Problem 1

a) Expand the energy out to fourth order in the dimensionless parameter $\beta \equiv v/c$ about the point $\beta = 0$.

Solution: We can write this equation as $E(\beta) = mc^2(1 - \beta^2)^{-1/2}$. Then, we can just do a standard taylor series expansion about $\beta = 0$. The Taylor series is defined as:

$$E(\beta) = E(0) + E'(0)\beta + \frac{1}{2}E''(0)\beta^2 + \cdots = \sum_{n=0}^{\infty} \frac{E^{(n)}(0)\beta^n}{n!}$$

The first term in this series is $E(0) = mc^2(1) = mc^2$. Now, we compute derivatives:

$$E'(\beta) = mc^2 \cdot -\frac{1}{2}(1 - \beta^2)^{-3/2} \cdot (-2\beta) = mc^2\beta(1 - \beta^2)^{-3/2} \implies E'(0) = 0$$

Now for the second order term:

$$E''(\beta) = mc^2\left[(1 - \beta^2)^{-3/2} + \beta \cdot -\frac{3}{2}(1 - \beta^2)^{-5/2} \cdot (-2\beta)\right] = mc^2\left[(1 - \beta^2)^{-3/2} + 3\beta^2(1 - \beta^2)^{-5/2}\right]$$

Substituting in $\beta = 0$ gives $E''(0) = mc^2$. The rest of this process is just taking derivatives:

$$E'''(\beta) = -\frac{3}{2}(1 - \beta^2)^{-5/2}(-2\beta) + 3\left[(2\beta)(1 - \beta^2)^{-5/2} + \beta^2\left(-\frac{5}{2}(1 - \beta^2)^{-7/2}(-2\beta)\right)\right]$$

$$= 9(1 - \beta^2)^{-5/2}\beta + 15\beta^3(1 - \beta^2)^{-7/2}$$

Substituting $\beta = 0$, we again get $E'''(0) = 0$. For the fourth-order derivative, we don't actually have to compute all of it. Notice that the derivative of the second term in $E'''(0)$ will always have a β factor, hence when we substitute zero we will get zero. The only part that contributes to $E'''(0)$ is the term in the first product rule where the outside factor of β is removed:

$$E^{(4)}(\beta) = 9(1 - \beta^2)^{-5/2} + \text{stuff that equals zero}$$

Hence, $E^{(4)}(0) = 9$. Now we are ready to put it all together. Recalling the earlier Taylor Expansion:

$$E(\beta) \approx mc^2 + \frac{1}{2}mc^2\beta^2 + \frac{9}{24}\beta^4 mc^2$$

$$\approx mc^2 + \frac{1}{2}mc^2\left(\frac{v}{c}\right)^2 + \frac{3}{8}mc^2\left(\frac{v}{c}\right)^4$$

$$\approx mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mc^2v^4$$

As explained by the hint, the first two terms are familiar, and the third nonzero term is the relativistic correction. $\qquad\square$

b) What is z's polar form? What is the inverse z^{-1} in Cartesian form? Draw and label on the complex plane the points representing the complex numbers z, z^*, $z + z^*$.

Solution: The number z in polar form is given by $z = |z|e^{i\text{Arg}(z)}$. Computing the magnitude of z:

$$|z| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$$

And now for $\text{Arg}(z)$:

$$\text{Arg}(z) = \tan^{-1}\left(\frac{6}{8}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

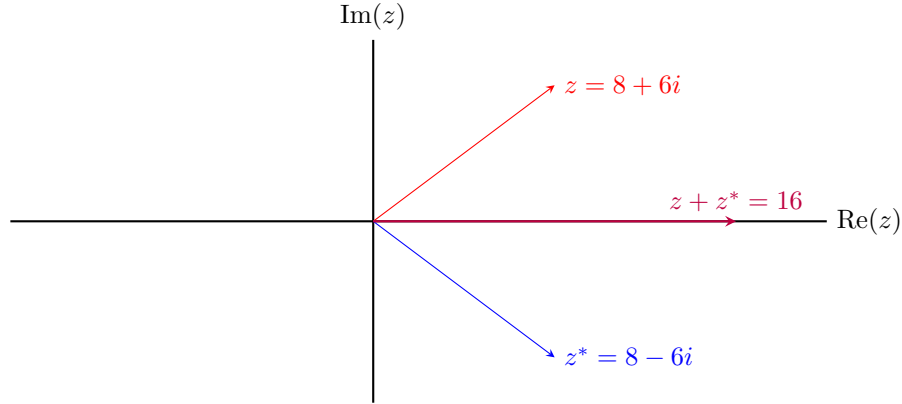therefore, in polar form, we have:

