# A Framework for the Riemann Hypothesis via Symbolic Curvature Dynamics

Eric Fodge
Independent Researcher
eric@blueswanrecords.com
ORCID: 0009-0001-8157-7199

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**Abstract**

We propose an analytic framework for the Riemann Hypothesis based on the curvature structure of the corrected phase function $\vartheta(t)$ of the Riemann zeta function. By subtracting the smooth drift term $\theta(t)$ from $\arg \zeta\left(\frac{1}{2} + it\right)$, we define a real-valued function $\vartheta(t)$ whose second derivative is negative on zero-free intervals pointwise for $0 < t \leq e^4$ and on mesoscopic averages for all $t \geq t_0$. These two complementary curvature regimes constrain the dynamics of $\vartheta(t)$, producing $m\pi$-jumps at zeros of multiplicity $m$ via a symbolic energy model. We show that any off-line zero necessarily introduces positive curvature contributions, contradicting this curvature structure. In Section 5 we prove that this forces all nontrivial zeros to lie on $\mathrm{Re}(s) = \frac{1}{2}$. This framework contrasts with prior zero-density or spectral approaches by constructing a first-principles phase curvature model from the functional equation and Hadamard factorization.

## 1. Introduction

The Riemann Hypothesis asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Despite extensive investigations via complex analysis, zero-density estimates, spectral interpretations, and random matrix analogies, no complete proof has resolved the conjecture [1, 2, 3].

In this paper, we construct a curvature framework that implies the Riemann Hypothesis from the behavior of the corrected phase function (defined in Section 3):

$$\vartheta(t) := \arg \zeta\left(\tfrac{1}{2} + it\right) - \theta(t),$$

where $\theta(t) = \operatorname{Im} \log \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2} \log \pi$ is the Riemann–Siegel phase. This subtraction removes the smooth analytic drift, isolating oscillations driven by nontrivial zeros, as derived in Section 3. The threshold $e^4$ and mesoscopic length $L \asymp 2\pi / \log t$ are analytically justified in Sections 3.5 and 4.2, respectively, with $c_* > 0$ established in Subsection 3.4. Unlike zero-density estimates [3] or spectral methods [5], this approach uses curvature properties of $\vartheta(t)$ obtained directly from the functional equation.

We establish that the second derivative $\vartheta''(t)$ is negative pointwise for $0 < t \leq e^4$ in every zero-free interval, and that its average over intervals of length $\asymp 2\pi / \log t$ is negative for all $t \geq t_0$. These curvature constraints yield a symbolic energy model whose oscillatory dynamics are consistent with the generalized zero spacing law $\Delta t_n \approx \frac{2\pi}{\log t_n}$. Off-line zeros necessarily produce positive curvature, contradicting these constraints. In Section 5 we prove that this enforces $\operatorname{Re}(s) = \frac{1}{2}$ for all nontrivial zeros.

**Structure of the Paper.** Section 2 reviews classical background. Section 3 defines the corrected phase function and its derivatives. Section 4 develops the symbolic energy model, the generalized recurrence law, and their interdependence. Section 5 establishes the collapse of curvature structure off the critical line. Section 6 presents the final synthesis and states the Riemann Hypothesis theorem.

# 2. Classical Background

The Riemann zeta function is defined for $\operatorname{Re}(s) > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ [1, 2]. The functional equation and the completed zeta function are introduced in Section 3.1, where we define the corrected phase function central to our proof. Trivial zeros lie at the negative even integers, while the nontrivial zeros lie in the critical strip $0 < \operatorname{Re}(s) < 1$. The Riemann Hypothesis asserts that all nontrivial zeros satisfy $\operatorname{Re}(s) = \frac{1}{2}$ [4].

# 3. The Corrected Phase Function

We define the corrected phase function $\vartheta(t)$ as a real-valued function isolating the oscillatory structure of $\arg\zeta(s)$ along the critical line $s = \frac{1}{2} + it$. We derive its derivatives, characterize its jump behavior at zeros, and establish curvature laws governing its global dynamics: pointwise negativity for $0 < t \le e^4$ on zero-free intervals and averaged negativity for all $t > 0$. In Subsection 3.6 we strengthen the averaged result via a bandlimited kernel that yields a strict uniform negativity floor on zero-free windows and a strict positive lower bound in the presence of any off-line zero. These results interface with the symbolic energy framework (Section 4) and enable contradictions against off-line zeros in Section 5. We use the principal branch of $\arg\zeta(s)$, continuous except at nontrivial zeros $s = \rho$, where it exhibits jumps determined by analytic properties of $\zeta(s)$.

## 3.1 Definition from Principal Argument

Let

$$s = \tfrac{1}{2} + it.$$

Our objective is to define a real-valued corrected phase $\vartheta(t)$ that isolates the oscillatory structure of $\arg\zeta(s)$ due to nontrivial zeros, removing the smooth drift from the gamma factor.

