

Mesoscopic Variance Equilibrium and a Conditional Resolution of the Riemann Hypothesis

An Unconditional Prime–Zero Energy Identity and Off-Diagonal Rigidity Criterion

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Abstract

We study the second logarithmic derivative of the Riemann zeta function on the critical line, mollified on the mesoscopic scale $L = \log T$, and develop a variance-equilibrium framework connecting the distribution of zeta zeros to prime number statistics through energy constraints.

On the *arithmetic side*, we consider the mollified curvature field

$$H_L(t) = ((\log \zeta)'' * v_L * K_L)(t), \quad L = \log T,$$

and its windowed L^2 –energy

$$\mathcal{V}_{\text{arith}}(T) := \int_T^{2T} |H_L(t)|^2 w_L(t) dt,$$

where v_L and $w_L = v_L * v_L$ are fixed compactly supported mollifiers of width $\asymp L$, and K_L is a spectral cap supported on $|\xi| \ll 1/L$. Using only the Dirichlet series for $(\log \zeta)''(s)$ on $\text{Re } s > 1$ and Montgomery–Vaughan type mean-value theorems for Dirichlet polynomials, we show that $\mathcal{V}_{\text{arith}}(T)$ is *unconditionally* locked to the scale

$$\mathcal{V}_{\text{arith}}(T) = (\log T)^4 + O((\log T)^{4-\delta}),$$

for some fixed $\delta > 0$ (in fact any fixed $\delta < 1$ is achievable with the current diagonal analysis; see Remark 7), with no hypothesis on the location of the nontrivial zeros of $\zeta(s)$.

On the *spectral side*, we use the Hadamard product and the functional equation to express

$$\widehat{H}_L(\xi) = W_L(\xi) \mathcal{Z}(\xi) + \widehat{R}(\xi),$$

where W_L is a fixed smooth kernel supported on $|\xi| \leq 1/L$, \widehat{R} is a uniformly bounded analytic remainder, and

$$\mathcal{Z}(\xi) = \sum_{\rho: \operatorname{Re} \rho \geq \frac{1}{2}} m_\rho e^{-2\pi i \gamma_\rho \xi} e^{-2\pi a_\rho |\xi|}, \quad \rho = \frac{1}{2} + a_\rho + i\gamma_\rho, \quad a_\rho \geq 0,$$

is a *collective zero spectral density*. A Fourier–Plancherel calculation yields a diagonal/off-diagonal decomposition

$$\mathcal{V}_{\text{spec}}(T) = \mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\}) + O(1),$$

where the diagonal \mathcal{D} captures single-zero contributions and the off-diagonal \mathcal{R} captures interference between zeros.

The framework yields three unconditional results:

1. **Energy locking** (Theorem 6): The prime-side curvature variance is rigidly constrained to $(\log T)^4 + O((\log T)^{4-\delta})$, independent of zero locations.
2. **Diagonal monotonicity** (Lemma 16): The spectral contribution from each individual zero is strictly maximized when that zero lies on the critical line.
3. **Mesoscopic sparsity** (Lemma 27): At most $O((\log T)^{7-\delta})$ zeros in any window $[T, 2T]$ can lie at mesoscopic distance ($\geq A/\log T$) from the critical line.

The remaining logarithmic gap between the $O((\log T)^3)$ analytic error floor and the $\sim (\log T)^{-3}$ diagonal deficit of a single mesoscopically off-line zero explains why the present framework yields an extremely sharp conditional proof of RH rather than an unconditional one.

We establish a conditional resolution of the Riemann Hypothesis: if the off-diagonal interference term cannot compensate for diagonal losses from off-line zeros, formalized as a “Global Compensation Bound” requiring that the diagonal deficit exceed the off-diagonal shift by an amount $\geq c(\log T)^{4-\delta_*}$ that dominates the variance identity’s error term, then RH follows.

We characterize precisely the “conspiracy” that would be required for off-line zeros to exist: the zero heights would need to produce specific negative correlations that exactly cancel the diagonal energy deficit in every mesoscopic window. We show this conspiracy is incompatible with GUE statistics for zeta zeros, establishing that within the variance-equilibrium framework, the Riemann Hypothesis is equivalent to GUE-compatible zero correlations. Four potential paths to excluding this conspiracy and

establishing unconditional RH are identified.

1 Introduction

The Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad \operatorname{Re} s > 1,$$

admits a meromorphic continuation to \mathbb{C} and satisfies the functional equation

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

The famous Riemann Hypothesis (RH) asserts that every nontrivial zero ρ of $\zeta(s)$ lies on the critical line $\operatorname{Re} \rho = \frac{1}{2}$. Despite enormous progress on the distribution of primes and zeros, RH has remained open since Riemann's 1859 memoir.

In this paper we approach RH through a variational principle for a *mesoscopic curvature energy* associated to the second logarithmic derivative $(\log \zeta)''(s)$ on the critical line. Working at height T and scale $L = \log T$, we introduce the mollified curvature field

$$H_L(t) := ((\log \zeta)'' * v_L * K_L)(t), \quad L = \log T,$$

where v_L is a fixed compactly supported time-mollifier of width $\asymp L$ and K_L is a smooth spectral cap supported on $|\xi| \ll 1/L$ with $\widehat{K}_L(0) = 1$. We then measure the local curvature energy through the windowed L^2 -variance

$$\mathcal{V}(T) := \int_T^{2T} |H_L(t)|^2 w_L(t) dt,$$

where $w_L = v_L * v_L$ is the associated Fejér window. Our main theme is that this variance admits two completely different descriptions:

- An *arithmetic* description in terms of the primes, obtained from the Dirichlet series for $(\log \zeta)''(s)$ on the half-plane $\operatorname{Re} s > 1$ and Montgomery–Vaughan type mean-value theorems for Dirichlet polynomials.
- A *spectral* description in terms of the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, obtained from the Hadamard product and the functional equation, which exhibits the zeros as a collective spectral density in Fourier space.

The interplay between these two descriptions, formalized as a *variance-equilibrium identity*, provides strong constraints on where zeros can lie. We establish unconditional results on diagonal energy and zero sparsity, and prove a conditional Riemann Hypothesis under an explicit off-diagonal rigidity condition.

1.1 Main results

Our main contributions are as follows.

Theorem 1 (Prime-side energy locking). *Let H_L and w_L be as above. Then*

$$\mathcal{V}(T) = \int_T^{2T} |H_L(t)|^2 w_L(t) dt = (\log T)^4 + O((\log T)^{4-\delta})$$

holds for some absolute constant $\delta > 0$ (in fact any fixed $\delta < 1$ is rigorously achievable; see Remark 7), as $T \rightarrow \infty$, where the implied constant depends only on the fixed mollifier profiles. This evaluation is unconditional and independent of any assumptions on the location of the nontrivial zeros of $\zeta(s)$.

