Physics 212B - Part 2 - Homework 1

Eric Weise
A09642187

For a non-relativistic free particle $L = \frac{m\dot{x}}{2}$

Part 1

Show that the stationary (classical) action $S[x]$ corresponding to the classical motion of a free particle travelling from (x_0, t_0) to (x_1, t_1) is $S[x] = \frac{m(x_1-x_0)^2}{2(t_1-t_0)}$

$$
\begin{aligned}
S[x] &= \int_{t_0}^{t_1} \frac{m}{2}\dot{x}^2 dt \\
&= \frac{m}{2}\int_{t_0}^{t_1} v(t)v(t)dt \\
&= \frac{m}{2}\Big[\big[v(t)x(t)\big]_{t_0}^{t_1} - \int_{t_0}^{t_1} x(t)dv(t)\Big] && \text{Using itegration by parts.} \\
&= \frac{m}{2}\Big[\big[v(t)x(t)\big]_{t_0}^{t_1} - \int_{t_0}^{t_1} x(t)\ddot{x}(t)dt \\
&= \frac{m}{2}\big[v(t)x(t)\big]_{t_0}^{t_1} && \text{since } \ddot{x} = 0 \text{ (Free particle)} \\
&= \frac{m}{2}\big[v(t_1)x(t_1) - v(t_0)x(t_0)\big] \\
&= \frac{m}{2}\big[\frac{x_1-x_0}{t_1-t_0}x(t_1) - \frac{x_1-x_0}{t_1-t_0}x(t_0)\big] \\
&= \frac{m}{2}\frac{(x_1-x_0)^2}{t_1-t_0}
\end{aligned}
$$

Part 2

Show that the spatial derivative of the action $\partial_{x_1}S[x]$ is the momentum of the particle.

$$
\begin{aligned}
\partial_{x1}S[x] &= \partial_{x_1}\frac{m}{2}\frac{(x_1-x_0)^2}{t_1-t_0} \\
&= \frac{m}{2}\frac{2(x_1-x_0)(\partial_{x_1}(x_1-x_0))}{t_1-t_0} && \text{By the chain rule} \\
&= \frac{m}{2}\frac{2(x_1-x_0)}{t_1-t_0} \\
&= m\frac{x_1-x_0}{t_1-t_0} \\
&= m \cdot v
\end{aligned}
$$

Part 3

Show that the (negative) temporal derivative of the action, $-\partial_{t_1}S[x]$, is the energy of the particle.

$$
\begin{aligned}
-\partial_{t_1}S[x] &= -\partial_{t_1}\frac{m}{2}\frac{(x_1-x_0)^2}{t_1-t_0} \\
&= -\frac{m}{2}(x_1-x_0)^2\big[\partial_{t_1}(t_1-t_0)^{-1}\big] \\
&= -\frac{m}{2}(x_1-x_0)^2\big[(-1)(t_1-t_0)^{-2}\big] \\
&= \frac{m}{2}\big(\frac{x_1-x_0}{t_1-t_0}\big)^2 \\
&= \frac{m}{2}\ddot{x}^2
\end{aligned}
$$

Physics 212B - Part 2 - Homework 2

Eric Weise
A09642187

Consider state $|\psi\rangle = \int dx\ \psi(x)\,|x\rangle$ described by the following wave function with a tunable parameter σ:

$$\psi(x) = \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\Big\{-\frac{x^2}{2\sigma^2}\Big\}$$

Part 1

Check that the state is normalized.

