross section for backward scattering is found by integrating $d\sigma/d\Omega = \sigma_o/\sin^4(\theta/2)$ over the backward hemisphere, $\theta \geq \pi/2$:

$$\sigma(\theta \geq \pi/2) = \int_{\theta\geq\pi/2} d\Omega \frac{d\sigma}{d\Omega} = \sigma_o \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} \frac{\sin\theta \, d\theta}{\sin^4(\theta/2)} = 4\pi\sigma_o.$$

[One way to do the integral over θ is to write $\sin\theta$ as $2\sin(\theta/2)\cos(\theta/2)$ and make the substitution $u = \sin(\theta/2)$.]

The number of α's scattered into the backward hemisphere should be $N_{sc}(\theta \geq \pi/2) = N_{inc}n_{tar}\sigma(\theta \geq \pi/2)$. Thus the requested ratio is $N_{sc}(\theta \geq \pi/2)/N_{inc} = n_{tar}\sigma(\theta \geq \pi/2)$. With

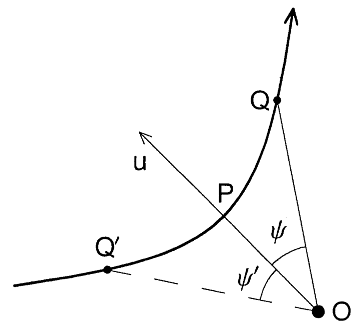
$$n_{tar} = \frac{\varrho t}{m_{Pt}} = \frac{(21.4 \times 10^3 \text{ kg/m}^3) \times (3 \times 10^{-6} \text{ m})}{195 \times 1.66 \times 10^{-27} \text{ kg}} = 1.98 \times 10^{23} \text{ m}^{-2}$$

and

$$\sigma_o = \left(\frac{kqQ}{4E}\right)^2 = \left(\frac{(8.99 \times 10^9 \text{ N}\cdot\text{m}^2/\text{ C}^2) \times 2 \times 78 \times (1.60 \times 10^{-19} \text{ C})^2}{4 \times (7.8 \times 1.60 \times 10^{-13} \text{ J})}\right)^2 = 5.17\times10^{-29} \text{ m}^2$$

the required ratio is $n_{tar}\sigma(\theta \geq \pi/2) = 4\pi n_{tar}\sigma_o = 1.29 \times 10^{-4}$. The reciprocal of this is 7750, so the Rutherford model predicts that 1 particle in 7750 would be "reflected" into the backward hemisphere, in remarkable agreement with Geiger and Marsden's observed "about 1 in 8000."

14.19 ★★ The figure shows a typical orbit of a projectile. The point O is the force center, P is the point of closest approach, and let us choose the time $t = 0$ to be the time at which the projectile passes through P. The point Q is the projectile's position at an arbitrary time t and Q' the position at time $-t$. To prove that the orbit is symmetric about the line OP, I shall prove that the points Q and Q' are equidistant from O (that is, $OQ = OQ'$) and make equal angles with the direction OP (that is, $\psi = \psi'$). Since Q is any point on the orbit, this does it.



(a) Under the assumptions on the effective PE, the projectile must have $E \geq 0$, corresponding to the upper dashed line in Fig.8.5. Since $U_{\text{eff}}(r) \to \infty$ as $r \to 0$, there has to be an r_{min} at which $E = U_{\text{eff}}(r_{\text{min}})$. The projectile moves steadily inward from $r = \infty$ until it reaches r_{min} and then moves steadily outward toward $r = \infty$ again. The distance r_{min} defines the point of closest approach labeled P in the figure here.

(b) Since $E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r)$, it follows that $\dot{r}^2$ is a single-valued function of r on any one orbit. Therefore, the magnitude of $\dot{r}$ at any distance r on the inward trip is the same as that at the same distance r on the outward trip: $\dot{r}(r)_{\text{in}} = -\dot{r}(r)_{\text{out}}$. It immediately follows that the time to move in from r to r_{min} is the same as that to move out again from r_{min} to r. Turning this around, if the times from Q' to P and from P to Q are equal, then the distances OQ' and OQ are equal.

Using conservation of angular momentum ℓ, we can prove a similar result for the angle ψ. Since $\ell = mr^2\dot{\psi}$, it follows that $\dot{\psi}$ is a single-valued function of r. Therefore $\dot{\psi}$ is the same at any point Q and at the corresponding Q', which in turn means that the change in ψ from Q' to P is the same as that from P to Q. That is $\psi = \psi'$.

(c) The symmetry of the orbit now follows as in the first paragraph above.

14.20 ★★ As argued in the problem statement, $\theta = \pi - 2\psi_{\text{o}}$ and $\psi_{\text{o}} = \int(\dot{\psi}/\dot{r})dr$. Thus all that remains is to rewrite this integral in terms of r and the impact parameter b. To do this, we use three tricks: First when the projectile is far from the target, the angular momentum is easily evaluated as $\ell = pb = \sqrt{2mE}\, b$. Second, in general $\ell = mr^2\dot{\psi}$, so

$$\dot{\psi} = \frac{\ell}{mr^2} = \frac{\sqrt{2mE}\, b}{mr^2} = \sqrt{2E/m}\, b/r^2. \tag{i}$$

Third, since $\frac{1}{2}m\dot{r}^2 = E - U_{\text{eff}} = E - U - \ell^2/(2mr^2)$,

$$\dot{r} = \sqrt{2/m}\,\sqrt{E - U - Eb^2/r^2} = \sqrt{2E/m}\,\sqrt{1 - U/E - b^2/r^2}. \tag{ii}$$

Substituting the results (i) and (ii) into the two equations at the start of this solution, we find the advertized expression (14.61) for θ.

14.21 ⋆⋆ For a hard sphere, $r_{\text{min}} = R$ and, since $U(r) = 0$ for $r > R$, Eq.(14.61) reduces to

$$\theta = \pi - 2\int_R^\infty \frac{(b/r^2)dr}{\sqrt{1-(b/r)^2}} = \pi + 2\int_{b/R}^0 \frac{du}{\sqrt{1-u^2}} = \pi - 2\arcsin(b/R),$$

where to get the second integral I substituted $u = b/r$. This is quickly solved to give $b = R\cos(\theta/2)$ as in (14.24).

14.22 ⋆⋆⋆ With $U(r) = \gamma/r$, Eq.(14.61) becomes

$$\theta = \pi - 2\int_{r_{\text{min}}}^\infty \frac{(b/r^2)dr}{\sqrt{1-(b/r)^2-\gamma/Er}} = \pi - 2\int_{u_{\text{min}}}^0 \frac{-du}{\sqrt{1-u^2-\gamma u/Eb}}$$

where the second integral results from the substitution $u = b/r$. One way to do this integral is to complete the square for the last two terms in the denominator to give

$$\theta = \pi - 2\int_0^{u_{\text{min}}} \frac{du}{\sqrt{1+(\gamma/2Eb)^2-(u+\gamma/2Eb)^2}} = \pi - 2\,\text{arccot}(\gamma/2Eb).$$

(In doing the last integral, it helps to notice that the denominator is zero at the limit u_{min}.) Solving for b, we find that $b = (\gamma/2E)\cot(\theta/2)$ in agreement with (14.31).

14.23 ⋆⋆⋆ Before we use Eq.(16.61) we need to find r_{min}. This is determined by the condition that $U_{\text{eff}} = E$ where

$$U_{\text{eff}} = U + \frac{\ell^2}{2mr^2} = \frac{\gamma}{r^2} + \frac{p^2b^2}{2mr^2} = \left(\gamma + Eb^2\right)\frac{1}{r^2}.$$

Therefore, the minimum value of r (where $U_{\text{eff}} = E$) is

$$r_{\text{min}} = \sqrt{b^2+\gamma/E}.$$

Notice that this let's us rewrite U_{eff} as $U_{\text{eff}} = E\,(r_{\text{min}}/r)^2$.

Returning to Eq.(14.61), we can write

$$\pi - \theta = 2\int_{r_{\text{min}}}^\infty \frac{(b/r^2)\,dr}{\sqrt{1-U_{\text{eff}}(r)/E}} = 2b\int_{r_{\text{min}}}^\infty \frac{dr/r^2}{\sqrt{1-(r_{\text{min}}/r)^2}} = \frac{\pi b}{r_{\text{min}}} = \frac{\pi b}{\sqrt{b^2+\gamma/E}},$$

which we can solve to give $b^2 = \dfrac{\gamma}{E}\cdot\dfrac{(\pi-\theta)^2}{\theta(2\pi-\theta)}$. Finally, as you can easily check,

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{1}{2\sin\theta}\left|\frac{d(b^2)}{d\theta}\right| = \frac{\gamma}{E}\frac{\pi^2(\pi-\theta)}{\theta^2(2\pi-\theta)^2\sin\theta}.$$

14.24 $\star\star$ **(a)** We are told that, if $m_1 = m_2$, then $\theta_{\text{lab}} = \frac{1}{2}\theta_{\text{cm}}$. To relate the corresponding differential cross sections we need to find the derivative

$$\frac{d(\cos\theta_{\text{cm}})}{d(\cos\theta_{\text{lab}})} = \frac{d(\cos 2\theta_{\text{lab}})}{d(\cos\theta_{\text{lab}})} = \frac{d(2\cos^2\theta_{\text{lab}} - 1)}{d(\cos\theta_{\text{lab}})} = 4\cos\theta_{\text{lab}}.$$

Substitution into (14.45) yields the claimed result (14.63).

(b) Since $(d\sigma/d\Omega)_{\text{cm}} = R^2/4$, it follows that $(d\sigma/d\Omega)_{\text{lab}} = (4\cos\theta_{\text{lab}})R^2/4 = R^2\cos\theta_{\text{lab}}$. This is for $0 \le \theta_{\text{lab}} \le \pi/2$. Since no particles are scattered with $\theta_{\text{lab}} > \pi/2$, it follows that $(d\sigma/\Omega)_{\text{lab}} = 0$ for $\pi/2 < \theta_{\text{lab}} \le \pi$. Therefore,

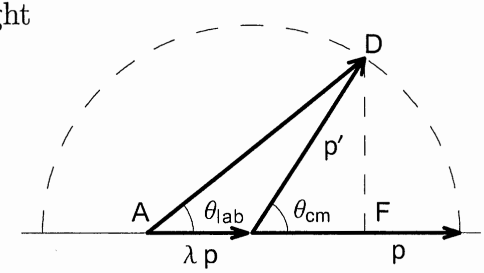
$$\sigma_{\text{tot}} = \int d\Omega_{\text{lab}} \left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = R^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos(\theta_{\text{lab}})\sin(\theta_{\text{lab}}) d\theta_{\text{lab}} = R^2 \cdot 2\pi \cdot \int_0^1 u\, du = \pi R^2,$$

where the last integral results from the substitution $u = \sin\theta_{\text{lab}}$.

14.25 $\star\star$ Consider the right triangle ADF, whose height is $DF = p'\sin\theta_{\text{cm}} = p\sin\theta_{\text{cm}}$ (recall that $p = p'$) and whose base is $AF = \lambda p + p\cos\theta_{\text{cm}}$. Therefore

$$\tan\theta_{\text{lab}} = \frac{DF}{AF} = \frac{p\sin\theta_{\text{cm}}}{\lambda p + p\cos\theta_{\text{cm}}} = \frac{\sin\theta_{\text{cm}}}{\lambda + \cos\theta_{\text{cm}}},$$

as claimed.



14.26 $\star\star$ From Eq.(14.26) it follows that

$$\sec^2\theta_{\text{lab}} = 1 + \tan^2\theta_{\text{lab}} = 1 + \frac{\sin^2\theta_{\text{cm}}}{(\lambda + \cos\theta_{\text{cm}})^2} = \frac{\lambda^2 + 2\lambda\cos\theta_{\text{cm}} + 1}{(\lambda + \cos\theta_{\text{cm}})^2}$$

and hence

$$\cos\theta_{\text{lab}} = \frac{1}{\sec\theta_{\text{lab}}} = \frac{\lambda + \cos\theta_{\text{cm}}}{\sqrt{\lambda^2 + 2\lambda\cos\theta_{\text{cm}} + 1}}.$$

Therefore

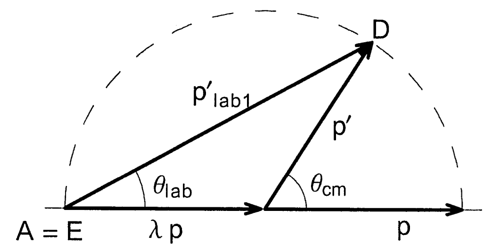
$$\frac{d(\cos\theta_{\text{lab}})}{d(\cos\theta_{\text{cm}})} = \frac{1}{\sqrt{\lambda^2 + 2\lambda\cos\theta_{\text{cm}} + 1}} - \frac{(\lambda + \cos\theta_{\text{cm}})\lambda}{(\lambda^2 + 2\lambda\cos\theta_{\text{cm}} + 1)^{3/2}} = \frac{1 + \lambda\cos\theta_{\text{cm}}}{(\lambda^2 + 2\lambda\cos\theta_{\text{cm}} + 1)^{3/2}}$$

14.27 $\star\star$ **(a)** With $\lambda = 1$, Eq.(14.64) reads

$$\tan\theta_{\text{lab}} = \frac{\sin\theta_{\text{cm}}}{1 + \cos\theta_{\text{cm}}} = \frac{2\sin(\theta_{\text{cm}}/2)\cos(\theta_{\text{cm}}/2)}{1 + \cos^2(\theta_{\text{cm}}/2) - \sin^2(\theta_{\text{cm}}/2)} = \frac{\sin(\theta_{\text{cm}}/2)}{\cos(\theta_{\text{cm}}/2)} = \tan(\theta_{\text{cm}}/2).$$

Therefore, $\theta_{\text{lab}} = \frac{1}{2}\theta_{\text{cm}}$.

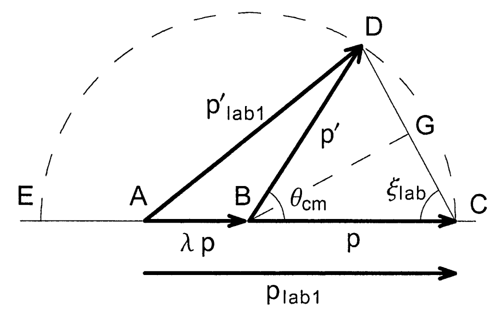
(b) Compare the figure here with Figure 14.15. Here, with $\lambda = 1$, the point A has moved out and coincides with E. If $m_1 < m_2$ (Fig. 14.15), then as θ_{cm} increases toward π the point D moves around toward E and $\theta_{\text{lab}} \to \pi$. But here, with equal masses, as θ_{cm} increases toward π the line AD becomes tangent to the circle at A and $\theta_{\text{lab}} \to \pi/2$.



14.28 ⋆⋆ **(a)** In the lab frame, the initial momentum of particle 2 is zero, $\mathbf{p}_{\text{lab2}} = 0$. Therefore, conservation of momentum implies that

$$\mathbf{p}_{\text{lab1}} = \mathbf{p}'_{\text{lab1}} + \mathbf{p}'_{\text{lab2}}. \tag{iii}$$

That is, these three vectors form a triangle in the appropriate order. Since the first two vectors are represented by the sides AC and AD, it follows that $\mathbf{p}'_{\text{lab2}}$ is represented by the side DC. The angle ξ_{lab} is the angle between $\mathbf{p}'_{\text{lab2}}$ and the incident direction, and this is equal to the angle BCD. By looking at the right triangle BCG, you should be able to convince yourself that $\xi_{\text{lab}} = (\pi/2) - (\theta_{\text{cm}}/2)$, as claimed.



(b) If the masses are equal, then $\lambda = 1$ and the point A in Fig.14.15 coincides with E; that is, AC is a diameter, and, by a well known theorem of geometry, the angle ADC is 90°.

(c) If you square Eq.(iii) above (conservation of momentum) and compare the result with the equation of conservation of kinetic energy, $T_1 = T'_1 + T'_2$, you will find that $\mathbf{p}'_{\text{lab1}} \cdot \mathbf{p}'_{\text{lab2}} = 0$, which says that the two final momenta are perpendicular.

14.29 ⋆⋆ **(a)** In the CM frame the total momentum is zero, so the two initial momenta are equal in magnitude, $p_1 = p_2$ and likewise the two final momenta, $p'_1 = p'_2$. This let's us write the conservation of kinetic energy (elastic collision) as

$$E = p_1^2 \left(\frac{1}{2m_1} + \frac{1}{2m_2} \right) = E' = p_1'^2 \left(\frac{1}{2m_1} + \frac{1}{2m_2} \right)$$

which implies that $p_1 = p'_1$ and hence that the KE of particle 1 is separately conserved, and likewise for particle 2.

(b) In the lab frame, particle 2 is initially at rest, so that $T_2 = 0$. If any kind of collision occurs, particle 2 must recoil with $T_2 > 0$. Therefore, particle 2 *gains* kinetic energy. By conservation of energy, the projectile must lose KE, and the separate energies are definitely *not* conserved.

(c) In Fig.14.15 you can see that the final momentum of particle 2 is represented by the line DC, which has magnitude $p'_{\rm lab2} = 2p\sin(\theta_{\rm cm}/2)$. Therefore the energy gained by particle 2 (and lost by particle 1) is

$$\Delta E = \frac{(p'_{\rm lab2})^2}{2m_2} = \frac{2p^2\sin^2(\theta_{\rm cm}/2)}{m_2}.$$

This is to be compared with the original energy of particle 1, which is (in the lab frame)

$$E = T_1 = \frac{(p_{\rm lab1})^2}{2m_1} = \frac{(1+\lambda)^2p^2}{2m_1}$$

Therefore

$$\frac{\Delta E}{E} = \frac{2p^2\sin^2(\theta_{\rm cm}/2)}{m_2}\cdot\frac{2m_1}{(1+\lambda)^2p^2} = \frac{4\lambda\sin^2(\theta_{\rm cm}/2)}{(1+\lambda)^2}.$$

(d) For given λ this fractional loss is greatest if $\theta_{\rm cm}/2 = 90°$ or $\theta_{\rm cm} = 180°$ — what one would normally expect in a direct head-on collision. Differentiating with respect to λ, you can easily check that the corresponding fractional loss is maximum if $\lambda = 1$, that is, if the two particles have equal masses.

14.30 ⋆⋆⋆ From Example 14.5 we know that the CM differential cross section for the hard spheres is $(d\sigma/d\Omega)_{\rm cm} = R^2/4$ (where R is the sum of the two separate radii). Thus, according to Eq.(14.63) the lab cross section is $d\sigma/d\Omega = 4\cos\theta(d\sigma/d\Omega)_{\rm cm} = R^2\cos\theta$. (I'll denote lab variables without any subscript.) Thus the number of A particles that emerge into solid angle $d\Omega$ at angle Θ should be

$$N(A \text{ into } d\Omega \text{ at } \Theta) = N_{\rm inc}n_{\rm tar}R^2\cos(\Theta)d\Omega. \qquad \text{(iv)}$$

Now, we know that the A and B particles emerge at 90° to one another. Thus a B emerges at angle Θ if and only if an A emerges at $\Theta' = \pi/2 - \Theta$, and

$$N(B \text{ into } d\Omega \text{ at } \Theta) = N(A \text{ into } d\Omega' \text{ at } \Theta') \qquad \text{(v)}$$

where $d\Omega'$ is the solid angle at $\Theta' = \pi/2 - \Theta$ corresponding to $d\Omega$ at Θ:

$$d\Omega' = \sin(\Theta')d\Theta'd\phi' = \sin(\pi/2-\Theta)d\Theta d\phi = \cos(\Theta)d\Theta d\phi = \frac{\cos\Theta}{\sin\Theta}d\Omega \qquad \text{(vi)}$$

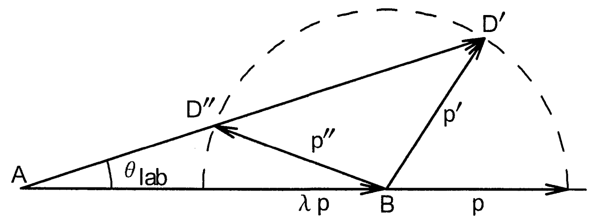
Substituting (iv) and (vi) into (v), we find

$$\begin{aligned} N(B \text{ into } d\Omega \text{ at } \Theta) &= N_{\rm inc}n_{\rm tar}R^2\cos(\Theta')d\Omega' = N_{\rm inc}n_{\rm tar}R^2\sin(\Theta)\frac{\cos\Theta}{\sin\Theta}d\Omega \\ &= N(A \text{ into } d\Omega \text{ at } \Theta), \end{aligned}$$

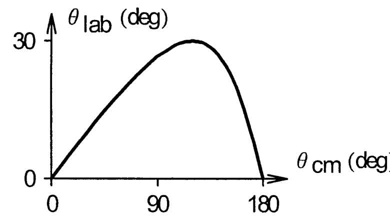
where for the second equality I used the identity $\cos(\Theta') = \cos(\pi/2-\Theta) = \sin(\Theta)$ and for the final line I used (iv) again. We see that the number of B particles emerging into $d\Omega$ at Θ should be exactly the same as the number of A particles emerging into the same solid angle in the same direction.

A somewhat slicker way to do this problem is to move to the CM frame, as I'll leave you to check.

14.31 ⋆⋆⋆



Problem 14.31 (a)



(c)

(a) The two final CM momenta labeled $\mathbf{p}'$ and $\mathbf{p}''$ are in different directions, but their corresponding lab momenta, represented by the lines AD' and AD'' are in the same direction.

(b) The lab angle θ_{lab} is 0 if $\theta_{\text{cm}} = 0$ and if $\theta_{\text{cm}} = \pi$. In the case $\theta_{\text{cm}} = 0$, the projectile passes the target unscattered, and the same is true as seen in the lab; that is, $\theta_{\text{lab}} = 0$. In the case $\theta_{\text{cm}} = \pi$, the projectile bounces straight back (as seen in the CM frame) — typically in a head-on collision. However, in the lab frame, a head-on collision with $m_1 > m_2$ only slows the projectile, which emerges in the forward direction with $\theta_{\text{lab}} = 0$.

(c) See figure.

(d) From the figure of part (a), the lab angle θ_{lab} is maximum when the line $AD''D'$ is tangent to the circle. Therefore $\theta_{\text{lab}}(\text{max}) = \arcsin(p/\lambda p) = \arcsin(1/\lambda)$. If $\lambda = 1$, this gives the known result that $\theta_{\text{lab}}(\text{max}) = 90°$.

