j

Problem 1

Prove the following proposition:

Proposition 1. *A set $A \subset \mathbb{R}$ is bounded if and only if there exists a real number $M \geq 0$ such that*

$$|x| \leq M \text{ for every } x \in A$$

Solution: We first show that if M exists, then A is bounded. By the definition of supremum and infimum, we know that the following two relations hold:

$$\sup A \leq M \quad \inf A \geq -M$$

Therefore, since the supremum and infimum are finite, then A is bounded.

Now we show that if A is bounded then M exists. We know that if A is bounded, then $\sup A$ and $\inf A$ exist. We can choose M to be:

$$M = |\sup A| + |\inf A|$$

Since all $a \leq \sup A$, then $a \leq |\sup A| + |\inf A|$. On the contrary, since $a \geq \inf A$, then $a \geq -|\inf A| - |\sup A|$. Therefore, a finite M exists. □

Problem 2

Consider each of the following sets:

$$A = (0, \infty), \qquad B = \{\frac{1}{m} + \frac{1}{n} : m, n, \in \mathbb{N}\}, \qquad C = \{x^2 - x - 1 : x \in \mathbb{R}\}$$

$$D = [0, 1] \cup [2, 3], \qquad E = \cup_{n=1}^{\infty}[2n, 2n + 1], \qquad F = \cap_{n=1}^{\infty}(1 - \frac{1}{n})(1 + \frac{1}{n})$$

For each set, determine its minimum and maximum if they exist. In addition, determine each set's infimum and supremum, writing your answers in terms of infinity for unbounded sets. Detailed proofs are not required.

Solution:

(a) neither max nor min exists, $\inf A = 0$ and $\sup A = \infty$.

(b) $\max B = 2$, $\min B$ does not exist, $\sup B = 2$ and $\inf B = 0$.

(c) This is a parabola, so $\min C = -5/4$, $\max C$ does not exist, $\inf C = -5/4$ and $\sup C = \infty$.

(d) This is an interval including the endpoints, so $\min D = \inf D = 0$ and $\max D = \sup D = 3$.

(e) The smallest element here is $2(1) = 2$, so therefore $\min E = \inf E = 2$, but $\max E$ does not exist and $\sup E = \infty$.

(f) Here $\min F$ and $\max F$ both do not exist, while $\inf F = \sup F = 1$.

□

Problem 3

Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A+B$ be the set of all sums $a+b$ where $a \in A$ and $b \in B$.

(a) Prove $\sup(A+B) = \sup A + \sup B$. *Hint:* To show $\sup A + \sup B \leq \sup(A+B)$, show that for each $b \in B$, $\sup(A+B) - b$ is an upper bound for A, hence $\sup A \leq (\sup(A+B)) - b$. Then, show $\sup(A+B) - \sup A$ is an upper bound for B.

Solution: We know that for all $a \in A$, $a \leq \sup A$. Likewise, we know that for all $b \in B$, $b \leq \sup B$. Therefore, for any sum $a+b$, we have

$$a + b \leq \sup A + \sup B \tag{1}$$

Since this equation is true for all a, b, then we can also write:

$$a + b \leq \sup(A+B) \leq \sup A + \sup B \tag{2}$$

Note we can insert $\sup(A+B)$ in between since we define $\sup(A+B)$ to be the least upper bound, and we assume that $\sup A + \sup B$ might be larger than $\sup(A+B)$.

We now use the hint: we first show that $\sup(A+B) - b$ is an upper bound for A. We can do this because from equation 1, we get

$$a \leq \sup(A+B) - b$$

And so therefore $\sup A \leq \sup(A+B) - b$. Then rearranging this last equation, we get that for all $b \in B$,

$$b \leq \sup(A+B) - \sup A$$

Then since this is true for all B, then we can write

$$\sup A + \sup B \leq \sup(A+B)$$

But then combining this with equation 2, we get the inequality

$$\sup A + \sup B \leq \sup(A+B) \leq \sup A + \sup B$$

And so therefore $\sup(A+B) = \sup A + \sup B$. $\square$

(b) Prove $\inf(A+B) = \inf A + \inf B$.

Solution: We know from the previous homework that $\inf A = -\sup(-A)$ so we know then that $\inf(A+B) = -\sup(-A + (-B)) = -\sup(-A) - \sup(-B) = \inf A + \inf B$ $\square$

Problem 4

Given a sequence (x_n), prove: $\lim_{n\to\infty} x_n = x$ is equivalent to $\lim_{n\to\infty} |x_n - x| = 0$.

