PHYSICS 105 NOTES

Typeset notes for Physics 105: Analytic Mechanics

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Introduction

- Instructor: Prof. Dan McKinsey
- Office Hours: Tuesdays, 2:30pm in 441 Physics South
- Grading:
  - 30% Homework
  - 20% Midterm 1
  - 20% Midterm 2
  - 30% Final

  The grade bins for this class are fixed, and they will not be altered throughout the semester.

Lecture 1 (01/17)

This lecture was held on **January 17, 2023**. It covered equations for simple harmonic motion in one and two dimensions.

1.1 Why Do We Study Oscillations?

We study oscillations because they are very common in physics − they happen any time we have a system with a stable equilibrium point. When we nudge our system away from this point, a restoring force $F_x(x)$ tries to bring our system back to equilibrium.

Although $F_x(x)$ could potentially be functions of more variables, we can let it be x for now. We will also assume that $F_x(x)$ has continuous derivtaives everywhere so that we can expand it as a Taylor series. Thus:

$$F_x(x) = F_0 + x\left(\frac{dF_x}{dx}\right)_0 + \frac{1}{2}x^2\left(\frac{d^2F_x}{dx^2}\right)_0 + \dots$$

Then, since the origin is also an equilibrium point, then F_0 must equal 0 at the equilibrium point. Then, neglecting higher order terms, we get the approximate relation that:

$$F_x(x) = -kx$$

where $k \equiv -\left(\frac{dF_x}{dx}\right)$. Since the restoring force always points toward the equilibrium positio, then dF/dx is alwaysnegative, so k is a positive constant.

Alternatively, we can also write the force in terms of the potential:

$$U(x) = \frac{1}{2}kx^2$$

where $U(x) = U(0) + U'(0) + \frac{1}{2}U''(0)x^2 + \dots$, with $U'(0) = 0$ using the same logic as before. These oscillations can be damped or driven, which we will revisit later.

1.2 Simple Harmonic Oscillator

Here we will look at different ways to represent the oscillatory equations of motion for simple harmonic oscillators. To start, let's use Newton's second law to get the differential equation:

$$-kx = m\ddot{x}$$

Then, we can define $\omega^2 \equiv \frac{k}{m}$, then we get the equation

$$\ddot{x} + \omega^2 x = 0$$

which is the standard differential equation for simple harmonic motion. This differential equation has the solutions:

$$\begin{aligned} x(t) &= A\sin(\omega t - \varphi) \\ x(t) &= A\cos(\omega t - \delta) \end{aligned}$$

where $|\delta - \phi| = \pi/2$. The kinetic energy can also be calculated:

$$\begin{aligned} T = \frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m\left[A\omega\cos(\omega t - \phi)\right]^2 \\ &= \frac{1}{2}mA^2\frac{k}{m}\cos^2(\omega t - \varphi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t - \varphi) \end{aligned}$$

Since $U(x) = \frac{1}{2}kx^2$, then we get:

$$U(x) = \frac{1}{2}kA^2\sin^2(\omega t - \varphi)$$

Adding the two, we get:

$$T + U = \frac{1}{2}kA^2(\sin^2(\omega t - \varphi) + \cos^2(\omega t - \varphi)) = \frac{1}{2}kA^2$$

which is a constant for all θ. We expect this result, since we know that the total energy of an isolated oscillatory system doesn't change. The period τ can also be expressed as:

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

And we also have the relation that $\nu = 2\pi\omega$.

1.2.1 Method 2: Summation of sines and cosines

Looking at the second solution more closely, we note that

$$\begin{aligned} A\cos(\omega t - \delta) &= A\left[\cos(\delta)\cos(\omega t) + \sin(\delta)\sin(\omega t)\right] \\ &= A\left[\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right] \\ &= B_1\cos(\omega t) + B_2\sin(\omega t) \end{aligned}$$

with $A = \sqrt{B_1^2 + B_2^2}$. This form of the solution is particularly nice since it allows us to deal with initial conditions very easily. For instance, if the oscillation started at the peak, then we know that $B_2 = 0$ and so we're left with a pure sine wave which is really easy to deal with.