Chapter 15

Special Relativity

In the planning of this book, this chapter on special relativity was probably the most difficult. Should there be a chapter on relativity? And, if so, what should it contain? Although a majority of physics departments do not teach relativity as a part of classical mechanics, an appreciable minority do. Thus for the sake of that minority it seemed necessary to include the chapter, and certainly a majority of the colleagues we surveyed thought so. Therefore, we decided to include the chapter. For some of the possible users, this will be the first serious encounter with relativity, so it was essential that the chapter begin at the beginning. On the other hand, an upper division course should surely take the students to a reasonably sophisticated level, so it was essential that the chapter aim higher than the typical freshman or sophomore introduction. Therefore, the chapter has to begin at the beginning and take the reader at least through the concepts of four-vectors, four-tensors, and a little relativistic electrodynamics. The result is that the chapter is easily the longest in the book.

Fortunately, it is not essential to cover the whole chapter. For an introduction approximately like a sophomore course you could stop at Section 15.13. In fact, since the most glamorous part of relativity for the beginner is the kinematics (time dilation, length contraction, and so on) you could even just cover Sections 15.1 through 15.6 (or maybe 15.10 so that your students would at least meet the notion of space-time and the light cone). Of course, if you have the time, I hope you will cover the whole chapter so your students meet some relativistic dynamics, massless particles, tensors, and electrodynaimics.

There is one notational feature I should mention. It is fairly generally perceived that there are just two ways to handle the four-dimensional "length squared" $x \cdot x = \mathbf{x} \cdot \mathbf{x} - c^2 t^2$ of a vector x with spatial part $\mathbf{x}$ and time component ct. The first is to introduce the imaginary fourth component $x_4 = ict$, so that $x \cdot x = \sum_{\mu=1}^4 x_\mu^2$. The second is to introduce the elaborate machinery of covariant and contravariant vectors, with components x_μ and x^μ respectively, where $x^i = x_i$ for $i = 1, 2, 3$ but $x^4 = -x_4 = ct$, so that the length squared becomes $x \cdot x = \sum_{\mu=1}^4 x_\mu x^\mu$.[1] The first approach has the undeniable advantage of simplicity,

[1]There is a third possility, to define $x \cdot x$ as the negative of the definition I'm using here. Although, many physicists feel passionately about this choice of sign, the difference between the second and third approaches is not really an important philosophical or pedagogical issue, and I shan't worry about it here.

but most physicists (though certainly not all) feel that it is dishonest, in that it makes a genuinely more complicated situation *seem* like a simpler one. The method of covariant and contravariant indices is honest but costs a distressing amount of time and effort to teach and learn. I was surprised to realise that one can go a surprising distance without either device. I define the real four-vector x as $x = (\mathbf{x}, ct)$ (or, when the distinction becomes important, as the *column matrix* made up of these four numbers). For a while, one has only to define the four-dimensional scalar product $x{\cdot}y$ as $x{\cdot}y = x_1y_1 + y + x_2y_2 + x_3y_3 - x_4y_4$ and no machinery at all is required. If you want to go a little further, it is only necessary to recognize that the scalar product can equally be seen as[2] $x \cdot y = \tilde{x}Gy$, where now y denotes the *column* comprising y_1, y_2, y_3, and y_4, while $\tilde{x}$ is the *row* made up of x_1, x_2, x_3, and x_4, and G is the diagonal "metric matrix" with diagonal elements 1,1,1,−1. In other words, to form a dot product, we simply sandwich the metric matrix G between the appropriate matrices. It turns out that as long as our ambitions go no further than second rank tensors, all of the machinery of four-vectors and four-tensors can be handled using matrix notation, without introducing either the imaginary fourth component or covariant and contravariant indices, and this is what I elected to do.

Solutions to Problems for Chapter 15

15.1 ⋆ Suppose that the first law holds in a certain frame $\mathcal{S}$. Now consider a second frame $\mathcal{S}'$ obtained from $\mathcal{S}$ by a Galilean transformation and suppose that the net force $\mathbf{F}'$ on a body is zero as measured in $\mathcal{S}'$. By the invariance of force, this means that $\mathbf{F} = \mathbf{0}$ as measured in $\mathcal{S}$. By the first law (true in $\mathcal{S}$), this means that the acceleration $\mathbf{a}$ as measured in $\mathcal{S}$ is zero. Finally, since $\mathbf{v}' = \mathbf{v} - \mathbf{V}$, with $\mathbf{V}$ constant, it follows that $\mathbf{a}' = \mathbf{a} = 0$. In summary, we've proved that

$$(\mathbf{F}' = 0) \implies (\mathbf{F} = 0) \implies (\mathbf{a} = 0) \implies (\mathbf{a}' = 0)$$

and we've proved that the first law holds in $\mathcal{S}'$.

Similarly, we can prove that if the third law holds in $\mathcal{S}$, then

$$\mathbf{F}'_{12} = \mathbf{F}_{12} = -\mathbf{F}_{21} = -\mathbf{F}'_{21}$$

and the third law holds in $\mathcal{S}'$.

15.2 ⋆⋆ Conservation of momentum in a frame $\mathcal{S}$ implies that

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m_C\mathbf{v}_C + m_D\mathbf{v}_D. \tag{i}$$

If this law is invariant under the Galilean transformation, this it is also true in any other frame $\mathcal{S}'$,

$$m_A\mathbf{v}'_A + m_B\mathbf{v}'_B = m_C\mathbf{v}'_C + m_D\mathbf{v}'_D, \tag{ii}$$

[2]This is, of course, just the matrix equivalent of the familiar statement that $x{\cdot}y = \sum x_\mu g^{\mu\nu} y_\nu$.

where all velocities in $\mathcal{S}'$ are related to those in $\mathcal{S}$ by $\mathbf{v}'_A = \mathbf{v}_A - \mathbf{V}$ and so on (and $\mathbf{V}$ is the relative velocity of the two frames). Subtracting (2) from (1) we find that

$$(m_A + m_B)\mathbf{V} = (m_C + m_D)\mathbf{V} \tag{iii}$$

and hence that $(m_A + m_B) = (m_C + m_D)$. That is, the Galilean invariance of momentum conservation implies that mass must be conserved. Conversely, if (i) and (iii) are true, then by subtracting (iii) from (i) we can prove (ii); that is, conservation of mass guarantees the invariance of momentum conservation.

15.3 ⋆ If $v = 8000$ m/s, then $\beta = 8000/(3 \times 10^8) = (8/3) \times 10^{-5}$ and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1 + \tfrac{1}{2}\beta^2 = 1 + 3.56 \times 10^{-10},$$

where in the third expression I have used the binomial approximation. Since $\Delta t = \gamma \Delta t_o$, the difference is

$$\Delta t_o - \Delta t = (1-\gamma)\Delta t_o \approx -\tfrac{1}{2}\beta^2 \Delta t = -(3.56 \times 10^{-10}) \times 3600 \text{ s} = -1.28\ \mu\text{s}.$$

where in the third expression I have used the binomial approximation for γ and replaced Δt_o by Δt, since the difference is extremely small. The fractional difference is just $-\frac{1}{2}\beta^2 = -3.56 \times 10^{-8}\%$.

15.4 ⋆ If $\beta = 0.99$, then $\gamma = 1/\sqrt{1-\beta^2} = 7.09$. The time elapsed on the moving clock is $\Delta t_o = \Delta t/\gamma = 1$ hour$/7.09 = 8.5$ min. The difference is $\Delta t_o - \Delta t = -51.5$ min.

15.5 ⋆ With $\beta = 0.95$, the γ factor for both the outward and return trips is $\gamma = 1/\sqrt{1-\beta^2} = 3.20$. The times for the two halves of the journey satisfy

$$\Delta t_B^{\text{out}} = \gamma \Delta t_A^{\text{out}} \quad \text{and} \quad \Delta t_B^{\text{back}} = \gamma \Delta t_A^{\text{back}},$$

so, by addition, the times for the whole journey satisfy the same relation $\Delta t_B = \gamma \Delta t_A$. Therefore $\Delta t_A = \Delta t_B/\gamma = (80 \text{ yr})/3.20 = 25$ yr, which is the amount by which twin A has aged.

15.6 ⋆ Clearly $\gamma = 1/\sqrt{1-\beta^2} = 3$, so $\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{8/9} = 0.94$, and $v = 0.94c = 2.8 \times 10^8$ m/s.

15.7 ⋆⋆ With $v = 0.99c$, $\gamma = 7.09$. The half-life (measured in the muons' rest frame) is $t_{1/2}(\text{proper}) = 1.5\ \mu$s, and that measured in the earth frame is $t_{1/2}(\text{earth}) = \gamma t_{1/2}(\text{proper})$. The time of flight measured in the earth frame is $T(\text{earth}) = h/v$, so the number of half-lives that have elapsed is

$$n = \frac{T(\text{earth})}{t_{1/2}(\text{earth})} = \frac{h}{v\gamma t_{1/2}(\text{proper})} = 0.63. \tag{iv}$$

Therefore, the number that survive to the ground should be about $N = N_o/2^n = 650/2^{0.63} = 420$.

To find the classical answer, we must delete the factor of γ in the expression (iv) for n, to give $n(\text{clas}) = \gamma n(\text{rel}) = 4.49$ half-lives, and $N(\text{clas}) = 650/2^{4.49} \approx 29$.

15.8 ⋆⋆ **(a)** With $\beta = 4/5$, $\gamma = 5/3$. The half-life measured in the lab is $t_{1/2}(\text{lab}) = \gamma t_{1/2}(\text{proper}) = (5/3) \times (1.8 \times 10^{-8}\text{ s}) = 3.0 \times 10^{-8}$ s.

(b) The time of flight (measured in the lab) is $T(\text{lab}) = d/v$, where $d = 36$ m is the length of the pipe. Thus the number of half-lives elapsed is

$$n = \frac{T(\text{lab})}{t_{1/2}(\text{lab})} = \frac{d/v}{\gamma t_{1/2}(\text{proper})} = 5.00. \tag{v}$$

Therefore, the number of pions that survive the journey is $N = N_o/2^n = 32,000/2^5 = 1000$.

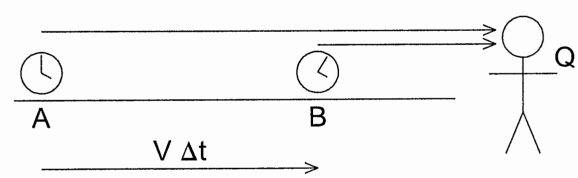
(c) To find the classical answer, we must delete the factor of γ in the expression (v) for n, to give $n(\text{clas}) = \gamma n(\text{rel}) = 8.33$ half-lives, and $N(\text{clas}) = 32,000/2^{8.33} \approx 100$.

15.9 ⋆⋆ The time for the helper to reach his assigned position (as measured by the chief observer) is $t_c = d/V$. Meanwhile, the time elapsed on the helper's clock is $t_h = t_c/\gamma = d/(V\gamma)$. Thus the difference is $\delta = t_c - t_h = (d/V)(1 - 1/\gamma)$. We wish to see what happens to this as $V \to 0$, and this requires some care (because of the factor of V in the denominator). However, as $V \to 0$ we can use the binomial expansion to give

$$\delta = \frac{d}{V}\left(1 - \frac{1}{\gamma}\right) = \frac{d}{\beta c}\left(1 - [1 - \beta^2]^{1/2}\right) \approx \frac{d}{\beta c}\left(1 - [1 - \tfrac{1}{2}\beta^2]\right) = \frac{d}{2c}\beta,$$

which approaches zero as $\beta \to 0$.

15.10 ⋆⋆⋆ **(a)** The picture shows two signals travelling from the clock to the observer Q. The first leaves the clock at position A and time t_A (as measured in the rest frame of Q), the second at position B and time t_B. The times at which Q receives these signals are:



$$(\text{time at which } Q \text{ sees clock at } B) = t_B + BQ/c$$

and

$$(\text{time at which } Q \text{ sees clock at } A) = t_A + AQ/c$$

The difference of these is the time Δt_{see} between Q's seeing the clock at A and at B:

$$\Delta t_{\text{see}} = t_B - t_A - AB/c = \Delta t - V\Delta t/c = \Delta t(1 - \beta). \tag{vi}$$

Since $\Delta t = \gamma \Delta t_o$, where Δt_o is the time elapsed on the clock itself, we can rewrite this as

$$\Delta t_{\text{see}} = \gamma \Delta t_o(1 - \beta) = \frac{1}{\sqrt{(1-\beta)(1+\beta)}}\Delta t_o(1 - \beta) = \Delta t_o\sqrt{\frac{1-\beta}{1+\beta}}.$$

(b) Once the clock is past Q, we can use the same picture except that Q is now to the left of both A and B, so AQ is less than BQ. This means that in Eq.(vi) the term $-AB/c$ is replaced by $+AB/c$ and $\Delta t_{\text{see}} = \Delta t(1+\beta)$.

15.11 ⋆ The stick's proper length is $l_o = 100$ cm, whereas I measure it to be $l = 80$ cm. Since $l = l_o/\gamma$, we see that $\gamma = 5/4$. Because $\gamma = 1/\sqrt{1-\beta^2}$,

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{4^2}{5^2}} = \sqrt{\frac{9}{25}} = \frac{3}{5}.$$

That is, $v = (3/5)c$.

15.12 ⋆⋆ The half-life in the pions' rest frame is the proper half-life, $t_{1/2}(\text{proper}) = 1.8 \times 10^{-8}$ s. The length of the pipe as "seen" by the pions is given by the length-contraction formula

$$(\text{length of pipe in pions' frame}) = d/\gamma = 21.6 \text{ m},$$

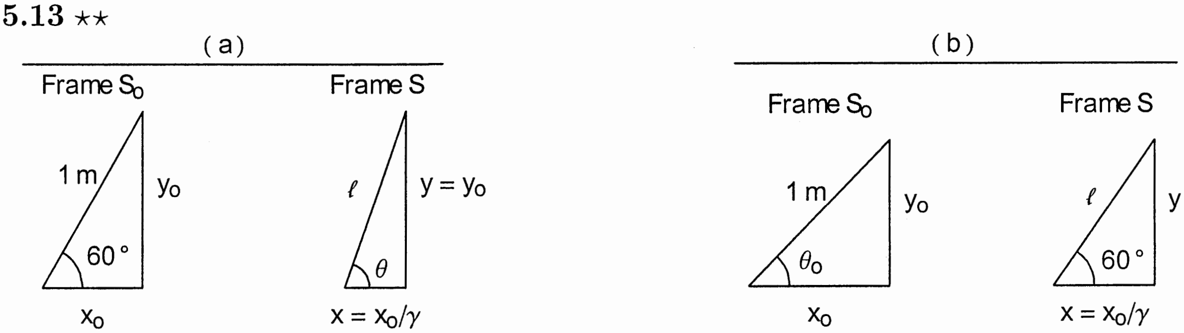
where $d = 36$ m is the length measured in the lab. The time for the pipe to pass the pions is therefore $T(\pi \text{ frame}) = (d/\gamma)/v = d/(\gamma v)$, and the number of half-lives that elapse is

$$n = \frac{T(\pi \text{ frame})}{t_{1/2}(\pi \text{ frame})} = \frac{d}{\gamma v t_{1/2}(\text{proper})} = 5.00. \tag{vii}$$

Therefore the number of pions that survive is $N = N_o/2^n = 32{,}000/32 = 1000$, the same answer as in Problem 15.8.

The two arguments, in this problem and Problem 15.8, lead to the same formula, (vii) here and (v) in the solution to Problem 15.8. Here the factor of γ comes from the length contraction of the tube, there the same factor came from the time dilation of the time of flight.

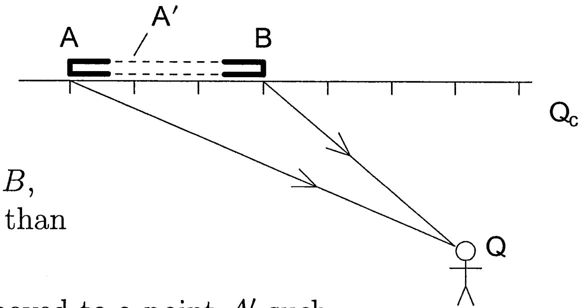
15.13 ⋆⋆



(a) With $\beta = 4/5$, $\gamma = 5/3$. In the frame $\mathcal{S}_o$, we know the length $l_o = 100$ cm and the angle $\theta_o = 60°$, so we can calculate $x_o = 50$ cm and $y_o = 86.6$ cm. In the frame $\mathcal{S}$, x is contracted ($x = x_o/\gamma = 30$ cm) but y is not ($y = y_o = 86.6$ cm). Thence $l = \sqrt{x^2+y^2} = 91.7$ cm and $\theta = \arctan(y/x) = 70.9°$.

(b) The angle 60° is given in the frame $\mathcal{S}$, so $\tan 60° = y/x = y_o/(x_o/\gamma)$ and $\tan\theta_o = y_o/x_o = (\tan 60°)/\gamma$, whence $\theta_o = 46.1°$. From this we find $x_o = 69.3$ cm and $y_o = 72.1$ cm, and from these we can calculate $x = x_o/\gamma$ and $y = y_o$ and thence $l = 83.2$ cm.

15.14 ⋆⋆⋆



(a) The observer Q sees the two ends of the rod by means of light striking his eye at one time t_Q. If $AQ > BQ$, as in the figure, the light from A had farther to travel that that from B, so the light from A must have left the rod earlier than that from B; that is, $t_A < t_B$.

(b) By the time t_B, the back of the rod has moved to a point A' such that the distance $A'B = l$ (the length of the rod as measured at one time, t_B, in frame $\mathcal{S}$). Since $A'B = l$, the distance AB between the two points at which Q *sees* the two ends is greater than l.

(c) If the observer Q places himself at the point Q_c close to the x axis, the arithmetic is quite straightforward. On the one hand, $AB = l + v\Delta t$, where $\Delta t = t_B - t_A$. On the other hand, $AB = c\Delta t$, because the light leaving A must reach Q at the same time as that leaving B a time Δt later. Eliminating Δt from these two equations, we find that $AB = l/(1 - v/c)$. But AB is precisely the length l_{see} seen by Q. Therefore

$$l_{\text{see}} = \frac{l}{1-\beta} = \frac{l_o/\gamma}{1-\beta} = \frac{l_o}{1-\beta}\sqrt{(1-\beta)(1+\beta)} = l_o\sqrt{\frac{1+\beta}{1-\beta}}$$

which is certainly greater than l_o.

15.15 ⋆ If we multiply the fourth of Eqs.(15.20) by V and add it to the first, we find that

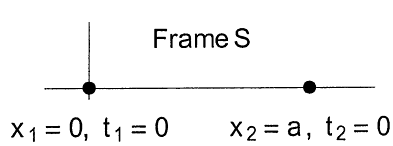
$$x' + Vt' = \gamma x - \gamma V^2 x/c^2 = \gamma(1-\beta^2)x = \sqrt{1-\beta^2}\, x$$

and hence that $x = \gamma(x' + Vt')$. In the same way, we can eliminate x to give $t = \gamma(t' + Vx'/c^2)$. The equations $y = y'$ and $z = z'$ follow trivially, and we have derived the inverse Lorentz transformation (15.21).

15.16 ⋆ If you write down the Lorentz-transformation equations (15.20) for $\mathbf{r}_1, t_1$ and for $\mathbf{r}_2, t_2$ and take their difference, you should find that

$$\Delta x' = \gamma(\Delta x - V\Delta t), \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z, \quad \text{and} \quad \Delta t' = \gamma(\Delta t - V\Delta x/c^2).$$

15.17 ⋆



(a) Frame $\mathcal{S}'$ has velocity V relative to $\mathcal{S}$. Therefore

$$t_1' = \gamma(t_1 - \beta x_1/c) = 0 \quad \text{and} \quad t_2' = \gamma(t_2 - \beta x_2/c) = -\gamma\beta a/c.$$

(b) Frame $\mathcal{S}''$ has velocity $-V$ relative to $\mathcal{S}$.
Therefore

$$t_1' = \gamma(t_1 + \beta x_1/c) = 0 \quad \text{and} \quad t_2' = \gamma(t_2 + \beta x_2/c) = \gamma\beta a/c.$$

In frame $\mathcal{S}$ the two events are simultaneous. In $\mathcal{S}'$ event 1 is later than 2, and in $\mathcal{S}''$ event 1 is earlier than 2.

15.18 ⋆⋆ The inverse Lorentz transformation (15.21) applied to the flash and beep of Figure 15.3 implies that

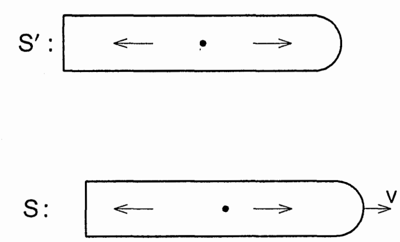
$$t_{\text{beep}} = \gamma(t'_{\text{beep}} + \beta x'_{\text{beep}}/c) \qquad \text{and} \qquad t_{\text{flash}} = \gamma(t'_{\text{flash}} + \beta x'_{\text{flash}}/c)$$

Now, in frame $\mathcal{S}'$ the flash and beep occur at the same place, so $x'_{\text{beep}} = x'_{\text{flash}}$. Therefore, if we subtract the first of these equations from the second, we find

$$\Delta t = \gamma \Delta t'$$

which is the time-dilation formula.

15.19 ⋆⋆ **(a)** $x'_F = d$, $t'_F = d/c$; $x'_B = -d$, $t'_B = d/c$.



(b)

$$x_F = \gamma(x'_F + vt'_F) = \gamma(1+\beta)d$$
$$t_F = \gamma(t'_F + vx'_F/c^2) = \gamma(1+\beta)d/c$$
$$x_B = \gamma(x'_B + vt'_B) = -\gamma(1-\beta)d$$
$$t_B = \gamma(t'_B + vx'_B/c^2) = \gamma(1-\beta)d/c$$

Although the two events are simultaneous as measured in S', they are *not* simultaneous in S. As observed in S, the two signals start out from the middle of the rocket, but while they are traveling the rocket is also traveling to the right at speed v. Thus the front is receding from its signal, which must travel more than half the rocket's length. Meanwhile the back of the rocket is approaching its signal, which needs to travel only a shorter distance. Therefore this signal arrives first; that is, $t_B < t_F$. (In S' the signals again start from the middle of the rocket; but since the rocket is not moving they naturally arrive simultaneously.)

