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Lecture Notes

Signals and Systems

EE 120
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1 Lecture 1

1.1 Motivations

- Why study this class?
  - Given a "black box" circuit, with input and output leads, we can determine what's within the "black box".
  - In this particular case, if our black box contains a voltage divider, and the output voltage is given by the equation:

    $$v_{\text{out}}(t) = \frac{R_2}{R_1 + R_2} v_{\text{in}}(t)$$

    In principle though, the signal can be anything that we want: for facial recognition software, the input signal could be the configuration of the intensity the camera picks up. There's many more we went over, don't really want to write it all down.
- In essence, there's a lot of systems that can be modeled by a system that takes in a signal $x(t)$, and outputs a signal $y(t) = f(x(t))$.
  - The signals are usually functions of time, location, in any number of dimensions.
  - The systems does some sort of transformation on an input signal. In particular, we will study linear systems, shift-invariant systems, etc.

    We'll talk about mathematical operations that we use to perform these transformations: Fourier, Laplace, Z-transformations, convolutions, correlation, etc.
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1.2 Types of Signals

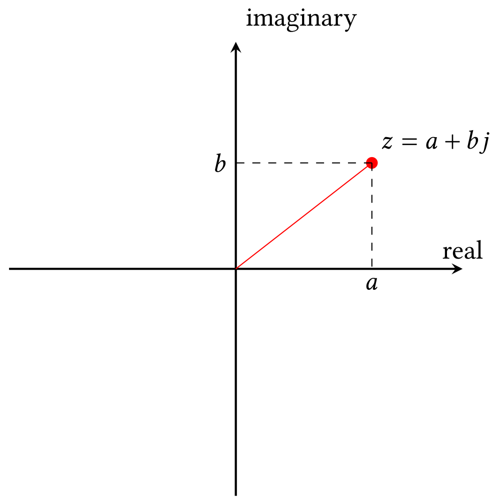
- **Continuous-time:** signals defined over continuous variables (e.g. position, time). For instance, a signal $x(t)$ is continuous for our purposes, since time is a continuous variable.

  Further, because t is continuous, then x must also be continuous. If the signal is differentiable, then the derivative $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ also exists.

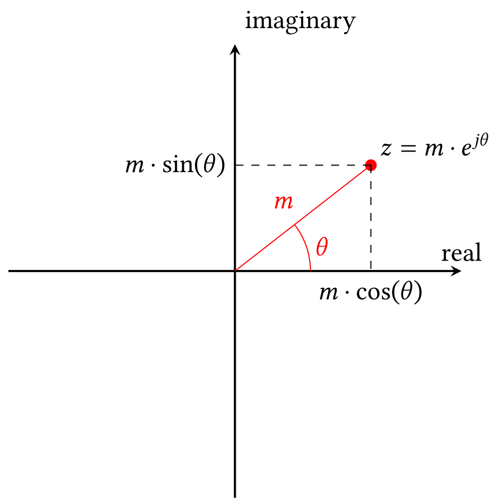
  t being continuous does not imply that $x(t)$ is continuous (e.g. Thomae function), but is it true for this class?
- **Discrete-Time:** These are signals defined over discrete variables. For instance, if we had $x[n]$ as a signal, where n is an integer.

  We don't have a concept of differentiability, but we can compute the difference: $x[n] - x[n-1]$, and talk about that quantity.
- **Real-Valued:** A signal $x(t)$ is real-valued if $x(t) \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of all real numbers.
- **Complex-Valued:** A signal $x(t)$ is complex-valued if $x(t) \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers.
- Note that while we're using the continuous-time notation here, the same concepts apply with discrete-time signals.
  - Quick recap on complex numbers: denoted by $a + bj$ or $a + bi$, where i and j denote the imaginary unit.
  - They are defined as $i^2 = -1$ or $j^2 = -1$.
  - a is the real part, and b is the imaginary part.

– We can plot these values in the complex plane, using the real and imaginary representation:



Or using the magnitude-phase representation:



We represent the magnitude as $m = |z|$, and the phase angle θ is the angle made with the real axis.