$$\begin{aligned}
\langle\psi|\psi\rangle &= \int_{-\infty}^{\infty} dx\, \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\Big\{\frac{-x^2}{2\sigma^2}\Big\} \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\Big\{\frac{-x^2}{2\sigma^2}\Big\} \\
&= \frac{1}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} dx\, \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} \\
\big|\langle\psi|\psi\rangle\big|^2 &= \Big(\frac{1}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} dx\, \exp\Big\{\frac{-x^2}{\sigma^2}\Big\}\Big)\Big(\frac{1}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} dy\, \exp\Big\{\frac{-y^2}{\sigma^2}\Big\}\Big) \\
&= \frac{1}{\pi\sigma^2} \iint_{-\infty}^{\infty} dx\, dy\, \exp\Big\{-\frac{x^2+y^2}{\sigma^2}\Big\}\Big) \\
&= \frac{1}{\pi\sigma^2} \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r \exp\Big\{\frac{-r^2}{\sigma^2}\Big\} && \text{Change of to polar coordinates} \\
&= \frac{2\pi}{\pi\sigma^2} \int_0^{\infty} dr\, r \exp\Big\{\frac{-r^2}{\sigma^2}\Big\} \\
&= \int_0^{\infty} dr\, \frac{2r}{\sigma^2} \exp\Big\{\frac{-r^2}{\sigma^2}\Big\} \\
&= -\int_0^{-\infty} ds\, \exp\{s\} && \text{Changing Coordinates: } s = \frac{-r^2}{\sigma^2} \text{ and } ds = \frac{-2r}{\sigma^2} dr \\
&= -\exp\{-\infty\} + \exp\{0\} \\
&= 1 \\
\big|\langle\psi|\psi\rangle\big|^2 &= 1 \implies \langle\psi|\psi\rangle = 1
\end{aligned}$$

Part 2

Evaluate the expectation values: $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{x}^2\rangle$, and $\langle\hat{p}^2\rangle$ in terms of σ.

