Problem 1

a) Find the Eigenvalues of A. Verify that $\vec{v}_1$ is indeed an eigenvector and find its eigenvalue. Pick *one* of the remaining two eigenvalues and find the eigenvector (you don't need to normalize)

Solution: To show that $\vec{v}_1$ is an eigenvector, we can just compute $A\vec{v}_1$:

$$A\vec{v}_1 = \begin{pmatrix} 1 & 0 & 2i \\ 0 & 3 & 0 \\ 2i & 0 & -2 \end{pmatrix} \begin{pmatrix} -2i \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2i - 2i \\ 0 \\ -4i^2 - 2 \end{pmatrix} = \begin{pmatrix} -4i \\ 0 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -2i \\ 0 \\ 1 \end{pmatrix}$$

so we see that we get back $\vec{v}_1$, meaning it is an eigenvector, and its associated eignenvalue is 2.

To find the other eigenvalues, we find the characteristic polynomial using $\det(A - \lambda\mathbb{1}) = 0$:

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & 0 & 2i \\ 0 & 3-\lambda & 0 \\ 2i & 0 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda)(-2-\lambda) + (-2i)(-(3-\lambda)(2i)) \\ &= -(1-\lambda)(6+\lambda-\lambda^2) - 12 + 4\lambda \\ &= -6 - \lambda + \lambda^2 + 6\lambda + \lambda^2 - \lambda^3 - 12 + 4\lambda \\ &= -\lambda^3 + 2\lambda^2 + 9\lambda - 18 \end{aligned}$$

We can use a root finder here, which will give us eigenvalues of $\lambda = -3, 3, 2$. Since $\lambda = 2$ is given by $\vec{v}_1$, we'll do $\lambda = 3$. To find the eigenvector, we now solve the equation $\det(A - 3\mathbb{1}) = \vec{0}$. This can be done by augmenting our matrix with zeroes on the right column:

$$\tilde{A} = \begin{pmatrix} -2 & 0 & -2i & 0 \\ 0 & 0 & 0 & 0 \\ 2i & 0 & -5 & 0 \end{pmatrix}$$

Now we can begin the process of row reduction. First, swap rows 2 and 3, and also divide row 1 by 2:

$$\begin{pmatrix} 1 & 0 & i & 0 \\ 2i & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now subtract row 2 by $2i$ times row 1, $(R_2 - 2iR_1)$:

$$\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

At this point, we can divide row 2 by 2, thereby making it a pivot, then use the pivot to kill the third entry in the first row. Doing so, we get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, this gives the equations $a = 0$ and $c = 0$, but there's no condition on b. Thus, any vector of the form $\vec{v} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$ would be an eigenvector. In its simplest form, $b = 1$, so our eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Also, as per the exam policy, I admit that I did use Mathematica to verify that this is indeed the correct eigenvector. □

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b) Find the rotation matrix $R_{[e]}(\theta)$.

Solution: We'll first find $R_{[f]}$, since the matrix $L_{[f]}$ is diagonal in this basis, so it allows for easy application of the exponential. Recall the convenient relation for diagonal matrices:

$$f(D) = \begin{pmatrix} f(d_{11}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & f(d_{nn}) \end{pmatrix}$$

provided that f is a Taylor-expandable function. Since e^x is such a function, we can apply this result here to make our lives very easy:

$$R_{[f]} = e^{\theta L_{[f]}} = \exp\left(\begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}\right) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Now we have to find $R_{[e]}$, which can be found by performing a change of basis, $R_{[e]} = C^\dagger R_{[f]} C$. Therefore:

$$\begin{aligned} R_{[e]}(\theta) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & ie^{i\theta} \\ e^{-i\theta} & -ie^{-i\theta} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{i\theta} + e^{-i\theta} & i(e^{i\theta} - e^{-i\theta}) \\ i(e^{-i\theta} - e^{i\theta}) & e^{i\theta} + e^{-i\theta} \end{pmatrix} \end{aligned}$$

Here's where we remember our good old formulas for sine and cosine:

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Therefore, this simplifies to:

$$R_{[e]}(\theta) = \frac{1}{2} \begin{pmatrix} 2\cos\theta & -2\sin\theta \\ 2\sin\theta & 2\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Which just so happens to be our familiar formula for 2D rotation. □

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c) Solve Eq. 1 with the initial condition $h(0) = h_0$ to find $h(t)$. For what range of t does our solution make physical sense?

