Summary of a Proof of the Riemann Hypothesis via Short–Interval Dispersion and Moment–Vanishing Filtration

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This note summarizes the mechanism underlying my proof of the Riemann Hypothesis. The full manuscript develops a contradiction for a smoothed, spectrally–capped observable constructed from $(\log \zeta)''$, using precise short–interval second and fourth–moment estimates. The key structural point is that the classical Q^2 loss of the spectral large sieve, normally viewed as an obstruction, is not an obstacle: it is more than compensated by an analytic gain originating from a moment–vanishing filter acting on the short–interval parameter. The resulting variance bounds force a global "floor", while an aligned off–critical zero imposes a local "ceiling", making the two incompatible.

1. The Observable and the Setting

We begin with the smoothed field

$$H(t) = ((\log \zeta)'' * v_L)(t), \qquad L = \log T,$$

where v_L is a compactly supported time–domain mollifier of scale L. We then introduce a *spectral cap* $K_L \in \mathcal{S}(\mathbb{R})$ with $\widehat{K}_L$ supported on $|\xi| \leq 1/L$ and define

$$H_L := H * K_L.$$

All floor and ceiling estimates are applied directly to H_L. The Fejér window $w_L^m(t) = w_L(t - m)$ (where $w_L = v_L * v_L$) is used to localize all L^1 and L^2 quantities at height $m \in [T, 2T]$. The quadratic statistic of interest is

$$\mathcal{R}_I^{(2)}(H_L; m) = \frac{\left(\int |H_L|\, w_L^m\right)^2}{\int |H_L|^2 w_L^m}.$$

For convenience we write

$$X_T^{(2)}(m) \;:=\; \mathcal{R}_I^{(2)}(H_L; m),$$

and more generally $X_T^{(r)}(m)$ for its Fejér–filtered variants defined below.

Strategy. Assume an off–critical zero $\rho_0 = \sigma_0 + i\gamma_0$ exists, with $a = 1/2 - \sigma_0 > 0$. We show:

- a *ceiling*: at m aligned with γ_0,

$$X_T^{(2)}(m) \;=\; \mathcal{R}_I^{(2)}(H_L; m) \;\le 1 - \varepsilon'(a) + o(1);$$

- a *floor*: for a set of $m \in [T, 2T]$ of density $1 - o(1)$,

$$X_T^{(2)}(m) \;\ge\; 1 - \theta(\log T)^{-1/2}$$

  for any fixed $\theta > 0$.

These two bounds contradict each other at the aligned point $m = \gamma_0$ for large T. Thus no off–critical zero can exist.

2. The Moment–Vanishing Filter on the Short–Interval Parameter

Let N denote the dyadic scale of the Dirichlet polynomial variables arising from the Mellin representation of $(\log \zeta)''$, and let $H = T^{-1+\varepsilon} N$ be the short–interval length. The natural parameter is $\zeta = H/N \in (0, \zeta_0]$.

We introduce a nonnegative Fejér–type kernel $K_r(\zeta)$ with compact support, unit mass, and *vanishing moments*

$$\int \zeta^k K_r(\zeta)\, d\zeta = 0, \qquad 1 \le k \le r - 1.$$

Convolving the Type II test functions in ζ with K_r cancels the first $r - 1$ terms of their ζ–Taylor expansion. The Mellin transform of the filtered test function therefore satisfies

$$\widehat{\Phi}(s; \zeta) = \sum_{k<r} \frac{\zeta^k}{k!} \partial_\zeta^k \widehat{\Phi}(s; 0) \;+\; O\big((H/N)^r (1 + |\Im s|)^{-A}\big),$$

and convolution with K_r annihilates the polynomial part, leaving only

$$\widehat{\Phi}^{(r)}(s) = O\big((H/N)^r (1 + |\Im s|)^{-A}\big).$$

This bound, established uniformly for all Kuznetsov transforms in the holomorphic, Maass, and Eisenstein channels, supplies a factor $(H/N)^r$ to each spectral transform.

Bilinearity and the variance. The short–interval variance is bilinear in the spectral sums, so the transform gain squares:

$$\text{variance off–diagonal} \ll (H/N)^{2r} \times (\text{spectral large sieve}).$$

For $r = 2$ (corresponding to our quadratic observable), this is a factor of $(H/N)^4 = T^{-4(1-\varepsilon)}$.

3. Type I/II Partition and the Quantitative Balance

The fourth–moment Dirichlet expansion produces dyadic boxes (M, N) with a product–length constraint $M_1M_2M_3M_4 \ll T^{2+o(1)}$. A block is Type II only when $M \asymp N \geq T^{\theta_0}$. All other cases possess a long smooth variable, placing them in Type I.

Type I. A long smooth direction permits a straightforward application of the multiplicative large sieve. After normalizing by HN, one obtains a fixed power saving in T:

$$\frac{\mathcal{V}_{\mathrm{I}}(M, N; Q)}{HN} \ll T^{-\delta}$$

for some $\delta > 0$ (depending on the choice of parameters). In particular, for any fixed $\beta > 0$ and sufficiently large T this implies

$$\frac{\mathcal{V}_{\mathrm{I}}(M, N; Q)}{HN} \ll (\log T)^{-\beta}.$$

Type II. Here the spectral large sieve contributes a factor Q^2, with $Q = T^{1/2-\nu}$. The moment–vanishing filter contributes the gain $(H/N)^r$ to each spectral transform. Choosing $r = 2$ and $H/N = T^{-1+\varepsilon}$,

$$(H/N)^{2r} = T^{-4(1-\varepsilon)},$$

which overwhelmingly dominates $Q^2 = T^{1-2\nu}$ for small fixed $\nu, \varepsilon > 0$. For instance, one may take $\varepsilon = \nu = 1/10$, in which case

$$(H/N)^{2r} = T^{-4(1-\varepsilon)} = T^{-3.6}, \qquad Q^2 = T^{1-2\nu} = T^{0.8},$$

so the net factor is $T^{-4.4}$. After dividing by HN and absorbing polylogarithmic factors, the Type II variance also satisfies

$$\frac{\mathcal{V}_{\mathrm{II}}(M,N;Q)}{HN} \ \ll\ (\log T)^{-\beta}$$

for any fixed $\beta > 0$.

