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Lecture Notes

Signals and Systems

EE 120
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1 Introduction

1.1 Motivations

- Why study this class?
  - Given a "black box" circuit, with input and output leads, we can determine what's within the "black box".
  - In this particular case, if our black box contains a voltage divider, and the output voltage is given by the equation:
    $$v_{\text{out}}(t) = \frac{R_2}{R_1 + R_2} v_{\text{in}}(t)$$
    In principle though, the signal can be anything that we want: for facial recognition software, the input signal could be the configuration of the intensity the camera picks up. There's many more we went over, don't really want to write it all down.
- In essence, there's a lot of systems that can be modeled by a system that takes in a signal $x(t)$, and outputs a signal $y(t) = f(x(t))$.
  - The signals are usually functions of time, location, in any number of dimensions.
  - The systems does some sort of transformation on an input signal. In particular, we will study linear systems, shift-invariant systems, etc.

    We'll talk about mathematical operations that we use to perform these transformations: Fourier, Laplace, Z-transformations, convolutions, correlation, etc.
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1.2 Types of Signals

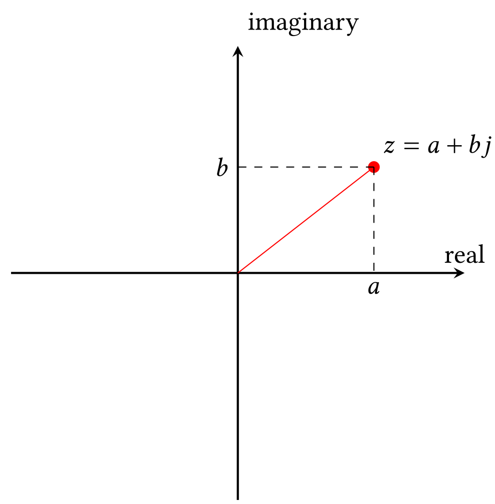
- **Continuous-time:** signals defined over continuous variables (e.g. position, time). For instance, a signal $x(t)$ is continuous for our purposes, since time is a continuous variable.

  Further, because t is continuous, then x must also be continuous. If the signal is differentiable, then the derivative $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ also exists.

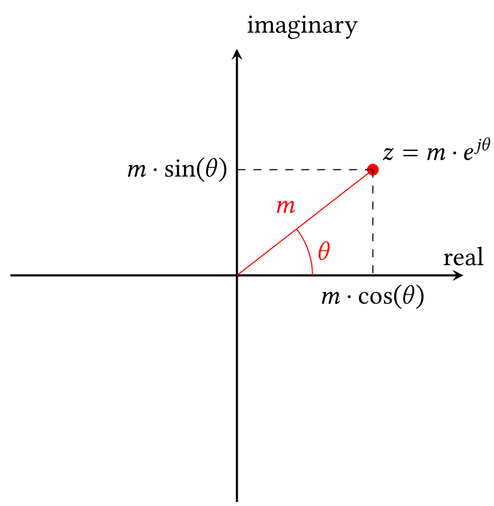
  t being continuous does not imply that $x(t)$ is continuous (e.g. Thomae function), but is it true for this class?
- **Discrete-Time:** These are signals defined over discrete variables. For instance, if we had $x[n]$ as a signal, where n is an integer.

  We don't have a concept of differentiability, but we can compute the difference: $x[n] - x[n-1]$, and talk about that quantity.
- **Real-Valued:** A signal $x(t)$ is real-valued if $x(t) \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of all real numbers.
- **Complex-Valued:** A signal $x(t)$ is complex-valued if $x(t) \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers.
- Note that while we're using the continuous-time notation here, the same concepts apply with discrete-time signals.

- Quick recap on complex numbers: denoted by $a + bj$ or $a + bi$, where i and j denote the imaginary unit.
- They are defined as $i^2 = -1$ or $j^2 = -1$.
- a is the real part, and b is the imaginary part.
- We can plot these values in the complex plane, using the real and imaginary representation:



Or using the magnitude-phase representation:



We represent the magnitude as $m = |z|$, and the phase angle θ is the angle made with the real axis.

- **Periodic Signal:** Two quantities we'll introduce here: the period T is the time it takes for the signal to repeat itself. T is measured in units of time, generally seconds.

  The frequency f is the "inverse" of period, defined by $f = \frac{1}{T}$. We will also use the angular frequency ω, defined by $\omega = \frac{2\pi}{T} = 2\pi f$. Angular frequency is mainly going to be used when we involve complex numbers. We will see:
$$e^{j\omega t} = e^{j(2\pi ft)} = \cos(2\pi ft) + i\sin(2\pi ft)$$

- **Dimensionality:** We will deal with multi-dimensional signals: an example of a 2D signal are images, which determine the color of a pixel based on a row and column. The spaces that we'll be working with are either $\mathbb{R}^n$ or $\mathbb{C}^n$.