Solution: First, divide both sides by B:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\mu A}{B}\sqrt{2g}\cdot\sqrt{h}$$

Let's let $-k = -\frac{\mu A}{B}\sqrt{2g}$ to clear up the clutter of constants. This now means we have the differential equation:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -k\sqrt{h}$$

This differential equation can be solved by separation of variables:

$$\begin{aligned}\frac{\mathrm{d}h}{\sqrt{h}} &= -k\mathrm{d}t\\ \int\frac{\mathrm{d}h}{\sqrt{h}} &= -\int k\ \mathrm{d}t\\ 2\sqrt{h(t)} &= -kt + C\end{aligned}$$

Solving for $h(t)$, we get:

$$h(t) = \left(-\frac{kt}{2} + C\right)^2$$

The constant C can be solved using the initial condition $h(0) = h_0$:

$$h_0 = C^2 \implies C = \sqrt{h_0}$$

We reject the negative solution $C = -\sqrt{h_0}$ since having a negative height makes no sense. Therefore, our full solution is:

$$h(t) = \left(-\frac{\mu A t}{2B}\sqrt{2g} + \sqrt{h_0}\right)^2$$

Further, this equation only makes sense starting from $t = 0$ until the moment when $h(t)$ reaches zero (since our tank can't drain further). The root occurs at:

$$-\frac{\mu A t}{2B}\sqrt{2g} + \sqrt{h_0} = 0 \implies t = \frac{2B\sqrt{h_0}}{\mu A\sqrt{2g}}$$

Therefore, our equation only makes sense from $t = 0$ to $t = \frac{2B}{\mu A}\sqrt{\frac{h_0}{2g}}$. In reality, after this time, the equation $h(t)$ begins to rise (since it's quadratic), which again isn't physical in our context since this would imply that the water height is increasing despite not adding any water into the bucket. □

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d) First, starting from an ansatz, find a pair of linearly independent solutions to the homogeneous version of Eq. 1. Then find a particular solution to this inhomogeneous ODE.

Solution: The homogeneous equation is:

$$\ddot{y} + 2b\dot{y} + \omega_0 y = 0$$

This differential equation can be solved using the ansatz [1] of $y = e^{\lambda t}$. Computing derivatives:

$$\begin{aligned}\dot{y} &= \lambda e^{\lambda t}\\ \ddot{y} &= \lambda^2 e^{\lambda t}\end{aligned}$$

[1] Technically speaking, I think there should be an $Ae^{\lambda t}$ to account for initial conditions, but this constant can be handled also by the linear combination, which is also a solution to the homogeneous ODE.

Plugging this into the homogeneous ODE:

$$\lambda^2 e^{\lambda t} + 2b\lambda e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0 \implies \lambda^2 + 2b\lambda + \omega_0^2 = 0$$

Therefore, we can use the quadratic equation, to find the solutions:

$$\begin{aligned}\lambda_\pm &= \frac{-2b \pm \sqrt{(4b)^2 - 4\omega_0^2}}{2} \\ &= -b \pm \sqrt{b^2 - \omega_0^2}\end{aligned}$$

Since $b < \omega_0$, this quantity is complex, and using $\widetilde{\omega}$ from the problem statement, we can write this as:

$$\lambda_\pm = -b \pm i\widetilde{\omega}$$

We also confirm that these two solutions are linearly independent, since they are different exponentials (we proved this in homework). To find the particular solution, we can use variation of parameters, since our inhomogeneity isn't an element of our table of guesses. For the sake of clarity, I will use $\lambda_1 = -b + i\widetilde{\omega}$ and $\lambda_2 = -b - i\widetilde{\omega}$ to refer to the exponents of the two homogeneous solutions. Our general solution with the variation of parameters approach is to have solution of the form $y(t) = c_1(t)y_{h1}(t) + c_2(t)y_{h2}(t)$. Before we get to calculating c_1 and c_2, let's first find the Wronskian, which will be useful later:

$$W(t) = \begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t}\end{vmatrix} = \lambda_2 e^{(\lambda_1+\lambda_2)t} - \lambda_1 e^{(\lambda_1+\lambda_2)t} = e^{(\lambda_1+\lambda_2)t}(\lambda_2 - \lambda_1)$$