$$\begin{aligned}
\langle\hat{x}\rangle &= \langle\psi|\hat{x}|\psi\rangle \\
&= \int dx\, dy\, dz\, \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\Big\{\frac{-y^2}{2\sigma^2}\Big\} \langle y|x\rangle\, x\, \langle x|z\rangle \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\Big\{\frac{-z^2}{2\sigma^2}\Big\} \\
&= \frac{1}{\sqrt{\pi}\sigma} \int dx\, dy\, dz\, \delta(y-x)\delta(x-z) x \exp\Big\{\frac{-y^2-z^2}{2\sigma^2}\Big\} \\
&= \frac{1}{\sqrt{\pi}\sigma} \int dx\, x \exp\Big\{\frac{x^2}{\sigma^2}\Big\} \\
&= \frac{1}{\sqrt{\pi}\sigma} \Big[\int_{-\infty}^{0} dx\, x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} + \int_0^{\infty} dx\, x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\}\Big] \\
&= \frac{1}{\sqrt{\pi}\sigma} \Big[\int_{\infty}^{0} (-dx)(-x) \exp\Big\{\frac{-(-x)^2}{\sigma^2}\Big\} + \int_0^{\infty} dx\, x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\}\Big] && \text{Change variables from } x \text{ to } -x \text{ in first integral} \\
&= \frac{1}{\sqrt{\pi}\sigma} \Big[-\int_0^{\infty} x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} + \int_0^{\infty} x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\}\Big] && \text{Swap integration limits} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\langle \hat{x}^2 \rangle &= \int dw\, dx\, dy\, dz\, \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\Big\{\frac{-w^2}{2\sigma^2}\Big\} \langle w|x\rangle\, x\, \langle x|y\rangle\, y\, \langle y|z\rangle\, \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\Big\{\frac{-z^2}{2\sigma^2}\Big\} \\
&= \frac{1}{\sqrt{\pi}\sigma} \int dx\, dy\, dz\, \delta(w-x)\delta(x-y)\delta(y-z) xy\, \exp\Big\{\frac{-w^2-z^2}{2\sigma^2}\Big\} \\
&= \frac{1}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} dx\, x^2\, \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} \\
&= \frac{2}{\sqrt{\pi}\sigma} \int_{0}^{\infty} dx\, x^2\, \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} && \text{Integrand is symmetric around 0} \\
&= \frac{\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} \Big(dx\, \frac{2x}{\sigma^2}\Big)\Big(\frac{x}{\sigma}\Big) \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} \\
&= \frac{\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} dt\, t^{1/2} \exp\Big\{-t\Big\} && t = \frac{x^2}{\sigma^2} \text{ and } dt = \frac{2x}{\sigma^2} dx \\
&= \frac{\sigma^2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) \\
&= \frac{\sigma^2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \\
&= \frac{\sigma^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{p} \rangle &= \langle \psi | -i\hbar\partial_x | \psi \rangle \\
&= \frac{-i\hbar}{\sqrt{\pi}\sigma} \int dx\, \exp\Big\{\frac{-x^2}{2\sigma^2}\Big\} \partial_x \exp\Big\{\frac{-x^2}{2\sigma^2}\Big\} \\
&= \frac{-i\hbar}{\sqrt{\pi}\sigma} \int dx\, \frac{-2x}{\sigma^2} \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} \\
&= \frac{2i\hbar}{\sqrt{\pi}\sigma^3} \int dx\, x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} \\
&= \frac{2i\hbar}{\sqrt{\pi}\sigma^3} \Big[\int_{-\infty}^{0} dx\, x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} + \int_{0}^{\infty} dx\, x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} \Big] \\
&= \frac{2i\hbar}{\sqrt{\pi}\sigma^3} \Big[\int_{-\infty}^{0} dx\, x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} + \int_{0}^{\infty} dx\, x \exp\Big\{\frac{-x^2}{\sigma^2}\Big\} \Big] \\
&= \frac{i\hbar}{\sqrt{\pi}\sigma} \Big[\int_{+\infty}^{0} du\, \exp\Big\{-u\Big\} + \int_{0}^{\infty} du\, \exp\Big\{-u\Big\} \Big] && u = \frac{x^2}{\sigma^2} \text{ and } du = dx\, \frac{2x}{\sigma^2} \\
&= \frac{i\hbar}{\sqrt{\pi}\sigma} \Big[-\int_{0}^{\infty} du\, \exp\Big\{-u\Big\} + \int_{0}^{\infty} du\, \exp\Big\{-u\Big\} \Big] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{p}^2 \rangle &= \langle \psi | (-i\hbar\partial_x)(-i\hbar\partial_x) | \psi \rangle \\
&= \frac{(-i\hbar)^2}{\sqrt{\pi}\sigma} \int dx\ \exp\{\frac{-x^2}{2\sigma^2}\} \partial_x \partial_x \exp\{\frac{-x^2}{2\sigma^2}\} \\
&= \frac{-\hbar^2}{\sqrt{\pi}\sigma} \int dx\ \exp\{\frac{-x^2}{2\sigma^2}\} \partial_x \Big(\frac{-x}{\sigma^2} \exp\{\frac{-x^2}{2\sigma^2}\} \Big) \\
&= \frac{\hbar^2}{\sqrt{\pi}\sigma^3} \int dx\ \exp\{\frac{-x^2}{2\sigma^2}\} \partial_x \Big(x \exp\{\frac{-x^2}{2\sigma^2}\} \Big) \\
&= \frac{\hbar^2}{\sqrt{\pi}\sigma^3} \int dx\ \exp\{\frac{-x^2}{2\sigma^2}\} \Big[1 - \frac{x^2}{\sigma^2} \Big] \exp\{\frac{-x^2}{2\sigma^2}\} \\
&= \frac{\hbar^2}{\sqrt{\pi}\sigma^3} \int dx\ \Big[1 - \frac{x^2}{\sigma^2} \Big] \exp\{\frac{-x^2}{\sigma^2}\} \\
&= \frac{\hbar^2}{\sqrt{\pi}\sigma^3} \int dx\ \exp\{\frac{-x^2}{\sigma^2}\} - \frac{\hbar^2}{\sqrt{\pi}\sigma^3} \int dx\ \frac{x^2}{\sigma^2} \exp\{\frac{-x^2}{2\sigma^2}\} \\
&= \frac{\hbar^2}{\sqrt{\pi}\sigma^3} \Big[\sqrt{\pi}\sigma \Big] - \frac{\hbar^2}{\sqrt{\pi}\sigma^3} \int dx\ \frac{x^2}{\sigma^2} \exp\{\frac{-x^2}{2\sigma^2}\} && \text{Gaussian integral (same as part 1)} \\
&= \frac{\hbar^2}{\sigma^2} - \frac{\hbar^2}{\sigma^4} \Big[\frac{1}{\sqrt{\pi}\sigma} \int dx\, x^2 \exp\{\frac{-x^2}{2\sigma^2}\} \Big] \\
&= \frac{\hbar^2}{\sigma^2} - \frac{\hbar^2}{\sigma^4} [i \frac{\sigma^2}{2}] && \text{Same as } \langle \hat{x}^2 \rangle \\
&= \frac{\hbar^2}{2\sigma^2}
\end{aligned}
$$

