Modeling Expected Reaching Error and Behaviors for Motor Adaptation

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Abstract— **Motor adaptation studies can provide insight into how the brain handles ascending and descending neural signals during motor tasks, revealing how neural pathologies affect the capacity to learn and adapt to movement errors. Such studies often involve reaches towards prompted target locations, with adaptation and learning quantified according to Euclidean distance between reach endpoint and target location. This paper describes methods to calculate steady-state error using knowledge of the distribution of univariate, bivariate, and multivariate steady-state reaches. Additionally, in cases where steady-state error is known or estimated, it does not fully describe underlying reach distributions that could be observed at steady-state. Thus, this paper also investigates methods to describe univariate, bivariate, and multivariate steady-state reaching behavior using knowledge of the estimated steady-state error. These methods may yield a clearer understanding of adaptation and steady-state reaching behavior, allowing greater opportunities for inter-study comparison and modeling.**

I. INTRODUCTION

Motor adaptation is the process by which motor errors are gradually reduced over time via modifications of movements on a trial-by-trial basis [1]. This process is critical for learning new tasks, as well as dexterously performing learned tasks. Numerous studies investigate motor adaptation to understand how learning is affected by neural pathologies [2], sensory feedback [3]–[6], and brain-computer interfaces [7].

To quantify adaptation, many studies utilize reaching paradigms including side-to-side [8] and center-out reaches [9], [10]. Subjects typically move a cursor towards a target, attempting to land as close to the target as possible. Errors are calculated as the Euclidean distance between the final cursor position and the target location.

Though error gradually decreases over repeated trials, inherent variability in reaches guarantees errors never converge to zero. Instead, as reach behavior becomes more consistent, errors converge to some positive value. This steady-state error can be estimated from the distribution of cursor positions, defined simply by their mean and (co)variance. This method allows for direct calculation of steady-state error based on steady-state reach behavior in similar studies, or reach behavior as predicted by motor learning models.

The purpose of this paper is to detail methods of calculating the steady-state level of reaching errors during adaptation studies using the distribution of steady-state reaches. These methods provide an alternative to using Monte Carlo methods to approximate steady-state reaching behavior.

Some studies, such as those investigating models of motor learning, simulate reaching task errors over time for different conditions [11]. In these cases, steady-state error remains a somewhat arbitrary metric and does not describe all aspects of reaching behavior; however, it is possible to use steady-state error to estimate the mean and (co)variance of steady-state reaches.

In this paper we also explore the inverse process of describing the expected reaching behavior at convergence using steady-state error. Both processes are described in three categories of reaching tasks: univariate reaches, bivariate reaches, and generalized multivariate reaches.

II. METHODS

A. Univariate Normal Reaches

We start with the simplest reaching tasks: univariate, or 1-dimensional reaching tasks. These include experimental protocol such as oscillating side-to-side reaches [8] and cursor movement along a circular track [7]. In these tasks, error (ε) is defined as the absolute value of the distance between the cursor and the target, and steady-state error (ε_∞) is defined as the average error achieved when the distribution of reaches has stabilized [12]. Even if this distribution is centered over the target, the average error cannot be zero unless variance is also zero.

Errors for univariate reaches are described by the folded normal distribution [13]. Given a univariate normal distribution $X \sim N(\mu, \sigma^2)$ with probability density function (PDF) $f(x|\mu, \sigma^2)$, the PDF of the folded normal distribution is defined as:

$$f_f(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}} + e^{-\frac{(x+\mu)^2}{2\sigma^2}}\right), x \geq 0 \quad (1)$$

where μ and σ^2 are the mean and variance of the underlying normal distribution, respectively. Simply, the

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likelihood of negative values is added to the likelihood of their corresponding positive values.

Figure 1 shows the relationship between the normal PDF and the folded normal PDF. Importantly, it shows that the mean of the folded normal distribution is always greater than that of the underlying distribution. This mean is calculated as:

$$\mu_f = \varepsilon_\infty = \sigma\sqrt{\frac{2}{\pi}}e^{\frac{-\mu^2}{2\sigma^2}} + \mu\left(1 - 2\Phi\left(\frac{-\mu}{\sigma}\right)\right) \tag{2}$$

where $\Phi(x)$ is the standard normal cumulative density function (CDF) $F(x|0,1)$ [13]. This mean represents the average error of reaches following the underlying distribution. In other words, it is the same as the expected steady-state error. Likewise, the variance of steady-state errors is calculated as:

$$\sigma_f{}^2 = \mu^2 + \sigma^2 - \mu_f{}^2 \tag{3}$$

In summary, given a mean and variance of steady-state reaches, the mean and variance of reach errors can be explicitly calculated.