1.2.2 Method 3: Exponentials

We can also formulate the solution as

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

This form is useful since integrals and derivatives are especially easy. You can also check that this solution satisfies the differential equation by plugging in $x(t)$ into our differential equation. To show that it's consistent with our previous form, we use Euler's identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$:

$$\begin{aligned} x(t) &= C_1 e^{i\omega t} + C_2 e^{-i\omega t} \\ &= (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\sin(\omega t) \\ &= B_1 \cos(\omega t) + B_2 \sin(\omega t) \end{aligned}$$

And so naturally in this form we assume $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$.

1.2.3 Real Part of Expoonentials

Since $x(t)$ is a real quantity, then we can actually make a couple simplifications to our solution in the previous section. Firstly, we note that since $x(t)$ is real, then B_1, B_2 must also be real. Therefore, this enforces $C_1 = C_2^*$, so we now have

$$x(t) = C_1 e^{i\omega t} + C_1^* e^{-i\omega t}$$

And since we know that $z + z^* = 2\Re(z)$, then letting $z = C_1 e^{i\omega t}$, we get:

$$x(t) = 2\Re\left(C_1 e^{i\omega t}\right)$$

Then as one final simplification, if we let $C = 2C_1$, then $C = B_1 - iB_2 = Ae^{i\delta}$ so we can write:

$$x(t) = \Re(Ce^{i\omega t}) = A\cos(\omega t - \delta)$$

As an illustration of this expresion, we can imagine moving around a unit circle:

[INSERT TIKZ HERE]

1.2.4 Summary

To summarize, we have the following equivalent ways of writing solutions to these oscillations:

$$\begin{aligned} x(t) &= A\cos(\omega t - \delta) \\ &= B_1\cos(\omega t) + B_2\sin(\omega t) \\ &= C_1 e^{i\omega t} + C_2 e^{i\omega t} \\ &= C_1 e^{i\omega t} + C_1^* e^{i\omega t} \\ &= \Re(Ce^{i\omega t}) \\ &= \Re(Ae^{i\omega t - \delta}) \end{aligned}$$

These solutions are all equivalent, each having its own benefits when it comes to solving problems. It's our job to figure out which form is the most convenient for our problem at hand.

1.3 Oscillations in 2 Dimensions

How do our equations for oscillations generalize in 2 dimensions? Well, now our restoring force is slightly more complicated to account for a new dimension:

$$\vec{F} = -k\vec{r}$$

And we can split this up into component form:

$$\begin{aligned} F_x &= -kr\cos\theta = -kx \\ F_y &= -kr\sin\theta = -ky \end{aligned}$$

This then generates the same set of differential equations as before, and they are independent so we can solve them separately. Therefore, we generate the equations:

$$\begin{aligned} x(t) &= A_x\cos(\omega t - \delta_x) \\ y(t) &= A_y\cos(\omega t - \delta_y) \end{aligned}$$

Here, we can "zero out" one of these phases (by simply starting at one of the phases), so this changes our equations to:

$$\begin{aligned} x(t) &= A_x\cos(\omega t) \\ y(t) &= A_y\cos(\omega t - \delta) \end{aligned}$$

where δ now refers to some *relative phase* between the two oscillations. Now we ask ourselves, what is the path of this particle? To do this we eliminate t. First, we can expand out $y(t)$ without introducing the relative phase:

$$\begin{aligned} y(t) &= A_y\cos(\omega t - \delta_x + (\delta_x - \delta_y)) \\ &= A_y\cos(\omega t - \delta_x)\cos(\delta_x - \delta_y) - A_y\sin(\omega t - \delta x)\sin(\delta_x - \delta_y) \\ &= A_y\cos(\omega t - \delta_x)\cos(\delta_x - \delta_y) + A_y\sin(\omega t - \delta x)\sin(\delta_y - \delta_x) \end{aligned}$$