$$z = 10e^{i\tan^{-1}(3/4)}$$

As for the inverse of z we want to find the number such that $zz^{-1} = 1$. Since z is just a number, we can take the reciprocal:

$$z^{-1} = \frac{1}{8 + 6i}$$

And we can make this look a bit nicer:

$$\begin{aligned}
z^{-1} &= \frac{8 - 6i}{(8 + 6i)(8 - 6i)} \\
&= \frac{8 - 6i}{100} \\
&= \frac{4 - 3i}{50}
\end{aligned}$$

Now for the diagram:

The purple arrow is a bit hard to see since it lies on the real axis, but it's meant to show that $z + z^* = 16$.

$\square$

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c) Prove this relationship using index notation using the Kronecker delta, Levi-Civita symbol, and manipulations.

Solution: To do this, we first write this down in index notation:

$$\nabla \cdot (\vec{E} \times \vec{B}) = \partial_i\left[\epsilon_{ijk}E_jB_k\right]$$

Now, we take note of the fact that this is a derivative, meaning that we have to use product rule. Specifically, we need to take the i-th derivative of E as well as for B, which gives us two terms:

$$\begin{aligned}
\partial_i[\epsilon_{ijk}E_iB_k] &= \epsilon_{ijk}(\partial_iE_j)B_k + \epsilon_{ijk}E_j(\partial_iB_k) \\
&= B_k\epsilon_{kij}\partial_iE_j + E_j\epsilon_{jki}\partial_iB_k
\end{aligned}$$

The first term we've cyclically permuted i, j, k, so the Levi-Civita symbol doesn't change sign. However, in order for the second term to become the curl, we need to swap k and i. Using the fact that $\epsilon_{jki} = -\epsilon_{jik}$ we have to flip the sign of that term:

$$B_k\epsilon_{kij}\partial_iE_j + E_j\epsilon_{jki}\partial_iB_k = B_k\underbrace{\epsilon_{kij}\partial_iE_j}_{\nabla\times E} - E_j\underbrace{\epsilon_{jik}\partial_iB_k}_{\nabla\times B}$$

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Now we can get rid of indices:

$$\therefore \nabla \cdot (\vec{E} \times \vec{B}) = B_k \epsilon_{kij} \partial_i E_j - E_j \epsilon_{jik} \partial_i B_k = \vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times B)$$

which is exactly the expression we wanted to prove. $\square$

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d) Arrange these vectors into a 3×4 matrix D and use row reduction to argue that this set spans the space $\mathbb{R}^3$ but is not linearly independent. Then, use whatever method you like to explicitly find v_4 as a linear combination of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

Solution: Given the vectors, we can construct D:

$$D = \begin{pmatrix} 1 & 9 & 8 & 6 \\ 1 & 0 & 1 & 10 \\ 8 & 9 & 8 & 1 \end{pmatrix}$$

First, we can subtract row 1 from row 3 ($R_3 - R_1$):

$$\begin{pmatrix} 1 & 9 & 8 & 6 \\ 1 & 0 & 1 & 10 \\ 7 & 0 & 0 & -7 \end{pmatrix}$$

Now divide row 3 by 7 and swap rows 1 and 3:

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 10 \\ 1 & 9 & 8 & 6 \end{pmatrix}$$

Subtract both row 2 and row 3 by row 1:

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 11 \\ 0 & 9 & 8 & 7 \end{pmatrix}$$

Now divide row 3 by 9, and also swap rows 2 and 3:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 8/9 & 7/9 \\ 0 & 0 & 1 & 11 \end{pmatrix}$$

