Topos Theory - Exercise Sheet 1

1. Show that a map $m : A \rightarrowtail B$ is a monomorphism if and only if the square

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, tail, "m"] \\ A \arrow[r, tail, "m"'] & B \end{tikzcd}$$

is a pullback. (And so a map is an epi if and only if ...)

2. Show that every equalizer is a mono (and therefore, every coequalizer is an epi).

3. Conversely, in a topos check that any mono $m : A \rightarrowtail B$ is an equalizer

$$\begin{tikzcd} A \arrow[r, tail, "m"] & B \arrow[r, shift left, "\mathrm{true}_B"] \arrow[r, shift right, "\chi_m"'] & \Omega \end{tikzcd}$$

where true_B is the composite $B \to 1 \xrightarrow{\mathrm{true}} \Omega$ and $\chi_m : B \to \Omega$ is the classifying map of m.

4. Suppose we have an equalizer diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

for the same map f, then e is an isomorphism.

5. For a category $\mathcal{E}$, let $M(\mathcal{E})$ be the category whose objects are the monomorphisms of $\mathcal{E}$ and whose morphisms are pullback squares of monos. Show that a subobject classifier is the same thing as a terminal object of the category $M(\mathcal{E})$.

6. Let $f : A \to B$ be a morphism.
   (a) Show that $(1, f) : A \to A \times B$ is mono
   (b) Define the *graph* G_f of f to be the subobject corresponding to this mono. Show that if $f' : A \to B$ is another morphism such that $G_f = G_{f'}$ as subobjects of $A \times B$, then $f = f'$.