Eric Du | | EE 120
HW 03 | Signals and Systems | February 12, 2024

Header styling inspired by CS 70: https://www.eecs70.org/

Collaborators

I worked with the following people to complete this assignment:

- Teja Nivarthi:
- Nikhil Maserang:

Problem 1

Here is a list of statements about systems. Determine if each of them is true or false with your justification.

a) If an LTI system is causal, it's stable.

Solution: False. A system being causal – the fact that it only depends on past inputs, has nothing to do with whether it is bounded or not. ☐

---

b) If a discrete time LTI system has an impulse response $h[n]$ of finite duration, the system is stable.

Solution: Based on Ed, we are allowed to assume that $h[n]$ also has finite amplitude. Then, we know that the signal response for an LTI system is characterized by the convolution of the impulse response with $x[n]$:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

If $x[n]$ is bounded and $h[n]$ is bounded, then we know that the system is stable. Alternatively, one could argue this by using the fact that $h[n]$ is absolutely integrable, so it is stable. ☐

---

c) If $h(t)$ is the impulse response of an LTI system and $h(t)$ is periodic and non-zero, the system is unstable.

Solution: True. If $h(t)$ is nonzero and nonperiodic, then it is not absolutely integrable. Hence, it cannot be stable. ☐

---

d) If $|h[n]| \leq K$ for each n, where K is a given number, then the LTI system with $h[n]$ as its impulse response is stable.

Solution: False. If $h[n]$ does not have finite support, then an LTI system with $h[n]$ as its impulse response is still not absolutely integrable, so the system is not stable. ☐

---

e) A discrete-time LTI system is causal if and only if its step response $s[n]$ is zero for $n < 0$.

Solution: True. An LTI system is causal if and only if its impulse response is a causal function (i.e. it's value is 0 for all $t < 0$), which is true in this case. ☐

Problem 2

Here are some impulse responses $h[n]$ and $h(t)$ of several discrete-time and continuous-time LTI systems, where $u[n]$ and $u(t)$ are the discrete-time and continuous-time unit step functions, respectively. Determine wheter each system is 1) stable or not, 2) causal or not. Show your justification.

a) $h[n] = \alpha^n u[n]$ for $|\alpha| < 1$

Solution: The function $h[n]$ is absolutely summable (it's a geometric series over α, and since $|\alpha| < 1$ then the series converges), so it is stable. It is also causal, since the impulse response is causal. $\square$

---

b) $h[n] = \beta^n u[3-n]$ for $|\beta| > 1$

Solution: This system is also absolutely summable – it's the same geometric series as in (a) except it's support is over the interval $(-\infty, 3]$. Therefore, it is stable. It is not causal, however since $h[n] \neq 0$ for $n < 0$. $\square$

---

c) $h(t) = e^{-4t|t|}$

Solution: This function is in fact absolutely integrable: for $t > 0$, the function $h(t) = e^{-4t^2}$, which converges over the interval $[0, \infty)$. For $t < 0$, then we have $|t| = -t$, so therefore $h(t) = e^{4t^2}$, which is also absolutely integrable over the interval $(-\infty, 0]$. Therefore, since $h(t)$ is absolutely integrable over the whole space $(-\infty, \infty)$, then we know that $h(t)$ is absolutely integrable, hence it is BIBO stable.

The system is not causal however, since $h(t)$ is nonzero for $t < 0$. $\square$

---

d) $h(t) = te^{-t}u(t)$

Solution: For $t > 0$, this function is $h(t) = te^{-t}$, and computing the integral on $[0, \infty)$:

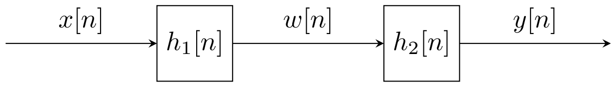
$$\int_0^\infty |h(t)|dt = \int_0^\infty te^{-t}dt = 1$$

(I used mathematica because I'm lazy.) $h(t) = 0$ for $t < 0$, so $h(t)$ is absolutely summable, hence it is BIBO stable.