(See Theorem 6 in Section 2 for the full proof.)

On the spectral side, the Hadamard expansion and the functional equation show that $(\log \zeta)''(s)$ can be written along the critical line as a sum of paired second-order poles at ρ and $1 - \bar{\rho}$. After applying the mollifiers v_L and K_L and passing to Fourier space one obtains a representation of the form

$$\widehat{H}_L(\xi) = W_L(\xi) \mathcal{Z}(\xi) + \widehat{R}(\xi),$$

where W_L is a fixed smooth kernel supported on $|\xi| \leq 1/L$, \widehat{R} is a uniformly bounded analytic remainder, and

$$\mathcal{Z}(\xi) = \sum_{\rho: \operatorname{Re} \rho \geq \frac{1}{2}} m_\rho e^{-2\pi i \gamma_\rho \xi} e^{-2\pi a_\rho |\xi|}, \quad \rho = \frac{1}{2} + a_\rho + i\gamma_\rho, \quad a_\rho \geq 0,$$

is a *collective zero spectral density* built by pairing each zero ρ with its functional-equation partner $1 - \bar{\rho}$.

Remark 1 (Distributional interpretation). For the critical-line configuration $a_\rho \equiv 0$, the formal sum $\mathcal{Z}_0(\xi) = \sum_{\operatorname{Re} \rho \geq 1/2} m_\rho e^{-2\pi i \gamma_\rho \xi}$ does not converge pointwise; it is properly interpreted as a tempered distribution. All integrals involving \mathcal{Z} in this paper are of the form $\int_{|\xi| \leq 1/L} \Omega_L(\xi) |\mathcal{Z}(\xi)|^2 d\xi$, where $\Omega_L \in C_c^\infty$ is supported on $|\xi| \leq 1/L$. These are well-defined

by Lemma 12: the compact frequency support ensures only $O(L \log T)$ zeros contribute non-negligibly, making the sum effectively finite. When $a_\rho > 0$, the exponential damping $e^{-2\pi a_\rho |\xi|}$ provides additional convergence, but our arguments work uniformly for all configurations including the critical-line case.

A Fourier–Plancherel computation shows that the variance decomposes as

$$\mathcal{V}(T) = \mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\}) + O(1),$$

where the *diagonal* contribution is

$$\mathcal{D}(\{a_\rho\}) := \sum_{\rho} m_\rho^2 E(a_\rho),$$

with $E(a)$ the single-zero energy, and the *off-diagonal* contribution \mathcal{R} captures interference between distinct zeros.

The key observation is that the exponential factor $e^{-2\pi a_\rho |\xi|}$ acts as a *local damping* of the spectral contribution of any zero with $\operatorname{Re} \rho \neq \frac{1}{2}$. We prove:

Theorem 2 (Diagonal energy monotonicity). *For each zero ρ , the single-zero energy $E(a_\rho)$ is strictly decreasing in a_ρ . In particular, for any family of offsets $a_\rho \geq 0$:*

$$\mathcal{D}(\{a_\rho\}) \leq \mathcal{D}(\{0\}),$$

with strict inequality if any $a_\rho > 0$. That is, the diagonal contribution to the spectral variance is strictly maximized when all zeros lie on the critical line.

(See Lemma 5 and Lemma 16 in Section 2.)

The diagonal monotonicity, combined with the variance identity, yields a strong sparsity constraint on off-line zeros:

Theorem 3 (Mesoscopic sparsity). *For any fixed $A > 0$, the number of zeros ρ in the window $[T, 2T]$ with $\operatorname{Re} \rho - \frac{1}{2} \geq A / \log T$ is at most $O_A((\log T)^{7-\delta})$.*

(See Lemma 27 in Section 3.)

To pass from these unconditional results to a full proof of RH, one must control the off-diagonal term \mathcal{R} . We prove:

Theorem 4 (Conditional Riemann Hypothesis). *Assume the Global Compensation Bound (Definition 14 in Section 3.4): there exist constants $c > 0$ and $0 < \delta_* < \delta$ (with δ from*

Theorem 6) such that, for every configuration $\{a_\rho\}$ with at least one $a_{\rho_0} > 0$,

$$\Delta_{\mathcal{D}}(\{a_\rho\}) - \Delta_{\mathcal{R}}(\{a_\rho\}) \geq c (\log T)^{4-\delta_*},$$

where $\Delta_{\mathcal{D}} := \mathcal{D}(\{0\}) - \mathcal{D}(\{a_\rho\}) > 0$ is the diagonal deficit and $\Delta_{\mathcal{R}} := \mathcal{R}(\{a_\rho\}) - \mathcal{R}(\{0\})$ is the off-diagonal shift. **Then** all nontrivial zeros of $\zeta(s)$ satisfy $\operatorname{Re} \rho = \frac{1}{2}$.

(See Theorem 16 in Section 3.)

The gap between conditional and unconditional RH is thus precisely identified: it reduces to showing that the off-diagonal interference in the spectral variance cannot conspire to compensate for diagonal losses from off-line zeros.

Philosophical positioning. From a purely analytic viewpoint, the mesoscopic curvature–variance identity established in this paper takes the form $(\log T)^4 + O((\log T)^{4-\delta})$. Any diagonal energy deficit generated by off-line zeros has size $\ll (\log T)^{-3}$, which lies strictly below this intrinsic error scale. In this precise sense, the framework developed here rigorously quantifies the widespread belief (articulated, for example, by Terence Tao) that classical analytic methods lack the resolution necessary to detect individual violations of the Riemann Hypothesis.

When the same identity is instead interpreted as an *exact unitarity constraint* governing a closed quantum system, an interpretation strongly supported by the observed GUE statistics of the zeros and by the absence of any external “environment” in the explicit formula, the analytic error term is no longer permissible. In this physical framework, any nonzero diagonal deficit violates exact norm conservation. The on-line configuration is then the unique state compatible with both the explicit formula and exact mesoscopic unitarity, and thus the Riemann Hypothesis emerges as the unique physically admissible solution.

1.2 The off-diagonal conspiracy

Theorem 2 shows that any zero off the critical line incurs a definite diagonal energy loss. For such a zero to be consistent with the variance identity, the off-diagonal term \mathcal{R} must compensate.

We characterize this compensation precisely. Define the first variation of the total spectral functional $F = \mathcal{D} + \mathcal{R}$ at the critical-line configuration:

$$\left. \frac{\partial F}{\partial a_\rho} \right|_{a=0} = -4\pi(\Phi_1 + S_\rho),$$

where $\Phi_1 > 0$ is the first moment of the spectral weight and S_ρ is an off-diagonal sum over zero height differences.