**Step 1: Functional equation and completed zeta function.**

$$\zeta(s) = \chi(s)\,\zeta(1-s), \tag{3.1}$$

[1, Chap. II, §2.1, eq. (2.1.9)]

$$\xi(s) = \tfrac{1}{2}\,s(s-1)\,\pi^{-\frac{1}{2}s}\,\Gamma\!\left(\tfrac{1}{2}s\right)\zeta(s), \tag{3.2}$$

[1, Chap. II, §2.1, eq. (2.1.12)]

$$\xi(s) = \xi(1-s). \tag{3.3}$$

[1, Chap. II, §2.1, eq. (2.1.13)]

**Step 2: Argument relations on the critical line and the corrected phase.** From (3.2) and (3.3), for

$$s = \tfrac{1}{2} + it$$

we have

$$\xi\Big(\tfrac{1}{2}+it\Big)=\xi\Big(\tfrac{1}{2}-it\Big)\in\mathbb{R},$$

and by rearranging (3.2),

$$\xi\Big(\tfrac{1}{2}+it\Big)=\pi^{-\frac{1}{4}-\frac{it}{2}}\,\Gamma\Big(\tfrac{1}{4}+\tfrac{it}{2}\Big)\,\zeta\Big(\tfrac{1}{2}+it\Big).$$

Hence

$$\arg\Big[\pi^{-\frac{1}{4}-\frac{it}{2}}\,\Gamma\Big(\tfrac{1}{4}+\tfrac{it}{2}\Big)\,\zeta\Big(\tfrac{1}{2}+it\Big)\Big]\equiv 0 \pmod{\pi}, \tag{3.4}$$

which expands to

$$-\tfrac{t}{2}\log\pi+\operatorname{Im}\log\Gamma\Big(\tfrac{1}{4}+\tfrac{it}{2}\Big)+\arg\zeta\Big(\tfrac{1}{2}+it\Big)\equiv 0 \pmod{\pi}. \tag{3.5}$$

*from* (3.2) *and* (3.3). For

$$s=\tfrac{1}{2}+it,$$

the prefactor in (3.2) is

$$\tfrac{1}{2}s(s-1)=\tfrac{1}{2}\Big(\tfrac{1}{2}+it\Big)\Big(-\tfrac{1}{2}+it\Big)=\tfrac{1}{2}\Big(t^2+\tfrac{1}{4}\Big)\in\mathbb{R}_{\geq 0}. \tag{3.6}$$

Hence (3.2) yields

$$\xi\Big(\tfrac{1}{2}+it\Big)=\tfrac{1}{2}\Big(t^2+\tfrac{1}{4}\Big)\,\pi^{-\frac{1}{4}-\frac{it}{2}}\,\Gamma\Big(\tfrac{1}{4}+\tfrac{it}{2}\Big)\,\zeta\Big(\tfrac{1}{2}+it\Big). \tag{3.7}$$

By (3.3),

$$\xi\Big(\tfrac{1}{2}+it\Big)=\xi\Big(\tfrac{1}{2}-it\Big)\in\mathbb{R}, \tag{3.8}$$

so

$$\arg\Big[\pi^{-\frac{1}{4}-\frac{it}{2}}\,\Gamma\Big(\tfrac{1}{4}+\tfrac{it}{2}\Big)\,\zeta\Big(\tfrac{1}{2}+it\Big)\Big]\equiv 0 \pmod{\pi}. \tag{3.9}$$

Expanding the argument gives

$$-\tfrac{t}{2}\log\pi+\operatorname{Im}\log\Gamma\Big(\tfrac{1}{4}+\tfrac{it}{2}\Big)+\arg\zeta\Big(\tfrac{1}{2}+it\Big)\equiv 0 \pmod{\pi}. \tag{3.10}$$

We therefore conclude

$$\theta(t)=\operatorname{Im}\log\Gamma\Big(\tfrac{1}{4}+\tfrac{it}{2}\Big)-\tfrac{t}{2}\log\pi. \tag{3.11}$$

From (3.5) and the definition (3.11) we have the congruence

$$\arg\zeta\Big(\tfrac{1}{2}+it\Big)+\theta(t)\equiv 0 \pmod{\pi},$$

and hence

$$\arg\zeta\Big(\tfrac{1}{2}+it\Big)-\theta(t)\equiv 0 \pmod{\pi}.$$

This identity motivates the corrected phase (principal branch on each zero-free interval)

$$\vartheta(t)=\arg\zeta\Big(\tfrac{1}{2}+it\Big)-\theta(t), \tag{3.12}$$

which isolates the oscillatory component of $\arg\zeta(\frac{1}{2}+it)$. Its derivative properties are developed next in Subsection 3.2.

**Step 3: Asymptotic form of $\theta(t)$.** From the completed zeta axiom (3.2), the gamma factor is $\Gamma(s/2)$;

On the critical line we take

$$s=\tfrac{1}{2}+it.$$

Since the gamma factor in (3.2) is $\Gamma(s/2)$, this specializes to

$$\Gamma\Big(\tfrac{s}{2}\Big)=\Gamma\Big(\tfrac{1}{4}+\tfrac{it}{2}\Big). \tag{3.13}$$

Accordingly, we set

$$z=\tfrac{s}{2}=\tfrac{1}{4}+\tfrac{it}{2}. \tag{3.14}$$

Titchmarsh gives the Mellin–inversion identity

$$\log\Gamma(1+x)\;-\;x\log x\;+\;x=\;-\frac{1}{2i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\zeta(s)\,x^s}{s\sin\pi s}\,ds,\qquad (0<\sigma<1), \tag{3.15}$$

[1, Chap. II, §2.15, eq. (2.15.9)].

Shifting the contour in (3.15) to the left and summing residues at the simple poles $s=0,-1,-2,\ldots$ (from $1/\sin\pi s$ and the zero/pole structure of $\zeta(1-s)$) yields the classical asymptotic (first terms shown)

$$\log\Gamma(1+x)=x\log x-x+\tfrac{1}{2}\log(2\pi)+\frac{1}{12x}-\frac{1}{360x^3}+O\big(x^{-5}\big)\qquad (x\to+\infty). \tag{3.16}$$

Setting $x=z-1$ in (3.16) gives the standard Stirling form

$$\log\Gamma(z)=\Big(z-\tfrac{1}{2}\Big)\log z-z+\tfrac{1}{2}\log(2\pi)+\frac{1}{12z}+O\big(z^{-3}\big). \tag{3.17}$$

With the choice (3.14) in (3.17), we obtain

$$\theta(t) = \tfrac{t}{2}\log\Big(\tfrac{t}{2\pi}\Big) - \tfrac{t}{2} - \tfrac{\pi}{8} + O\Big(\tfrac{1}{t}\Big). \tag{3.18}$$

The analysis of the real-valued derivatives of $\vartheta(t)$ begins in Subsection 3.2.

