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HW 07

Introduction to Quantum Information

Physics C191
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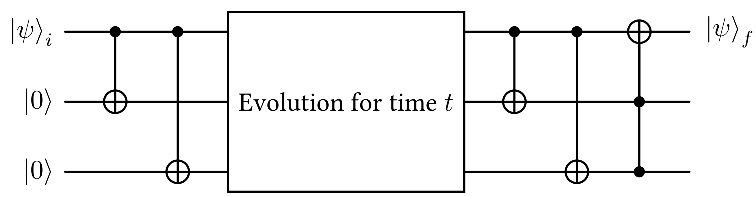
Problem 1

Suppose we have 3 identical physical qubits that experience exponential longitudinal relaxation with $1/e$ lifetimes of $T_1 = 1/\gamma$ for each of the qubits so that the probability of error on a single qubit $p(t) = 1 - e^{-\gamma t}$, which may be approximated to lowest non-zero order as $p(t) = \gamma t$. (For the sake of simplicity we will assume that the relaxation process is symmetric with respect to $|0\rangle$ and $|1\rangle$). We use the three-qubit bit-flip error correction code you learned about in lecture (see below) to encode a single logical qubit, $|\psi\rangle_i$, in the three physical qubits, wait for time t, and decode it to get $|\psi\rangle_f$.

Note: For the entire problem, you can assume perfect initial state preparation at the start of the circuit, and instantaneous and error free gates throughout the circuit.



a) To lowest nonzero order, at approximately what wait time t would a single unencoded physical qubit have a 1% probability of experiencing a bit-flip error?

Solution: To start, we can show that our initial qubit is in the state:

$$|\psi\rangle = a\,|000\rangle + b\,|111\rangle$$

We're given that we should use $p(t) = \gamma t$ here, so this means that if we want a 1% probability error, then we want:

$$p(t) = \frac{1}{100} \implies t = \frac{1}{100\gamma}$$

□

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b) To lowest nonzero order, at approximately what wait time t would the logical qubit (before decoding) have a 1% probability of experiencing a bit-flip error?

Solution: Since this is a three qubit repetition code, then there are three places that our bit-flip could happen. Therefore, the probability of a bit-flip is $3p(t) = 3\gamma t$, so:

$$3\gamma t = \frac{1}{100} \implies t = \frac{1}{300\gamma}$$

This t is smaller than that of the previous part, which makes intuitive sense since we have thrice the probability of having an error. □

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c) To lowest nonzero order, at approximately what wait time t will the corrected logical state (after decoding) $|\psi\rangle_f$ have a 1% probability of having experienced a bit-flip error?

Solution: To lowest-order, this would occur when we have two bit flips happen simultaneously. Firstly, there are three ways that two bit flips could happen: $(1,1,0), (1,0,1), (0,1,1)$. Therefore, the probability of a bit flip is given by $3(p(t))^2 = 3\gamma^2 t^2$, so therefore:

$$3\gamma^2 t^2 = \frac{1}{100} \implies t = \frac{1}{3\gamma\sqrt{100}}$$

□

d) At what wait time t will the probability of a single physical qubit bit-flip error be equal to the probability of a bit-flip error on the decoded state $|\psi\rangle_f$?

Solution: The probability of a single bit-flip error is $p(t) = 1 - e^{-\gamma t}$, and the probability of a logical error is $3(1 - e^{-\gamma t})^2$, so setting them equal:

$$\begin{aligned} 1 - e^{-\gamma t} &= 3(1 - e^{-\gamma t})^2 \\ e^{-\gamma t} &= \frac{2}{3} \\ \therefore t &= -\frac{1}{\gamma} \ln\left(\frac{2}{3}\right) \end{aligned}$$

□

e) If we define the $T_{1,L}$ lifetime of the encoded logical qubit $|\psi\rangle_f$ in the same manner as for a single qubit, meaning $T_{1,L}$ is the time at which the probability for $|\psi\rangle_f$ to *not* have experienced a bit-flip is equal to $1/e$, then is $T_{1,L}$ longer or shorter than the single qubit lifetime T_1?

Solution: This problem basically asks us to find the time t such that:

$$3(1 - e^{-\gamma t})^2 = 1 - \frac{1}{e} \implies T_{1,L} = -\frac{1}{\gamma} \ln\left(1 - \sqrt{\frac{1}{3}(1 - e^{-1})}\right)$$

whereas for a single qubit, we have $T_1 = \frac{1}{\gamma}$. The factor attached to the $\frac{1}{\gamma}$ in the above expression is equal to -0.61, meaning that $T_{1,L}$ is shorter than T_1.