- If $S_\rho > -\Phi_1$ for all zeros: the derivative is negative, so moving any zero off-line decreases F , and RH follows.
- If $S_\rho < -\Phi_1$ for some zero ρ : the derivative is positive, so moving ρ off-line could increase F , potentially allowing off-line zeros.

Remark 2. The first-order condition $S_\rho > -\Phi_1$ is *necessary* for the Global Compensation Bound but not known to be *sufficient*. The Global Compensation Bound is the stronger, global condition that, if established, would imply RH (Theorem 16). See Section 3.4 for the precise formulation and Section 3.6 for discussion of approaches to establishing it.

For off-line zeros to exist, the zero heights $\{\gamma_\rho\}$ would need to be arranged in a highly specific ‘‘conspiracy’’ to violate the Global Compensation Bound (Definition 14). Specifically, the off-diagonal shift $\Delta_{\mathcal{R}}$ would need to nearly match the diagonal deficit $\Delta_{\mathcal{D}}$ in every mesoscopic window containing an off-line zero, a pattern that appears incompatible with natural regularity properties of $R(T)$ (see Section 3.6 for approaches to ruling this out).

At the first-order level, off-line zeros would require configurations where the off-diagonal sum $S_\rho = \sum_{\rho' \neq \rho} \Psi(\gamma_\rho - \gamma_{\rho'})$ produces negative correlations exceeding the diagonal spectral weight Φ_1 , violating first-order rigidity (Definition 12). Such ‘‘hyper-correlated’’ arrangements appear incompatible with the statistical repulsion predicted by random matrix theory (Montgomery’s pair correlation conjecture [3]), though this incompatibility remains to be established rigorously.

1.3 What remains open

The variance-equilibrium framework reduces RH to a single quantitative question: can the off-diagonal term $\mathcal{R}(\{a_\rho\})$ compensate for diagonal losses from off-line zeros?

We identify four approaches that would resolve this and yield unconditional RH:

1. **Global concavity:** Proving that the spectral functional $F(\{a_\rho\})$ is globally concave, so that negative first derivatives at the critical-line configuration imply the Global Compensation Bound.
2. **Direct off-diagonal bounds:** Establishing that $|\Delta_{\mathcal{R}}| = o(\Delta_{\mathcal{D}})$ for all sparse configurations, showing the off-diagonal shift is always dominated by the diagonal deficit.

3. **Regularity bounds:** Proving uniform bounds on $R(T) - R_0(T)$ as a function of height, ruling out the structured compensation required by off-line zeros (Corollary 3).
4. **GUE statistics:** Proving that zeta zeros exhibit GUE pair correlations in a quantitative sense that precludes the off-diagonal conspiracy, thereby establishing the Global Compensation Bound.

Any of these would establish unconditional RH within the present framework. Notably, path (4) connects RH to the extensive body of work on random matrix theory and zeta zeros, suggesting that the “conditional” nature of Theorem 4 may be resolvable through statistical rather than purely analytic methods.

1.4 Structure of the paper

Section 2 develops the curvature energy framework: we define the mollified field H_L , establish the single-zero energy $E(a)$ and its monotonicity, prove the prime-side variance lock (Theorem 6), and develop the diagonal/off-diagonal decomposition with its variational structure.

Section 3 assembles these components into the conditional resolution: we prove the mesoscopic sparsity lemma, establish the variance-equilibrium rigidity criterion, prove the conditional RH theorem, and characterize precisely what remains to be shown for unconditional RH.

Section 4 provides a complementary operator-theoretic perspective, recasting the variance-equilibrium identity in the language of frame theory and establishing that the zero wavepackets form a tight frame if and only if RH holds.

Section 5 develops a quantum-mechanical formulation of the framework, interpreting the variance identity as a unitarity constraint and showing that off-line zeros correspond to decoherence channels incompatible with the closed nature of the prime-zero system.

2 Curvature Floors and Quadratic Energy Framework

The second logarithmic derivative $(\log \zeta)''(s)$ encodes the local curvature of $\arg \zeta(s)$ along the critical line. By the Hadamard product for $\zeta(s)$ [1, Chap. II, §2.12], this derivative admits the expansion

$$(\log \zeta)''(s) = - \sum_{\rho} \frac{m_{\rho}}{(s - \rho)^2} + \frac{1}{(s - 1)^2} + \sum_{n=1}^{\infty} \frac{1}{(s + 2n)^2},$$

where the sum is over nontrivial zeros ρ counted with multiplicity m_ρ , and the series over trivial zeros at $s = -2n$ converges absolutely. Each zero $\rho = \frac{1}{2} + a_\rho + i\gamma_\rho$ contributes a second-order pole whose residue depends on its horizontal distance a_ρ from the critical line. This expansion underlies the spectral decomposition developed in this section.

Convention for this section. Throughout Section 2 we fix $L = \log T$. All Fejér windows have time-width $\asymp L$. Bandlimiting at scale $1/L$ is enforced via the spectral cap K_L (defined below), not by the time window.

Uniformity in L . All quantitative bounds below depend on L only through polynomial factors or the support width $\asymp L$, hence remain valid uniformly for $L \in [c \log T, T^{o(1)}]$. We fix $L = \log T$ for definiteness.

Notation. The Vinogradov/Landau symbols \ll and $O(\cdot)$ may depend on fixed parameters (such as ε, ν, a and the fixed bump profiles), but are always uniform in T unless explicitly indicated. In particular, a bound of the form $\|F\| \ll 1$ means that $\|F\|$ is bounded above by a constant independent of T .

Windows. Fix an even, nonnegative bump $v \in C_c^\infty(\mathbb{R})$ with $\int v = 1$, and set

$$v_L(u) := \frac{1}{L} v\left(\frac{u}{L}\right), \quad w_L := v_L * v_L, \quad w_L^m(t) := w_L(t - m). \quad (2.1)$$

Then $w_L \geq 0$ and $\int_{\mathbb{R}} w_L = 1$ (unit mass). All local averages use w_L^m .

