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HW 01 **Mathematical Methods in Physics** June 13, 2023

# Problem 1.1

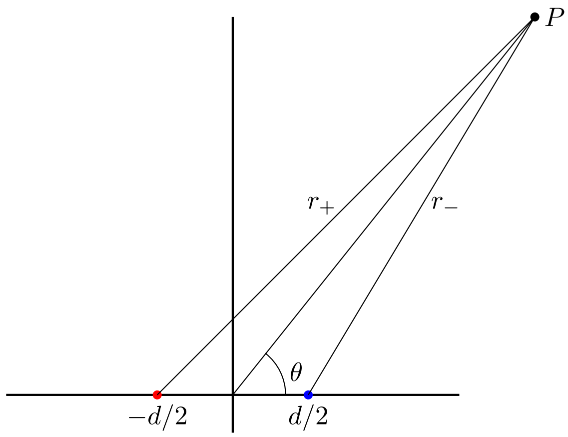
In lecture we saw how to use Taylor approximations to find the potential along the axis of an ***electric dipole***. In this problem we will expand on this analysis to study ***electric multipoles***. Recall that the potential due to a *single* point charge $q$ a distance $r$ away is given by

$$V = \frac{kq}{r}$$

Consider an electric dipole consisting of a positive charge $+q$ on the $x$-axis at $x = d/2$ and an equal and opposite negative charge $-q$ on the $x$-axis at $x = d/2$ as shown.

a) Find the lowest-order approximation to the potential at a general point $(x, y) = (r\cos\theta, r\sin\theta)$ when $r \gg d$.

*Solution:* Let us define two quantities $r_+$ and $r_-$ defined as shown in the diagram below




From the diagram, we can use cosine law:

$$r_-^2 = \left(\frac{d}{2}\right)^2 + r^2 - 2r\frac{d}{2}\cos\theta = \left(\frac{d}{2}\right)^2 + r^2 - rd\cos\theta$$
$$r_+^2 = \left(\frac{d}{2}\right)^2 + r^2 - 2r\frac{d}{2}\cos(\pi-\theta) = \left(\frac{d}{2}\right)^2 + r^2 + rd\cos\theta$$

The first order approximation is sufficient here so we can neglect the $(\frac{d}{2})^2$ term, which then allows us to factor out $r^2$ from both expressions:

$$r_+^2 = r^2 + rd\cos\theta = r^2\left(1 + \frac{d}{r}\cos\theta\right) \implies r_+ = r\left(1 + \frac{d}{r}\cos\theta\right)^{1/2}$$
$$r_-^2 = r^2 - rd\cos\theta = r^2\left(1 - \frac{d}{r}\cos\theta\right) \implies r_- = r\left(1 - \frac{d}{r}\cos\theta\right)^{1/2}$$

Now let's go back to the potential and add up the contributions due to the positive and negative charges:

$$V = \frac{kq}{r_-} - \frac{kq}{r_+} = kq\left(\frac{1}{r_-} - \frac{1}{r_+}\right)$$

Now we can substitute what we have for $r_+$ and $r_-$, while letting $\epsilon = \frac{d}{r}$:

$$\begin{aligned} V &= kq\left[\frac{1}{r(1-\epsilon\cos\theta)^{1/2}} - \frac{1}{r(1+\epsilon\cos\theta)^{1/2}}\right] \\ &= \frac{kq}{r}\left[(1-\epsilon\cos\theta)^{-1/2} - (1+\epsilon\cos\theta)^{-1/2}\right] \\ &= \frac{kq}{r}\left[1+\frac{1}{2}\epsilon\cos\theta - (1-\frac{1}{2}\epsilon\cos\theta)\right] \\ &= \frac{kq}{r}(\epsilon\cos\theta) \\ &= \frac{kqd\cos\theta}{r^2} \end{aligned}$$

□

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Next consider a ***linear electric quadrupole***, with a point charge $+2q$ at the origin, a point charge $-q$ at $x = +d$ and a point charge $-q$ at $x = -d$. Consider a point $P$ at position $x$ on the $x$-axis and a good distance to the right of the quadrupole so $x \gg d$.

b) Find the lowest-order approximation to the potential at point $P$ when $x \gg d$.

