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Problem 1

Show that the mode TE$_{00}$ cannot occur in a rectangular wave guide. [*Hint:* In this case $\omega/c = k$, so Eqs. 9.182 are indeterminate, and you must go back to Eq. 9.181. show that B_z is constant, and hence – applying Faraday's law in integral frrom to a cross section – that $B_z = 0$, so this would be a TEM mode.]

Solution: Using the hint, we go back to Eq. 9.181, which are the differential equations obtained via Maxwell's equations. TE waves have $E_z = 0$, so the equations we are interested in are:

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \tag{1}$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x \tag{2}$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y \tag{3}$$

Further, from equations (i), (ii) and (iii) we also have the relations:

$$-ikE_y = i\omega B_x$$

$$ikE_x = i\omega B_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z$$

Finally, using the relation that $\omega/c = k$, we get:

$$i\frac{\omega}{c^2}E_x = i\frac{\omega}{c}B_y$$

$$-i\frac{\omega}{c^2}E_y = i\frac{\omega}{c}B_x$$

As such, equations 2 and 3 become:

$$\frac{\partial B_z}{\partial y} = 0$$

$$\frac{\partial B_z}{\partial x} = 0$$

So along the xy direction, we have $B_z = $ const., so using Faraday's law we indeed get $B_z = 0$, and hence this is a TEM mode. Therefore, the TE$_{00}$ mode doesn't exist. □

Problem 2

Confirm that the energy in the TE$_{mn}$ mode travels at the group velocity. [*Hint:* Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle u \rangle$ (use Prob. 9.12 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]

Solution: Problem 9.12 gives us formulas formulas for computing $\langle u \rangle$ and $\langle \mathbf{S} \rangle$:

$$\langle u \rangle = \frac{1}{4} \operatorname{Re} \left(\epsilon_0 \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^* + \frac{1}{\mu_0} \tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}}^* \right) \quad \langle \mathbf{S} \rangle = \frac{1}{2\mu_0} \operatorname{Re} \left(\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^* \right)$$

From the book, we know that the equation for B_z is given as:

$$B_z = B_0 \cos \left(\frac{m\pi x}{a} \right) \cos \left(\frac{n\pi y}{b} \right)$$

So, for the x and y directions of the waves, we know that $E_z = 0$, so using equation 9.180 we have:

$$E_x = -\frac{i\omega}{\left(\frac{\omega}{c} \right)^2 - k^2} B_0 \frac{n\pi}{b} \cos \left(\frac{m\pi x}{a} \right) \sin \left(\frac{n\pi y}{b} \right)$$

$$E_y = \frac{i\omega}{\left(\frac{\omega}{c} \right)^2 - k^2} B_0 \frac{m\pi}{a} \sin \left(\frac{m\pi x}{a} \right) \cos \left(\frac{n\pi y}{b} \right)$$

$$B_x = -\frac{ik}{\left(\frac{\omega}{c} \right)^2 - k^2} B_0 \frac{m\pi}{a} \sin \left(\frac{m\pi x}{a} \right) \cos \left(\frac{n\pi y}{b} \right)$$

$$B_y = -\frac{ik}{\left(\frac{\omega}{c} \right)^2 - k^2} B_0 \frac{n\pi}{b} \cos \left(\frac{m\pi x}{a} \right) \sin \left(\frac{n\pi y}{b} \right)$$

And of course $E_z = 0$ because we're dealing with a TE wave. Now, we can compute $\langle u \rangle$:

$$\langle u \rangle = -\frac{\epsilon_0 \omega^2 B_0^2}{4 \left[(\omega/c)^2 - k^2 \right]^2} \left(\left(\frac{n\pi}{b} \right)^2 \sin^2 \left(\frac{n\pi y}{b} \right) \cos^2 \left(\frac{m\pi x}{a} \right) - \left(\frac{m\pi}{a} \right)^2 \sin^2 \left(\frac{m\pi x}{a} \right) \cos^2 \left(\frac{n\pi y}{b} \right) \right)$$

$$-\frac{k^2 B_0^2}{4\mu_0 [(\omega/c)^2 - k^2]^2} \left(\left(\frac{m\pi}{a} \right)^2 \sin^2 \left(\frac{m\pi x}{a} \right) \cos^2 \left(\frac{n\pi y}{b} \right) + \left(\frac{n\pi}{b} \right)^2 \cos^2 \left(\frac{m\pi x}{a} \right) \sin^2 \left(\frac{n\pi y}{b} \right) \right) + \frac{B_0^2}{\mu_0} \cos^2 \left(\frac{m\pi x}{a} \right) \sin^2 \left(\frac{n\pi y}{b} \right)$$