In the last line we've used the identity that $\sin(-x) = -\sin(x)$. Now, we can use the relative phase $\delta \equiv \delta_y - \delta_x$ and $\cos(\omega t - \delta_x) = \frac{x}{A_x}$ to get:

$$y = \frac{A_y}{A_x} x \cos\delta + A_y \sqrt{1 - \left(\frac{x^2}{A_x^2}\right)}$$

We can alternatively write this as

$$A_x y - A_y x \cos\delta = A_y \sin\delta \sqrt{A_x^2 - x^2}$$

So squaring this, and simplifying, we get:

$$A_y^2 x^2 - 2A_x A_y xy \cos(\delta) + A_x^2 y^2 = A_x^2 A_y^2 \sin^2\delta$$

Then, if $\delta = \pm\frac{\pi}{2}$, then we get an ellipse:

$$\frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1$$

If $\delta = 0$ (i.e. no phase), then we get $(A_y x - A_x y)^2 = 0 \implies y = \frac{A_y}{A_x} x$, which is a straight line! Visually, it looks like this:

[INSERT TIKZ HERE]

1.3.1 Lissajous Curves

Note that in the previous derivation we used the same k in both the x and y directions. However, in the most general case this isn't actually required! Therefore, the general set of equations are:

$$\begin{aligned} x(t) &= A_x \cos(\omega_x t) \\ y(t) &= A_y \cos(\omega_y t - \delta) \end{aligned}$$

If $\frac{\omega_x}{\omega_y}$ is rational, then the motion is periodic, called a Lissajous figure (or a Lissajous curve). If it is irrational, then the curve will eventually fill out a rectangle over time.

Lecture 2 (01/19)

This lecture was held on **January 19th, 2023**. It covered the equations of motion for damped and driven oscillators, as well as their applications in modern circuits.

2.1 Last time: The Free Oscillator

On Tuesday we explored oscillatory mechanics where there were no other forces besides the restoring force. However, in most systems, we will always have some kind of *daming force* which impedes motion. This doesn't always have to be the case, but we will first explore a damping force which is proportional to the velocity:

$$\vec{f} = b\vec{v}$$

Under this, we now have the restoring force and the damping force, so Newton's second law now reads:

$$m\ddot{x} + b\dot{x} + kx = 0$$

And so if we let $\beta = \frac{b}{2m}$ (we'll see later why this substitution is useful), then we can write

$$\ddot{x} + 2\beta x + \omega_0^2 x = 0$$

The nature of these differential equations is that due to their linearity, if we find two independent solutions $x_1(t)$ and $x_2(t)$, then in general their solution will be a linear combination of the two:

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

We saw that exponentials worked before, let's have that as our main guess. Let

$$x(t) = e^{rt} \implies \dot{x}(t) = re^{rt}, \ddot{x}(t) = r^2 e^{rt}$$

Plugging this in, we get:

$$\begin{aligned} r^2 e^{rt} + 2\beta r e^{rt} + \omega_0^2 e^{rt} &= 0 \\ \therefore r^2 + 2\beta r + \omega_0^2 &= 0 \end{aligned}$$

This is quadratic in r, so therefore we have solutions $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$. Now, we can then write

$$\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned}$$

so the general solution now becomes:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}t}\right)$$

This equation makes sense intuitively, since a large value of β generates a faster decay, which makes sense since β refers to the damping constant.

Now we have 3 cases that we want to analyze:

(a) Underdamped: $\omega_0^2 > \beta^2$

(b) Critical damping: $\omega_0^2 = \beta^2$

(c) Overdamped: $\omega_0^2 < \beta^2$

As it will turn out, only the overdamping will give us oscillatory motion.

