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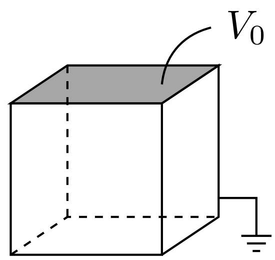
HW 06	Electrodynamics and Optics	March 9, 2023

Collaborators

I worked with **Andrew Binder, Christine Zhang, Teja Nivarthi, Nathan Song** and **Nikhil Maserang** to complete this assignment.

Problem 1

A hollow cube has six square faces. Five faces are grounded ($V = 0$), while the top face is held at a constant potential $V = V_0$. Find the potential at the center of the cube. *Hint:* Duplicate five other such cubes. And then use the superposition principle.



Solution: Following the hint, construct a cube with all five sides with potential V_0. Since there is no charge inside the cube, then it is impossible for there to be any electric field present. Now, since $\vec{E} = -\nabla V$, then the condition that $\vec{E} = 0$ everywhere in the cube requires that there is no gradient within the cube, or in other words $V = V_0$ everywhere inside the cube. Therefore, $V = V_0$ in the center of the cube as well.

Now, we can use the superposition principle. We expect that every plate contributes equally to the potential at the center of the cube because the cube has rotational symmetry. Therefore, we conclude that the contribution of potential due to a single plate (which is also the potential the problem statement asks us to find) is $V = \frac{V_0}{6}$.

Alternatively, one could solve Laplace's equation with $\nabla^2 V = 0$ and fit the boundary conditions, but that approach is longer, and I frankly find this one to be much cleaner and more conceptual. $\square$

Problem 2

A sphere centered at the origin has radius R. The electric field within the sphere is given by

$$E = -\frac{V_0 x}{R^2}\hat{x} - \frac{V_0 y}{R^2}\hat{y} + \frac{2V_0 z}{R^2}\hat{z} \text{ for } x^2 + y^2 + z^2 \leq R^2$$

Find the volume charge density $\rho(r,\theta,\phi)$ (confined to $r < R$) and the surface charge density $\sigma(\theta)$ (confined to $r = R$) that produce the electric field given above. We assume there is no charge outside the sphere for this problem. Express your answer in terms of spherical coordinates. Is your answer unique? If not, find the general charge distributions that produce such electric field.

Solution: First, we can find the volumetric charge density by taking the divergence of this vector field. We can take it in cartesian coordinates since our choice of coordinate system here really doesn't matter. Doing this, we find:

$$\nabla \cdot E = -\frac{V_0}{R^2}(0+0+0) = 0 = \frac{\rho(x,y,z)}{\epsilon_0}$$

So this means that $\rho(x,y,z) = 0$. This also implies that $\rho(r,\theta,\phi) = 0$, since changing the coordinate system does not change the volume we're interested in. To find the surface charge density, we first note that the general solution to the potential is:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Outside, we know that there are no charges so the potential should equal 0 at infinity, meaning that outside, the potential is:

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

To find the potential inside the sphere, notice that we're given the electric field, so using $E = -\nabla V$ and a little bit of algebra we find that:

$$V_{in}(r,\theta) = \frac{V_0}{R^2}\left(\frac{x^2}{2} + \frac{y^2}{2} - z^2\right)$$

So now using the substitution that $x = r\sin\theta\sin\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$, then we get:

$$\begin{aligned}
V_{in}(r,\theta) &= \frac{V_0}{R^2}\left(\frac{r^2\sin^2\theta\cos^2\phi}{2} + \frac{r^2\sin^2\theta\sin^2\phi}{2} - r^2\cos^2\theta\right) \\
&= \frac{V_0 r^2}{R^2}\left(\frac{\sin^2\theta}{2} - \cos^2\theta\right) \\
&= \frac{V_0 r^2}{2R^2}(1 - 3\cos^2\theta)
\end{aligned}$$