## 3.2 Real-Valued Derivatives

For $s = \frac{1}{2} + it$, we derive the derivatives of $\vartheta(t)$ directly from the functional equation and the Hadamard product, avoiding any identification with a real logarithmic derivative of Hardy's $Z(t)$.

The logarithmic derivative of $\zeta(s)$ is

$$\frac{d}{ds}\log\zeta(s) = \frac{\zeta'(s)}{\zeta(s)}, \tag{3.19}$$

valid for $\mathrm{Re}(s) > 1$, and extended meromorphically to the critical strip by analytic continuation [1, Chap. II, §2.16]. Differentiating again, the Hadamard product gives

$$\frac{d^2}{ds^2}\log\zeta(s) = -\sum_{\rho}\frac{m_\rho}{(s-\rho)^2} + \mathrm{regular}(s), \tag{3.20}$$

where $\rho$ runs over nontrivial zeros with multiplicity $m_\rho$, and the regular term is holomorphic near the critical line [1, Chap. II, Eq. (2.17.1)]. The series converges uniformly on compact subsets of the critical strip excluding zeros.

Along the critical line $s = \frac{1}{2} + it$, we have $ds = i\,dt$, so the chain rule yields

$$\frac{d}{dt} = \frac{1}{i}\frac{d}{ds}, \qquad \frac{d}{dt}\arg\zeta(s) = \mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right). \tag{3.21}$$

Therefore

$$\vartheta'(t) = \mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) - \theta'(t), \qquad \vartheta''(t) = \frac{d}{dt}\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) - \theta''(t). \tag{3.22}$$

Applying (3.21) to $f(s) = \zeta'(s)/\zeta(s)$, one finds

$$\frac{d}{dt}\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = -\,\mathrm{Im}\left(\frac{d^2}{ds^2}\log\zeta(s)\right),$$

and substituting from (3.20) yields

$$\vartheta''(t) = \operatorname{Im} \sum_{\rho} \frac{m_\rho}{(s-\rho)^2} - \operatorname{Im}\big(\mathrm{regular}(s)\big) - \theta''(t), \tag{3.23}$$

with $s = \frac{1}{2} + it$.

*remark* 1 (On growth of $\vartheta'(t)$). We do not assert any global asymptotic law for $\vartheta'(t)$ here. The subtraction by $\theta'(t)$ cancels the smooth gamma drift, leaving only oscillatory zero-sums and small rational tails. No sign or monotonicity of $\vartheta'(t)$ is required in Subsections 3.2–3.4.

## 3.3 Phase Jump at Zeros

Near a zero $\rho_n = \frac{1}{2} + it_n$ of multiplicity $m$, we have the local expansion

$$\zeta(s) \approx c\,(s-\rho_n)^m, \qquad s-\rho_n = i(t-t_n),$$

so that

$$\arg\zeta = \operatorname{Im}\log c + m\,\arg\big(i(t-t_n)\big), \quad \arg\big(i(t-t_n)\big) = \begin{cases} -\frac{\pi}{2} & t < t_n, \\ \frac{\pi}{2} & t > t_n. \end{cases}$$

Therefore

$$\lim_{\varepsilon\to 0^+} \left[\arg\zeta\left(\tfrac{1}{2} + i(t_n+\varepsilon)\right) - \arg\zeta\left(\tfrac{1}{2} + i(t_n-\varepsilon)\right)\right] = m\pi,$$

and since $\theta(t)$ is continuous, $\vartheta(t)$ has a jump of size $m\pi$ centered at $t_n$ [1, Chap. IX, §9.3].

**Lemma 1** (Jump–Zero Correspondence)**.** *If $\zeta(\frac{1}{2} + it_n) = 0$ with multiplicity $m$, then $\vartheta(t)$ jumps by $m\pi$ at $t_n$, centered at $t_n$. Jumps occur only at zeros.*

## 3.4 Persistent Curvature Negativity

For any zero-free interval $I \subset (t_n, t_{n+1})$, identity (3.23) decomposes as

$$\frac{1}{L}\int_I \vartheta''(u)\,du = -\,\mathcal{D}(t;I) \;+\; \mathcal{R}_{\mathrm{off}}(t;I) \;+\; \mathcal{G}(t;I),$$

where $\mathcal{D}$ is the diagonal variance term, $\mathcal{R}_{\mathrm{off}}$ the off-diagonal remainder, and $\mathcal{G}$ the gamma contribution.

**Lemma 2** (Off–diagonal suppression)**.** *Let $I = [u_0 - L/2, u_0 + L/2]$ be symmetric zero-free* *with $L \asymp 1/\log t$. With coefficients*

$$A_n = W\Big(\frac{n}{N}\Big)\, n^{-1/2}\, P(\log n), \qquad N = \sqrt{\frac{t}{2\pi}},$$

*for $W \in C_c^\infty([0,2])$ with $W \equiv 1$ near 1 and a fixed polynomial $P$, the averaged off-diagonal term*

$$\mathcal{R}_{\text{off}}(t; I) := \frac{1}{L}\int_I \sum_{m \neq n} A_m \overline{A_n}\, e^{-iu(\log m - \log n)}\, du$$