So we see that there are three pivots, meaning that the rank of D is 3. This implies that $\dim(\mathrm{Im}(D)) = 3$, and hence these vectors span $\mathbb{R}^3$. However, they cannot be linearly independent, since we can only have at most three vectors that are linearly independent in $\mathbb{R}^3$. To find the linear combination of vectors that give us $\vec{v}_4$, we can set up the system of equations:

$$a + 9b + 8c = 6$$
$$a + c = 10$$
$$8a + 9b + 8c = -1$$

Turning this into a matrix and augmenting:

$$\begin{pmatrix} 1 & 9 & 8 & 6 \\ 1 & 0 & 1 & 10 \\ 8 & 9 & 8 & -1 \end{pmatrix}$$

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This is exactly D from before, so row reducing this is the exact same, and we get:

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 8/9 & 7/9 \\ 0 & 0 & 1 & 11 \end{pmatrix}$$

So this actually allows us to read off the solutions nicely:

$$a = -1$$
$$c = 11$$
$$b = \frac{7}{9} - \frac{8}{9}c = -9$$

Therefore, the linear combination is:

$$\vec{v}_4 = -\vec{v}_1 - 9\vec{v}_2 + 11\vec{v}_3$$

Indeed, if we check this:

$$-\begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix} - 9\begin{pmatrix} 9 \\ 0 \\ 9 \end{pmatrix} + 11\begin{pmatrix} 8 \\ 1 \\ 8 \end{pmatrix} = \begin{pmatrix} -1 - 81 + 88 \\ -1 + 11 \\ -8 - 81 + 88 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ = 1 \end{pmatrix}$$

This actually also serves as a good check of the fact that our row reduction was done properly, since had there been a mistake we wouldn't be able to find the correct linear combination. □

e) Write down the matrix representation of E (that is, the matrix that acts on the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and returns the sheared vector). Make sure to explain your work. What does this transformation do to the vector $\vec{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$? Add a sketch showing the original vector $\vec{v}$ and the sheared vector.

Solution: The rectangle in the diagram is formed with two vectors:

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

From the diagram after the transformation, we can see that:

$$E\vec{v}_1 = \vec{v}_1, \ E\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We know that E takes in a vector in $\mathbb{R}^2$ and spits out a vector in $\mathbb{R}^2$, so this implies that E must be a 2×2 matrix. Thus, we can write out the above two equations in terms of matrix multiplication:

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Multiplying this out, we get:

$$E_{11}(0) + E_{12}(1) = 0 \implies E_{12} = 0$$
$$E_{21}(0) + E_{22}(1) = 1 \implies E_{21} = 1$$

Now for the other vector:

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Multiplying:

$$E_{11}(2) + E_{12}(0) = 2 \implies E_{11} = 1$$
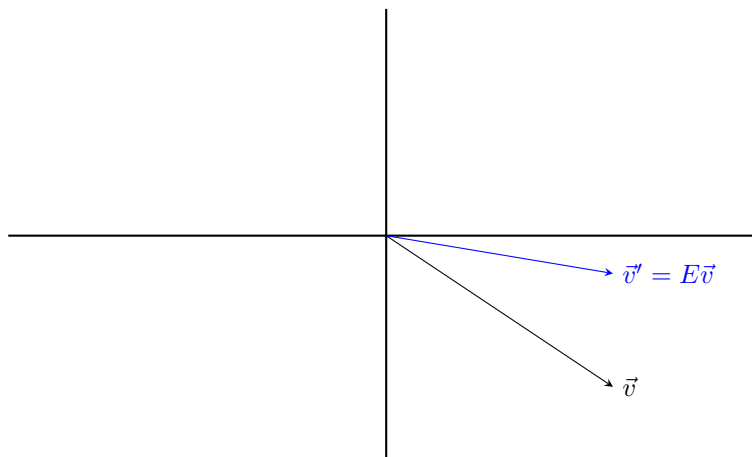$$E_{21}(2) + E_{22}(0) = 1 \implies E_{21} = \frac{1}{2}$$

Therefore, the matrix E is:

$$E = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$$

Acting this matrix on the vector $\vec{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, we get:

$$E\vec{v} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{3}{2} - 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{1}{2} \end{pmatrix}$$

As a diagram:

$\square$

f) Find the antisymmetric part of F. Find a pair of constants ρ and σ such that the matrix F is orthogonal?

