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Collaborators

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Problem 1

This problem will have you explore a basic idea in phase estimation algorithms. Imagine that we have a quantum computer that implemenets a Z-rotation of some unknown angle

$$U = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

where θ is an unknown phase in the range $[0, 2\pi]$ that we seek to estimate. We will consider repeatedly applying U a number k times, written U^k.

a) What is $U^k \left|+\right\rangle$ written in the computational basis?

Solution: First, we know that since U is a diagonal matrix, then:

$$f(U) = \begin{bmatrix} f(u_{11}) & 0 \\ 0 & f(u_{22}) \end{bmatrix}$$

So, we can define a function $f(x) = x^k$, which gives us:

$$f(U) = U^k = \begin{bmatrix} 1 & 0 \\ 0 & e^{ki\theta} \end{bmatrix}$$

Now, applying this to the state $\left|+\right\rangle$:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{ki\theta} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ e^{ki\theta} \end{bmatrix}$$

□

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b) Identify two projective measurements $\left\langle m_c\right|$ and $\left\langle m_s\right|$ such that the output distributions take the forms

$$|\left\langle m_c|U^k|+\right\rangle|^2 = \frac{1+\cos(k\theta)}{2}$$

and

$$|\left\langle m_s|U^k|+\right\rangle|^2 = \frac{1+\sin(k\theta)}{2}$$

(hint: you may wish to refer back to HW 1 Problem 3).

Solution: Looking at Homework 1 problem 3, we see that measuring this in the Hadamard basis (that is $\left|+\right\rangle, \left|-\right\rangle$) will give us the desired probabilities. Specifically, we have a state in the form:

$$U^k \left|+\right\rangle = \frac{1}{\sqrt{2}} \left|0\right\rangle + \frac{e^{ki\theta}}{\sqrt{2}} \left|1\right\rangle$$

Transforming this into the Hadamard basis, this state becomes:

$$\begin{aligned} U^k |+\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right) + \frac{e^{ki\theta}}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right) \\ &= \frac{1+e^{ki\theta}}{2}|+\rangle + \frac{1-e^{ki\theta}}{2}|-\rangle \end{aligned}$$

Then, if we let $\langle m_c| = \langle +|$, then the probability of measuremen tis:

$$|\langle m_c|U^k|+\rangle|^2 = \left|\frac{1+e^{ki\theta}}{2}\right|^2 = \cos^2\left(\frac{k\theta}{2}\right) = \frac{1+\cos(k\theta)}{2}$$

Then, we can use $\langle -| S$ as the second measurement. This turns out state into:

$$SU^k|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}1\\ e^{ki\theta}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ ie^{ki\theta}\end{pmatrix}$$

Now, we convert this into the Hadamard basis:

$$|\psi\rangle = \frac{1+ie^{ki\theta}}{2}|+\rangle + \frac{1-ie^{ki\theta}}{2}$$

So the probability of measuring $|-\rangle$ is:

$$|\langle m_s|U^k|+\rangle|^2 = \frac{1+\sin(k\theta)}{2}$$

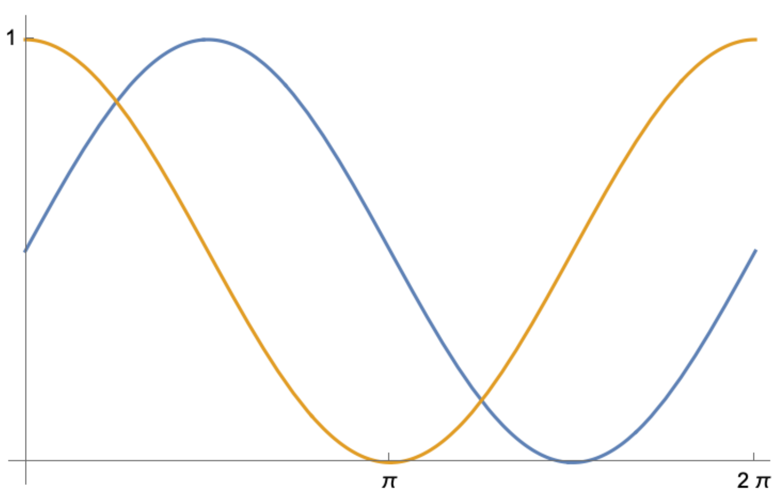
as desired. □

c) How can you estimate θ for $k = 1$? Why can we not just use a single measurement $\langle m_c|$ or $\langle m_s|$ (hint: think in terms of the unit circle)?

