Eric Du		EE 120
HW 09	Signals and Systems	April 7, 2024

Header styling inspired by CS 70: https://www.eecs70.org/

Collaborators

I worked with the following people on this assignment:

- Teja Nivarthi: 3036508567
- Nikhil Maserang: 3036978230

Problem 1

In this question, we explore the impulse response and frequency response of 2D filters.

a) Find the impulse response of the 2D filter whose output is

$$y[n_1, n_2] = \frac{1}{5}(x[n_1, n_2] + x[n_1 - 1, n_2] + x[n_1 + 1, n_2] + x[n_1, n_2 - 1] + x[n_1, n_2 + 1])$$

Solution: I imagine the impulse response of such a filter is the same as in 1D, so all we have to do is replace all the x's with delta functions:

$$h[n_1, n_2] = \frac{1}{5}(\delta[n_1, n_2] + \delta[n_1 - 1, n_2] + \delta[n_1 + 1, n_2] + \delta[n_1, n_2 - 1] + \delta[n_1, n_2 + 1])$$

□

---

b) Calculate the frequency response $H(e^{j\omega_1}, e^{j\omega_2})$ and its magnitude $|H(e^{j\omega_1}, e^{j\omega_2})|$. Plot $H(e^{j\omega_1}, e^{j\omega_2})$ and $|H(e^{j\omega_1}, e^{j\omega_2})|$ in the range $(\omega_1, \omega_2) \in [-\pi, \pi] \times [-\pi, \pi]$ using appropriate software (e.g. MATLAB with `surf` or Python with `plot_surface` Attach your code and plot (PDF printout or screenshots are fine).

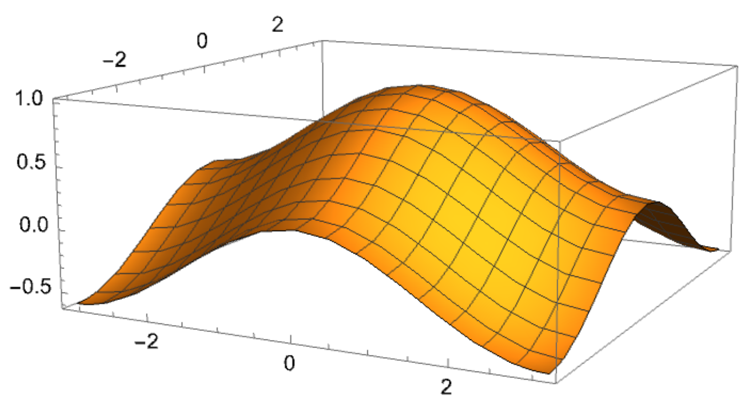
Solution: The Fourier transform of delta functions is very nice, and especially because each delta function can be separated, the integral is also very nice. We use the Fourier transform of delta functions to get:

$$H(e^{j\omega_1}, e^{j\omega_2}) = \frac{1}{5}\left(1 + e^{-j\omega_x} + e^{j\omega_x} + e^{-j\omega_y} + e^{j\omega_y}\right) = \frac{1}{5}\left(1 + 2\cos(\omega_x) + 2\cos(\omega_y)\right)$$

All of these are real valued quantities, so the magnitude and phase are as follows:

$$|H| = \frac{1}{5}(1 + 2\cos(\omega_x) + 2\cos(\omega_y)) \quad \angle H = 0$$

As for the plot, I've attached one below:



□

Problem 2

In general, computers store images as a 2-dimensional array of pixels, which we will represent as a 2-dimensional signal $z[n_1, n_2]$. Given an image, we can apply filters to change its appearance (for instance, blurring or sharpening an image). In this problem, we will examine one of these filters in the frequency domain.

Let's consider the following 4-point moving average filter.

$$h_1[n_x, n_y] = \frac{1}{4}(\delta[x, y] + \delta[x, y-1] + \delta[x-1, y] + \delta[x-1, y-1])$$

a) Is the function $h_1[n_x, n_y]$ separable?

Solution: This is indeed separable:

$$h_1[n_x, n_y] = \frac{1}{4}(\delta_x[x] + \delta_x[x-1])(\delta_y[y] + \delta_y[y-1])$$

□

---

b) Find the frequency response $H(e^{j\omega_x}, e^{j\omega_y})$ of the filter.