15.20 ⋆ The statement that an object is isolated is surely Lorentz invariant — if one inertial observer sees that the object is isolated, all other inertial observers should surely agree. If Newton's first law holds in frame $\mathcal{S}$, this means that an isolated object's velocity $\mathbf{v}$, as measured in $\mathcal{S}$, is constant. Now consider the same object's velocity $\mathbf{v}'$ as measured in another frame $\mathcal{S}'$ traveling with velocity $\mathbf{V}$ relative to $\mathcal{S}$. Inspection of the velocity addition formulas (15.26) and (15.27), shows that, since $\mathbf{v}$ and $\mathbf{V}$ are both constant, the same is true of $\mathbf{v}'$. That is, the first law holds in $\mathcal{S}'$ as well.

15.21 ⋆ Let us take our x axis in the direction of the two velocities. Then the velocity of the rocket's frame $\mathcal{S}'$ has $V = \frac{1}{2}c$ and that of the bullets relative to the rocket has $v'_x = \frac{3}{4}c$, with all other components zero. According to the inverse of the velocity-addition formula (15.26),

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2} = \frac{\frac{1}{2} + \frac{3}{4}}{1 + \frac{3}{8}} c = \frac{5/4}{11/8} c = \frac{10}{11} c,$$

with all other components zero.

15.22 ⋆ According to the inverse velocity transformation,

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2} = V = 0.9c \qquad \text{and} \qquad v_y = \frac{v'_y}{\gamma(1 + v'_x V/c^2)} = \frac{0.9c}{\gamma}$$

Therefore the speed v satisfies

$$v^2 = (0.9c)^2(1 + 1/\gamma^2) = (0.9c)^2[1 + (1 - \beta^2)] = (0.9c)^2[2 - \beta^2] = 0.81c^2 \times 1.19 = 0.96c^2.$$

Therefore $v = 0.98c$, and the direction makes an angle with the x axis of

$$\theta = \arctan(v_y/v_x) = \arctan(1/\gamma) = \arctan(\sqrt{1 - 0.81}) = 23.6°.$$

15.23 ⋆ If we let $\mathcal{S}'$ denote the rest frame of the left rocket, then the velocity of $\mathcal{S}'$ relative to $\mathcal{S}$ is $V = 0.9c$. The velocity of the right rocket relative to $\mathcal{S}$ has $v_x = -0.9c$ and, relative to $\mathcal{S}'$,

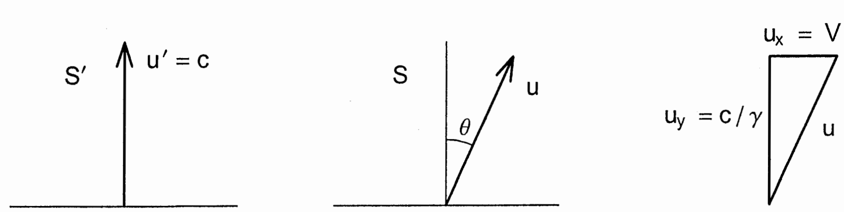
$$v'_x = \frac{v_x - V}{1 - v_x V/c^2} = \frac{-0.9c - 0.9c}{1 + 0.9 \times 0.9} = -\frac{1.8c}{1.81} = -0.994c.$$

15.24 ⋆ Let $\mathcal{S}$ be the frame fixed to the ground and $\mathcal{S}'$ the one fixed to the cop's car. The velocity of $\mathcal{S}'$ relative to $\mathcal{S}$ is $V = 0.4c$ and the velocity of the bullet relative to $\mathcal{S}'$ is $v = 0.5c$. By the inverse velocity transformation, the bullet's velocity relative to the ground is

$$v = \frac{v' + V}{1 + v'V/c^2} = \frac{0.4 + 0.5}{1 + 0.4 \times 0.5} c = 0.75c.$$

Since the robber's velocity relative to the ground is $0.8c$, the bullets do not catch the robber.

15.25 ⋆



According to the inverse velocity transformation,

$$u_x = \frac{u'_x + V}{1 + u'_x V/c^2} = V, \quad u_y = \frac{u'_y}{\gamma(1 + u'_x V/c^2)} = c/\gamma, \quad u_z = 0$$

As you would expect, the signal angles to the right of the y axis, as seen in $\mathcal{S}$. Its speed is given by

$$u^2 = u_x^2 + u_y^2 = V^2 + c^2/\gamma^2 = V^2 + c^2(1 - V^2/c^2) = c^2.$$

That is, $v = c$. This illustrates the general result that anything which has speed c in one frame has the same speed in *any* frame.

15.26 ⋆ The two positions are $x_A = v_A t$ and $x_B = d - v_B t$. The two objects meet when $x_A = x_B$ or $v_A t = d - v_B t$; that is, $t = d/(v_A + v_B)$.

15.27 ⋆⋆ Consider first the final x coordinate. This is $x'' = \gamma_2(x' - V_2 t')$, where γ_2 is the γ factor corresponding to the second velocity V_2. The coordinates x' and t' are given by the first Lorentz transformation (velocity V_1), and substituting these values we find that

$$\begin{aligned} x'' &= \gamma_1\gamma_2[(x - V_1 t) - V_2(t - V_1 x/c^2)] \\ &= \gamma_1\gamma_2[x(1 + V_1V_2/c^2) - (V_1 + V_2)t] \\ &= \gamma_1\gamma_2\left(1 + \frac{V_1V_2}{c^2}\right)\left[x - \left(\frac{V_1 + V_2}{1 + V_1V_2/c^2}\right)t\right] \\ &= \gamma_1\gamma_2(1 + \beta_1\beta_2)[x - Vt] \end{aligned} \qquad \text{(viii)}$$

In the last line I have made two substitutions. In the interests of tidiness, I have replaced V_1/c by β_1 and V_2/c by β_2. More important, I have recognized that the coefficient of t in the previous line is the relativistic "sum"

$$V = \frac{V_1 + V_2}{1 + V_1V_2/c^2}$$

of the two separate velocities V_1 and V_2. The form (viii) for x'' is very close to the standard Lorentz transformation for the single velocity V. All that remains to be shown is that the product to the left of the square bracket is equal to the γ factor for the velocity V. To show this, let's evaluate the latter:

$$\begin{aligned} \gamma_V = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - (\beta_1 + \beta_2)^2/(1 + \beta_1\beta_2)^2}} &= \frac{(1 + \beta_1\beta_2)}{\sqrt{(1 + \beta_1\beta_2)^2 - (\beta_1 + \beta_2)^2}} \\ &= \frac{(1 + \beta_1\beta_2)}{\sqrt{(1 - \beta_1^2)(1 - \beta_2^2)}}. \end{aligned}$$

Comparing with (viii) we see that indeed $x'' = \gamma_V(x - Vt)$; that is, the two successive Lorentz transformations with velocities V_1 and V_2 produce the same effect as a single Lorentz transformation with velocity V equal to the relativistic "sum" of V_1 and V_2. The transformations of y and z are trivial (for example, $y'' = y' = y$) and that of the time works just the same as that of x.

15.28 ⋆⋆ Consider first the vector $\mathbf{w} =$ "$\mathbf{u}+\mathbf{v}$". Since the first boost ($\mathbf{u}$) is along the x axis, our standard (inverse) velocity-addition formula applies, and we find (remember that $v_x = 0$)

$$w_x = \frac{u_x + v_x}{1+u_x v_y/c^2} = u_x = u, \qquad w_y = \frac{v_y}{\gamma_u(1+u_x v_x/c^2)} = \frac{v_y}{\gamma_u} = \frac{v}{\gamma_u}, \qquad w_z = 0.$$

If we consider instead $\mathbf{w}' =$ "$\mathbf{v}+\mathbf{u}$", the first boost ($\mathbf{v}$) is in the y direction, so we must rewrite the velocity-addition formulas exchanging the roles of x and y. This gives

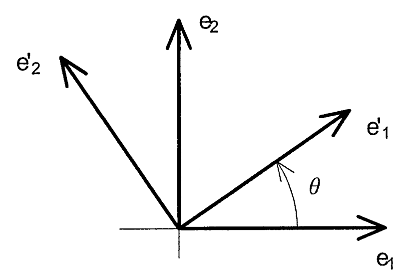
$$w'_x = \frac{u_x}{\gamma_v(1+u_y v_y/c^2)} = \frac{u_x}{\gamma_v} = \frac{u}{\gamma_v}, \qquad w'_y = \frac{u_y + v_y}{(1+u_y v_y/c^2)} = v_y = v, \qquad w'_z = 0.$$

The most obvious thing about $\mathbf{w}$ and $\mathbf{w}'$ is that they are not equal. However, it is easy to see that they are equal in magnitude:

$$w^2 = u^2 + \frac{v^2}{\gamma_u^2} = u^2 + v^2\left(1 - \frac{u^2}{c^2}\right) = u^2 + v^2 - \frac{u^2 v^2}{c^2}.$$

and, as you can easily check, w'^2 is the same. Since $\mathbf{w}$ and $\mathbf{w}'$ have the same magnitude and both lie in the xy plane, they differ only by a rotation about the z axis.

15.29 ⋆ **(a)** Bearing in mind that $R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$, we can read off the elements of $\mathbf{R}(\theta)$ from the picture:



$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Using the standard rules of matrix multiplication to multiply $\mathbf{R}(\theta)$ by itself, we find

$$[\mathbf{R}(\theta)]^2 = \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta & 0 \\ -2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}(2\theta).$$

This expresses the obvious result that two successive rotations of θ about a given axis produce the same effect as a single rotation of 2θ about the same axis.

15.30 ⋆ If we define ϕ so that $\cosh\phi = \gamma$, then

$$\sinh\phi = \sqrt{\cosh^2\phi - 1} = \sqrt{\gamma^2 - 1} = \sqrt{\frac{1}{1-\beta^2} - 1} = \sqrt{\frac{\beta^2}{1-\beta^2}} = \frac{\beta}{\sqrt{1-\beta^2}} = \gamma\beta,$$

and $\tanh\phi = (\sinh\phi)/(\cosh\phi) = \gamma\beta/\beta = \beta$.

15.31 ⋆ The speed $v = \beta c$ of C relative to A is given by the inverse velocity addition formula:

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + (\tanh\phi_1)(\tanh\phi_2)} = \tanh(\phi_1 + \phi_2)$$

(In the last step I used the "well known" addition formula for the hyperbolic tangent. One way to prove this is to start from the corresponding formula for the more familiar trigonometric tangent.) That is, the velocity of C relative to A is given by $\beta = \tanh\phi$ where the rapidity ϕ is just $\phi = \phi_1 + \phi_2$.

15.32 ⋆ Suppose that the 4×4 matrix Λ has the block form (15.44). Then we can divide its elements into three classes as follows:

$$\Lambda_{ij} = R_{ij} \;\; [i \text{ and } j = 1,2,3]\,; \qquad \Lambda_{i4} = \Lambda_{4i} = 0 \;\; [1 = 1,2,3]\,; \qquad \Lambda_{44} = 1.$$

Now consider the transformation $x' = \Lambda x$. In view of the above properties, the ith component of this equation simplifies to give

$$x'_i = \sum_{j=1}^{4} \Lambda_{ij} x_j = \begin{cases} \sum_{i=1}^{3} R_{ij} x_j & [i = 1,2,3] \\ x_4 & [i = 4]. \end{cases}$$

That is, the spatial and time components of x transform independently. The spatial part $\mathbf{x}$ (comprising x_1, x_2, and x_3) is transformed to $\mathbf{x}' = \mathbf{R}\mathbf{x}$, a rotation of space, while the time component is unchanged, $x'_4 = x_4$. It is easy to see that the converse is also true. If only the spatial part $\mathbf{x}$ of x is changed, then Λ must have the block form (15.44).

15.33 ⋆⋆

(a)

$$\left.\begin{aligned} x'_1 &= x_1 \\ x'_2 &= \gamma(x_2 - \beta x_4) \\ x'_3 &= x_3 \\ x'_4 &= \gamma(x_4 - \beta x_2) \end{aligned}\right\} \quad \text{whence} \quad \Lambda_{B2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -\gamma\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma\beta & 0 & \gamma \end{bmatrix}$$

(b)

$$\Lambda_{R+} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda_{R-} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\Lambda_{R-}\Lambda_{B1}\Lambda_{R+} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \gamma & 0 & -\gamma\beta \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma\beta & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -\gamma\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma\beta & 0 & \gamma \end{bmatrix} = \Lambda_{\mathrm{B2}}$$

The rotation $\Lambda_{\mathrm{R}+}$ changes what was the y axis into the new x axis. Then the boost Λ_{B1} boosts to velocity V along the new x axis, and finally, the rotation $\Lambda_{\mathrm{R}-}$ converts the new x axis back to being the y axis. The net effect of all three is therefore a boost along the y axis.

15.34 ⋆⋆

The effect of the three matrices in the product $\Lambda_{\mathrm{B}}(\theta) = \Lambda_{\mathrm{R}}(-\theta)\Lambda_{\mathrm{B}}(0)\Lambda_{\mathrm{R}}(\theta)$ is this: The first, $\Lambda_{\mathrm{R}}(\theta)$, moves the x axis into the direction with angle θ; the second, $\Lambda_{\mathrm{B}}(0)$, boosts the system along the new x axis; and the third, $\Lambda_{\mathrm{R}}(-\theta)$, restores the original orientation. The net effect is a boost in the direction θ.

The matrices $\Lambda_{\mathrm{R}}(\theta)$ and $\Lambda_{\mathrm{B}}(0)$ are

$$\mathbf{\Lambda}_{\mathrm{R}}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{\Lambda}_{\mathrm{B}}(0) = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix}.$$

The product of interest is (as you can check)

$$\Lambda_{\mathrm{B}}(\theta) = \Lambda_{\mathrm{R}}(-\theta)\Lambda_{\mathrm{B}}(0)\Lambda_{\mathrm{R}}(\theta) = \begin{bmatrix} \gamma\cos^2\theta + \sin^2\theta & (\gamma-1)\cos\theta\sin\theta & 0 & -\gamma\beta\cos\theta \\ (\gamma-1)\cos\theta\sin\theta & \cos^2\theta + \gamma\sin^2\theta & 0 & -\gamma\beta\sin\theta \\ 0 & 0 & 1 & 0 \\ -\gamma\beta\cos\theta & -\gamma\beta\sin\theta & 0 & \gamma \end{bmatrix}$$

This transformation is supposed to boost to a frame traveling with velocity

$$\mathbf{V} = (V\cos\theta, V\sin\theta, 0).$$

Thus a body at rest at the origin of the original frame should be observed to be moving with velocity $-\mathbf{V}$. To check this claim, we act on the four-vector $x = (0, 0, 0, ct)$ (actually a 4×1 column) with $\Lambda_{\mathrm{B}}(\theta)$ and find:

$$\text{If } x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ct \end{bmatrix}, \quad \text{then } x' = \Lambda_{\mathrm{B}}(\theta)x = \begin{bmatrix} -(\gamma\beta\cos\theta)ct \\ -(\gamma\beta\sin\theta)ct \\ 0 \\ \gamma ct \end{bmatrix} = \begin{bmatrix} -(V\cos\theta)t' \\ -(V\sin\theta)t' \\ 0 \\ ct' \end{bmatrix}$$

where, in the last expression, I have used the fact that $t' = \gamma t$ (an expression of time dilation) to replace γt by t' in the other components. The final expression shows that a body resting at the origin of the original frame is seen in the new frame to move with velocity $-\mathbf{V}$.

15.35 ⋆⋆ Suppose that $q_4 = 0$ in all inertial frames. Now consider any frame $\mathcal{S}$ and a second one $\mathcal{S}'$ related to $\mathcal{S}$ by the standard boost, so that $q_4' = \gamma(q_4 - \beta q_1)$. Since $q_4 = q_4' = 0$, it follows that $q_1 = 0$. But the frame $\mathcal{S}$ was arbitrary. Therefore $q_1 = 0$ in all frames. If we repeat the argument using boosts along the y and z axes, we can prove similarly that q_2 and q_3 are zero in all frames.

15.36 ⋆ Let x and y be any two four-vectors and consider the four-vector $w = x + y$. The invariance of $w{\cdot}w$ implies that

$$x{\cdot}x + y{\cdot}y + 2x{\cdot}y = x'{\cdot}x' + y'{\cdot}y' + 2x'{\cdot}y'$$

Since the first two terms on the left are equal to the correponding terms on the right, it follows that $x{\cdot}y = x'{\cdot}y'$ and we're home.

15.37 ⋆ If x and y are four-vectors, then under the standard boost $x_1' = \gamma(x_1 - \beta x_4), x_2' = x_2$, and so on. Therefore

$$\begin{aligned} x'{\cdot}y' &= x_1'y_1' + x_2'y_2' + x_3'y_3' - x_4'y_4' \\ &= \gamma(x_1 - \beta x_4)\gamma(y_1 - \beta y_4) + x_2y_2 + x_3y_3 - \gamma(x_4 - \beta x_1)\gamma(y_4 - \beta y_1) \\ &= \gamma^2(1 - \beta^2)x_1y_1 + x_2y_2 + x_3y_3 - \gamma^2(1 - \beta^2)x_4y_4 \end{aligned}$$

since the cross terms involving x_1y_4 and x_4y_1 in the second line all cancelled. Finally, notice that $\gamma^2(1 - \beta^2) = 1$, so the last line is just $x{\cdot}y$, and we've proved that $x'{\cdot}y' = x{\cdot}y$.

15.38 ⋆⋆ In the observer's rest frame, $dx = (0, 0, 0, c\,dt)$. On the other hand, since P and Q are simultaneous, $t_P = t_Q$ and $x_P - x_Q = (\Delta\mathbf{x}, 0)$. Therefore, $(x_P - x_Q){\cdot}dx = 0$.

15.39 ⋆ If $x{\cdot}x < 0$ in frame $\mathcal{S}$, then $x'{\cdot}x' < 0$ in any other frame $\mathcal{S}'$, since $x{\cdot}x$ has the same value in all frames. The condition $x{\cdot}x = \mathbf{x}^2 - x_4^2 < 0$ implies that $|\mathbf{x}| < |x_4|$. Now suppose, in addition, that $x_4 < 0$ in frame $\mathcal{S}$ and let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by a standard boost. Then

$$x_4' = \gamma(x_4 - \beta x_1) < 0$$

since $|\beta| < 1$ and $|x_1| \le |\mathbf{x}| < |x_4|$. That is, $x_4' < 0$ in $\mathcal{S}'$. Since x_4 is unchanged by any rotation, the same conclusion holds in all frames $\mathcal{S}'$.

15.40 ⋆ A point x in space-time lies on the forward light cone if and only if $x{\cdot}x = 0$ and $x_4 > 0$. We have to show that if these two conditions hold in a frame $\mathcal{S}$, they automatically hold in any other frame $\mathcal{S}'$. This is certainly true of the condition $x{\cdot}x = 0$ since $x{\cdot}x$ is Lorentz invariant. To check the second condition, note that because $x{\cdot}x = \mathbf{x}^2 - x_4^2 = 0$, it follows that $|\mathbf{x}| = |x_4|$. Now suppose $x_4 > 0$ (in frame $\mathcal{S}$) and let's consider a frame $\mathcal{S}'$ related to $\mathcal{S}$ by the standard boost. In $\mathcal{S}'$

$$x_4' = \gamma(x_4 - \beta x_1) > 0$$

because $|\beta| < 1$ and $|x_1| \leq |\mathbf{x}| = |x_4|$. That is, $x_4' > 0$ in $\mathcal{S}'$. Since x_4 is unchanged by any rotation, the same conclusion holds in all frames $\mathcal{S}'$.

15.41 ⋆ To simplify our notation, I'll put the origin at point Q. Then statement (2) says that the point P has coordinates x satisfying $x^2 > 0$ and that there is a frame (call it $\mathcal{S}$) in which $x_4 = 0$. We have to prove that there are other frames where $x_4' < 0$ and still others where $x_4'' > 0$. By rotating our coordinates, if necessary, we can arrange that $\mathbf{x}$ lies on the positive x_1 axis and $x = (x_1, 0, 0, 0)$, with $x_1 > 0$. Now consider a frame $\mathcal{S}'$ obtained from $\mathcal{S}$ by a standard boost. In this frame

$$x_4' = \gamma(x_4 - \beta x_1) = -\gamma\beta x_1 < 0.$$

Similarly, by boosting in the opposite direction we could obtain a frame $\mathcal{S}''$ in which $x_4'' = +\gamma\beta x_1 > 0$.

15.42 ⋆ That x is time-like means that $|\mathbf{x}| < |x_4|$. That $x{\cdot}y = 0$ means that $\mathbf{x}{\cdot}\mathbf{y} - x_4 y_4 = 0$ or equivalently

$$|\mathbf{x}|\,|\mathbf{y}|\cos\theta = x_4 y_4.$$

Since $|\mathbf{x}| < |x_4|$, this implies that $|\mathbf{y}|\,|\cos\theta| > |y_4|$ which guarantees that $|\mathbf{y}| > |y_4|$ and hence that y is space-like. (In the special case that $|\mathbf{x}| = 0$, it is clear that y_4 must be zero, and again y is space-like.)

15.43 ⋆ **(a)** Suppose that the body moves from $\mathbf{x}$ to $\mathbf{x} + d\mathbf{x}$ as the time advances from t to $t + dt$. Let dx denote the four-vector displacement $dx = (d\mathbf{x}, c\,dt) = (\mathbf{v}, c)dt$ and consider the following two equivalent statments:

$$|\mathbf{v}| < c \iff dx^2 < 0. \tag{ix}$$

Since dx^2 is Lorentz invariant, the second condition, if true in one frame, must be true in all frames. The same must therefore apply to the first; that is, if $|\mathbf{v}| < c$ in one frame, then $|\mathbf{v}| < c$ in all frames.

(b) The argument for a signal with speed c is the same except that the two conditions (ix) are replaced by $|\mathbf{v}| = c \iff dx^2 = 0$.

15.44 ⋆⋆ **(a)** If q is time-like, then $q{\cdot}q = |\mathbf{q}|^2 - q_4^2 < 0$, which implies that $|\mathbf{q}| < |q_4|$. First rotate the coordinates so that $\mathbf{q}$ points along the x axis and $q = (q_1, 0, 0, q_4)$, with $|q_1| < |q_4|$. Now apply the standard boost to give $q_1' = \gamma(q_1 - \beta q_4)$. We can choose $\beta = q_1/q_4$ (since $|q_1| < |q_4|$, this makes $|\beta| < 1$, as it has to be) and then $q_1' = 0$ and $q' = (0, 0, 0, q_4')$.