*Solution:* First we write the potential out fully:

$$V = k\left(-\frac{q}{x-d} + \frac{2q}{x} - \frac{q}{x+d}\right) = kq\left(\frac{2}{x} - \frac{1}{x-d} - \frac{1}{x+d}\right)$$

Then since $\frac{d}{x} \ll 1$, then let $\epsilon = \frac{d}{x} \implies d = \epsilon x$. Substituting this in:

$$V = \frac{kq}{x}\left(2 - \frac{1}{1-\epsilon} - \frac{1}{1+\epsilon}\right)$$

The taylor expansions of the two fractions are as follows:

$$\begin{aligned} \frac{1}{1+\epsilon} &= (1+\epsilon)^{-1} = 1 - \epsilon + \epsilon^2 + O(\epsilon^3) \\ \frac{1}{1-\epsilon} &= (1+(-\epsilon))^{-1} = 1 + \epsilon + \epsilon^2 + O(\epsilon^3) \end{aligned}$$

we only take these to second order since that's the first nonzero term. Now combining everything together:

$$V \approx \frac{kq}{x}(2 - (1+\epsilon+\epsilon^2) - (1-\epsilon+\epsilon^2)) = \frac{kq}{x}(-2\epsilon^2) = -\frac{2kqd^2}{x^3}$$

□

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## Problem 1.2

In class we presented the Taylor expansions for $\sin(x)$ and $\cos(x)$ about the point $x = 0$,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

a) Let $f(x) = \sin(x)$ and explicitly carry out the Taylor expansion about the point $x = 0$. Show that your answer reproduces the formula for $\sin(x)$ shown above.

*Solution:* Taking the derivatives, we get the following:

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= \cos(0) = 1 \\ f''(0) &= -\sin(0) = 0 \\ f'''(0) &= -\cos(0) = -1 \\ &\vdots \end{aligned}$$

The nonzero derivatives occur whenever we have a $\cos(0)$ term. This pattern occurs at every other term, since taking the derivative of $\sin(x)$ gives us $\cos(x)$ (giving us a nonzero term), and taking the derivative of $\cos(x)$ gives us $\sin(x)$, giving us a zero. Taking the derivative again returns us to $\cos(x)$, which is nonzero again hence there is a pattern that every other term is nonzero. Therefore, we get:

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + O(x^9)$$

This matches the series expansion, since for $n = 0$ we would get $x$, then $n = 1$ gives us the $-x^3/3!$ term, and so on. In the expansion, we are neglecting the indices that give us zero by using $2n + 1$ as the index in the summation rather than just $n$, meaning that our index "step size" is 2. In other words, we are allowed to "skip over" the even indices because they are all zero. $\square$

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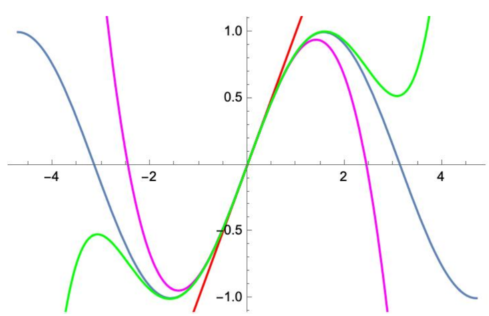
b) Graph $\sin(x)$, and the first, third, and fifth order Taylor series approximation to $\sin(x)$ in the range $-3\pi/2 \leq x \leq 3\pi/2$.

*Solution:* Below I've plotted the sine curve over the $x \in [-3\pi/2, 3\pi/2]$. The blue curve is the full sine curve, and the other colors are the approximations.




$\square$

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Some functions can't be Taylor expanded about certain points. For example, if we try to Taylor expand $1/x$ about the point $x = 0$ we would fail since $1/x = \infty$! In such cases, it may be possible to expand the function using both positive and *negative* powers of $x$. For example, we can express the function $\sin(x)/x^2$ as,

$$\frac{\sin x}{x^2} \approx \frac{x - x^3/3! + x^5/5! + \mathcal{O}(x^7)}{x^2} = \frac{1}{x} - \frac{x^2}{3!} + \frac{x^3}{5!} + \mathcal{O}(x^5)$$

Such a series is a generalization of the Taylor series called the ***Laurent Series***, which is an enormously useful tool in complex analysis.

c) Use the two series for $\sin(x)$ and $\cos(x)$ to find the Laurent series/small-angle approximation for $\cot(x)$ to the lowest two non-zero terms. Check your approximation by finding $\cot(0.1)$ and comparing with your approximation. How about for $\cot(1)$? $\cot(\pi/2)$?

