Physics W89 - Summer 22 - MT Solns

Problem 1 - Nuts & Bolts

a) Taylor Series

$$\arctan(1) = \frac{\pi}{4} \qquad \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$(1+\epsilon)^p = 1 + p\epsilon + \frac{p(p-1)}{2}\epsilon^2 + \cdots$$

$$\Rightarrow (1+x^2)^{-1} = 1 - x^2 + \frac{(-1)(-2)}{2}(x^2)^4 + \mathcal{O}((x^2)^3)$$

$$= 1 - x^2 + x^4 + \mathcal{O}(x^6)$$

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^1 \frac{d}{dx}\arctan(x)\,dx = \arctan(x)\Big|_0^1 = \arctan(1) = \frac{\pi}{4}$$

$$= \int_0^1 \left(1 - x^2 + x^4 + \mathcal{O}(x^6)\right) dx$$

$$= \left(x - \frac{x^3}{3} + \frac{x^5}{5} + \mathcal{O}(x^7)\right)\Bigg|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5} + \cdots \qquad \frac{15-5+3}{15}$$

$$\approx 0.867$$

$$\text{vs. } \frac{\pi}{4} \approx 0.785$$

~10% off.

b) Complex Numbers - cube roots of i

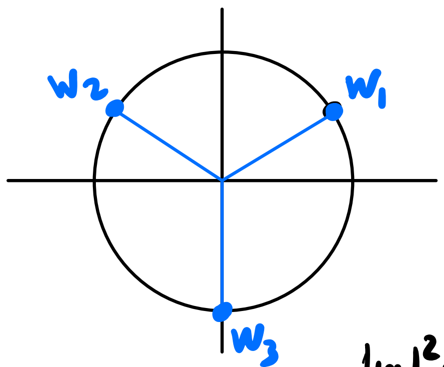
$$w_1 = e^{i\pi/6} \qquad w_2 = e^{5i\pi/6} \qquad w_3 = e^{3i\pi/2} \qquad w_i^3 = i$$

$$\begin{aligned}
w_1 + w_2 + w_3 &= e^{i\pi/6} + e^{5i\pi/6} + e^{3i\pi/2} \\
&= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} + \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) + \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) \\
&= \left(\cos\frac{\pi}{6} + \cos\frac{5\pi}{6} + \cos\frac{3\pi}{2}\right) + i\left(\sin\frac{\pi}{6} + \sin\frac{5\pi}{6} + \sin\frac{3\pi}{2}\right) \\
&= \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 0\right) + i\left(\frac{1}{2} + \frac{1}{2} - 1\right) \\
&= 0 \checkmark
\end{aligned}$$

$\frac{\pi}{6} = 30°$

$\frac{5\pi}{6} = 150°$

$\frac{3\pi}{2} = 270°$



$$z \equiv w_1 + 2w_2$$

$$\begin{aligned}
|z|^2 &= (w_1 + 2w_2)^*(w_1 + 2w_2) \\
&= w_1^* w_1 + w_1^* 2w_2 + 2w_2^* w_1 + 4w_2^* w_2 \\
&= |w_1|^2 + 2(w_1^* w_2 + w_2^* w_1) + 4|w_2|^2
\end{aligned}$$

$|w_1|^2 = |w_2|^2 = 1$

$$\begin{aligned}
&= 1 + 2\left(e^{-i\pi/6} e^{5i\pi/6} + e^{-5i\pi/6} e^{i\pi/6}\right) + 4 \\
&= 5 + 2\left(e^{4i\pi/6} + e^{-4i\pi/6}\right)
\end{aligned}$$

$e^{ix} + e^{-ix} = 2\cos x$

$$\begin{aligned}
&= 5 + 4\cos\left(\frac{2\pi}{3}\right) \\
&= 5 + 4\left(-\frac{1}{2}\right) = 3
\end{aligned}$$

$$\Rightarrow \boxed{|z| = \sqrt{3}}$$

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c) Vector Identities in 3D

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \times \vec{B})_i (\vec{C} \times \vec{D})_i$$

$$= (\epsilon_{ijk} A_j B_k)(\epsilon_{ilm} C_l D_m)$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) A_j B_k C_l D_m$$

$$= \delta_{jl}\delta_{km} A_j B_k C_l D_m - \delta_{jm}\delta_{kl} A_j B_k C_l D_m$$

$$= (A_j C_j)(B_k D_k) - (A_j D_j)(B_k C_k)$$

$$= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \quad \checkmark$$

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d) Linear Systems of Equations

