CS 170 Homework 12

Due **Monday 11/20/2023, at 10:00 pm (grace period until 11:59pm)**

1 Study Group

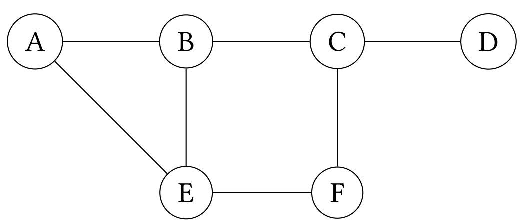
List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write "none".

Solution: None in particular. Got a bit of guidance on problem 4 by TAs.

2 Vertex Cover to Set Cover

To help jog your memory, here are some definitions:

> **Vertex Cover:** given an undirected unweighted graph $G = (V, E)$, a vertex cover C_V of G is a subset of vertices such that for every edge $e = (u, v) \in E$, at least one of u or v must be in the vertex cover C_V.
>
> **Set Cover:** given a universe of elements U and a collection of sets $\mathcal{S} = \{S_1, \dots, S_m\}$, a set cover is any (sub)collection C_S whose union equals U.
>
> In the *minimum vertex cover problem*, we are given an undirected unweighted graph $G = (V, E)$, and are asked to find the smallest vertex cover. For example, in the following graph, $\{A, E, C, D\}$ is a vertex cover, but not a minimum vertex cover. The minimum vertex covers are $\{B, E, C\}$ and $\{A, E, C\}$.



> Then, recall in the *minimum set cover problem*, we are given a set U and a collection $\mathcal{S} = \{S_1, \dots, S_m\}$ of subsets of U, and are asked to find the smallest set cover. For example, given $U := \{a, b, c, d\}$, $S_1 := \{a, b, c\}$, $S_2 := \{b, c\}$, and $S_3 := \{c, d\}$, a solution to the problem is $C_S = \{S_1, S_3\}$.

Give an efficient reduction from the minimum vertex cover problem to the minimum set cover problem. Briefly justify the correctness of your reduction (i.e. 1-2 sentences).

Solution: So it turns out that this is the same problem as homework 10 problem 7. I took inspiration from the official solution, but wrote it in my own language and made sure I understood the solution properly before typing up my own solution.

We can create a set cover instance where for every vertex v, its set consists of v and all vertices connected to v. In other words, first label the vertices $v_1, v_2, \dots, v_n$, and have the sets S_i contain v_i and all vertices that share an edge with v_i.

Then, once the set cover instance is solved, the indices of the sets that it chooses in C_S will be the minimum vertex cover (in other words, if $C_S = \{S_1, S_2, S_3\}$, then our vertex cover is $C_V = \{v_1, v_2, v_3\}$.

Proof of Correctness: Based on the way we constructed our sets, each set chosen in our set cover corresponds directly to the vertex that would be chosen for a vertex cover. In other words, the set cover we choose will also be a vertex cover.

To prove this, consider a given set cover C_S, and its corresponding vertex cover C_V. Suppose there is an edge (u, v) that isn't covered by C_V. This implies that both vertices u, v did not exist within the vertex cover, but this is impossible, since this would imply that C_S was not a valid set cover (as a set cover covers all vertices). Hence, each set cover corresponds to some vertex cover. Therefore, the minimum set cover also corresponds directly to the minimum vertex cover, as desired.

3 Reduction to 3-Coloring

Given a graph $G = (V, E)$, a valid 3-coloring assigns each vertex in the graph a color from {red, green, blue} such that for any edge (u, v), u and v have different colors. In the 3-coloring problem, our goal is to find a valid 3-coloring if one exists. In this problem, we will give a reduction from 3-SAT to the 3-coloring problem. Since we know that 3-SAT is NP-Hard (there is a reduction to 3-SAT from every NP problem), this will show that 3-coloring is NP-Hard (there is a reduction to 3-coloring from every NP problem).