Part 3

Based on the result of Part 2, calculate (stdx) and (stdp) in terms of σ. Do they satisfy the uncertainty relation?

$$
\begin{aligned}
(stdx) &= \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} \\
&= \sqrt{\frac{\sigma^2}{2} - 0} \\
&= \frac{\sigma}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
(stdp) &= \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} \\
&= \sqrt{\frac{\hbar^2}{2\sigma^2} - 0} \\
&= \frac{\hbar}{\sqrt{2}\sigma}
\end{aligned}
$$

$$
(stdx)(stdp) = \frac{\sigma}{\sqrt{2}} \cdot \frac{\hbar}{\sqrt{2}\sigma} = \frac{\hbar}{2} \geq \frac{\hbar}{2}
$$

So, the uncertainty is satisfied (barely).

Physics 212B - Part 2 - Homework 3

Eric Weise
A09642187

Consider $\hat{H} = \frac{1}{2}\big(\hat{p}^2 + \hat{x}^2\big)$, derive the Heisenberg equation for operator $\hat{a} = \frac{1}{\sqrt{2}}\big(\hat{x} + i\hat{p}\big)$

Operator Relationships

Define $\hat{b} = \frac{1}{\sqrt{2}}\big(\hat{x} - i\hat{p}\big)$. Then the following are true:

$$\begin{aligned}
\hat{a}\,\hat{b} &= \frac{1}{2}\big(\hat{x}^2 - i\hat{x}\hat{p} + i\hat{p}\hat{x} + \hat{p}^2\big) \\
&= \frac{1}{2}\big(\hat{x}^2 + \hat{p}^2 - i[\hat{x},\hat{p}]\big) \\
&= \frac{1}{2}\big(\hat{x}^2 + \hat{p}^2 + \hbar\big) \\
&= \hat{H} + \frac{\hbar}{2}
\end{aligned}$$

$$\begin{aligned}
\hat{b}\,\hat{a} &= \frac{1}{2}\big(\hat{x}^2 - i\hat{p}\hat{x} + i\hat{x}\hat{p} + \hat{p}^2\big) \\
&= \frac{1}{2}\big(\hat{x}^2 + \hat{p}^2 + i[\hat{x},\hat{p}]\big) \\
&= \frac{1}{2}\big(\hat{x}^2 + \hat{p}^2 - \hbar\big) \\
&= \hat{H} - \frac{\hbar}{2}
\end{aligned}$$

So $\hat{H} = \hat{a}\,\hat{b} - \frac{\hbar}{2} = \hat{b}\,\hat{a} + \frac{\hbar}{2}$