- **Periodic Signal:** Two quantities we'll introduce here: the period T is the time it takes for the signal to repeat itself. T is measured in units of time, generally seconds.

  The frequency f is the "inverse" of period, defined by $f = \frac{1}{T}$. We will also use the angular frequency ω, defined by $\omega = \frac{2\pi}{T} = 2\pi f$. Angular frequency is mainly going to be used when we involve complex numbers. We will see:

  $$e^{j\omega t} = e^{j(2\pi f t)} = \cos(2\pi f t) + i\sin(2\pi f t)$$

- **Dimensionality:** We will deal with multi-dimensional signals: an example of a 2D signal are images, which determine the color of a pixel based on a row and column. The spaces that we'll be working with are either $\mathbb{R}^n$ or $\mathbb{C}^n$.

1.3 Signal Transformations

- **Shifts:** Essentially just shifts the signal along one dimension: $x(t) \to x(t-T)$. T is some constant. If $T > 0$, then the shift is to the *right*, and if $T < 0$ then the shift is to the *left*.
- **Scaling:** We can multiply a signal $x(t)$ by some constant $a : x(t) \to a \cdot x(t)$. If $a < 1$, then we shrink $x(t)$, and if $a > 1$ then we amplify the signal.
- **Reversal:** Given $x(t)$, we can "reverse time" by adding a negative to the argument: $x(t) \to x(-t)$. Visually, all we do is flip the signal around the y-axis.

1.4 Signal Properties

- **Even:** Functions which satisfy $x(t) = x(-t)$. In other words, if we perform a reversal, the signal stays the same.
- **Odd:** Functions which satisfy $x(t) = -x(-t)$. If we perform a reversal, the signal becomes the negative of itself.
- **Periodic:** If T is the period, then nT is also a period for any $n \in \mathbb{Z}$. However, we will call T the fundamental period; the smallest T for which the function repeats.

  For the function $\sin(2\pi f t)$, the fundamental period is $1/f$.

1.5 Model Functions

- These are called model functions because they're idealized models to analyze.
- **Heaviside Step function:** For the continuous-time case it's usually modeled by:

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

  In the discrete-time case, it's written as:

$$u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

2 Lecture 2

2.1 Model Functions Continued

- **Ramp Function:** The continuous-time is expressed as:

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

  Similarly in discrete time:

$$\text{ramp}[n] = \begin{cases} 0 & \text{for } n < 0 \\ n & \text{for } n \geq 0 \end{cases}$$

  Note that we can express the ramp function in terms of the step function, in many ways:

  - $r(t) = t \cdot u(t)$
  - $r(t) = \int_{-\infty}^{t} u(t)\, dt$, the discrete case is just a sum over the same bound.

- **Rectangular Function:** In continuous-time:

$$\text{rect}(t) = \sqcap(t) = \begin{cases} 1 & \text{for } |t| \leq 1/2 \\ 0 & \text{else} \end{cases}$$

  In discrete time:

$$\text{rect}\left[\frac{n}{N}\right] = \begin{cases} 1 & \text{for } |n| \leq N \\ 0 & \text{for } |n| > N \end{cases}$$

  We can also express $\text{rect}(t)$ in terms of $u(t)$:

$$\sqcap(t) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

- **Triangle Function:** In continuous-time:

$$\Lambda(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

And in discrete-time:

$$\Lambda\left[\frac{n}{N}\right] = \begin{cases} 1 - \left|\frac{n}{N}\right| & \text{for } |n| \leq N \\ 0 & \text{else} \end{cases}$$

- **Delta Function:** In continuous time, it's called the Dirac delta function. It has the property that $\delta(t) = 0$ for all $t \neq 0$, but

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

So in essence, this is an infinitesimally "thin" function, that extends to infinity. There are also other ways to represent the Delta function:

  - Derivative of the Heaviside step function: $\delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$
  - The integral of a complex exponential:

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft}\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