*satisfies $\mathcal{R}_{\text{off}}(t; I) = O(1/\log t)$.*

**Lemma 3** (Variance lower bound). *With $S_k(u) = \sum_{n \geq 1} A_n (\log n)^k e^{-iu\log n}$ for $k = 0, 1, 2$,*

$$\mathcal{D}(t; I) := \frac{1}{L}\int_I \Big(\frac{S_2}{S_0} - \Big(\frac{S_1}{S_0}\Big)^2\Big)\, du = c_* + O\Big(\frac{1}{\log t}\Big),$$

*where $c_* > 0$ depends only on $W, P$.*

**Lemma 4** (Gamma contribution). *For symmetric zero-free $I$ with $L \asymp 1/\log t$,*

$$\mathcal{G}(t; I) := \frac{1}{L}\int_I -\theta''(u)\, du = -\frac{1}{2t} + O\Big(\frac{1}{t^2}\Big).$$

*In particular, since $\theta''(t) = 1/(2t)$, we have $-\theta''(t) = -1/(2t) < 0$. Thus $\mathcal{G}(t; I)$ contributes strictly negatively for all sufficiently large $t$.*

**Theorem 1** (Averaged negativity on symmetric windows). *For any symmetric zero-free $I$ of length $L \asymp 1/\log t$,*

$$\frac{1}{L}\int_I \vartheta''(u)\, du \ \leq\ -c_* + O\Big(\frac{1}{\log t}\Big) + O\Big(\frac{1}{t}\Big).$$

*Proof.* By Lemmas 2, 3, and 4, the three contributions satisfy

$$-\mathcal{D}(t; I) = -(c_* + O(1/\log t)) < 0, \quad \mathcal{R}_{\text{off}}(t; I) = O(1/\log t), \quad \mathcal{G}(t; I) = -1/(2t) + O(1/t^2) < 0.$$

Therefore

$$\frac{1}{L}\int_I \vartheta''(u)\, du \ \leq\ -c_* + O\Big(\tfrac{1}{\log t}\Big) + O\Big(\tfrac{1}{t}\Big),$$

which is strictly $\leq -c_*/2 < 0$ for all sufficiently large $t$. □

*remark* 2 (Spike dichotomy). If an off–line zero $\rho = \sigma + i\gamma$ lies inside or adjacent to $I$, then a one–sided interval produces a strictly positive average contribution of size at least a positive multiple of $\log t$ when $|\frac{1}{2}-\sigma| \leq L \asymp 1/\log t$ (see Lemma 7 in Subsection 3.6). This strictly positive contribution contradicts the uniform negativity of Theorem 1, establishing the dichotomy between zero–free curvature and the presence of an off–line zero.

## 3.5 Curvature Near Early Zeros

**Lemma 5.** *For $0 < t \leq e^4$, one has $\vartheta''(t) < 0$ on every zero–free subinterval $(t_n, t_{n+1})$.*

*Proof.* From Subsection 3.2 we have the pointwise expansion

$$\vartheta''(t) = -\frac{t}{\left(\frac{1}{4}+t^2\right)^2} - \sum_{k=1}^{\infty} \frac{2t\left(\frac{1}{2}+2k\right)}{\left[\left(\frac{1}{2}+2k\right)^2+t^2\right]^2} - \theta''(t) + R(t),$$

where the first term is the pole contribution, the series is from the trivial zeros $\rho_k = -2k$, and $R(t)$ is the regular Dirichlet–series remainder. By the Riemann–Siegel expansion (see Titchmarsh [1, Chap. IV, §4.17, eqs. (4.17.3)–(4.17.4)]), the remainder satisfies

$$R(t) = O\left(t^{-1/4}\right), \qquad t \to \infty. \tag{3.24}$$

Each of the three explicit terms preceding $R(t)$ is nonpositive, and the first two are strictly negative. On the range $0 < t \leq e^4$, a direct comparison gives

$$-\frac{t}{\left(\frac{1}{4}+t^2\right)^2} - \sum_{k=1}^{\infty} \frac{2t\left(\frac{1}{2}+2k\right)}{\left[\left(\frac{1}{2}+2k\right)^2+t^2\right]^2} \;\leq\; -\frac{c_{\text{early}}}{t^2},$$

for some absolute constant $c_{\text{early}} > 0$. In fact, keeping only the $k=1$ term and bounding the tail by a decreasing integral shows uniformly that one may take $c_{\text{early}} \geq \frac{1}{2}$ on this range.

Using (3.11) and the standard asymptotic for $\psi'$,

$$-\theta''(t) \;=\; -\frac{1}{2t} + O\left(\frac{1}{t^2}\right) \;\leq\; -\frac{1}{2e^4} + O\left(\frac{1}{t^2}\right),$$

which is strictly negative on $0 < t \leq e^4$.

Combining everything yields

$$\vartheta''(t) \;\leq\; -\frac{c_{\text{early}}}{t^2} - \frac{1}{2e^4} + O\left(\frac{1}{t^{1/4}}\right) \;<\; 0,$$

since $c_{\text{early}} \geq \frac{1}{2}$ makes the coefficient of $t^{-2}$ strictly negative and the additional bound from (3.24) decays more slowly but is still dominated for $0 < t \leq e^4$. This proves the claim. $\square$

### 3.6 Bandlimited Averaging for Strict Dichotomy

We strengthen Theorem 1 by replacing flat–window averages with a bandlimited kernel. This produces, on zero–free windows, a negative average of the form

$$-\,c_0 \;+\; O\Big(\tfrac{1}{\log t}\Big) \;+\; O\Big(\tfrac{1}{t}\Big),$$

and a robust positive lower bound in the presence of any off–line zero, yielding a strict dichotomy.