(b) A vector q is forward time-like if and only if $q^2 < 0$ and $q_4 > 0$. The first condition is Lorentz invariant and implies that $|\mathbf{q}| < q_4$. Now suppose, in addition, the second condition holds in a frame $\mathcal{S}$ and imagine applying a standard boost so that, in the new frame $\mathcal{S}'$, $q_4' = \gamma(q_4 - \beta q_1)$. Now, $|\beta| < 1$ and $|q_1| \leq |\mathbf{q}| < q_4$. Therefore, $q_4' > 0$ and the second condition is valid in $\mathcal{S}'$ also. This conclusion would certainly not be changed if we made any rotation of our coordinates, and since any Lorentz transformation can be built up from standard boosts and rotations, $q_4' > 0$ in any inertial frame. Therefore a vector that is forward time-like in one frame is forward time-like in all frames.

15.45 ⋆ If k and x are both four-vectors, then for any two frames $\mathcal{S}$ and $\mathcal{S}'$ $k' = \Lambda k$ and $x' = \Lambda x$. Now since $k = \lambda x$,

$$\Lambda k = \Lambda \lambda x = \lambda \Lambda x = \lambda x'.$$

On the other hand, $\Lambda k = k' = \lambda' x'$. Comparing these two expressions for Λk, we conclude that $\lambda = \lambda'$ and we've proved that λ is a four-scalar. (The only exception is if all four components of x' are zero, but this case is of little interest since then $x = k = 0$.)

15.46 ⋆ **(a)** With $\theta = 0$, Eq.(15.64 becomes

$$\omega = \frac{\omega_o}{\gamma(1-\beta\cos\theta)} = \frac{\omega_o}{\gamma(1-\beta)} = \omega_o \frac{\sqrt{1-\beta^2}}{1-\beta} = \omega_o \sqrt{\frac{1+\beta}{1-\beta}}$$

(b) With $\theta = 180°$, we get $\omega = \omega_o \sqrt{(1-\beta)/(1+\beta)}$.

15.47 ⋆ Since he had to be approaching the light head-on (or nearly so), $\theta = 0$ and Eq.(15.64) becomes $\omega = \omega_o\sqrt{(1+\beta)/(1-\beta)}$ (as in Problem 1.46). Solving for β we find that

$$\beta = \frac{\omega^2 - \omega_o^2}{\omega^2 + \omega_o^2} = \frac{\lambda_o^2 - \lambda^2}{\lambda_o^2 + \lambda^2} = \frac{65^2 - 53^2}{65^2 + 53^2} = 0.20.$$

His speed had to be about $0.20c$.

15.48 ⋆⋆ **(a)** With $\theta = 90°$, Eq.(15.64) reduces to $\omega = \omega_o/\gamma$. If $\beta = 0.2$, then $1/\gamma = \sqrt{1-\beta^2} = \sqrt{1-0.04} \approx 1 - 0.02$. Therefore the percent shift is -2%.

(b) If the source approaches head-on, the observed frequency is $\omega = \omega_o/\gamma(1-\beta) \approx \omega_o(1-0.02)/(1-0.2) \approx 1.22\omega_o$, and the percent shift is $+22\%$.

15.49 ⋆ Since $u = \gamma(\mathbf{v}, c)$, its square is $u^2 = \gamma^2(v^2 - c^2) = \dfrac{1}{1-\beta^2}(\beta^2 - 1)c^2 = -c^2$.

15.50 ⋆ Because $u_a \cdot u_b$ is invariant, we can evaluate it in any convenient frame, and one good choice is the rest frame of particle a. In this frame, $u_a = (\mathbf{0}, c)$ and $u_b = \gamma(v_{\rm rel})(\mathbf{v}_{\rm rel}, c)$, so $u_a \cdot u_b = \gamma(v_{\rm rel})(\mathbf{0}, c) \cdot (\mathbf{v}_{\rm rel}, c) = -\gamma(v_{\rm rel})c^2$.

15.51 ⋆⋆ **(a)** The two incoming particles have the same mass, $m_a = m_b = m$. Since their velocities are equal and opposite, $\mathbf{v}_a = -\mathbf{v}_b = \mathbf{v}$, say, they have the same value of γ, that is, $\gamma_a = \gamma_b = \gamma$, say. It follows that $p_a = (\gamma m\mathbf{v}, \gamma mc)$ and $p_b = (-\gamma m\mathbf{v}, \gamma mc)$. Therefore the total initial four-momentum is $\sum p_{\text{in}} = (\mathbf{0}, 2\gamma mc)$. The same argument gives the same value for the final total four-momentum, and we've shown that $\sum p_{\text{in}} = \sum p_{\text{fin}}$ in frame $\mathcal{S}$.

(b) Since the two sides of the last equation are four-vectors, the truth of the equation in one inertial frame automatically assures its truth in all such frames.

15.52 ⋆⋆ **(a)** Let Q be the change in the total four-momentum (between any two times of interest), $Q = P_{\text{fin}} - P_{\text{in}}$. We are told that the space part $\mathbf{Q}$ iz zero in all frames, and we have to prove that the same is true of Q_4. To do this, consider an arbitrary frame $\mathcal{S}$ and a second frame $\mathcal{S}'$, obtained from $\mathcal{S}$ by a standard boost. Then $Q_1' = \gamma(Q_1 - \beta Q_4)$, and, since $Q_1' = Q_1 = 0$, it follows that $Q_4 = 0$. Since the frame $\mathcal{S}$ was arbitrary, this proves that $Q_4 = 0$ in all frames.

(b) Using the same notation as in part (a), we are told that one component of Q is zero in all frames. By the zero-component theorem, this guarantees that all components are zero in all frames, and we're home.

15.53 ⋆⋆ The quantity $p_a \cdot p_b$ is invariant, so can be evaluated in any conveniently chosen frame. In the rest frame of a, $p_a = (\mathbf{0}, m_a c)$ and $p_b = (\mathbf{p}_b, E_b/c)$, so $p_a \cdot p_b = -m_a E_b$. Still in the same frame, the speed of b is v_{rel}, so its energy is $E_b = \gamma(v_{\text{rel}}) m_b c^2$, and $p_a \cdot p_b = -m_a m_b c^2 \gamma(v_{\text{rel}})$. Finally, working in the same way, but in the rest frame of b, we find that $p_a \cdot p_b = -m_b E_a$.

15.54 ⋆⋆⋆ The velocity of frame $\mathcal{S}'$ relative to $\mathcal{S}$ is $V = \xi$ along the x axis. Thus, applying the relativistic velocity-addition formula to the initial velocity of ball a, we find

$$v_x' = \frac{v_x - V}{1 - v_x V/c^2} = \frac{\xi - \xi}{1 - \beta^2} = 0 \quad \text{and} \quad v_y' = \frac{v_y}{\gamma(1 - v_x V/c^2)} = \frac{\eta}{\gamma(1 - \beta^2)}$$

(where $\beta = \xi/c$) and similarly with the other three velocities. The initial and final values of $\sum m\mathbf{v}'$ are shown in last column of the table below (where I have omitted the z components, all of which are zero). While the x components are equal, the y components have opposite signs and are *not* equal. Thus $\sum m\mathbf{v}_y'$ is *not* conserved.

	First particle ($\mathbf{v}_1'$)	Second particle ($\mathbf{v}_2'$)	$m\mathbf{v}_1' + m\mathbf{v}_2'$
Before:	$\left(0, \dfrac{\eta}{\gamma(1-\beta^2)}\right)$	$\left(\dfrac{-2\xi}{(1+\beta^2)}, \dfrac{-\eta}{\gamma(1+\beta^2)}\right)$	$\left(\dfrac{-2m\xi}{(1+\beta^2)}, \dfrac{2m\eta\beta^2}{\gamma(1-\beta^4)}\right)$
After:	$\left(0, \dfrac{-\eta}{\gamma(1-\beta^2)}\right)$	$\left(\dfrac{-2\xi}{(1+\beta^2)}, \dfrac{\eta}{\gamma(1+\beta^2)}\right)$	$\left(\dfrac{-2m\xi}{(1+\beta^2)}, \dfrac{-2m\eta\beta^2}{\gamma(1-\beta^4)}\right)$

15.55 ⋆⋆⋆ Because there are three different velocities in this problem, we must be careful to distinguish the different corresponding factors of β and γ. Thus I'll write $u = \gamma(v)(\mathbf{v}, c)$, and the velocity transformation as

$$u_1' = \gamma(V)(u_1 - \beta(V)u_4), \quad u_2' = u_2, \quad u_3' = u_3, \quad u_4' = \gamma(V)(u_4 - \beta(V)u_4). \tag{x}$$

We must now rewrite these in terms of the three-velocity $\mathbf{v}$. The first gives

$$\gamma(v')v_1' = \gamma(V)[\gamma(v)v_1 - \beta(V)\gamma(v)c] = \gamma(v)\gamma(V)[v_1 - V] \tag{xi}$$

and the last gives

$$\gamma(v')c = \gamma(V)[\gamma(v)c - \beta(V)\gamma(v)v_1] = \gamma(v)\gamma(V)c[1 - v_1V/c^2]. \tag{xii}$$

Dividing Eq.(xi) by Eq.(xii), we find

$$v_1' = \frac{v_1 - V}{1 - v_1V/c^2}$$

which is the first component of the velocity-addition formula. Similarly, the second of Eqs.(x) gives $\gamma(v')v_2' = \gamma(v)v_2$ and, dividing this by Eq.(xii), we find

$$v_2' = \frac{v_2}{\gamma(V)(1 - v_1V/c^2)}$$

which is the second component. The third works in the same way, and we're home.

15.56 ⋆ **(a)** Since $M_i c^2 + T_i = M_f c^2 + T_f$, we see that $\Delta M c^2 = -\Delta T = -5$ eV. Thus

$$\Delta M = -5 \text{ eV}/c^2 = -5.4 \times 10^{-9} \text{ u}.$$

(b) Since the initial mass of two H_2 and one O_2 molecules is 36 u, the fractional change in mass is $\Delta M/M = -(5.4 \times 10^{-9})/36 = -1.5 \times 10^{-10}$.

(c) Whatever the initial mass, the fractional change will be the same, so, with 10 grams initially, the change will be $\Delta M = -1.5 \times 10^{-9}$ gram. Pretty small!

15.57 ⋆ With the initial atom at rest, conservation of energy implies that $M_i c^2 = M_f c^2 + T_f$, so $T_f = (M_i - M_f)c^2 = (m_{At} - m_{Bi} - m_{He})c^2 = (0.0087 \text{ u})c^2 = 8.1 \text{ MeV} = 1.3 \times 10^{-12}$ J.

15.58 ⋆ **(a)** If $T = mc^2$, then $E = T + mc^2 = 2mc^2$ and $\gamma = 2$. Therefore $\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 1/4} = 0.866$.

(b) If $E = nmc^2$, then $\gamma = n$ and $\beta = \sqrt{1 - 1/n^2}$.

15.59 ⋆ The correct relativistic KE is $T = E - mc^2 = (\gamma - 1)mc^2$. The question is: Could this be the same as $\frac{1}{2}m_{\text{var}}v^2 = \frac{1}{2}\gamma mv^2$? If these were the same, then it would have to be that $(\gamma - 1)c^2 = \frac{1}{2}\gamma v^2$ or $\gamma = 1/(1 - \frac{1}{2}v^2/c^2)$ which is certainly false.

15.60 ⋆ The reaction is $a \to b + b$. Since particle a is at rest, conservation of momentum tells us that the two particles b have equal and opposite momenta. Therefore they have equal speeds and hence equal values of γ. Thus conservation of energy tells us that $m_a c^2 = 2\gamma m_b c^2$ and hence that $\gamma = m_a/2m_b$. Therefore each particle b has speed given by $\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - (2m_b/m_a)^2}$.

15.61 ⋆ $E = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{(4 \text{ MeV})^2 + (3 \text{ MeV})^2} = 5$ MeV.
$\beta = pc/E = (4 \text{ MeV})/(5 \text{ MeV}) = 0.8$. Therefore $v = 0.8c$.

15.62 ⋆ $E = T + mc^2 = 13$ MeV. Therefore $pc = \sqrt{E^2 - mc^2} = 5$ MeV or $p = 5$ MeV/c, and $\beta = pc/E = 5/13 = 0.38$, so $v = 0.38c$.

15.63 ⋆ **(a)** $1 \dfrac{\text{MeV}}{c^2} = \dfrac{1.60 \times 10^{-13} \text{ J}}{(3.00 \times 10^8 \text{ m/s})^2} = 1.78 \times 10^{-30}$ kg.

(b) $1 \dfrac{\text{MeV}}{c} = \dfrac{1.60 \times 10^{-13} \text{ J}}{3.00 \times 10^8 \text{ m/s}} = 5.33 \times 10^{-22}$ kg·m/s. (The final 3 here contains a rounding error and should really be a 4.)

15.64 ⋆ The quantity $u \cdot p$ is invariant, so has the same value in all inertial frames. In the frame $\mathcal{S}'$, $u' = (\mathbf{0}, c)$ and $p' = (\mathbf{p}', E'/c)$. Therefore $u \cdot p = u' \cdot p' = -E'$ and $E' = -u \cdot p$, as claimed.

15.65 ⋆⋆

$$\begin{aligned} T = mc^2(\gamma - 1) &= mc^2[(1 - \beta^2)^{-1/2} - 1] \\ &= mc^2\left[\left(1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 + \cdots\right) - 1\right] \\ &= mc^2\left[\tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 + \cdots\right] \end{aligned}$$

(a) The first term is $\frac{1}{2}mc^2\beta^2 = \frac{1}{2}mv^2 = T_{\text{nr}}$. The difference is $T_{\text{rel}} - T_{\text{nr}} = mc^2[\frac{3}{8}\beta^4 + \cdots]$.

(b) The fractional difference is $\dfrac{T_{\text{rel}} - T_{\text{nr}}}{T_{\text{rel}}} \approx \dfrac{\frac{3}{8}\beta^4}{\frac{1}{2}\beta^2} = \frac{3}{4}\beta^2$, and this is less than 1% as long as $\beta < 0.12c$.

15.66 ⋆⋆ That p is a four-vector requires, among other things, that $p_1' = \gamma(p_1 - \beta E/c)$. If we add an arbitrary constant to E, this adds a constant to p_1', which is certainly not permissible. (For instance, if the body is stationary in the frame $\mathcal{S}'$, p_1' must certainly be zero.)

15.67 ⋆ With $v = 0.8c$, $\gamma = 5/3$. Since the initial velocities are equal and opposite, the same is true of the momenta. Therefore the total momentum is zero, and the final body is at rest. Thus $M = E_{\text{fin}}/c^2 = E_{\text{in}}/c^2 = 2\gamma m = 3.33m$.

15.68 ⋆ In the CM frame $\mathbf{p}_a^{\text{in}} = -\mathbf{p}_b^{\text{in}}$, so the two initial momenta are equal in magnitude, $p_a^{\text{in}} = p_b^{\text{in}} = p^{\text{in}}$, say. (I'll use the italic p to denote the magnitude of the three-momentum here.) By conservation of momentum, the final total momentum is also zero, so the same argument applies to the final momenta and $p_a^{\text{fin}} = p_b^{\text{fin}} = p^{\text{fin}}$, say. Now the initial total energy is

$$E^{\text{in}} = \sqrt{(p^{\text{in}}c)^2 + (m_a c^2)^2} + \sqrt{(p^{\text{in}}c)^2 + (m_b c^2)^2}$$

with a similar expression for E^{fin}. Notice that E^{in} is a monotonically increasing function of p^{in} (and likewise E^{fin}). Thus conservation of energy, $E^{\text{fin}} = E^{\text{in}}$, requires that $p^{\text{fin}} = p^{\text{in}}$ and hence that $\mathbf{p}_a^{\text{fin}} = \pm\mathbf{p}_a^{\text{in}}$. Here the plus sign corresponds to the initial state, before the collision. After the collision the minus sign must apply: In the CM frame the three-momentum of particle a (and likewise b) simply reverses itself.

15.69 ⋆ **(a)** A four-vector q is forward time-like if and only if $|\mathbf{q}| < q_4$. The four-momentum of a massive particle is defined as $p = mu = \gamma m(\mathbf{v}, c)$, and, since $m > 0$, $|\mathbf{v}| < c$ and hence $|\mathbf{p}| < p_4$. Therefore p is forward time-like.

(b) If p and q are forward time-like, $|\mathbf{p}| < p_4$ and $|\mathbf{q}| < q_4$. It follows that $|\mathbf{p} + \mathbf{q}| \leq |\mathbf{p}| + |\mathbf{q}| < p_4 + q_4 = (p + q)_4$. Therefore, $p + q$ is forward time-like.

15.70 ⋆ **(a)** From Problem 15.69 it follows that the total four-momentum p^{tot} is forward time-like and so $|\mathbf{p}_1^{\text{tot}}| < p_4^{\text{tot}}$. By rotating our axes if necessary we can put $\mathbf{p}^{\text{tot}}$ along the positive x axis, so that $p^{\text{tot}} = (p_1^{\text{tot}}, 0, 0, p_4^{\text{tot}})$ with $p_1^{\text{tot}} < p_4^{\text{tot}}$. Now consider a standard boost to a frame $\mathcal{S}'$ in which

$$p_1'^{\text{tot}} = \gamma(p_1^{\text{tot}} - \beta p_4^{\text{tot}}).$$

If we choose $\beta = p_1^{\text{tot}}/p_4^{\text{tot}}$ (which is less than 1 because $p_1^{\text{tot}} < p_4^{\text{tot}}$), then in the frame $\mathcal{S}'$ the total three-momentum is zero.

(b) It is already clear from the above that in the original frame $\mathcal{S}$ the velocity of the CM frame has to be given by $\boldsymbol{\beta} = \mathbf{p}^{\text{tot}}/p_4^{\text{tot}} = \sum \mathbf{p}c/\sum E$.

15.71 ⋆ **(a)** The total energy needed to produce any given final set of particles must satisfy $E \geq \sum m_{\text{fin}}c^2$. The unique feature of the CM frame is that with $\mathbf{p}_{\text{tot}} = 0$, the final particles can all be at rest and the inequality can actually be an equality. Thus the threshold energy in the CM frame is $\sum m_{\text{fin}}c^2$. If the particle d is much heavier than all the others, this gives $E_{\text{cm}} \approx m_d c^2$ for the threshold.

(b) In the lab frame, with particle b at rest initially, the threshold energy is given by (15.98) as

$$E_{\text{lab}} = \frac{\sum m_{\text{fin}})^2 - m_a^2 - m_b^2}{2m_b} c^2 \approx \frac{m_d^2 c^2}{2m_b}$$

where the final expression holds if particle d is much heavier than all the others. (Strictly speaking this is the minimum energy for the incident particle a — as opposed to the minimum total energy — but if m_d is much bigger than all other masses, this difference is unimportant.)

(c) Putting in $m_b = m_e = 0.5\ \text{MeV}/c^2$ and $m_d = m_\psi = 3100\ \text{MeV}/c^2$, we find for the two threshold energies

$$E_{\text{cm}} \approx m_\psi c^2 = 3100\ \text{MeV} \quad \text{whereas} \quad E_{\text{lab}} \approx \frac{m_\psi^2 c^2}{2m_e} = 9,600,000\ \text{MeV}.$$

15.72 ⋆ However much energy the mad physicist gives the mass M in his frame, there is nothing to stop us watching the process from M's rest frame, where the initial energy is just $E_{\text{in}} = Mc^2$. Since the final energy satisfies $E_{\text{fin}} \geq 2mc^2 > Mc^2$, the process cannot satisfy conservation of energy.

15.73 ⋆⋆ **(a)** According to (15.95), the final velocity of b is $\mathbf{v}_b = 2\boldsymbol{\beta}c/(1+\beta^2)$ where $\boldsymbol{\beta}$ is the (dimensionless) velocity of the CM frame relative to the lab, $\boldsymbol{\beta} = \mathbf{p}_a c/(E_a + mc^2)$ and $m = m_a = m_b$. Putting these together, we find, after a little algebra, $\mathbf{v}_b = \mathbf{p}_a c^2/E_a = v_a$.

(b) In the limit that $\beta \to 0$,

$$\mathbf{v}_b = \frac{2\boldsymbol{\beta}c}{1+\beta^2} \to 2\boldsymbol{\beta}c = \frac{2\mathbf{p}_a c^2}{E_a + m_b c^2} \to \frac{2m_a \mathbf{v}_a}{m_a + m_b}.$$

As you can easily check, this is the answer you would get using conservation of nonrelativistic energy and momentum.

15.74 ⋆⋆ **(a)** In the CM frame (the rest frame of the original particle a), the two final particles move with equal and opposite three-momenta and equal energies, $E_{b1} = E_{b2} = \frac{1}{2}m_a c^2 = \frac{5}{4}m_b c^2$. Thus either b particle has three-momentum of magnitude given by

$$|\mathbf{p}_b|c = \sqrt{E_b^2 - (m_b c^2)^2} = \sqrt{\left(\tfrac{5}{4}\right)^2 - 1}\ m_b c^2 = \tfrac{3}{4} m_b c^2$$

and speed $v_b = |\mathbf{p}_b|c^2/E_b = 0.6c$.

(b) The two b particles travel in opposite directions with velocities $\pm 0.6c$ along the x axis of the CM frame, and the CM frame travels at speed $0.5c$ relative to the frame $\mathcal{S}$. Thus the velocities relative to $\mathcal{S}$ are given by the inverse velocity transformation as

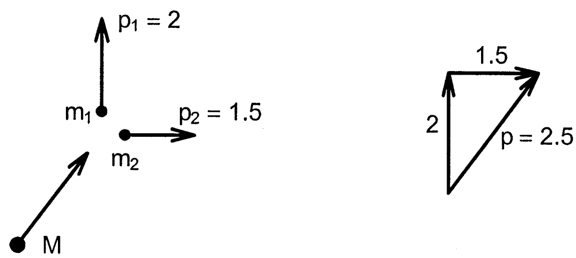
$$v_{b1} = \frac{0.6c + 0.5c}{1 + (0.6)\times(0.5)} = \frac{1.1c}{1.3} = 0.85c \quad \text{and} \quad v_{b2} = \frac{-0.6c + 0.5c}{1 - (0.6)\times(0.5)} = \frac{-0.1c}{0.7} = -0.14c$$

15.75 ⋆⋆ Using the useful relation (15.85) we can find the energies of the two final particles

$$E_1 = \sqrt{(p_1c)^2 + (m_1c^2)^2} = \sqrt{2^2 + (0.5)^2} = 2.06 \text{ GeV}$$

and

$$E_2 = \sqrt{(p_2c)^2 + (m_2c^2)^2} = \sqrt{(1.5)^2 + 1^2} = 1.80 \text{ GeV}$$



By conservation of energy and momentum, the original particle had

$$E = E_1 + E_2 = 3.86 \text{ GeV}$$

and

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = 2.5 \text{ GeV}/c, \text{ in the direction shown.}$$

Finally, $M = \sqrt{E^2 - (pc)^2} = \sqrt{3.86^2 - 2.5^2} = 2.95 \text{ GeV}/c^2$, and $\beta = pc/E = 2.5/3.86 = 0.65$.