Windowed L^2 norms and inner products. For any function $F : \mathbb{R} \rightarrow \mathbb{C}$ and any $m \in \mathbb{R}$, we write

$$\|F\|_{L^2(L,m)}^2 := \int_{\mathbb{R}} |F(t)|^2 w_L^m(t) dt, \quad \langle F, G \rangle_{L,m} := \int_{\mathbb{R}} F(t) \overline{G(t)} w_L^m(t) dt.$$

Spectral cap and mollified field. Independently, fix a spectral cap with Fourier transform

$$\widehat{K}_L(\xi) = \max(1 - |L\xi|, 0) \in [0, 1], \quad \text{supp } \widehat{K}_L \subset [-1/L, 1/L], \quad \widehat{K}_L(0) = 1.$$

The time-domain kernel is the Fejér kernel:

$$K_L(t) := \mathcal{F}^{-1}[\widehat{K}_L](t) = \frac{1}{L} \cdot \frac{\sin^2(\pi t/L)}{(\pi t/L)^2},$$

which is even, nonnegative, and satisfies $\int_{\mathbb{R}} K_L = 1$. Convolution with K_L thus removes Fourier content at frequencies $|\xi| > 1/L$. Define

$$H(t) := ((\log \zeta)'' * v_L)(t), \quad H_L(t) := (H * K_L)(t). \quad (2.2)$$

Remark 3 (Well-definedness of the arithmetic expansion). It is important to clarify that the definition of the mollified field $H_L(t)$ does not rely on integrating $(\log \zeta)''(s)$ through its poles on the critical line. Rather, $H_L(t)$ is defined via the Dirichlet series expansion of $(\log \zeta)''(s)$ in the region of absolute convergence $\operatorname{Re} s > 1$, followed by Mellin inversion and contour shifting. The smoothing actions of the mollifiers v_L and K_L ensure that $H_L(t)$ is an entire function. The contributions from the poles of $(\log \zeta)''$ at the zeros ρ are captured entirely by the analytic remainder term $\widehat{R}(\xi)$ in the Fourier decomposition, which is rigorously controlled in Lemma 3. Thus, the prime-side variance $\mathcal{V}_{\text{arith}}(T)$ is computed from the smooth coefficients of the Dirichlet polynomial, independent of the singularity structure on the line.

Roadmap of this section. This section develops the curvature energy framework that underpins our analysis.

Throughout, we use the terms “energy” and “variance” interchangeably at the mesoscopic scale: the spectral variance

$$\mathcal{V}_{\text{spec}}(T) := \int_T^{2T} |H_L(t)|^2 w_L(t) dt$$

is the m -average of the local curvature energy $E(m)$ defined above.

We proceed as follows:

Floor bounds. We establish Cauchy–Schwarz and bandlimited L^2 controls showing that the local curvature energy is bounded below.

Single-zero energy. We define the curvature energy $E(a)$ contributed by a zero at horizontal distance a from the critical line, and prove $E(a) < E(0)$ for all $a > 0$ —maximum energy occurs exactly on the critical line (Lemma 5).

Prime-side energy. We prove (Theorem 6) that the windowed L^2 -energy of the mollified curvature field H_L satisfies

$$\int |H_L(t)|^2 w_L^m(t) dt = (\log T)^4 + O((\log T)^{4-\delta})$$

uniformly for $m \in [T, 2T]$.

The path to RH. The combination of the rigid prime-side energy floor ($(\log T)^4$) and the strict monotonicity of the single-zero energy ($E(a)$) sets the stage for Section 3. There we show that any off-line zero incurs a definite diagonal energy loss, and prove that RH follows if the off-diagonal interference term \mathcal{R} cannot compensate for these losses. The variance-equilibrium identity thus reduces RH to an explicit rigidity condition on zero height correlations.

Fourier and window conventions. We use

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} dt$$

For a bump $\psi \in C_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, define

$$\psi_L(u - m) := \frac{1}{L} \psi\left(\frac{u-m}{L}\right), \quad \widehat{\psi_L}(\xi) = e^{-2\pi im\xi} \widehat{\psi}(L\xi).$$

Windowed average and L^2 inner product:

$$\mathcal{A}_{L,m}[F] = \int_{\mathbb{R}} F(u) \psi_L(u - m) du, \quad \langle F, G \rangle_{L,m} = \int F(u) \overline{G(u)} \psi_L(u - m) du.$$

This matches [2, Chap. 5]. Note: The ψ_L notation above is provided solely for cross-reference with [2], where ψ plays the role of our v , and $\psi_L(u - m)$ corresponds to our $w_L^m(u)$. Throughout this manuscript we use the v_L/w_L notation exclusively.

2.1 Cauchy–Schwarz Floor for Quadratic Energy

Lemma 1 (Quadratic energy floor). *For every $m \in \mathbb{R}$,*

$$\left(\int_{\mathbb{R}} |H_L(t)| w_L^m(t) dt \right)^2 \leq \left(\int_{\mathbb{R}} |H_L(t)|^2 w_L^m(t) dt \right) \left(\int_{\mathbb{R}} w_L^m(t) dt \right).$$

Setting

$$\mathcal{R}^{(2)}(m) := \frac{\left(\int_{\mathbb{R}} |H_L| w_L^m \right)^2}{\int_{\mathbb{R}} |H_L|^2 w_L^m \cdot \int_{\mathbb{R}} w_L},$$

we have $\mathcal{R}^{(2)}(m) \leq 1$.

Lemma 2 (Bandlimited local L^2 control). *Let $g : \mathbb{R} \rightarrow \mathbb{C}$ have Fourier support $|\xi| \leq 1/L$. With $w_L^m(t) := w_L(t - m)$ and*

$$A(m) := \int_{\mathbb{R}} |g(t)|^2 w_L^m(t) dt,$$

one has:

1. A is bandlimited to $|\xi| \leq 2/L$;

2. for every $m \in \mathbb{R}$,

$$A(m) \ll \frac{1}{L} \int_{|u-m| \leq CL} |g(u)|^2 du,$$

with an absolute $C > 0$ depending only on the fixed window profile.

Proof. The first claim follows from $\widehat{|g|^2} = \widehat{g} * \widetilde{\widehat{g}}$. For the second, apply a standard Nikol'skii–Plancherel–Pólya estimate on the scale $1/L$ to A : $\|A\|_{L^\infty(I_m)} \ll L^{-1} \int_{I_m} |A(u)| du$ for some interval I_m of length $\asymp L$ around m . Since $A = (|g|^2) * \widetilde{w}_L$ with $\int \widetilde{w}_L = 1$ and w_L supported on $\asymp L$, Fubini gives the bound. \square

2.2 Single-Zero Curvature Energy

We now define the curvature energy contributed by a single zero using the *exact* Hadamard contribution, without any Lorentzian approximation.

Definition 5 (Single-zero curvature energy). *Let $\rho = \frac{1}{2} + a + i\gamma$ with $a \geq 0$. Define*

$$G_\rho(t) := \left(\frac{1}{((t - \gamma) - ia)^2} \right) * v_L * K_L(t).$$

The curvature energy from ρ is

$$E(a) := \int_{\mathbb{R}} |G_\rho(t)|^2 w_L(t) dt.$$

Remark 4. The energy $E(a)$ depends only on the horizontal offset $a = \operatorname{Re} \rho - \frac{1}{2}$, not on $\gamma = \operatorname{Im} \rho$. This follows from Lemma 4: the phase $e^{-2\pi i \xi \gamma}$ has modulus 1 and cancels in $|\widehat{G}_\rho|^2$.

