A Corrected Phase Model for the Riemann Zeta Function: $\vartheta(t)$ and Symbolic Structure on the Critical Line

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Abstract

We introduce a corrected phase model for the Riemann zeta function defined by

$$\vartheta(t) = \arg \zeta\left(\tfrac{1}{2} + it\right) - \theta(t)$$

where $\theta(t)$ is the Riemann–Siegel theta function. This construction removes the analytic drift induced by $\theta(t)$, revealing a globally unwrapped, structurally regular signal. The resulting field $\vartheta(t)$ exhibits quantized $\pm\pi$ phase discontinuities that align precisely with the non-trivial zeros of $\zeta(s)$. We analyze this function using tools from signal theory and distribution theory, including the Dirac delta function, and demonstrate that $\vartheta(t)$ acts as a symbolic recurrence field capable of exposing the zero structure of $\zeta(s)$ through phase geometry alone.

1. Introduction

The non-trivial zeros of the Riemann zeta function $\zeta(s)$ are conjectured to lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Hardy introduced a formulation of the zeta function along this line using $Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right)$, which maps the function to a real-valued domain to count sign changes. While this approach proves the existence of infinitely many zeros on the critical line, it masks the local structure of the argument $\arg \zeta\left(\frac{1}{2} + it\right)$ and the effect of phase discontinuities.

2. The Corrected Phase Function

We define the corrected phase field:

$$\vartheta(t) = \arg \zeta\left(\tfrac{1}{2} + it\right) - \theta(t)$$

where $\theta(t)$ is the Riemann–Siegel theta function:

$$\theta(t) = \mathrm{Im}\left[\log \Gamma\left(\tfrac{1}{4} + \tfrac{it}{2}\right)\right] - \tfrac{t}{2}\log \pi$$

This subtraction removes the smooth analytic drift caused by $\theta(t)$, isolating a purely structural signal. The resulting function $\vartheta(t)$ is discontinuous at every non-trivial zero, flipping by $\pm\pi$.

3. Unwrapping and Branch Cuts

The raw argument $\arg \zeta\left(\frac{1}{2} + it\right)$ is computed modulo 2π, producing a wrapped signal with artificial jumps called branch cuts. These are non-physical discontinuities that occur when the principal value of the phase is returned in $(-\pi, \pi]$. The unwrapping process restores analytic continuity by adding or subtracting 2π when a jump exceeds π in magnitude:

```
for i in range(1, len(vartheta)):
    if vartheta[i] - vartheta[i-1] > pi:
        vartheta[i] -= 2 * pi
```

```
    elif vartheta[i] - vartheta[i-1] < -pi:
        vartheta[i] += 2 * pi
```

This operation recovers a globally smooth function that shows consistent $\pm\pi$ jumps exactly at non-trivial zeros.

4. Symbolic Structure and Delta Interpretation

Each $\pm\pi$ jump in $\vartheta(t)$ defines a zero-centered singularity. The derivative $\vartheta'(t)$ approximates an impulse-like spike at these locations, and $\vartheta''(t)$ reveals the curvature basin. In the limit, this behavior resembles the Dirac delta function $\delta(t - t_n)$, supporting a distributional model:

$$\vartheta'(t) \sim \sum_n \pi \cdot \delta(t - t_n)$$

where t_n are the Riemann zeros. The integral of each spike over an infinitesimal region is exactly π, structurally encoding the zero location.

5. Analytic Continuation Perspective

Although $\vartheta(t)$ is defined using phase subtraction, it acts as an analytic continuation of the argument field in a structural sense. The unwrapped $\vartheta(t)$ eliminates the principal branch limitation and tracks the true analytic geometry of the phase. Rather than continuing $\zeta(s)$ itself, this continuation operates on the signal field derived from it, producing a continuous, real-valued landscape across the critical line.

6. Figures

7. Conclusion

The corrected phase function $\vartheta(t)$ provides a real-valued, discontinuous representation of the Riemann zeta field that aligns exactly with non-trivial zeros. Its construction is both structurally clean and numerically observable, enabling the detection of zeros without root-solving. This model may serve as a basis for future symbolic or oscillator-driven recurrence frameworks.

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References

1. E.C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford University Press, 1986.
2. H.M. Edwards, *Riemann's Zeta Function*, Dover Publications, 2001.
3. G.H. Hardy, "Sur les Zéros de la Fonction $\zeta(s)$ de Riemann," *Comptes Rendus*, vol. 158, 1914, pp. 1012–1014.
4. B.C. Berndt, "The Number of Zeros for $\zeta(s)$ in the Critical Strip," *Journal of the London Mathematical Society*, 1970.
5. R. Beals and R. Wong, *Special Functions and Orthogonal Polynomials*, Cambridge University Press, 2016.

fig1_phase_flip.pdf

Figure 1: Wrapped vs Unwrapped Phase $\vartheta(t)$. The $\pm\pi$ flip occurs at the Riemann zero around $t \approx 14.13$, with unwrapping revealing continuous phase behavior.

6. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1927.

7. L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.

8. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1987.

9. J. Sondow and D. Marques, "The Euler–Mascheroni constant and the Riemann zeta function," *The American Mathematical Monthly*, 2010.

10. A. Zygmund, *Trigonometric Series*, Cambridge University Press, 2002.

fig2_extended_phase.pdf

Figure 2: Corrected Phase Field $\vartheta(t)$ from $t = 14$ to $t = 50$. The flip structure and symbolic recurrence are clearly visible, demonstrating phase quantization and regular zero spacing.