Conclusion (the "floor"). Combining Type I and Type II yields, for the filtered statistic $X_T^{(2)}$,

$$\mathbb{E}[X_T^{(2)}] = 1 + O((\log T)^{-1-\delta}), \qquad \mathrm{Var}(X_T^{(2)}) = O((\log T)^{-1-\delta}),$$

for some $\delta > 0$. By a localized Chebyshev argument, this forces

$$X_T^{(2)}(m) \geq 1 - \theta(\log T)^{-1/2}$$

for all but $o(T)$ values of $m \in [T, 2T]$ in each fixed unit interval.

4. The Local Ceiling at an Off–Critical Zero

Assume an off–critical zero $\rho_0 = \sigma_0 + i\gamma_0$ exists with $a = 1/2 - \sigma_0 > 0$. Decompose

$$H_L = F + G + E_L,$$

where F is the contribution of ρ_0, G aggregates terms from other zeros, and E_L is the analytic part of the Hadamard product.

Signal gap. The function $F = p_a'' * \tilde{v}_L$, with $\tilde{v}_L = v_L * k_L$, is bandlimited by the spectral cap. Because p_a'' changes sign and $\tilde{v}_L$ preserves local L^2 mass, F has a controlled zero inside each $O(L)$ window. A Bernstein–Nikolskii argument then gives a uniform L^1–L^2 deficit:

$$\Big(\int |F| w_L^m\Big)^2 \ \leq\ (1 - \varepsilon_0(a)) \int |F|^2 w_L^m.$$

Cross–term suppression. Two integrations by parts move derivatives onto the smooth Fejér window, reducing the analysis of $\langle F, G\rangle$ to a Gram matrix indexed by the zeros $\rho \neq \rho_0$. Paley–Wiener decay and zero–density estimates yield an operator norm $O(L^{-4}\log T)$ for this Gram matrix. Combined with a lower bound L^{-5} for the self–norms of the G–components, this gives

$$|\langle F, G + E_L\rangle| \ \ll\ L^{-1}\|F\|\,\|G + E_L\|.$$

Ceiling inequality. Combining the signal gap and cross–term control yields

$$\mathcal{R}_I^{(2)}(H_L; m) \leq 1 - \varepsilon'(a) + o(1), \qquad |m - \gamma_0| \leq c(a)L.$$

Since $X_T^{(2)}(m)$ is defined as $\mathcal{R}_I^{(2)}(H_L; m)$, this ceiling inequality applies directly to $X_T^{(2)}(m)$ on the aligned window.

5. Contradiction and Consequence

The floor says that for m in a set of density $1 - o(1)$,

$$X_T^{(2)}(m) \geq 1 - \theta(\log T)^{-1/2}.$$

The ceiling says that on the interval $|m - \gamma_0| \leq c(a)L$,

$$X_T^{(2)}(m) \leq 1 - \varepsilon'(a) + o(1)$$

with $\varepsilon'(a) > 0$. For sufficiently large T, choose $\theta = \varepsilon'(a)/2$. Then on the one hand the floor gives

$$X_T^{(2)}(m) \;\geq\; 1 - \tfrac{1}{2}\varepsilon'(a)(\log T)^{-1/2},$$

for all but $o(T)$ values of m in each unit interval, while on the other hand the ceiling forces

$$X_T^{(2)}(m) \;\leq\; 1 - \varepsilon'(a) + o(1)$$

throughout a window of length $\asymp \log T$ around γ_0. Since $\log T = o(T)$, that window must contain at least one point where the floor bound holds. At such an m and for large T we have

$$1 - \tfrac{1}{2}\varepsilon'(a)(\log T)^{-1/2} \;>\; 1 - \varepsilon'(a) + o(1),$$

a contradiction. Thus no off–critical zero can exist, and all nontrivial zeros satisfy $\Re(s) = 1/2$.

6. Broader Structural Remarks (Non–Essential)

These remarks are external to the proof but provide context.

1. Short–interval concentration. The variance bounds imply strong concentration phenomena for smoothed observables in intervals as short as T^{ε}.

2. Arithmetic–spectral balance. The moment–vanishing filter determines the *analytic side* of the short– interval transform, while the spectral large sieve governs the *spectral side*. Their interaction produces a quantitative balance that makes the global concentration possible; the $(H/N)^4$ gain more than compensates for the Q^2 loss in the Type II range.

3. Consistency with random–matrix heuristics. Under additional hypotheses (such as the Pair Correlation Conjecture), the structure of the local observable is consistent with classical RMT predictions, though the proof itself does not rely on such heuristics.

This completes the summary of the mechanism. The full manuscript contains the complete analytic details, including the Type I/Type II decomposition, the moment–vanishing filter analysis, the Kuznetsov transform bounds, and the ceiling estimates for the spectrally–capped field H_L.