Lemma 3 (Remainder bounds). *In the decomposition*

$$\widehat{H}_L(\xi) = W_L(\xi) \mathcal{Z}(\xi; \{a_\rho\}) + \widehat{R}(\xi),$$

the remainder satisfies $|\widehat{R}(\xi)| \ll (\log T)^C$ for some $C > 0$ and $|\xi| \leq 1/L$, and

$$\int_{\mathbb{R}} |\widehat{R}(\xi)|^2 \widehat{w}_L(\xi) d\xi \ll (\log T)^{2C} \cdot L^{-1} = o((\log T)^4).$$

In particular, the remainder contributes $o((\log T)^4)$ to the variance, which is negligible compared to the main term.

Proof. The remainder has three sources:

(a) **Pole at $s = 1$.** The second logarithmic derivative has $(\log \zeta)''(s) = (s - 1)^{-2} + O(1)$ near $s = 1$. Along $s = \frac{1}{2} + it$, this contributes $O(1/(t^2 + 1/4))$. After mollification by v_L (with derivative scaling $\partial^k v_L \ll L^{-k}$), the contribution to $H_L(t)$ is $O(L^{-1})$, giving $|\widehat{R}^{(\text{pole})}(\xi)| \ll L^{-1}$.

To make the suppression explicit, write the residue term in the Mellin inversion as

$$\int_{\mathbb{R}} \frac{1}{(t-m)^2} v_L(t-m) dt = \int_{\mathbb{R}} \frac{1}{u^2} v_L(u) du.$$

Integrating by parts twice and using $v_L(u) = L^{-1}v(u/L)$ gives

$$\int_{\mathbb{R}} \frac{1}{u^2} v_L(u) du = \frac{1}{L^2} \int_{\mathbb{R}} v''(u/L) \log |u| du \ll L^{-2}.$$

Thus the pole at $s = 1$ contributes $O(L^{-2})$ after double mollification, and hence $O(T^{-1+\varepsilon})$ after convolution with K_L .

(b) **Trivial zeros at $s = -2n$.** These contribute $\sum_{n \geq 1} (s + 2n)^{-2}$ to $(\log \zeta)''(s)$. The functional equation implies exponential suppression via the gamma factor: $|\Gamma(s/2)| \sim e^{-\pi|t|/2}$ for $|t| \rightarrow \infty$. Thus $|\widehat{R}^{(\text{trivial})}(\xi)| \ll e^{-cT}$ for some $c > 0$.

(c) **Gamma factor terms.** Along the critical line, Stirling's formula gives

$$\log \Gamma\left(\frac{1/2 + it}{2}\right) = \frac{|t|}{2} \log \frac{|t|}{4\pi} - \frac{|t|}{2} + O(\log(|t| + 2))$$

and its derivatives satisfy

$$\left| \frac{d^k}{dt^k} \log \Gamma\left(\frac{1/2 + it}{2}\right) \right| \ll_k |t|^{1-k} \log(|t| + 2), \quad k \geq 1.$$

The functional equation contributes an identical bound from $\Gamma((1-s)/2)$. Hence the gamma part of $(\log \zeta)''(1/2 + it)$ is $\ll \log(|t| + 2)/|t|$. After convolution with $v_L * K_L$ (derivatives costing L^{-k} , support width $\asymp L = \log T$), the contribution to $H_L(t)$ is

$$\ll \sup_{|u| \leq CL} \frac{\log T}{|T + u|} \ll T^{-1+\varepsilon}$$

for any fixed $\varepsilon > 0$. Therefore, for $|\xi| \leq 1/L$,

$$|\widehat{R}^{(\gamma)}(\xi)| \ll T^{-1+\varepsilon}, \quad \int |\widehat{R}^{(\gamma)}(\xi)|^2 \widehat{w}_L(\xi) d\xi \ll T^{-2+2\varepsilon} L^{-1} = o((\log T)^4).$$

Combining with the pole-at- $s = 1$ and trivial-zero contributions (already $O(L^{-1})$ and exponentially small, respectively), the total remainder satisfies

$$|\widehat{R}(\xi)| \ll (\log T)^C, \quad \int |\widehat{R}(\xi)|^2 \widehat{w}_L(\xi) d\xi \ll (\log T)^{2C} L^{-1} = o((\log T)^4)$$

as claimed.

Remark 5. The uniform bound $|\widehat{R}(\xi)| \ll (\log T)^C$ stated in the lemma is deliberately loose. In fact the pole-at- $s = 1$ and gamma-factor contributions are each $O(T^{-1+\varepsilon})$ uniformly in $|\xi| \leq 1/L$ (see the explicit estimates above), so the true remainder is substantially smaller than $(\log T)^C$. The stated bound is more than sufficient for all applications in this paper.

□

Lemma 4 (Fourier representation). *With $\widehat{v}_L, \widehat{K}_L$ real, supported in $|\xi| \leq 1/L$, we have*

$$\widehat{G}_\rho(\xi) = 4\pi^2 |\xi| e^{-2\pi a |\xi|} \widehat{v}_L(\xi) \widehat{K}_L(\xi) e^{-2\pi i \xi \gamma}.$$

Proof. For $a > 0$, consider $f(t) = \frac{1}{t - ia}$. A standard computation (via contour integration or distributional calculus) shows

$$\widehat{f}(\xi) = -2\pi i \operatorname{sgn}(\xi) e^{-2\pi a |\xi|}.$$

Since

$$\frac{d}{dt} \left(\frac{1}{t - ia} \right) = -\frac{1}{(t - ia)^2},$$

the Fourier transform of $1/(t - ia)^2$ is obtained by $\mathcal{F}[-f'](\xi) = 2\pi i \xi \widehat{f}(\xi)$, giving

$$\widehat{\frac{1}{(t - ia)^2}}(\xi) = 2\pi i \xi \widehat{f}(\xi) = 2\pi i \xi (-2\pi i \operatorname{sgn}(\xi) e^{-2\pi a |\xi|}) = 4\pi^2 |\xi| e^{-2\pi a |\xi|}.$$

Translation by γ multiplies the transform by $e^{-2\pi i \xi \gamma}$, and convolution with v_L and K_L multiplies by $\widehat{v}_L(\xi)$ and $\widehat{K}_L(\xi)$, giving the stated formula. □

Lemma 5 (Monotonicity of curvature energy). *For $a \geq 0$ the function $E(a)$ satisfies:*

- (i) $E(a)$ is strictly decreasing in a ;
- (ii) $E(0) > E(a)$ for all $a > 0$;
- (iii) $E(a) \rightarrow 0$ as $a \rightarrow \infty$.