The delta function allows us to approximate the integral $\int_{-\infty}^{\infty} \cos(\omega t)\, dt$. We can do the following:

$$\begin{aligned}\int_{-\infty}^{\infty} \cos(\omega t)\, dt &= \mathrm{Re}\left[\int_{-\infty}^{\infty} \cos(\omega t) + i\sin(\omega t)\right] dt \\ &= \mathrm{Re}\left[\int_{-\infty}^{\infty} e^{i\omega t}\, dt\right] \\ &= \mathrm{Re}\left[2\pi\delta(\omega)\right]\end{aligned}$$

Looking at the delta function, we know that when $\omega = 0$, then $\cos(\omega t) = 1$, so the integral diverges, as expected. When $\omega \neq 0$, our integral result implies that the integral evaluates to 0. This is not exactly true since the integral will oscillate between ± 1, which is relatively small compared to $\omega = 0$, so it can effectively be taken as 0.

How does this compare with the definition we use in physics that $\delta(t)$ is defined as the function which satisfies:

$$\int_{-\infty}^{\infty} f(t)\delta(t) = f(0)$$

Do both work?

See below bullet point, the definition allows you to derive this property.

- Let's explore some properties of the Delta function:

  - **Scaling:**

$$\int_{-\infty}^{\infty} \delta(\alpha t) dt = \int_{-\infty}^{\infty} \delta(t)\frac{\mathrm{d}\tau}{\mathrm{d}\alpha} = \frac{1}{|\alpha|}$$

In other words, $\delta(\alpha t) = \frac{\delta(t)}{|\alpha|}$

  - **Sifting:** If we have $f(t)$ and multiply it by a Delta function:

$$\int_{-\infty}^{\infty} f(t)\delta(t - T)\, dt = f(T)$$

  - **Delta function of a function:** We can take the delta function of a function as well:

$$\delta(f(t)) = \frac{\delta(t - t_0)}{|f'(t_0)|},\ f(t_0) = 0$$

We take the derivative in the denominator.

- In discrete time, the delta function is represented as the Kronecker delta:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

The function attempts to model the Dirac delta but for discrete time intervals:

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-10] = x[10]$$

- **Shah function:** It's basically a bunch of Dirac deltas:

$$\text{III}(t) = \sum_{k=0}^{\infty} \delta(t-k)$$

In discrete time, it also is a sum of all deltas:

$$\text{III}[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

2.2 System Characterization

- Systems perform operations on input signals, like functions $F : x \to y$. For instance, the following is a moving average filter:

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$

- We will be particularly interested in **linear systems**, systems which satisfy the following two properties:
  - **Scaling:** If for any input-output pair $x(t) \to y(t)$, then for any constant a, $ax(t) \to ay(t)$
  - **Addition:** Given any two input-output pairs

  $$x_1(t) \to y_1(t)$$
  $$x_2(t) \to y_2(t)$$

  Then it's also true that $x_1(t) + x_2(t) \to y_1(t) + y_2(t)$

Combining these two properties, given two general signals $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, then $ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$.

Note that a function like $y(t) = x(t) + b$ is not a linear function, becuase it doesn't satisfy the second property. Even though the function is linear, doesn't mean that the transformation is linear.

- **Shift Invariant:** A shift-invariant system is a system where if we shift the input, the output is also shifted. Given $x(t)$ and its output $y(t)$, then $x(t-T)$ should produce $y(t-T)$ for any T.

- **Memoryless:** A function whose output at any given time only depends on the input at that time. For instance, machine learning algorithms are not memoryless, since their output depends on previous inputs.

Does "current" here refer to a given input, or does it refer to past inputs? For instance, is the moving average function considered memoryless?

Most systems that take time to react are not considered memoryless, since

- **Causality:** A system is causal if the output depends on the input at the present and past times only, not on future times. A system defined by:

$$y[n] = \frac{1}{3}(x[n] + x[n+1])$$

is not considered causal, because $y[n]$ depends on the $n+1$-th input.