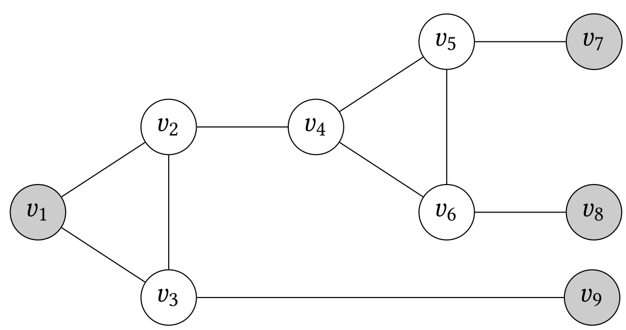
In our reduction, the graph will start with three special vertices, labelled v_{TRUE}, v_{FALSE}, and v_{BASE}, as well as the edges $(v_{\mathsf{TRUE}}, v_{\mathsf{FALSE}})$, $(v_{\mathsf{TRUE}}, v_{\mathsf{BASE}})$, and $(v_{\mathsf{FALSE}}, v_{\mathsf{BASE}})$.

(a) For each variable x_i in a 3-SAT formula, we will create a pair of vertices labeled x_i and $\neg x_i$. How should we add edges to the graph such that in any valid 3-coloring, one of $x_i, \neg x_i$ is assigned the same color as v_{TRUE} and the other is assigned the same color as v_{FALSE}?

Hint: any vertex adjacent to v_{BASE} must have the same color as either v_{TRUE} or v_{FALSE}. Why is this?

Solution: I'll assign x_i to v_{TRUE} and $\neg x_i$ to v_{FALSE}. We can do this by connecting x_i to v_{BASE} and v_{FALSE}, and $\neg x_i$ to v_{BASE} and v_{TRUE}. This way, x_i doesn't share the same color as v_{BASE} or v_{FALSE}, and since we only have 3 colors, it must then share the same color as v_{TRUE}. A similar approach shows that $\neg x_i$ must have the same color as v_{FALSE}.

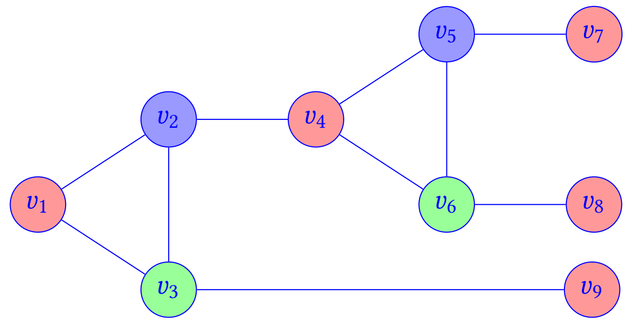
(b) Consider the following graph, which we will call a "gadget":



Consider any valid 3-coloring of this graph that does *not* assign the color blue to any of the gray vertices (v_1, v_7, v_8, v_9). Show that if v_1 is assigned the color green, then at least one of $\{v_7, v_8, v_9\}$ is assigned the color green.

Hint: it's easier to prove the contrapositive!

Solution: The contrapositive statement is that if none of v_7, v_8, v_9 are colored green, then v_1 is also not colored green. Given this statement, we'll work backwards from v_7, v_8, v_9. Because v_7, v_8, v_9 cannot be colored blue, then they must be colored red. The following is a valid coloring of the graph in this case:



Alternatively, one could also swap the green and blue colors around, but that doesn't change the overall coloring of the graph (you could just swap all greens and blues here the coloring would still be valid).

This is also an exhaustive list of possible colorings: v_6 and v_5 are forced to be green and blue because v_7 and v_8 are red, which forces v_4 to also be red. This then forces v_2 to be either green or blue. On the other hand, because v_9 is red, this also forces v_3 to be green or blue. WLOG if v_2 is blue, this means that v_3 is green, forcing v_1 to be red.

(c) We have now observed the following about the graph we are creating in the reduction:

(i) For any vertex, if we have the edges (v, v_{FALSE}) and (v, v_{BASE}) in the graph, then in any valid 3-coloring v will be assigned the same color as v_{TRUE}.

(ii) Through brute force one can also show that in a gadget, if all the following hold:

(1) All gray vertices are assigned the color red or green.

(2) v_1 is assigned the color green.

(3) At least one of $\{v_7, v_8, v_9\}$ is assigned the color green.

Then there is a valid coloring for the white vertices in the gadget.

Using these observations and your answers to the previous parts, **give a reduction from 3-SAT to 3-coloring. Prove that your reduction is correct (you do not need to prove any of the observations above).**

Hint: create a new gadget per clause!