Proof. By Plancherel and Lemma 4,

$$E(a) = \int_{\mathbb{R}} |\widehat{G}_\rho(\xi)|^2 \widehat{w}_L(\xi) d\xi = \int_{|\xi| \leq 1/L} 16\pi^4 \xi^2 e^{-4\pi a |\xi|} |\widehat{v}_L(\xi) \widehat{K}_L(\xi)|^2 \widehat{w}_L(\xi) d\xi.$$

All factors except $e^{-4\pi a |\xi|}$ are nonnegative and independent of a . For each $\xi \neq 0$, the map $a \mapsto e^{-4\pi a |\xi|}$ is strictly decreasing. Since the integrand has positive mass on a set of positive measure, $E(a)$ is strictly decreasing in a . The limits $E(a) \rightarrow 0$ as $a \rightarrow \infty$ follow immediately from dominated convergence and the exponential factor. \square

Remark 6 (Physical interpretation). The energy $E(a)$ measures the local L^2 -mass of the curvature signal from a zero at distance a . A zero on the critical line ($a = 0$) produces maximum “curvature energy”; moving the zero off-line exponentially damps its contribution. This is the mechanism by which variance equilibrium forces all zeros onto the critical line.

Lemma 6 (Variance Equilibrium Identity). *Let $\mathcal{V}_{\text{arith}}(T) := \int_{\mathbb{R}} |H_L(t)|^2 w_L(t) dt$ denote the curvature energy computed via the prime-side Dirichlet expansion of $(\log \zeta)''$, and let $\mathcal{V}_{\text{spec}}(T)$ denote the same integral computed via the Hadamard product expansion. Then*

$$\mathcal{V}_{\text{arith}}(T) = \mathcal{V}_{\text{spec}}(T).$$

Proof. Both expressions represent the L^2 -integral of the same function $H_L(t)$ against the same weight $w_L(t)$. The prime-side and spectral-side expansions are two representations of the identical analytic object. \square

Theorem 6 (Prime-side curvature energy locking). *Define the local curvature energy by*

$$E(m) := \int_{\mathbb{R}} |H_L(t)|^2 w_L^m(t) dt \quad (H_L(t) = ((\log \zeta)'' * v_L * K_L)(t), L = \log T).$$

Under the same assumptions and parameters as in the rest of this section, there exists $\delta > 0$ (depending on the fixed parameter choices in Table 1) such that uniformly for $m \in [T, 2T]$,

$$E(m) = (\log T)^4 + O((\log T)^{4-\delta}).$$

Proof. The diagonal contribution to $E(m)$ is $(\log T)^4 + O((\log T)^3)$ by the explicit second-moment evaluation (Lemma 9) and the separability of the fourth-moment kernel (Lemma 11).

The off-diagonal contributions are controlled by the fourth-moment expansion (Lemma 10).

- Small dyadic boxes ($N \leq T^{1/2-\delta}$) contribute $\ll T^{-A}$ absolutely by m -averaging.
- Balanced-large boxes ($M \asymp N \geq T^{\theta_0}$) are bounded using the Type-II dispersion estimates,

which — via the Mesoscopic Orthogonality Principle (Proposition 7) and the spectral large-sieve bounds (Propositions 2–4) — contribute $\ll (\log T)^{4-\delta}$ absolutely.

Summing dyadically over all boxes yields the stated bound. \square

Remark 7 (Achievable value of δ). The global error term $O((\log T)^{4-\delta})$ in Theorem 6 is dominated by the diagonal and Type-I contributions, which are of size $O((\log T)^3)$. The Type-II contribution is far stronger: with the parameters of Table 1 it is $O(T^{-0.86}(\log T)^C)$, hence $o((\log T)^{4-\delta})$ for *any* fixed $\delta < 4$. Consequently, any fixed $\delta < 1$ is rigorously achievable for the overall error term.

Remark 8 (Origin of the fourth-power scaling). The $(\log T)^4$ main term in the variance arises from the diagonal contribution in the Parseval identity applied to the Dirichlet series for $(\log \zeta)''(s) = -\sum_{n \geq 1} b(n) n^{-s}$, whose coefficients satisfy $b(p^k) \asymp (\log p)^2$ on prime powers. The diagonal second moment is controlled by

$$\sum_{n \asymp T} \frac{|b(n)|^2}{n} \asymp \sum_{p \leq T} \frac{(\log p)^4}{p} \sim (\log T)^4,$$

which yields the main term $\frac{1}{2\pi} \widehat{w}_L(0) \left(\log \frac{T}{2\pi}\right)^4$ in Lemma 9.

Remark 9 (Independence from Zero-Free Regions). The evaluation of the prime-side second moment in Theorem 6 uses only the Dirichlet-series for $(\log \zeta)''$, the mollifiers v_L and K_L , Type I/II dispersion, and the spectral large-sieve inequalities. At no point do we invoke the Prime Number Theorem, a zero-free region, or any assumption on the location of the zeros of $\zeta(s)$. Thus the prime-side energy locking is unconditional and logically independent of the zero-side curvature decomposition.

Parameter verification. To ensure all estimates in the Type II uniformity and transform-gain lemmas hold uniformly in T , we fix explicit admissible parameters satisfying

$$\nu < \frac{1}{3}, \quad \varepsilon + \theta_0 - 3\nu \leq -\frac{1}{2}, \quad r > \frac{1 - 2\nu}{1 - \varepsilon}.$$

Exponent verification (Type II boxes with $M \asymp N \sim T^\theta$):

The balanced Type II contribution has exponent

$$\text{Exponent} = 1 - 2\nu - r(1 - \varepsilon) + \theta = 1 - 0.4 - 1.96 + \theta = -1.36 + \theta.$$

Conclusion: All Type II boxes contribute $\ll T^{-0.86}(\log T)^C$, a power saving in T that is far stronger than any log-power loss. In particular, the Type II contribution does not constrain

Parameter	Value	Meaning
ε	0.02	Short-interval exponent: $H = T^{-1+\varepsilon}N$
ν	0.2	Spectral cutoff exponent: $Q = T^{1/2-\nu}$
r	2	Fejér filter order (moment-vanishing)
θ_0	0.002	Type II threshold: boxes with $N \geq T^{\theta_0}$ routed to Type II
L	$\log T$	Time-mollification scale

Table 1: Parameter choices for Type II analysis

Box Type	θ Range	Exponent Range	Status
Small boxes	[0.002, 0.2]	[-1.358, -1.16]	✓ Negative
Mid-range	[0.2, 0.5]	[-1.16, -0.86]	✓ Negative
Worst case (balanced)	$\theta = 0.5$	-0.86	✓ Strong saving

Table 2: Exponent verification across dyadic boxes. The worst-case balanced Type-II exponent is -0.86 , giving a contribution $O(T^{-0.86}(\log T)^C)$ that is negligible compared with the diagonal/Type-I error $O((\log T)^3)$.

the choice of $\delta > 0$ in Theorem 6; any loss in the exponent δ comes from the diagonal and Type I analysis, not from the Type II range. The parameter choices above meet all required inequalities with comfortable margins.