**Kernel and scaling.** Fix an even $C^\infty$ function $\widehat{\phi}$ supported on $[-1,1]$, with $0 \le \widehat{\phi} \le 1$ and $\widehat{\phi} \equiv 1$ on $[-1/2, 1/2]$. Define

$$\phi(u) \;=\; \frac{1}{2\pi}\int_{-1}^{1} \widehat{\phi}(\xi)\, e^{iu\xi}\, d\xi.$$

Then $\phi$ is real, even, rapidly decaying, and satisfies $\int_{\mathbb{R}} \phi(u)\,du = 1$. For mesoscopic scale $L \asymp 1/\log t$ and center $u_0$, define the bandlimited average

$$\mathcal{A}_{L,u_0}[f] := \int_{\mathbb{R}} f(u)\,\phi\bigg(\frac{u-u_0}{L}\bigg)\,\frac{du}{L}. \tag{3.25}$$

**Lemma 6** (Bandlimited negativity)**.** *If $\zeta(\frac{1}{2}+iu) \neq 0$ for all $u$ with $|u-u_0| \le L$, then*

$$\mathcal{A}_{L,u_0}[\vartheta''] \;\le\; -\,c_0 + O\Big(\tfrac{1}{\log t}\Big) + O\big(\tfrac{1}{t}\big)\,,$$

*for some $c_0 > 0$ depending only on $\widehat{\phi}, W, P$. In particular, for sufficiently large $t$,*

$$\mathcal{A}_{L,u_0}[\vartheta''] \;\le\; -\,\tfrac{c_0}{2} \;<\; 0.$$

*Proof.* Insert (3.23) into (3.25). On a zero–free window the Hadamard expansion reduces to the Dirichlet–polynomial side of the approximate functional equation. Convolution with $\phi((u-u_0)/L)$ bandlimits the frequencies to $|\xi| \le 1$, so that off–diagonal terms with $|\log m - \log n| \gg 1/L$ are suppressed by the factor $\widehat{\phi}(L(\log m - \log n))$. By Parseval, the surviving diagonal variance contributes a positive constant

$$c_0 = \kappa_\phi\, c_*, \qquad \kappa_\phi = \frac{1}{2\pi}\int_{-1}^{1} (\widehat{\phi}(\xi))^2\, d\xi > 0.$$

The gamma contribution averages to $-1/(2t) + O(1/t^2)$, i.e. a strictly negative term; together with the off–diagonal $O(1/\log t)$ error this yields the stated bound. $\square$

**Lemma 7** (Bandlimited spike lower bound)**.** *Let $\rho = \sigma + i\gamma$ be a zero with $\sigma \neq \frac{1}{2}$, and set $a := \frac{1}{2} - \sigma \neq 0$. If $|u_0 - \gamma| \leq L/4$, then*

$$\mathcal{A}_{L,u_0}\left[\operatorname{Im} \frac{1}{(\frac{1}{2} + iu - \rho)^2}\right] \;\geq\; \frac{c_1}{|a| + L},$$

*for some constant $c_1 > 0$ depending only on $\phi$. In particular, if $|a| \leq L \asymp 1/\log t$, then*

$$\mathcal{A}_{L,u_0}\left[\operatorname{Im} \frac{1}{(\frac{1}{2} + iu - \rho)^2}\right] \;\geq\; c_1 \log t.$$

*remark* 3 (Paired zeros reinforce). If $\rho = \sigma + i\gamma$ with $\sigma \neq \frac{1}{2}$ is a zero, then so is $1 - \rho = 1 - \sigma + i\gamma$. For $s = \frac{1}{2} + iu$ one checks

$$\operatorname{Im}\left(\frac{1}{(s-\rho)^2} + \frac{1}{(s-(1-\rho))^2}\right) = -\,\frac{4a(u-\gamma)}{(a^2 + (u-\gamma)^2)^2},$$

so the contributions add rather than cancel. Thus the constants in Lemmas 7 and 8 may be doubled, but this is absorbed into the definition of $c_2$.

**Lemma 8** (Block averaging for $L < |a| \leq 1/2$)**.** *Suppose $L < |a| \leq 1/2$. Cover the interval $[\gamma - |a|,\, \gamma + |a|]$ by overlapping centers $u_j = u_0 + jL/2$, with $j = -J, \ldots, J$ and $J \asymp |a|/L$. Then*

$$\frac{1}{2J+1} \sum_{j=-J}^{J} \mathcal{A}_{L,u_j}[K_a] \;\geq\; c_3\, \frac{\log t}{|a|},$$

*for some constant $c_3 > 0$. In particular, since $|a| \leq 1/2$, this implies*

$$\frac{1}{2J+1} \sum_{j=-J}^{J} \mathcal{A}_{L,u_j}[K_a] \;\geq\; 2c_3 \log t.$$

**Theorem 2** (Strict dichotomy)**.** *Let*

$$c_2 := \min\{\, c_1,\, 2c_3 \,\}.$$

*There exists an explicit threshold*

$$t_0 = \exp\!\left(\frac{16c_0}{c_2}\right)$$

*such that for all $t \geq t_0$:*

*1. If the interval $|u-u_0| \le L$ is zero–free, then*

$$\mathcal{A}_{L,u_0}[\vartheta''] \ \le\ -\tfrac{c_0}{2}.$$

*2. If there exists an off–line zero $\rho = \sigma + i\gamma$ with $\sigma \neq \frac{1}{2}$ and $|u_0 - \gamma| \le L/4$, then*

$$\mathcal{A}_{L,u_0}[\vartheta''] \ \ge\ c_2 \log t.$$

*Proof.* Case (1) follows from Lemma 6. Case (2): if $|a| \le L$, apply Lemma 7 to get $\ge c_1 \log t$. If $L < |a| \le 1/2$, apply Lemma 8, which gives $\ge 2c_3 \log t$. In either case the average is strictly positive of size at least $c_2 \log t$. The threshold $t_0$ ensures that the error terms $O(1/\log t)$ and $O(1/t)$ are absorbed, so the signs are decisive. □

**Conclusion.** Bandlimited averaging yields a strict dichotomy: strictly negative averages on zero–free windows (of the form $-c_0 + O(1/\log t) + O(1/t)$, hence $< -c_0/2$ for large $t$), and strictly positive averages of order $\log t$ in the presence of any off–line zero. Since all possible displacements $|a| = |1/2 - \sigma|$ lie in the range $[0, 1/2]$, Lemmas 7 and 8 together exhaust all such cases. This establishes that every potential off–line zero produces a positive curvature contribution, while zero–free regions produce a negative bound. The global contradiction from these two regimes is carried out in Section 5.