Deriving Heisenberg's Equation

$\hat{a}\,(t) = e^{i\hat{H}t/\hbar}\,\hat{a}\;e^{-i\hat{H}t/\hbar}$

$$\begin{aligned}
\frac{d}{dt}\hat{a}\,(t) &= \frac{i}{\hbar}\hat{H}\,e^{i\hat{H}t/\hbar}\,\hat{a}\;e^{-i\hat{H}t/\hbar} + e^{i\hat{H}t/\hbar}\,\frac{\partial \hat{a}}{\partial t}\,e^{-i\hat{H}t/\hbar} - \frac{i}{\hbar}\,e^{i\hat{H}t/\hbar}\,\hat{a}\,\hat{H}\,e^{-i\hat{H}t/\hbar} \\
&= \frac{i}{\hbar}\,e^{i\hat{H}t/\hbar}\left(\hat{H}\hat{a} - \hat{a}\,\hat{H}\right)e^{-i\hat{H}t/\hbar} + e^{i\hat{H}t/\hbar}\,\frac{\partial \hat{a}}{\partial t}\,e^{-i\hat{H}t/\hbar} \\
&= \frac{i}{\hbar}\,e^{i\hat{H}t/\hbar}\left(\hat{H}\hat{a} - \hat{a}\,\hat{H}\right)e^{-i\hat{H}t/\hbar} \qquad \text{since } \hat{x} \text{ and } \hat{p} \text{ are time independent.} \\
&= \frac{i}{\hbar}\,e^{i\hat{H}t/\hbar}\left((\hat{a}\,\hat{b} - \frac{\hbar}{2})\hat{a} - \hat{a}\,(\hat{b}\,\hat{a} + \frac{\hbar}{2})\right)e^{-i\hat{H}t/\hbar} \\
&= \frac{i}{\hbar}\,e^{i\hat{H}t/\hbar}\left(\hat{a}\,\hat{b}\,\hat{a} - \hat{a}\,\hat{b}\,\hat{a} - \hbar\hat{a}\right)e^{-i\hat{H}t/\hbar} \\
&= -i\;e^{i\hat{H}t/\hbar}\left(\hat{a}\right)e^{-i\hat{H}t/\hbar} \\
&= -i\,\hat{a}\,(t)
\end{aligned}$$

Physics 212B - Part 2 - Homework 4

Eric Weise
A09642187

Part 1

Show that $[\,\hat{x}\,,\,\hat{p}^{\,n}] = i\hbar n\,\hat{p}^{\,n-1}$ for $n \in \mathbb{N}$

Suppose that $[\,\hat{x}\,,\,\hat{p}^{\,n}] = i\hbar n\,\hat{p}^{\,n-1}$. Then:

$$\begin{aligned}
[\,\hat{x}\,,\,\hat{p}^{\,n+1}] &= \hat{x}\,\hat{p}^{\,n+1} - \hat{p}^{\,n+1}\,\hat{x} \\
&= \hat{x}\,\hat{p}^{\,n+1} - \hat{p}\,\hat{x}\,\hat{p}^{\,n} + \hat{p}\,\hat{x}\,\hat{p}^{\,n} - \hat{p}^{\,n+1}\,\hat{x} \\
&= (\,\hat{x}\,\hat{p} - \hat{p}\,\hat{x}\,)\,\hat{p}^{\,n} + \hat{p}\,(\,\hat{x}\,\hat{p}^{\,n} - \hat{p}^{\,n}\,\hat{x}\,) \\
&= [\,\hat{x}\,,\,\hat{p}\,]\,\hat{p}^{\,n} + \hat{p}\,[\,\hat{x}\,,\,\hat{p}^{\,n}] \\
&= i\hbar\,\hat{p}^{\,n} + \hat{p}\,i\hbar n\,\hat{p}^{\,n-1} \\
&= i\hbar\,\hat{p}^{\,n} + i\hbar n\,\hat{p}^{\,n} \\
&= i\hbar(n+1)\,\hat{p}^{\,n}
\end{aligned}$$

Part 2

Show that $[\,\hat{x}\,, F(\,\hat{p}\,)] = i\hbar\partial_{\hat{p}}\,F(\,\hat{p}\,)$ for generic function F.