The system is also causal because for $t < 0$ we have $h(t) < 0$. $\square$

Problem 3

Consider the cascade of two LTI systems:



where we have

$$h_1[n] = \sin[8n]$$

and

$$h_2[n] = a^n u[n] \;\; |a| < 1$$

and the input is

$$x[n] = \delta[n] - a\delta[n-1]$$

Determine the output $y[n]$ using convolution properties. *Hint:* Use the commutative property of convolution.

Solution: We know that for two LTI systems in series, then the signal output $y[n]$ is defined as

$$y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$

where we've used both associativity and commutativity. Now we just carry out the algebra:

$$\begin{aligned}
y[n] &= \left(\sum_{k=-\infty}^{\infty} x[n-k] h_2[k] \right) * h_1[n] \\
&= \left(\sum_{k=-\infty}^{\infty} (\delta[n-k] - a\delta[n-k-1]) a^k u[k] \right) * h_1[n] \\
&= \underbrace{[a^n u[n] - a^n u[n-1]]}_{a^n \delta[n]} * h_1[n] \\
&= \sum_{k=-\infty}^{\infty} a^{n-k} \delta[n-k] \sin[8n] \\
&= \sin[8n]
\end{aligned}$$

so we have $y[n] = h_1[n] = \sin[8n]$, what a wonderful coincidence. $\square$

Problem 4

This problem will explore the connection between LCCDEs and the frequency response. The LCCDE for system F is

$$3y[n] - 4y[n-1] - 5y[n-2] = 6x[n] + 7x[n-1] \ \forall n \in \mathbb{Z}$$

a) Using the eigenfunction property, find the frequency response $F(\omega)$.

Solution: The Eigenfunction property says that given a signal $x[n] = Ae^{i\omega n}$, then the output signal $y[n] = F(\omega)x[n]$. Using this property, we can write:

$$3F(\omega)Ae^{i\omega n} - 4F(\omega)Ae^{i\omega(n-1)} - 5F(\omega)Ae^{i\omega(n-2)} = 6Ae^{i\omega n} + 7Ae^{i\omega(n-1)}$$
$$F(\omega)\left(4e^{i\omega n} - 5e^{i\omega(n-1)} - 5e^{i\omega(n-2)}\right) = e^{i\omega(n-1)}(6e^{i\omega} + 7)$$

Isolating for $F(\omega)$ gives:

$$F(\omega) = \frac{e^{i\omega(n-1)(6e^{i\omega}+7)}}{e^{i\omega(n-2)(3e^{2i\omega}-4e^{i\omega}-5}} = \frac{e^{i\omega}(6e^{i\omega}+7)}{3e^{2i\omega} - 4e^{i\omega} - 5}$$

$\square$

---

b) Now given a general LCCDE in this form:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_k y[n-k] = b_0 x[n] + b_1 x[n-1] + \cdots + b_l x[n-l]$$

Find the frequency response $F(\omega)$.

Solution: Following the pattern derived in the previous part, we then know that:

$$F(\omega) = \frac{\sum_{m=0}^{l} b_m e^{-i\omega m}}{\sum_{m=0}^{k} a_m e^{-i\omega m}}$$

$\square$

---

c) Based on your results from above, find the LCCDE given the frequency response for a new LTI system H.

$$H(\omega) = \frac{3 + 8e^{3i\omega} - e^{5i\omega}}{1 + e^{i\omega}}$$

Solution: Based on this, we can conclude that $b_i = \{3, 0, 0, 8, 0, -1\}$, and $a_i = \{1, 1\}$. Therefore:

$$y[n] + y[n-1] = 3x[n] + 8x[n-3] - x[n-5]$$

$\square$

Problem 5

The following input signal is applied to a real, continuous-time LTI system:

$$x : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R}, \quad x(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

For each LTI system described below, determine the output signal y in response to the input signal x; do this by providing a well-labeled plot of y as a function of t or finding an expression for y in terms of x.

a) The impulse response h of the LTI system is

$$h : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R} \quad h(t) = \begin{cases} 1 & \text{if } 1 \leq t \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution: We use the formula for the convolution:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau)\, d\tau$$