2.1.1 Case 1: Underdamped Oscillation

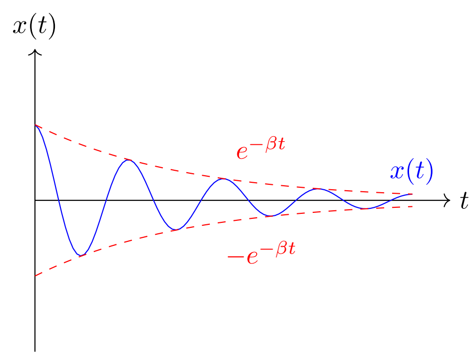
Here we look at the case where $\omega_0^2 > \beta^2$. If this is the case, then we can write $\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\omega_1$, with $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

Insight: if $\beta = 0$, then we exactly recover the solution that we got last time:

$$x(t) = C_1 e^{\sqrt{-\omega_0^2}t} + C_2 e^{-\sqrt{-\omega_0^2}t} = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

This is a good way to check that what we're doing still makes sense.

There's also the case where we get *weak underdamping*, where essentially we have $\beta \ll \omega_0$, so we get instead $x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$. Here, the $e^{-\beta t}$ term goes to zero over time, and the second term is just oscillation with $\omega_0 \to \omega_1$, so we can interpret this as an oscillation with an envelope of a decaying exponential. A diagram representation:



Here, the red dashed lines represent the envelope, and the $x(t)$ represents the actual motion. As we can see, the oscillation goes to zero over time, which makes sense since the damping force is constantly taking away energy.

2.1.2 Case 2: Damped Oscillation

Now we look at $\omega_0^2 < \beta^2$. Here, the exponentials are real, so we can write:

$$x(t) = C_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}$$

On large time scales, we can see that the oscillation is dominated by the slower of the two exponentials (the faster one just dies out quicker, so we don't see it for large times). Interestingly, the rate of decay is lower if β is larger, and we can see that since the exponentials get larger.

When solving these problems, there are three cases we need to specifically look at if we let $x(0) = x_0 > 0$:

- $\dot{x_0} > 0$ such that we reach the maximum then the pendulum comes back.
- $\dot{x_0} < 0$ but we approach $x = 0$ but we do not go past.
- Same as the previous case, but we do end up going past $x = 0$.

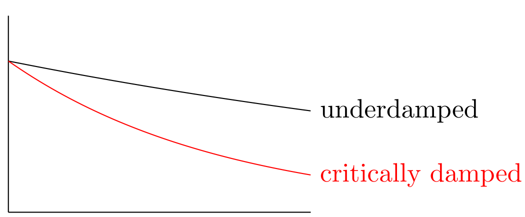
2.1.3 Case 3: Critical Damping

Now we look at $\omega_0^2 = \beta^2$, the most interesting case. Here, this means that the roots of $r^2 + 2\beta r + \omega_0^2 = 0$ are equal, so our solution becomes

$$x(t) = e^{\beta t}$$

Well we know that this differential equation has two solutions, and there is indeed a second solution, $x(t) = xe^{\beta t}$. We can check that it satisfies the differential equation by plugging it in (you can do this on your own time)

A critically damped oscillator will approach equilibrium the fastest. This has important design consequences − in times where we want to reduce oscillation, critically damped systems are especially useful. Below is an illustration of this:



Both of these curves are exponential decay curves, but the critically damped one goes to zero much faster than any other damping.

2.2 Driving Forces

Now imagine instead of a resistive force, that we have a driven force instead. That is, our equation of motion now looks like this:

$$F = -kx - bx + F_0 \cos(\omega t)$$

where $F_0 \cos(\omega t)$ denotes the driving force, with ω being the angular frequency of the driving force. Therefore, we have the differential equation:

$$\ddot{x} + 2\beta x + \omega_0^2 x = f_0 \cos(\omega t)$$

where $f_0 = \frac{F_0}{m}$, and β is defin in the same way as before. This differential equation has two solutions:

- A homogeneous solution $x_h(t)$, which solves the differential equation when the right hand side is 0.
- A particular soluiton, $x_p(t)$, which solves the differential equation while replicating the right hand side.