With that out of the way, we are ready to construct the particular solution. From lecture, we know that for an ODE with two homogeneous solutions, we have the set of equations (this form comes directly as a result of Cramer's rule):

$$\dot{c}_1(t) = \frac{\det\begin{pmatrix} 0 & y_{h2}(t) \\ r(t) & \dot{y}_{h2}(t)\end{pmatrix}}{W(t)} \quad \dot{c}_2(t) = \frac{\det\begin{pmatrix} y_{h1}(t) & 0 \\ \dot{y}_{h1}(t) & r(t)\end{pmatrix}}{W(t)}$$

where $y_{h1}(t)$ and $y_{h2}(t)$ refer to the homogeneous solutions, and $r(t)$ is our inhomogeneity. Thus, plugging everything in, we get:

$$\dot{c}_1(t) = \frac{\det\begin{pmatrix} 0 & e^{\lambda_2 t} \\ A\delta(t-t_0) & \lambda_2 e^{\lambda_2 t}\end{pmatrix}}{W(t)} = -\frac{Ae^{\lambda_2 t}\delta(t-t_0)}{e^{(\lambda_2+\lambda_1)t}(\lambda_2 - \lambda_1)}$$

Likewise for $c_2(t)$, skipping the algebra here:

$$\dot{c}_2(t) = \frac{Ae^{\lambda_1 t}\delta(t-t_0)}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)t}}$$

Then to find $c_1(t)$ and $c_2(t)$, we integrate both of these from 0 to t (this also comes from the lecture notes):

$$\begin{aligned} c_1(t) &= \int_0^t -\frac{Ae^{\lambda_2 t}\delta(t-t_0)}{e^{(\lambda_2+\lambda_1)t}(\lambda_2-\lambda_1)}dt \\ c_2(t) &= \int_0^t \frac{Ae^{\lambda_1 t}\delta(t-t_0)}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)t}}dt\end{aligned}$$

Now t_0 is some positive value. Recall that for a Dirac delta,

$$\int_a^b f(u)\delta(u-u_0) = \begin{cases} f(u_0) & a < u_0 < b \\ 0 & \text{else}\end{cases}$$

So in order for our integral to be nonzero, we require that $t > t_0$. When $t < t_0$, then we have $c_1(t) = c_2(t) = 0$. Therefore, we now have our solutions $c_1(t)$ and $c_2(t)$:

$$c_1(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ -\frac{Ae^{\lambda_2 t_0}}{(\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)t_0}} & t > t_0 \end{cases}$$
$$c_2(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ \frac{Ae^{\lambda_1 t_0}}{(\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)t_0}} & t > t_0 \end{cases}$$

We can now plug back in the values $\lambda_1 = -b + i\sqrt{\omega_0^2 - b^2}$ and $\lambda_2 = -b - i\widetilde{\omega}$. First, we can compute their sum and difference:

$$\lambda_2 - \lambda_1 = -b - i\widetilde{\omega} - (-b + i\widetilde{\omega}) = -2i\widetilde{\omega}$$
$$\lambda_2 + \lambda_1 = -b - i\widetilde{\omega} + (-b + i\widetilde{\omega}) = -2b$$

Therefore, our $c_1(t)$ and $c_2(t)$ simplify to:

$$c_1(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ \dfrac{Ae^{(-b-i\widetilde{\omega})t_0}}{2i\widetilde{\omega}e^{-2bt_0}} & t > t_0 \end{cases}$$
$$c_2(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ -\dfrac{Ae^{(-b+i\widetilde{\omega})t_0}}{2i\widetilde{\omega}e^{-2bt_0}} & t > t_0 \end{cases}$$

Therefore, our particular solution is of the form $y(t) = c_1(t)e^{(-b+i\widetilde{\omega})t} + c_2(t)e^{-(b+i\widetilde{\omega})t}$ where $c_1(t)$ and $c_2(t)$ are defined piecewise above. $\square$

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e) Plug the power series ansatz of Frobenius, $y(x) = \sum_{m=0} a_m x^{m+r}$, into this ODE. Find the indical equation to determine r. Find the recursion relation from the coefficient of the x^{r+m} term ($m \geq 0$). Finally, find the solution for $n = 2$.