We know that $x(t)$ is only nonzero on the interval $[-\frac{1}{2}, \frac{1}{2}]$, so we restrict τ to this interval. Further, since $x(t)$ is constant, we can use that to siplify the expression too:

$$y(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t - \tau)\, d\tau$$

Consider a particular t. Then, the integral sweeps over a region of $t - \frac{1}{2}$ to $t + \frac{1}{2}$. Since $h(t)$ is nonzero on $[1, 2]$, this means that for any $t < \frac{1}{2}$ or $t > \frac{5}{2}$, we have $y(t) = 0$. Then, for $\frac{1}{2} < t < \frac{3}{2}$, then our integral goes from 1 to $t + \frac{1}{2}$:

$$y(t) = \int_{1}^{t+\frac{1}{2}} h(\tau)\, d\tau = t - \frac{1}{2}$$

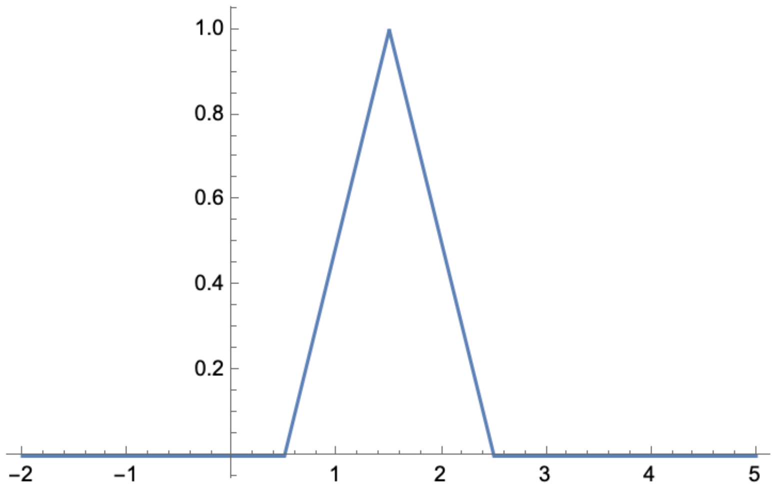
Then, for $\frac{3}{2} < t < \frac{5}{2}$, the integral goes from $t - \frac{1}{2}$ to 2 :

$$y(t) = \int_{t-\frac{1}{2}}^{2} h(\tau)\, d\tau = -t + \frac{5}{2}$$

So in essence, $y(t)$ is a triangle function. Written out explicitly, we have:

$$y(t) = \begin{cases} 0 & t < \frac{1}{2} \text{ or } t > \frac{5}{2} \\ t - \frac{1}{2} & \frac{1}{2} < t < \frac{3}{2} \\ \frac{5}{2} - t & \frac{3}{2} < t < \frac{5}{2} \end{cases}$$

Plotting:



□

---

b) the impulse response h of the LTI system is

$$h : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R}, \quad h(t) = \begin{cases} 1 & \text{if } 1 \leq t \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

Solution: This is very similar to the previous problem esxcept our limits are different. For $t < \frac{1}{2}$ and $t > \frac{7}{2}$, we have $y(t) = 0$. For $\frac{1}{2} < t < \frac{3}{2}$, the integral is the same, over the interval 1 to $t + \frac{1}{2}$

$$y(t) = \int_1^{t+\frac{1}{2}} h(\tau)\, d\tau = t - \frac{1}{2}$$

For $\frac{3}{2} < t < \frac{5}{2}$, $h(t)$ is constant on this interval, so the value over this interval is just the area under a rectangle of width 1 and height 1:

$$y(t) = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} h(\tau)\, d\tau = 1$$