- **Stability:** There are many different ways to define stability, here are some of them:

– A system is called BIBO stable if bounded inputs generate boudned outputs. Mathematically, this means:

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

And in discrete time:

$$\sum_{-\infty}^{\infty} |x[n]| < \infty$$

– We can also look at the energy and power:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \quad P = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

- **System Response function:** These are particular outputs for systems when given an impulse response of a delta function. They are calculated by substituting $x(t) = \delta(t)$ in the continuous case, and $x[n] = \delta[n]$ in the discrete case. Watch lecture for this.

  To calculate the impulse response for the moving average filter:

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$

  To find the impulse response, we substitute $x[n] = \delta[n]$ to get $h[n]$. Here, notice that for $n < -2$, then $h[n] = 0$, since $x[-2+1] = x[-1] = 0$, and same goes for the other terms. Then, refer to the following table:

n	$h[n]$
-2	0
-1	1/3
0	1/3
1	1/3
2	0

3 Lecture 3

3.1 Step Response

- The step response function is the function $y_{\text{step}}(t)$ when a step function $u(t)$ is fed into the system. In discrete-time: we feed $u[n]$ into the system, and get $y_{\text{step}}[n]$ as an output.
- For instance, for the moving average filter defined earlier, we have the following result:

n	$y_{\text{step}}[n]$
-2	0
-1	1/3
0	2/3
1	1
2	1

  Note that this resembles a ramp function, and is called a ramp-step function.

- **Harmonic Response:** The harmonic response is the response by the system when presented with a harmonic function, of the form $Ae^{i\omega t}$.

  In discrete time, we feed in $Ae^{i\omega n}$ where n is an integer.

- For the moving average filter, let's write out $y[n]$:

$$\begin{aligned}
y[n] &= \frac{1}{3}\left(Ae^{i\omega(n-1)} + Ae^{i\omega n} + Ae^{i\omega(n+1)}\right) \\
&= \frac{1}{3}\left(e^{-i\omega} + 1 + e^{i\omega}\right) \\
&= \frac{1}{3}(2\cos\omega + 1)Ae^{i\omega n}
\end{aligned}$$

  The interesting thing here is that when given a harmonic function, the system response just scales the signal by a constant amount!

3.2 LCCDE

- In this class, we will deal with lots of differential equations, so it's going to be very useful to look at their form, and how to solve them.
- There are two solutions to any differential equation:
  - **Particular Solution:** $y_p(t)$ is called a particular solution if it satisfies:

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y_p(t)}{\mathrm{d}t^k} = \sum_{k=0}^{N} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

  - **Homogeneous Solution:** $y_h(t)$ is called a homogeneous solution if it satisfies:

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y_p(t)}{\mathrm{d}t^k} = 0$$

- In general, the solution will be a linear combination of the two:

$$y(t) = y_p(T) + a y_h(t)$$

  the value of a is generally going to be given by some initial condition.

- For the homogeneous solution, an ansats of the form Ae^{st} where s is an undetermined constant will solve the differental equation. We can then determine the value of s by solving the resulting polynomial.

  To determine the value of A, these are determined by the initial conditions, and depending on the number of initial conditions given, that would correspond directly to the number of distinct values of A.

3.2.1 Example

- Given a first order LCCDE, with a step function input (this means that the right hand side is a step function). This means we're solving for a solution $y(t)$ to:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a y(t) = b x(t) = b u(t)$$

- First we look for the homogeneous solution, which will give us:

$$\frac{\mathrm{d}y_h(t)}{\mathrm{d}t} + a y_h(t) = 0$$

  This is a separable DE, so we move $a y_h(t)$ to the right hand side and integrate, which gives us a solution of the form:

$$y_h(t) = Ae^{-at}$$

- The particular solution is the function $y_p(t)$ that satisfies:

$$\frac{\mathrm{d}y_p(t)}{\mathrm{d}t} + ay_p(t) = bu(t)$$

Now, we break this up into what values $u(t)$ takes along the real line. For $t < 0$, we have $u(t) = 0$, which gives us a solution of $y_p(t) = 0$. Note that even though the right hand side is zero doesn't mean this is homogeneous, since the homogeneous solution has 0 on the right hand side *for all* t.