Solution: The frequency response is given by the Fourier transform of $h_1[n_x, n_y]$. Because the Fourier transform of delta functions is just an exponential, the algebra is quite easy:

$$H(e^{j\omega_x}, e^{j\omega_y}) = \frac{1}{4}(1 + e^{-j\omega_x x})(1 + e^{-j\omega_y y}) = e^{-j\omega_x/2}e^{-j\omega_y/2}\cos(\omega_x/2)\cos(\omega_y/2)$$

□

---

c) What type of filter is this? What frequencies pass through without being attenuated?

Solution: Because this is a discrete-time signal, the frequencies are 2π-periodic, meaning that we only need to consider the region between 0 and 2π in order to find out what kind of filter this is. Because they're cosines, we can then conclude that this is a low-pass filter. □

---

d) Find the magnitude and phase of $H_1(e^{j\omega_x}, e^{j0})$.

Solution: Here we plug in $y = 0$, so the function looks like:

$$H_1(e^{j\omega_x}, e^{j0}) = e^{-j\omega_x/2}\cos(\omega_x/2)$$

Here, the cosine is real-valued so:

$$|H_1| = \cos(\omega_x/2) \quad \angle H_1 = \omega_x/2$$

□

---

e) Let's say you have an image represented by the signal

$$z[n_x, n_y] = \frac{1}{2}\cos\left[\frac{\pi}{2}n_x\right]\sin\left[\frac{\pi}{3}n_y\right] + \frac{1}{2}$$

If you apply the filter $h_1[n_x, n_y]$ to z, what is the resulting image $z_1[n_x, n_y] = (z * h_1)[n_x, n_y]$?

Solution: The 2D convolution in question is:

$$\sum_{m_1, m_2} \frac{1}{2}\left(\cos\left[\frac{\pi}{2}m_1\right]\sin\left[\frac{\pi}{3}m_2\right] + 1\right) \cdot \frac{1}{4}(\delta_x[x - m_1] + \delta_x[x-1-m_1])(\delta_y[y-m_2] + \delta_y[y-1-m_2])$$

We can expand everything out, and notice that we will get 4 terms, one for every combination of delta functions.

$$z_1[n_x, n_y] = \frac{1}{8}\left[\left(\cos\left[\frac{\pi}{2}n_x\right]\sin\left[\frac{\pi}{3}n_y\right]+1\right)+\left(\cos\left[\frac{\pi}{2}(n_x-1)\right]\sin\left[\frac{\pi}{3}n_y\right]+1\right)\right.$$
$$\left.+\left(\cos\left[\frac{\pi}{2}n_x\right]\sin\left[\frac{\pi}{3}(n_y-1)\right]+1\right)+\left(\cos\left[\frac{\pi}{2}(n_x-1)\right]\sin\left[\frac{\pi}{3}(n_y-1)\right]+1\right)\right]$$

This simplifies a little if we combine some terms:

$$z_1[n_x, n_y] = \frac{1}{8}\left[\cos\left[\frac{\pi}{2}n_x\right]\sin\left[\frac{\pi}{3}n_y\right]+\cos\left[\frac{\pi}{2}(n_x-1)\right]\sin\left[\frac{\pi}{3}n_y\right]+\right.$$
$$\left.\cos\left[\frac{\pi}{2}n_x\right]\sin\left[\frac{\pi}{3}(n_y-1)\right]+\cos\left[\frac{\pi}{2}(n_x-1)\right]\sin\left[\frac{\pi}{3}(n_y-1)\right]+4\right]$$

□

---

f) Describe qualitatively what a moving average filter does to an image.

Solution: As far as I can tell based on the equations, a moving average filter basically just smoothens out the image, and (as its name suggests), every pixel's intensity now reflects the average intensity of the pixels around it.