Problem 1 Solutions

Wednesday, July 13, 2022 4:26 AM

1) Nuts and Bolts

a) Taylor series

$\arctan(1) = \frac{\pi}{4}$ $\qquad \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

$$\frac{1}{(1+x^2)} \approx 1 - x^2 + \frac{(-1)(-1-1)}{2}x^4 = \boxed{1 - x^2 + x^4}$$

$$\int_0^1 (1 - x^2 + x^4) = 1 - \frac{1}{3} + \frac{1}{5} \approx \frac{6}{5} - \frac{1}{3} = \frac{13}{15} \approx 0.87$$

$$\frac{\pi}{4} \approx 0.785 \qquad \frac{13}{15} - \frac{\pi}{4} = 0.08126\ldots$$

b) Complex

$$\left.\begin{aligned} w_1 &= e^{i\pi/6} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2} \\ w_2 &= e^{i5\pi/6} = \text{---} = -\frac{\sqrt{3}}{2} + i\frac{1}{2} \\ w_3 &= e^{i3\pi/2} = \sim\sim = -i \end{aligned}\right\} w_1 + w_2 + w_3 = 0$$

$$z = w_1 + 2w_2 = i + w_2 = -\frac{\sqrt{3}}{2} + i\frac{3}{2}, \quad |z| = \sqrt{\frac{3}{4} + \frac{9}{4}} = \sqrt{3}$$

c)

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

$$[\epsilon_{ijk} A_j B_k][\epsilon_{ilm} C_l D_m] = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(A_j B_k C_l D_m)$$
$$= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

d)

$$\begin{aligned} 2x - y + 6z &= \alpha \\ 3x - 3y + 9z &= \beta \\ -4x + 3y - 12z &= \gamma \end{aligned} \qquad A\vec{x} = \vec{b}$$

$$D = \begin{pmatrix} 2 & -1 & 6 \\ 3 & -3 & 9 \\ -4 & 3 & -12 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 6 & \alpha \\ 3 & -3 & 9 & \beta \\ & & & \gamma \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1/2 & 3 & \alpha/2 \\ 0 & -3/2 & 0 & \beta - \frac{3\alpha}{2} \end{array}\right) \Rightarrow \quad \begin{aligned} &\text{rank}(A) = 2 \\ &\text{nullity}(A) = 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 6 & \alpha \\ 3 & -3 & 9 & \beta \\ -4 & 3 & -12 & \gamma \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1/2 & 3 & \alpha/2 \\ 0 & -3/2 & 0 & \beta - \frac{3\alpha}{2} \\ 0 & 1 & 0 & \gamma + 2\alpha \end{array}\right) \Rightarrow$$

$\text{rank}(A) = 2$

$\text{nullity}(A) = 1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & \frac{3\alpha}{2} + \frac{\gamma}{2} \\ 0 & 1 & 0 & \gamma + 2\alpha \\ 0 & -3/2 \to 1 & 0 & \beta - \frac{3\alpha}{2} \to \alpha - \frac{2}{3}\beta \end{array}\right)$$

$$x + 3z = \frac{3\alpha}{2} + \frac{\gamma}{2}$$

$$y = \gamma + 2\alpha$$

$$y = \alpha - \frac{2}{3}\beta$$

$$\Rightarrow \boxed{\beta = -\frac{3}{2}(\alpha + \gamma)}$$

For $\alpha = 3,\ \beta = -6,\ \gamma = 1 \Rightarrow y = 7,\ x + 3z = 5 \Rightarrow$ solutions are $\boxed{\begin{pmatrix} 5 - 3z \\ 7 \\ z \end{pmatrix} = \begin{pmatrix} x \\ 7 \\ \frac{5}{3} - \frac{x}{3} \end{pmatrix}}$

also $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} + k\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ //

e) • lower-tri, $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & i \end{pmatrix}$

• $A^{\dagger} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & -i \end{pmatrix}$

$$A^{\dagger}A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & i \end{pmatrix} = \begin{pmatrix} 21 & \cdots \\ \vdots & \ddots \end{pmatrix}$$

$$\neq$$

$$AA^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & i \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & \cdots \\ \vdots & \ddots \end{pmatrix}$$