15.76 ⋆⋆ The two initial four-momenta are

$$p_a = \gamma_a m_a(\mathbf{v}_a, c) \qquad \text{and} \qquad p_b = \gamma_b m_b(\mathbf{v}_b, c)$$

where I am using the abbreviation $\gamma_a = \gamma(v_a)$ and so on, and the two vectors $\mathbf{v}_a$ and $\mathbf{v}_b$ both point along the positive x axis. By conservation of four-momentum, the momentum of the final mass m is $p = p_a + p_b$. Therefore

$$m^2c^2 = -p^2 = -(p_a + p_b)^2 = m_a^2c^2 + m_b^2c^2 - 2p_a{\cdot}p_b = m_a^2c^2 + m_b^2c^2 + 2\gamma_a\gamma_b m_a m_b(c^2 - v_a v_b).$$

Dividing by c^2 we get the advertised result. The final velocity is

$$v = \frac{pc^2}{E} = \frac{\gamma_a m_a v_a + \gamma_b m_b v_b}{\gamma_a m_a + \gamma_b m_b}.$$

15.77 ⋆⋆⋆ **(a)** In the CM frame we know that the two incoming momenta are equal and opposite and that, in a head-on collision, they simply exchange roles. Therefore $\mathbf{p}_b^{\text{fin}} = \mathbf{p}_a^{\text{in}}$, and $p_a^{\text{fin}} - p_b^{\text{in}} = (\mathbf{0}, E_a^{\text{in}} - E_b^{\text{fin}})$, which is pure time-like. Therefore $(p_a^{\text{fin}} - p_b^{\text{in}})^2 < 0$. (The one exception is if $m_a = m_b$, in which case $E_a^{\text{in}} = E_b^{\text{fin}}$, so $p_a^{\text{fin}} - p_b^{\text{in}} = 0$.) Now, in the lab frame (particle b initially at rest) $p_b^{\text{in}} = (\mathbf{0}, m_b c)$, so the condition $(p_a^{\text{fin}} - p_b^{\text{in}})^2 < 0$ can be written as

$$-m_a^2c^2 - m_b^2c^2 - 2p_a^{\text{fin}}{\cdot}p_b^{\text{in}} = -m_a^2c^2 - m_b^2c^2 + 2E_a^{\text{fin}}m_b < 0.$$

If we write $E_a^{\rm fin} = m_a c^2 + T_a^{\rm fin}$, this gives the desired result, $T_a^{\rm fin} < (m_a - m_b)^2 c^2 / 2m_b$. (Notice that if $m_a = m_b$, the corresponding result is that $T_a^{\rm fin} = 0$, giving a nice proof that, in this case, particle a comes to a dead stop.)

(b) Using nonrelativistic mechanics, it's easy to show that $\mathbf{v}_a^{\rm fin} = \mathbf{v}_a^{\rm in}(m_a - m_b)/(m_a + m_b)$, whence $T_a^{\rm fin} = T_a^{\rm in}(m_a - m_b)^2/(m_a + m_b)^2$.

15.78 ⋆⋆⋆ **(a)** In the lab frame $\mathcal{S}$, the total four-momentum is $p = \big(\mathbf{p}_a, [E_a + m_b c^2]/c\big)$. The velocity of the CM frame $\mathcal{S}'$ relative to $\mathcal{S}$ is the velocity of a boost that makes $\mathbf{p}' = 0$, and this is easily seen to be $\mathbf{V} = \mathbf{p}_a c^2/[E_a + m_b c^2]$.

(b) Let us denote the two final four-momenta by q_a and q_b in the lab frame (and, of course, q'_a and q'_b in the CM frame). Then by the inverse Lorentz transformation,

$$q_{ax} = \gamma(q'_{ax} + VE_{q'_a}/c^2) \qquad \text{and} \qquad q_{ay} = q'_{ay}.$$

Dividing the second of these by the first, we get

$$\tan\theta = \frac{q_{ay}}{q_{ax}} = \frac{q'_{ay}}{\gamma(q'_{ax} + VE_{q'_a}/c^2)} = \frac{|\mathbf{q}'_a|\sin\theta'}{\gamma(|\mathbf{q}'_a|\cos\theta' + V|\mathbf{q}'_a|/v'_a)} = \frac{\sin\theta'}{\gamma(\cos\theta' + V/v'_a)} \tag{xiii}$$

where for the third equality I used the fact that $\mathbf{v}'_a = \mathbf{q}'_a c^2/E_{q'_a}$, so that $E_{q'_a}/c^2 = |\mathbf{q}'_a|/v'_a$.

(c) In the nonrelativistic limit, $\gamma \to 1$. Also, since $V = v'_b$, $V/v'_a = v'_b/v'_a$, which, in the nonrelativistic limit, is just the mass ratio $m_a/m_b = \lambda$. Therefore, Eq.(xiii) becomes $\tan\theta = (\sin\theta')/(1 + \cos\theta')$ as in Eq.(14.53).

(d) As we've already seen, $V = v'_b$ which, in the case of equal masses, equals v'_a. Therefore $V/v'_a = 1$. An argument parallel to that of part (b) gives $\tan\psi = (\sin\theta')/\gamma(1 - \cos\theta')$.

(e) With $m_a = m_b$, Eq.(xiii) reduces to $\tan\theta = (\sin\theta')/\gamma(1 + \cos\theta')$. Combining these two results in the formula for $\tan(\theta + \psi)$ and perfoming a little algebra, we find that

$$\tan(\theta + \psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi} = \frac{2}{\gamma\beta^2\sin\theta'}.$$

In the nonrelativistic limit, $\gamma \to 1$ and $\beta \to 0$; therefore, $\tan(\theta + \psi) \to \infty$ and $(\theta + \psi) \to 90°$.

15.79 ⋆ First, $\mathbf{F} = d\mathbf{p}/dt = d(\gamma m\mathbf{v})/dt = \gamma m\mathbf{a} + (d\gamma/dt)m\mathbf{v}$. To evaluate $d\gamma/dt$, note that $\gamma = E/mc^2$ and, by the work-KE theorem (15.101), $dE/dt = \mathbf{F}\cdot\mathbf{v}$. Combining all of these, we get $\mathbf{F} = \gamma m\mathbf{a} + (\mathbf{F}\cdot\mathbf{v})\mathbf{v}/c^2$.

15.80 ⋆ Since $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$ is perpendicular to $\mathbf{v}$, the work-KE theorem implies that E and γ are constant and the equation of motion (as in Problem 15.79) becomes $\gamma m\dot{\mathbf{v}} = q\mathbf{v}\times\mathbf{B}$. This is exactly the same as the non-relativistic equation (2.63) except that the constant m has been replaced by the constant γm. Thus all the same conclusions apply (with the appropriate replacements). In particular, if $\mathbf{v}$ starts out perpendicular to $\mathbf{B}$, it remains so, and the particle moves in a circle of radius $r = |\gamma mv/qB| = |\mathbf{p}/qB|$.

15.81 ⋆ The electron mass is $m \approx 0.5\ \text{MeV}/c^2 \approx 9 \times 10^{-31}$ kg, and with $v = 0.7c$, $\gamma = 1.4$. Thus

$$r = \left|\frac{\gamma m v}{qB}\right| = \frac{1.4 \times (9 \times 10^{-31}\ \text{kg}) \times (0.7 \times 3 \times 10^8\ \text{m/s})}{(1.6 \times 10^{-19}\ \text{C}) \times (0.02\ \text{T})} = 8.3\ \text{cm}.$$

The nonrelativistic answer is less by a factor of γ, $r(\text{nonrel}) = 5.9$ cm.

15.82 ⋆ **(a)** Starting from (15.105), we find

$$\mathbf{x} = \int_0^t \mathbf{v}(t')dt' = \frac{\mathbf{F}}{m}\int_0^t \frac{t'\,dt'}{\sqrt{1+(Ft'/mc)^2}}$$

$$= \frac{\mathbf{F}}{2m}\left(\frac{mc}{F}\right)^2 \int \frac{du}{\sqrt{1+u}} = \frac{\mathbf{F}}{m}\left(\frac{mc}{F}\right)^2 \left(\sqrt{1+\left(\frac{Ft}{mc}\right)^2} - 1\right)$$

(b) When $t \to 0$, we can use the binomial approximation for the square root to give

$$\mathbf{x} \approx \frac{\mathbf{F}}{m}\left(\frac{mc}{F}\right)^2 \left(\left[1 + \frac{1}{2}\left(\frac{Ft}{mc}\right)^2\right] - 1\right) = \tfrac{1}{2}\mathbf{a}t^2.$$

(c) When $t \to \infty$, the square root approaches FT/mc and

$$\mathbf{x} \to (\mathbf{F}/F)(ct + \text{const}) = \hat{\mathbf{F}}(ct + \text{const}).$$

As $t \to \infty$, $v \to c$ and the particle moves with essentially constant velocity in the direction of $\mathbf{F}$.

15.83 ⋆ The force in frame $\mathcal{S}$ is $\mathbf{F} = d\mathbf{p}/dt$ and that in $\mathcal{S}'$ is $\mathbf{F}' = d\mathbf{p}'/dt'$. To relate these we have only to use the Lorentz transformation:

$$dp_1' = \gamma(dp_1 - \beta dE/c),\quad dp_2' = dp_2,\quad dp_3' = dp_3,\quad \text{and}\quad dt' = \gamma(dt - \beta dx_1/c).$$

Thus

$$F_1' = \frac{dp_1'}{dt'} = \frac{\gamma(dp_1 - \beta dE/c)}{\gamma(dt - \beta dx_1/c)} = \frac{F_1 - \beta(dE/dt)/c}{1 - \beta v_1/c} = \frac{F_1 - \beta \mathbf{F}\cdot\mathbf{v}/c}{1 - \beta v_1/c}$$

where in the last equality I used the work-KE theorem to replace dE/dt by $\mathbf{F}\cdot\mathbf{v}$. Similarly

$$F_2' = \frac{dp_2'}{dt'} = \frac{dp_2}{\gamma(dt - \beta dx_1/c)} = \frac{F_2}{\gamma(1 - \beta v_1/c)}.$$

with a similar result for F_3'.

15.84 ⋆⋆ The constant force is $\mathbf{F} = F\hat{\mathbf{x}}$ and the initial momentum is $\mathbf{p}_o = p_o\hat{\mathbf{y}}$. Because $\mathbf{F} = d\mathbf{p}/dt$, we can find $\mathbf{p}$ immediately by integrating $\mathbf{F}$ to give $\mathbf{p} = \int \mathbf{F}dt = \mathbf{F}t + \mathbf{p}_o$. We can next find γ as

$$\gamma = \frac{E}{mc} = \frac{\sqrt{(\mathbf{p}c)^2 + (mc^2)^2}}{mc^2} = \frac{\sqrt{F^2t^2 + p_o^2 + m^2c^2}}{mc} = \frac{F}{mc}\sqrt{t^2 + k^2}$$

where $k^2 = (p_o^2 + m^2c^2)/F^2$. Therefore

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{(\mathbf{F}t + \mathbf{p}_o)c}{F\sqrt{t^2+k^2}}.$$

Therefore

$$\mathbf{x} = \int_0^t \mathbf{v}(t')dt' = \hat{\mathbf{F}}c\int_0^t \frac{t'dt'}{\sqrt{t'^2+k^2}} + \frac{\mathbf{p}_o c}{F}\int_0^t \frac{dt'}{\sqrt{t'^2+k^2}}$$

$$= \hat{\mathbf{F}}c\left(\sqrt{t^2+k^2} - k\right) + \frac{\mathbf{p}_o c}{F}\operatorname{arcsinh}\left(\frac{Ft}{\sqrt{p_o^2+m^2c^2}}\right)$$

In the nonrelativistic limit, mc is much greater than the other relevant parameters ($mc \gg p_o$ and $mc \gg Ft$) and hence $k \approx mc/F \gg t$. Hence

$$\mathbf{x} \approx \hat{\mathbf{F}}ck\left(\left[1 + \frac{t^2}{2k^2}\right] - 1\right) + \frac{\mathbf{p}_o c}{F}\frac{Ft}{mc} = \tfrac{1}{2}\mathbf{a}t^2 + \mathbf{v}_o t.$$

15.85 ⋆⋆ **(a)** Differentiating the relation $E^2 = (\mathbf{p}c)^2 + (mc^2)^2$ we find that

$$E\frac{dE}{dt_o} = \mathbf{p}c^2\cdot\frac{d\mathbf{p}}{dt_o} + mc^4\frac{dm}{dt_o}$$

whence

$$-c^2\frac{dm}{dt_o} = \frac{\mathbf{p}}{m}\cdot\frac{d\mathbf{p}}{dt_o} - \frac{E}{mc^2}\frac{dE}{dt_o} = \gamma\mathbf{v}\cdot\frac{d\mathbf{p}}{dt_o} - \gamma\frac{dE}{dt_o} = \gamma(\mathbf{v}, c)\cdot\frac{d}{dt_o}(\mathbf{p}, E/c) = u\cdot K.$$

(b) Because $u\cdot K$ is invariant, we can evaluate it in any frame. In particular, in the particle's instantaneous rest frame $u = (\mathbf{0}, c)$ and $K = (d/dt)(\mathbf{p}, E/c)$, so $u\cdot K = dE/dt$, which we have been told is zero. Therefore the mass of a charged particle exposed only to electromagnetic fields is constant.

15.86 ⋆ **(a)** Let us call the frame where the initial pion is at rest $\mathcal{S}'$. Since the pion is at rest, the two final photons must have equal and opposite three-momenta, $\mathbf{p}'_{\gamma 1} = -\mathbf{p}'_{\gamma 2}$, and hence hence equal energies, $E'_{\gamma 1} = E'_{\gamma 2}$. By conservation of energy, $E'_{\gamma 1} + E'_{\gamma 2} = m_\pi c^2$. Therefore, $E'_{\gamma 1} = E'_{\gamma 2} = m_\pi c^2/2 = 67.5\ \mathrm{MeV}/c^2$.

(b) Let $\mathcal{S}$ be the frame where the pion has velocity V. Since a photon's energy is $E = pc$, the transformations that gives the photon energies in $\mathcal{S}$ are

$$E_{\gamma 1} = \gamma(E'_{\gamma 1} + \beta p'_{\gamma 1}c) = \gamma E'_{\gamma 1}(1+\beta) \text{ and similarly } E_{\gamma 2} = \gamma E'_{\gamma 2}(1-\beta).$$

Dividing the first of these by the second, we find $3 = (1+\beta)/(1-\beta)$, whence $\beta = 0.5$. That is, the pion's speed in $\mathcal{S}'$ is $0.5c$.

15.87 ⋆ Let's take take the direction of the initial pion to be our x axis and the plane of the two emerging photons to be the xy plane. Conservation of four-momentum implies that $p_\pi = p_{\gamma 1} + p_{\gamma 2}$, the y component of which says that $|\mathbf{p}_{\gamma 1}| \sin\theta = |\mathbf{p}_{\gamma 2}| \sin\theta$. Therefore $|\mathbf{p}_{\gamma 1}| = |\mathbf{p}_{\gamma 2}|$ and $E_{\gamma 1} = E_{\gamma 2} = |\mathbf{p}_{\gamma 1}|c$ (the last because the photon is massless). The x and t components of momentum conservation now tell us that

$$|p_\pi| = 2|\mathbf{p}_{\gamma 1}| \cos\theta \qquad \text{and} \qquad E_\pi = 2E_{\gamma 1} = 2|\mathbf{p}_{\gamma 1}|c,$$

Dividing the first of these by the second we find that $v_\pi = |p_\pi|c^2/E_\pi = c\cos\theta$.

15.88 ⋆ The total four-momentum of the two particles is $p = p_a + p_b$. We have to show that there is a frame, $\mathcal{S}'$, where the spatial part $\mathbf{p}' = \mathbf{0}$; that is, that p is time-like. Since p_a is forward light-like and p_b forward time-like, we know that $p_{a4} > 0$ and $p_{b4} > 0$ and that $p_{a4} = |\mathbf{p}_a|$ and $p_{b4} > |\mathbf{p}_b|$. Combining these, we find

$$p_4 = p_{a4} + p_{b4} > |\mathbf{p}_a| + |\mathbf{p}_b| \geq |\mathbf{p}_a + \mathbf{p}_b| = |\mathbf{p}|;$$

that is, p is forward time-like.

15.89 ⋆ Two particles have a CM frame if and only if there exists a frame where their total three-momentum is zero, and this is the case if and only if their total four-momentum is time-like. Now, a four-vector p is forward time-like if and only if $p_4 > |\mathbf{p}|$ (and forward light-like if and only if $p_4 = |\mathbf{p}| > 0$). Since our two particles are massless their four-momenta are forward light-like, so $p_{a4} = |\mathbf{p}_a| > 0$ and $p_{b4} = |\mathbf{p}_b| > 0$. Therefore their total momentum satisfies

$$p_4 = p_{a4} + p_{b4} = |\mathbf{p}_a| + |\mathbf{p}_b| \geq |\mathbf{p}_a + \mathbf{p}_b| = |\mathbf{p}|.$$

This proves that $p = p_a + p_b$ is forward time-like, *provided* the "$\geq$" sign is actually "$>$". Now, we know for any two three-vectors $\mathbf{A}$ and $\mathbf{B}$, that $|\mathbf{A}| + |\mathbf{B}| > |\mathbf{A} + \mathbf{B}|$ unless $\mathbf{A}$ and $\mathbf{B}$ are parallel. Therefore p is time-like unless $\mathbf{p}_a$ and $\mathbf{p}_b$ are parallel.

15.90 ⋆⋆ **(a)** The four-momentum of a single photon satisfies $p^2 = 0$. The total four-momentum of an electron and positron is the sum of two forward time-like vectors and so is forward time-like (Problem 15.69b), satisfying $p^2 < 0$. Therefore, the process $\gamma \to \mathrm{e}^+ + \mathrm{e}^-$ cannot conserve four-momentum.

(b) The derivation of (15.98) assumed that the incident particle had $m > 0$, but all that mattered was that the two initial particles have a CM frame, which is certainly true as long as one of them had $m > 0$. According to (15.98) the minimum energy for the photon is

$$E_\gamma^{\min} = \frac{(\sum m_{\text{fin}})^2 - m_{\mathrm{N}}^2}{2m_{\mathrm{N}}} c^2 = \frac{(m_{\mathrm{N}} + 2m_{\mathrm{e}})^2 - m_{\mathrm{N}}^2}{2m_{\mathrm{N}}} c^2 \approx \frac{4m_{\mathrm{e}}m_{\mathrm{N}}}{2m_{\mathrm{N}}} c^2 = 2m_{\mathrm{e}}c^2.$$

15.91 ⋆⋆ In the process $X^* \to X + \gamma$, the initial X^* is at rest so has four-momentum $p^* = (\mathbf{0}, M^*c)$. The four-momentum of the outgoing photon is $p_\gamma = (\mathbf{p}_\gamma, E_\gamma/c)$. By conservation of four-momentum $p^* = p + p_\gamma$ or $p = p^* - p_\gamma$, which, if we square both sides, implies that

$$-M^2c^2 = -M^{*2}c^2 - 2p^*{\cdot}p_\gamma + 0 = -M^{*2}c^2 + 2M^*E_\gamma$$

or, solving for E_γ,

$$E_\gamma = \frac{M^{*2} - M^2}{2M^*}c^2 = (M^* - M)c^2\frac{(M^* + M)}{2M^*} = \Delta E\left(1 - \frac{\Delta E}{2M^*c^2}\right)$$

where in the last equality I used the fact that $M = M^* - \Delta E/c^2$. The fractional difference between ΔE and E_γ is $\Delta E/(2M^*c^2) \approx$ (a few eV)/(at least 2 GeV), of order one part in a billion or less.

15.92 ⋆⋆ By conservation of four-momentum, $p_\mu = p_\pi - p_\nu$, where, since the pion is at rest, $p_\pi = (\mathbf{0}, m_\pi/c)$ and $p_\nu = (\mathbf{p}_\nu, E_\nu/c)$, where $E_\nu = |\mathbf{p}_\nu|c$ because we are treating the neutrino as massless. Squaring the first equation, we find

$$-m_\mu^2c^2 = -m_\pi^2c^2 + 2m_\pi E_\nu \quad \text{whence} \quad E_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}c^2.$$

From this we find,

$$|\mathbf{p}_\mu| = |\mathbf{p}_\nu| = E_\nu/c = (m_\pi^2 - m_\mu^2)c/2m_\pi \qquad \text{and} \qquad E_\mu = m_\pi c^2 - E_\nu = (m_\pi^2 + m_\mu^2)c^2/2m_\pi$$

so that $\beta_\mu = |\mathbf{p}_\mu|c/E_\mu = (m_\pi^2 - m_\mu^2)/(m_\pi^2 + m_\mu^2)$, as claimed. Putting in the numbers for the muon, we get $\beta_\mu = 0.27$. For an electron, $\beta_e = 0.99997$.

15.93 ⋆⋆⋆ The four-momenta of the initial and final electron and photon can be written as follows:

	initial	final
electron:	$p_o = (\beta_o, 0, 0, 1)E_o/c$	p
photon:	$p_{\gamma o} = (-1, 0, 0, 1)E_{\gamma o}/c$	$p_\gamma = (1, 0, 0, 1)E_\gamma/c$

According to (15.123), $p_o{\cdot}(p_{\gamma o} - p_\gamma) = p_{\gamma o}{\cdot}p_\gamma$, which becomes

$$E_oE_{\gamma o}(-\beta_o - 1) - E_oE_\gamma(\beta_o - 1) = E_{\gamma o}E_\gamma(-1 - 1).$$

Solving for E_γ we get the advertised answer

$$E_\gamma = E_o\frac{1 + \beta_o}{2 + (1 - \beta_o)E_o/E_{\gamma o}}. \tag{xiv}$$

The numerator of the fraction on the right of Eq.(xiv) is clearly less than 2, while the denominator is greater than 2. Therefore $E_\gamma < E_o$; the energy of the outgoing photon is less than the energy of the incident electron. But as $E_o \to \infty$ the proportion given to the photon approaches 100%. To see this, note that as $E_o \to \infty$, $\beta_o \to 1$ and the numerator in (xiv) approaches 2. The denominator is a bit harder, because the factor $(1 - \beta_o)$ approaches zero while the factor E_o approaches infinity. However,

$$(1-\beta_o)E_o = (1-\beta_o)\frac{mc^2}{\sqrt{1-\beta_o^2}} = mc^2\sqrt{\frac{1-\beta_o}{1+\beta_o}} \to 0. \qquad \text{(xv)}$$

Thus the denominator approaches 2 and the ratio $E_\gamma/E_o \to 1$. In the limit $E_o \to \infty$ the proportion of the electron's energy transferred to the photon approaches 100%.

