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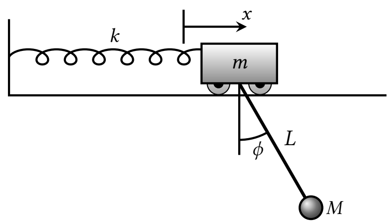
HW 09	Analytic Mechanics	September 7, 2023

Problem 1

A cart of mass m is constrained to move along the x-axis. It is attached by a spring (force constant k) so that the spring is unstretched at $x = 0$. Moreover, a mass M is hangs from the cart by a rigid, massless rod of length L. Gravity acts downwards.



a) Write down the Lagrangian for this system

Solution: The Lagrangian is just the kinetic energy minus potential, so writing out these two terms:

$$T = \dot{x}^2(m + M) + \frac{M}{2}\left[2\dot{x}\dot{\phi}L\cos\phi + L^2\dot{\phi}^2\right]$$
$$U = \frac{1}{2}kx^2 + MgL(1 - \cos\phi)$$

Therefore, the Lagrangian is:

$$\mathcal{L} = \frac{\dot{x}^2}{2}(m + M) + \frac{M}{2}\left(2\dot{x}\dot{\phi}L\cos\phi + L^2\dot{\phi}^2\right) - MgL(1 - \cos\phi) - \frac{1}{2}kx^2$$

□

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b) Simplify your Lagrangian in the case where x and ϕ (and derivatives) are small, and derive the equations of motion in this approximation

Solution: We use the approximation that $\cos\phi \approx 1 - \frac{\phi^2}{2}$, and also we can get rid of terms which are of order 3 or higher in the coordinates. Therefore, our Lagrangian simplifies to:

$$\mathcal{L} = \frac{\dot{x}^2}{2}(m + M) + \frac{M}{2}\left(2\dot{x}\dot{\phi}L + L^2\dot{\phi}^2\right) - MgL(\frac{\phi^2}{2}) - \frac{1}{2}kx^2$$

Now we find the E-L equations:

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}}$$
$$-kx = \ddot{x}(m + M) + ML\ddot{\phi}$$
$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$
$$-MgL\phi = ML\ddot{x} + ML^2\ddot{\phi}$$

This implies the equations of motion:

$$-kx = \ddot{x}(m+M) + ML\ddot{\phi}$$
$$-MgL\phi = ML\ddot{x} + ML^2\ddot{\phi}$$

$\square$

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c) Find the normal frequencies.

Solution: Writing this equation in matrix form, this gives us the matrices:

$$\mathbf{K} = \begin{bmatrix} k & \\ & MgL \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m+M & ML \\ ML & ML^2 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ \phi \end{bmatrix}$$

Therefore, so now we need to calculate $\det\left(\mathbf{K} - \omega^2\mathbf{M}\right)$:

$$KMgL - 2\omega^2L^2k - \omega^2(m+M)MgL + 2\omega^4(m+M)L^2 - (\omega^2ML)^2 = 0$$

This gives solutions:

$$\omega^2 = \frac{g(m+M) + kL \pm \left[-4gkLm + (-kL - gm - gM)^2\right]^{1/2}}{2Lm}$$

$\square$

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d) Find the normal modes. Sketch the motion of each. You do not have to normalize your modes, and you may write your answers in terms of the corresponding frequencies from the previous part.

Solution: Since we need only two solutions for ω to fully describe our system, we can choose the positive root for ω in the solution above. Further, let $\omega_\pm$ refer to these solutions, and a_1 and a_2 be the components of the eigenvector. Therefore, we are solving the system:

$$ka_1 - \omega_\pm^2(m+M)a_1 - \omega_\pm^2 MLa_2 = 0$$
$$-\omega_\pm^2 MLa_1 + MgLa_2 - \omega_\pm^2(ML^2)a_2 = 0$$