Use

$$F(\,\hat{p}\,) = \sum_{n=0}^{\infty} \frac{f_n}{n!}\,\hat{p}^{\,n}$$

then:

$$\begin{aligned}
[\,\hat{x}\,, F(\,\hat{p}\,)] &= \sum_{n=0}^{\infty} \frac{f_n}{n!}[\,\hat{x}\,,\,\hat{p}^{\,n}] \\
&= [\,\hat{x}\,,\mathbb{1}] + \sum_{n=1}^{\infty} \frac{f_n}{n!}[\,\hat{x}\,,\,\hat{p}^{\,n}] \\
&= \sum_{n=1}^{\infty} \frac{f_n}{n!}[\,\hat{x}\,,\,\hat{p}^{\,n}] \\
&= \sum_{n=1}^{\infty} \frac{f_n}{n!} i\hbar n\,\hat{p}^{\,n-1} \\
&= i\hbar \sum_{n=1}^{\infty} \frac{f_n}{(n-1)!} i\hbar\,\hat{p}^{\,n-1} \\
&= i\hbar \sum_{n=0}^{\infty} \frac{f_{n+1}}{n!}\,\hat{p}^{\,n} \\
&= i\hbar\partial_{\hat{p}}\,F(\,\hat{p}\,)
\end{aligned}$$

Part 3

Show that $[\,\hat{x}\,,\hat{T}(a)] = -a\hat{T}(a)$.

$$\begin{aligned}
[\,\hat{x}\,,\hat{T}(a)] &= i\hbar\partial_{\hat{p}}\; e^{i\,\hat{p}\,a/\hbar} \\
&= i\hbar\frac{ia}{\hbar}\,e^{i\,\hat{p}\,a/\hbar} \\
&= -a\,e^{i\,\hat{p}\,a/\hbar} \\
&= -a\hat{T}(a)
\end{aligned}$$

Physics 212B - Part 2 - Homework 5

Eric Weise
A09642187

Show that $|N\rangle = \frac{1}{\sqrt{2\pi}} \int d\theta \; e^{iN\theta} \; |\theta\rangle$ is normalized

$$\begin{aligned}
\langle N|N\rangle &= \frac{1}{2\pi} \int d\theta_1 \, d\theta_2 \; e^{-iN\theta_1} \; e^{iN\theta_2} \; \langle \theta_1|\theta_2\rangle \\
&= \frac{1}{2\pi} \int d\theta \; e^{iN(\theta_2-\theta_1)} \, \delta(\theta_2-\theta_1) \\
&= \frac{1}{2\pi} \int d\theta_1 \; e^0 \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta_1 \\
&= \frac{1}{2\pi} 2\pi \\
&= 1
\end{aligned}$$

Physics 212B - Part 2 - Homework 6

Eric Weise
A09642187

part 1

From equation 149:

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum e^{iN\theta} \; |N\rangle$$

$$|-\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum e^{-iN\theta} \; |N\rangle$$

$$\langle\theta| = \frac{1}{\sqrt{2\pi}} \sum e^{-iN\theta} \; \langle N|$$

Then

$$\begin{aligned}
\int |-\theta\rangle \langle\theta| &= \frac{1}{2\pi} \int d\theta \Big(\sum_M e^{iM\theta} \; |M\rangle \Big) \Big(\sum_N e^{iN\theta} \; \langle N| \Big) \\
&= \frac{1}{2\pi} \int d\theta \sum_{M,N} e^{i(M+N)\theta} \; |M\rangle \langle N| \\
&= \frac{1}{2\pi} \Big(\int d\theta \sum_{M=-N} |M\rangle \langle N| + \int d\theta \sum_{M \neq -N} e^{i(M+N)\theta} \; |N\rangle \langle M| \Big) \\
&= \frac{1}{2\pi} \Big(2\pi \sum_{n \in \mathbb{Z}} |-N\rangle \langle N| + \sum_{M \neq -N} \Big[e^{i(M+N)\theta} \Big]_{\theta=0}^{2\pi} |N\rangle \langle M| \Big) \\
&= \sum_{n \in \mathbb{Z}} |-N\rangle \langle N| + 0
\end{aligned}$$

part 2

Show that $\hat{P}$ commutes with $\hat{H}$.