1.3 Signal Transformations

- **Shifts:** Essentially just shifts the signal along one dimension: $x(t) \to x(t-T)$. T is some constant. If $T > 0$, then the shift is to the *right*, and if $T < 0$ then the shift is to the *left*.
- **Scaling:** We can multiply a signal $x(t)$ by some constant $a : x(t) \to a \cdot x(t)$. If $a < 1$, then we shrink $x(t)$, and if $a > 1$ then we amplify the signal.
- **Reversal:** Given $x(t)$, we can "reverse time" by adding a negative to the argument: $x(t) \to x(-t)$. Visually, all we do is flip the signal around the y-axis.

1.4 Signal Properties

- **Even:** Functions which satisfy $x(t) = x(-t)$. In other words, if we perform a reversal, the signal stays the same.
- **Odd:** Functions which satisfy $x(t) = -x(-t)$. If we perform a reversal, the signal becomes the negative of itself.
- **Periodic:** If T is the period, then nT is also a period for any $n \in \mathbb{Z}$. However, we will call T the fundamental period; the smallest T for which the function repeats.

  For the function $\sin(2\pi f t)$, the fundamental period is $1/f$.

1.5 Model Functions

- These are called model functions because they're idealized models to analyze.
- **Heaviside Step function:** For the continuous-time case it's usually modeled by:

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

  In the discrete-time case, it's written as:

$$u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

2 More on Model Functions, System Characterization

2.1 Model Functions Continued

- **Ramp Function:** The continuous-time is expressed as:

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

  Similarly in discrete time:

$$\text{ramp}[n] = \begin{cases} 0 & \text{for } n < 0 \\ n & \text{for } n \geq 0 \end{cases}$$

  Note that we can express the ramp function in terms of the step function, in many ways:

  - $r(t) = t \cdot u(t)$
  - $r(t) = \int_{-\infty}^{t} u(t)\, dt$, the discrete case is just a sum over the same bound.

- **Rectangular Function:** In continuous-time:

$$\text{rect}(t) = \sqcap(t) = \begin{cases} 1 & \text{for } |t| \leq 1/2 \\ 0 & \text{else} \end{cases}$$

In discrete time:

$$\text{rect}\left[\frac{n}{N}\right] = \begin{cases} 1 & \text{for } |n| \leq N \\ 0 & \text{for } |n| > N \end{cases}$$

We can also express $\text{rect}(t)$ in terms of $u(t)$:

$$\sqcap(t) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

- **Triangle Function:** In continuous-time:

$$\Lambda(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

And in discrete-time:

$$\Lambda\left[\frac{n}{N}\right] = \begin{cases} 1 - \left|\frac{n}{N}\right| & \text{for } |n| \leq N \\ 0 & \text{else} \end{cases}$$

- **Delta Function:** In continuous time, it's called the Dirac delta function. It has the property that $\delta(t) = 0$ for all $t \neq 0$, but

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

So in essence, this is an infinitesimally "thin" function, that extends to infinity. There are also other ways to represent the Delta function:

  – Derivative of the Heaviside step function: $\delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$

  – The integral of a complex exponential:

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft}\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

The delta function allows us to approximate the integral $\int_{-\infty}^{\infty} \cos(\omega t)\, dt$. We can do the following:

$$\begin{aligned} \int_{-\infty}^{\infty} \cos(\omega t)\, dt &= \text{Re}\left[\int_{-\infty}^{\infty} \cos(\omega t) + i\sin(\omega t)\right]\, dt \\ &= \text{Re}\left[\int_{-\infty}^{\infty} e^{i\omega t}\, dt\right] \\ &= \text{Re}\left[2\pi\delta(\omega)\right] \end{aligned}$$

Looking at the delta function, we know that when $\omega = 0$, then $\cos(\omega t) = 1$, so the integral diverges, as expected. When $\omega \neq 0$, our integral result implies that the integral evaluates to 0. This is not exactly true since the integral will oscillate between ± 1, which is relatively small compared to $\omega = 0$, so it can effectively be taken as 0.

How does this compare with the definition we use in physics that $\delta(t)$ is defined as the function which satisfies:

$$\int_{-\infty}^{\infty} f(t)\delta(t) = f(0)$$

Do both work?

See below bullet point, the definition allows you to derive this property.

- Let's explore some properties of the Delta function:
  - **Scaling:**
  $$\int_{-\infty}^{\infty} \delta(\alpha t)dt = \int_{-\infty}^{\infty} \delta(t)\frac{\mathrm{d}\tau}{\mathrm{d}\alpha} = \frac{1}{|\alpha|}$$
  In other words, $\delta(\alpha t) = \frac{\delta(t)}{|\alpha|}$
  - **Sifting:** If we have $f(t)$ and multiply it by a Delta function:
  $$\int_{-\infty}^{\infty} f(t)\delta(t-T)\,dt = f(T)$$
  - **Delta function of a function:** We can take the delta function of a function as well:
  $$\delta(f(t)) = \frac{\delta(t-t_0)}{|f'(t_0)|},\ f(t_0) = 0$$
  We take the derivative in the denominator.
- In discrete time, the delta function is represented as the Kronecker delta:
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$
The function attempts to model the Dirac delta but for discrete time intervals:
$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-10] = x[10]$$
- **Shah function:** It's basically a bunch of Dirac deltas:
$$\text{III}(t) = \sum_{k=0}^{\infty} \delta(t-k)$$
In discrete time, it also is a sum of all deltas:
$$\text{III}[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