NOT normal

• $\begin{pmatrix} -2 & i \\ 3i & 2 \end{pmatrix}\begin{pmatrix} -2 & i \\ 3i & 2 \end{pmatrix} = \begin{pmatrix} 4 - 3 & -2i + 2i \\ -6i + 6i & -3 + 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓ is involutary

• $\dfrac{\begin{pmatrix} -2 & i \\ 3i & 2 \end{pmatrix} - \begin{pmatrix} -2 & -3i \\ -i & 2 \end{pmatrix}}{2} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}$ $\left(\frac{A - A^{\dagger}}{2} = A_{\text{anti-herm}}\right)$

f) $\begin{pmatrix} 3 & 6 & 2 \\ 2 & 2 & 0 \\ 2 & 4 & 1 \end{pmatrix} = F$

• Compute $\det(F)$.

$$2(2\cdot 4 - 2\cdot 2) + 1(3\cdot 2 - 6\cdot 2) = 2(8-4) + 1(6 - 12) = 2\cdot 4 + (-6) = 8 - 6 = \boxed{2}$$

or

$$3(2\cdot 1 - 0\cdot 4) - 6(2\cdot 1 - 0\cdot 2) + 2(2\cdot 4 - 2\cdot 2) = 3(2) - 6(2) + 2(4) = -6 + 8 = \boxed{2}$$

• Compute inverse.

$$\left(\begin{array}{ccc|ccc} 3 & 6 & 3 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R1/3} \left(\begin{array}{ccc|ccc} 1 & 2 & 2/3 & 1/3 & 0 & 0 \\ 0 & -2 & -4/3 & -2/3 & 1 & 0 \\ 0 & 0 & -1/3 & -2/3 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 2/3 & 1/3 & 0 & 0 \\ 0 & 1 & 2/3 & 1/3 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & 0 & -3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 2/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1/2 & 2 \\ 0 & 0 & 1 & 2 & 0 & -3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & -1 & -1/2 & 2 \\ 0 & 0 & 1 & 2 & 0 & -3 \end{array}\right)$$

$$\boxed{\begin{pmatrix} 1 & 1 & -2 \\ -1 & -1/2 & 2 \\ 2 & 0 & -3 \end{pmatrix}} = H^{-1}$$

• Verify.

$$H H^{-1} = \begin{pmatrix} 3 & 6 & 3 \\ 2 & 2 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ -1 & -1/2 & 2 \\ 2 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

g)

$$\underset{v_1}{\begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}} \quad \underset{v_2}{\begin{pmatrix} 7 \\ 14 \\ 6 \\ 0 \end{pmatrix}} \quad \underset{v_3}{\begin{pmatrix} 8 \\ 9 \\ 8 \\ 9 \end{pmatrix}}$$

• 3 vectors are not enough to span $\mathbb{R}^4$

• $a v_1 + b v_2 + c v_3 = 0 \Rightarrow$

$$\begin{aligned} 2a + 7b + 8c &= 0 \quad \rightarrow 2a + 7b - \tfrac{14 \cdot 8}{9} b = 2a - \tfrac{49}{9} b = 0 \\ 14b + 9c &= 0 \quad \Rightarrow c = -\tfrac{14}{9} b \\ 2a + 6b + 8c &= 0 \\ 2a + 9c &= 0 \quad \rightarrow 2a - 14b = 0 \end{aligned}$$

Can't be true unless $a = b = c = 0$

lin. indep. ✓

h) $\vec{A} \cdot \vec{B} = \frac{1}{2} tr(A^{\dagger} B) = \frac{1}{2} A^{\dagger}_{ij} B_{ji}$

• $\vec{B} \cdot \vec{A} = \frac{1}{2} tr(B^{\dagger} A) = \frac{1}{2} B^{\dagger}_{ij} A_{ji} = \frac{1}{2} A_{ji} B^{\dagger}_{ij} = \frac{1}{2} A^{T}_{ij} B^{*}_{ji} = \frac{1}{2} (A^{\dagger}_{ij} B_{ji})^{*}$ ✓

• $\vec{A} \cdot (c\vec{B} + \vec{D}) = \frac{1}{2} tr(A^{\dagger}(cB + D)) = \frac{1}{2}[c\, tr(A^{\dagger}B) + tr(A^{\dagger}D)]$ since trace is linear
$= c(\vec{A} \cdot \vec{B}) + (\vec{A} \cdot \vec{D})$