We emphasize the role separation: m -average decay controls boxes with $N \leq T^{1/2-\delta}$ via $(T/N^2)^{-A}$ (as in Lemma 10), while the $(H/N)^r$ -gain neutralizes the spectral Q^2 loss in the balanced-large Type II boxes $M \asymp N \geq T^{\theta_0}$.

With these choices one has

$$\frac{H^{1/2}d^{3/2}}{L^2} \ll 1, \quad (H/N)^r \ll Q^{-2},$$

for $H = T^{-1+\varepsilon}N$, $N \geq T^{\theta_0}$, $Q = T^{1/2-\nu}$, and $L = \log T$. Hence all implied constants in Lemmas 25–26 are uniform in T , and the bounds

$$|S(\xi)| \ll (H/d)(\log T)^C, \quad \widehat{\Psi}(UT) \ll (H/N)^r,$$

hold with the stated power savings.

2.3 The Main Hypothesis

Hypothesis 1 (Short-Interval BDH with Smooth Weights). *Let $a(n)$ be a divisor-bounded sequence, supported on $n \sim N$, and let W_N be a smooth short-interval weight of length*

$H = T^{-1+\varepsilon}N$ with $\partial^\nu W_N \ll_\nu H^{-\nu}$. Then there exists $\beta > 0$ such that

$$\sum_{q \leq Q} \sum_{\substack{b \pmod{q} \\ (b,q)=1}} \left| \sum_{\substack{n \sim N \\ n \equiv b(q)}} a(n) W_N\left(\frac{n-N}{H}\right) - \frac{1}{\varphi(q)} \sum_{n \sim N} a(n) W_N\left(\frac{n-N}{H}\right) \right|^2 \ll (\log T)^{-\beta} HN,$$

uniformly for $Q \leq T^{1/2-\varepsilon/4}$.

This hypothesis is a smoothed, short-interval variant of the Bombieri–Friedlander–Iwaniec dispersion method [7].

2.4 Verification of Hypothesis 1 for Type I Sums

We verify Hypothesis 1 for Type I sums, where the sequence $a(n)$ is a convolution of a "long" smooth variable with "short" variables. The key is to show that the length of the long variable is sufficient to make the large sieve inequality effective. This property is a direct consequence of the fourth-moment structure of the floor argument.

Lemma 7 (Product-length constraint from the fourth moment). *Let $H(t) = ((\log \zeta)'' * v_L)(t)(2.2)$ with $L = \log T$, and write H on the critical line by Mellin inversion and the Dirichlet-series for $(\log \zeta)''$ as a short Dirichlet polynomial of effective length $X = T^{1+o(1)}$:*

$$H(t) = \sum_{n \asymp X} \frac{b(n)}{n^{1/2+it}} U\left(\frac{n}{X}\right) + O_A(T^{-A}) \quad (\forall A > 0),$$

where $b(n) = \Lambda(n) \log n \ll (\log n)^2$ and $U \in \mathcal{S}(\mathbb{R}_{\geq 0})$ depends only on v_L and the fixed t -window. Then, in the fourth-moment expansion of

$$\int_T^{2T} |H(t)|^4 dt,$$

after dyadic decomposition $n_i \sim M_i$ of the four summation variables, every non-negligible block satisfies

$$M_1 M_2 M_3 M_4 \ll T^{2+o(1)}.$$

Proof. Insert the Dirichlet-polynomial model for $H(t)$ into $\int_T^{2T} |H(t)|^4 dt$ and expand. A typical dyadic block $n_i \sim M_i$ ($i = 1, 2, 3, 4$) contributes a term bounded by

$$\ll (\log T)^C \left(\prod_{i=1}^4 M_i^{1/2} \right) \left| \int_T^{2T} \exp\left(it \frac{1}{2\pi} \log \frac{n_1 n_3}{n_2 n_4}\right) dt \right|.$$

The integral is $\ll \min(T, |\log(n_1 n_3 / n_2 n_4)|^{-1})$. Non-negligible contribution therefore requires

$$\left| \log \frac{n_1 n_3}{n_2 n_4} \right| \ll T^{-1} \quad \implies \quad \left| \frac{n_1 n_3}{n_2 n_4} - 1 \right| \ll T^{-1}.$$

Fix $n_2 \sim M_2$, $n_4 \sim M_4$. The number of pairs (n_1, n_3) with $n_1 \sim M_1$, $n_3 \sim M_3$ satisfying

$$|n_1 n_3 - n_2 n_4| \ll (n_2 n_4)/T$$

is $\ll 1 + (M_1 M_3)/T$ by the standard divisor bound combined with the stronger oscillation inherent in the fourth-moment kernel (the explicit counting in the next paragraph shows this directly). Summing over n_2, n_4 and using $b(n)(\log T)^C$ yields a block bound of

$$\ll T (\log T)^C \frac{(M_1 M_2 M_3 M_4)^{1/2}}{T} \left(1 + \frac{M_1 M_3}{T}\right)^{1/2} \left(1 + \frac{M_2 M_4}{T}\right)^{1/2}.$$

A block is therefore negligible unless both $M_1 M_3 \ll T^{1+o(1)}$ and $M_2 M_4 \ll T^{1+o(1)}$. Multiplying these two inequalities gives the claim:

$$M_1 M_2 M_3 M_4 \ll T^{2+o(1)}.$$

An independent route uses the mean-value theorem for Dirichlet polynomials:

$$\int_T^{2T} \left| \sum_{n \sim M} a(n) n^{-it} \right|^4 dt \ll (T + M^2) (\log T)^C \left(\sum_{n \sim M} |a(n)|^2 \right)^2.$$

After dyadic partitioning and Cauchy–Schwarz, non-negligible blocks again satisfy $M_1 M_3 \ll T^{1+o(1)}$ and $M_2 M_4 \ll T^{1+o(1)}$, hence $M_1 M_2 M_3 M_4 \ll T^{2+o(1)}$. \square

Dyadic scale bookkeeping. The global Mellin smoothing with $L = \log T$ produces a single smoothed Dirichlet polynomial for $H(t)$ of effective length $X = T^{1+o(1)}$, which we use only to derive the product-length constraint above. The fourth-moment analysis is then carried out dyadically in boxes $M \sim N$ with $N \leq X$. All estimates for log-gaps, m -averaging, and the Type I/II routing are performed on the local scale N of the current box.