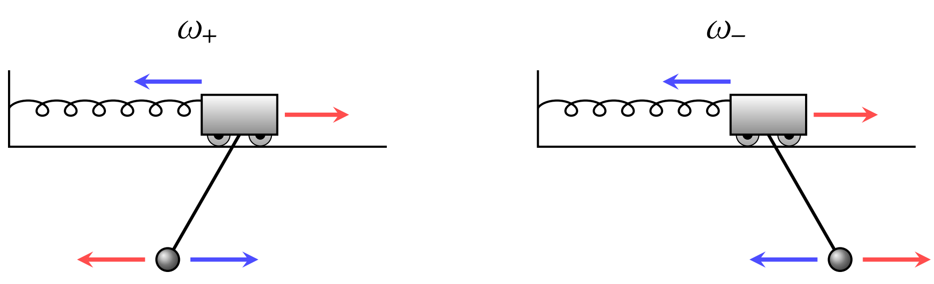
Therefore, from the first equation, we derive a_2 in terms of a_1:

$$a_2 = \frac{(k - \omega_\pm^2(m+M))a_1}{\omega_\pm^2 ML}$$

Then, since a_1 and a_2 are constants we can write them as $Ae^{i\delta}$, and using the relation between a_1 and a_2, we get the modes:

$$x = A\cos(\omega_\pm t - \delta)$$
$$\phi = \frac{k - \omega_\pm^2(m+M)}{\omega^2 Ml}A\cos(\omega t - \delta)$$

Where A is a constant determined by initial conditions. Furthermore, one of the modes (the positive ω_+) will contribute to the masses oscillating in phase, the other (ω_-) will contribute to them oscillating out of phase. Thanks to Andrew Binder for the Ti*k*Z diagram:



□

e) Now suppose there is a small amount of friction acting on the cart, producing a resistive force $-b\dot{x}$. Describe the effect on the normal modes. Is one mode damped? Both?

Solution: We expect both modes to be damped here. The reason is because while the friction only acts on the cart, the equations of motion are coupled together, so changes to the acceleration of the cart are also felt by the mass. This occurs in both modes, since there is nothing fundamentally different about the masses oscillating in phase or out of phase. □

Problem 2

A thin loop of mass M and radius R oscillates in its own plane. It hangs vertically from a single fixed point, with gravity acting downwards. Moreover, a bead of the same mass, M, is constrained to move (without friction) along the hoop. Find the normal frequencies, normal modes and sketch the motion for each mode. You should find that the eigenfrequencies are:

$$\omega_1 = \sqrt{\frac{2g}{R}} \quad \omega_2 = \sqrt{\frac{g}{2R}}$$

Solution: First, we set the point of zero potential to be at the fixed point on the ring. This allows us to express the potential in the easiest way. This potential is equal to:

$$U = -mgR(\cos\phi_1) - mgR(\cos\phi_1) - mgR(\cos\phi_2)$$

The kinetic energy of the hoop can be expressed easily as $T_1 = \frac{1}{2}I\dot{\phi}_1^2$, but the kinetic energy of the bead is slightly trickier. We can make it slightly easier by writing it out in cartesian coordinates, from which we can get:

$$x = R\sin\phi_1 + R\sin\phi_2 \quad y = -R\cos\phi_1 - R\cos\phi_2$$

Therefore, the kinetic energy is (skipping some algebra)

$$T = \frac{m}{2}(R^2(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2)^2 + R^2(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2)^2) = \frac{mR^2}{2}\left[\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right]$$

So the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}I\dot{\phi}_1^2 + \frac{MR^2}{2}\left[\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right] + mgR(2\cos\phi_1 + \cos\phi_2)$$

Using the small angle approximation of $\cos\theta \approx 1 - \frac{\theta^2}{2}$, we now get:

$$\mathcal{L} = \frac{1}{2}I\dot{\phi}_1^2 + \frac{MR^2}{2}(\dot{\phi}_1 + \dot{\phi}_2)^2 + 2MgR(1 - \frac{\phi_1^2}{2})$$

Then, computing Euler-Lagrange equations, we get the following system:

$$\begin{aligned} -2MgR\phi_1 &= 3MR^2\ddot{\phi}_1 + MR^2\ddot{\phi}_2 \\ -MgR\phi_2 &= MR^2\ddot{\phi}_1 + MR^2\ddot{\phi}_2 \end{aligned}$$