As a total effect, this basically corresponds to the image "smoothening out" and looking overall more uniform. □

Problem 3

The theorem of computed tomography is based on "Projection-slice Theorem" (a.k.a. "Central-slide Theorem", a.k.a. "Fourier-slice Theorem"). It is a widely used application of 2d FT. In the 2D case, Projection-slice Theorem states: The 1D Fourier transform of a projection at angle θ is the 2D Fourier transform of the object evaluated at angle θ.

$$\mathcal{F}_{1D}\{p(l;\theta)\} = \mathcal{F}_{2D}\{f(x,y)\}(\rho,\theta)$$

Prove this theorem.

Solution: I will provide the solution I private-posted on EdStem, not sure if it's correct but it's been 3 days since the post without reply (I'm also inclined to give myself full points for this solution until I'm told otherwise that the proof is wrong).

In essence, instead of performing the computations in the (x, y) plane, we will instead consider the Fourier transform and computations in a rotated (x', y') frame, tilted at the angle θ. In this frame, the projection is computed as:

$$\rho = \int_{-\infty}^{\infty} f(x', y') dy'$$

then the Fourier transform of this is computed as:

$$\mathcal{F}_{\mathrm{1D}}\{\rho\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\ dy' e^{-j\omega_x x'}\ dx' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y') e^{-j\omega_x x'}\ dx'\ dy'$$

As for the 2DFT, because this basis is also orthogonal, the form of the Fourier transform doesn't change:

$$\mathcal{F}_{\mathrm{2D}}\{f(x', y')\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y') e^{-j\omega_x x'} e^{-j\omega_y y'}\ dx'\ dy'$$

Now, evaluation at an angle θ now means evaluating about the line $\omega_y = 0$, so therefore we have:

$$\mathcal{F}_{\mathrm{2D}}\{f(x', y')\}(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y') e^{-j\omega_x x'}\ dx'\ dy'$$

which matches the equation for $\mathcal{F}_{\mathrm{1D}}\{\rho\}$, and completes the proof. $\square$

Problem 4

In the olden days of Spring 2021, the professor asked what te Fourier transform of the curtain behind DiCaprio would look like. We will address this now.

It is a 2 dimensional (2D) image, so we need to use the 2DFT. In this question, we use linear frequency without loss of generality. Recall the 2DFT of a 2D function $f(x, y)$ is:

$$F(f_x, f_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(f_x x + f_y y)} \, dx \, dy \tag{2DFT}$$

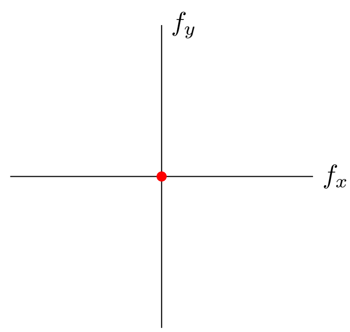
$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(f_x, f_y) e^{j2\pi(f_x x + f_y y)} \, df_x \, df_y \tag{inverse 2DFT}$$

In this problem, we consider real-valued $f(x, y)$.

a) Compute and sketch the magnitude of 2D FT of the following patterns:

- $f(x, y) = 1$

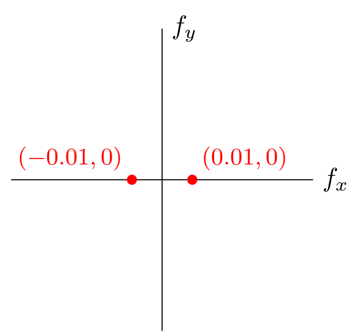
  Solution:



$\square$

- $f(x, y) = \sin \frac{2\pi}{100} x$

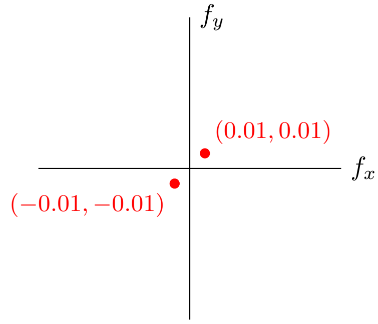
  Solution: This is a sinusoid, so we just have two delta functions along x at the appropriate frequencies:



  Note that we have 0.01 here because the frequency is $\frac{1}{100}$. $\square$

- $f(x, y) = \sin\left(\frac{2\pi}{100} x + \frac{2\pi}{100} y\right)$

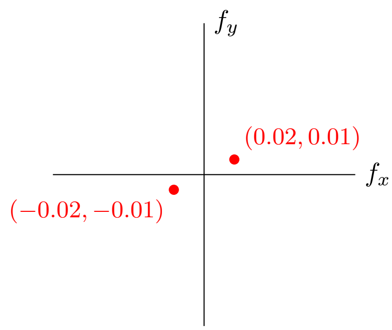
  Solution: Here, we will have two delta functions shifted by the combination $\frac{2\pi}{100} x + \frac{2\pi}{100y}$ and $-\frac{2\pi}{100} x - \frac{2\pi}{100} y$, so this will give us:



I included the coordinates in decimals just becuase it was easier for me, and that remains true for the preceding subparts. $\square$

- $f(x, y) = \sin\left(\frac{2\pi}{50}x + \frac{2\pi}{100}y\right)$

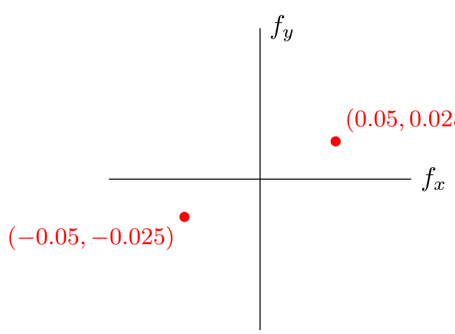
  Solution:



$\square$

- $f(x, y) = \sin\left(\frac{2\pi}{20}x + \frac{2\pi}{40}y\right)$

  Solution:



$\square$

b) Now we consider the illumination situation in the picture. We then model the luminosity of one light source follows:

$$I(r) = I_0 e^{-ar^2}$$

where I_0 is the intensity of light at the source, r is the distance from the light source, $r^2 = x^2 + y^2$.

Find the 2D FT of Gaussian $h(x, y) = e^{-a(x^2+y^2)}$ where $a > 0$. What shape is this?

Solution: This is a separable equation:

$$e^{-a(x^2+y^2)} = e^{-ax^2}e^{-ay^2}$$

meaning that the 2D Fourier transform will also be a product. Also, because they're the same form, we'll just get two copies of the same function. Calculating one of them:

$$\mathcal{F}\{e^{-ax^2}\} = \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi j f_x x}\, dx$$

This is a well known integral, I proved in homework 6 that:

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2+bx} = e^{\frac{b^2}{2a}}\sqrt{\frac{2\pi}{a}}$$

(the proof just involves change of variables) so applying it to this equation, we have:

$$\mathcal{F}\{e^{-ax^2}\} = e^{-\pi^2 f_x^2/a}\sqrt{\frac{\pi}{a}}$$

Thus, the full Fourier transform is:

$$H(\omega_x, \omega_y) = \frac{\pi}{a} e^{-\pi^2 f_x^2/a} e^{-\pi^2 f_y^2/a}$$

This is a 2-dimensional Gaussian. □

---

c) We model the curtain behind him to be $f(x, y) = \sin\left(\frac{2\pi}{10}x\right)\cos\left(\frac{2\pi}{50}y\right)$. What's the 2D FT of this curtain pattern?

Solution: Again, this is separable, so we just have a product over x and y. The Fourier transform of sines and cosines are just linear combinations of plane waves with the positive and negative frequency (with appropriate prefactors), so we have:

$$F(f_x, f_y) = \frac{1}{4i}\left(\delta\left(f_x - \frac{2\pi}{10}\right) - \delta\left(f_x + \frac{2\pi}{10}\right)\right)\left(\delta\left(f_y - \frac{2\pi}{50}\right) + \delta\left(f_y + \frac{2\pi}{50}\right)\right)$$

□

---

d) Suppose the light source is at the center of the frame. What is the 2D FT of the pattern from part 3. with centered light source with $a = 1$ from part (b)?

i) Show that

$$\mathcal{F}\{f(x, y) \cdot g(x, y)\} = F(f_x, f_y) * G(f_x, f_y)$$

for any aribtrary $f(x, y)$ and $g(x, y)$ with corresponding Fourier transform $F(f_x, f_y)$ and $G(f_x, f_y)$.