Solution: First, we'll take derivatives of the ansatz:

$$y'(x) = \sum_{m=0} a_m(r+m)x^{r+m-1}$$
$$y''(x) = \sum_{m=0} a_m(r+m)(r+m-1)x^{r+m-2}$$

Therefore, our differential equation is:

$$x\sum_{m=0} a_m(r+m)(r+m-1)x^{r+m-2} + (1-x)\sum_{m=0} a_m(r+m)x^{r+m-1} + n\sum_{m=0} a_m x^{r+m} = 0$$

Expanding out the $1-x$ term, we get:

$$\sum_{m=0} a_m(r+m)(r+m-1)x^{r+m-1} + \sum_{m=0} a_m(r+m)x^{r+m-1} - \sum_{m=0} a_m(r+m)x^{r+m} + n\sum_{m=0} a_m x^{r+m} = 0 \quad (1)$$

To find the indical equation, we substitute in $m = 0$ to get:

$$a_0 r(r-1)x^{r-1} + a_0 r x^{r-1} - a_0 r x^r + n a_0 x^r = 0$$

We look specifically in front of the x^{r-1} term, which gives us the equation:

$$a_0 r(r-1) + a_0 r = 0 \implies r^2 = 0 \implies r = 0$$

Now to find the recursion relation, we have to shift the first two terms in equation 1 over by 1 so that the exponents match up (while simultaneously letting $a_{-1} = 0$). At this point we can also substitute in $r = 0$, meaning that we are looking at the coefficient of the x^m term now:

$$\sum_{m=-1} a_{m+1}(m+1)mx^m + \sum_{m=-1} a_{m+1}(m+1)x^m - \sum_{m=0} a_m m x^m + n\sum_{m=0} a_m x^m = 0$$

We can now throw this under one large summation:

$$\sum_{m=-1} \left[a_{m+1}(m+1)m + a_{m+1}(m+1) - a_m m + n a_m\right] x^m = 0$$

The coefficient must be zero, so therefore we get the equation:

$$\left[m(m+1) + (m+1)\right] a_{m+1} - \left[m - n\right] a_m = 0$$

Now solving for a_{m+1}:

$$a_{m+1} = \frac{m-n}{m(m+1)+m+1} a_m = \frac{m-n}{(m+1)^2} a_m$$

For $n = 2$, the solution becomes:

$$a_{m+1} = \frac{m-2}{(m+1)^2} a_m$$

We'll express the solution in terms of a_0. For a_1, we plug in $m = 0$:

$$a_1 = -\frac{2}{1} = -2a_0$$

Then for a_2, we plug in $m = 1$:

$$a_2 = -\frac{1}{2^2} = -\frac{1}{4} a_1 = \frac{1}{2} a_0$$

Then for a_3, we plug in $n = 2$, but we find that $a_3 = 0a_2 = 0$. Since all subsequent terms depend on a_3 recursively, we conclude that the sequence terminates. Therefore, our polynomial is:

$$y(x) = \frac{1}{2} a_0 x^2 - 2a_0 x + a_0$$

□

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f) Find the exponential Fourier coefficients c_n of this function.

Solution: The coefficients are calculated using the formula:

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} e^{-in\omega_0 t} f(t) dt$$

We can choose $t_0 = 0$, giving us:

$$c_n = \frac{1}{T} \int_0^T e^{-in\omega_0 t} A_0 e^{-\alpha(t-mT)} dt = \frac{A_0}{T} \int_0^T e^{-(in\omega_0+\alpha)t+\alpha mT} dt = \frac{A_0}{T} e^{\alpha mT} \int_0^T e^{-(in\omega_0+\alpha)t} dt$$

This integral evaluates to:

$$\begin{aligned}
c_n &= \frac{A_0}{T} e^{\alpha m T} \cdot \frac{1}{-(in\omega_0 + \alpha)} \left[e^{-(in\omega_0+\alpha)t} \right]_0^T \\
&= \frac{A_0 e^{\alpha m T}}{T(in\omega_0 + \alpha)} \left[1 - e^{-(in\omega_0+\alpha)T} \right] \\
&= -\frac{A_0 e^{\alpha m T}}{T(in\omega_0 + \alpha)} \left[e^{-(in\omega_0+\alpha)T} - 1 \right]
\end{aligned}$$

I don't see any real point in making the substitution $\omega_0 = \frac{2\pi}{T}$ since it doesn't really lead to any meaningful simplifications, so this is my final answer. □

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g) Find the convolution $\Lambda_a(x) \equiv (\Pi_a * \Pi_a)(x)$ of the rectangular function with itself. Then find $\lambda_a(k)$, the Fourier transform of $\Lambda_a(x)$.

