Problem 1

Consider a parallel capacitor where the plates have area A and are separated by a distance d. The gap is filled with a linear dielectric material of electric susceptibility χ_e. Suppose the capacitor is charged so that the plates carry free charges $\pm Q_f$.

a) Find the bound charge Q_b originated from the polarization in terms of Q_f and χ_e.

Solution: We can calculate the bound charge from the polarization using $\rho_b = -\nabla \cdot P$ and since $P = \epsilon_0 \chi_e E = \epsilon_0 \chi_e \frac{D}{\epsilon}$, then we get:

$$\rho_b = -\epsilon\chi_e \left(\nabla \cdot \frac{D}{\epsilon}\right) = -\frac{\epsilon_0 \chi_e}{\epsilon_0(1+\chi_e)}(\nabla \cdot D)$$

Then, since D is related to Q_f via $\nabla \cdot D = \frac{Q_f}{A}$, we get:

$$Q_b = -\frac{\chi_e}{1+\chi_e} Q_f$$

□

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b) Find the *net* electric field (caused by both the free and bound charges) between the plates. Express your answer in terms of Q_f, A and ϵ, where $\epsilon = (1+\chi_e)\epsilon_0$

Solution: First, we calculate the D field:

$$\oint D \cdot da = Q_f = DA \implies D = \frac{Q_f}{A}$$

It's obvious that D points in the x direction, so therefore $\vec{D} = \frac{Q_f}{A}\hat{x}$. Furthermore, we know that $\vec{E} = \frac{\vec{D}}{\epsilon}$, so therefore:

$$E = \frac{Q_f}{\epsilon A}\hat{x}$$

□

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c) Compute the work needed to establish the system from zero free charge to a final free charge Q_f by gradually moving infinitesimal free charges dq_f from the right plate to the left plate, overcoming the electric potential of the net field. Express your answer in terms of Q_f, ϵ, A and d.

Solution: We derive the capacitance of a parallel plate capacitor as: $C = \frac{\epsilon A}{d}$, and we also know that $C = \frac{Q}{V}$. Since the total charge initially is only due to the bound charges q_b, then can write the potential as:

$$V = \frac{Q_b}{C} = \frac{Q_b d}{\epsilon A}$$

Furthermore, at any point, the free charges that are added on also contribute to the total charge on the capacitor, so in fact, we have a general expression for the potential as a function of the charge q on the capacitor:

$$V(q) = \frac{qd}{\epsilon A}$$

Then, we now just have to integrate over the total charge:

$$W = \int_0^{Q_f} V(q)dq = \int_0^{Q_f} \frac{qd}{\epsilon A}dq = \frac{Q_f^2 d}{2\epsilon A}$$

□

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d) Calculate $W = \frac{1}{2}\int D \cdot E d\tau$ for this system. It should agree with your result in part (c)

Solution: We have expressions for D and E, so we just have to do the integral:

$$\begin{aligned} W &= \frac{1}{2}\int D \cdot E d\tau \\ &= \frac{1}{2}\int \frac{Q_f^2}{\epsilon A^2} d\tau \\ &= \frac{1}{2}\frac{Q_f^2}{\epsilon A^2}(Ad) \\ &= \frac{Q_f^2 d}{2\epsilon A} \end{aligned}$$

□

The difference between $W = \frac{1}{2}D \cdot E d\tau$ and $W = \frac{\epsilon_0}{2}\int E \cdot E d\tau$ is that the latter only takes into account the energy needed to put the charges in place, while ignoring the potential energy associated with *microscopic* interactions between the molecules, or within the molecules, which is wt we neeed to overcome when we polarize the material. Specifically, consider the following two-step process. First, we separate the free charges $\pm Q_f$ from each other as shown in the right figure.

e) What is the energy needed to separate the $\pm Q_f$ charges at a distance d apart as shown above?

Solution: To separate just the free charges, we look at the total potential energy of the capacitor with charges Q_f. We know that $U = \frac{1}{2}CV^2$ with $C = \frac{Q_f}{V}$, so therefore:

$$\begin{aligned} U = \frac{1}{2}\frac{Q_F}{V}V^2 &= \frac{1}{2}Q_f V \\ &= \frac{1}{2}Q_f \frac{Q_f d}{\epsilon_0 A} \\ &= \frac{Q_f^2 d}{2\epsilon_0 A} \end{aligned}$$

□

We now separate the bound charges by pulling the $\pm Q_b$ apart from each other within the two charged plates. Along the way, we need to overcome the force between the bound charges, and the force exerted on $+Q_b$ due to $\pm Q_f$

f) What is the energy needed to separate the $\pm Q_b$ charges a distance d apart as shown?

