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HW 04

Introduction to Analysis

Math 104
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Problem 1

Prove the following proposition:

Proposition 1. *A series* $\sum a_n$ *with positive terms* $a_n \geq 0$ *converges if and only if its partial sums*

$$\sum_{k=1}^{n} a_k \leq M$$

are bounded from above, otherwise, it diverges to ∞.

Solution: First, we prove the reverse direction: if $\sum_{k=1}^{n} a_k \leq M$ then we prove that $\sum a_n$ converges. Consider the sequence s_k defined as the partial sums of a_n. That is,

$$s_k = \sum_{n=1}^{k} a_n$$

Since $s_k \leq M$ by definition, we know that s_k is bounded above by M. Furthermore, since a_n only contains positive terms, the partial sums s_k must be monotonically increasing. Therefore, s_k is a monotonically increasing sequence which is bounded above by M, which implies that s_k converges, and so $\sum a_n$ converges. Otherwise, if no such M exists, then the sequence of partial sums is unbounded, implying that $\sum a_n$ also diverges.

For the forward direction: we prove that if $\sum a_n$ converges, then s_k is bounded. Let $L \in \mathbb{R}$ be defined as:

$$L = \sum_{n=1}^{\infty} a_n = \lim s_k$$

Since the limit of s_k exists, then it is a bounded sequence, therefore $\sum_{k=1}^{n} a_k$ is also bounded from above, as desired. If $\sum a_n$ diverges, then we know that

$$\sum_{n=1}^{\infty} a_n = \lim s_k = \infty$$

which implies that the partial sums are unbounded. $\square$

Problem 2

Prove $\sum_{n=1}^{\infty} \frac{1}{(n+1)(\log(n+1))^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Solution: We use the integral test. This series converges if and only if:

$$\int_0^\infty \frac{1}{(n+1)(\log(x+1))^p} dx$$

converges. We let $u = \log(x+1)$ so $du = \frac{1}{x+1}\, dx$. This turns our integral into:

$$\int_1^t \frac{1}{u^p} du$$

and this integral converges if and only if the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges. As we know, this series only converges when $p > 1$ and diverges when $p \leq 1$, meaning that our original series also converges and diverges under these values for p. $\square$

Problem 3

Given two sequences (a_n) and (b_n). Assume there exists N such that for any $n > N$, $a_n = b_n$. Prove: $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Solution: We prove only one direction since the argument is entirely symmetric in the other direction. Consider the sums:

$$x_N = \sum_{k=N}^{\infty} a_n \qquad\qquad y_N = \sum_{k=N}^{\infty} b_n$$

Since $a_n = b_n$ when $n > N$, then $x_N = y_N$. If $\sum a_n$ is convergent, then its partial sum x_N must also be convergent, and thus y_N is also convergent. Now we break up $\sum b_n$ into two portions:

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{N-1} b_n + \sum_{k=N}^{\infty} b_n$$

The second term is just y_N, which we know converges to a real value. The first term only contains a finite number of elements, so that sum also converges. Therefore, since both of these two terms are convergent, then their sum is convergent, and thus $\sum b_n$ is also convergent.

As mentioned already, the argument is exactly the same in the other direction, since $a_n = b_n$. $\square$

Problem 4

Determine the convergence or divergence of each of the following series defined for $n \in \mathbb{N}$:

(a) $\sum_n \frac{n^3}{2^n}$

Solution: By the ratio test:

$$\left| \frac{\frac{(n+1)^3}{2^{n+1}}}{\frac{n^3}{2^n}} \right| = \frac{1}{2} \left| \frac{(n+1)^3}{n^3} \right|$$

This converges to $\frac{1}{2} < 1$, so therefore the original series converges. $\square$

(b) $\sum_n \sqrt{n+1} - \sqrt{n}$

Solution: By telescoping, we can see that for any finite n, this sequence is equal to:

$$\sqrt{n+1} - \sqrt{n} = \sqrt{2} - \sqrt{1} + (\sqrt{3} - \sqrt{2}) \cdots + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - 1$$

So this sequence diverges. $\square$

(c) $\sum_n \frac{1}{\sqrt{n!}}$

Solution: For all $n > 7$, we see that

$$\frac{1}{\sqrt{n!}} < \frac{1}{n^2}$$

And the series $\sum \frac{1}{n^2}$ converges by the p-series, so by the comparison test, the original series also converges. $\square$

(d) $\sum_n 2^{-3n+(-1)^n}$

Solution: By the root test:

$$\limsup \left| 2^{-3n+(-1)^n} \right|^{\frac{1}{n}} = \left| 2^{-3+\frac{(-1)^n}{n}} \right| = \frac{1}{8}$$

And since $0 \leq \limsup |a_n|^{\frac{1}{n}} = \frac{1}{8} < 1$, our original series converges. $\square$

(e) $\sum_n \frac{n!}{n^n}$

Solution: Expanding this out, we see that:

$$\frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right) < \frac{2}{n^2}$$

Since the series $\frac{2}{n^2}$ converges as it is a p-series, then our original series converges also.

$\square$

Problem 5

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove that there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

Solution: Since $\liminf |a_n| = 0$, then we know that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$|\inf\{|a_n|, n > N\}| < \epsilon$$

Or equivalently, within the sequence $\{a_n | n > N\}$, there exists an element a_{n_k} such that $|a_{n_k}| < \epsilon$ for any choice of $\epsilon > 0$. Notice that this is equivalent to writing

$$\lim a_{n_k} = 0$$

so there exists a subsequence a_{n_k} which has a limit of 0. This is useful, because we can now write that for any $\epsilon > 0$, there exists an $n_K \in \mathbb{N}$ such that for all $n_k > n_K$:

$$a_{n_k} < \epsilon$$

We can define ϵ however we'd like, so we can choose $\epsilon = \frac{1}{m^2}$, and select any a_{n_k} that satisfies $a_{n_k} < \frac{1}{m^2}$. Since the series $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges, then $\sum_{k=1}^{\infty} a_{n_k}$ also converges, as desired. $\square$

Problem 6

Find a sequence (a_n) such that $\sum_{n=1}^{2N} a_n$ and $\sum_{n=1}^{2N+1} a_n$ both converge as $N \to \infty$, but $\sum a_n$ is divergent.

Solution: Consider the sequence $a_n = (-1)^n$. Then, the sequence of partial sums

$$\sum_{n=1}^{2N} a_n = 0$$

for all N, so this sequence of partial sums converges to 0. On the other hand,

$$\sum_{n=1}^{2N+1} a_n = -1$$

for all N, so this sequence converges to -1. Therefore, both partial sums $\sum_{n=1}^{2N} a_n$ and $\sum_{n=1}^{2N+1} a_n$ both converge, but we know that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n$$

diverges. $\square$