Solution: The convolution of two functions is given as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(s) g(x - s) ds$$

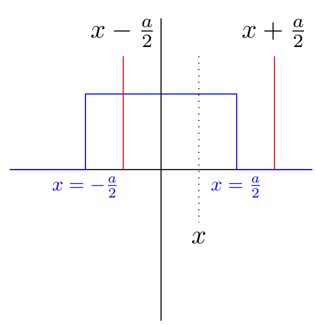
Substituting in the rectangular function, we have:

$$(\Pi_a * \Pi_a)(x) = \int_{-\infty}^{\infty} \Pi_a(s) \Pi_a(x - s) ds$$

First thing to note is that this integral is only nonzero when $|s| < \frac{a}{2}$, since we need $\Pi_a(s)$ to be nonzero. This means we can change our integral bounds to:

$$\Lambda_a(x) = \int_{-\frac{a}{2}}^{\frac{a}{2}} \Pi_a(x - s) ds$$

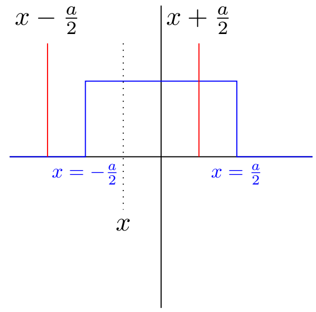
Now look at this visual diagram:



As shown in the diagram, the integration is actually only nonzero over the interval $x \in [x - \frac{a}{2}, \frac{a}{2}]$. Further, when $x \geq a$, we get an integral of zero, since there will be no overlap. We can see this from the fact that when $x = a$, the aforementioned interval becomes $x \in [\frac{a}{2}, \frac{a}{2}]$ which is of zero width (resulting in a zero integral), and any value $x > a$ gives us an interval where the first value is larger than the second, meaning it's an invalid interval. Thus, there when $x > a$ there is no interval where we have a nonzero integral. So now we can restrict ourselves to $0 < x < a$, which is the integral:

$$\Lambda_a(x) = \int_{x-\frac{a}{2}}^{\frac{a}{2}} 1 ds = \frac{a}{2} - \left(x - \frac{a}{2} \right) = a - x$$

On the other hand, for $x < 0$, it's kind of the same story: when $x < -a$, the integral becomes zero since there's no overlap (for the same reason as mentioned earlier), and when $-a < x < 0$, we have this diagram:



Now the integral is only nonzero over the interval $x \in [-\frac{a}{2}, x + \frac{a}{2}]$, so our integral becomes:

$$\Lambda_a(x) = \int_{-\frac{a}{2}}^{x+\frac{a}{2}} 1 ds = x + \frac{a}{2} - \left(-\frac{a}{2}\right) = x + a$$

Therefore, our $\Lambda_a(x)$ can be defined piecewise:

$$\Lambda_a(x) = \begin{cases} x + a & -a \leq x \leq 0 \\ -x + a & 0 \leq x \leq a \\ 0 & \text{else} \end{cases}$$

To find the Fourier transform of $\Lambda_a(x)$, we can use the convolution theorem from homework, which states that the Fourier transform of a convolution of two functions is the product of the Fourier transforms of the functions individually, or equivalently:

$$\mathcal{F}[f * g](k) = \mathcal{F}[f](k)\ \mathcal{F}[g](k)$$

Since the Fourier transform of the rectangular function is already given to us, this becomes:

$$\lambda_a(k) = \mathcal{F}[\Lambda_a](k) = \mathcal{F}[\Pi_a * \Pi_a](k) = \pi_a(k)\pi_a(k) = \frac{4}{k^2}\sin^2\left(\frac{ka}{2}\right)$$

$\square$

Problem 2

a) Find the quadratic form matrix Q for the potential $U(x, y)$ of Eq. 3 and verify that it gives the correct potential energy function.

Solution: We're given:

$$U(x, y) = 3\alpha x^2 + 12\alpha xy - 4\alpha y^2$$

We know that a matrix Q of the form (this is from the lecture notes):

$$Q = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

gives an equation of the form

$$U(x, y) = \frac{1}{2}\vec{x}^\top Q x = \frac{1}{2}(ax^2 + 2bx + dy^2)$$

meaning that we can just read off the entries of Q from the equation itself, then multiply by 2 to account for the $\frac{1}{2}$:

$$Q = 2\alpha \begin{pmatrix} 3 & 6 \\ 6 & -4 \end{pmatrix}$$

Checking that this indeed gives us the equation we want:

$$\begin{aligned}
\frac{1}{2}\vec{x}^\top Q\vec{x} &= \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix} 2\alpha \begin{pmatrix} 3 & 6 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \alpha \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3x + 6y \\ 6x - 4y \end{pmatrix} \\
&= \alpha[x(3x + 6y) + y(6x - 4y)] \\
&= 3\alpha x^2 + 12\alpha x_6 - 4\alpha y^2
\end{aligned}$$

indeed, it works out. □

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b) Find the eigenvalues and normalized eignevectors of the matrix. Make a copy of the contour plot and draw in the eigenvectors.