- For $t > 0$, the differential equation becomes:

$$\frac{\mathrm{d}y_p(t)}{\mathrm{d}t} + ay_p(t) = b$$

We can rearrange this slightly:

$$\frac{\mathrm{d}\left(y_p(t) - \frac{b}{a}\right)}{\mathrm{d}t} + a\left(y_p(t) - \frac{b}{a}\right) = 0$$

You can verify that this is actually the same differential equation, since the derivative of a constant is zero. But now, we can define $z(t) = y_p(t) - \frac{b}{a}$, and solve instead a homogeneous differential equation for $z(t)$. Going through the same steps as before, we have $y_p(t) = \frac{b}{a} + Be^{-at}$.

- We can combine both $t < 0$ and $t > 0$ with a nice analytical form:

$$y_p(t) = \left(\frac{b}{a} + Be^{-at}\right)u(t)$$

Is this necessary when possible?

- The general solution is then:

$$y(t) = y_p(t) + y_h(t) = \frac{b}{a} + Be^{-at} + Ae^{-at}$$

We need an initial condition to solve for A and B :

  - Initial rest condition: at $t = 0$, no input, so the output should also be 0. This gives the equation:

$$A + B = -\frac{b}{a}$$

4 Lecture 4

4.1 System Block Diagram

- Now we'll look at how to convert an LCCDE into a block diagram.
- Suppose we're given a system of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

This implies the equation:

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] - \sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k]$$

4.2 Linear Time Invariant (LTI), Linear Shift Invariant (LSI)

- What is an LTI system? Firstly, it's linear, so it satisfies the superposition rule: given two signals $x_1(t)$ and $x_2(t)$, then an input of $ax_1(t) + bx_2(t)$ will generate an output of $ay_1(t) + by_2(t)$.
- An LSI is also a linear system, and given an input signal $x(t)$ with an output $y(t)$, then we can shift the system $x(t-T)$ to generate an output $y(t-T)$, but $y(t-T) = y(t)$. In other words, the output will look like $y(t)$, except shifted by T.

- As an example, the continuous LCCDE is a linear time invariant system. This is because the derivative is linear:
$$\sum_{k=0}^{M} \frac{\mathrm{d}^k(ax_1(t) + bx_2(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t-T)}{\mathrm{d}(t-T)^k}$$
And then we can substitute $u = t - T$:
$$\sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(u)}{\mathrm{d}u^k} = \sum_{k=0}^{M} \frac{\mathrm{d}^k y(t-T)}{\mathrm{d}(t-T)^k}$$
- The same principle also holds for discrete time signals.
- The most important property of an LTI system is that **the system response is fully characterized by an impulse response function.**

  What this means is that if we feed the system a $\delta(t)$ or $\delta[n]$, it gives us an impulse response function $h(t)$ or $h[n]$, and this gives us enough information to characterize the entire system.
- In the continuous time case, suppose we had the following:
$$\begin{tikzcd} x(t) \arrow[r] & \boxed{h(t)} \arrow[r] & y(t) \end{tikzcd}$$
  Then, $y(t)$, the signal generated by an arbitrary $x(t)$ is generated by:
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$
  In discrete time, the formula is:
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
  This is called a *convolution*, we will come back to this later.

4.2.1 Why a convolution?

- Again, consider the diagram:
$$\begin{tikzcd} \delta(t) \arrow[r] & \boxed{\text{LTI}} \arrow[r] & y(t) = h(t) \end{tikzcd}$$
  If we send a signal $\delta(t-\tau)$ into the system, then due to linear time invariance, the system should output $y(t-\tau) = h(t-\tau)$.
- If we now send the signal $x(\tau)\delta(t-\tau)$, then because $x(\tau)$ is a constant, then we invoke linearity to get that the output is $x(\tau)h(t-\tau)$.
- Now, consider what happens when we send in the signal that is just a combination of all possible τ. Each $x(\tau)$ is a constant, so the output signal is of the form
$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) \mapsto \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$
  But now notice that this signal can also be written as:
$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau = x(t)$$
  And so if we're sending in a signal $x(t)$, then the output should be $y(t)$! Thus, we've proven that the impulse response is all we need in order to characterize $y(t)$.
- For future reference, a convolution, denoted by $x(t) * h(t)$, is defined as:
$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\, d\tau$$
  this last equality shows that convolution is a commutative operation.