A special case of the folded normal distribution, known as the half-normal distribution, arises when the mean of steady-state reaches is also the position of the target [13]. Solving (1) for $\mu = 0$, the PDF of the half-normal distribution is defined as:

$$f_h(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}, x \geq 0 \tag{4}$$

Solving (2) and (3) for $\mu = 0$, the steady-state error for reaches centered over the target is calculated as:

$$\mu_h = \varepsilon_\infty = \sigma\sqrt{\frac{2}{\pi}} \tag{5}$$

and the variance of these errors is calculated as:

$$\sigma_h{}^2 = \sigma^2\left(1 - \frac{2}{\pi}\right) \tag{6}$$

This distribution can be used to describe a potential best-case scenario, where reaching errors are solely attributed to inherent variability in reaches.

Rearranging (5) to solve for σ^2 yields an explicit calculation of variance of steady-state reaches, given an estimated steady-state error ε_∞ and assuming $\mu = 0$:

$$\sigma^2 = \frac{\pi}{2}\varepsilon_\infty{}^2 \tag{7}$$

Though (5) and (6) provide explicit formulae to describe steady-state behavior, (2) and (3) cannot be rearranged in provide an analytical solution for reach behavior at steady-state. However, numerical solutions can reveal the valid combinations of μ and σ^2 for a given ε_∞ or $\sigma_f{}^2$. Some possible solutions are detailed in Section III (A).

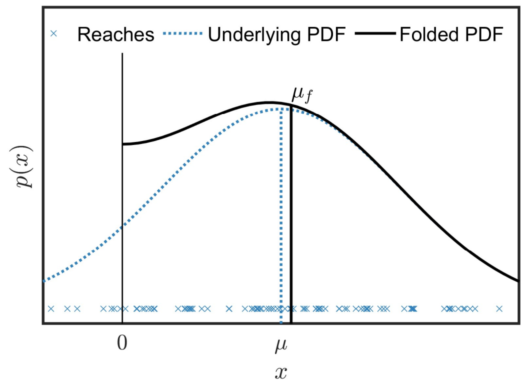

Fig. 1. Univariate folded normal PDF. 1D reaches (blue x) are normally distributed (blue line), but the absolute error is distributed according to a folded normal PDF (black line). As a result, $\mu < \mu_f$

B. Bivariate Normal Reaches

Having discussed the simplest reaching tasks, we now move to perhaps the most common task in testing motor adaptation – the center-out reaching task [9], [10]. Like univariate reaches, errors in reach are calculated as the Euclidean distance between the cursor and the target. If bivariate reaches are distributed according to a bivariate normal distribution $X \sim N_2(\mu, \Sigma)$ with PDF $f(x|\mu, \Sigma)$, defined by mean μ and covariance matrix Σ, the PDF of absolute reach locations can be calculated using a bivariate folded normal distribution [14], as can the mean reach location [15]. Figure 2 shows the relationship between the bivariate normal PDF and the bivariate folded normal PDF.

Here, reaching errors are quantified via the Euclidean distance, which is constrained to positive values. However, although we can calculate the mean reach of a bivariate folded normal PDF, calculating the Euclidean distance between the origin and the mean reach underestimates the mean error of all reaches, as demonstrated by Jensen's inequality [16]:

$$\sqrt{E[X_1]^2 + E[X_2]^2} \leq E\left[\sqrt{X_1{}^2 + X_2{}^2}\right] \tag{8}$$

$E[X]$ is the expected value of bivariate random variable $X \sim N_2(\mu, \Sigma)$, defined as:

$$E[X] = \int_{-\infty}^{\infty} xF(x|\mu, \Sigma)\, dx \tag{9}$$

where $F(x|\mu, \Sigma)$ is the CDF of X. It is thus necessary to directly calculate the expected value of the Euclidean distance from the bivariate normal PDF, requiring the following theorem [17]:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x|\mu, \Sigma)\, dx \tag{10}$$

Applying (10) to Euclidean distance yields the expected error:

$$E\left[\sqrt{X_1{}^2 + X_2{}^2}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x_1{}^2 + x_2{}^2} f(x|\mu, \Sigma)\, dx_2\, dx_1 \tag{11}$$