Solution: The antisymmetric part of F is given by

$$F_{\text{antisym}} = \frac{F - F^\top}{2}$$

We're given F, so we can just throw it into the formula:

$$F_{\text{antisym}} = \frac{1}{2}(F - F^\top) = \frac{1}{2}\left[\begin{pmatrix} \sqrt{3} & -1 \\ \rho & \sigma \end{pmatrix} - \begin{pmatrix} \sqrt{3} & \rho \\ -1 & \sigma \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & -1-\rho \\ \rho+1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -(\rho+1) \\ \rho+1 & 0 \end{pmatrix}$$

Now to find the constants ρ and σ, we can multiply $FF^\top$ and set it equal to the identity:

$$FF^\top = \frac{1}{4}\begin{pmatrix} \sqrt{3} & -1 \\ \rho & \sigma \end{pmatrix}\begin{pmatrix} \sqrt{3} & \rho \\ -1 & \sigma \end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix} 4 & \sqrt{3}\rho - \sigma \\ \sqrt{3}\rho - \sigma & \rho^2 + \sigma^2 \end{pmatrix}$$

If we want this to equal the identity, we require the following conditions:

$$\frac{\sqrt{3}\rho - \sigma}{4} = 0$$

$$\frac{\rho^2 + \sigma^2}{4} = 1$$

From the first equation we get that $\sqrt{3}\rho = \sigma$ so $3\rho^2 = \sigma^2$. Substituting this into the other equation, we get:

$$\rho^2 + 3\rho^2 = 4\rho^2 = 4 \implies \rho^2 = 1$$

This gives the solutions $\rho = \pm 1$, and consequently, $\sigma = \pm\sqrt{3}$ from the equation $\sqrt{3}\rho = \sigma$. Thus, one of the pairs is $(\rho, \sigma) = (1, \sqrt{3})$. $\qquad\square$

g) Explicitly compute $\det G$ by hand using the Laplace expansion. Be sure to fully explain your steps and identify which row or column you are using for the expansion. Then, using whatever (by-hand) calculation method you would like, solve for the inverse matrix G^{-1}. Again, be sure to explain your steps. Finally, explicitly verify that you have found the inverse by computing GG^{-1} or $G^{-1}G$.

Solution: I'm going to use the first row for the Laplace expansion since we have a zero in the first row. The determinant is given by taking the dot product between the first row and the first row of the cofactor matrix, whose entries are the minors of every element. In other words, the determinant of G can be written as:

$$\det(G) = 0\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = -2(-1) - 4(1-1) = 2$$

Now to solve for the inverse matrix. I'll use the cofactor method which first asks us to compute the cofactor matrix:

$$C_G = \begin{pmatrix} +\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -4 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & -4 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 2 & -4 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -4 \\ -1 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} \end{pmatrix}$$

A quick description of how this process is done: The magnitude of each entry is given by the minor of that entry (i.e. finding the determinant of the matrix formed by ignoring the row and column that minor is in), then we go along the matrix and alternatingly multiply by ± 1 along each row or column. Sorry if this explanation is a bit confusing; it's the best description I can give of this process while still being relatively concise. Now we recall that to compute a 2×2 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We can now simplify the cofactor matrix:

$$C_G = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 2 \\ 4 & 4 & 2 \end{pmatrix}$$

The formula for the inverse is:

$$G^{-1} = \frac{C_G^\top}{\det(G)}$$

Therefore, computing the transpose of C_G:

$$C_G^\top = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 4 & 4 \\ 0 & 2 & 2 \end{pmatrix}$$

Thus since $\det(G) = 2$ from earlier:

$$G^{-1} = \frac{1}{2}C_G^\top = \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ \frac{1}{2} & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