This also makes sense, since the probability of having a logical bit flip is higher than the probability of a single bit flip, so we expect the lifetime to be shorter. □

Problem 2

This question will help you understand Pauli commutation relations, an important part of syndrome extraction analysis. First, we summarize a convenient and standard short-hand notation for n-qubit Pauli operators: subscript the non-trivial Pauli operators (i.e. X, Y, Z excluding I) with the qubit on which they act. For example, on 5-qubits and indexing the first qubit at 1, $X_1 = X \otimes I \otimes I \otimes I \otimes I$, $X_1 X_2 = X \otimes X \otimes I \otimes I \otimes I$, $X_2 Y_4 = I \otimes X \otimes I \otimes Y \otimes I$, etc…

Recall that two operators, F_i and F_j commute if $F_i F_j = F_j F_i$, and anti-commute if $F_i F_j = (-1) F_j F_i$. consider a register of 8 qubits. Do the following pairs of operators commute or anti-commute? Show your work and/or explain your reasoning.

a) X_1, Y_1.

Solution: I'm using the relations given in discussion section. There we mentioned that $X_1 Z_1 = -Z_1 X_1$, since they act on the the same qubit. The same logic applies here, so since both subscripts are 1, then these two anticommute. ☐

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b) $Z_1 Z_2, Y_1 Y_2$

Solution: Here, we are asked to compare the operators $Z_1 Z_2 Y_1 Y_2$ against $Y_2 Y_1 Z_2 Z_1$, which do commute:

$$Y_1 Y_1 Z_2 Z_1 = (-1) Z_2 Y_2 (-1) Z_1 Y_1 = Z_2 Y_2 Z_1 Y_1 = Z_2 Z_1 Y_2 Y_1 = Z_1 Z_2 Y_1 Y_2$$

☐

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c) $Z_1 X_3 Y_4, Y_1 Z_2 Y_4$

Solution: These operators anticommute:

$$\begin{aligned} Y_4 Z_2 Y_1 Y_4 X_3 Z_1 &= (-1) Z_1 Y_4 Z_2 Y_1 Y_4 X_3 \\ &= (-1) Z_1 X_3 Y_4 Z_2 Y_1 Y_4 \\ &= (-1) Z_1 X_3 Y_4 Y_1 Z_2 Y_4 \end{aligned}$$

☐

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d) $Z_1 X_2 Y_3 Y_5 Z_6 X_7 Y_8$, $X_1 X_2 Z_3 Z_4 Y_5 Z_7 X_8$

Solution: We can do the math out for this, but notice a pattern in the previous two solutions: we look at the operations on each qubit, and tally up the commutativity of each one:

Qubit	Operators	Commutativity
1	Z_1, X_1	-1
2	X_2, X_2	1
3	Y_3, Z_3	-1
4	I, I	1
5	Y_5, Y_5	1
6	Z_6, I	1
7	X_7, Z_7	-1
8	Y_8, X_8	-1

Here, we see that there are an even number of anticommuting operators, so therefore this pair of operators commute. ☐

Problem 3

This question will have you analyze properties of the 9-qubit Shor code. For convenience, its logical states and its stabilizers are provided below.

Logical states for the 9-qubit Shor code:

$$
\begin{aligned}
|0\rangle_L &= \frac{1}{2\sqrt{2}}(\,|000\rangle + |111\rangle)(\,|000\rangle + |111\rangle)(\,|000\rangle + |111\rangle) \\
|1\rangle_L &= \frac{1}{2\sqrt{2}}(\,|000\rangle - |111\rangle)(\,|000\rangle - |111\rangle)(\,|000\rangle - |111\rangle)
\end{aligned}
$$

Stabilizers of the 9-qubit Shor code:

$$
\begin{array}{c|c}
X_1X_2X_3X_4X_5X_6 & X_4X_5X_6X_7X_8X_9 \\
Z_1Z_2 & Z_2Z_3 \\
Z_4Z_5 & Z_5Z_6 \\
Z_7Z_8 & Z_8Z_9
\end{array}
$$

a) Show that the 9-qubit Shor code is distance 3. That is, show that a three qubit error can take one codeword to the other codeword.

Solution: From lecture, we define the distance d to be the smallest weight logical operator that takes $|0\rangle_L$ to $|1\rangle_L$. It is then distance 3 because the operator $Z_1Z_4Z_7$ does precisely that. ☐

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b) What are the syndromes of the following errors: Z_1, Z_2, Z_3?