*Solution:* We know that $\cot(x) = \frac{\cos(x)}{\sin(x)}$, so therefore we can take the first two terms of the Taylor expansion for $\sin(x)$ and $\cos(x)$:

$$\begin{aligned}\cot(x) &= \left(1 - \frac{x^2}{2!}\right)\left(x - \frac{x^3}{3!}\right)^{-1} \\ &= \left(1 - \frac{x^2}{2!}\right) \cdot \frac{1}{x} \cdot \left(1 - \frac{x^2}{3!}\right)^{-1}\end{aligned}$$

Now we can use the binomial expansion to simplify the term with the negative exponent:

$$(1 - x)^n = 1 - nx + O(x^2)$$

Doing so, we get:

$$\begin{aligned}\cot(x) &= \left(1 - \frac{x^2}{2!}\right) \cdot \frac{1}{x} \cdot \left(1 - \frac{x^2}{3!}\right) \\ &= \frac{1}{x}\left[1 - \frac{x^2}{3!} - \frac{x^2}{2!} - \frac{x^4}{2!3!}\right]\end{aligned}$$

Keeping the first two terms only, we get:

$$\cot(x) \approx \frac{1}{x} - \frac{x}{3}$$

Using Mathematica, we find that $\cot(0.1) = 9.96664$, and our Taylor approximation gives 9.96667, which indicates that it's a very good approximation. Then, with the other values, we get:

| Input | Exact | Taylor Approximation |
|---|---|---|
| $\cot(1)$ | 0.6420... | 0.6666... |
| $\cot(\pi/2)$ | 0 | 0.1130 |

We can see that since we centered our taylor approximations around $x = 0$, that our approximations for $\cot(x)$ around $x = 0$ is quite good (difference of 0.00003), but the approximation starts to fail once we get to $x = \frac{\pi}{2}$, where the difference jumps to over 0.1. □

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d) Use the Taylor series for $e^x$, $\sin(x)$ and $\cos(x)$ to show ***Euler's formula***

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

*Solution:* We have the series of $\sin(x)$ from part (a), but I'll repeat it here:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + O(x^9)$$

The expansion for $\cos(x)$ is:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + O(x^8)$$

The expansion for $e^x$ is:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5)$$

then plugging in $x = i\theta$ into the last series, we have:

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right)}_{=\cos(\theta)} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}_{=\sin(\theta)} = \cos(\theta) + i \sin\theta$$

$\square$

## Problem 1.3

a) Let $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ and $z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}$. Find the real part and imaginary part of the product $z_1 z_2$ and the quotient $z_1/z_2$ in terms of the Cartesian components $x_1, x_2, y_1$ and $y_2$. Then find the real and imaginary part of the product $z_1 z_2$ and the quotient $z_1/z_2$ in terms of the polar components $r_1, r_2, \theta_1$ and $\theta_2$.

*Solution:* The product is:

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2)$$

so the real part is $\text{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2$, and the complex part is $\text{Im}(z_1 z_2) = x_1 y_2 + y_1 x_2$. The quotient is:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2 - i(x_1 y_2 + x_2 y_1)}{x_2^2 + y_2^2}$$

Here, the real part is:

$$\text{Re}\left(\frac{z_1}{z_2}\right) = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}$$

and the imaginary part is:

$$\text{Im}\left(\frac{z_1}{z_2}\right) = -\frac{x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2}$$

With the other representation, the product $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$. We then have to break up the complex exponential into its real and imaginary part (using $e^{i\theta} = \cos\theta + i\sin\theta$), so we get:

$$\text{Re}(z_1 z_2) = r_1 r_2 \cos(\theta_1 + \theta_2) \qquad \text{Im}(z_1 z_2) = r_1 r_2 \sin(\theta_1 + \theta_2)$$

