# Topos Theory - Exercise Sheet 3

1. Let $\mathcal{C}$ be a smal category and $F : \mathcal{C}^{op} \to \mathcal{S}et$ a presheaf. Define the *category of elements of* $F$ to be the category whose objects are pairs $(c, x)$ where $c \in \mathcal{C}$ and $x \in F(c)$. A morphism $(c, x) \to (d, y)$ is a morphism $f : c \to d$ such that $F(f)(y) = x$. We will write $\int F$ for this category. Note that there is a canonical projection functor

$$\pi_F : \int F \to \mathcal{C}$$

   (a) Show that we have an equivalence of categories

   $$\mathcal{S}et^{\mathcal{C}^{op}}/F \cong \mathcal{S}et^{\int F}$$

   We say that presheaf categories are closed under slicing.

   (b) Show that any presheaf $F$ may be recovered as the colimit of the composite

   $$\int F \xrightarrow{\pi_F} \mathcal{C} \xrightarrow{y} \mathcal{S}et^{\mathcal{C}^{op}}$$

   (c) The previous construction can be generalized: suppose we have a functor $G : \mathcal{C} \to \mathcal{E}$ to some category $\mathcal{E}$ which has all colimits. Show that we have an induced functor

   $$\hat{G} : \mathcal{S}et^{\mathcal{C}^{op}} \to \mathcal{E}$$

   defined by

   $$\hat{G}(F) := \operatorname{colim}(G \circ \pi_F)$$

   Find a right adjoint for $\hat{G}$.