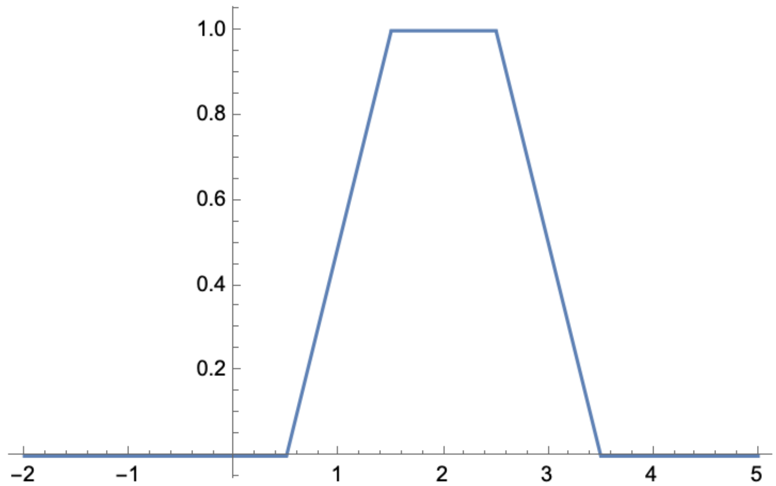
For $\frac{5}{2} < t < \frac{7}{2}$, our integral goes from $t - \frac{1}{2}$ to 3, so therefore:

$$y(t) = \int_{t-\frac{1}{2}}^{3} h(\tau)\, d\tau = \frac{7}{2} - t$$

Therefore, we have:

$$y(t) = \begin{cases} 0 & t < \frac{1}{2} \text{ or } t > \frac{7}{2} \\ t - \frac{1}{2} & \frac{1}{2} < t < \frac{3}{2} \\ 1 & \frac{3}{2} < t < \frac{5}{2} \\ \frac{7}{2} - t & \frac{5}{2} < t < \frac{7}{2} \end{cases}$$

Plotting:



□

---

c) The impulse response h of the LTI system is

$$h : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R}, \quad h(t) = \delta(t) - \delta\left(t - \frac{1}{2}\right)$$

where δ is the Dirac delta function.

Solution: We know that convolutions are distributive:

$$y(t) = x(t) * h(t) = x(t) * \left[\delta(t) - \delta\left(t - \frac{1}{2}\right)\right] = x(t) * \delta(t) - x(t) * \delta\left(t - \frac{1}{2}\right)$$

Then, we use the identity that $x(t) * \delta(t - T) = x(t - T)$. Therefore, we have:

$$y(t) = x(t) - x\left(t - \frac{1}{2}\right)$$

□

---

d) The impulse response h of the LTI system is

$$h : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R} \quad h(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

where $1 < T$.

Solution: This is a generalization of the previous part,so we have:

$$y(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x(t - kT)$$

Since $T > 1$, this function basically is a series of square pulses separated T apart from one another. Since we're not given a value of T, it's impossible to plot it. □

Problem 6

Consider the signal

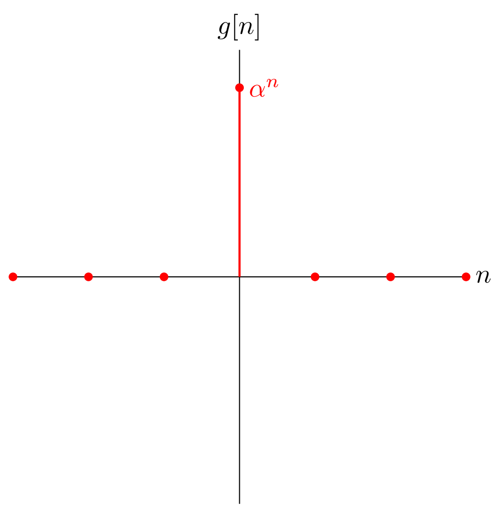
$$x[n] = \alpha^n u[n]$$

a) Sketch the signal $g[n] = x[n] - \alpha x[n-1]$. Write $g[n]$ as a convolution related to $x[n]$.

Solution: Firstly, to plot $g[n]$, we simplify the expression for g :

$$\begin{aligned} g[n] &= x[n] - \alpha x[n-1] \\ &= \alpha^n u[n] - \alpha^n u[n-1] \\ &= \alpha^n (u[n] - u[n-1]) \\ &= \alpha^n \delta[n] \end{aligned}$$

So this is basically a delta function with height α^n at $n = 0$, and zero everywhere else. Plotting this:



Now, we write $g[n]$ as a convolution related to $x[n]$. To do this, we go back to the original form of $g[n] = x[n] - \alpha x[n-1]$. Becuase we want to write $g[n] = x[n] * h[n]$ for some $h[n]$, it helps to think of $h[n]$ as a sum of two functions, so that we may write $g[n] = x * (h_1 + h_2) = x * h_1 + x * h_2$.