Before we put in numbers, let's rewrite Eq.(xiv) as $E_\gamma = E_o(1+\beta_o)/(2+\xi)$, where according to Eq.(xv)

$$\xi = \frac{mc^2}{E_{\gamma o}}\sqrt{\frac{1-\beta_o}{1+\beta_o}}.$$

With the given numbers, $\beta_o = 1 - 1.25 \times 10^{-15}$ and $\xi = 0.004167$. Thus

$$E = E_o + E_{\gamma o} - E_\gamma = E_o\frac{1-\beta_o+\xi}{2+\xi} + E_{\gamma o}.$$

Because both $(1-\beta_o)$ and $E_{\gamma o}/E_o$ are much much less than ξ, this gives (almost exactly) $E/E_o = \xi/(2+\xi) = 0.002 = 0.2\%$.

15.94 ⋆ The elements of the matrix $C = AB$ are defined by $C_{ik} = \sum_j A_{ij}B_{jk}$ and those of the transpose $\tilde{C}$ by $\tilde{C}_{ki} = C_{ik}$. Therefore,

$$\tilde{C}_{ki} = C_{ik} = \sum_j A_{ij}B_{jk} = \sum_j \tilde{A}_{ji}\tilde{B}_{kj} = \sum_j \tilde{B}_{kj}\tilde{A}_{ji} = (\tilde{B}\tilde{A})_{ki}.$$

That is, $\tilde{C} = \tilde{B}\tilde{A}$.

15.95 ⋆ The condition $\tilde{\mathbf{a}}\mathbf{C}\mathbf{b} = \tilde{\mathbf{a}}\mathbf{D}\mathbf{b}$ is equivalent to the condition $\tilde{\mathbf{a}}(\mathbf{C}-\mathbf{D})\mathbf{b} = 0$. Thus what we have to prove is that if $\tilde{\mathbf{a}}\mathbf{E}\mathbf{b} = 0$ for all vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{E} = 0$. To prove this, choose $\mathbf{a}$ and $\mathbf{b}$ with the form on the right with the single nonzero entry of $\mathbf{a}$ in the kth row and that of $\mathbf{b}$ in the lth row. Then it is easy to see that $\tilde{\mathbf{a}}\mathbf{E}\mathbf{b} = E_{kl}$. Since this has to be zero for any choice of k and l, we conclude that $\mathbf{E} = 0$.

$$\mathbf{a} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

15.96 ⋆ Since a and T are respectively a four-vector and a four-tensor, $a' = \Lambda a$ and $T' = \Lambda T\tilde{\Lambda}$. Thu $b' = T'\cdot a' = T'Ga' = (\Lambda T\tilde{\Lambda})G(\Lambda a) = \Lambda T(\tilde{\Lambda}G\Lambda)a = \Lambda TGa = \Lambda T\cdot a = \Lambda b$, where, in writing the fifth equality, I have used the condition (15.136) that $\tilde{\Lambda}G\Lambda = G$.

15.97 ⋆ **(a)** Suppose that T is symmetric, that is, $\tilde{T} = T$. Then $(T')^{\sim} = (\Lambda T\tilde{\Lambda})^{\sim} = \Lambda\tilde{T}\tilde{\Lambda} = T'$, where I have used the rules that $(AB)^{\sim} = \tilde{B}\tilde{A}$ and that $(\tilde{A})^{\sim} = A$. This shows that T' is also symmetric.

(b) By the same argument, if $\tilde{T} = -T$, then also $(T')^{\sim} = -T'$.

15.98 ⋆⋆ **(a)** The invariance of $a{\cdot}b$ requires that $a{\cdot}b = a'{\cdot}b'$ or, in matrix notation,

$$\tilde{a}Gb = (a')\tilde{}\,G\,b' = (\Lambda a)\tilde{}\,G\,(\Lambda b) = \tilde{a}(\tilde{\Lambda}G\Lambda)b$$

Since this must be true for any choice of a and b, it follows (Problem 15.95) that $G = \tilde{\Lambda}G\Lambda$.

(b) The matrices Λ and G are given by (15.43) and (15.135). Therefore, as you can check,

$$\tilde{\Lambda}G\Lambda = \tilde{\Lambda}(G\Lambda) = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix}\begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & -\gamma \end{bmatrix} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = G.$$

15.99 ⋆⋆ **(a)** That $\mathbf{a}$ and $\mathbf{b}$ are vectors means that, given any two frames $\mathcal{S}$ and $\mathcal{S}'$, related by a rotation $\mathbf{R}$, $\mathbf{a}' = \mathbf{Ra}$ and $\mathbf{b}' = \mathbf{Rb}$ (or, in reverse, $\mathbf{a} = \tilde{\mathbf{R}}\mathbf{a}'$, etc.). Thus $\mathbf{b}' = \mathbf{T}'\mathbf{a}'$, but also $\mathbf{b}' = \mathbf{Rb} = \mathbf{RTa} = \mathbf{RT}\tilde{\mathbf{R}}\mathbf{a}'$. Since these two expressions for $\mathbf{b}'$ must be equal for any choice of $\mathbf{a}'$, it follows that $\mathbf{T}' = \mathbf{RT}\tilde{\mathbf{R}}$; that is, $\mathbf{T}$ is a tensor.

(b) Suppose that p and q are known to be four-vectors and suppose that for every inertial frame there is a 4×4 matrix T with the property that $q = T{\cdot}p$ for every choice of p, then T is a four-tensor. The proof parallels the three-dimensional case, except that where $p' = \Lambda p$, $p = G\tilde{\Lambda}Gp'$. Then $q' = T'{\cdot}p' = T'Gp'$, but also $q' = \Lambda q = \Lambda T{\cdot}p = \Lambda TGp = \Lambda TG(G\tilde{\Lambda}Gp') = \Lambda T\tilde{\Lambda}Gp'$. Since these are equal for any choice of p' we see that $T' = \Lambda T\tilde{\Lambda}$; that is, T is a tensor.

15.100 ⋆⋆⋆ **(a)** By the chain rule,

$$\frac{\partial\phi}{\partial x_i} = \sum_{j=1}^{3}\frac{\partial x'_j}{\partial x_i}\frac{\partial\phi}{\partial x'_j} = \sum_j R_{ji}\frac{\partial\phi}{\partial x'_j} \tag{xvi}$$

where in the last equality I used the transformation equation $x'_j = \sum_k R_{jk}x_k$ to make the replacement $\partial x'_j/\partial x_i = R_{ji}$. If agree to regard $\nabla\phi$ as the column vector comprising the three derivatives $\partial\phi/\partial x_i$, then Eq.(xvi) takes the matrix form

$$\nabla\phi = \tilde{\mathbf{R}}\nabla'\phi. \tag{xvii}$$

Finally, the property (15.129), $\tilde{\mathbf{R}}\mathbf{R} = \mathbf{1}$, shows that $\tilde{\mathbf{R}} = \mathbf{R}^{-1}$. Therefore, multiplying Eq.(xvii) by $\mathbf{R}$, we get the desired result

$$\nabla'\phi = \mathbf{R}\nabla\phi.$$

(b) The corresponding four dimensional problem is closely similar, though, as you might expect, it is complicated by the presence of the metric matrix G in the scalar product. First, corresponding to Eq.(xvi), we have

$$\frac{\partial\phi}{\partial x_i} = \sum_{j=1}^{4}\frac{\partial x'_j}{\partial x_i}\frac{\partial\phi}{\partial x'_j} = \sum_j \Lambda_{ji}\frac{\partial\phi}{\partial x'_j} \tag{xviii}$$

since $x' = \Lambda x$. Next, if we regard $\partial\phi$ as the column vector comprising the four derivatives $\partial\phi/\partial x_i$, then the definition (15.156) of $\Box\phi$ becomes $\Box\phi = G\partial\phi$. Thus we can rewrite Eq.(xviii) in matrix form as

$$\Box\phi = G\partial\phi = G\tilde{\Lambda}\partial'\phi = G\tilde{\Lambda}G\Box'\phi \tag{xix}$$

where in the last equality I used the facts that $G^2 = 1$ and hence that $\partial'\phi = G\Box'\phi$. Finally, the property (15.136), $\tilde{\Lambda}G\Lambda = G$, implies that $G\tilde{\Lambda}G = \Lambda^{-1}$. Therefore, multiplying (xix) by Λ, we get the desired result

$$\Box'\phi = \Lambda\Box\phi.$$

15.101 ⋆ **(a)** If $\mathcal{S}$ is obtained from $\mathcal{S}'$ by the standard boost, then, according to (15.146)

$$\begin{aligned}\mathbf{E}'\cdot\mathbf{B}' &= E_1'B_1' + E_2'B_2' + E_3'B_3' \\ &= E_1B_1 + \gamma^2(E_2 - \beta cB_3)(B_2 + \beta E_3/c) + \gamma^2(E_3 + \beta cB_2)(B_3 - \beta E_2/c) \\ &= E_1B_1 + E_2B_2 + E_3B_3 = \mathbf{E}\cdot\mathbf{B}\end{aligned}$$

[Remember that $\gamma^2(1-\beta^2) = 1$.] Similarly

$$\begin{aligned}E'^2 &= E_1'^2 + E_2'^2 + E_3'^2 = E_1^2 + \gamma^2(E_2 - \beta cB_3)^2 + \gamma^2(E_3 + \beta cB_2)^2 \\ &= E_1^2 + \gamma^2(E_2^2 + E_3^2) + \gamma^2\beta^2c^2(B_2^2 + B_3^2) - 2\gamma^2\beta c(E_2B_3 - E_3B_2)\end{aligned}$$

and, as you can check,

$$c^2B'^2 = \gamma^2\beta^2(E_2^2 + e_3^2) + c^2\left[B_1^2 + \gamma^2(B_2^2 + B_3^2)\right] - 2\gamma^2\beta c(E_2B_3 - E_3B_2).$$

Taking the difference of these two and remembering that $\gamma^2(1-\beta^2) = 1$, we find that

$$E'^2 - c^2B'^2 = E^2 - c^2B^2.$$

We have proved that neither of the named quantities is changed by a standard boost. They are clearly unchanged by any rotation, so, since any Lorentz transformation can be made up from rotations and a standard boost, they are actually invariant under any Lorentz transformation.

(b) If $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in $\mathcal{S}$, then $\mathbf{E}\cdot\mathbf{B} = 0$, and by the invariance of $\mathbf{E}\cdot\mathbf{B}$ the same is true in any other frame.

(c) If $E > cB$ in $\mathcal{S}$, then $E^2 - c^2B^2 > 0$, and by the invariance of $E^2 - c^2B^2$ the same is true in any other frame.

15.102 ⋆ **(a)** To find the requested boost, we have only to make the replacements $1 \to 3 \to 2 \to 1$ in Eq.(15.146) to give

$$\begin{aligned}E_3' &= E_3, \quad & E_1' &= \gamma(E_1 - \beta cB_2), \quad & E_2' &= \gamma(E_2 + \beta cB_1) \\ B_3' &= B_3, \quad & B_1' &= \gamma(B_1 + \beta E_2/c), \quad & B_2' &= \gamma(B_2 - \beta E_1/c)\end{aligned}$$

(b) The inverse transformation is given in (15.150) and the derivations are given in almost complete detail between (15.150) and (15.152).

15.103 ⋆ **(a)** If $\mathbf{E} = 0$, then, according to (15.146)

$$B_1' = B_1, \quad B_2' = \gamma B_2, \quad B_3' = \gamma B_3$$

and

$$E_1' = 0, \quad E_2' = -\gamma\beta c B_3 = -vB_3', \quad E_3' = \gamma\beta c B_2 = -vB_2'.$$

Combining these last three (and recalling that $\mathbf{v}$ is along the x axis), we see that $\mathbf{E}' = \mathbf{v}' \times \mathbf{B}'$.

(b) In exactly the same way, if $\mathbf{B} = 0$, then $\mathbf{B}' = -\mathbf{v} \times \mathbf{E}'/c^2$.

15.104 ⋆⋆ **(a)** From its definition (15.107), the four-force K is $K = \gamma(\mathbf{F}, \mathbf{v}\cdot\mathbf{F}/c)$. In the case of the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ this becomes

$$\begin{aligned} K = \gamma q\big(\mathbf{E} + \mathbf{v} \times \mathbf{B}, \mathbf{v}\cdot\mathbf{E}/c\big) = \gamma q\big(&E_1 + v_2B_3 - v_3B_2, \\ &E_2 + v_3B_1 - v_1B_3, \\ &E_3 + v_1B_2 - v_2B_1, \\ &[v_1E_1 + v_2E_2 + v_3E_3]/c\big) \end{aligned}$$

Since $u = \gamma(\mathbf{v}, c)$, we can replace $\gamma\mathbf{v}$ with $\mathbf{u}$ and γ with u_4/c to give

$$\begin{aligned} K = q\big(&B_3u_2 - B_2u_3 + (E_1/c)u_4, \\ &-B_3u_1 + B_1u_3 + (E_2/c)u_4, \\ &B_2u_1 - B_1u_2 + (E_3/c)u_4, \\ &(E_1/c)u_1 + (E_2/c)u_2 + (E_3/c)u_3\big) \end{aligned}$$

(b) Comparing this with the equation $K = q\mathcal{F}Gu$, we can read off the elements of the matrix $\mathcal{F}G$ to give exactly the results shown in Eq.(15.142), and, multiplying on the right by G, we find $\mathcal{F}$ itself in agreement with (15.143).

15.105 ⋆⋆ The matrices Λ and $\mathcal{F}$ are given in Eqs.(15.43) and (15.143). Using them you can easily check that

$$\Lambda\mathcal{F}\tilde{\Lambda} = \begin{bmatrix} 0 & \gamma(B_3 - \beta E_2/c) & -\gamma(B_2 + \beta E_3/c) & -E_1/c \\ \cdot & 0 & B_1 & -\gamma(E_2 - \beta c B_3)/c \\ \cdot & \cdot & 0 & -\gamma(E_3 + \beta c B_2)/c \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

(I didn't bother to write the elements below the diagonal because we know that the matrix is antisymmetric.) But this must be equal to $\mathcal{F}'$,

$$\mathcal{F}' = \begin{bmatrix} 0 & B_3' & -B_2' & -E_1'/c \\ \cdot & 0 & B_1' & -E_2'/c \\ \cdot & \cdot & 0 & -E_3'/c \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

Comparing these two you can read off the transformed fields as in Eqs.(15.146).

15.106 $\star\star$ **(a)** We can orient our axes in frame $\mathcal{S}$ so that the velocity $\mathbf{v}$ of the charge is along the positive x axis. Then the charge's rest frame $\mathcal{S}'$ is in standard configuration with respect to $\mathcal{S}$, with relative velocity $\mathbf{v}$. Since the charge is at rest in $\mathcal{S}'$ the force on it is $\mathbf{F}' = q\mathbf{E}'$. We now use the inverse force transformation [the inverse of Eqs.(15.155)] to write down $\mathbf{F}$:

$$F_1 = \frac{F_1' + \beta \mathbf{F}' \cdot \mathbf{v}'/c}{1 + \beta v_1'/c} = F_1' = qE_1'$$

(remember $\mathbf{v}' = 0$) and

$$F_2 = \frac{F_2'}{\gamma(1 + \beta v_1'/c)} = \frac{F_2'}{\gamma} = qE_2'/\gamma$$

and, similarly, $F_3' = qE_3'/\gamma$.

(b) We can now use the field transformations (15.146) to rewrite $\mathbf{E}'$ in terms of $\mathbf{E}$ and $\mathbf{B}$ to give

$$F_1 = qE_1, \quad F_2 = q(E_2 - vB_3). \quad F_3 = q(E_3 + vB_2).$$

Bearing in mind that $\mathbf{v}$ is along the x axis, you should recognize this as $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

15.107 $\star\star$ We are given that $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$, $\mathbf{B} = \nabla \times \mathbf{A}$ and $A = (\mathbf{A}, \phi/c)$. I'll examine the components of $\mathcal{F}$ in turn. First, if $i = 1$, 2, or 3,

$$\mathcal{F}_{i4} = -\frac{E_i}{c} = \frac{\partial}{\partial x_i}\frac{\phi}{c} + \frac{1}{c}\frac{\partial A_i}{\partial t} = \frac{\partial}{\partial x_i}A_4 + \frac{\partial A_i}{\partial x_4} = \Box_i A_4 - \Box_4 A_i.$$

and

$$\mathcal{F}_{12} = B_3 = \frac{\partial}{\partial x_1}A_2 - \frac{\partial}{\partial x_2}A_1 = \Box_1 A_2 - \Box_2 A_1.$$

All other elements follow similarly, and we've proved that $\mathcal{F}_{\mu\nu} = \Box_\mu A_\nu - \Box_\nu A_\mu$, for any μ and ν equal to 1,2,3, or 4.

15.108 $\star\star$ **(a)** Let $\mathcal{S}$ and the rest frame $\mathcal{S}_o$ be related by the standard boost and consider a small rectangular volume with sides $\Delta x = \Delta x_o/\gamma, \Delta y = \Delta y_o$, and $\Delta z = \Delta z_o$, containing charge $\Delta Q = \Delta Q_o$. (Remember that charge is invariant.) Then

$$\varrho = \frac{\Delta Q}{\Delta x \Delta y \Delta z} = \frac{\Delta Q_o}{(\Delta x_o/\gamma)\Delta y_o \Delta z_o} = \gamma \varrho_o.$$

(b) The three-current density is $\mathbf{J} = \varrho\mathbf{v}$, and the corresponding four-current is

$$J = (\mathbf{J}, c\varrho) = \varrho(\mathbf{v}, c) = \gamma\varrho_o(\mathbf{v}, c) = \varrho_o u,$$

which is the product of a four-scalar and a four-vector. Therefore J is a four-vector.

(c) $$\nabla \cdot \mathbf{J} + \frac{\partial \varrho}{\partial t} = \frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = \Box_1 J_1 + \Box_2 J_2 + \Box_3 J_3 - \Box_4 J_4 = \Box \cdot J.$$

15.109 ⋆⋆⋆ **(a)** In $\mathcal{S}'$ the two charges are at rest on the y axis a distance r apart (this transverse distance is the same in either frame), so $\mathbf{F}' = (0, kq^2/r^2, 0)$. The inverse of the force transformation (15.155) gives the force $\mathbf{F}$ measured in $\mathcal{S}$ as

$$F_1 = \frac{F_1' + \beta \mathbf{F}'\cdot\mathbf{v}'/c}{1+\beta v_1'/c} = 0, \quad F_2 = \frac{F_2'}{\gamma(1+\beta v_1'/c)} = \frac{F_2'}{\gamma} = \frac{kq^2}{\gamma r^2}, \quad F_3 = 0.$$

(b) The fields of the lower charge at the location of the upper one, as measured in $\mathcal{S}'$, are $\mathbf{E}' = (0, kq/r^2, 0)$ and $\mathbf{B}' = 0$. Using the inverse field transformations, we find the fields in $\mathcal{S}$ to be $\mathbf{E} = (0, \gamma kq/r^2, 0)$ and $\mathbf{B} = (0, 0, \gamma\beta kq/cr^2)$. (Note well how each charge feels a magnetic field due to the other moving charge.) Given that the velocity of either particle in $\mathcal{S}$ is $\mathbf{v} = (v, 0, 0)$, we can now evaluate the foce on the upper charge as

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \frac{\gamma kq^2}{r^2}\hat{\mathbf{y}} - \frac{v\gamma\beta kq^2}{cr^2}\hat{\mathbf{y}} = \frac{\gamma(1-\beta^2)kq^2}{r^2}\hat{\mathbf{y}} = \frac{kq^2}{\gamma r^2}\hat{\mathbf{y}}$$

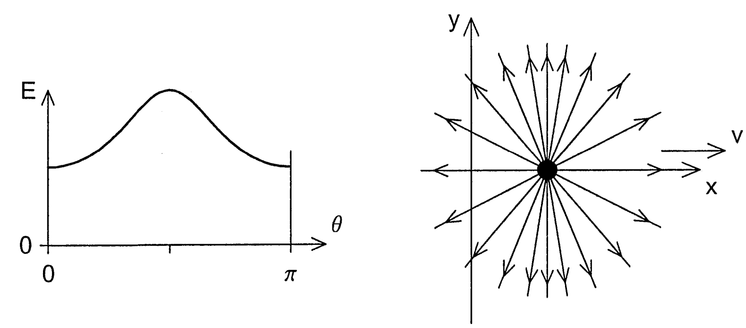
since $\gamma(1-\beta^2) = 1/\gamma$. This is the same as we found in part (a).

15.110 ⋆⋆⋆ **(a)** In the charge's rest frame $\mathcal{S}'$, $\mathbf{E}' = \dfrac{kq}{r'^3}(x', y', z')$ and $\mathbf{B}' = 0$.

(b) The field transformation (15.146) gives the E field in $\mathcal{S}$ as $\mathbf{E} = (kq/r'^3)(x', \gamma y', \gamma z')$. Since $x' = \gamma(x - vt) = \gamma R_x$, while $y' = y = R_y$ and $z' = z = R_z$, this gives

$$\mathbf{E} = \frac{\gamma kq}{(\gamma^2 R_x^2 + R_y^2 + R_z^2)^{3/2}}\mathbf{R} = \frac{(1-\beta^2)kq}{[\cos^2\theta + (1-\beta^2)\sin^2\theta]^{3/2}}\frac{\mathbf{R}}{R^3} = \frac{kq(1-\beta^2)}{[1-\beta^2\sin^2\theta]^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}.$$

(c)



Two noteworthy features: The electric field is peaked in the transverse direction, $\theta = \pi/2$, and its direction is everywhere radially outward from the charge's present position, that is, in the direction of the vector $\mathbf{R} = \mathbf{r} - \mathbf{v}t$ pointing from the charge's position $\mathbf{v}t$ to the point of observation $\mathbf{r}$.