To verify that this is indeed the inverse, we compute GG^{-1}:

$$\begin{aligned}
GG^{-1} &= \begin{pmatrix} 0 & 2 & -4 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ \frac{1}{2} & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 4-4 & 4-4 \\ -\frac{1}{2}+\frac{1}{2} & -1+2-1 & -2+2 \\ \frac{1}{2}-\frac{1}{2} & 1-2+1 & 2-2+1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}
\end{aligned}$$

And since $GG^{-1} = \mathbb{1}$, then we can safely say that G^{-1} was indeed computed correctly. As per the exam policy, I will also admit that I did use Mathematica to verify this inverse. □

Problem 2

a) Show that the space of periodic functions of period L is a vector subspace of the vector space of V. Show or argue that we need $k = 2\pi n/L$ with $n = z$ in order for the vector $\vec{h}_k = e^{ikx}$ to be an element of this subspace.

Solution: To show that this is a vector subspace, we show that it's closed under vector addition and scalar multiplication. First, we show vector addition. Suppose f and g are two functions with period L. That is, $f(x + L) = f(x)$ and $g(x + L) = g(x)$. Then, we construct $h(x) = f(x) + g(x)$. Checking periodicity:

$$h(x + L) = f(x + L) + g(x + L) = f(x) + g(x) = h(x)$$

so since $h(x + L) = h(x)$, this means that $h = f + g$ is also periodic, and thus this space is closed under vector addition. Now under scalar multiplication, let f be defined the same way, but now consider $h(x) = cf(x)$ for some constant c. Then:

$$h(x + L) = cf(x + L) = cf(x) = h(x)$$

hence, since $h(x + L) = h(x)$, we conclude that this space is also closed under scalar multiplication. With these two properties verified, we conclude that it is indeed a vector subspace.

In order for $\vec{h}_k$ to belong to V_L, we require that $h_k(x) = h_k(x + L)$. There are two ways this can be true. The first way:

$$e^{ikx} = e^{ik(x+L)}$$

However, this is only true when $L = 0$. However, because we're dealing with a complex phase, we note that adding and subtracting by integer multiples of 2π doesn't change the value of $h_k(x)$. Therefore, the following is another way this $h_k(x) = h_k(x + L)$ can be satisfied:

$$e^{ikx} = e^{ik(x+L)+2\pi n}$$

comparing the exponents directly, we get:

$$k(x + L) = kx + 2\pi n$$
$$kx + kL - kx = 2\pi n$$
$$kL = 2\pi n$$
$$\therefore k = \frac{2\pi n}{L}$$

Since we deal with nonzero L (a zero L implies the trivial result that $V_L = V$ which is to say, the functions don't have to be periodic), then in order for $\vec{h}_k \in V_L$, it is a requirement that $k = 2\pi n/L$. $\square$

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b) Verify that the vectors $\hat{f}_n$ and $\hat{f}_m$ are orthogonal when $n \neq m$.

Solution: To do this, we need to show that $\hat{f}_n \cdot \hat{f}_m = 0$ with respect to the defined inner product. The inner product is:

$$\vec{f} \cdot \vec{g} = \frac{1}{2\pi} \int_0^{2\pi} f^*(\theta) g(\theta) d\theta$$

so therefore:

$$\hat{f}_n \cdot \hat{f}_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} e^{im\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} d\theta$$

$$= \frac{1}{2\pi} \left[\frac{1}{i(m-n)} e^{i(m-n)\theta} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi i(m-n)} \left[e^{2\pi i(m-n)} - 1 \right]$$

$$= \frac{1}{2\pi i(m-n)} \left[\cos(2\pi(m-n)) + i\sin(2\pi(m-n)) - 1 \right]$$

Now we use the fact that the sine and cosine functions have a period of 2pi, and also the fact that $m-n$ is a nonzero integer when $n \neq m$. Because of these two facts, we conclude that $\cos(2\pi k) = \cos(0) = 1$ for any integer k, and $\sin(2\pi k) = \sin(0) = 0$. Thus, the above equation evaluates to:

$$\hat{f}_n \cdot \hat{f}_m = \frac{1}{2\pi i(m-n)} [1-1] = 0$$

as desired. Note also that we didn't have to convert the exponnetial into sines and cosines, but could also have argued that it equals 1 from the term $e^{2\pi i(m-n)}$ since this term means the same thing as rotating around $\theta = 0$ an integer multiple of 2π times, which would return us right back to $\theta = 0$. Hence, $e^{2\pi i(m-n)} = e^0 = 1$, and the rest follows in the exact same way. $\quad\square$

c) Find the expansion coefficients c_0 and c_{+1}. Argue that, since $T(\theta)$ must be a real function, we must have $c_{-n} = c_n^*$.

Solution: To find c_0, we dot $\vec{f}_0$ with T:

$$c_0 = \vec{h}_0 \cdot \vec{T} = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{e^{-i(0)\theta}}_{=1} T(\theta) d\theta$$

$$= \frac{1}{2\pi} \left[\int_0^{\pi} T_0 d\theta + \int_{\pi}^{2\pi} 0 d\theta \right]$$

$$= \frac{1}{2\pi} T_0 \pi$$

$$= \frac{T_0}{2}$$

Now we dot f_{+1} with $\vec{T}$:

$$c_{+1} = \frac{1}{2\pi}\left[\int_0^\pi e^{-i\theta}T(\theta)d\theta + \int_\pi^{2\pi} e^{-i\theta}\underbrace{T(\theta)}_{=0}d\theta\right]$$

$$= \frac{1}{2\pi}\left[T_0\int_0^\pi e^{-i\theta}d\theta\right]$$

$$= \frac{T_0}{2\pi}\left[-ie^{-i\theta}\right]_0^\pi$$

$$= \frac{T_0}{2\pi}\left[-i\underbrace{e^{-i\pi}}_{=-1} - i\right]$$

$$= \frac{T_0}{2\pi}(-2i)$$

$$= -\frac{iT_0}{\pi}$$

To show that $c_{-n} = c_n^*$, consider how we compute c_n:

$$c_n = \hat{f}_n \cdot \vec{T} = e^{in\theta} \cdot \vec{T}$$

Now we take the conjugate of this:

$$c_n^* = (e^{in\theta} \cdot \vec{T})^* = (e^{in\theta})^* \cdot \vec{T}^* = e^{-in\theta} \cdot \vec{T}^*$$

Now, we note that since T must be a real function, then $\vec{T}^* = \vec{T}$. Thus:

$$c_n^* = e^{-in\theta} \cdot \vec{T}$$

Writing this in a more suggestive way:

$$c_n^* = e^{i(-n)\theta} \cdot \vec{T}$$

We can see that the complex exponential here is exactly $\hat{f}_{-n}$ so therefore:

$$c_n^* = \hat{f}_{-n} \cdot \vec{T} = c_{-n}$$

as desired. $\qquad\qquad\square$

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d) Give a *brief* (no more than one or two sentences) explanation as to why G is in fact a linear transformation. Then, express the result of acting G on the basis vectors $\hat{c}_3$ and $\hat{s}_3$ as linear combinations of $\hat{c}_3$ and $\hat{s}_3$. Use this to write down a (2×2) matrix representation of G acting on the subspace spanned by $\{\hat{c}_3, \hat{s}_3\}$

Solution: Firstly, G does take in a vector f (in our space) and spit out another vector $\frac{df}{d\theta}$, so therefore it satisfies the aspect of it being a transformation. Further, the operation of taking a derivative is linear with respect to linear combinations of functions, or in other words:

$$G(\alpha f(\theta) + \beta h(\theta)) = \alpha G f(\theta) + \beta G h(\theta)$$

thus satisfying the definition of a linear transformation. Now to find the representation of G, we first act G on the basis vectors. Using the definition given above:

$$G\hat{c}_3 = \frac{d}{d\theta}\frac{1}{\sqrt{2}}\cos(3\theta) = -\frac{3}{\sqrt{2}}\sin(3\theta) = -3\hat{s}_3$$

$$G\hat{s}_3 = \frac{d}{d\theta}\frac{1}{\sqrt{2}}\sin(3\theta) = \frac{3}{\sqrt{2}}\cos(3\theta) = 3\hat{c}_3$$