Solution: I will instead prove the inverse relation:

$$f(x, y) \cdot g(x, y) = \mathcal{F}^{-1}\{F(f_x, f_y) * G(f_x, f_y)\}$$

The inverse Fourier transform of the convolution is written as:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(f_x', f_y')G(f_x - f_x', f_y - f_y')e^{2\pi j f_x} e^{2\pi j f_y}\, df_x\, df_y\, df_x'\, df_y'$$

We can then rearrange it as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(f_x', f_y')\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(f_x - f_x', f_y - f_y')e^{j2\pi f_x} e^{2\pi j f_y}\, df_x\, df_y\right) df_x'\, df_y'$$

We can then use the property that: $X(f - f_0) \leftrightarrow e^{2\pi j f_0 t}x(t)$ to simplify the expression in parentheses:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(f_x - f_x', f_y - f_y')e^{2\pi j f_x} e^{2\pi j f_y}\, df_x\, df_y = g(x, y)e^{2\pi j f_x'} e^{2\pi j f_y'}$$

Therefore, the equation now becomes:

$$g(x, y)\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(f_x', f_y')e^{2\pi j f_x'} e^{2\pi j f_y'}\, dx'\, dy'}_{f(x,y)} = f(x, y) \cdot g(x, y)$$

where the $\cdot$ denotes pointwise multiplication. □

ii) Find the 2D FT of $f(x, y) \cdot h(x, y)$

Solution: Here, we will use the 2D convolution theorem to help us, since the Fourier transform is just a convolution in frequency space.

Because $F(f_x, f_y)$ is a series of delta functions, this is relatively easy. We know that the convolution has the property that $x(t) * \delta(t - T) = x(t - T)$, so this means that we will basically have four terms that are just a combination of the shifts:

$$Y(f_x, f_y) = \frac{\pi}{4ai}\Big[e^{-\pi^2\left(f_x - \frac{2\pi}{10}\right)^2/a} e^{-\pi^2\left(f_y - \frac{2\pi}{50}\right)^2} + e^{-\pi^2\left(f_x - \frac{2\pi}{10}\right)^2} e^{-\pi^2\left(f_y + \frac{2\pi}{50}\right)^2}$$
$$- e^{-\pi^2\left(f_x + \frac{2\pi}{10}\right)^2} e^{-\pi^2\left(f_y - \frac{2\pi}{50}\right)^2} - e^{-\pi^2\left(f_x + \frac{2\pi}{10}\right)^2} e^{-\pi^2\left(f_y + \frac{2\pi}{50}\right)^2}\Big]$$

□

e) Finally, we can answer the question: "What would the Fourier transform of the curtain behind DiCaprio look like" Hint: The light source seems to be on the right hand side of the image.

Solution: If the light source is not centered, then this is the same as basically shifting $h(x, y)$, so that we have $h(x - x_0, y - y_0)$. In terms of a Fourier transform, we can use the relation that:

$$\mathcal{F}\{h(x - x_0, y - y_0)\} = e^{-j\omega x_0} e^{-j\omega y_0} \mathcal{F}\{h(x, y)\}$$

this is the two-dimensional version of the time-shift property. So depending on the location (say, the source was located at (x_0, y_0)), we basically have:

$$Y(f_x, f_y) = e^{-2\pi j(f_x x_0 + f_y y_0)} \frac{\pi}{4ai}\Big[e^{-\pi^2\left(f_x - \frac{2\pi}{10}\right)^2/a} e^{-\pi^2\left(f_y - \frac{2\pi}{50}\right)^2} + e^{-\pi^2\left(f_x - \frac{2\pi}{10}\right)^2} e^{-\pi^2\left(f_y + \frac{2\pi}{50}\right)^2}$$
$$- e^{-\pi^2\left(f_x + \frac{2\pi}{10}\right)^2} e^{-\pi^2\left(f_y - \frac{2\pi}{50}\right)^2} - e^{-\pi^2\left(f_x + \frac{2\pi}{10}\right)^2} e^{-\pi^2\left(f_y + \frac{2\pi}{50}\right)^2}\Big]$$

□