15.111 ⋆⋆⋆ The proposed equation $\Box\cdot\mathcal{F} = \mu_o\tilde{J}$ is

$$\Box\cdot\mathcal{F} = \tilde{\Box}G\mathcal{F} = [\partial_x, \partial_y, \partial_z, \partial_t/c]\begin{bmatrix} 0 & B_3 & -B_2 & -E_1/c \\ -B_3 & 0 & B_1 & -E_2/c \\ B_2 & -B_1 & 0 & -E_3/c \\ E_1/c & E_2/c & E_3/c & 0 \end{bmatrix}$$

$$= [-\partial_y B_z + \partial_z B_y + \partial_t E_x/c^2,\ \ \partial_x B_z - \partial_z B_x + \partial_t E_y/c^2,$$
$$-\partial_x B_y + \partial_y B_x + \partial_t E_z/c^2,\ \ -(\partial_x E_x + \partial_y E_y + \partial_z E_z)/c^2]$$
$$= -[\mu_o J_x, \mu_o J_y, \mu_o J_z, \varrho/c\epsilon_o].$$

Here I have used the abbreviations $\partial_x = \partial/\partial x$ and so on. In writing the second equality, I noted that $\tilde{\Box}G = [\partial_x, \partial_y, \partial_z, \partial_t/c]$ and in the last line I used the fact that $\mu_o c = 1/c\epsilon_o$. If you look carefully at the last equality, you will recognize its first three components as the first of the Maxwell equations (15.159) and the time component as the second.

Chapter 16

Continuum Mechanics

Like the other chapters of Part II, Chapter 16 is optional and is independent of the other four chapters. The first four sections are about the wave equation, a theme to which I return in Section 16.11 and again in 16.13. Sections 16.5 through 16.11 are mostly about the mechanics of continuous solids and the main targets are the stress and strain tensors. Sections 16.12 and 16.13 are about the mechanics of inviscid fluids and introduce several important concepts — the material derivative, the equation of continuity, and Bernouilli's theorem. All of the topics treated here are obviously important, but you may well not have time to cover them. I never got to this chapter in lecture, though several students studied it for their term project and seemed to enjoy it. By including just the first four sections, you would at least introduce the wave equation — so important in electromagnetism and elsewhere. If you covered through Section 16.11, you would also give your students a fairly thorough introduction to three-dimensional tensors.

Concerning my treatment of the wave equation, I decided that it is more important that our students see its derivation for a genuinely continuous system (specifically a taut string) than that they be dragged through the awkward derivation as the limit of a discrete set of masses connected by springs. If you want them to go through that limiting process, it's covered in Problem 16.2. My treatments of solid and fluid mechanics are necessarily brief and rather theoretical. Nevertheless, they go far enough to establish the properties of longitudinal and transverse waves in solids and of longitudinal waves in fluids. My one possible regret was that I couldn't find space for any treatment of viscous fluids, so there is no mention of the Navier-Stokes equation (which probably doesn't belong in a course at this level anyway).

Solutions to Problems for Chapter 16

16.1 $\star$ The SI units of the tension T and linear density μ are $[T]$ = [force] = kg·m/s^2 and $[\mu]$ = [mass/length] = kg/m. Therefore, those of $c = \sqrt{T/\mu}$ are

$$[c] = \sqrt{\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\cdot\frac{\text{m}}{\text{kg}}} = \frac{\text{m}}{\text{s}}$$

as required.

16.2 ⋆⋆ The equation of motion of the ith mass is

$$m\ddot{u}_i = T(\sin\theta_i - \sin\theta_{i-1}) = \left(\frac{u_{i+1}-u_i}{b} - \frac{u_i - u_{i-1}}{b}\right) \tag{i}$$

where m is the mass of any one of the masses, dots denote differentiation with respect to t, and θ_i is the angle between the horizontal and the string to the right of the ith mass. If now we let $b \to 0$ holding μ constant, then we can replace m by $m = \mu b$, and u_i by $u_i = u(x)$ [actually $u(x,t)$ but I'll omit the t], and $(u_i - u_{i-1})/b \to u'(x)$, where a prime denotes differentiation with respect to x. With these replacements, Eq.(i) becomes (after dividing by b)

$$\mu\ddot{u}(x) = T\,\frac{u'(x) - u'(x-b)}{b} \to Tu''(x), \quad \text{as } b \to 0$$

which is the wave equation.

16.3 ⋆ Let $f'(\xi) = df(\xi)/d\xi$ and likewise $f''(\xi) = d^2f(\xi)/d\xi^2$. By the chain rule

$$\frac{\partial}{\partial t}f(x-ct) = -cf'(x-ct) \quad \text{and} \quad \frac{\partial^2}{\partial t^2}f(x-ct) = c^2 f''(x-ct)$$

and

$$\frac{\partial}{\partial x}f(x-ct) = f'(x-ct) \quad \text{and} \quad \frac{\partial^2}{\partial x^2}f(x-ct) = f''(x-ct).$$

Comparing these two, we see that

$$\frac{\partial^2}{\partial t^2}f(x-ct) = c^2\frac{\partial^2}{\partial x^2}f(x-ct)$$

which is the wave equation.

16.4 ⋆ If we make the suggested substitutions, then $x = (\xi + \eta)/2$ and $t = (\eta - \xi)/2c$. Therefore,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2c}\frac{\partial}{\partial t}$$

and

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \eta}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t}.$$

Therefore

$$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} = \frac{1}{4c^2}\left(c\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\left(c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) = \frac{1}{4c^2}\left(c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)$$

which is the claimed identity.

16.5 ⋆ **(a)** Let $g'(\xi) = dg(\xi)/d\xi$ and likewise $g''(\xi) = d^2g(\xi)/d\xi^2$. By the chain rule

$$\frac{\partial}{\partial t}g(x+ct) = cg'(x+ct) \quad \text{and} \quad \frac{\partial^2}{\partial t^2}g(x+ct) = c^2g''(x+ct)$$

and

$$\frac{\partial}{\partial x}g(x+ct) = g'(x+ct) \quad \text{and} \quad \frac{\partial^2}{\partial x^2}g(x+ct) = g''(x+ct).$$

Comparing these two, we see that

$$\frac{\partial^2}{\partial t^2}g(x+ct) = c^2\frac{\partial^2}{\partial x^2}g(x+ct)$$

which is the wave equation.

(b) If $u(x,t) = g(x+ct)$, then at time zero, $u(x,0) = g(x)$. Now suppose, for example, that the function $g(x)$ has a crest (or trough, or any other feature) at the point $x = x_o$. Then at a later time t, $u(x,t) = g(x+ct)$ has the same crest at the point $x + ct = x_o$ or $x = x_o - ct$. That is, as time goes by, the crest moves to the left with speed c. Because the same applies to all other features, we conclude that the whole initial disturbance moves bodily to the left with speed c.

16.6 ⋆ The initial condition $f'(x) - g'(x) = 0$ implies that $f(x) - g(x) = k$, a constant, and, from (16.13), we know that $f(x) + g(x) = u_o(x)$. Solving these two for $f(x)$ and $g(x)$, we find

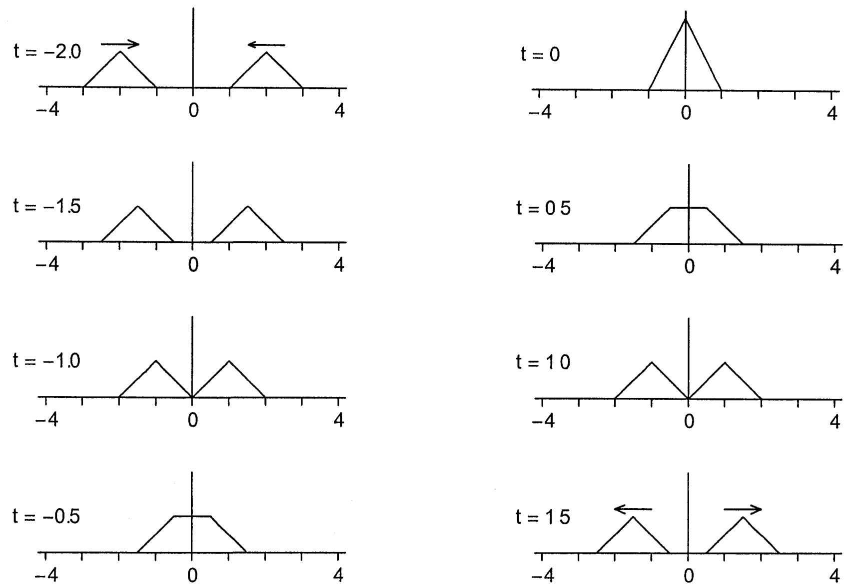
$$f(x) = \tfrac{1}{2}[u_o(x) + k] \quad \text{and} \quad g(x) = \tfrac{1}{2}[u_o(x) - k]$$

whence

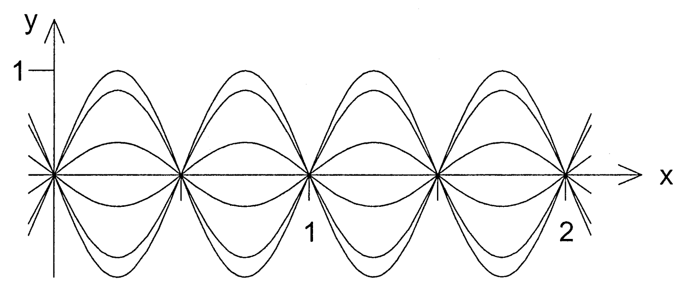
$$\begin{aligned} u(x,t) = f(x-ct) + g(x+ct) &= \tfrac{1}{2}[u_o(x-ct) + k] + \tfrac{1}{2}[u_o(x+ct) - k] \\ &= \tfrac{1}{2}u_o(x-ct) + \tfrac{1}{2}u_o(x+ct). \end{aligned}$$

Because the terms involving k cancel, we get the same answer as before.

16.7 ⋆⋆ On the next page is a subset of the requested graphs. Initially ($t < -1$), the two separate triangles move inward from the left and right. At $t = -1$ they start to overlap and reinforce one another, producing a truncated triangle. At $t = 0$ they overlap completely, producing a triangle of twice their separate heights, and then they start to move apart again.



16.8 ⋆⋆ The picture shows the standing wave (16.18) at times $t = 0, 0.1, \cdots, 1$. (The last five curves coincide with the first five — in reverse order — so you see only six distinct curves.) The string is stationary at the nodes at $x = 0.5, 1, 1.5$, and 2. At any other point, it oscillates in the y direction only.



16.9 ⋆⋆ **(a)** We are certainly free to try for a solution of the form $u(x,t) = X(x)T(t)$. If we substitute this into Eq.(16.19), $\partial^2 u/\partial t^2 = c^2 \partial^2 u/\partial x^2$, we find $X(x)T''(t) = c^2 T(t)X''(x)$, from which it follows that

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} \tag{ii}$$

where, as usual, primes indicate differentiation with respect to the argument. This equation must hold for all t and all x in the ranges of interest. If we temporarily fix x, then Eq.(ii) says that the left side is independent of t. Similarly, if we fix t, we see that the right side is independent of x. Since the two sides are equal, this implies that both sides are equal to a constant K, independent of t and x. As discussed in the footnote, this constant has to be negative, so we can call it $-\omega^2$. Therefore, $T''(t) = -\omega^2 T(t)$, which implies that $T(t) = A\cos(\omega t - \delta)$ and hence that $u(x,t)$ has the form (16.21).

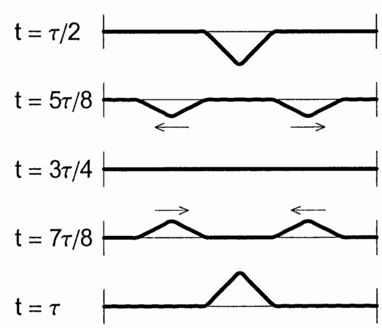
16.10 ⋆⋆ The function $u_o(x)$ is symmetric under reflection about the point $x = 4$. For n even, $\sin(n\pi x/8)$ changes sign under the same reflection, so the same is true of the whole integrand; thus the integral is zero, since each contribution from the left of $x = 4$ is cancelled by the corresponding contribution from the right.

If n is odd, the integrand is symmetric (under the same reflection) and we can replace the integral by twice the integral from 4 to 8. If we change variables to $u = x - 4$ and write $n = 2m + 1$, this gives

$$B_n = \frac{(-1)^m}{2}\int_0^1 (1-u)\cos\left(\frac{n\pi u}{8}\right) du = (-1)^m \frac{32}{n^2\pi^2}[1 - \cos(n\pi/8)]$$

in agreement with (16.34). (You can do the integral by integration by parts.)

16.11 ⋆⋆ Here is the story from $t = \tau/2$ to τ, at the same intervals as in Fig. 16.8. At $t = \tau/2$ the two traveling waves have formed a single inverted triangle. They then separate and move out toward the walls, where they reflect, invert, and travel inward again. By the time $t = \tau$ they have coalesced and the string is back to its original position.



16.12 ⋆⋆ **(a)** Since $f(\xi)$ is localized around $\xi = 0$, the function $f(x+ct)$ is localized around $x = -ct$. When $t = t_o$, with t_o large and negative, this means that $f(x + ct)$ is localized far to the right and traveling in toward the origin.

(b) Consider the wave $u(x,t) = f(x + ct) - f(-x + ct)$. The second term is called the "image" because it is the result of reflecting the first term in the plane $x = 0$ (and inverting it). Because both terms satisfy the wave equation, so does $u(x,t)$. When t is large and negative, the second term in $u(x,t)$ is zero everywhere that $x \geq 0$ (that is, where the string is actually located). Therefore, $u(x,t)$ is exactly equal to the wave of part (a) everywhere on the string. If we put $x = 0$, we find $u(0,t) = 0$.

(c) Because $u(x,t)$ satisfies the wave equation, the initial conditions, and the boundary conditions, it is the solution. As long as t remains large and negative, the second term is zero (where the string is), so the wave continues to move in from the right toward $x = 0$. As t nears 0, the "image" wave begins to emerge onto the string, and the two terms interfere. Once t is large and positive, the original wave has disappeared into the region $x < 0$, and we are left with just the reflected "image" wave traveling to the right.

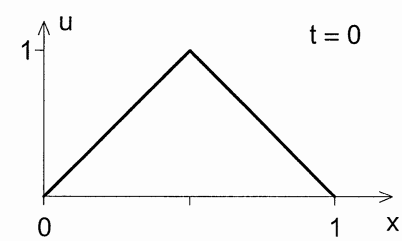
16.13 ⋆⋆ The first condition in (16.143) defines $f(x)$ in the interval $-L \le x \le 0$, and the second condition then defines it for all x. The second condition guarantees that it is $2L$-periodic and the first condition that it is odd. Clearly the new $f(x)$ coincides with the old in the original interval $0 \le x \le L$. Since the new function is periodic with period $\lambda = 2L$, it can be expanded in a standard Fourier series as

$$f(x) = \sum_{n=0}^{\infty}(A_n \cos k_n x + B_n \sin k_n x).$$

However, since we cunningly defined $f(x)$ to be an odd function, the coefficients A_n of the cosine terms are all zero, and since the period is $\lambda = 2L$ the wave numbers k_n are $2n\pi/\lambda = n\pi/L$ as in (16.27). This establishes the claimed expansion (16.31). Finally the usual integral for the coefficients is $B_n = (2/\lambda)\int_{-L}^{L} f \sin(k_n x)dx$. If we make the replacement $\lambda = 2L$, the factors of 2 cancel, but we can then replace the integral from $-L$ to L by twice the inegral from 0 to L, and we wind up with $B_n = (2/L)\int_0^L f \sin(k_n x)dx$ as in (16.33).

16.14 ⋆⋆⋆

(a) The disturbance $u(x, 0)$ has the triangular shape sketched on the right. Because the string is initially at rest the coefficients C_n in the Fourier series (16.30) for $u(x,t)$ are all zero and, as in (16.33) the coefficients B_n are given by



$$B_n = \int_0^1 u(x, 0)\sin(n\pi x)dx.$$

The function $u(x, 0)$ is unchanged under reflection in the line $x = 0.5$, whereas the function $\sin n\pi x$ with n even changes sign under the same reflection. Therefore the coefficients B_n with n even are all zero. If n is odd, the integrand is unchanged under reflection and (the integral is easily done by integration by parts)

$$B_n = 4\int_0^{0.5}(2x)\sin(n\pi x)dx = (-1)^m 8/(n\pi)^2$$

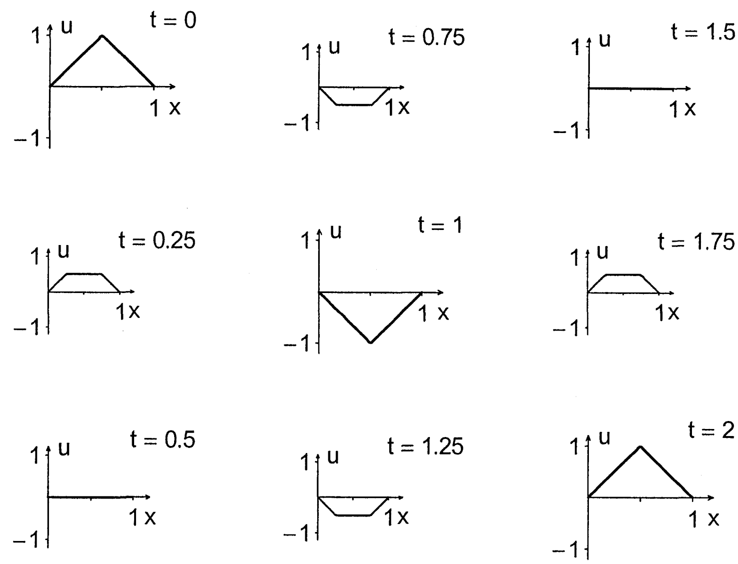
where I have set $n = 2m + 1$.

(b) With the given numbers, the period of the oscillations is $\tau = 2\pi/\omega_1 = 2$, and the nine pictures on the next page show the sums of the first 19 terms of the Fourier series,

$$u(x,t) = \sum_{m=0}^{19} B_n \sin(n\pi x)\cos(n\pi t),$$

for $t = 0, 0.25, \cdots, 2$.

The original disturbance moves downward, starting at the top, so as to form a truncated triangle ($t = 0.25$). It continues down until it is completely inverted ($t = 1$), and then moves upward until it is back at its initial configuration ($t = 2$).



16.15 ⋆⋆ **(a)** By the chain rule,

$$\frac{\partial}{\partial x} f(\mathbf{n}\cdot\mathbf{r} - ct) = f'(\mathbf{n}\cdot\mathbf{r} - ct)\frac{\partial}{\partial x}(\mathbf{n}\cdot\mathbf{r}) = n_x f'(\mathbf{n}\cdot\mathbf{r} - ct). \tag{iii}$$

Combining this with the two corresponding results for $\partial f/\partial y$ and $\partial f/\partial z$, we conclude that $\nabla f(\mathbf{n}\cdot\mathbf{r} - ct) = \mathbf{n}f'(\mathbf{n}\cdot\mathbf{r} - ct)$.

(b) Differentiating Eq.(iii) again we see that

$$\frac{\partial^2}{\partial x^2} f(\mathbf{n}\cdot\mathbf{r} - ct) = n_x^2 f''(\mathbf{n}\cdot\mathbf{r} - ct)$$

and adding this to the two corresponding results for the y and z derivatives, we find that $\nabla^2 f(\mathbf{n}\cdot\mathbf{r} - ct) = f''(\mathbf{n}\cdot\mathbf{r} - ct)$, because $n_x^2 + n_y^2 + n_z^2 = 1$. On the other hand, differentiating twice with respect to t, we find

$$\frac{\partial^2}{\partial t^2} f(\mathbf{n}\cdot\mathbf{r} - ct) = c^2 f''(\mathbf{n}\cdot\mathbf{r} - ct) = c^2\nabla^2 f(\mathbf{n}\cdot\mathbf{r} - ct)$$

by the previous result. This is the wave equation.

(c) At any fixed time t, $f(\mathbf{n}\cdot\mathbf{r} - ct)$ is constant provided $\mathbf{n}\cdot\mathbf{r} = \text{const}$; that is, $f(\mathbf{n}\cdot\mathbf{r} - ct)$ is constant in any plane perpendicular to $\mathbf{n}$. Suppose now that $f(\xi)$ has a maximum (or any other feature) at $\xi = \xi_o$. Then $f(\mathbf{n}\cdot\mathbf{r} - ct)$ will share this feature at any point satisfying $\mathbf{n}\cdot\mathbf{r} = \xi_o + ct$, and this defines a plane perpendicular to $\mathbf{n}$ that travels at speed c in the direction of $\mathbf{n}$.

16.16 ⋆⋆ **(a)** We wish to prove that, if f is spherically symmetric, $f = f(r)$, then

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf). \tag{iv}$$

We'll look at the two sides of this equation in turn. First,

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(f'(r)\frac{\partial r}{\partial x}\right)=f''(r)\left(\frac{\partial r}{\partial x}\right)^2+f'(r)\frac{\partial^2 r}{\partial x^2}$$
$$=f''(r)\frac{x^2}{r^2}+\frac{f'(r)}{r}\left(1-\frac{x^2}{r^2}\right) \qquad \text{(v)}$$

where in pasing to the second line I used that $\partial r/\partial x = x/r$ [first derived in (4.42)] and hence $\partial^2 r/\partial x^2 = (1 - x^2/r^2)/r$. Adding (v) to the two corresponding results in y and z, we find that

$$\nabla^2 f = f''(r) + \frac{2f'(r)}{r}. \qquad \text{(vi)}$$

Meanwhile the right side of Eq(iv) is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf)=\frac{1}{r}\frac{\partial}{\partial r}(rf'+f)=\frac{1}{r}(rf''+2f').$$

This is the same as (vi), and we've proved (iv).

(b) The formula inside the back cover for ∇^2 in spherical polars contains three terms, but two of them involve derivatives with respect to θ or ϕ. Acting on $f(r)$ these two give zero, so we're left with a single term that is precisely the right side of Eq.(iv).

16.17 $\star\star$ **(a)** The increase in length of a segment of length l near position x is $dl = u(x+l) - u(x) \approx l\,\partial u/\partial x$. Thus, according to (16.55) the (additional) tension in the string is $F = A\,\mathrm{YM}\,dl/l = A\,\mathrm{YM}\,\partial u/\partial x$.