For the quotient, we have:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Again, splitting this up using the identity $e^{i\theta} = \cos\theta + i\sin\theta$, we get:

$$\text{Re}\left(\frac{z_1}{z_2}\right) = \frac{r_1}{r_2}\cos(\theta_1 - \theta_2) \qquad \text{Im}\left(\frac{z_1}{z_2}\right) = \frac{r_1}{r_2}\sin(\theta_1 - \theta_2)$$

$\square$

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b) Let $z = x + iy = re^{i\theta}$. Find $|z|^2$ and $z^2$ first in terms of $x$ and $y$ and then in terms of $r$ and $\theta$. Under what circumstances does $|z|^2 = z^2$?

*Solution:* The modulus square $|z|^2 = x^2 + y^2 = r^2$, whereas the square $z^2 = x^2 + 2ixy - y^2 = r^2 e^{2i\theta}$. Equality only occurs when the imaginary part disappears, so when $y = 0$. This also corresponds to $\theta = 0, \pi, 2\pi, \ldots$ (every integer multiple of $\pi$). $\square$

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c) Use Euler's formula on both sides of $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ to derive the formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$

*Solution:* On the left hand side we have

$$e^{i(\alpha+\beta)} = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

and on the right hand side we have to do some multiplication:

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \end{aligned}$$

So comparing the real and imaginary parts, we get the identities:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

□

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Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

d) Show that $|z_1 + z_2| = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}$

*Solution:* We can do this by finding the norm-squared:

$$\begin{aligned} |z_1 + z_2|^2 &= |r_1 e^{i\theta_1} + r_2 e^{i\theta_2}|^2 \\ &= (r_1 e^{i\theta_1} + r_2 e^{i\theta_2})(r_1 e^{-i\theta_1} + r_2 e^{-i\theta_2}) \\ &= r_1^2 + r_2^2 + r_1 r_2 e^{i(\theta_1-\theta_2)} + r_1 r_2 e^{i(\theta_2-\theta_1)} \\ &= r_1^2 + r_2^2 + r_1 r_2 \left(e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)}\right) \end{aligned}$$

Now we use the identity:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

with $x = \theta_1 - \theta_2$ in this case, so therefore:

$$|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

taking the square root:

$$|z_1 + z_2| = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

□

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e) (Extra) Use the result from part (d) to show the ***triangle inequality*** $|z_1 + z_2| \leq |z_1| + |z_2|$. Under what conditions is the inequality ***saturated***.

*Solution:* We expand $|z_1| + |z_2| = r_1 + r_2 = \sqrt{(r_1 + r_2)^2} = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2}$. Note that since $-1 \leq \cos(\theta_1 - \theta_2) \leq 1$, the quantity $2r_1 r_2 \geq 2r_1 r_2 \cos(\theta_1 - \theta_2)$, and so overall the triangle inequality holds. Equality is reached when $\theta_1 - \theta_2 = 0, 2\pi, \ldots$ or in general: $\theta_1 - \theta_2 = 2n\pi$ where $n = 1, 2, \ldots$. □

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f) Show ***De Moivre's formula***, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

*Solution:* On the left hand side, we can rewrite $\cos\theta + i\sin\theta = e^{i\theta}$, so therefore

$$(e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

as desired. □

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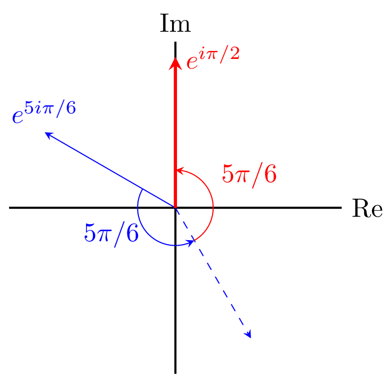
Let $n \in \mathbb{N}$ (that is, $n$ is a positive whole number). The ***n-th root*** of a complex number $z$ is $z^{1/n}$. When dealing with roots we have to be careful when the $n$-th root function is ***multi-valued***. In particular, there are $n$ different possible values of $w$ such that $w^n = z$, all of which could properly be considered $n$-th roots of $z$. Let's explore this.

