Physics 5A: Lecture 2

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1 Brief Review - 1D Motion

You can graph the position of an object over time as a graph of $y(t)$ (for instance) over a time t. Some general things that you should keep at the back of your head:

- **Average velocity:** $< v >= \frac{y(t_2)-y(t_1)}{t_2-t_1}$. It has the same quantity as the slope of the line connecting $(t_1, y(t_1)$ and $(t_2, y(t_2)$
- **Instantaneous Velocity:** also known as $\frac{dy}{dt}$ or $\dot{y}$, is the instantaneous case of the average velocity. It's also defined as the tangent slope.
  - Formula:
  $$\lim_{\delta \to 0} \frac{y(t+\delta) - y(t)}{\delta}$$
  - Velocity curve is essentially a graph of the tangent slope at all given t
- **Average acceleration:** $\bar{a} = \frac{v(t_2)-v(t_1)}{t_2-t_1}$
- **Instantaneous Acceleration:**
$$\frac{dv}{dt} = \ddot{x} = \dot{v} = \lim_{\delta \to 0} \frac{v(t_\delta) - v(t)}{\delta}$$

Average and instantaneous velocity have units of m/s, and average and instantaneous acceleration have units of m/s^2

Example 1.1

If $y = y_0 \sim (\omega t)$, compute $\dot{y}$ and $\ddot{y}$

Solution. Taking the derivative twice we get:

$$\dot{y}(t) = y_0 \omega \cos(\omega t)$$
$$\ddot{y}(t) = -y_0 \omega^2 \sin(\omega t)$$

Example 1.2

If $y = ct^2 + dt + e$, find $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

Solution. We have:

$$\begin{aligned} \frac{dy}{dt} &= 2ct + d \\ \frac{d^2y}{dt^2} &= 2c \end{aligned}$$

2 Vector Calculus

Same terms but defined in vector form:

Average Velocity

We can write the average velocity as:

$$\begin{aligned} < \vec{v} > &= \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} \\ &= \frac{[x(t_2)\hat{\imath} - y(t_2)\hat{\jmath}] - [x(t_1)\hat{\imath} - y(t_1)\hat{\jmath}]}{t_2 - t_1} \end{aligned}$$

With some clever grouping we can rearrange this into:

$$< \vec{\vec{v}} >= v_x\hat{\imath} + v_y\hat{\jmath}$$

Instantaneous Velocity

You can do the limit definition, but in the end you get the same thing:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$$

Instantaneous Acceleration

Again, the same limit definition:

$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$$

General Path

Assume you have three vectors $\vec{r}$, $\vec{v}$ and $\vec{a}$, which are the radial, velocity and acceleration vectors respectively. (meaning that they are derviatives of each other) We can establish a relationship between the vectors using differentiation and integration:

$$\int \vec{a}(t)dt = \int \frac{d\vec{v}}{dt}$$

If you split into te components:

$$\begin{aligned}
\int \vec{a}(t)dt = \int \frac{d\vec{v}}{dt} &= \int a_x(t)\hat{\imath}\ dt + \int a_y(t)\hat{\jmath}\ dt + \int a_z(t)\hat{k}\ dt \\
&= \hat{\imath}\int \frac{dx}{dt}\ dt + \hat{\jmath}\int \frac{dy}{dt}\ dt + \hat{k}\int \frac{dz}{dt}\ dt \\
&= \hat{\imath}[v_x(t) + c_x] + \hat{\jmath}[v_y(t) + c_y] + \hat{k}[v_z(t) + c_z]
\end{aligned}$$

Notice that $v_x(t)$, $v_y(t)$ and $v_z(t)$ are just the velocity vectors in their respective directions, which all add up to $\vec{v}(t)$. So we conclude:

$$\int \vec{a}(t)dt = \vec{v}(t) + \vec{c}$$

Remark 2.1 (constant acceleration $\vec{a}$)**.** If you integrate from 0 to a constant t:

$$\begin{aligned}
\int_0^t \vec{a}dt' &= \int_0^t \frac{d\vec{v}}{dt}(t')dt' \\
\vec{a}\int dt &= \vec{v}(t) - \vec{v}(0) \\
\vec{a}(t) &= \vec{v}(t) - \vec{v}(0) \implies \vec{v}(t) = \vec{v}(0) + at
\end{aligned}$$

Where we arrive at one of the kinematic formulas. You can dot each of these solutions by $\hat{\imath}, \hat{\jmath}$, or $\hat{k}$ to derive the $a_x(t), a_y(t)$ and $a_z(t)$ formulas.

We can do the same process for velocity to position:

$$\begin{aligned}
\int_0^t \vec{v}(t)\ dt' &= \int_0^t \frac{d\vec{v}}{dt}\ dt \\
\int_0^t \vec{a}t' + \vec{v}(0)\ dt' &= \vec{x}(t) - \vec{x}(0)
\end{aligned}$$

If we have $\vec{a}(t)$ constant, we get:

$$\vec{a}\int_0^t t'\ dt + \vec{v}(0)\int_0^t\ dt' = \vec{x}(t) - \vec{x}(0) \implies \vec{x}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}(0)t + \vec{x}(0)$$

2.1 That one monkey problem

Considering the y-position of the monkey through time:

$$\vec{x}_m(t) = -g\frac{1}{2}t^2\hat{\jmath} + x_m(0)\hat{\imath} + v_x(0)\hat{\imath}$$
$$\vec{x}_b(t) = -g\frac{1}{2}t^2\hat{\jmath} + [v_{0x}\hat{\imath} + v_{0y}\hat{\jmath}]t$$