Then, if we can find these two solutions, then we can combine them together:

$$x(t) = x_h(t) + x_p(t)$$

We already know $x_h(t)$ from the earlier section, so if we can find an $x_p(t)$, then we have the full solution to the problem. We've seen before that sines and cosines seem to work well with oscillations, so why not try the solution $x_p(t) = A\cos(\omega t - \delta)$. Plugging this in, we get:

$$\begin{aligned} -A\omega^2 \cos(\omega t - \delta) - 2\beta A\omega \sin(\omega t - \delta) + \omega_0^2 A \cos(\omega t - \delta) &= f_0 \cos \omega t \\ f_0 \cos(\omega t) - A(\omega_0^2 - \omega^2) \cos(\omega t - \delta) + 2\beta A \cos(\omega t - \delta) &= 0 \end{aligned}$$

Now we use the trigonometric identities:

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha \cos\beta + \sin\alpha \sin\beta \\ \sin(\alpha - \beta) &= \sin\alpha \cos\beta - \cos\alpha \sin\beta \end{aligned}$$

Doing so we get the equation:

$$\left\{ f_0 - A\left[(\omega_0^2 - \omega^2)\cos\delta + 2\omega\beta \sin\delta \right] \right\} \cos(\omega t) - \left\{ A\left[(\omega_0^2 - \omega^2)\sin\delta - 2\omega\beta\cos(\omega t) \right] \right\} \sin(\omega t) = 0$$

Since $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent functions, this euqation is only satisfied when both terms are zero. From the first term, we get:

$$\begin{aligned} \tan\delta &= \frac{2\omega\beta}{\omega_0^2 - \omega^2} \\ \therefore \delta &= \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2} \right) \end{aligned}$$

Therefore,

$$\sin\delta = \frac{2\omega\beta}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} \qquad \cos\delta = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}$$

From the cosine term, we get:

$$\begin{aligned} A &= \frac{f_0}{(\omega_0^2 - \omega^2)\cos\delta + 2\omega\beta\sin\delta} \\ &= \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2) + 4\omega^2\beta^2}} \end{aligned}$$

so now we finally have the particular solution:

$$x_p(t) = A\cos(\omega t - \delta) = \frac{f_0\cos(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2) + 4\omega^2\beta^2}}$$

So our general solution, (again $x(t) = x_h(t) + x_p(t)$) is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + A\cos(\omega t - \delta)$$

Note that for large time, the $x_h(t)$ terms die out, but the $x_p(t)$ terms don't, and dominate for large time $t \gg \frac{1}{\beta}$. In other words:

$$x(t \gg \frac{1}{\beta}) = x_p(t)$$

2.2.1 The phase difference

Earlier we said that δ represents the phase differene between the action and the resulting motion:

$$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

For a fixed ω_0, as ω increases from 0, we get that the phase increases from $\delta = 0$ to $\delta = \pi/2$ at $\omega = \omega_0$ and $\delta \to \pi$ as $\omega \to \infty$.

2.2.2 Amplitude

At large times, it's useful to look at the effect of ω on the amplitude by looking at $x_p(t)$. To find where A is maximized, we simply take $\frac{dA}{d\omega} = 0$, giving us $\omega_2 = \sqrt{\omega_0^2 - 2\beta^2}$, so the resonance frequency is lowered as damping β is increased. This makes sense intuitively, since the larger value of β means that our oscillation decays much faster.

> **Insight:** Note that we don't get any resonance frequencies for $\beta^2 > \omega_0^2/2$, since that's when the square root becomes complex-valued.

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2.3 Summary

In summary, so far we've looked at free, damped and driven oscillations:

- Free oscillations: $\omega_0 = \frac{k}{m}$
- Damped oscillations: $\omega_1 = \omega_0^2 - \beta^2$
- Driven oscillations: $\omega_2 = \omega_0 - 2\beta^2$

This set of relations gives us $\omega_0 > \omega_1 > \omega_2$. The maximum amplitude is given by

$$A_{max} = \frac{f_0}{\sqrt{4\omega^2\beta^2}} = \frac{f_0}{2\omega\beta}$$

This equation tells us that if we make β smaller, then A_{max} increases and also becomes taller and narrower. We define the *full width half maximum* (FWHM $= 2\beta$) to quanitfy this value.