2.2 System Characterization

- Systems perform operations on input signals, like functions $F : x \to y$. For instance, the following is a moving average filter:
$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$
- We will be particularly interested in **linear systems**, systems which satisfy the following two properties:
  - **Scaling:** If for any input-output pair $x(t) \to y(t)$, then for any constant a, $ax(t) \to ay(t)$
  - **Addition:** Given any two input-output pairs
  $$x_1(t) \to y_1(t)$$
  $$x_2(t) \to y_2(t)$$
  Then it's also true that $x_1(t) + x_2(t) \to y_1(t) + y_2(t)$

Combining these two properties, given two general signals $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, then $ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$.

Note that a function like $y(t) = x(t) + b$ is not a linear function, becuase it doesn't satisfy the second property. Even though the function is linear, doesn't mean that the transformation is linear.

- **Shift Invariant:** A shift-invariant system is a system where if we shift the input, the output is also shifted. Given $x(t)$ and its output $y(t)$, then $x(t-T)$ should produce $y(t-T)$ for any T.
- **Memoryless:** A function whose output at any given time only depends on the input at that time. For instance, machine learning algorithms are not memoryless, since their output depends on previous inputs.

  Does "current" here refer to a given input, or does it refer to past inputs? For instance, is the moving average function considered memoryless?

  Most systems that take time to react are not considered memoryless, since
- **Causality:** A system is causal if the output depends on the input at the present and past times only, not on future times. A system defined by:

$$y[n] = \frac{1}{3}(x[n] + x[n+1])$$

  is not considered causal, because $y[n]$ depends on the $n+1$-th input.
- **Stability:** There are many different ways to define stability, here are some of them:
  - A system is called BIBO stable if bounded inputs generate boudned outputs. Mathematically, this means:

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

    And in discrete time:

$$\sum_{-\infty}^{\infty} |x[n]| < \infty$$

  - We can also look at the energy and power:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \quad P = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

- **System Response function:** These are particular outputs for systems when given an impulse response of a delta function. They are calculated by substituting $x(t) = \delta(t)$ in the continuous case, and $x[n] = \delta[n]$ in the discrete case. Watch lecture for this.

  To calculate the impulse response for the moving average filter:

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$

  To find the impulse response, we substitute $x[n] = \delta[n]$ to get $h[n]$. Here, notice that for $n < -2$, then $h[n] = 0$, since $x[-2+1] = x[-1] = 0$, and same goes for the other terms. Then, refer to the following table:

n	$h[n]$
-2	0
-1	1/3
0	1/3
1	1/3
2	0

3 Characterization Continued

3.1 Step Response

- The step response function is the function $y_{\text{step}}(t)$ when a step function $u(t)$ is fed into the system. In discrete-time: we feed $u[n]$ into the system, and get $y_{\text{step}}[n]$ as an output.
- For instance, for the moving average filter defined earlier, we have the following result:

n	$y_{\text{step}}[n]$
-2	0
-1	1/3
0	2/3
1	1
2	1

  Note that this resembles a ramp function, and is called a ramp-step function.

- **Harmonic Response:** The harmonic response is the response by the system when presented with a harmonic function, of the form $Ae^{i\omega t}$.

  In discrete time, we feed in $Ae^{i\omega n}$ where n is an integer.

- For the moving average filter, let's write out $y[n]$:

$$\begin{aligned} y[n] &= \frac{1}{3}\left(Ae^{i\omega(n-1)} + Ae^{i\omega n} + Ae^{i\omega(n+1)}\right) \\ &= \frac{1}{3}\left(e^{-i\omega} + 1 + e^{i\omega}\right) \\ &= \frac{1}{3}(2\cos\omega + 1)Ae^{i\omega n} \end{aligned}$$

  The interesting thing here is that when given a harmonic function, the system response just scales the signal by a constant amount!

3.2 LCCDE

- In this class, we will deal with lots of differential equations, so it's going to be very useful to look at their form, and how to solve them.
- There are two solutions to any differential equation:
  - **Particular Solution:** $y_p(t)$ is called a particular solution if it satisfies:

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y_p(t)}{\mathrm{d}t^k} = \sum_{k=0}^{N} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

  - **Homogeneous Solution:** $y_h(t)$ is called a homogeneous solution if it satisfies:

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y_p(t)}{\mathrm{d}t^k} = 0$$

- In general, the solution will be a linear combination of the two:

$$y(t) = y_p(T) + ay_h(t)$$

  the value of a is generally going to be given by some initial condition.

- For the homogeneous solution, an ansats of the form Ae^{st} where s is an undetermined constant will solve the differental equation. We can then determine the value of s by solving the resulting polynomial.

  To determine the value of A, these are determined by the initial conditions, and depending on the number of initial conditions given, that would correspond directly to the number of distinct values of A.