Furthermore, the expected variance of X is defined as [17]:

$$Var(X) = E[X^2] - E[X]^2 \quad (12)$$

Applying (10) to (12) yields the expected variance of reaching errors:

$$Var\left(\sqrt{{X_1}^2 + {X_2}^2}\right) = E\left[{X_1}^2 + {X_2}^2\right] - E\left[\sqrt{{X_1}^2 + {X_2}^2}\right]^2 \quad (13)$$

$$E\left[{X_1}^2 + {X_2}^2\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}({x_1}^2 + {x_2}^2) f(x|\mu, \Sigma)\, dx_2\, dx_1 \quad (14)$$

Numerical integration allows for calculation of the expected mean and variance of Euclidean distance reaching errors following any arbitrary mean and covariance. However, solutions for reach behavior given an estimated steady-state error are not unique. Some possible solutions assuming independent random variables are detailed in Section III (B).

C. *Multivariate Normal Reaches*

Multivariate reaching tasks may involve endpoint reaching in three dimensions, endpoint or joint orientation, end effector state, or some combination thereof. Like the bivariate case, formulae have been proposed for the multivariate folded normal distribution [18], [19] and their mean [15]. Also like the bivariate case, the folded normal distribution cannot be used to calculate expected error, and it is necessary to directly calculate the expected error from the multivariate PDF.

The derivation of expected error is simply the multivariate extension of (11). Given a k-dimensional multivariate normal distribution $X \sim N_k(\mu, \Sigma)$, the expected error is calculated from the multivariate normal PDF $f(x|\mu, \Sigma)$ as follows:

$$E\left[\sqrt{\sum_{i=1}^{k} {X_i}^2}\right] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \sqrt{\sum_{i=1}^{k} {x_i}^2}\, f(x|\mu, \Sigma)\, dx_k \dots dx_1 \quad (15)$$

And the expected variance of errors is calculated as:

$$Var\left(\sqrt{\sum_{i=1}^{k} {X_i}^2}\right) = E\left[\sum_{i=1}^{k} {X_i}^2\right] - E\left[\sqrt{\sum_{i=1}^{k} {X_i}^2}\right]^2 \quad (16)$$

$$E\left[\sum_{i=1}^{k} {X_i}^2\right] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\sum_{i=1}^{k} {x_i}^2\right) f(x|\mu, \Sigma)\, dx_k \dots dx_1 \quad (17)$$

As with the bivariate case, solutions for reach behavior given an estimated steady-state error are not unique.

D. *Validation*

To validate the approaches presented in this paper to calculate expected reach error, given arbitrary reach distributions, proposed solutions were compared to solutions estimated via Monte Carlo methods. MATLAB code validating these methods are freely available for download on the Open Science Framework [20].

For univariate validation, 10,000 conditions were tested, consisting of 100 values for μ evenly distributed between 0 and 10, and 100 values for σ evenly distributed between 0.25 and 10. For each condition, 1,000,000 data were drawn from a univariate normal distribution $X \sim N(\mu, \sigma^2)$. The mean and variance of the error between generated data and the origin were calculated, and the difference between the Monte Carlo solution and those obtained from equations (2) and (3) were normalized by the mean error of the simulated data.

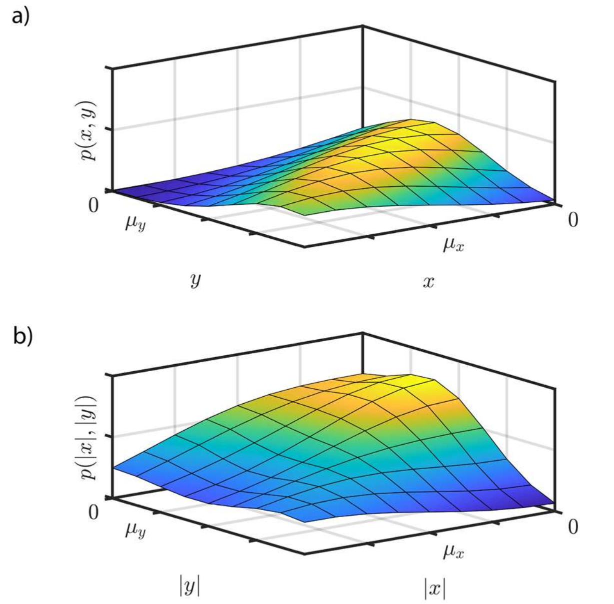

Fig. 2. Bivariate folded normal PDF. (a) 2D reaches are bivariate normally distributed. (b) The absolute positions of 2D reaches are distributed according to a bivariate folded normal PDF.