2.3.1 Aside: Quality Factor

We sometimes define a quality factor Q as the ration of the resonance position ω_0 to its width 2β:

$$Q = \frac{\omega_0}{2\beta}$$

This means that large values of Q correspond to narrow resonance, and small values correspond to a wide resonance.

2.4 Application: LC Circuits

We've learned in electricity and magnetism (and also 5BL) that a simple LC circuit also behaves like an oscillator, since it follows the equation:

$$L\frac{dI}{dt} + \frac{1}{c}\int I\ dt = 0$$

or in terms of the charge q:

$$L\ddot{q} + \frac{1}{c}q = 0$$

The solutions are of the same form since the only thing we've changed here are the variable names: $q(t) = q_0\cos(\omega t)$. Overall, we can just perform the substtution of letters:

$$m \to L \quad x \to q \quad \frac{1}{k} \to c \quad \dot{x} \to I$$

Similarly, adding a resistance will give us the equivalent of a damping term, and adding an AC power generator is the same as adding a driving force to the system. Again, the nature of these system is the exact same as what we just solved for, so the equations of motion remain the same as well.

Lecture 9

Lecture 9 was held on [INSERT DATE HERE], and it covered [INSERT TOPIC HERE]

3.1 Orbital Motion Revisited

Let's look back at our expression for the path of a particle $\phi(r)$:

$$\phi(r) = \int \frac{\frac{\ell}{r^2}dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}$$

If we assume that we're talking about the gravitational force here, the we can write $U = -\frac{Gm_1m_2}{r} \equiv -\frac{\gamma}{r}$. And so we instead just have to solve the integral:

$$\phi(r) = \int \frac{\frac{\ell}{r^2}dr}{\sqrt{2\mu\left(E + \frac{\gamma}{r} - \frac{\ell^2}{2\mu r^2}\right)}}$$

To make this integral easier, we perform a u-substitution of $u = \frac{1}{r}$ and $du = -\frac{1}{r^2}$ so we have:

$$\phi(r) = -\int \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu\gamma}{\ell^2}u - u^2}}$$

The solution to this integral can be calculated using an integral table:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{-a}}\sin^{-1}\left[\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right] + C$$

Insight: Here, it's important to note that since $a = -1$ in our original equation, so that the fraction $\frac{1}{-\sqrt{a}}$ is a real quantity. Since we need to enforce that $\phi(r)$ must also be real then we also require that $2ax + b < \sqrt{b^2 - 4ac}$, otherwise the $\sin^{-1}$ term would also return a complex value.

Using this solution, we then have:

$$\phi + C = \sin^{-1}\left[\frac{-\frac{2}{r} + \frac{2\mu\gamma}{\ell^2}}{\sqrt{\left[\frac{2\mu\gamma}{\ell^2}\right]^2 + 8\frac{\mu E}{\ell^2}}}\right]$$

Taking the sine now, we can get:

$$\sin(\phi + C) = \frac{-\frac{2}{r} + \frac{2\mu\gamma}{\ell^2}}{\sqrt{\left[\frac{2\mu\gamma}{\ell^2}\right]^2 + 8\frac{\mu E}{\ell^2}}}$$

Then we can cancel the constant by allowing our ϕ to start at a moment where $C = \frac{\pi}{2}$, and so we get instead:

$$\cos\phi = \frac{\frac{\ell^2}{\mu\gamma}\frac{1}{r} - 1}{1 + \frac{2E\ell^2}{\mu\gamma^2}}$$

Here, we can define the constants:

$$c \equiv \frac{\ell^2}{\gamma\mu} \qquad \epsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu\gamma^2}}$$

so we get the simple looking equation:

Theorem

For two particles moving in orbit with one another under a force that varies with $\frac{1}{r^2}$, the orbit of one particle in terms of the center of mass frame is:

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi}$$

This gives us an equation for r in terms of the angle ϕ, which is measured as the angle between the x axis and the line connecting the focus to the orbiting body. This equation also happens to be the equation of a conic section with one focus at the origin, directly implying that planetary motion always traces out conic sections.