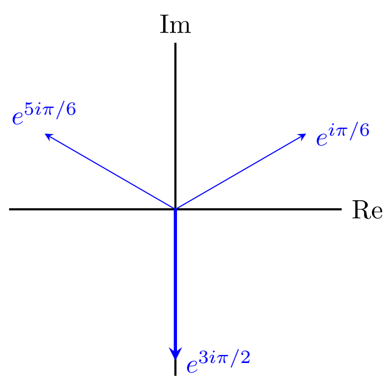
g) Consider the three different cube-roots of $z = i$ (which, using Euler's formula, we can also write as $e^{i\pi/2}$. Show that $w = e^{i\pi/6}, e^{5i\pi/6}$, and $e^{3i\pi/2}$ all cube to $z = i$. Plot these roots on the complex plane and demonstrate graphically how $e^{5i\pi/6}$ cubes to $i$.

*Solution:* Let $w_1, w_2, w_3$ be the solutions. Then, we have $w_1^3 = e^{3i\pi/6} = e^{i\pi/2} = i$, $w_2^3 = e^{15i\pi/6} = e^{5i\pi/2} = e^{2i\pi+i\pi/2} = e^{i\pi/2} = i$ and finally $w_3^3 = e^{9i\pi/2} = e^{8i\pi+i\pi/2} = e^{i\pi/2} = i$. Graphically speaking, when we cube $e^{5i\pi/6}$ we are rotating the vector $e^{5i\pi/6}$ by the angle $5\pi/6$ two times, which comes out to a total angle of $15\pi/6$, which is the same as rotating by $\pi/2$, as shown below:




Here, we see that the solid blue arrow represents the initial vector $e^{5i\pi/6}$, and when we multiply it by $e^{5i\pi/6}$ once, we get the dashed blue line, and when we multiply it again we get the red vector, which is also represented by $e^{i\pi/2}$.

The problem also asks for each of the roots to be plotted, so here they are:




□

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h) (Extra) Show that the $n$ distinct values of the $n$-th root of $z = re^{i\theta}$ are $w_k = r^{1/n}e^{i(\theta+2\pi k)/n}$, with $k = 0, \cdots, n-1$. As a check, test that your formula implies that if $w$ is a square root of $z$ then so is $-w$.

*Solution:* Taking any of the roots $w_k$ to the $n$-th power, we get:

$$w_k^n = re^{i(\theta+2\pi k)} = re^{i\theta}e^{2\pi ik}$$

the term $e^{2\pi ik}$ represents an integer multiple of $2\pi$ revolutions, which is the same thing as multiplying by 1, and thus we've proven equality. We can also see this explicitly by remembering the identity $e^{2\pi i} = 1$, so therefore

$$e^{2\pi ik} = (e^{2\pi i})^k = 1^k = 1$$

which is another way to show that $re^{i\theta}e^{2\pi ik} = re^{i\theta}$. $\square$

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The multi-valued nature of the $n$-th root ultimately comes from the multi-valued nature of the logarithm. Recall that

$$\ln z = \ln |z| + i\text{Arg}(z) + 2\pi ik$$

where $\text{Arg}(z) = \tan^{-1}(y/x)$ is the ***principle value*** of the argument, so $-\pi < \text{Arg}(z) \leq \pi$, and $k \in \mathbb{Z}$ is the ***branch index***. Before solving part (i), be sure to convince yourself that the exponentian of the right-hand side of Eq.1 does indeed produce $z = |z|e^{i\text{Arg}(z)}$ for any integer $k$.

i) Use Eq.1 to solve the formula $w^n = z$ for $w$ by first taking the logarithm of both sides, solving for $\ln w$, and then exponentiating. You should get back your formula from part (h)!