Solution: For $k = 1$, then the measurements are:

$$|\langle m_c|U^k|+\rangle|^2 = \frac{1+\cos(k\theta)}{2} \qquad |\langle m_s|U^k|-\rangle|^2 = \frac{1+\sin(k\theta)}{2}$$

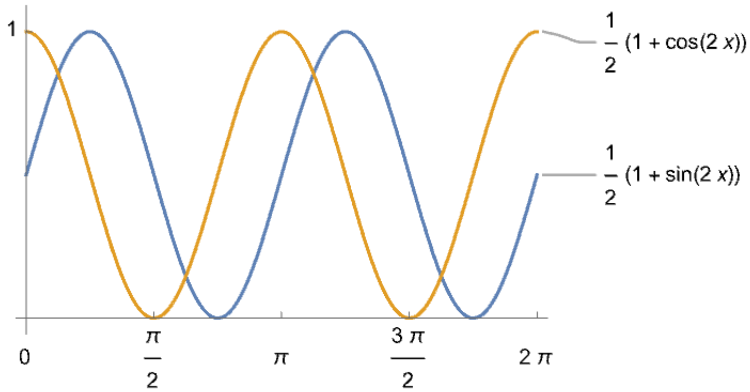
We can estimate θ for $k = 1$ by repeatedly creating the state and measuring, eventually we will get the distribution of measuring $|+\rangle$ and $|-\rangle$, from which we can determine the angle θ we're working with. Plotting these two curves together:



We can see that for any particular value of θ, the distributions of $\langle m_c|$ and $\langle m_s|$ are distinct, and so with enough repeated measurements we are able to estimate θ. □

d) What problems arise when trying to estimate θ for $k > 1$ (hint: it may be useful to draw a picture)?

Solution: For $k > 1$, we run into the issue that on the interval $[0, 2\pi]$, the distribution for m_c and m_s look the same for different values of θ. Here's the plot for $k = 2$:



Consider the distribution at $\theta = \frac{\pi}{2}$. Here, we have that the probability of measuring m_c is zero and the probability of measuring m_s is $\frac{1}{2}$. Notice that this same distribution could also correspond to $\theta = \frac{3\pi}{2}$. There is no way to tell these two angles apart, which is the issue. □

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e) It tursn out that more applications of U can actually increase the sensitivity of an estimator. However, as you just saw, there is a problem with just using long circuits. How can you use information you gain from taking measurements at $k = 1$ to solve this problem at $k = 2$?

Solution: We can use the measurement at $k = 1$ to get oureslves a rough idea of what the angle θ is, then use the values ta higher k in order to be very precise about the value. In other words, measuring at $k = 1$ would get rid of the ambiguity presented by multiple identical distributions in $k = 2$, and we then leverage the accuracy given at $k = 2$ in order to precisely determine θ. □

Problem 2

The von Neumann entropy of a density matrix is

$$S(\rho) = -\operatorname{Tr} \rho \ln \rho$$

where $\ln$ is the natural logarithm of the matrix. Equivalently, the von Neumann entropy may be written

$$S(\rho) = -\sum_j \lambda_j \ln \lambda_j$$

where λ_j are the eigenvalues of ρ. Note that $0 \ln 0 = 0$. The von Neumann entropy quantifies the amount of classical uncertainty in a quantum state.

Consider the following parametrized state

$$\rho(x) = x \left|00\right\rangle \left\langle 00\right| + (1-x) \left|\Phi^+\right\rangle \left\langle \Phi^+\right|$$

where, as usual,

$$\left|\Phi^+\right\rangle = \frac{\left|00\right\rangle + \left|11\right\rangle}{\sqrt{2}}$$

a) Calculate $S(\rho(x))$.