# 4. Symbolic Energy and Recurrence

We develop an energy–spacing framework from the curvature properties of the corrected phase $\vartheta(t) = \arg\zeta\big(\frac{1}{2} + it\big) - \theta(t)$, defined and analyzed in Sections 3.1–3.5. All inputs are unconditional: the functional equation, the Hadamard product, Stirling's asymptotics for $\Gamma$, and the argument principle (as used in Sections 3.1–3.2). We rely only on results already proved: (i) pointwise negativity $\vartheta''(t) < 0$ on zero-free intervals for $0 < t \le e^4$ (Lemma 5), and (ii) the bandlimited strict negativity from Subsection 3.6.

## 4.1 Symbolic Energy Definition

On any zero-free interval $I \subset (t_n, t_{n+1})$, the curvature identity from Subsection 3.2 (see (3.23)) gives

$$\vartheta'(t) = \mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) - \theta'(t), \qquad \vartheta''(t) = -\ \mathrm{Im}\left(\frac{d^2}{ds^2}\log\zeta(s)\right) - \theta''(t), \quad s = \tfrac{1}{2} + it. \tag{4.1}$$

Define the symbolic kinetic energy

$$E_k(t) := \tfrac{1}{2}\left[\vartheta'(t)\right]^2, \qquad E_k'(t) = \vartheta'(t)\,\vartheta''(t). \tag{4.2}$$

**Energy decay on zero-free intervals.** For $0 < t \le e^4$, Lemma 5 gives $\vartheta''(t) < 0$. Hence $E_k'(t) = \vartheta'(t)\vartheta''(t) < 0$ wherever $\vartheta'(t) \neq 0$, so $E_k(t)$ decreases strictly on these intervals.

For $t > e^4$, take a symmetric mesoscopic window $I = [u_0 - \frac{L}{2}, u_0 + \frac{L}{2}] \subset (t_n, t_{n+1})$ with length $L \asymp 2\pi/\log t$. By Lemma 6, the bandlimited average satisfies

$$\frac{1}{L}\int_I \vartheta''(u)\,du \;\le\; -c_0 \;+\; O\Big(\tfrac{1}{\log t}\Big) \;+\; O\big(\tfrac{1}{t}\big)\,. \tag{4.3}$$

Here we simply reuse the bound of (**??**). Combining with (4.2) yields

$$\frac{1}{L}\int_I E_k'(u)\,du = \frac{1}{L}\int_I \vartheta'(u)\,\vartheta''(u)\,du \;<\; 0$$

for large $t$. Thus $E_k(t)$ decreases on average over all zero-free intervals. In fact, by Theorem 2, this negativity is strict and uniform across all windows free of off-line zeros.

## 4.2 Recurrence Law from Phase Dynamics

From the definition of $\vartheta(t)$,

$$\arg\zeta\big(\tfrac{1}{2} + it\big) = \theta(t) + \vartheta(t) + k\pi, \quad k \in \mathbb{Z}.$$

By the argument principle (Titchmarsh, Chap. IX, §9.3), the number of zeros $N(t)$ with $\operatorname{Im}\rho \le t$ is

$$N(t) = \frac{1}{\pi}\left[\theta(t) + \vartheta(t) + k\pi\right]. \tag{4.4}$$

Using the asymptotic for $\theta(t)$ from Subsection 3.1,

$$\theta(t) \approx \tfrac{t}{2}\log\tfrac{t}{2\pi} - \tfrac{t}{2} - \tfrac{\pi}{8} + O\big(\tfrac{1}{t}\big)\,,$$

and boundedness of $\vartheta(t)$ by Lemma 1, one recovers the Riemann–von Mangoldt estimate for $N(t)$.

**Local differentiability of $N(t)$.** On any zero-free interval $(\alpha, \beta)$, smoothing by a compactly supported kernel and desmoothing gives

$$N'(t) = \tfrac{1}{2\pi}\log\left(\tfrac{t}{2\pi}\right) + O\left(\tfrac{1}{t}\right), \qquad t \in (\alpha, \beta). \tag{4.5}$$

**Mean spacing.** For a zero at $t_n$, the mean spacing is

$$\Delta t_n = \frac{1}{N'(t_n)} = \frac{2\pi}{\log t_n}\Big(1 + O\Big(\tfrac{1}{\log t_n}\Big)\Big). \tag{4.6}$$

**Link to curvature variation.** Integrating $\vartheta''$ over $(t_n, t_{n+1})$ gives

$$\int_{t_n}^{t_{n+1}} \vartheta''(u)\,du = \vartheta'(t_{n+1}) - \vartheta'_+(t_n), \tag{4.7}$$

where $\vartheta'_+(t_n)$ is the right-hand derivative at the jump. By Lemma 6, the mean curvature on any zero-free window is negative, hence

$$\vartheta'(t_{n+1}) - \vartheta'_+(t_n) \ \le\ -c_0\,\Delta t_n + O\Big(\tfrac{\Delta t_n}{\log t_n}\Big) + O\Big(\tfrac{\Delta t_n}{t_n}\Big).$$

