Homotopy Type Theory - Exercise Sheet 2

1. In the last exercise session, we defined the type of n-cells in a type by mutal recursion. Here is a (slightly modified) version of that definition:

$$\begin{aligned}
&\mathsf{bdry} : \mathbb{N} \to \mathsf{Type} \to \mathsf{Type} \\
&\mathsf{bdry}\, 0\, X = X \times X \\
&\mathsf{bdry}\, (S\, n)\, X = \sum_{x\, y : X} \mathsf{bdry}\, n (x \equiv y)
\end{aligned}$$

$$\begin{aligned}
&\mathsf{disc} : (n : \mathbb{N})\, (X : \mathsf{Type}) \to \mathsf{bdry}\, n\, X \to \mathsf{Type} \\
&\mathsf{disc}\, 0\, X\, (x\, ,\, y) = x \equiv y \\
&\mathsf{disc}\, (S\, n)\, X\, (x\, , y\, , \partial) = \mathsf{disc}\, n\, (x \equiv y)\, \partial
\end{aligned}$$

   Show that for a type X we have

$$(S^n \to X) \simeq \mathsf{bdry}\, n\, X$$

   That is, the spheres *represent* the boundary of an n-disc in X.

2. Let $f : X \to Y$ be a map between types X and Y. The type of *null-homotopies* of f is defined as

$$\mathsf{null}\, f := \sum_{y : Y} \prod_{x : X} f\, x \equiv y$$

   That is, a null-homotopy of f is a point of y and a proof that f is equal to the constant function at y.

   Given a map $\phi : S^n \to X$ as in the last exercise, prove that the type of disc's defined above can be identified with the space of null-homotopies of ϕ.

3. Suppose given types $A\, B\, C : \mathsf{Type}$ and maps $f : A \to B$ and $g : A \to C$. Define the *pushout* as a higher inductive type by giving it introduction, elimination and computation rules. Recall that the pushout can be described as the type obtained from the disjoint union of the types B and C by identifying, for every $a : A$, the point $f\, a$ and the point $g\, a$.