For bivariate validation, 160,000 conditions were tested, consisting of 20 values each for μ_x and μ_y evenly distributed between 0 and 10, 20 values for σ_x evenly distributed between 0.25 and 10, and 20 values for the ratio $\frac{\sigma_y}{\sigma_x}$ between 1 and 10. For each condition, 1,000,000 data were drawn from a bivariate normal distribution $X \sim N_2(\mu, \Sigma)$. The mean and variance of the Euclidean distance error between generated data and the origin were calculated, and the difference between the Monte Carlo solutions and those obtained from equations (11) and (13) were normalized by the mean error of the simulated data. Validation results are presented in Section III (C).

III. RESULTS

This section covers some numerical solutions of the univariate and bivariate reaches as detailed in Sections II (B) and (C), as well as the validations detailed in Section II (D).

A. *Univariate Normal Reaches*

There are two conditions for which steady-state reach behavior can be determined analytically. If the mean of reaches is 0 (i.e. centered over the target), the variance of the underlying distribution given an estimated steady-state reaching error is calculated using (7). Alternatively, if the variance of reaches is 0, the mean of the underlying distribution is the same as the estimated reaching error. However, solving (2) for parameters of the underlying distribution requires numerical methods. Figure 3 shows the valid combinations of μ and σ^2 for a given ε_∞. Importantly, it

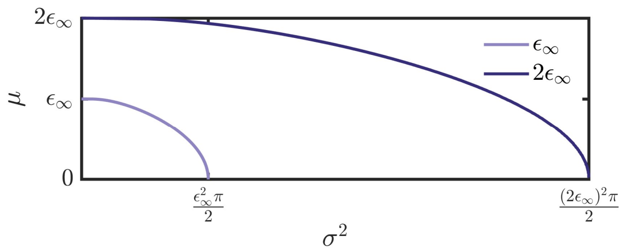

Fig. 3. Visualization of possible univariate reaching behaviors. Plot shows valid means and variances of the underlying normal distribution resulting in an estimated steady-state error (light), as well as how these valid combinations change when steady-state error is doubled (dark).

shows that a linear increase in ε_∞ results in an equivalent linear increase in maximum μ, but an exponential increase in maximum σ^2.

B. *Bivariate Normal Reaches*

For bivariate steady-state reaches, (11) contains 6 parameters which can affect estimated error: μ_x, μ_y, σ_{xx}, σ_{yy}, σ_{xy}, and σ_{yx}. Rotating the basis vectors to align with the eigenvectors of the underlying distribution sets $\sigma_{xy} = 0$ and $\sigma_{yx} = 0$ and removes redundant solutions for these two parameters, reducing the possibility space to 4 parameters. For a given ε_∞, Figure 4(a) shows possible distribution means with defined variances, while Figure 4(b) shows possible distribution variances with defined means. Importantly, the dashed black lines show maximum possible values corresponding to zero variance (Figure 4(a)) and zero mean (Figure 4(b)). Figure 4 also shows diagonal symmetry when variances or means are constrained to be equivalent, but asymmetry otherwise.

C. *Validation*

For univariate reaches, the mean error for data generated via Monte Carlo methods was within 0.062% of that calculated by equation (2), and the variance of error was within 0.457% of that calculated by equation (3).

For bivariate reaches, the mean error for data generated via Monte Carlo methods was within 0.040% of that calculated by equation (11), and the variance of error was within 0.260% of that calculated by equation (13).

Taken together, reaching error and variance estimates procured via Monte Carlo methods closely matched those obtained by the methods proposed in this paper.

IV. Discussion

In this paper, we explore methods for estimating steady-state error during univariate, bivariate, and multivariate reaching tasks using the distribution of steady-state reaches. We also describe the inverse process, providing guidelines for estimating steady-state reaching behavior given an estimated steady-state error. Together, these may yield a clearer picture of adaptation and steady-state reaching behavior.

The methods presented in this paper can also be modified based on predicted changes in reach behavior. For example, it is possible to calculate how errors will change if the distribution means shift, or if the covariance of reaches changes. Furthermore, applying various assumptions to reaching behavior can reveal the maxima of means or covariances, given a particular reaching error. This may provide insight into how changes in reaching error (for example, as a result of a visuomotor rotation) impacts reaching behavior.