4.2.2 Impulse Response of 1st order LCCDE

- Recall the step response to LCCDE:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) = bx(t) = bu(t) \implies y_{\text{step}}(t) = \left(\frac{b}{a}(1 - e^{-at})\right) u(t)$$

- Given an impulse, which in this case can be written as:

$$\delta(t) = \lim_{\epsilon \to 0} \frac{u(t) - u(t-\epsilon)}{\epsilon}$$

  This implies that the response $h(t)$ is given by:

$$h(t) = \lim_{\epsilon \to 0} \frac{y_{\text{step}}(t) - y_{\text{step}}(t-\epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\frac{b}{a}\left(e^{-a(t-\epsilon)}u(t-\epsilon) - e^{-at}u(t)\right)}{\epsilon} = be^{-at}u(t)$$

  (verify this at home, the simplification makes use of the fact that $e^{a\epsilon} \approx 1 + a\epsilon + a^2\frac{\epsilon^2}{2} + \cdots$, but the higher order terms die).

4.3 Harmonic Response of an LTI system

- The response of an LTI system to a complex signal $x(t) = Ae^{j\omega t}$ is always going to be another complex exponential signal $y(t) = H(\omega)Ae^{j\omega t}$
- Given the input signal $x(t) = Ae^{j\omega t}$, we can write:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} Ae^{j\omega t}h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} Ae^{j\omega(t-\tau')}h(\tau')\, d\tau' \\
&= Ae^{j\omega t} \underbrace{\int_{-\infty}^{\infty} e^{-j\omega\tau'}h(\tau')\, d\tau'}_{H(\omega)} \\
&= H(\omega)Ae^{j\omega t}
\end{aligned}$$

- By definition:

$$H(\omega) \equiv \int_{-\infty}^{\infty} e^{-j\omega\tau'}h(\tau')\, d\tau' \quad H(f) \equiv \int_{-\infty}^{\infty} e^{-j2\pi ft}h(t)\, dt$$

  You'll recognize $H(\omega)$: it's the Fourier transform equation.

  When given an harmonic input, and we're asked to measure it, are we measuring the real part of the signal?

4.3.1 Example: Frequency response of an RC Circuit

- Given the following circuit:
- The impulse response is given by the differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

  This is a first order LCCDE, so therefore the impulse response $h(t)$ is given by $h(t) = be^{-at}u(t)$.

- For the frequency response, we have a function of the form $x(t) = e^{j\omega t}$, which we know has an output signal of the form $y(t) = H(\omega)e^{j\omega t}$. So all that remains now is to find $H(\omega)$:

$$\begin{aligned} y(t) &= e^{j\omega t} \int_{-\infty}^{\infty} e^{-j\omega t} h(\tau)\, d\tau \\ &= e^{j\omega t} \int_{-\infty}^{\infty} be^{-a\tau} u(\tau) e^{-j\omega\tau}\, d\tau \\ &= be^{j\omega t} \int_{0}^{\infty} e^{-a\tau} e^{-j\omega\tau}\, d\tau \\ &= \left(-\frac{1}{a + j\omega} e^{-a\tau} e^{-j\omega t} \Big|_0^{\infty} \right) be^{j\omega t} \\ &= \frac{b}{a + j\omega} e^{j\omega t} \end{aligned}$$