Solution: Firstly, note that we can simplify the state:

$$\rho(x) = \frac{1+x}{2} \left|00\right\rangle \left\langle 00\right| + \frac{1-x}{2} (\left|00\right\rangle \left\langle 11\right| + \left|11\right\rangle \left\langle 00\right| + \left|11\right\rangle \left\langle 11\right|)$$

I did the matrix computation in Mathematica, and it gives me the following matrix out:

$$\rho(x) = \frac{1}{2} \begin{pmatrix} 1+x & 0 & 0 & 1-x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-x & 0 & 0 & 1-x \end{pmatrix}$$

The (nonzero) eigenvalues of this matrix are:

$$\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{1 - 2x + 2x^2})$$

So, the von Neumann entropy is given by:

$$S(\rho) = -\frac{1}{2}(1 + \sqrt{1-2x+2x^2}) \ln \left(\frac{1}{2}(1 + \sqrt{1-2x+2x^2}) \right) \\ - \frac{1}{2}(1 - \sqrt{1-2x+2x^2}) \ln \left(\frac{1}{2}(1 - \sqrt{1-2x+2x^2}) \right)$$

□

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b) What is the von Neumann entropy of a pure state? (Hint: evaluate $S(\rho(0))$ and $S(\rho(1))$, then generalize)

Solution: For $x = 0$, we know that the second term goes to 0, and so we have:

$$S(\rho(0)) = \frac{1}{2}(2) \ln(1) = 0$$

For $x = 1$:

$$S(\rho(1)) = \frac{1}{2}(2) \ln(1) + \frac{1}{2}(0) \ln(0) = 0$$

Therefore, for pure states, we have $S(\rho) = 0$. □

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c) Next, calculate the reduced density matrix $\rho(x)_B$ by tracing out the first subsystem

$$\rho(x)_B = \mathrm{Tr}_A(\rho(x))$$

Solution: Here, we're basically just asked to take the trace over one of the qubits. We know that the original state ρ is written as:

$$\rho(x) = \frac{1+x}{2} |00\rangle \langle 00| + \frac{1-x}{2}(|00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11|)$$

Taking the trace across the first subsystem is the same as if we looked at this in terms of the first qubit:

$$\rho'(x) = \frac{1+x}{2} |0\rangle \langle 0| + \frac{1-x}{2}(|0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|)$$

The trace is just the terms along the diagonal, so therefore:

$$\rho(x)_B = \mathrm{Tr}_A(\rho(x)) = \frac{1+x}{2} |0\rangle \langle 0| + \frac{1-x}{2}(|1\rangle \langle 1|)$$

□

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d) What is $S(\rho(x)_B)$

Solution: The matrix $\rho(x)_B$ can be represented as:

$$\rho(x)_B = \begin{pmatrix} \frac{1+x}{2} & 0 \\ 0 & \frac{1-x}{2} \end{pmatrix}$$

This matrix has eigenvalues:

$$\lambda_{1,2} = \frac{1 \pm x}{2}$$

Therefore:

$$S(\rho(x)_B) = -\left(\frac{1+x}{2} \ln\left(\frac{1+x}{2}\right) + \frac{1-x}{2} \ln\left(\frac{1-x}{2}\right)\right)$$

□

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e) How does the idea that "complete knowledge of the whole does not imply knowledge of the parts" apply to entangled states?

Solution: If we plug in $x = 0$ into $S(\rho(x)_B)$, we see that we get a nonzero value, implying that even though we have full knowledge of the full state (i.e. a pure state), the parts themselves, expressed by $S(\rho(x)_B)$ aren't necessarily pure states themselves. □

Problem 3

Similarly to pure states, mixed states may also be represented as points on the Bloch sphere. This is done in the same way as for pure states, by calculating expectation values of Pauli X, Y, Z observable under particular states and plotting on the Bloch sphere. However, as you saw in class, the expectation vlaue of an observable under a mixed state is written

$$\langle O \rangle_\rho = \mathrm{Tr}(O\rho)$$

where O is some observable.

