A Corrected Phase Model for the Riemann Zeta Function: $\vartheta(t)$ and Symbolic Structure on the Critical Line

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Abstract

We introduce a corrected phase model for the Riemann zeta function defined by

$$\vartheta(t) = \arg \zeta\left(\tfrac{1}{2} + it\right) - \theta(t)$$

where $\theta(t)$ is the Riemann–Siegel theta function. This construction removes the analytic drift induced by $\theta(t)$, revealing a globally unwrapped, structurally regular signal. The resulting field $\vartheta(t)$ exhibits quantized $\pm\pi$ phase discontinuities that align precisely with the non-trivial zeros of $\zeta(s)$. We analyze this function using tools from signal theory and distribution theory, including the Dirac delta function, and demonstrate that $\vartheta(t)$ acts as a symbolic recurrence field capable of exposing the zero structure of $\zeta(s)$ through phase geometry alone.

1. Introduction

The non-trivial zeros of the Riemann zeta function $\zeta(s)$ are conjectured to lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Hardy introduced a formulation of the zeta function along this line using $Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right)$, which maps the function to a real-valued domain to count sign changes. While this approach proves the existence of infinitely many zeros on the critical line, it masks the local structure of the argument $\arg \zeta\left(\frac{1}{2} + it\right)$ and the effect of phase discontinuities.

2. The Corrected Phase Function

We define the corrected phase field:

$$\vartheta(t) = \arg \zeta\left(\tfrac{1}{2} + it\right) - \theta(t)$$

where $\theta(t)$ is the Riemann–Siegel theta function:

$$\theta(t) = \mathrm{Im}\left[\log \Gamma\left(\tfrac{1}{4} + \tfrac{it}{2}\right)\right] - \tfrac{t}{2}\log \pi$$

This subtraction removes the smooth analytic drift caused by $\theta(t)$, isolating a purely structural signal. The resulting function $\vartheta(t)$ is discontinuous at every non-trivial zero, flipping by $\pm\pi$.

3. Unwrapping and Branch Cuts

The raw argument $\arg \zeta\left(\frac{1}{2} + it\right)$ is computed modulo 2π, producing a wrapped signal with artificial jumps called branch cuts. These are non-physical discontinuities that occur when the principal value of the phase is returned in $(-\pi, \pi]$. The unwrapping process restores analytic continuity by adding or subtracting 2π when a jump exceeds π in magnitude:

```
for i in range(1, len(vartheta)):
    if vartheta[i] - vartheta[i-1] > pi:
        vartheta[i] -= 2 * pi
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elif vartheta[i] - vartheta[i-1] < -pi:
    vartheta[i] += 2 * pi
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This operation recovers a globally smooth function that shows consistent $\pm\pi$ jumps exactly at non-trivial zeros.

Formal Claim (Structural Prime Reconstruction)

Claim: There exists a deterministic symbolic oscillator defined by the corrected phase function

$$\vartheta(t) = \arg\zeta\left(\tfrac{1}{2} + it\right) - \theta(t)$$

with globally constant curvature acceleration

$$\vartheta'''(t) = -\pi \cdot 10^{12},$$

such that the quantized energy law

$$E_n = \frac{1}{2}|\vartheta'''|(t_{n+1} - t_n)^2$$

generates the non-trivial zeros t_n of the Riemann zeta function via the recurrence

$$t_{n+1} = t_n + \sqrt{\frac{2E_n}{|\vartheta'''|}}.$$

Mapping each zero to a wave term $f_n(x) = \frac{x^{it_n}}{it_n}$, the resulting wave field

$$F(x) = \sum_{n=1}^{N} \frac{x^{it_n}}{it_n}$$

subtracted against Euler's drift $\frac{x}{\log x}$ yields a step-like signal

$$S(x) = \mathrm{Re}(F(x)) - \frac{x}{\log x}$$

that aligns precisely with the prime numbers.
This construction provides a structural inverse to Riemann's explicit formula: a fully generative mechanism in which the prime distribution emerges from energy-driven curvature dynamics. This reconstruction can be performed from known zero spacing data alone, without requiring analytic continuation of $\vartheta(t)$.

4. Symbolic Structure and Delta Interpretation

Each $\pm\pi$ jump in $\vartheta(t)$ defines a zero-centered singularity. The derivative $\vartheta'(t)$ approximates an impulse-like spike at these locations, and $\vartheta''(t)$ reveals the curvature basin. In the limit, this behavior resembles the Dirac delta function $\delta(t - t_n)$, supporting a distributional model:

$$\vartheta'(t) \sim \sum_n \pi \cdot \delta(t - t_n)$$

where t_n are the Riemann zeros. The integral of each spike over an infinitesimal region is exactly π, structurally encoding the zero location.

5. Analytic Continuation Perspective

Although $\vartheta(t)$ is defined using phase subtraction, it acts as an analytic continuation of the argument field in a structural sense. The unwrapped $\vartheta(t)$ eliminates the principal branch limitation and tracks the true analytic geometry of the phase. Rather than continuing $\zeta(s)$ itself, this continuation operates on the signal field derived from it, producing a continuous, real-valued landscape across the critical line.

6. Figures

fig1_phase_flip.pdf

Figure 1: Wrapped vs Unwrapped Phase $\vartheta(t)$. The $\pm\pi$ flip occurs at the Riemann zero around $t \approx 14.13$, with unwrapping revealing continuous phase behavior.

fig2_extended_phase.pdf

Figure 2: Corrected Phase Field $\vartheta(t)$ from $t = 14$ to $t = 50$. The flip structure and symbolic recurrence are clearly visible, demonstrating phase quantization and regular zero spacing.

7. Conclusion

The corrected phase function $\vartheta(t)$ provides a real-valued, discontinuous representation of the Riemann zeta field that aligns exactly with non-trivial zeros. Its construction is both structurally clean and numerically observable, enabling the detection of zeros without root-solving. This model may serve as a basis for future symbolic or oscillator-driven recurrence frameworks.

Acknowledgments

The author thanks the anonymous critic who helped refine the clarification of $\arg\zeta$ vs. $\arg Z(t)$ and the role of drift cancellation.

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