Thus, we want $x[n] * h_1[n] = x[n]$, which is easily done by letting $h_1[n] = \delta[n]$. Then, we want $x[n] * h_2[n] = \alpha x[n-1]$, so we can let $h_2[n] = \alpha\delta[n-1]$. Here we've used the identity that $x[n] * \delta[n-N] = x[n-N]$. Therefore, we have:

$$g[n] = x[n] * (\delta[n] - \alpha\delta[n-1])$$

□

---

b) Use the result of part (a) along with the properties of convolution to determine $h[n]$ such that

$$x[n] * h[n] = \left(\frac{1}{2}\right)^n (u[n+2] - u[n-2])$$

Hint: Use the fact that $x[n] * \delta[n] = x[n]$.

Solution: In the previous part, notice we wrote $g[n] = \alpha^n(u[n] - u[n-1])$, and we could also write $g[n] = x[n] * (\delta[n] - \alpha\delta[n-1])$. In this problem, we also have a difference of two step functions, so the logic is actually quite similar. Let's first define a new function:

$$g'[n] = \alpha^{-2}x[n+2] - \alpha^2 x[n-2] = \alpha^n(u[n+2] - u[n-2])$$

Then, using the same logic as the previous part, we notice that we can write:

$$g'[n] = x[n] * (\alpha^{-2}\delta[n+2] - \alpha^2\delta[n-2]) = \alpha^n(u[n+2] - u[n-2])$$

Now, we can just choose $\alpha = \frac{1}{2}$ to finish the problem. Therefore, we identify:

$$h[n] = \frac{1}{4}\delta[n+2] - 4\delta[n+2]$$

$\square$

Problem 7

Consider two signals,

$$x(t) = \begin{cases} 0 & t < 0 \text{ and } t > 3 \\ 2 & 0 \leq t \leq 3 \end{cases}$$

and

$$g(t) = e^{-t/2}u(t)$$

a) Compute the convoluiton of $x(t)$ and $g(t)$, and sketch your result. Denote the convolution result as $y(t)$.

Solution: The convolution of $x(t)$ and $g(t)$ is given by:

$$x(t) * g(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau)\,d\tau$$

Due to the fact that $x(t) \neq 0$ only on the interval $[0,3]$, then we can restrict our domain to that region too. Further, since $x(t)$ is constant on this interval, then:

$$x(t) * g(t) = 2\int_{0}^{3} g(t-\tau)\,d\tau$$

Since $g(t)$ has a step function, then we know that for $t < 0$ we have $y(t) = 0$. For $0 \leq t \leq 3$, we see that since the integral sweeps between the interval $[t, t-3]$, then we know that this integral is equivalent to:

$$y(t) = 2\int_{0}^{t} g(\tau)\,d\tau = 2\int_{0}^{t} e^{-\tau/2}\,d\tau = 4(1-e^{-t/2})$$

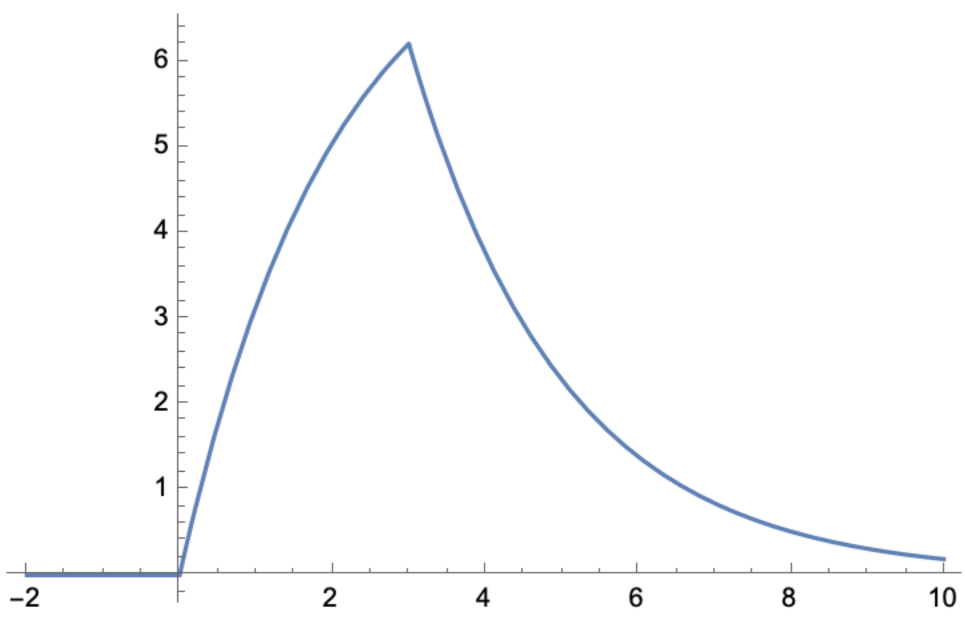
For $t > 3$, then our integral just sweeps over the entire interval $[t, t-3]$, so we have;