Now, if we impose that $a = b = \frac{1}{RC}$, then we get the equation:

$$\frac{\frac{1}{jc\omega}}{\frac{1}{jc\omega} + R} e^{j\omega t}$$

Now, $\frac{1}{jc\omega}$ is the impedance of a capacitor, and this overall equation takes the form of a voltage divider for a circuit with known impedance:

$$y(t) = \frac{z(\omega)}{z(\omega) + R} e^{j\omega t}$$

4.4 Sinusoidal Input

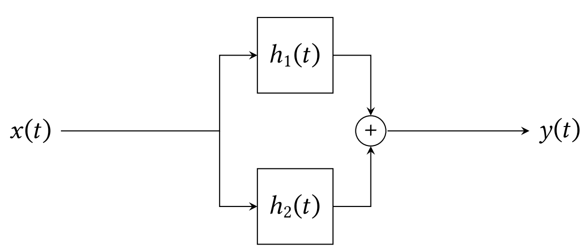
- With the harmonic response tools, we can now evaluate the system response when given a sinusoidal input, since we know that

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Does the same work with sine, where there's a complex number is in the denominator?

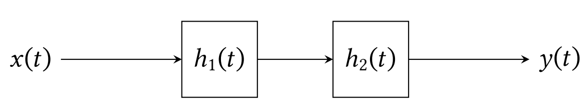
4.5 LTI systems in Parallel and Series

- For a basic system with a single input and output, we've already discussed that $y(t) = x(t) * h(t)$. Now, what if we connect these systems in parallel?



then, the result $y(t)$ is given by $y(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$.

- If we connect them in series:



then

5 Lecture 5

- Recall that last time, given a system equation $y[n] - ay[n-1] = x[n]$, an impulse response function $h[n] = a^n u[n]$ and a step input signal $x[n] = u[n]$, then we can calculate the general signal response $y[n] = x[n] * h[n]$.

- see lecture for notes

5.1 Causality and BIBO stability

- An LTI system is causal if and only if the impulse response function is a causal function. This means that $h(t) = 0$ for all $t < 0$. Under this condition, then the system will be causal.
- For BIBO stability, the same thing applies: it's BIBO stable if and only if its impulse response $h(t)$ is absolute integrable:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

- The same thing applies for discrete-time: it's causal iff $h[n] = 0$ for all $n < 0$.

  Proof. First, we prove the forward case: if $h[n] = 0$ for all $n < 0$, then:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

  But then this means that $y[n]$ only depends on the present and the past, hence it's a causal system.

  Now for the reverse case by contrapositive: for any $m < 0$, if $h[m] \neq 0$, then there is one $n - k < 0$ such that $h[n-k] \neq 0$. If this is true, then the system depends on at least one value $k > n$, hence it's non-causal. □

- For BIBO stability, the equivalence here that the impulse response is absolute summable, so:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

  Proof. First, we show that if $h[n]$ is absolute summable, then given a bounded input:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \leq c \sum_{k=-\infty}^{\infty} |h[k]| \leq \infty$$

  Now the reverse case: if the impulse response by contraposition: if our system is not absolute summable, then our system is not BIBO stable. To do this, let $x[n] = \text{sgn}(h[-n])$, then since:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  then for instance, we can evaluate $y[0]$:

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$

  And since $x[k] = \text{sgn}(h[-k])$, then:

$$y[0] = \sum_{k=-\infty}^{\infty} |h[-k]| = \infty$$

  This is equal to infinity, because this is this is precisely the fact that h is not absolute absolute sumamble. □

5.2 Convolution and Correlation

- As we've seen already, the convolution is essential to an LTI system. Given a signal $x(t)$ and system $h(t)$ and output $y(t)$, then we can calcualte $y(t)$ based on the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

- In discrete-time:

$$y[n] = \sum_{k=\infty}^{\infty} x[k]h[n-k]$$

- A **convolution** takes in two signals x, y and computes:

$$\text{cov}(x, y) = (x * y)(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\, d\tau$$

  Notice that we're integrating with respect to τ, meaning that the resulting function should be a function of t.

- A **cross-correlation** is defined as r_{xy}, and is computed as:

$$\text{corr}(x, y) = (x \circ y)(t) = x(t) \circ y(t) = \int_{-\infty}^{\infty} x(\tau)y(t+\tau)\, d\tau$$

  The main difference between the two is that with convolution, we're flipping the second function and computing the integral of the product. With cross correlation, we're not flipping anything, and instead just "sliding" the product across.