Similarly, we can compute $\langle \mathbf{S} \rangle$ using these formulas. Before we do that, one simplification we can make: because $E_z = 0$, then only the z component of S survives, and hence we have:

$$\langle \mathbf{S} \rangle = \frac{k\omega B_0^2}{2\mu_0 [(\omega/c)^2 - k^2]^2} \left(\left(\frac{n\pi}{b} \right)^2 \cos^2 \left(\frac{m\pi x}{a} \right) \sin^2 \left(\frac{n\pi y}{b} \right) + \left(\frac{m\pi}{a} \right)^2 \sin^2 \left(\frac{m\pi x}{a} \right) \cos^2 \left(\frac{n\pi y}{b} \right) \right) \hat{\mathbf{z}}$$

Now, we are ready to compute the integrals. Integrating $\langle u \rangle$ first, which I did using a calculator (thank you wolfram), I get:

$$\int \langle u \rangle \ da = -\frac{\epsilon \omega^2 B_0^2}{4 \left[(\omega/c)^2 - k^2 \right]^2} \frac{(a^2 n^2 - b^2 m^2)\pi^2}{4ab} - \frac{k^2 B_0^2}{4\mu_0 [(\omega/c)^2 - k^2]^2} \frac{(b^2 m^2 - a^2 n^2)\pi^2}{4ab} + \frac{B_0^2}{4\mu_0} \frac{ab}{4}$$

For $\langle \mathbf{S} \rangle$ (again, wolfram), I get:

$$\int \langle \mathbf{S} \rangle \ d\mathbf{a} = \frac{k\omega B_0^2}{2\mu_0 [(\omega/c)^2 - k^2]^2} \frac{(b^2 m^2 + a^2 n^2)\pi}{4ab}$$

For the $\langle u \rangle$ integral, we can simplify it using $c = \frac{\omega}{k}$, which after some algebra gets us:

$$\int \langle u \rangle \ da = \frac{B_0^2}{4[(\omega/c)^2 - k^2]^2} \frac{-2k^2}{\mu_0} \frac{(a^2 n^2 - b^2 m^2)\pi^2}{4ab} + \frac{B_0^2}{4\mu_0} \left(\frac{ab}{4} \right)$$

From here, we're supposed to take the ratio of the energy per unit time versus the energy per unit length, which means to take the integral of $\langle \mathbf{S} \rangle$ over the integral of $\langle u \rangle$. I couldn't finish the writeup in time though sadly. □

Problem 3

Work out the theory of TM modes for a rectangular wave guide. In particular, find the longitudinal electric field, the cutoff frequencies, and the wave and group velocities. Find the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given wave guide. [*Caution:* What is the lowest TM mode?]

Solution: For TM modes, we have $B_z = 0$, but we follow much of the logic from the derivation of the TE waves. We let $E_z(x, y) = X(x)Y(y)$, and since B_z satisfies the same wave equation as E_z in the TE case, then we also get solutions:

$$X(x) = A \sin(k_x x) + B \cos(k_x x)$$

The boundary condition for conductors requires that $E_z = 0$ inside conductors, so first $B = 0$, so we're left with $X(x) = A \sin(k_x x)$. The same boundary condition then implies:

$$k_x = \frac{m\pi}{a}$$

just like the TE case. Here, $m = 1, 2, 3, \ldots$, and the same goes for the y direction, so we get:

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

with $m, n = 1, 2, 3, \ldots$. Note that we cannot allow either $m, n = 0$ because that would mean zero wave, which is a trivial solution. The formula for the wave number is the same as that in the TE wave case:

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$$

The cutoff frequency then follows the same equation:

$$\omega_c = c\pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

But here, the lowest cutoff frequency is the lowest value of $m = n = 1$, so:

$$\omega_c = c\pi \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

So the ratio of cutoff frequencies is just the ratio of this to ω_{10} in the TE wave case:

$$\frac{\omega_c}{\omega_{10}} = a\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

and that concludes the problem. $\qquad\square$