Insight: There is nothing really special about the gravitational force either – any central force that varies like $\frac{1}{r^2}$ will naturally give us these orbit equations, since we didn't really assume anything besides substituting $U = -\frac{\gamma}{r}$.

3.2 Alternate Method

Alternatively, we could have also determined $r(\phi)$ by analyzing forces, by using the equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = -\frac{u}{\ell^2}\frac{1}{u^2}F(u)$$

If we assume that F follows an inverse-square relation, then we ge;

$$F(r) = -\gamma u^2$$

(here γ eats up all the constants), and so we get the differential equation:

$$u''(\phi) = -u(\phi) + \frac{\gamma\mu}{\ell^2}$$

Note that here, we have a differential equation that almost looks like simple harmonic motion, but we have a constant term now. To get rid of this, we introduce the substitution

$$w(\phi) = u(\phi) - \frac{\gamma\mu}{\ell^2}$$

so we get:

$$w''(\phi) = -w(\phi)$$

which has a solution:

$$w(\phi) = A\cos(\phi - \delta)$$

We can then cancel δ with ah appropriate choice for $\phi = 0$, so therefore our general solution $u(\phi)$ is:

$$u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi = \frac{\gamma\mu}{\ell^2}(1 + \epsilon\cos\phi)$$

So finally:

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi}$$

which is the same equation we've recovered before. This derivation using forces and solving the differential equation is what Taylor goes through, but either method is obviously valid. To find ϵ, we can also notice that $E = U_{eff}(r_{min})$ to derive an expression for ϵ in terms of the constants we had before. The expression for E then becomes:

$$E = U(r_{min}) + \frac{\ell^2}{2\mu r_{min}^2} = \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right)$$

We also know that $r = r_{min}$ when $\cos\phi = 1$, so therefore $r_{min} = \frac{c}{1+\epsilon}$. Plugging this in, we get:

$$\begin{aligned} E &= \frac{1}{2\left(\frac{c}{1+\epsilon}\right)}\left(\frac{\ell^2}{\mu\left(\frac{c}{1+\epsilon}\right)} - 2\gamma\right) \\ &= \frac{1}{2\left(\frac{\ell^2/\gamma\mu}{1+\epsilon}\right)}\left(\frac{\ell^2}{\mu\left(\frac{\ell^2/\gamma\mu}{1+\epsilon}\right)} - 2\gamma\right) \\ &= \frac{\gamma\mu(1+\epsilon)}{2\ell^2}[\gamma(1+\epsilon) - 2\gamma] \\ &= \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) \end{aligned}$$

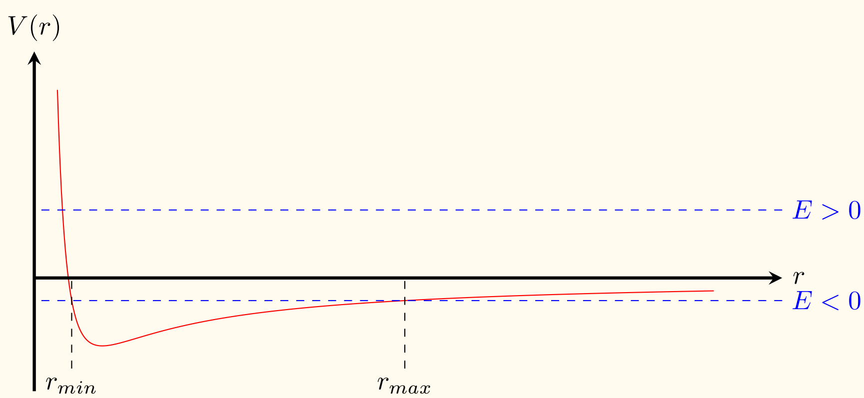
So this then gives us the relation that we're looking for:

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu\gamma^2}}$$

Notation (Eccentricity): The quantity ϵ is also called the *eccentricity* of orbit, and roughly translates to how "squashed" the orbit looks.