Solution: We can write the matrix $Q_{[e]}$ as:

$$Q_{[e]} = 2\beta \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

The eigenvalues are given by $\det(A - \lambda_i \mathbb{1}) = \vec{0}$:

$$\begin{aligned}
0 &= \begin{vmatrix} 1 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix} = (1 - \lambda)(-2 - \lambda) - 4 \\
&= \lambda^2 + \lambda - 6 \\
&= (\lambda + 3)(\lambda - 2)
\end{aligned}$$

Therefore we get eigenvalues of $\lambda = -3, 2$. Multiplying by 2β, we get eigenvalues of $\lambda_1 = -6\beta, \lambda_2 = 4\beta$. To solve for the eigenvectors, we do the same process as part a). I'll stick to eigenvalues of $\lambda = -3, 2$

since the 2β is just a constant factor outside the matrix, and doesn't change the eigenvectors. Setting up the row reduction and augmented matrix with $\lambda = -3$:

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

We can first do $R_1 - 2R_2$ then swap the two rows, giving us:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives the equation $2a + b = 0$, or $b = -2a$. If we let $a = t$, then the vector is of the form:

$$\vec{v}_1 = t \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Setting $t = 1$ gives

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Normalizing this vector means $\hat{v} = \frac{\vec{v}}{|v|}$, so:

$$\hat{v}_1 = \frac{1}{\sqrt{(-2)^2 + 1^2}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

A similar process follows for $\lambda = 2$:

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{pmatrix}$$

We can do $R_2 + 2R_1$ and also multiply row 1 by -1:

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives the equation $a - 2b = 0$, so $a = 2b$. Like before, letting $b = t$ gives:

$$\vec{v}_2 = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

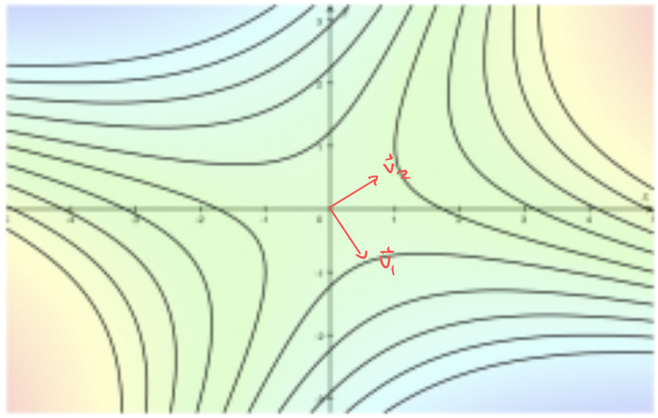
Setting $t = 1$:

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Normalizing:

$$\hat{v} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

As for the plot, I did it on my iPad and copied it over:



Again, as per the exam policy, I will admit that I did use Mathematica to verify my answer. □

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c) Find the change of basis matrix C that diagonalizes $Q_{[e]}$. What is the diagonalized matrix $Q_{[f]}$? Show that in the eigenbasis, Eq. 5 decouples and find the ODE for the variable c_1. Use the change-of-basis matrix to find the components $\vec{r}_{[f]}$ for the position vector $\vec{r}_{[e]} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution: The matrix that will diagonalize $Q_{[e]}$ is going to be composed of the eigenvectors of Q, so

$$C = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$$

Furthermore, we know that the diagonalized form of the matrix $Q_{[f]}$ is going to be a diagonal matrix with eigenvalues on the diagonal:

$$Q_{[f]} = \begin{pmatrix} \gamma & 0 \\ 0 & 5\gamma \end{pmatrix}$$

In the eigenbasis, the differential matrix equation still holds:

$$\ddot{\vec{r}}_{[f]} = -\frac{1}{m}Q_{[f]}\vec{r}_{[f]} = -\frac{1}{m}\begin{pmatrix} \gamma & 0 \\ 0 & 5\gamma \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} \frac{\gamma}{m}c_1 \\ \frac{5\gamma}{m}c_2 \end{pmatrix}$$