Therefore, our matrices are:

$$\mathbf{K} = MgR\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}, \ \mathbf{M} = MR^2\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Now, the determinant of this system is:

$$\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = 2M^2g^2R^2 - \omega^2 MgR + 2M\omega^4 = 0$$

This gives the solutions:

$$\omega^2 = \frac{g}{2R}, \frac{2g}{R} \implies \omega_1 = \sqrt{\frac{g}{2R}}, \omega_2 = \sqrt{\frac{2g}{R}}$$

as expected from the problem statement. The corresponding eigenvectors are:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

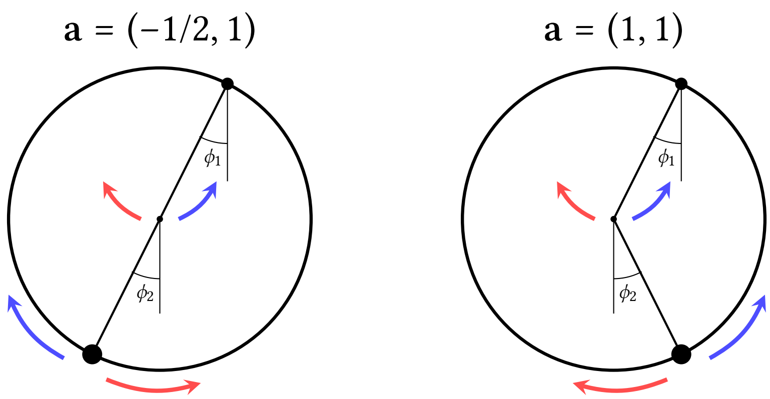
For the $a_1 = a_2 = 1$, the oscillations are:

$$\phi_1(t) = A\cos\left(\sqrt{\frac{g}{2R}} - \delta\right)$$
$$\phi_2(t) = A\cos\left(\sqrt{\frac{g}{2R}}t - \delta\right)$$

In other words, they oscillate in phase with one another. In the other mode, the oscillations are:

$$\phi_1(t) = \frac{1}{2}A\cos\left(\sqrt{\frac{2g}{R}}t - \delta\right)$$
$$\phi_2(t) = -A\cos\left(\sqrt{\frac{2g}{R}}t - \delta\right)$$

Here, the oscillations are out of phase with one another, and further the bead oscillates with an amplitude which is twice that of the hoop. Here's a diagram:



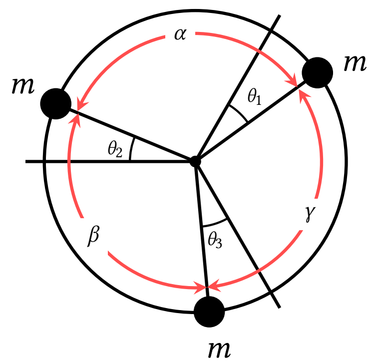
In the left figure, we can see that the masses oscillate out of phase with respect to one another, and specifically the second mass oscillates with an amplitude twice that of the hoop. This corresponds to the eigenvector $\mathbf{a} = (-1/2, 1)$. In the other case, we see that the masses oscillate in phase relative to one another, with the same amplitude of oscillation, corresponding to the eigenvector $\mathbf{a} = (1, 1)$ $\square$

Problem 3

Three particles of equal mass m move on a circle with radius a under forces that can be derived from the potential

$$V(\alpha, \beta, \gamma) = V_0 \left(e^{-2\alpha} + e^{-2\beta} + e^{-2\gamma} \right)$$

where α, β, γ are the angular separations of the masses (in radians) as shown in the figure below. An equilibrium position is indicated by the solid lines extending beyond the circle and has $\alpha = \beta = \gamma = 2\pi/3$. In the following parts, θ_i represents the deviation from this equilibrium of the ith mass in the clockwise direction (see figure)



a) Find the normal mode frequencies using the small amplitude approximation for oscillations about equilibrium.