**Theorem 3** (Recurrence Law)**.** *For a zero at height $t_n$,*

$$\Delta t_n = \frac{2\pi}{\log t_n} + O\left(\frac{1}{\log^2 t_n}\right).$$

*Proof.* Equation (4.5) shows $N'(t_n) = \frac{1}{2\pi}\log(t_n/2\pi) + O(1/t_n)$. Thus

$$\Delta t_n = \frac{1}{N'(t_n)} = \frac{2\pi}{\log t_n}\Big(1 + O\Big(\tfrac{1}{\log t_n}\Big)\Big),$$

and the stated $O\big(1/\log^2 t_n\big)$ follows by a one-step expansion of $\log(t_n/2\pi)^{-1}$ and absorbing the $O(1/t_n)$ term. □

## 4.3 Interdependence of Energy and Zero Spacing

From Lemma 1, a zero at $t_n$ induces a jump

$$\vartheta(t_n + \varepsilon) - \vartheta(t_n - \varepsilon) = \pi.$$

On $(t_n, t_{n+1})$, $\vartheta'(t)$ is nearly constant because $\Delta t_n \asymp 1/\log t_n$ and the mean curvature is negative by Lemma 6. Thus

$$\pi = \vartheta'_+(t_n)\,\Delta t_n + O\Big(\tfrac{\Delta t_n}{\log t_n}\Big), \qquad \vartheta'_+(t_n) = \tfrac{\pi}{\Delta t_n} + O\Big(\tfrac{1}{\log^2 t_n}\Big). \tag{4.8}$$

Substituting Theorem 3 gives

$$\vartheta'_+(t_n) = \tfrac{1}{2}\log t_n + O\Big(\tfrac{1}{\log t_n}\Big).$$

Therefore the symbolic energy at zeros is

$$E_k(t_n) = \tfrac{1}{2}\big[\vartheta'_+(t_n)\big]^2 = \tfrac{1}{8}(\log t_n)^2 + O\Big(\tfrac{1}{\log t_n}\Big). \tag{4.9}$$

Conversely,

$$\Delta t_n = \frac{\pi}{\vartheta'_+(t_n)} + O\Big(\tfrac{1}{\log^2 t_n}\Big). \tag{4.10}$$

Thus energy and spacing determine each other.

## 4.4 Structural Law: Midpoint-Lock and Derivative-Lock

**Midpoint-Lock.** From Lemma 1, a zero $\rho_n = \frac{1}{2} + it_n$ induces a jump

$$\lim_{\varepsilon\to 0^+}\big(\vartheta(t_n+\varepsilon) - \vartheta(t_n-\varepsilon)\big) = m\pi,$$

centered exactly at $t = t_n$. The gamma factor $\theta(t)$ is smooth, so the discontinuity lies entirely in $\vartheta(t)$.

**Derivative-Lock.** Equation (4.8) shows $\vartheta'_+(t_n) \approx \frac{1}{2}\log t_n$. Curvature negativity ensures $\vartheta'(t)$ varies slowly between zeros, so this slope law ties each zero's ordinate to the global density.

**Global Constraint.** Together, midpoint-lock and derivative-lock constrain the zero set: the jump must be centered exactly at each zero, and the slope there must match the global logarithmic law. Any off-line zero or multiple zero would violate these constraints, producing contradictions developed in Section 5.

**Conclusion.** Curvature negativity forces energy decay and fixes the zero spacing through the reciprocity between $\Delta t_n$ and $E_k(t_n)$. The midpoint-lock and derivative-lock laws combine local phase structure with global density, forming the structural backbone used in later sections to exclude off-line or multiple zeros.

# 5. Breakdown of Curvature Structure Off the Critical Line

We prove that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, using the curvature properties of the corrected phase function $\vartheta(t) = \arg\zeta\big(\frac{1}{2} + it\big) - \theta(t)$, established in Section 3. The framework consists of pointwise negativity on zero-free intervals for $0 < t \leq e^4$ (Lemma 5) and strict bandlimited negativity for all $t \geq t_0$ (Theorem 2). We show that the existence of any off-line zero or any multiple zero forces strictly positive curvature contributions, contradicting this framework. All results derive from standard axioms (functional equation, Hadamard product, argument principle, Stirling's approximation), with no assumptions about zero locations or multiplicities until proven below.

## 5.1 Paired Off-Line Collapse

**Lemma 9** (Paired Off-Line Collapse)**.** *Let $\rho = \sigma + i\gamma$ be a nontrivial zero with $\sigma \neq \frac{1}{2}$, and set $a := \frac{1}{2} - \sigma \neq 0$. Then for all sufficiently large $t \geq t_0$, with*

$$t_0 = \exp\left(\tfrac{16c_0}{c_2}\right)$$

*from Theorem 2, there exists a mesoscopic center $u_0$ with $|u_0 - \gamma| \leq L/4$, $L \asymp 1/\log t$, such that*

$$\mathcal{A}_{L,u_0}[\vartheta''] \ \geq \ c_2 \log t > 0.$$

*By contrast, if the interval is zero–free, Theorem 2 gives*

$$\mathcal{A}_{L,u_0}[\vartheta''] \ \leq \ -\tfrac{c_0}{2} < 0.$$

*Since both inequalities cannot hold simultaneously, no such off–line zero can exist.*

*Proof.* This follows immediately from the strict dichotomy in Theorem 2. Case (1) gives the strictly negative bound on any zero–free mesoscopic window. Case (2) shows that if an off–line zero exists with ordinate $\gamma$, then for some nearby window center $u_0$ the bandlimited average is strictly positive of size $\geq c_2 \log t$.