This gives us two independent differential equations:

$$\ddot{c}_1(t) = -\frac{\gamma}{m}c_1$$
$$\ddot{c}_2(t) = -\frac{5\gamma}{m}c_2$$

hence they are decoupled. Finally, to find $\vec{r}_{[f]}$ for $\vec{r}_{[e]} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we use the relation $\vec{r}_{[f]} = C^{-1}\vec{r}_{[e]}$. First, we notice that the set $\{\hat{f}_i\}$ is actually an orthonormal set:

$$\hat{f}_1 \cdot \hat{f}_2 = \frac{1}{2}\left(\sqrt{3} - \sqrt{3}\right) = 0$$

and we know that the set $\{\hat{e}_i\}$ is also orthonormal. Since both sets are orthonormal, then we have the result from lecture that C (the change of basis matrix) must be unitary. Therefore, $C^{-1} = C^\dagger$:

$$C^{-1} = C^\dagger = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$$

(As per the exam policy, I admit that I did use Mathematica to check that this is indeed the inverse.) Therefore, we can now calculate $\vec{r}_{[f]}$:

$$\vec{r}_{[f]} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3}+2 \\ 1-2\sqrt{3} \end{pmatrix}$$

$\square$

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d) Find all possible values of w_1 and w_2 in terms of γ such that $\vec{r}_1(t) = e^{iw_1 t}\hat{f}_1$ and $\vec{r}_2(t) = e^{iw_2 t}\hat{f}_2$ solve Eq. 5. From this, determine the most general real solution $r(t)$ in terms of $\hat{f}_i$ and γ. Finally, briefly describe how to find $\vec{r}_{[e]}(t)$ given initial conditions $\vec{r}_{[e]}(0)$ and $\vec{v}_{[e]}(0) = \dot{\vec{r}}_{[e]}(0)$.

Solution: The most general real solution $\vec{r}(t)$ is composed of a linear combination of these two vectors $\vec{r}_1(t)$ and $\vec{r}_2(t)$:

$$\vec{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix}$$

From the earlier decoupling, we know that $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are a part of two independent differential equations, since the eigenbasis decouples these two equations:

$$\ddot{r}_1(t) = -\frac{\gamma}{m} r_1(t)$$
$$\ddot{r}_2(t) = -\frac{5\gamma}{m} r_2(t)$$

Letting $r_1(t) = e^{iw_1 t}$ (we neglect the $\hat{f}_1$ since it's our basis vector, and here I'm only looking at the magnitude), we find:

$$-w_1 e^{iw_1 t} = -\frac{\gamma}{m} e^{iw_1 t} \implies w_1^2 = \frac{\gamma}{m} \implies w_1 = \pm\sqrt{\frac{\gamma}{m}}$$

Similarly for $r_2(t)$:

$$-w_2^2 e^{iw_2 t} = -\frac{5\gamma}{m} e^{iw_2 t} \implies w_2^2 = \frac{5\gamma}{m} \implies w_2 = \pm\sqrt{5\frac{\gamma}{m}}$$

Now, since both differential equations are homogeneous and there are two solutions for w_1 and w_2, the most general solution for $r_1(t)$ is a linear combination of the two, and the most general solution for $r_2(t)$ would also be the same. Thus, the general solution for $r(t)$ is the combination of these two:

$$\vec{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix} = \begin{pmatrix} Ae^{i\sqrt{\frac{\gamma}{m}}t} + Be^{-i\sqrt{\frac{\gamma}{m}}t} \\ Ce^{i\sqrt{\frac{5\gamma}{m}}t} + De^{-i\sqrt{\frac{5\gamma}{m}}t} \end{pmatrix}$$

where A, B, C, D represent the constants for the linear combinations. Written out in terms of $\hat{f}_i$, this is:

$$r_{[f]}(t) = \left(Ae^{i\sqrt{\frac{\gamma}{m}}t} + Be^{-i\sqrt{\frac{\gamma}{m}}t}\right)\hat{f}_1 + \left(Ce^{i\sqrt{\frac{5\gamma}{m}}t} + De^{-i\sqrt{\frac{5\gamma}{m}}t}\right)\hat{f}_2$$

We can solve for A, B, C, D using initial conditions. In order to find $\vec{r}_{[e]}(t)$, there are a couple ways we can do this. Since we have the solution $\vec{r}_{[f]}(t)$, we can transform $\vec{r}_{[f]}(t) \to \vec{r}_{[e]}(t)$ via the change of basis matrix, this time using the equation $\vec{r}_{[e]}(t) = C\vec{r}_{[f]}(t)$. Then, once we have $\vec{r}_{[e]}(t)$, we can solve for the constants A, B, C, D using the given initial conditions. For each of $\hat{f}_1$ and $\hat{f}_2$, there are two equations for the initial condition, meaning that there are sufficiently many initial conditions to solve for the four unknowns A, B, C, D. $\square$

Problem 3

a) Given Eq. 6, what are the possibilities for the rank and type of tensor that P can be? For each type possibility, write one possible expression for the piezoelectric equation Eq. 6 in index notation. Be sure to explicitly state which contractions is/are taking place. What units must D have?