Now for the matrix representation. Let a vector in the subspace spanned by $\{\hat{c}_3, \hat{s}_3\}$ be represented as:

$$\begin{pmatrix} a \\ b \end{pmatrix} \equiv a\hat{c}_3 + b\hat{s}_3$$

Then, the basis vectors $\hat{c}_3$ and $\hat{s}_3$ are represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. Then, we can write:

$$G \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$$G \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Representing G by the most general 2×2 matrix, the first equation becomes:

$$\begin{pmatrix} G_{11} & g_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

Doing the matrix multiplication on the left hand side yields the following two equations:

$$G_{11}(1) + G_{12}(0) = 0 \implies G_{11} = 0$$
$$G_{21}(1) + G_{22}(0) = -3 \implies G_{21} = -3$$

Now for the second equation:

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Again, doing the matrix multiplication gives:

$$G_{11}(0) + G_{12}(1) = 3 \implies G_{12} = 3$$
$$G_{21}(0) + G_{22}(1) = 0 \implies G_{22} = 0$$

Therefore the matrix G is:

$$G = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

After completing this problem, I now realize that we could've just skipped all this and just written $G = \begin{pmatrix} G\hat{c}_3 & G\hat{s}_3 \end{pmatrix}$, but I wasn't entirely sure when this property holds and when it doesn't [1], so I decided to do it the long way. $\qquad\qquad\square$

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[1] I remember asking this question in one of the tap-backs, but I don't think that question was ever answered

Problem 3

a) What condition(s) relating the two phase angles δ_x and δ_y will give linearly polarized light? What is the intensity of light described by the Jones vector $\vec{J} = \begin{pmatrix} 3E_0 \\ 4iE_0 \end{pmatrix}$?

Solution: As described in the problem statement, in order for the light to be linearly polarized we require that J_x/J_y to be a real quantity. Computing this quantity:

$$\frac{J_x}{J_y} = \frac{E_{0,x}e^{i\delta_x}}{E_{0,y}e^{i\delta_y}} = \frac{E_{0,x}}{E_{0,y}}e^{i(\delta_x - \delta_y)}$$

Thus, in order for this quantity to be real, we require the phase to be an integer multiple of π. Therefore, the condition is:

$$\delta_x - \delta_y = k\pi, \ k \in \mathbb{Z}$$

The intensity of light from the vector $\vec{J}$ is given by $I = \alpha\vec{J} \cdot \vec{J}$, so we just need to compute $\vec{J} \cdot \vec{J}$:

$$I = \alpha\vec{J} \cdot \vec{J} = \alpha J^\dagger J$$

$$= \alpha \begin{pmatrix} 3E_0 & -4iE_0 \end{pmatrix} \begin{pmatrix} 3E_0 \\ 4iE_0 \end{pmatrix}$$

$$= \alpha(9E_0^2 + 16E_0^2)$$

$$= 25\alpha E_0^2$$

□

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b) Find the rank and nullity of the matrix P. Find a general form for an element of the image of P. Then find a general form for an element of the kernel of P. In particular, for an element of the kernel, what is the condition on the ratio of amplitudes $E_{0,y}/E_{0,x}$?

Solution: To find the rank of P first let's row reduce P. As a first step, we'll divide the first row by $\cos^2(\theta)$ and divide the second row by $\sin^2(\theta)$. This leaves us with:

$$\begin{pmatrix} 1 & \tan\theta \\ \cot\theta & 1 \end{pmatrix}$$

Now we do $R_2 - R_1 \cot\theta$, giving:

$$\begin{pmatrix} 1 & \tan\theta \\ 0 & 0 \end{pmatrix}$$

Therefore, the rank of P is 1, since there is only a single pivot. The dimension of P is 2 (given by 2 columns), so therefore by the rank nullity theorem, we get that the nullity is: [2]

$$\text{null}(P) = 2 - \text{rank}(P) = 2 - 1 = 1$$

The general form for an element of the image is just acting P on an arbitrary matrix $\vec{J}$:

$$P\vec{J} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} E_{0,x}e^{i\delta_x} \\ E_{0,y}e^{i\delta_y} \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta E_{0,x}e^{i\delta_x} + \cos\theta\sin\theta E_{0,y}e^{i\delta_y} \\ \cos\theta\sin\theta E_{0,x}e^{i\delta_x} + \sin^2\theta E_{0,y}e^{i\delta_y} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta(\cos\theta E_{0,x}e^{i\delta_x} + \sin\theta E_{0,y}e^{i\delta_y}) \\ \sin\theta(\cos\theta E_{0,x}e^{i\delta_x} + \sin\theta E_{0,y}e^{i\delta_y}) \end{pmatrix}$$

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[2]In doing this part I had to go into the notes to verify that the proper notation was indeed $\text{null}(P) = \dim(\ker(P))$. That was all I had to look up though, I did remember the formula correctly.

We can now see that the terms in parentheses are actually the same, so we can factor that out, giving:

$$P\vec{J} = (\cos\theta E_{0,x}e^{i\delta_x} + \sin\theta E_{0,y}e^{i\delta_y})\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

I could also write this in terms of J_y and J_x just to make this expression look a bit more visually appealing. Therefore, vectors in the image of P are of the form:

$$\vec{w} = P\vec{J} = (\cos\theta J_x + \sin\theta J_y)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

As for an element of the kernel, we are looking for a vector $\vec{J}$ that satisfies:

$$\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Augmenting this matrix, we have:

$$\tilde{P} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & \sin^2\theta & 0 \end{pmatrix}$$

The zeroes in the third column don't affect our row reduction, and we notice that the other two columns are exactly just P, row reducing $\tilde{P}$ is the same as row reducing P then tacking on a oolumn of zeroes as the rightmost column. Therefore, we're allowed to use the row reduction from earlier:

$$\tilde{P} = \begin{pmatrix} 1 & \tan\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the first row, we get that $J_x + J_y\tan\theta = 0$, or in other words $J_x = -J_y\tan\theta$. If we let $J_y = t$, then we are looking for vectors of the form:

$$\vec{J} = t\begin{pmatrix} 1 \\ -\tan\theta \end{pmatrix}$$

Note that this is a real vector, and thus linearly polarized, as suggested by the hint. Now recall the general form for $\vec{J}$:

$$J = \begin{pmatrix} E_{0,x}e^{i\delta_x} \\ E_{0,y}e^{i\delta_y} \end{pmatrix} = \begin{pmatrix} t \\ -t\tan\theta \end{pmatrix}$$

Equating the entries gives us the system:

$$E_{0,x}e^{i\delta_x} = t$$
$$E_{0,y}e^{i\delta_y} = -t\tan\theta$$

We know that the complex exponentials don't contribute anything to the magnitude (since they have magnitude 1, and are there to represent the complex phase), we can drop them from these two equations, giving us:

$$E_{0,x} = t$$
$$E_{0,y} = -t\tan\theta$$

And thus, the ratio $E_{0,y}/E_{0,x}$ is:

$$\frac{E_{0,y}}{E_{0,x}} = \frac{-t\tan\theta}{t} = -\tan\theta$$

$\square$

c) Show that P is an idempotent matrix.

Solution: To show that P is idempotent, we show that $P^2 = PP = P$. Computing P^2:

$$P^2 = PP = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^4\theta + \cos^2\theta\sin^2\theta & \cos^3\theta\sin\theta + \cos\theta\sin^3\theta \\ \cos^3\theta\sin\theta + \cos\theta\sin^3\theta & \cos^2\theta\sin^2\theta + \sin^4\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta(\cos^2\theta + \sin^2\theta) & \sin\theta\cos\theta(\cos^2\theta + \sin^2\theta) \\ \sin\theta\cos\theta(\cos^2\theta + \sin^2\theta) & \sin^2\theta(\sin^2\theta + \cos^2\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

$$= P$$

And with $P^2 = P$ verified, we can conclude that P is indeed idempotent. $\square$