$$\begin{aligned}
\hat{H}\hat{P} &= \frac{1}{2} \hat{N}^2 \sum |-N\rangle \langle N| \\
&= \frac{1}{2} \sum \hat{N}^2 \, |-N\rangle \langle N| \\
&= \frac{1}{2} \sum (-N)^2 \, |-N\rangle \langle N| \\
&= \frac{1}{2} \sum N^2 \, |-N\rangle \langle N|
\end{aligned}$$

and

$$\begin{aligned}
\hat{P}\hat{H} &= \frac{1}{2} \sum |-N\rangle \langle N| \, \hat{N}^2 \\
&= \frac{1}{2} \sum |-N\rangle \langle N| \, (N)^2 \\
&= \hat{H}\hat{P}
\end{aligned}$$

Physics 212B - Part 2 - Homework 7

Eric Weise
A09642187

Find the expectation value of the raising operator as a function of t.

$$\begin{aligned}|\psi(0)\rangle &= \frac{1}{\sqrt{4\pi}}\int d\theta\,(1+e^{i\theta})\,|\theta\rangle \\ &= \frac{1}{\sqrt{4\pi}}\left(\int d\theta\ |\theta\rangle + \int d\theta\ e^{i\theta}\ |\theta\rangle\right)\end{aligned}$$

Converting this to the $|N\rangle$ basis:

$$\begin{aligned}\int d\theta\ e^{i\theta}\ |\theta\rangle &= \int d\theta\ e^{i1\theta}\ |\theta\rangle \\ &= \sqrt{2\pi}\,|1\rangle\end{aligned}$$

$$\begin{aligned}\int d\theta\ |\theta\rangle &= \int d\theta\left(\frac{1}{\sqrt{2\pi}}\sum e^{-iN\theta}\ |N\rangle\right) \\ &= \frac{1}{\sqrt{2\pi}}\left[\int d\theta\ |0\rangle + \int d\theta\sum_{N\neq 0} e^{-iN\theta}\ |N\rangle\right] \\ &= \frac{1}{\sqrt{2\pi}}\Big[2\pi\,|0\rangle + 0\Big] \qquad \text{since, for nonzero } N,\ \int d\theta\ e^{-iN\theta} = 0 \\ &= \sqrt{2\pi}\,|0\rangle\end{aligned}$$

Then, in the momentum basis $|N\rangle$ the original wave function has the form:

$$|\psi(0)\rangle = \frac{\sqrt{2\pi}}{\sqrt{4\pi}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Now, applying the time evolution to find the wave function as a function of time:

$$\begin{aligned}|\psi(t)\rangle &= e^{-i\hat{H}t}\ |\psi(0)\rangle \\ &= \frac{1}{\sqrt{2}}\,e^{-i\hat{N}^2t/2}\,[\,|0\rangle + |1\rangle\,] \\ &= \frac{1}{\sqrt{2}}\,[\,e^{-i(0)^2t/2}\ |0\rangle + e^{-i(1)^2t/2}\ |1\rangle\,] \\ &= \frac{1}{\sqrt{2}}\,[\,|0\rangle + e^{-it/2}\ |1\rangle\,]\end{aligned}$$

Now, finding the expectation value of the raising operator as a function of time:

$$\begin{aligned}\langle\psi(t)|\ e^{i\hat{\theta}}\ |\psi(t)\rangle &= \frac{1}{2}\,[\,\langle 0| + e^{it/2}\ \langle 1|\,]\,e^{i\hat{\theta}}\,[\,|0\rangle + e^{-it/2}\ |1\rangle\,] \\ &= \frac{1}{2}\,[\,\langle 0| + e^{it/2}\ \langle 1|\,]\,[\,|1\rangle + e^{-it/2}\ |2\rangle\,] \\ &= \frac{1}{2}\,[\,\langle 0|1\rangle + e^{-it/2}\ \langle 0|2\rangle + e^{it/2}\ \langle 1|1\rangle + \langle 1|2\rangle\,] \\ &= \frac{1}{2}\,e^{it/2}\end{aligned}$$