This equation actually has very profound consequences. For one, it tells us that if $E < 0$ then we have $\epsilon < 1$, so this corresponds to bound orbits. Otherwise, if $E > 0$, then this corresponds instead to unbound orbits.

Insight: This result also matches our previous intuition about bounded orbits. Recall the potential energy curve:



Notice that for $E < 0$, then our energy intersects our curve at 2 locations, giving us an r_{min} and r_{max} matching the fact that the orbit is bounded. But, for $E > 0$, notice that we only have one intersection r_{min}, matching the fact that the orbit is unbounded.

Looking back at our equation $r(\phi)$,

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi}$$

notice that if $\epsilon < 1$, then the denominator in this expression never vanishes, so $r(\phi)$ is bounded for all ϕ. On the other hand, if $\epsilon \geq 1$, then the denominator vanishes for some angle and $r(\phi)$ approaches infinity as ϕ approaches that angle, corresponding to an unbounded orbit.

3.3 Bounded Orbits

Let's now consider the case where $\epsilon < 1$. Then, our denominator oscillates between:

$$r_{min} = \frac{c}{1+\epsilon} \quad \text{and} \quad r_{max} = \frac{c}{1-\epsilon}$$

Since this is a bounded orbit, then $r(\phi)$ clearly must be periodic with a period of 2π, and the orbit should close on itself after one revolution. If we then perform a change of variables and write $r = \sqrt{x^2+y^2}$ and $\cos\phi = x/r$, then we get:

$$\begin{aligned} r &= \frac{c}{1+\epsilon cos\phi} \\ r\left(1 + \frac{\epsilon x}{r}\right) &= c \\ \therefore r &= c - \epsilon x \end{aligned}$$

And so now completing the square and setting $r^2 = x^2 + y^2$:

$$\begin{aligned} r^2 = (c-\epsilon x)^2 &= x^2 + y^2 \\ c^2 + \epsilon^2 x^2 - 2c\epsilon x &= x^2 + y^2 \\ (1-\epsilon^2)x^2 + 2c\epsilon x + y^2 &= c^2 \\ (x+d)^2 + \frac{y^2}{1-\epsilon^2} &= \frac{c^2}{(1-\epsilon^2)} + d^2 \\ (x+d)^2 + \frac{y^2}{1-\epsilon^2} &= \frac{c^2}{(1-\epsilon^2)} + \frac{c^2\epsilon^2}{(1-\epsilon^2)^2} \\ (x+d)^2 + \frac{y^2}{1-\epsilon^2} &= \frac{c^2 - c^2\epsilon^2 + c^2\epsilon^2}{(1-\epsilon^2)^2} \\ (x+d)^2 + \frac{y^2}{1-\epsilon^2} &= \frac{c^2}{1-\epsilon^2} \\ \frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} &= 1 \end{aligned}$$

This is the equation for an ellipse, with $a = \frac{c}{1-\epsilon^2}$, $b = \frac{c}{\sqrt{1-\epsilon^2}}$ and $d = \frac{c\epsilon}{1-\epsilon^2} = a\epsilon$. The center of the ellipse is offset from the origin by d along the x-axis, and the ratio of major and minor axes is:

$$\frac{b}{a} = \sqrt{1-\epsilon^2}$$

This leads us to Kepler's first law:

Theorem

Planets and bound comets follow orbits that are ellipses with the sun at one focus.

Further, plugging in $c = \frac{\ell^2}{\gamma\mu}$ and $\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu\gamma^2}}$ then we can also see that:

$$a = \frac{c}{1-\epsilon^2} = \frac{\gamma}{2|E|}$$
$$b = \frac{c}{\sqrt{1-\epsilon^2}} = \frac{\ell}{2\mu|E|}$$