*Solution:* We follow the problem and take the logarithm of both sides and solving for $\ln w$:

$$\begin{aligned} n \ln w &= \ln z \\ \ln w &= \frac{1}{n} \ln z \end{aligned}$$

Now we then exponentiate:

$$\begin{aligned} w &= e^{\frac{1}{n} \ln z} \\ &= e^{\frac{1}{n}(\ln |z| + i\text{Arg}(z) + 2\pi ik)} \\ &= |z|^{1/n} e^{(i\text{Arg}(z) + 2\pi ik)/n} \\ &= r^{1/n} e^{i(\theta + 2\pi k)/n} \end{aligned}$$

as desired. $\square$

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## Problem 1.4

Consider a mass $m$ at the end of an ideal spring with spring constant $k$ (so the spring force is $F_{\text{spring}} = -kx$). In introductory physics when studying this system, you find that the mass undergoes *oscillatory* motion with angular frequency $\omega_0 = \sqrt{k/m}$. Now let's make the physical system a little more physically accurate by adding in the effects of ***damping***, a type of friction force (it removes mechanical energy from the system). The damping force is proportional to the velocity, $F_{\text{damping}} = -2m\gamma v$, where $\gamma \geq 0$ is the ***damping coefficient***. Therefore,

$$ma = -m\omega_0^2 x - 2m\gamma v$$

Our problem is ironically made less complicated by ***complexifying***. That is, we replace the real position $x \in \mathbb{R}$ with a "complex position" $z \in \mathbb{Z}$ such that $\text{Re}z \equiv x$. We intuitively know or can guess what our damped sprign will behave like - it should oscillate with a decreasing amplitude. Therefore, we will make an ***ansatz*** (a guess) at what the complex position will look like,

$$z(t) = Ae^{i\Omega t}$$

where $\Omega \in \mathbb{C}$ is a complex number with units of $[\Omega] = \text{s}^{-1}$ and $A = A_0 e^{i\phi_0}$ is some initial complex amplitude. We define $\omega \in \mathbb{R}$ and $\Gamma \in \mathbb{R}$ to be a real and imaginary parts of $\Omega$,

$$\Omega \equiv \omega + i\Gamma$$

a) Express $z(t) = Ae^{i\Omega t}$ in polar form $r(t)e^{i\phi(t)}$, with $r(t)$ and $\phi(t)$ are *real* functions expressed in terms of the real constants $\omega, \Gamma, A_0$ and $\phi_0$. Using your physical intuition (think about the actual motion of a damped oscillator or think about energy), what condition must $\Gamma$ satisfy for our solution to be physically reasonable?

*Solution:* We have $z(t) = Ae^{i\Omega t}$, so we can write $\Omega = \omega + i\Gamma$:

$$\begin{aligned} z(t) &= Ae^{i\Omega t} \\ &= A_0 e^{i\phi_0} e^{it(\omega + i\Gamma)} \\ &= A_0 e^{i\phi_0} e^{i\omega t - \Gamma t} \\ &= A_0 e^{-\Gamma t} e^{i(\phi_0 + \omega t)} \end{aligned}$$

And so we deduce that $r(t) = A_0 e^{-\Gamma t}$ and $\phi(t) = \phi_0 + \omega t$. In order for our solution to be physically reasonable, the motion it describes must either describe a decaying amplitude or a constant one, since the damping coefficient is a friction force, and thus it cannot add energy to the system (it could however, do nothing, which would result in a constant amplitude). Therefore, the condition on $\Gamma$ is that $\Gamma \geq 0$. □

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From the complex position $z$ we can define a complex velocity $\tilde{v} = \frac{\mathrm{d}z}{\mathrm{d}t}$ and a complex acceleration $\tilde{a} = \frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}$, in whihc case Eq. 2a becomes

$$m\tilde{a} = -m\omega_0^2 z - 2m\gamma\tilde{v}$$

At the end of the day, we get our physical solutions by taking the real part of the complex solution.

b) Consider the ansatz from Eq. 2b. Find the complex velocity $\tilde{v}(t)$ and complex acceleration $\tilde{a}(t)$ in terms of the complex constants $A$ and $\Omega$.

*Solution:* We have $z(t) = A_0 e^{i\Omega t}$, so therefore just taking derivatives the normal way:

$$\begin{aligned} \tilde{v}(t) &= \frac{\mathrm{d}z}{\mathrm{d}t} = A_0 i\Omega e^{i\Omega t} \\ \tilde{a}(t) &= \frac{\mathrm{d}\tilde{v}}{\mathrm{d}t} = -A_0 \Omega^2 e^{i\Omega t} \end{aligned}$$

$\square$

c) Plug your results from (b) into Eq. 2c. Use the fact that $e^{i\Omega t}$ is never zero to simplify your answer to a quadratic equation in $\Omega$.