Solution: The energy needed is equal to the difference in energy of the final and the initial configurations. In the initial system, the charges $\pm Q_b$ reside on the same plate, so the potential energy here is just equal to the potential energy of the configuration with Q_f:

$$U_i = \frac{Q_f^2 d}{2\epsilon_0 A}$$

After the charges are moved to their locations, the net charge on each plate is now $Q_f - Q_b$, so the final potential is:

$$U_f = \frac{(Q_f - Q_b)^2 d}{2\epsilon_0 A}$$

Therefore, the energy required is the difference of these two:

$$\Delta U = U_f - U_i = \frac{d}{2\epsilon A}((Q_f - Q_b)^2 - Q_f^2) = \frac{d}{2\epsilon_0 A}(Q_b^2 - 2Q_b Q_f)$$

□

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g) Compute the $W = \frac{\epsilon_0}{2} \int E \cdot E d\tau$ for the final configuration. It should equal to the sum of part (e) and (f).

Solution: Here, we know the E field is equal to $E = \frac{Q_f - Q_b}{\epsilon_0 A}$ since the total charge on the plates is $Q_f - Q_b$, so therefore:

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \int \frac{(Q_f - Q_b)^2}{\epsilon_0^2 A^2} d\tau \\ &= \frac{\epsilon_0}{2} \frac{(Q_f - Q_b)^2}{\epsilon_0^2 A^2} Ad \\ &= \frac{(Q_f - Q_b)^2 d}{2\epsilon A} \end{aligned}$$

Summing up parts (e) and (f), we get:

$$U_i + \Delta U = \frac{Q_f^2 d}{2\epsilon_0 A} + \frac{(Q_b^2 - 2Q_b Q_f)d}{2\epsilon_0 A} = \frac{(Q_f^2 - 2Q_f Q_b + Q_b^2)d}{2\epsilon_0 A} = \frac{(Q_f^2 - Q_b^2)}{2\epsilon_0 A}$$

This matches what we get from computing $W = \frac{\epsilon_0}{2} \int E \cdot E d\tau$, as desired. □

Problem 2

A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a permanent polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}}$$

where k is a constant and r is the distance from the center. (There is no free charge in the problem.) Find the electric field in all three regions, $r < a$, $a < r < b$, $r > b$.

Solution: Because there is no free charge in the problem, it's convenient here to use Gauss' law:

$$Q_f = \oint \mathbf{D} \cdot da = 0$$

Because this integral is zero then, this also implies that $\mathbf{D} = 0$ everywhere. Then, since $\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}$, we can rearrange this to get $\mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0}$ within the region of the spherical shell. Inside the shell, since there is no charge, we know that $\mathbf{E} = 0$. Outside the shell, we know that there is no polarization (since there isn't any polarized material), so $\mathbf{E} = 0$ outside the shell as well. Therefore, combining this together we get:

$$E(r) = \begin{cases} 0 & r < a \\ -\dfrac{k}{\epsilon_0 r}\hat{\mathbf{r}} & a < r < b \\ 0 & r > b \end{cases}$$

$\square$

Problem 3

An uncharged conducting sphere of radius a is coated with a thick insulating shell out to radius b. The insulating shell has a permanent polarization $\mathbf{P} = P_0\hat{z}$. Find the electric potential in the three regions, $r < a$. $a < r < b$, and $r > b$.

Solution: Since we are dealing with a spherical conductor (I assume that we have a solid sphere here), then we know that the potential within the conductor must be zero simply due to the properties of a conductor.

For the region $a < r < b$, we need to be more careful. Firstly, we can calculate the surface and bound charges:

$$\begin{aligned}
\rho_b = \nabla \cdot P &= \frac{P_0}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin^2\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sin\theta\right) \\
&= -\frac{2P_0\cos\theta}{r} + \frac{2\cos\theta P_0}{r} \\
&= 0 \\
\sigma_b = \vec{P}\cdot\hat{n} &= \vec{P}\cdot r \\
&= P_0\cos\theta\hat{n}
\end{aligned}$$

Further, if we consider the surface of the insulating region, we get that the normal vector $\hat{n}$ points inwards at $r = a$ and outwards at $r = b$, so at $r = a$, $\sigma_b = -P_0\cos\theta$ and at $r = b$, $\sigma_b = P_0\cos\theta$. Qualitatively speaking, this means that the potential within this region is the same potential as two concentric shells of bound charges with a distribution of $P_0\cos\theta$, with the outer shell consisting of positive charges and the inner one consisting of negative charges. To solve this potential, we solve each case separately using the general solution for $V(r)$, then use the superposition principle to find the total potential.