Solution: The second equation $\lim_{n\to 0} |x_n - x| = 0$ is the same thing as saying that for every $\epsilon > 0$, we can write:

$$||x_n - x| - 0| < \epsilon$$

And since $|x_n - x| \geq 0$, we can drop the outer set of absolute values. Therefore, this leaves us with

$$|x_n - x| < \epsilon$$

which is the standard definition of the limit $\lim_{n\to\infty} x_n = x$. $\square$

Problem 5

Prove the following theorem:

Theorem 1. *Suppose that (x_n) and (y_n) are convergence sequences of real numbers with the same limit L. If (z_n) is a sequence such that*

$$x_n \leq z_n \leq y_n \text{ for all } n \in \mathbb{N}$$

then (z_n) also converges to L.

Solution: By Theorem 4.5 we know that since $x_n \leq z_n$ [1], that $\lim x_n \leq \lim z_n$, and so therefore $L \leq \lim z_n$. Similarly, since $z_n \leq y_n$, then $\lim z_n \leq \lim y_n$ so $\lim z_n \leq L$. Therefore, we've arrived at the equation:

$$L \leq \lim z_n \leq L$$

which means that the only possible value for $\lim z_n = L$.

Alternatively, we can prove this final result by noting that because from this inequality, we can conclude that $\limsup z_n \leq L$ and $\liminf z_n \geq L$. But since we know that $\limsup z_n \geq \liminf z_n$, then we require:

$$L \leq \liminf z_n \leq \limsup z_n \leq L$$

from which we can conclude that

$$\limsup z_n = \liminf z_n = L$$

then by Theorem 6.2 we know that the sequence converges to L. $\square$

---

[1] I'm supressing the limits here, but it's implied that $n \to \infty$ is the limit that we're taking

Problem 6

Prove: $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Solution: We can define y_n and z_n as suprema and infima of the "tails" of x_n:

$$y_n = \inf\{x_k | k > m\} \; z_n = \sup\{x_k | k > n\}$$

Therefore, for all n, we know that $y_n \leq z_n$. Therefore, by theorem 4.3 (the theorem about monotonicity preservation), we know that

$$\lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n$$

and so therefore

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

□

Problem 7

Find the limits of each of the following sequences, defined for $n \in \mathbb{N}$:

(a) $\left(\frac{3n}{n+3}\right)^2$

(b) $\frac{1+2+\cdots+n^2}{n^2}$

(c) $\frac{a^n-b^n}{a^n+b^n}$, $a > b > 0$.

(d) $n^2/2^n$

(e) $\sqrt{n+1} - \sqrt{n}$

Defailed proofs are not required, but you should justify your answers.

Solution:

(a) We can write:

$$\lim_{n\to\infty} \left(\frac{3n}{n+3}\right)^2 = \lim_{n\to\infty} \frac{3n}{n\left(1+\frac{3}{n}\right)} = 9$$

(b) Rewrite the numerator:

$$\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \lim_{n\to\infty} \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)}{2n^2} = \frac{1}{2}$$

(c) Since we have $a > b > 0$, we can write:

$$\lim_{n\to\infty} \frac{a^n-b^n}{a^n+b^n} = \frac{a^n\left(1-\frac{b^n}{a^n}\right)}{a^n\left(1+\frac{b^n}{a^n}\right)} = \lim_{n\to\infty} \frac{1-\left(\frac{b}{a}\right)^n}{1+\left(\frac{b}{a}\right)^n} = 1$$

(d) We can rewire this term as

$$\lim_{n\to\infty} \frac{n^2}{2^n} = \lim_{n\to\infty} n^2 \cdot \frac{1}{2^n} = 0$$

since $\frac{1}{2^n} \to 0$ as $n \to \infty$.

(e) Let $x = \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$. Then we can write:

$$x(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} n+1-n$$

$$\therefore x = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

$\square$

Problem 8

Let $s_1 = t$ and for some $t \in \mathbb{R}$, define a sequence according to $s_{n+1} = 1 + \frac{s_n}{2}$ for $n \in \mathbb{N}$. Prove that for all t, $s_n \to 2$ as $n \to \infty$.

Solution: Analyzing the sequence, we see that

$$s_{n+2} = 1 + \frac{s_{n+1}}{2} = 1 + \frac{1 + \frac{s_n}{2}}{2}$$

As $n \to \infty$, we essentially keep "stacking" on the fraction here:

$$s_\infty \equiv \lim_{n\to\infty} s_n = 1 + \frac{1 + \frac{1+\cdots}{2}}{2}$$

Now notice that the numerator of the fraction, in the limit, is also equal to s_∞. Therefore, we now have the relation:

$$\begin{aligned} s_\infty &= 1 + \frac{s_\infty}{2} \\ \frac{s_\infty}{2} &= 1 \\ \therefore s_\infty &= 2 \end{aligned}$$

as desired. $\square$