Solution: For each clause, we will create a new gadget, and let v_7, v_8, v_9 represent the literals in that clause. Then, depending on their identity (whether the variable is x_i or $\neg x_i$), then we connect them in the way described in part (a). This way, all literals x_i are the same color, and $\neg x_i$ as well.

Further, we will connect all instances of v_1 to v_{BASE} and v_{FALSE}, so they have the same color as v_{TRUE}. If there is a valid coloring on this graph, then we know that this 3-SAT instance has a satisfying assignment.

Proof of Correctness: If this is a valid 3-coloring instance, then we know that each instance of v_1 is colored the same as v_{TRUE}. Then, since the literals x_i (corresponding to v_7, v_8, v_9) are colored either v_{TRUE} or v_{FALSE}, then we know that one of them must be colored the same as v_1, or v_{TRUE}. The coloring of the literals can then be interpreted as the assignment of literals x_i that satisfies the 3-SAT instance (those that have the same color as v_{TRUE} are the literals that need to be set to true to satisfy 3-SAT).

4 k-XOR

In the k-XOR problem, we are given n boolean variables $x_1, x_2, \ldots, x_n$, a list of m clauses each of which is the XOR of exactly k distinct variables (that is, the clause is true if and only if an odd number of the k variables in the clause are true), and an integer r. Our goal is to decide if there is some assignment of variables that satisfies at least r clauses.

(a) In the Max-Cut problem, we are given an undirected unweighted graph $G = (V, E)$ and integer c and want to find a cut $S \subseteq V$ such that at least c edges cross this cut (i.e. have exactly one endpoint in S). Give and argue correctness of a reduction from Max-Cut to 2-XOR.

Hint: every clause in 2-XOR is equivalent to an edge in Max-Cut.

Solution: Following the hint, we can construct a graph using the literals x_i such that for every clause $x_i \oplus x_j$ ($\oplus$ is the symbol for XOR apparently), there exists an edge between x_i and x_j. Then, we can set $c = r$ and find a max-cut on this graph.

If a max-cut is found, then we know that r edges cross the cut. Then, we can set all the variables on one side of the cut to be true, which satisfies r clauses, so we're done.

(b) Give and argue correctness of a reduction from 3-XOR to 4-XOR.

Solution: The key to this problem is to notice that the condition for satisfying 3-XOR is actually the same as 4-XOR, in the sense that the number of literals that are set to true in each clause is the same.

Given a 3-XOR instance, we can convert this into a 4-XOR instance by adding a dummy variable to every clause. Then, we solve the 4-XOR instance. To recover back to the 3-XOR instance, we check whether the dummy variables in each clause were set to true or false.

Case 1: If the dummy variable is set to true, then check if any other variable in that clause is set to true. If yes, then this clause is fine in the 3-XOR instance. Otherwise, this clause will fail the 3-XOR instance.

Case 2: If the dummy variable is set to false, we don't really care about it – it means that the clause was satisfied without the need of the dummy variable, so it will also be satisfied in the 3-XOR instance.

5 Dominating Set (Optional)

A dominating set of a graph $G = (V, E)$ is a subset D of V, such that every vertex not in D is a neighbor of at least one vertex in D. Let the Minimum Dominating Set problem be the task of determining whether there is a dominating set of size $\leq k$. Show that the Minimum Dominating Set problem is NP-Complete. You may assume that G is connected.

Hint: Try reducing from Vertex Cover or Set Cover.

6 Orthogonal Vectors (Optional)

In the 3-SAT problem, we have n variables and m clauses, where each clause is the OR of (at most) three of these variables or their negations. The goal of the problem is to find an assignment of variables that satisfies all the clauses, or correctly declare that none exists.

In the orthogonal vectors problem, we have two sets of vectors A, B. All vectors are in $\{0,1\}^m$, and $|A| = |B| = n$. The goal of the problem is to find two vectors $a \in A, b \in B$ whose dot product is 0, or correctly declare that none exists. The brute-force solution to this problem takes $O(n^2 m)$ time: We compute all $|A||B| = n^2$ dot products between two vectors in A, B, and each dot product takes $O(m)$ time.

Show that if there is a $O(n^c m)$-time algorithm for the orthogonal vectors problem for some $c \in [1, 2)$, then there is a $O(2^{cn/2} m)$-time algorithm for the 3-SAT problem. For simplicity, you may assume in 3-SAT that the number of variables must be even.

Hint: Try splitting the variables in the 3-SAT problem into two groups.