(b) The net force on a segment dx is

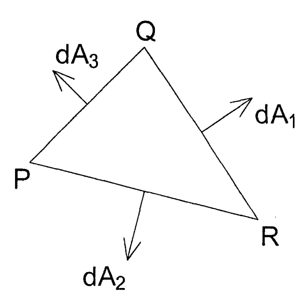
$$F_{\text{net}} = F(x+dx) - F(x) = A\,\mathrm{YM}\left[\frac{\partial u}{\partial x}(x+dx,t) - \frac{\partial u}{\partial x}(x,t)\right] = A\,\mathrm{YM}\frac{\partial^2 u}{\partial x^2}dx$$

But by Newton's second law, $F_{\text{net}} = ma = (\varrho A dx)\partial^2 u/\partial t^2$, and comparing these two expressions, we find

$$\frac{\partial^2 u}{\partial t^2}=\frac{\mathrm{YM}}{\varrho}\frac{\partial^2 u}{\partial x^2}$$

which is the wave equation with wave speed $c = \sqrt{\mathrm{YM}/\varrho}$.

16.18 $\star$ Let the height of the prism be h (measured into the page) and let's label the three visible corners as P, Q, and R as shown. The vector $d\mathbf{A}_1$ is normal to the line QR and has magnitude $(QR)h$, where I use (QR) to denote the length of side QR. Thus



$$dA_1 = (QR)h, \quad dA_2 = (RP)h, \quad \text{and } dA_3 = (PQ)h.$$

That PQR is a closed triangle implies that

$$\overrightarrow{QR} + \overrightarrow{RP} + \overrightarrow{PQ} = 0.$$

If we now multiply this equation by h and rotate all three vectors counterclockwise through $90°$, we get $d\mathbf{A}_1 + d\mathbf{A}_2 + d\mathbf{A}_3 = 0$ as claimed.

16.19 ⋆ The surface force on a small surface element $\mathbf{n}_1 dA$ is $\mathbf{F}(\mathbf{n}_1 dA) = \mathbf{\Sigma}\,\mathbf{n}_1 dA$ and the component of this force in the direction of $\mathbf{n}_2$ is $\mathbf{n}_2\cdot(\mathbf{\Sigma}\,\mathbf{n}_1)dA$. To emphasize the matrix products concerned, I'll temporarily use an underline to indicate a matrix. (For example, the dot product $\mathbf{a}\cdot\mathbf{b}$ of two vectors becomes $\mathbf{a}\cdot\mathbf{b} = \underline{\tilde{\mathbf{a}}}\,\underline{\mathbf{b}}$, where $\underline{\mathbf{b}}$ is the 3×1 column representing the vector $\mathbf{b}$ and $\underline{\tilde{\mathbf{a}}}$ is the 1×3 row representing $\mathbf{a}$.) With this notation

$$\mathbf{n}_2\cdot(\mathbf{\Sigma}\,\mathbf{n}_1) = \underline{\tilde{\mathbf{n}}}_2\,\underline{\mathbf{\Sigma}}\,\underline{\mathbf{n}}_1 = \underline{\tilde{\mathbf{n}}}_1\,\underline{\tilde{\mathbf{\Sigma}}}\,\underline{\mathbf{n}}_2 = \underline{\tilde{\mathbf{n}}}_1\,\underline{\mathbf{\Sigma}}\,\underline{\mathbf{n}}_2 = \mathbf{n}_1\cdot(\mathbf{\Sigma}\,\mathbf{n}_2)$$

where for the second equality I used the fact that any number is equal to its transpose and that $(\underline{\mathbf{u}}\,\underline{\mathbf{v}}\,\underline{\mathbf{w}})^{\sim} = \underline{\tilde{\mathbf{w}}}\,\underline{\tilde{\mathbf{v}}}\,\underline{\tilde{\mathbf{u}}}$, and for the third equality I use the symmetry of $\mathbf{\Sigma}$.

16.20 ⋆⋆ The given surface has the form $f = x^2 + y^2 + 2z^2 = 4$. The normal to this surface is in the direction of $\nabla f = (2x, 2y, 4z) = 2(1,1,2)$ at the point P with coordinates $(1,1,1)$. Thus the unit normal at P is $\mathbf{n} = (1,1,2)/\sqrt{6}$. Written as matrices, $\mathbf{\Sigma}$ (evaluated at P), $\mathbf{n}$, and $\mathbf{F} = \mathbf{\Sigma}\,\mathbf{n}\,dA$ are as follows:

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad \mathbf{n} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{F} = \mathbf{\Sigma}\,\mathbf{n}\,dA = \frac{dA}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

16.21 ⋆⋆ The theorem quoted guarantees that there is a set of Cartesian axes with respect to which $\mathbf{\Sigma}$ has the diagonal form shown below on the left. Now consider the surface force on a small element of area dA normal to the x axis of that set. The vector $d\mathbf{A}$ for this surface is $d\mathbf{A} = dA\,\hat{\mathbf{x}}$, whose matrix form is given in the middle expression below. Therefore the force on this small surface is as given on the right.

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}, \qquad d\mathbf{A} = dA\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{F} = \mathbf{\Sigma}\,d\mathbf{A} = dA\begin{bmatrix} \sigma_{11} \\ 0 \\ 0 \end{bmatrix}.$$

We see that the force on a surface normal to $\hat{\mathbf{x}}$ is in the direction of $\hat{\mathbf{x}}$ (that is, normal to the surface). Since the same happens with surfaces normal to $\hat{\mathbf{y}}$ or $\hat{\mathbf{z}}$, we've proved the requested result.

16.22 ⋆⋆⋆ We are promised that, with respect to any set of orthogonal axes, the stress tensor $\mathbf{\Sigma}$ has the diagonal form shown on the left below. Let us suppose that, with respect to a certain set of axes $\mathcal{S}$, two of the diagonal elements are not equal, $\sigma_{11} \neq \sigma_{33}$, say. Now consider a new set of axes $\mathcal{S}'$ obtained from $\mathcal{S}$ by a rotation about the y axis of $\mathcal{S}$. The matrix (15.36) for this rotation is shown on the right:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

With respect to the new axes, the stress tensor is $\mathbf{\Sigma}' = \mathbf{R\Sigma\tilde{R}}$, and, as you can check, its 1,3 element is $\sigma'_{13} = (\sigma_{33} - \sigma_{11})\cos\theta\sin\theta$, which is not zero if $\sigma_{11} \neq \sigma_{33}$. But we are promised that all off-diagonal elements *are* zero in all frames. Therefore σ_{11} and σ_{33} must be equal. By a similar argument σ_{11} and σ_{22} must be equal, and we conclude that, with all three elements equal, the matrix $\mathbf{\Sigma}$ on the left above is a multiple of the unit matrix.

16.23 ⋆ Suppose that $\mathbf{M} = \mathbf{M}_A + \mathbf{M}_S$, where $\mathbf{M}_A$ and $\mathbf{M}_S$ are respectively antisymmetric and symmetric. Now consider the following:

$$\mathbf{M} + \tilde{\mathbf{M}} = (\mathbf{M}_A + \mathbf{M}_S) + (\tilde{\mathbf{M}}_A + \tilde{\mathbf{M}}_S) = (\mathbf{M}_A + \mathbf{M}_S) + (-\mathbf{M}_A + \mathbf{M}_S) = 2\mathbf{M}_S$$

Therefore, for any decomposition of $\mathbf{M}$ into symmetric and antisymmetric matrices, the symmetric part $\mathbf{M}_S$ must satisfy $\mathbf{M}_S = \frac{1}{2}(\mathbf{M} + \tilde{\mathbf{M}})$. By considering $\mathbf{M} - \tilde{\mathbf{M}}$, we can prove similarly that $\mathbf{M}_A = \frac{1}{2}(\mathbf{M} - \tilde{\mathbf{M}})$. Therefore, both $\mathbf{M}_S$ and $\mathbf{M}_A$ are unique.

16.24 ⋆ The components of $\mathbf{u} = \boldsymbol{\theta} \times \mathbf{r}$ are

$$u_1 = \theta_2 r_3 - \theta_3 r_2, \quad u_2 = \theta_3 r_1 - \theta_1 r_3, \quad \text{and} \quad u_3 = \theta_1 r_2 - \theta_2 r_1.$$

The derivatives matrix defined in (16.76) is found by differentiating the components of $\mathbf{u}$ with respect to r_1, r_2, and r_3 with the exact result claimed in Eq.(16.78)

16.25 ⋆⋆⋆ **(a)** Because there is no rotation about P, the strain tensor $\mathbf{E} = \mathbf{D}$, the matrix made up of derivatives $D_{ij} = \partial u_i/\partial r_j$. Now consider a point Q originally located a small displacement $(dx, 0, 0)$ from P. The deformation represented by $\mathbf{E}$ moves Q away from P by an amount $du_1 = (\partial u_1/\partial r_1)dx = \epsilon_{11}dx$. (It may also cause Q to shear in the y and z directions, but that won't affect volumes.) Thus the length that was dx in the x direction has stretched to $dx(1 + \epsilon_{11})$, increasing by a factor $(1 + \epsilon_{11})$.

(b) Similarly, lengths in the y and z directions are stretched by factors $(1 + \epsilon_{22})$ and $(1 + \epsilon_{33})$ respectively. Therefore any small volume around P will increase by a factor

$$(1 + \epsilon_{11})(1 + \epsilon_{22})(1 + \epsilon_{33}) \approx 1 + \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 1 + \operatorname{tr}\mathbf{E},$$

where I have used the smallness of the elements of $\mathbf{E}$ to drop second order terms and higher. Thus a volume that was originally V will increase by $dV = V\operatorname{tr}\mathbf{E}$, as claimed.

An alternative and very slick proof uses the divergence theorem:

$$dV = \int_S \mathbf{u}\cdot d\mathbf{A} = \int_V \boldsymbol{\nabla}\cdot\mathbf{u}\, dV \approx V\boldsymbol{\nabla}\cdot\mathbf{u} = V\left(\frac{\partial u_1}{\partial r_1} + \frac{\partial u_2}{\partial r_2} + \frac{\partial u_3}{\partial r_3}\right) = V\operatorname{tr}\mathbf{E}$$

where the approximation depends on the smallness of V.

16.26 ⋆ According to (16.100), $\text{YM} = 9\,\text{BM}\cdot\text{SM}/(3\text{BM} + \text{SM})$. Using the given values of BM and SM we find (in GPa)

	YM (predicted)	YM (observed)	percent discrepancy
iron	104.5	100	5.0%
steel	201.6	200	0.8%
sandstone	16.1	16	0.7%
perovskite	379.7	390	2.6%
water	0	0	NA

16.27 ⋆⋆⋆ **(a)** The stress tensor $\mathbf{\Sigma}$ is as shown on the left below. From this we can calculate $\operatorname{tr}\mathbf{\Sigma} = \sigma_{11}$ and the strain tensor $\mathbf{E}$ (16.95) as shown on the right:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{E} = \frac{1}{3\alpha\beta}\Big[3\alpha\mathbf{\Sigma} + (\beta-\alpha)(\operatorname{tr}\mathbf{\Sigma})\,\mathbf{1}\Big] = \frac{\sigma_{11}}{3\alpha\beta}\begin{bmatrix} 2\alpha+\beta & 0 & 0 \\ 0 & \beta-\alpha & 0 \\ 0 & 0 & \beta-\alpha \end{bmatrix}.$$

(b) In the definition (16.55) of YM, the left side is $dF/A = \sigma_{11}$, and the factor dl/l on the right is ϵ_{11}. Therefore, $\text{YM} = \sigma_{11}/\epsilon_{11}$.

(c) Reading off ϵ_{11} from part (a), we find

$$\text{YM} = \frac{\sigma_{11}}{\epsilon_{11}} = \frac{3\alpha\beta}{2\alpha+\beta} = \frac{9\,\text{BM}\cdot\text{SM}}{3\,\text{BM}+\text{SM}}$$

because, according to (16.97) and (16.99), $\alpha = 3\text{BM}$ and $\beta = 2\text{SM}$.

16.28 ⋆⋆⋆ **(a)** By its definition, $\nu = (-\delta l_y/l_y)/(\delta l_x/l_x) = -\epsilon_{22}/\epsilon_{11}$.

(b) The strain tensor $\mathbf{E}$ is given in the solution to Problem 16.27, and from that we can read off

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = \frac{\alpha-\beta}{2\alpha-\beta} = \frac{3\text{BM}-2\text{SM}}{6\text{BM}+2\text{SM}}$$

Notice that ν is always less than 0.5 (or equal if $\text{SM} = 0$).

(c) Using the data from Problem 16.26, we find

material	Poisson's ratio
iron	0.31
steel	0.26
sandstone	0.34
perovskite	0.27
water	0.50

If $\text{SM} \ll \text{BM}$ (or, as with water, $\text{SM} = 0$), we see from part (b) that ν should be close to 0.5.

16.29 ⋆⋆⋆ $\mathrm{tr}(\mathbf{E}_R) = \mathrm{tr}(\mathbf{R}\mathbf{E}\tilde{\mathbf{R}}) = \sum_i (\mathbf{R}\mathbf{E}\tilde{\mathbf{R}})_{ii} = \sum_{i,j,k} R_{ij}E_{jk}\tilde{R}_{ki} = \sum_{i,j,k} \tilde{R}_{ki}R_{ij}E_{jk}$

$$= \sum_k (\tilde{\mathbf{R}}\mathbf{R}\mathbf{E})_{kk} = \sum_k E_{kk} = \mathrm{tr}(\mathbf{E}). \qquad \text{(vii)}$$

Suppose that we start in the reference frame $\mathcal{S}$, and decompose $\mathbf{E}$ as $\mathbf{E} = e\mathbf{1} + \mathbf{E}'$, where, by definition $e = \frac{1}{3}\mathrm{tr}(\mathbf{E})$. If, now, we make a rotation $\mathbf{R}$ to a new frame, $\mathcal{S}_R$ say, then we find for the new (rotated) strain tensor

$$\mathbf{E}_R = \mathbf{R}\mathbf{E}\tilde{\mathbf{R}} = \mathbf{R}(e\mathbf{1} + \mathbf{E}')\tilde{R} = e\mathbf{1} + \mathbf{R}\mathbf{E}'\tilde{\mathbf{R}}. \qquad \text{(viii)}$$

because $\mathbf{R}\tilde{\mathbf{R}} = \mathbf{1}$. On the other hand, if we start in the frame $\mathcal{S}_R$ and decompose the rotated matrix $\mathbf{E}_R$, we will get

$$\mathbf{E}_R = e_R\mathbf{1} + (\mathbf{E}_R)' \qquad \text{(ix)}$$

where by definition $e_R = \mathrm{tr}(\mathbf{E}_R)$. Now by the result (vii), $e_R = e$. Therefore the first terms on the right sides of Eqs.(viii) and (ix) are the same. It follows that the second terms are also equal, and we've proved that (1) the spherical parts of $\mathbf{E}_R$ and $\mathbf{E}$ are the same and (2) the deviatoric part of $\mathbf{E}_R$ is the result of rotating the deviatoric part of $\mathbf{E}$; that is $(\mathbf{E}_R)' = \mathbf{R}\mathbf{E}'\tilde{\mathbf{R}}$.

16.30 ⋆ Let us fix the value of i ($i = 1$ or 2 or 3) and consider the sum $\sum_{i=1}^{3} \delta_{ji}a_j$. This sum contains three terms, of which the two with $j \neq i$ are zero. This leaves just the term with $j = i$, so $\sum_{i=1}^{3} \delta_{ji}a_j = \delta_{ii}a_i = a_i$, which is the requested result.

16.31 ⋆ The times of travel of the two waves are $t_{\text{long}} = d/c_{\text{long}}$ and $t_{\text{tran}} = d/c_{\text{tran}}$, where d is the distance from the quake to the observer. Therefore the time between the arrivals of the two signals is $\Delta t = t_{\text{tran}} - t_{\text{long}} = d(1/c_{\text{tran}} - 1/c_{\text{long}})$, and

$$d = \frac{\Delta t}{1/c_{\text{tran}} - 1/c_{\text{long}}} = \frac{720\ \text{s}}{(1/3.0 - 1/5.25)\ \text{s/km}} = 5040\ \text{km}.$$

16.32 ⋆ Using Eqs. (16.119) and (16.120) one finds the following speeds in km/s:

	c_{long}	c_{tran}
iron	4.3	2.3
steel	5.6	3.2
sandstone	3.6	1.8
perovskite	10.7	6.0
water	1.5	0.0

16.33 ⋆ With $\mathbf{v} = 0$, Eq.(16.124), $\varrho d\mathbf{v}/dt = \varrho\mathbf{g} - \boldsymbol{\nabla} p$ reduces to $\varrho\mathbf{g} = \boldsymbol{\nabla} p$. Integrating both sides gives $\int \varrho\mathbf{g}\cdot d\mathbf{r} = \int \boldsymbol{\nabla} p\cdot d\mathbf{r}$ or $\varrho g \int dz = \int dp$ if we measure z vertically down. Therefore $\varrho g h = \Delta p$.

16.34 ⋆⋆ If we substitute $d/dt = \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla}$ into Eq.(16.129), we find

$$0 = \frac{d\varrho}{dt} + \varrho\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial\varrho}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\varrho + \varrho\boldsymbol{\nabla}\cdot\mathbf{v}.$$

Next note the identity $\boldsymbol{\nabla}\cdot(\varrho\mathbf{v}) = \mathbf{v}\cdot\boldsymbol{\nabla}\varrho + \varrho\boldsymbol{\nabla}\cdot\mathbf{v}$, which can be substituted into the right side of the above to give

$$0 = \frac{\partial\varrho}{\partial t} + \boldsymbol{\nabla}\cdot(\varrho\mathbf{v})$$

which is (16.130).

16.35 ⋆ If we make the substitution $\xi = \mathbf{n}\cdot\mathbf{r} - ct$, with $\mathbf{n}, \mathbf{r}$, and c all fixed, then $d\xi = -c\,dt$ and

$$\int f'(\mathbf{n}\cdot\mathbf{r} - ct)dt = \frac{-1}{c}\int f'(\xi)d\xi = \frac{-1}{c}f(\xi) = \frac{-1}{c}f(\mathbf{n}\cdot\mathbf{r} - ct.)$$

16.36 ⋆⋆ **(a)** The bulk modulus is defined by (16.56) so that $dp = -\text{BM}\cdot dV/V$. In an adiabatic change, $pV^\gamma = k$ (a constant), so $dp = -\gamma k V^{-(\gamma+1)}dV$. Comparing these two results, we see that $\text{BM} = \gamma k V^{-\gamma} = \gamma p$.

(b) The ideal gas law states that $pV = nRT$, where n is the number of moles of gas, $n = m/M$ where m is the mass of gas and M is the "molecular mass", the mass per mole. Thus $pV = mRT/M$ and the mean density of the gas is $\varrho_o = m/V = pM/RT$.

(c) From Eq.(16.140),

$$c = \sqrt{\frac{\text{BM}}{\varrho_o}} = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{1.4 \times (8.31\text{ J/mole·K}) \times (273\text{ K})}{29 \times 10^{-3}\text{ kg/mole}}} = 330.9\text{ m/s}.$$

This compares very favorably with the accepted value of 331 m/s.

16.37 ⋆⋆ The force exerted by a sliver of air of area dA on the air just ahead of it is $\mathbf{F} = p\mathbf{n}dA = (p_o+p')\mathbf{n}dA$ where $\mathbf{n}$ is the normal to the sliver, in the direction of propagation. The rate at which this force does work is $\mathbf{F}\cdot\mathbf{v}$ and the intensity I is found by dividing this by the area, $I = \mathbf{F}\cdot\mathbf{v}/dA = (p_o + p')\mathbf{n}\cdot\mathbf{v}$. According to (16.142), $\mathbf{v} = p'\mathbf{n}/c\varrho_o$. Therefore $I = (p_o + p')p'/c\varrho_o$. This gives I as the sum of two terms. When p' varies sinusoidally, the first term averages to zero, and we're left with $\langle I\rangle = \langle p'^2\rangle/c\varrho_o$, as claimed.

16.38 ⋆⋆⋆ **(a)** We need to show that the term $\varrho'\mathbf{g}$ on the right of the equation of motion (16.134)

$$\varrho_o \frac{\partial \mathbf{v}}{\partial t} = \varrho' \mathbf{g} - \nabla p'$$

is negligible compared to the second term $\nabla p'$. If λ is the distance over which p' changes appreciably, then $|\nabla p'| \sim p'/\lambda \sim \text{BM}\,\varrho'/\varrho_o\lambda$, where the symbol $\sim$ is to be read as "is of order" and in passing from the second to the third expression I used Eq.(16.139). Accordingly,

$$\left|\frac{\text{first term}}{\text{second term}}\right| \sim \frac{\varrho' g}{\text{BM}\,\varrho'/\varrho_o\lambda} = \frac{g\varrho_o\lambda}{\text{BM}} = \frac{(10\text{ m/s}^2)\times(10^3\text{ kg/m}^3)\times(1\text{ m})}{2\times 10^9\text{ Pa}} = \frac{1}{2}\times 10^{-5}$$

if we take $\lambda \sim 1$ m, as suggested. Evidently the first term can safely be neglected. If the medium were air instead of water, the density ϱ_o in the numerator would be about 10^3 times smaller, and the bulk modulus BM in the denominator would be about 10^4 times smaller, so the ratio would still be very small — of order 10^{-4}.

(b) In the equation of continuity (16.136),

$$\frac{\partial \varrho'}{\partial t} = -\varrho_o \nabla\cdot\mathbf{v} - \mathbf{v}\cdot\nabla\varrho_o\,,$$

we wish to show that the second term is negligible compared to the first. Using (16.138) and then (16.133) we can rewrite the second term as

$$\mathbf{v}\cdot\nabla\varrho_o = \mathbf{v}\cdot\nabla p_o\frac{\varrho_o}{\text{BM}} = \mathbf{v}\cdot\varrho_o\mathbf{g}\frac{\varrho_o}{\text{BM}} \sim vg\varrho_o^2/\text{BM}$$

while the first term is $\varrho_o\nabla\cdot\mathbf{v} \sim \varrho_o v/\lambda$. Thus their ratio is

$$\left|\frac{\text{second term}}{\text{first term}}\right| \sim \frac{vg\varrho_o^2/\text{BM}}{\varrho_o v/\lambda} = \frac{g\varrho_o\lambda}{\text{BM}} = \frac{1}{2}\times 10^{-5}$$

as before.