$(c\vec{A} + \vec{B}) \cdot \vec{D} = \frac{1}{2} tr((c^{*} A^{\dagger} + B^{\dagger}) D) = \frac{1}{2}[c^{*} tr(A^{\dagger} D) + tr(B^{\dagger} D)]$
$= c^{*}(\vec{A} \cdot \vec{D}) + (\vec{B} \cdot \vec{D})$ ✓

• $\vec{A} \cdot \vec{A} = \frac{1}{2} tr(A^{\dagger} A) = \frac{1}{2} A^{\dagger}_{ij} A_{ji} = \frac{1}{2} A^{*}_{ji} A_{ji} = \sum_{j,i} \frac{1}{2} |A_{ji}|^2 \geq 0$ ✓

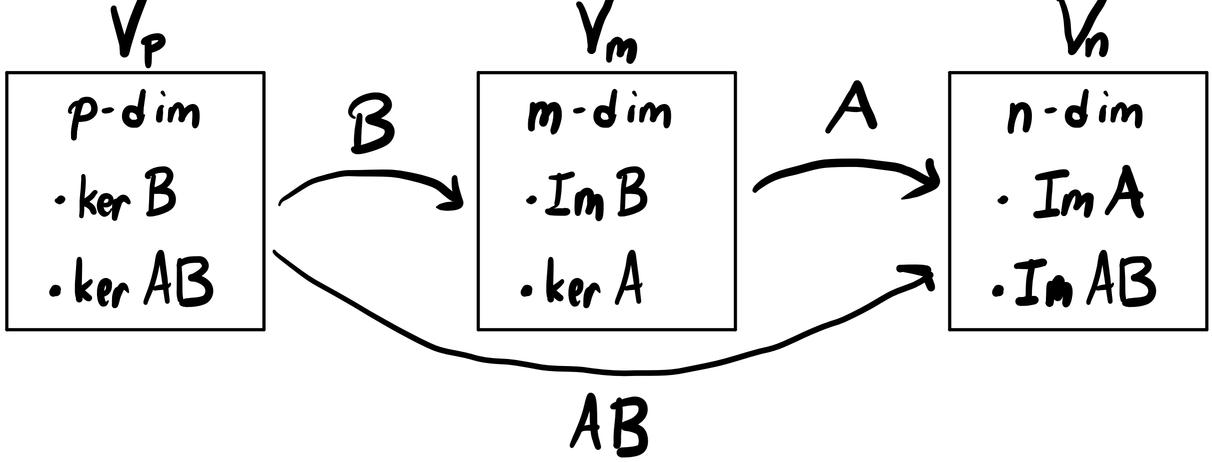
$$\vec{A} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \Rightarrow \sqrt{\vec{A} \cdot \vec{A}} = \frac{1}{\sqrt{2}} \sqrt{4 + 4} = 2 \quad \checkmark$$

$$\vec{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{A} \cdot \vec{B} = \frac{1}{2} \operatorname{tr} \left[\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \frac{1}{2} \operatorname{tr} \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix} = 0 \quad \checkmark$$

Problem 2 - An 89-Proof Problem

a) A an $m\times n$-matrix $\quad$ B an $n\times p$-matrix

$\Rightarrow AB$ an $(m\times n)(n\times p)$-matrix = an $(m\times p)$-matrix



• Let $\vec{v} \in \operatorname{Im} AB \Rightarrow \exists \vec{w} \in V_p$ s.t. $\vec{v} = (AB)\vec{w}$

$\Rightarrow \vec{v} = A(B\vec{w}) \Rightarrow \vec{v} \in \operatorname{Im} A \quad \boxed{\operatorname{Im} AB \subseteq \operatorname{Im} A}$

(note: Im AB is the vectors mapped by A from Im B

There may be vector in V_m not in Im B which won't necessarily be mapped to Im AB)