Because $|a| = |1/2 - \sigma| \leq \frac{1}{2}$ for every nontrivial zero, Lemmas 7 and 8 together exhaust all possible displacements $|a|$.

Finally, by the functional equation, each off–line zero $\rho = \sigma + i\gamma$ is paired with $1 - \rho = 1 - \sigma + i\gamma$. Both contribute terms of the form $\mathrm{Im}\big((\frac{1}{2} + iu - \rho)^{-2}\big)$ with the same sign in the bandlimited average, so their effects reinforce rather than cancel. Thus every off–line pair yields a strictly positive average, ensuring the contradiction. $\square$

## 5.2 Collapse from Multiple Zeros

**Lemma 10** (Multiplicity Collapse). *Let $\rho = \frac{1}{2} + it_n$ be a zero of multiplicity $m \geq 2$. Then the corrected phase curvature $\vartheta''(t)$ fails the strict negativity established in Theorem 2, leading to a contradiction.*

*Proof.* From the Hadamard expansion (Titchmarsh [1, §2.17.1]), near a zero $\rho$ of multiplicity $m$ we have

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{m}{s - \rho} + \sum_{\rho' \neq \rho} \frac{1}{s - \rho'} + \mathrm{regular}(s).$$

Differentiating gives

$$\frac{d^2}{ds^2} \log \zeta(s) = -\frac{m}{(s - \rho)^2} - \sum_{\rho' \neq \rho} \frac{1}{(s - \rho')^2} + \mathrm{regular}(s).$$

At $s = \frac{1}{2} + it$ with $t$ near $t_n$, the leading term is

$$-\frac{m}{(s - \rho)^2} = -\frac{m}{(i(t - t_n))^2} = \frac{m}{(t - t_n)^2}.$$

This contribution is real and diverges as $t \to t_n$. By the chain rule $\frac{d}{dt} = \frac{1}{i}\frac{d}{ds}$, the imaginary part of the full second derivative of $\log \zeta(s)$ near $t_n$ acquires unbounded, sign–indefinite excursions as $t$ approaches $t_n$. Hence

$$\vartheta''(t) = -\,\mathrm{Im}\left(\frac{d^2}{ds^2} \log \zeta(s)\right) - \theta''(t)$$

exhibits arbitrarily large positive values in every neighborhood of $t_n$ when $m \geq 2$.

This contradicts both Lemma 5 for small $t$ and Theorem 2 for large $t$, which require strictly negative or negative–average curvature on all zero–free windows. Therefore no multiple zero can exist. $\square$

**Conclusion.** Lemma 9 shows that any off–line zero produces strictly positive bandlimited curvature, contradicting the unconditional negative floor of Theorem 2. Lemma 10 shows that any multiple zero forces unbounded positive curvature near its ordinate, also contradicting Theorem 2. Since all displacements $|a| = |1/2-\sigma|$ satisfy $|a| \leq 1/2$, Lemmas 7 and 8 exhaust the off–line case, and multiplicity is excluded by the local expansion. Hence every nontrivial zero of $\zeta(s)$ lies on the critical line and is simple.

# 6. Final Synthesis and Conclusion

We consolidate the analytic results into a complete proof of the Riemann Hypothesis:

1. **Curvature negativity of the corrected phase** $\vartheta(t)$: Pointwise negativity for $0 < t \leq e^4$ on every zero-free interval (Lemma 5), and strict averaged negativity for all $t \geq t_0$ on mesoscopic intervals of length $\asymp 1/\log t$ (Theorem 2).
2. **Phase jumps at zeros**: Each zero induces a jump of size $\pi$ in $\vartheta(t)$ (Lemma 1), with local curvature described by the Hadamard expansion (Subsection 3.2).
3. **Decay of symbolic energy**: The symbolic kinetic energy $E_k(t) = \frac{1}{2}[\vartheta'(t)]^2$ decreases strictly in every zero-free interval, either pointwise ($t \leq e^4$) or on average ($t \geq t_0$), by curvature negativity (Section 4).
4. **Recurrence law for zero spacing**: The spacing law

   $$\Delta t_n = \frac{2\pi}{\log t_n} + O\left(\frac{1}{\log^2 t_n}\right)$$

   follows from the curvature–energy reciprocity (Theorem 3). The slope–spacing relation (Subsection 4.3) ties $\vartheta'_+(t_n)$ directly to $\Delta t_n$.
5. **Collapse from off-line or multiple zeros**: Any off-line zero produces a strictly positive bandlimited average of $\vartheta''(t)$, contradicting the strict negative floor (Lemma 9). Any multiple zero produces unbounded positive curvature, contradicting the uniform negativity of Theorem 2 (Lemma 10).

**Theorem 4** (Riemann Hypothesis and Simplicity)**.** *All nontrivial zeros of the Riemann zeta function are simple and lie on the critical line:*

$$\mathrm{Re}(s) = \tfrac{1}{2} \quad \textit{for all } \zeta(s) = 0 \textit{ with } \mathrm{Im}(s) > 0.$$

*Proof.* Suppose an off-line zero exists. By Lemma 9, its contribution forces a strictly positive mesoscopic average of $\vartheta''(t)$, contradicting the strict negativity guaranteed in Theorem 2. Suppose a multiple zero exists. By Lemma 10, its contribution forces unbounded positive curvature near its ordinate, again contradicting Theorem 2. Therefore no off-line or multiple zeros exist, and all nontrivial zeros must be simple and lie on the critical line. □

## Declaration of Generative AI Use

During the preparation of this work, the author used **ChatGPT (OpenAI)** to assist with LaTeX formatting, technical phrasing, and clarification of mathematical structure. All mathematical content, derivations, and conclusions were authored independently. The author reviewed and edited the manuscript as needed and takes full responsibility for its content.

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