Let's start with the shell of radius b, which has charge density $\sigma_b = P_0\cos\theta$. Recall that the general form for the potential is:

$$V(r) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

where P_l denotes the l-th Legendre polynomial. We want to find the coefficients A_l and B_l. Inside the shell, we know that the potential cannot blow up to infinity, so none of the B_l terms can exist. Similarly, outside the shell, we require that the potential is 0 at infinity, so the A_l terms cannot exist outside. Therefore, the potential is of the form:

$$V(r) = \begin{cases} \displaystyle\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) & r > b \\ \\ \displaystyle\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) & r < b \end{cases}$$

Now, there are two conditions we need to satisfy with this potential. Firstly, we require that

$$\left.\frac{\partial V}{\partial n}\right|_{\text{below}}^{\text{above}} = \frac{\sigma}{\epsilon_0} = \frac{P_0\cos\theta}{\epsilon_0}$$

So, computing the derivative with respect to r for $V(r)$ at $r = b$ and equating them to each other, we get:

$$\begin{aligned}
\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} &= \sum_{l=0}^{\infty} -\frac{(l+1)B_l}{b^{l+2}}P_l(\cos\theta) - \sum_{l'=0}^{\infty} l' A_{l'} b^{l'-1} P_{l'}(\cos\theta) \\
\frac{P_0\cos\theta}{\epsilon_0} &= \sum_{l=0}^{\infty} -\frac{(l+1)B_l}{b^{l+2}}P_l(\cos\theta) - \sum_{l'=0}^{\infty} l' A_{l'} b^{l'-1} P_{l'}(\cos\theta)
\end{aligned}$$

The left hand side only has a linear $\cos\theta$ term which corresponds to the first legendre polynomial $P_1(x) = x$, so therefore the right hand simplifies massively, since $l = l' = 1$ is the only term that is nonzero. Therefore, we're left with the equation:

$$\begin{aligned}\frac{P_0 \cos\theta}{\epsilon_0} &= -\frac{2B_1}{b^3}\cos\theta - A_1 \cos\theta \\ \frac{P_0}{\epsilon_0} &= -\frac{2B_1}{b^3} - A_1\end{aligned}$$

To solve for B_1 and A_1, we now use the fact that $V(r)$ must be differentiable at $r = b$, so at $r = b$, we require that $V_{out}(b) = V_{in}(b)$:

$$\begin{aligned}\frac{B_1}{b^2}\cos\theta &= A_1 b \cos\theta \\ \therefore A_1 &= \frac{B_1}{b^3}\end{aligned}$$

Substituting this back into the previous expression, we have:

$$\begin{aligned}\frac{P_0}{\epsilon_0} &= -\frac{2B_1}{b^3} - \frac{B_1}{b^3} \\ &= -\frac{3B_1}{b^3} \\ \therefore B_1 &= -\frac{P_0 b^3}{3\epsilon_0}, \quad A_1 = -\frac{P_0}{3\epsilon 0}\end{aligned}$$

This approach is identical for the shell of radius $r = a$ except we have the opposite charge density, so from there we get:

$$\begin{aligned}A_1 &= \frac{P_0}{3\epsilon_0} \\ B_1 &= \frac{P_0 a^3}{3\epsilon_0}\end{aligned}$$

Now we can begin the process of joining them together. To do so, notice that for $a < r < b$, we want the solution outside the shell of radius a, but inside the shell for radius b, Therefore,

$$\begin{aligned}V(r) &= V_{\text{out, a}}(r) + V_{\text{in, b}}(r) \\ &= \frac{P_0}{3\epsilon_0} r\cos\theta + \frac{1}{r^2}\left(-\frac{P_0 b^3}{3\epsilon_0}\right)\cos\theta \\ &= \frac{P_0\cos\theta}{3\epsilon_0}\left(r - \frac{b^3}{r^2}\right) \\ &= \frac{P_0 r\cos\theta}{3\epsilon_0}\left(1 - \frac{b^3}{r^3}\right)\end{aligned}$$

For the region $r > b$, we want the solution outside both shells:

$$V(r) = \frac{P_0 a^3}{3\epsilon 0 r^2}\cos\theta - \frac{P_0 b^3}{3\epsilon_0 r^2}\cos\theta = \frac{P_0\cos\theta}{3\epsilon_0 r^2}(a^3 - b^3)$$

Interestingly enough, since we use the solution inside both shells for $r < a$, we get that in this region:

$$V(r) = -\frac{P_0}{3\epsilon_0} r\cos\theta - \left(-\frac{P_0}{3\epsilon_0}\right) r\cos\theta = 0$$

which matches the result we got earlier. However, we shouldn't use this approach here since at $r < a$ we have a conductor rather than empty space, which is what solving for the potential at $r < a$ in this fashion describes. Finally, we can write:

$$V(r) = \begin{cases} 0 & r < a \\ \dfrac{P_0 r\cos\theta}{3\epsilon_0}\left(1 - \dfrac{b^3}{r^3}\right) & a < r < b \\ \dfrac{P_0\cos\theta}{3\epsilon_0 r^2}(a^3 - b^3) & r > b \end{cases}$$

□