$$y(t) = 8e^{-t/2}(e^{3/2}-1)$$

Therefore, written altogether:

$$y(t) = \begin{cases} 0 & t < 0 \\ 4(1-e^{-t/2}) & 0 \leq t \leq 3 \\ 4e^{-t/2}(e^{3/2}-1) & t > 3 \end{cases}$$

As a sketch, I did this in Mathematica:



The plot should also make sense intuitively: for $t > 3$, we are taking the convolution across a decaying exponential, so therefore the resulting $y(t)$ will look that way. For $t < 0$ the convolution is zero because there is no overlap. For $0 < t < 3$, the area is growing because as we increase t, more area between the two functions overlap, up to a maximum at $t = 3$. □

---

b) Compute the cross correlation of $x(t)$ and $g(t)$, $R_{x,g}(t) = x(t) \circ g(t)$, then sketch your result.

Solution: The definition of the cross correlation formula is

$$x(t) \circ g(t) = \int_{-\infty}^{\infty} x(\tau)g(t+\tau)\,d\tau$$

We will apply a very similar trick to the previous portion: since $x(\tau)$ is nonzero on $[0, 3]$, then we can restrict the domain, and use the fact that $x(\tau)$ is constant on this interval:

$$R_{x,g}(t) = 2\int_0^3 g(t+\tau)\,d\tau$$

Now, the integral sweeps over the interval $[t, t+3]$. Therefore, for $t < -3$, then we have $R_{x,g}(t) = 0$. For $-3 \le t \le 0$, then the integral goes from $[0, t+3]$, so the result is:

$$R_{x,g}(t) = 2\int_0^{t+3} g(\tau)\,d\tau = 4(1 - e^{-(t+3)/2})$$

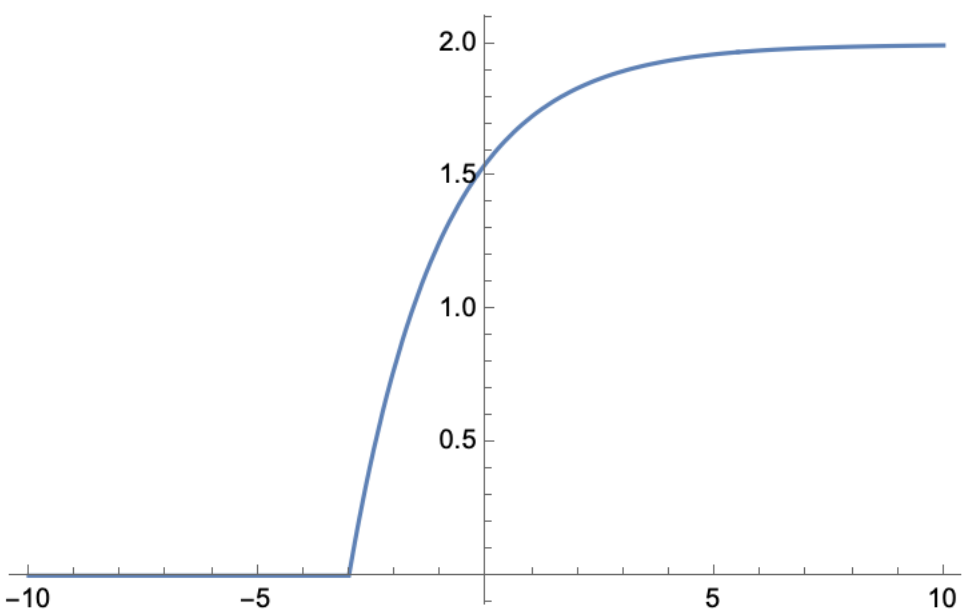
Then, for $t > 0$, we have:

$$R_{x,g}(t) = 2\int_t^{t+3} g(\tau)\,d\tau = 4e^{-(t+3)/2}(e^{3/2} - 1)$$

Therefore, we can write:

$$R_{x,g}(t) = \begin{cases} 0 & t < -3 \\ 4(1 - e^{-(t+3)/2}) & -3 \le t \le 0 \\ 4e^{-(t+3)/2}(e^{3/2} - 1) & t > 0 \end{cases}$$

As for the sketch:



□

---

c) Compute the cross correlation of $g(t)$ and $x(t)$, $R_{g,x}(t) = g(t) \circ x(t)$, then sketch your result.

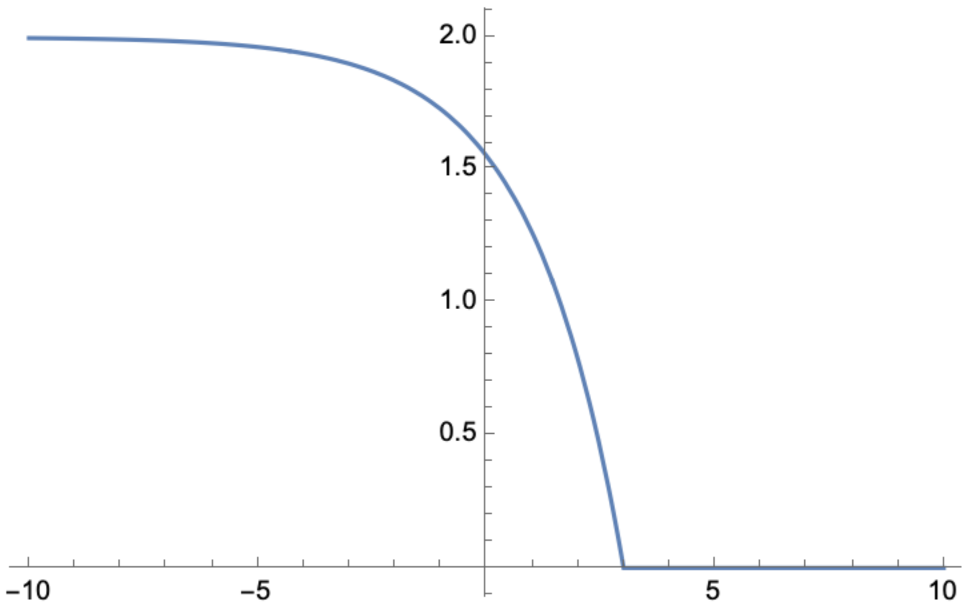
Solution: Instead of computing the entire thing again, we make use of the identity that

$$x(t) \circ g(t) = g(t) * x(-t)$$

So this means that $R_{g,x}(t) = R_{x,g}(-t)$. Therefore, making the substitution $t \to -t$ in the formula for $R_{x,g}(t)$ above, we get:

$$R_{g,x}(t) = \begin{cases} 0 & t > 3 \\ 4(1 - e^{-(3-t)/2}) & 0 \le t \le 3 \\ 4e^{-(3-t)/2}(e^{3/2} - 1) & t < 0 \end{cases}$$

Plotting:



□

d) Briefly describe how convolution and cross correlation of two signals are related but very different.

Solution: Convolution and correlation are the same in the sense that they both calculate some relationship between two functions f and g by taking pairwise products across the function. Convolution flips the second function g and slides it across f, whereas cross correlation doesn't do any flipping and just slides g across f and computes the product. □