*Solution:* We plug this in:

$$\begin{aligned} -mA_0\omega^2 e^{i\Omega t} &= -m\omega_0^2 A_0 e^{i\Omega t} - 2m\gamma A_0 i\Omega e^{i\Omega t} \\ -mA_0\Omega^2 &= -m\omega_0^2 A_0 - 2m\gamma A_0 i\Omega \\ \therefore \Omega^2 - 2i\gamma\Omega - \omega_0^2 = 0 \end{aligned}$$

this is the quadratic equation in $\Omega$ that we are asked to find. $\square$

We have a quadratic equation and we want to solve for $\Omega$! Gut instinct should tell you to use the quadratic formula to solve for $\Omega$, but is that valid for complex numbers?

d) (Extra) Show that the quadratic formula works to solve quadratic equations $az^2 + bz + c = 0$, even if the coefficients are complex!

e) Solve the quadratic equation from part (c) for $\Omega$. Does this $\Omega$ satisfy the condition you found in part (a)? Under what conditions (if any) on $\gamma$ and/or $\omega_0$ does $\Omega$ become purely real ($\Gamma = 0$)? Purely imaginary ($\omega = 0$)? Neither purely real or imaginary ($\omega, \Gamma \neq 0$)?

*Solution:* Using the quadratic formula, we get:

$$\begin{aligned} \Omega_{1,2} = \frac{2i\gamma \pm \sqrt{-4\gamma^2 - 4(1)(-\omega_0^2)}}{2} &= \frac{2i\gamma \pm 2\sqrt{\omega_0^2 - \gamma^2}}{2} \\ &= i\gamma \pm \sqrt{\omega_0^2 - \gamma^2} \end{aligned}$$

In order for these values to be puely real, then we must have $\gamma = 0$, in which case we get $\Omega = \pm\omega_0$. To be purely imaginary, then we require that $\omega_0 = \gamma$ and that $\gamma \neq 0$. Any other case would result in $\Omega_{1,2}$ having a nonzero real and imaginary part.

These values for $\Omega$ also make sense with part (a), since there we argued that the imaginary component of $\Omega$ (represented by $\Gamma$ must satisfy $\Gamma \geq 0$, and here we see that the imaginary part of the solutions for $\Omega$ are supplied by the damping coefficient, which by definition has the condition that $\gamma \geq 0$, so since these two match up these values for $\Omega$ make sense. $\square$

Now it's time for some reasonability checks! In the middle of a physics calculation it's always good to see if your answer is physically on the right track.

f) (Extra) First, using Eq. 2c, determine the units/dimensions of the constant $\gamma$. Next, make sure your expression for $\Omega$ is dimensionally consistent (we can oly add quantities if they have the same dimensions). What are the units of $\Omega$? It doesn't make sense to take the exponential (or sine or cosine) of a dimensionful quantity. Is this consistent for our soultion $z = Ae^{i\Omega t}$?

For the rest of the problem, let $\phi_0 = 0$ so that $A$ is real and positive.

g) (Extra) Find $x(t) = \text{Re}(z(t))$ when $\gamma = 0$. This is the ***undamped*** oscillator. Does this oscillator behave in the way you would expect?

h) For *one* of the two solutions for $\Omega$ you found in part (e), find $x(t) = \text{Re}(z(t))$ in the case $0 < \gamma < \omega_0$. This is the ***underdamped*** oscillator. Qualitatively sketch the solution $x(t)$.