• Let $\vec{w} \in \ker B \Rightarrow B\vec{w} = \vec{0}$

$\Rightarrow (AB)\vec{w} = A(B\vec{w}) = A\vec{0} = \vec{0}$

$\Rightarrow \vec{w} \in \ker AB \quad \boxed{\ker B \subseteq \ker AB}$

- $\ker B \subseteq \ker AB \Rightarrow \text{null } B \leq \text{null } AB$

B & AB have same # of columns, p

$\Rightarrow$ by rank-nullity thm:
$$p = \text{rank } B + \text{null } B$$
$$p = \text{rank } AB + \text{null } AB$$

$$\text{rank } B = p - \text{null } B \geq p - \text{null } AB = \text{rank } AB$$

$$\Rightarrow \boxed{\text{rank } AB \leq \text{rank } B}$$

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b.1) $\{\hat{f}_i\}$ an orthonormal set $\Rightarrow \hat{f}_i \cdot \hat{f}_j = \hat{f}_i^\dagger \hat{f}_j = \delta_{ij}$

$$U \equiv \begin{pmatrix} \hat{f}_1 & \hat{f}_2 & \cdots & \hat{f}_n \end{pmatrix} \Rightarrow U^\dagger = \begin{pmatrix} \hat{f}_1^\dagger \\ \hat{f}_2^\dagger \\ \vdots \\ \hat{f}_n^\dagger \end{pmatrix}$$

$$U^\dagger U = \begin{pmatrix} \hat{f}_1^\dagger \\ \hat{f}_2^\dagger \\ \vdots \\ \hat{f}_n^\dagger \end{pmatrix} \begin{pmatrix} \hat{f}_1 & \hat{f}_2 & \cdots & \hat{f}_n \end{pmatrix} = \begin{pmatrix} \hat{f}_1^\dagger \hat{f}_1 & \hat{f}_1^\dagger \hat{f}_2 & \cdots & \hat{f}_1^\dagger \hat{f}_n \\ \hat{f}_2^\dagger \hat{f}_1 & \hat{f}_2^\dagger \hat{f}_2 & \cdots & \hat{f}_2^\dagger \hat{f}_n \\ \vdots & \vdots & \ddots & \vdots \\ \hat{f}_n^\dagger \hat{f}_1 & \hat{f}_n^\dagger \hat{f}_2 & \cdots & \hat{f}_n^\dagger \hat{f}_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbb{1} \checkmark \Rightarrow \boxed{U \text{ is unitary}}$$

Alt: $(\mathcal{U}^\dagger \mathcal{U})^i{}_j = (\mathcal{U}^\dagger)^i{}_k \, \mathcal{U}^k{}_j = (\hat{f}_i^\dagger)_k (\hat{f}_j)^k = \hat{f}_i^\dagger \hat{f}_j = \delta_{ij}$

$\Rightarrow \mathcal{U}^\dagger \mathcal{U} = \mathbb{1}$ ✓

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b.2) If $[A,B]=0$ then $e^A e^B = e^{A+B}$ (given)

$$(e^A)^\dagger = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right)^\dagger = \sum_{n=0}^{\infty} \frac{(A^n)^\dagger}{n!} = \sum_{n=0}^{\infty} \frac{(A^\dagger)^n}{n!} = e^{(A^\dagger)} \checkmark$$

$$(A^n)^\dagger = (AA\cdots A)^\dagger = (A^\dagger)\cdots(A^\dagger)(A^\dagger) = (A^\dagger)^n$$

Let H be Hermitian $\Rightarrow H^\dagger = H$

Let $\mathcal{U} = e^{iH}$

$$\Rightarrow \mathcal{U}^\dagger = (e^{iH})^\dagger = e^{(iH)^\dagger} = e^{-iH^\dagger} = e^{-iH}$$

$$\mathcal{U}^\dagger \mathcal{U} = e^{-iH} e^{iH} = e^{-iH+iH} = e^{0} = \mathbb{1} \checkmark$$

$[-iH, iH] = H^2 - H^2 = 0$ ✓ $\qquad$ 0- matrix

$\Rightarrow$ $\boxed{\text{if } H \text{ is Hermitian, } \mathcal{U} = e^{iH} \text{ is unitary.}}$