  To be clear, it's the second function that we're sliding across the first one.

- Because the two equations are so similar, the convolution and cross correlation are related:

$$y(t) \circ x(t) = x(t) * y(-t)$$

  The proof is fairly trivial; too lazy to write it down. Note that the order matters though.

- See the lectures for a graphical representation of what a convolution does.

5.2.1 Example Functions

- Given two rectangle functions $x(t) = \sqcap(t-0.5), y(t) = \sqcap(t-0.5)$, their convolution is a triangle function $\wedge(t-1)$, and their correlation is $\wedge(t)$. Notice that the convolution is shifted over by 1, and the correlation is not. This is because the convolution flips the sign of one of them, so they intersect at a later time.
- The convolution of the delta function with itself is the delta function: $(\delta(t) * \delta(t)) = \delta(t)$.

5.3 Convolution Identities

- Convolution follows the distributive property: $x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$
- The convolution is commutative: $x(t) * h(t) = h(t) * x(t)$. This is different from the cross-correlation, where the resulting signal is time-reversed.
- The unit impulse is the identity element of the convolution:

$$x(t) * \delta(t) = x(t)$$

- The convolution of a signal with a shifted impulse shifts the signal:

$$x(t) * \delta(t-T) = x(t-T), \quad x[n] * \delta[n-N] = x[n-N]$$

- The convolution of two time-shifted signals will produce a signal that accounts for the shift of both functions:

$$x(t-T_1) * h(t-T_2) = y(t-T_1-T_2)$$

- When a function of time width T_1 is convolved with a function of width T_2, the resulting function has a width of $T_1 + T_2$.

  This makes sense, since the convolution computes the overlap – so if we don't have any overlap, then the result will be 0.

- Convolution also follows an area property: given a convolution $x(t) * h(t)$, then the area of $y(t)$ is given by the product of the area of $x(t)$ and $h(t)$ individually.

6 Lecture 6

6.1 Clarification on BIBO Stability

- When we say a "bounded" signal, we mean that the amplitude of the signal is bounded at all times:

$$|x(t)| < \infty \ \forall t \in \mathbb{R}$$

  The same definition follows for discrete-time signals.

- For LTI systems, we call the system BIBO stable if and only if its impulse $h(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$$

6.2 Cross-Correlation

- The cross correlation between two signals $r_{xy}(t) = r_{yx}(-t)$. To show this explicitly, we look at the cross-correlation equation:

$$\begin{aligned} r_{xy}(t) &= \int_{-\infty}^{\infty} x(\tau)y(t+\tau) \, d\tau \\ r_{yx}(t) &= \int_{-\infty}^{\infty} y(\tau)x(t+\tau) \, d\tau \end{aligned}$$

  But for the second equation, we can define a $\tau' = t + \tau$, so then we get:

$$r_{yx}(t) = \int_{-\infty}^{\infty} y(\tau' - t)x(\tau') \, d\tau' = \int_{-\infty}^{\infty} x(\tau')y(-t+\tau') \, d\tau'$$

  This looks like the first equation except we have $-t$ instead of t. Therefore, we have $r_{xy}(t) = r_{yx}(-t)$. The same works for discrete time: $r_{xy}[n] = r_{yx}[-n]$.

6.3 More Convolution Properties

- **Differentiation property:** Given $y(t) = x(t) * h(t)$, then:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = x(t) * \frac{\mathrm{d}h(t)}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} * h(t)$$

- **Intergration Property:** Given $y(t) = x(t) * h(t)$, we have:

$$\int_{-\infty}^{t'} y(t) \, dt = x(t) * \int_{-\infty}^{t'} h(\tau) \, d\tau$$

6.4 Fourier Transform

- Recall the frequency response of an LTI system:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

  Recall that we can characterize $y(t)$ via a convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau) \, d\tau$$

- Recall that the harmonic response of an LTI system is given by $H(\omega)e^{j\omega t}$.