Solution: The Lagrangian is:

$$\mathcal{L} = \frac{ma^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) - V_0 e^{-4\pi/3} \left[e^{-2(\theta_1 - \theta_2)} + e^{-2(\theta_2 - \theta_3)} + e^{-2(\theta_3 - \theta_1)} \right]$$

Therefore, the equations of motion from doing the Euler-Lagrange equations becomes:

$$\begin{aligned}
ma^2 \ddot{\theta}_1 &= -2V_0 e^{-4\pi/3} \left[e^{-2(\theta_3 - \theta_1)} - e^{-2(\theta_1 - \theta_2)} \right] \\
ma^2 \ddot{\theta}_2 &= -2V_0 e^{-4\pi/3} \left[e^{-2(\theta_1 - \theta_2)} - e^{-2(\theta_2 - \theta_3)} \right] \\
ma^2 \ddot{\theta}_3 &= -2V_0 e^{-4\pi/3} \left[e^{-2(\theta_2 - \theta_3)} - e^{-2(\theta_3 - \theta_1)} \right]
\end{aligned}$$

Then, using the approximation for small amplitude, we write $e^x \approx 1 + x$, which simplifies our equations to:

$$\begin{aligned}
ma^2 \ddot{\theta}_1 &= -2V_0 e{-}4\pi/3(4\theta_1 - 2\theta_2 - 2\theta_3) \\
ma^2 \ddot{\theta}_2 &= -2V_0 e^{-4\pi/3}(4\theta_2 - 2\theta_1 - 2\theta_3) \\
ma^2 \ddot{\theta}_3 &= -2V_0 e^{-4\pi/3}(4\theta_3 - 2\theta_1 - 2\theta_2)
\end{aligned}$$

This means that our matrices are:

$$\mathbf{M} = ma^2 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = ma^2 I, \ \mathbf{K} = 4V_0 e^{-4\pi/3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Solving this system gives:

$$\omega_1^2 = 0, \omega_{2,3}^2 = \frac{12e^{-4\pi/3} V_0}{a^2 m}$$

□

b) Find the corresponding normal modes.

Solution: Solving the eigenvectors from the solution in part (a) gives:

$$a = (1, 1, 1), (-1, 0, 1), (1, -1, 0)$$

Here, there are three modes to this system, which give the equations:

$$\theta_1(t) = (V_0 t + \theta_0)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\theta_2(t) = \begin{pmatrix}-1\\0\\1\end{pmatrix} A\cos(\omega_2 t - \delta)$$

$$\theta_3(t) = \begin{pmatrix}1\\-1\\0\end{pmatrix} A\cos(\omega_3 t - \delta)$$

Where the index on the left hand side refers to the mode of oscillation.

These normal modes make complete sense: in one of the modes, all the masses are shifted by the same angle θ, so it makes sense that no oscillatory motion occurs there. In the other two cases, we see that two masses oscillate such that they move out of phase with respect to each other, while the third mass remains stationary. The third mode is the same thing, except we have a different mass remaining stationary. □

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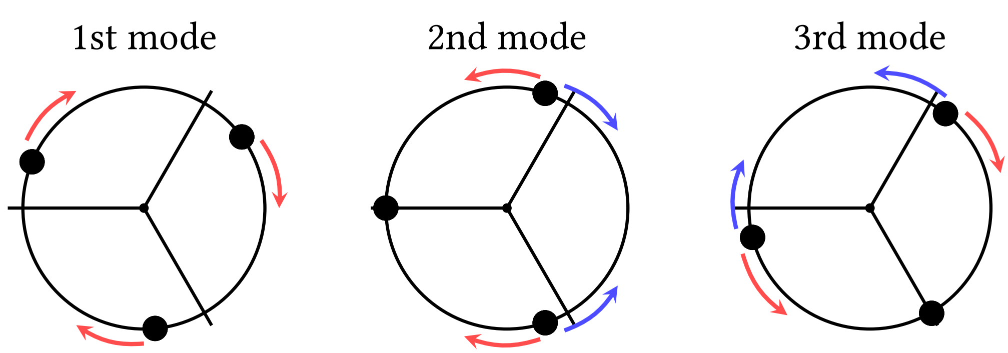
c) Sketch the motion for each of the normal modes you found in (b)