Problem 3 - Two Quantum System

a) Set of all fns a vector space

Subset: $\{f(x) \mid f(x)=0 \text{ if } x\le 0 \text{ or } x\ge L\}$

Let $f(x), g(x)$ be in this subset. $h(x) \equiv a f(x) + b g(x)$

$$h(x) = a\cdot 0 + b\cdot 0 = 0 \quad \text{if } x\le 0 \text{ or } x \ge L$$

$\Rightarrow$ $h(x)$ also in subset

$\Rightarrow$ set closed under vector addition & scalar multiplication

$\Rightarrow$ set is a <u>subspace</u>. ✓

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b) Basis: $\vec{\psi}_n \doteq \psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

orthonormal basis w.r.t inner product

$$\vec{\psi}_\gamma = \sum_{n=1}^{\infty} c_n \vec{\psi}_n$$

$$c_n = \vec{\psi}_n \cdot \vec{\psi}_\gamma \qquad\qquad \vec{\psi}_\alpha \cdot \vec{\psi}_\beta = \int_0^L \psi_\alpha^*(x)\,\psi_\beta(x)\,dx$$

$$\Rightarrow c_n = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\underbrace{\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right)}_{\psi_\gamma}dx$$

$$u = \frac{\pi x}{L} \qquad dx = \frac{L}{\pi}du$$

$$= \frac{2}{L}\cdot\frac{L}{\pi}\int_0^{\pi}\sin(nu)\cos u\,du$$

$$\frac{2}{\pi}\,\frac{n(\cos\pi n + 1)}{n^2-1} = \begin{cases} 0 & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)} & n \text{ even}\end{cases}$$

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c) $\vec{\Psi} \doteq \Psi(x)$

$P\vec{\Psi} \doteq -\Psi(L-x)$

note: well-defined on our subspace since

$x \leq 0 \Rightarrow L-x \geq L$

$x \geq L \Rightarrow L-x \leq 0$

$\Rightarrow P\vec{\Psi}$ will be 0 when $x \leq 0$ or $x \geq L$ ✓

Linear? Check $P(a\vec{\Psi}_1 + b\vec{\Psi}_2)$

$a\vec{\Psi}_1 + b\vec{\Psi}_2 \doteq a\Psi_1(x) + b\Psi_2(x)$

$\Rightarrow P(a\vec{\Psi}_1 + b\vec{\Psi}_2) \doteq -\left(a\Psi_1(L-x) + b\Psi_2(L-x)\right)$

$= a\left[-\Psi_1(L-x)\right] + b\left[-\Psi_2(L-x)\right]$

$\doteq a(P\vec{\Psi}_1) + b(P\vec{\Psi}_2)$ ✓

$M\vec{\Psi} \doteq |\Psi(x)|^2$

Check: $M(2\vec{\Psi}) \doteq |2\Psi(x)|^2 = 4|\Psi(x)|^2 \doteq 4M(\vec{\Psi})$

Since $M(2\vec{\Psi}) \neq 2M(\vec{\Psi})$, M is <u>not</u> linear. ◆

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d) Look at fns $\{x^2, e^{\pi i x/L}, \sin(\frac{2\pi x}{L})\}$

$\psi_1 \quad \psi_2 \quad \psi_3$

Check independence via Wronskian:

$$W(x) = \begin{vmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1' & \psi_2' & \psi_3' \\ \psi_1'' & \psi_2'' & \psi_3'' \end{vmatrix} = \begin{vmatrix} x^2 & e^{\pi i x/L} & \sin\frac{2\pi x}{L} \\ 2x & \left(\frac{i\pi}{L}\right)e^{\pi i x/L} & \frac{2\pi}{L}\cos\frac{2\pi x}{L} \\ 2 & -\frac{\pi^2}{L^2}e^{\pi i x/L} & -\frac{4\pi^2}{L^2}\sin\frac{2\pi x}{L} \end{vmatrix}$$

This doesn't obviously simplify cleanly but to prove independence we just need to show $W(x) \neq 0$ for at least one value of x.

Let's try an easy one - $x = 0$:

$$W(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & i\pi/L & 2\pi/L \\ 2 & -\pi^2/L^2 & 0 \end{vmatrix}$$

$$= \cancel{0 \cdot \begin{vmatrix} i\pi/L & 2\pi/L \\ -\frac{\pi^2}{L^2} & 0 \end{vmatrix}} - 1 \cdot \begin{vmatrix} 0 & 2\pi/L \\ 2 & 0 \end{vmatrix} + \cancel{0 \cdot \begin{vmatrix} 0 & i\pi/L \\ 2 & -\pi^2/L^2 \end{vmatrix}}$$

$$= -1 \cdot \left(0 - \frac{4\pi}{L}\right) = \frac{4\pi}{L} \neq 0$$

$\therefore$ our fns are linearly independent.