Solution: We know the form of σ, a type (0, 2) tensor:

$$\sigma = \sigma_{ij}\hat{e}^i \otimes \hat{e}^j$$

We also know the form of D, a type (3, 0) tensor:

$$D = D^{ijk}\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k$$

Therefore, the tensor product $P = D\sigma$ gives us a type (3, 2) tensor, in index notation:

$$P^{ijk}_{lm} = (D^{ijk}\sigma_{lm})\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}^l \otimes \hat{e}^m$$

As for the different ways we could contract:

- No contractions: We can leave it as is, meaning we have

$$P^{ijk}_{lm} = (D^{ijk}\sigma_{lm})\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}^l \otimes \hat{e}^m$$

- One contraction: I'll contract i and l, which will give us a type (2, 1) tensor:

$$P^{ijk}_{im} = (D^{ijk}\sigma_{im})\hat{e}_j \otimes \hat{e}_k \otimes \hat{e}^m$$

- Two contractions: I'll contract i, l as well as j, m, giving us a type (1, 0) tensor:

$$P^{ijk}_{ijm} = (D^{ijk}\sigma_{ij})\hat{e}_k$$

There are no other possibilities, since there is no dual basis vector to contract with $\hat{e}_k$, so this is a complete list. As for the units of D, we know that σ has units of force/area, and P has units charge/area. Therefore, D must have units of charge/force. □

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b) Explicitly show that $B^{ijk}\sigma_{jk} = 0$ (this is basically saying that the anti-symmetric part of D doesn't contribute to the piezoelectric equation Eq. 7).

Solution: We can show this by first writing out $B^{ijk}\sigma_{jk}$

$$\begin{aligned} B^{ijk}\sigma_{jk} &= \frac{1}{2}\left(D^{ijk} - D^{ikj}\right)\sigma_{jk} \\ &= \frac{1}{2}D^{ijk}\sigma_{jk} - \frac{1}{2}D^{ikj}\sigma_{jk} \end{aligned}$$

Now, since σ_{jk} is a symmetric tensor, we can flip the k and j on the σ in the second term without changing the expression, giving:

$$\frac{1}{2}D^{ijk}\sigma_{jk} - \frac{1}{2}D^{ikj}\sigma_{kj}$$

Now notice that these are actually the exact same summation since the indices line up perfectly. To be a bit more explicit, let's swap k and j on the second term (we can do this since they're dummy indices):

$$B^{ijk}\sigma_{jk} = \frac{1}{2}D^{ijk}\sigma_{jk} - \frac{1}{2}D^{ijk}\sigma_{jk} = 0$$

Therefore, we conclude that $B^{ijk}\sigma_{jk} = 0$. □

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c) How many independent components does D have? (Note that we are in three-dimensions here!)

Solution: Since $D^{ijk} = D^{ikj}$, this means that there's symmetry in j and k. Thus, for the pair (j,k), there are six independent components: $(j,k) = (1,1), (1,2), (1,3)(2,2), (2,3), (3,3)$. For each of these, we can have $i = 1,2,3$ since there aren't any restrictions on i, so therefore there are $3 \times 6 = 18$ independent components. $\square$

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d) How do the components of D^{ijk} and σ_{jk} of the piezoelectric and stress tensors transform under S?

Solution: S is an active transformation, and we know that for a general tensor, they transform as follows under an active transformation (from the big tensor table):

$$T'^{i_1,\ldots,i_p}_{j_1,\ldots,j_p} = S^{i_1}_{k_1} \cdots S^{i_p}_{k_p} T^{k_1,\ldots,k_p}_{l_1,\ldots l_q} (S^{-1})^{l_1}_{j_1} \cdots (S^{-1})^{l_q}_{j_q}$$

Therefore, applying this property to our tensors D and σ:

$$D'^{ijk} = S^i_l S^j_m S^k_n D^{lmn} \quad \sigma_{jk} = \sigma_{lm} (S^{-1})^l_j (S^{-1})^m_k$$

$\square$