Solution: I've described the oscillation in the previous part, here's the diagram (credit to Andrew Binder for the visualization)



Here, we can see that the first mode corresponds to all the masses moving together, where $\omega = 0$. In the other two cases, we have one mass stationary, while the other two masses oscillate. They are "in phase" in the sense that they both move toward and away from the stationary mass at the same time. Mathematically though, they are expressed as being out of phase. □

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d) Consider the following initial conditions at $t = 0$: $\theta_1 = \theta_2 = \theta_3 = 0$, $\dot{\theta}_1 = 3\omega_0$, $\dot{\theta}_2 = -2\omega_0$, and $\dot{\theta}_3 = -\omega_0$. If ω_0 is sufficiently small, then the small amplitude approximation will be valid. In this case, use your results above to find $\theta_i(t)$ for $i = 1, 2, 3$.

Solution: Since the initial conditions are $\theta_1 = \theta_2 = \theta_3 = 0$, so we expect our oscillatory solutions to be sinusoids. Further, our amplitudes are the initial velocities divided by ω_0. Therefore, the equations of motion are:

$$\theta_1(t) = \frac{3\omega_0}{\omega}\sin(\omega t)$$
$$\theta_2(t) = -\frac{2\omega_0}{\omega}\sin(\omega t)$$
$$\theta_3(t) = -\frac{\omega_0}{\omega}\sin(\omega t)$$

□

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e) Now suppose instead $\dot{\theta}_1 = 4\omega_0$, $\dot{\theta}_2 = -\omega_0$ and $\dot{\theta}_3 = 0$ (we have added ω_0 to each angular velocity). What are the new functions for $\theta_i(t)$? You should be able to answer this part without any new computations.

Solution: The oscillation will be exactly what we expect: the same normal modes, but this time with an added term which describes the addition of velocity:

$$\theta_i'(t) = \theta_i(t) + \omega_0 t$$

□

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f) How small should ω_0 be for (d) to hold? Write your answer in the form $\omega_0 \ll$ something.

Solution: In order for our approximations to hold, we require that $\omega_0 \ll \omega$, since otherwise the initial velocities would dominate over the oscillatory ones. □

Python Homework

The double pendulum is pictured below:

[inset tikz here]

a) Show that the Lagrangian of the system is given by

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_2 - \phi_1) - (m_1 + m_2)gL_1(1 - \cos\phi_1) - m_2gL_2(1 - \cos\phi_2)$$

Solution: Let the potential energy be measured relative to the configuration in which both masses are hanging straight down. Then, we can write the potential energy as:

$$U = m_1gL_1(1 - \cos\phi_1) + m_2gL_1(1 - \cos\phi_1) + m_2gL_2(1 - \cos\phi_2)$$

To calculate the kinetic energy term, we express the coordinates of the second mass in the following way:

$$\begin{aligned} X &= L_1\sin\phi_1 + L_2\sin\phi_2 \\ Y &= -L_1\cos\phi_1 - L_2\cos\phi_2 \end{aligned}$$

And since the kinetic energy is $\frac{1}{2}m_2(\dot{X}^2 + \dot{Y}^2)$, we get:

$$\begin{aligned} T_2 &= \frac{1}{2}m_2\left[L_1^2\dot{\phi}_1^2 + L_2^2\dot{\phi}_2^2 + 2L_1L_2\dot{\phi}_1\dot{\phi}_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)\right] \\ &= \frac{1}{2}m_2\left[L_1^2\dot{\phi}_1^2 + L_2^2\dot{\phi}_2^2 + 2L_1L_2\cos(\phi_2 - \phi_1)\right] \end{aligned}$$

Combining this with the kinetic energy term for the first mass, we have:

$$T = \frac{1}{2}m_1L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2\left[L_1^2\dot{\phi}_1^2 + L_2^2\dot{\phi}_2^2 + 2L_1L_2\cos(\phi_2 - \phi_1)\right]$$

Therefore, the Lagrangian is (with a little bit of rearrangement):

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_2 - \phi_1) - (m_1 + m_2)gL_1(1 - \cos\phi_1) - m_2gL_2(1 - \cos\phi_2)$$

as desired. □

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b) Find the equations of motion