Evaluating this expression at R gives:

$$V(r = R,\theta) = \frac{V_0}{2}(1 - 3\cos^2\theta) = -\frac{V_0}{2}P_2(\cos\theta)$$

where P_2 represents the second Legendre polynomial. Outside the sphere, we have:

$$V_{out}(R,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)$$

Since the Legendre polynomials are orthogonal to one another, this means that only the $l = 2$ term survives here, so we have $V_{out} = \frac{B_2}{R^3} P_2(\cos\theta)$. Now, we can equate the two together:

$$V_0 \cdot -P_2(\cos\theta) = \frac{B_2}{R^3} P_2(\cos\theta)$$
$$\therefore B_2 = -V_0 R^3$$

Therefore,

$$V_{out}(r,\theta) = -\frac{V_0 R^3}{2r^3}(3\cos^2\theta - 1)$$

Now we can proceed and find the surface charge density, using the relation that

$$\left.\frac{\partial V}{\partial r}\right|_{\text{below}}^{\text{above}} = -\frac{\sigma}{\epsilon_0}$$

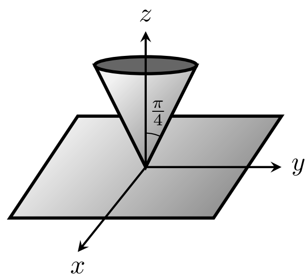
So calculating this:

$$\begin{aligned}
\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} &= \left(-\frac{V_0 R^3}{2} \cdot -\frac{3}{r^4}\right)(3\cos^2\theta - 1) - \left(\frac{2V_0 r}{2R^2}(1 - 3\cos^2\theta)\right) \\
&= \frac{3V_0}{2R}(3\cos^2\theta - 1) - \frac{V_0}{R}(1 - 3\cos^2\theta) \\
&= \frac{5V_0}{R} P_2(\cos\theta) = \frac{\sigma}{\epsilon_0} \\
\therefore \sigma(\theta) &= \frac{5\epsilon_0 V_0}{R} P_2(\cos\theta)
\end{aligned}$$

$\square$

Problem 3

Consider a capacitor formed by an infinitely large plate on $z = 0$ with $V = 0$, and an infinite, solid, conducting cone with an interior angle $\pi/4$ held at potential $V = V_0$. Note that the tip of the cone vertex and the infinitely large plate are insulated.



a) Based on symmetries, explain why $V(r, \theta, \phi) = V(\theta)$ in the space between the cone and the plate. (Note that here only the potential has this scale invariance. Other quantities such as the electric field and charge density do not.)

Solution: Firstly, there is rotational symmetry in the ϕ direction, so there can't be any ϕ dependence. Furthermore, since there is no characteristic length scale of the system, there also cannot be any r dependence. Therefore, V can only depend on θ. □

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b) Integrate Laplace's equation explicitly to find the potential between the cone and the plate. (Note that the general solution Eq. (3.65) in Griffiths does not apply to the case here, since we have charges, distributed at $\theta = \pi/4$ and $\pi/2$.

Solution: We know that $\nabla^2 V = 0$, so using spherical coordinates, this implies:

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) = 0$$

So this implies that $\sin\theta \frac{\partial V}{\partial \theta}$ is a constant, so integrating this:

$$\begin{aligned} \sin\theta \frac{\partial V}{\partial \theta} &= k \\ \int dV &= \int \frac{k}{\sin\theta} d\theta \\ \therefore V(\theta) &= k \left(\ln \left(\sin \frac{\theta}{2} \right) - \ln \cos \frac{\theta}{2} \right) + C \\ &= k \ln \left(\tan \frac{\theta}{2} \right) + C \end{aligned}$$

Substituting in $V(\frac{\pi}{2}) = 0$ gives us $C = 0$, and substituting $\theta = \frac{\pi}{2}$ gives:

$$\begin{aligned} k \log\left(\sqrt{2} - 1\right) &= V_0 \\ \therefore k &= \frac{V_0}{\ln(\sqrt{2} - 1)} \end{aligned}$$

Finally, we get the full expression for $V(\theta)$:

$$V(\theta) = \frac{V_0}{\ln(\sqrt{2} - 1)} \ln \left(\tan \frac{\theta}{2} \right)$$

□