Calculate $\langle X \rangle_\rho$, $\langle Y \rangle_\rho$, $\langle Z \rangle_\rho$ and plot the state on the Bloch sphere for each of the following:

a) $\rho_1 = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$

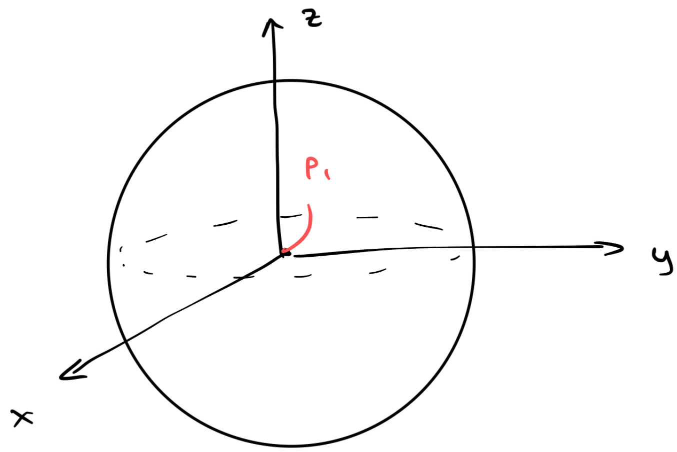
Solution: The state ρ_1 can be written as:

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We also know the matrix form of X, Y, Z, so we can just calculate it using the formula given in the problme statement. I did these computations in Mathematica:

$$\langle X \rangle_\rho = 0$$
$$\langle Y \rangle_\rho = 0$$
$$\langle Z \rangle_\rho = 0$$

Plotting these on the Bloch sphere is the same as just plotting it at the coordinates given by $\langle X \rangle_\rho$, $\langle Y \rangle_\rho$ and $\langle Z \rangle_\rho$, so therefore:



□

b) $\rho_2 = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}X|0\rangle\langle 0|X^\dagger$

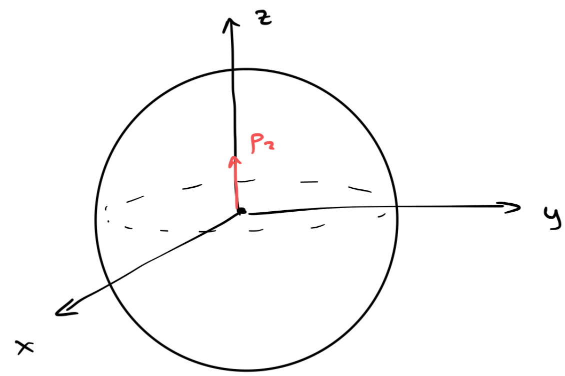
Solution: Again, following the same steps as earlier:

$$\rho_2 = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore:

$$\langle X \rangle_\rho = 0$$
$$\langle Y \rangle_\rho = 0$$
$$\langle Z \rangle_\rho = \frac{3}{4}$$

The diagram:



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c) $\rho_3 = \frac{3}{4} \left|+\right\rangle \left\langle+\right| + \frac{1}{4} Z \left|+\right\rangle \left\langle+\right| Z^\dagger$

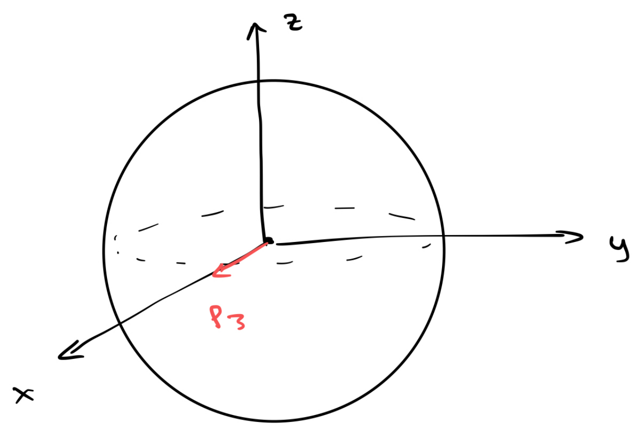
Solution: As before, we have:

$$\rho_3 = \begin{pmatrix} 1/2 & 3/8 \\ 3/8 & 1/2 \end{pmatrix}$$

Therefore:

$$\langle X \rangle_\rho = \frac{3}{4}$$
$$\langle Y \rangle_\rho = 0$$
$$\langle Z \rangle_\rho = 0$$

Diagram:



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