Solution: Because the Z errors only anticommute wtih X, then we only need to look at the first two stabilizers in the first row:

	Z_1	Z_2	Z_3
$X_1X_2X_3X_4X_5X_6$	-1	-1	-1
$X_4X_5X_6X_7X_8X_9$	1	1	1

All the other stabilizers will be just 1 because Z_i commutes with them. ☐

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c) What is the effect of the Z_1, Z_2 and Z_3 on the logical states of the Shor code? Provide a single correction operation that can correct each of the three individual error processes. This provides an example of why the Shor code is "degenerate," some of the errors in the set of correctable errors map to the same syndrome string.

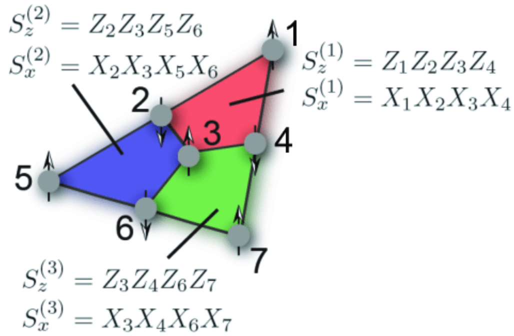
Solution: Because the Z error only flips the sign on the $|1\rangle$ state, this means that all three errors send the states as follows:

$$
\begin{aligned}
|0\rangle_L &\to \frac{1}{2\sqrt{2}}(\,|000\rangle - |111\rangle)(\,|000\rangle + |111\rangle)(\,|000\rangle + |111\rangle) \\
|1\rangle_L &\to \frac{1}{2\sqrt{2}}(\,|000\rangle + |111\rangle)(\,|000\rangle - |111\rangle)(\,|000\rangle - |111\rangle)
\end{aligned}
$$

what this means is that we just need to flip the sign back, whihc we can do with a simple Z_1 gate. ☐

Problem 4

This question will have you investigate properties of the 7-qubit Steane code. For your convenience, the Stabilizers are given below in a useful geometric mnemonic in the figure below. The stabilizers are denoted as $S_{x/z}^{(i)}$ for $i \in [1,3]$ and there are 6 in total.



a) Tabulate the syndromes of all single qubit Z-type errors of the form Z_i for i in the range $[1,7]$. Please use syndrome ordering so that the associated stabilizers appear in the order $S_x^{(1)}, S_x^{(2)}, S_x^{(3)}, S_z^{(1)}, S_z^{(2)}, S_z^{(3)}$.

Solution: Here's the table:

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
$S_x^{(1)}$	-1	-1	-1	-1	1	1	1
$S_x^{(2)}$	1	-1	-1	1	-1	-1	1
$S_x^{(3)}$	1	1	-1	-1	1	-1	-1
$S_z^{(1)}$	1	1	1	1	1	1	1
$S_z^{(2)}$	1	1	1	1	1	1	1
$S_z^{(3)}$	1	1	1	1	1	1	1

□

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b) Tabulate the syndromes of all single qubit X-type errors of the form X_i for i in the range $[1,7]$. Please use the same syndrome ordering as in part (a).

Solution: Again, here's the table:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
$S_x^{(1)}$	1	1	1	1	1	1	1
$S_x^{(2)}$	1	1	1	1	1	1	1
$S_x^{(3)}$	1	1	1	1	1	1	1
$S_z^{(1)}$	-1	-1	-1	-1	1	1	1
$S_z^{(2)}$	1	-1	-1	1	-1	-1	1
$S_z^{(3)}$	1	1	-1	-1	1	-1	-1

□

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c) How many errors are there of the form

$$Z_i^a X_j^b$$

for $a \in \{0,1\}, b \in \{0,1\}, i \in [1,7], j \in [1,7]$?

Solution: We only get an error if either one of $a, b = 1$, and there are seven options of choosing which qubit this error occurs to, and one option of not choosing a qubit at all. Therefore, there are $8 \times 8 = 64$ total errors. (this includes the case where we have no error as well) □

d) How many unique syndromes are there in the Steane code?

Solution: There are six stabilizers, meaning that our error space is size $2^6 = 64$, which is precisely the number of unique syndromes. ☐

e) Argue that the Steane code can correct all errors of the form

$$Z_i^a X_j^b$$

and no more (you may use the fact that the Steane code is non-degenerate).

Solution: Because the number of unique syndromes equals the number of unique errors, along with the fact that the Steane code is non-degenerate, this implies that there is a bijection between every error of the above form and its corresponding syndrome. This means that given any syndrome, we can automatically determine which weight-2 error caused it, and hence we can correct for it.

It also cannot correct for any more errors than this, since any other introduced error would have an identical syndrome to one of the weight-2 errors, and hence it is no longer correctable. ☐