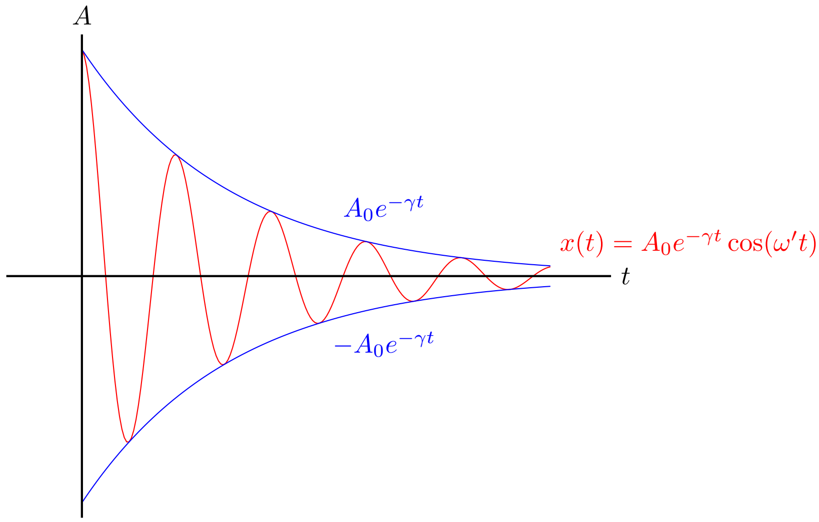
*Solution:* I'll take the positive case, $\Omega = i\gamma + \sqrt{\omega_0^2 - \gamma^2}$. Then, if $\omega_0 > \gamma$, then this means that the radical is a real quantity, and let's call this new quantity $\omega' = \sqrt{\omega_0^2 - \gamma^2}$. Then, we can write:

$$\begin{aligned} z(t) &= A_0 e^{it(i\gamma+\omega')} \\ &= A_0 e^{-\gamma t + i\omega' t} \\ &= A_0 e^{-\gamma t} e^{i\omega' t} \end{aligned}$$

Then if we take the real part, then we simply replace the phase with $\cos(\omega' t)$, so finally:

$$\mathrm{Re}(z(t)) = A_0 e^{-\gamma t} \cos(\omega' t)$$

where $\omega' = \sqrt{\omega_0^2 - \gamma^2}$. A sketch is shown below, and I've also plotted the exponential envelope to show that the amplitude follows an exponential decay, as expected.

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i) For *one* of the two solutions for $\Omega$ you found in part (e), find $x(t) = \mathrm{Re}(z(t))$ in the case $\omega_0 < \gamma$. This is the overdamped oscillator. Qualitatively sketch the solution $x(t)$.

*Solution:* Unlike the previous solution, here we have a case where the radical is complex-valued, so we can rewrite the radical as:

$$\sqrt{\omega_0^2 - \gamma^2} = \sqrt{-(\gamma^2 - \omega_0^2)} = i\sqrt{\gamma^2 - \omega_0^2}$$

Now, we define $\omega' = \sqrt{\gamma^2 - \omega_0^2}$, so we can write our solution $z(t)$ as:

$$z(t) = A_0 e^{i(i\gamma + i\omega')t}$$

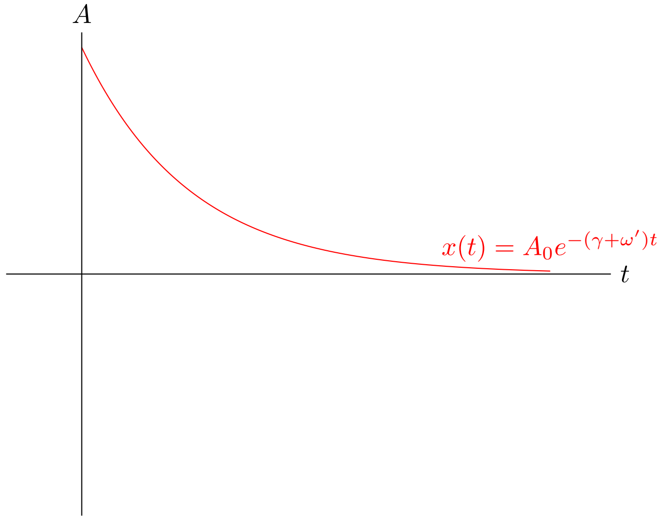
simplifying this, we get

$$z(t) = A_0 e^{(-\gamma - \omega')t}$$

And since there is no complex phase any longer, we have $\mathrm{Re}(z(t)) = z(t)$:

$$\mathrm{Re}(z(t)) = A_0 e^{-(\gamma + \omega')t}$$

Plotting this, we get:




there isn't really any fancy envelope I can draw for this curve so the plot looks kind of lame, but the lack of an oscillation is exactly what we expect for an overdamped oscillator. $\square$

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