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e) Inner Product: $\vec{\psi}_\alpha \cdot \vec{\psi}_\beta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f^*(x)\, g(x)\, e^{-x^2}\, dx$

$\{\vec{f}_0 \doteq 1,\ \vec{f}_1 \doteq x,\ \vec{f}_2 \doteq x^2\}$

- $\hat{e}_0 = \dfrac{\vec{f}_0}{|\vec{f}_0|}$ $\qquad |\vec{f}_0|^2 = \vec{f}_0 \cdot \vec{f}_0 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 1 \cdot 1 \cdot e^{-x^2} dx = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1.$

$$\Rightarrow \boxed{\hat{e} = \vec{f}_0 \doteq 1}$$

- $\hat{e}_1 = \dfrac{\vec{f}_1 - \cancel{(\hat{e}_0 \cdot \vec{f}_1)\hat{e}_0}}{\sqrt{|\vec{f}_1|^2 - \cancel{|\hat{e}_0 \cdot \vec{f}_1|^2}}}$

$$\hat{e}_0 \cdot \vec{f}_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 1 \cdot x \cdot e^{-x^2} dx = 0$$

$$|\vec{f}_1|^2 = \vec{f}_1 \cdot \vec{f}_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x \cdot x \, e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \frac{1}{2}$$

$$\Rightarrow \boxed{\hat{e}_1 = \sqrt{2}\,\vec{f}_1 \doteq \sqrt{2}\,x}$$

- $\hat{e}_2 = \dfrac{\vec{f}_2 - (\hat{e}_0 \cdot \vec{f}_2)\hat{e}_0 - \cancel{(\hat{e}_1 \cdot \vec{f}_2)}\hat{e}_1}{\sqrt{|\vec{f}_2|^2 - |\hat{e}_0 \cdot \vec{f}_2|^2 - \cancel{|\hat{e}_1 \cdot \vec{f}_2|^2}}}$

$$\hat{e}_0 \cdot \vec{f}_2 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 1 \cdot x^2 e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \frac{1}{2}$$

$$\hat{e}_1 \cdot \vec{f}_2 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x \cdot x^2 \cdot e^{-x^2} dx = 0$$

$$|\vec{f}_2|^2 = \vec{f}_2 \cdot \vec{f}_2 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 \cdot x^2 \cdot e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \cdot \frac{3\sqrt{\pi}}{4} = \frac{3}{4}$$

$$\Rightarrow \hat{e}_2 = \frac{\vec{f}_2 - \frac{1}{2}\hat{e}_0}{\sqrt{\frac{3}{4} - \left(\frac{1}{2}\right)^2}} = \sqrt{2}\left(\vec{f}_2 - \frac{1}{2}\hat{e}_0\right) \doteq \sqrt{2}\,x^2 - \frac{1}{\sqrt{2}} = \sqrt{2}\left(x^2 - \frac{1}{2}\right)$$

$$\boxed{\hat{e}_2 = \sqrt{2}\left(\vec{f}_2 - \frac{1}{2}\hat{e}_0\right) \doteq \frac{1}{\sqrt{2}}(2x^2 - 1)}$$

$\Rightarrow \left\{1, \sqrt{2}\,x, \frac{1}{\sqrt{2}}(2x^2-1), \dots\right\}$ 1st few Hermite polynomials.

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f) $$G(x,y) = e^{-y^2+2yx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n$$

$$G(x,y) = \sum_{n=0}^{\infty} \frac{\left(\frac{\partial^n G}{\partial y^n}\right)\Big|_{y=0}}{n!} y^n$$

$$H_n(x) = \left(\frac{\partial^n G}{\partial y^n}\right)\Big|_{y=0}$$

0^{th}: $G(x,0) = e^0 = 1$

1^{st}: $$\frac{\partial G}{\partial y} = (-2y+2x)e^{-y^2+2yx}$$

$$\left(\frac{\partial G}{\partial y}\right)\Big|_{y=0} = 2xe^0 = 2x$$

2^{nd}: $$\frac{\partial^2 G}{\partial y^2} = -2e^{-y^2+2yx} + (-2y+2x)^2 e^{-y^2+2yx}$$

$$\left(\frac{\partial^2 G}{\partial y^2}\right)\Big|_{y=0} = -2e^0 + (2x)^2 e^0 = 4x^2-2$$

$$G(x,y) = 1 + 2xy + (2x-1)y^2 + \cdots$$

$$\boxed{H_0(x) = 1 \qquad H_1(x) = 2x \qquad H_2(x) = 4x^2-2}$$