Solution: We have the Lagrangian, so we just have to use the Euler-Lagrange equations. There are two coordinates ϕ_1 and ϕ_2, so first for ϕ_1:

$$\frac{\partial\mathcal{L}}{\partial\phi_1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1}$$

This gets the equation:

$$\begin{aligned} &m_2L_1L_2\dot{\phi}_1\dot{\phi}_2\sin(\phi_2 - \phi_1) - (m_1 + m_2)gL_1\sin\phi_1 \\ &\qquad = (m_1 + m_2)L_1^2\ddot{\phi}_1 + m_2L_1L_2\left[\ddot{\phi}_2\cos(\phi_2 - \phi_1) - \dot{\phi}_2\sin(\phi_2 - \phi_1)(\dot{\phi}_2 - \dot{\phi}_1)\right] \end{aligned}$$

Similarly, the equation for ϕ_2 gets:

$$
\begin{aligned}
&- m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_2 - \phi_1) - m_2 g L_2 \sin\phi_2 \\
&\qquad\qquad = m_2 L_2^2 \ddot{\phi}_2 + m_2 L_1 L_2 \left[\ddot{\phi}_1 \cos(\phi_2 - \phi_1) - \dot{\phi}_1 \sin(\phi_2 - \phi_1)(\dot{\phi}_2 - \dot{\phi}_1)\right]
\end{aligned}
$$

□

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c) Under the small angle approximation $\phi_1, \phi_2 \ll 1$ (keeping only first-order terms), find the normal modes of oscillation, which is pictured below for a simple choice of parameters.

Solution: So if we keep only the first order terms, then we use the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. If we do this, the cross term with $(\dot{\phi}_2 - \dot{\phi}_1)$ term dies because it's of a higher order. Therefore, we get the equations:

$$
\begin{aligned}
-(m_1 + m_2) g L_1 \sin\phi_1 &= (m_1 + m_2) L_1^2 \ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2 \\
-m_2 g L_2 \sin\phi_2 &= m_2 L_2^2 \ddot{\phi}_2 + m_2 L_1 L_2 \ddot{\phi}_1
\end{aligned}
$$

Therefore, our matrices are:

$$
\mathbf{K} = \begin{bmatrix} (m_1 + m_2) g L_1 & \\ & m_2 g L_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} (m_1 + m_2) L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}
$$

Therefore, solving $\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = 0$ means to find solutions to ω that satisfy:

$$
m_2 L_1 L_2 g^2 \left[(m_1 + m_2) g^2 - g(L_1 + L_2)(m_1 + m_2)\omega^2 + L_1 L_2 m_1 \omega^4\right] = 0
$$

This then gives solutions (using WolframAlpha):

$$
\omega_{1,2}^2 = \frac{g(L_1 + L_2)(m_1 + m_2) \pm \left[g^2 (L_1 + L_2)^2 (m_1 + m_2)^2 - 4 g^2 L_1 L_2 m_1 (m_1 + m_2)\right]^{1/2}}{2 m_1 L_1 L_2}
$$

Again just like problem 1, we will only take positive solutions to this equation since there are only two normal modes. To find the components to the eigenvectors, we express a_1 in terms of a_2, which we can do from the equation:

$$
-\omega_{1,2}^2 m_2 L_1 L_2 a_1 + \left[m_2 g L_2 - \omega_{1,2} m_2 L_2^2\right] a_2 = 0
$$

This means that:

$$
a_1 = \frac{(m_2 g L_2 - \omega_{1,2}^2 m_2 L_2^2) a_2}{\omega_{1,2} m_2 L_1 L_2}
$$

This means that the oscillation modes are:

$$
\begin{aligned}
\phi_1(t) &= \frac{(m_2 g L_2 - \omega_{1,2}^2 m_2 L_2^2)}{\omega_{1,2} m_2 L_1 L_2} A \cos\left(\omega_{1,2} + \delta\right) \\
\phi_2(t) &= A \cos\left(\omega_{1,2} + \delta\right)
\end{aligned}
$$

Where A is a constant that is determined via initial conditions. □