The proposed methods are demonstrated and validated assuming normally-distributed reaching behavior. However, these methods are generalizable to any probability distribution with a known PDF. Thus, they are versatile for use in studies

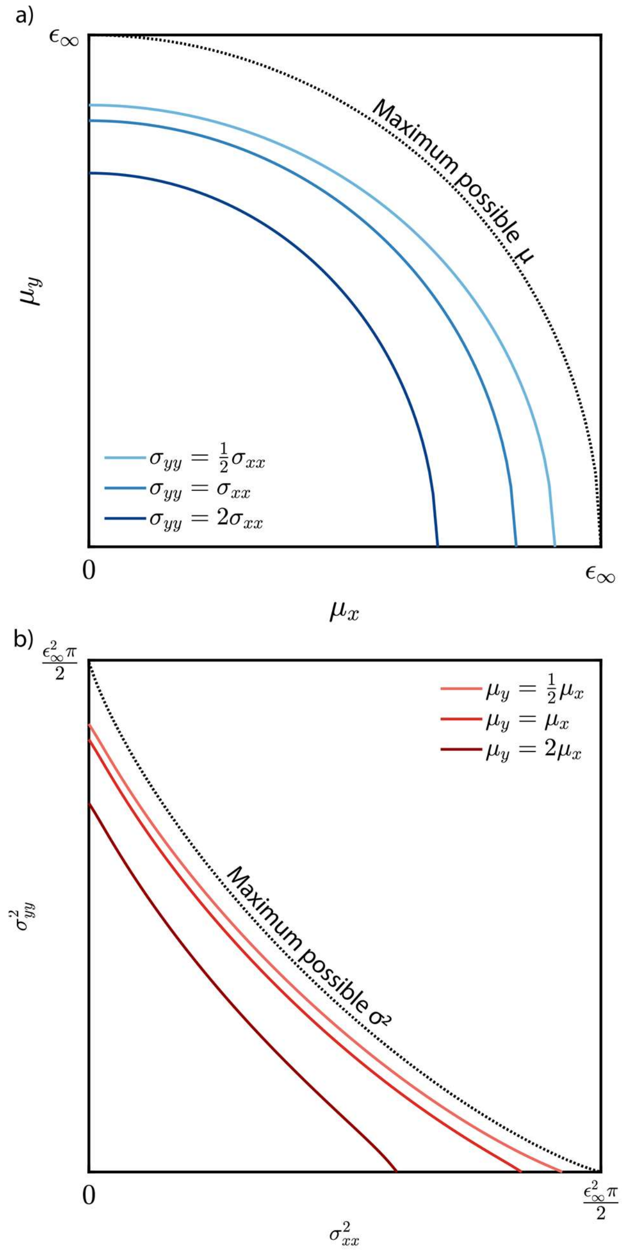

Fig. 4. Visualization of possible bivariate reaching behaviors. Some common assumptions are proposed to visualize valid mean and variance combinations (a) Valid means of the underlying normal distribution resulting in an estimated steady-state error for ${\sigma_{xx}}^2 = \frac{\varepsilon_\infty}{4}$. The maximum possible means are achieved when ${\sigma_{xx}}^2 = {\sigma_{yy}}^2 = 0$ (b) Valid variances of the underlying normal distribution resulting in an estimated steady-state error for $\mu_x = \frac{\varepsilon_\infty}{4}$. The maximum possible variances are achieved when ${\mu_x}^2 = {\mu_y}^2 = 0$.

and simulations which do not constrain reaching behavior to normal distributions.

In addition to the methods presented here, existing distributions may be valid in specific cases. For example, the Rayleigh distribution requires two independent random variables with zero mean and equal variance, as does its parent Chi-squared distribution. However, these do not generalize to arbitrary distribution means and covariances, whereas the methods described in this paper can handle reach distributions of any size, orientation, and dimensionality.

It is possible to estimate steady-state error through direct calculation of previously-collected steady-state data, or by using Monte Carlo methods. However, applicable steady-state data may not be available for the specific study at hand. Although Monte Carlo methods are viable for estimating steady-state error, inverting them to provide insight into steady-state reaching behavior given estimated steady-state error may be difficult.

The methods described in this paper provide a mathematical solution to estimating steady-state error of reaches given their probability distribution, and vice versa. Solutions derived from these methods provide an alternative to Monte Carlo methods. Furthermore, they may be used to gain a clearer understanding of adaptation and steady-state reaching behavior, ultimately allowing greater opportunities for inter-study comparison and modeling.

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