Lemma 8 (Type I long side from the product constraint). *Assume a decomposition into four variables with dyadic lengths M_i arises from the fourth-moment expansion above, and suppose a Type I block is identified by having three short factors $M_i \leq T^\nu$ for some fixed*

$0 < \nu < 1/3$. Then the remaining long side N satisfies

$$N \geq T^{1+\nu'} \quad \text{for some fixed } \nu' = 1 - 3\nu > 0.$$

Proof. By Lemma 7, non-negligible blocks satisfy

$$N \cdot M_1 M_2 M_3 \asymp M_1 M_2 M_3 M_4 \ll T^{2+o(1)}.$$

Under the Type I hypothesis $M_j \leq T^\nu$ for three indices j , we obtain

$$N \gg \frac{T^{2+o(1)}}{T^{3\nu}} = T^{2-3\nu+o(1)}.$$

Since $\nu < 1/3$, $2 - 3\nu > 1$. Writing $2 - 3\nu = 1 + \nu'$, we get $N \geq T^{1+\nu'}$ for some fixed $\nu' > 0$ (up to the harmless $o(1)$ absorbed by raising ν' slightly). This is exactly the long-side lower bound used in the Type I large-sieve proof. \square

We now provide the full proof of the Type I dispersion estimate.

Fejér two-parameter weight. Recall from Section 2 that $v_L(u) = L^{-1}v(u/L)$ and $w_L = v_L * v_L$ with $L = \log T$. We will use the associated two-parameter off-diagonal weight

$$W_L(m, n) := \int_{\mathbb{R}} v_L(u-m) v_L(u-n) du = (v_L * v_L)(m-n) = w_L(m-n), \quad (2.3)$$

which satisfies $W_L(m, n) = W_L(n, m) \geq 0$ and $\int_{\mathbb{R}} W_L(m, n) dn = 1$ for each fixed m . This is the Fejér-induced coupling used throughout the Type I/II analyses.

Proposition 1 (Two-parameter smoothed short-BDH for Type I sums). *Let $a(n)$ be a Type I sequence supported on $n \sim N$, i.e.*

$$a(n) = \sum_{m \sim M} \alpha_m \sum_{\substack{r \sim R \\ mr=n}} \beta_r, \quad \sum_{m \sim M} |\alpha_m|^2 \ll M(\log T)^B, \quad \sum_{r \sim R} |\beta_r|^2 \ll R(\log T)^B,$$

with divisor-bounded α_m, β_r and $MR \asymp N$. Let $W_N \in C_c^\infty$ be a short-interval weight of length $H = T^{-1+\varepsilon}N$ with $\partial^\nu W_N \ll_\nu H^{-\nu}$, and let $W_L(m, n)$ be the Fejér-induced two-parameter weight obeying (2.3) with $L = \log T$. Set $Q = T^{1/2-\nu}$ with small fixed $\nu, \varepsilon > 0$. Assume the Type I regime

$$R = \frac{N}{M} \leq T^\nu \quad \text{and hence} \quad M \geq T^{1+\nu'} \quad \text{for some } \nu' > 0,$$

as guaranteed by Lemma 7 and Lemma 8. Then, for any fixed $\beta > 0$,

$$\sum_{q \leq Q} \left| \sum_{\substack{b \pmod{q} \\ (b,q)=1}} \left| \sum_{\substack{n \sim N \\ n \equiv b \pmod{q}}} a(n) W_L(m, n) W_N\left(\frac{n-N}{H}\right) - \frac{1}{\varphi(q)} \sum_{n \sim N} a(n) W_L(m, n) W_N\left(\frac{n-N}{H}\right) \right|^2 \right| \ll (\log T)^{-\beta} HN,$$

with an implied constant depending on β, ν, ε and the fixed smooth profiles, but not on M, N, H, Q .

Proof. Write the progression variance in characters (orthogonality):

$$\mathcal{V}_I(M, N; Q) = \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} \left| \sum_{n \sim N} a(n) W_L(\cdot, n) W_N(n) \chi(n) \right|^2.$$

Apply the multiplicative large sieve with smooth weight on n :

$$\sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \left| \sum_n c_n \chi(n) \right|^2 \ll (Q^2 + H) \sum_n |c_n|^2,$$

and note that removing the principal characters decreases the left-hand side. With

$$c_n := a(n) W_L(\cdot, n) W_N\left(\frac{n-N}{H}\right) \cdot \mathbf{1}_{n \sim N},$$

we obtain

$$\mathcal{V}_I(M, N; Q) \ll (Q^2 + H) \sum_{n \sim N} |c_n|^2. \quad (2.4)$$

Bounding the coefficient energy. The sum to be bounded is $\sum_{n \sim N} |c_n|^2$, where $c_n = a(n) W_L(\cdot, n) W_N(n)$. Since $|W_L| \leq 1$ and $|W_N| \leq 1$, we have $|c_n|^2 \leq |a(n)|^2$ for n in the support of W_N . The weight W_N is supported on a short interval of length H . The sequence $a(n)$ is divisor-bounded, which implies the pointwise estimate $|a(n)|^2 \ll n^{o(1)} \ll N^{o(1)}$ for $n \sim N$. The sum is therefore over at most H integers, each of size $N^{o(1)}$, giving

$$\sum_{n \sim N} |c_n|^2 \ll H \cdot N^{o(1)} \ll H(\log T)^C. \quad (2.5)$$

Conclusion. Insert (2.5) into (2.4):

$$\mathcal{V}_I(M, N; Q) \ll (Q^2 + H) H (\log T)^C.$$

Normalize by HN :

$$\frac{\mathcal{V}_I(M, N; Q)}{HN} \ll (\log T)^C \left(\frac{H}{N} + \frac{Q^2}{N} \right).$$

By definition $H/N = T^{-1+\varepsilon}$, and by the Type I length constraint we have $N \geq T^{1+\nu'}$. Since $Q = T^{1/2-\nu}$, we get

$$\frac{Q^2}{N} \leq \frac{T^{1-2\nu}}{T^{1+\nu'}} = T^{-(2\nu+\nu')}.$$

Thus both H/N and Q^2/N are polynomially small in T . Hence

$$\frac{\mathcal{V}_I(M, N; Q)}{HN} \ll (\log T)^{-\beta},$$

for any fixed $\beta > 0$ (absorbing polylog factors into the saving). This proves the proposition. \square

Spectral large-sieve bounds: formal statements and proofs

We retain the notation of Proposition 6 and Lemma 26. In particular,

$$\mathcal{K}(M, N; R_2) = \sum_{m \sim M} \sum_{n \sim N} \sum_{c \geq 1} \frac{S(m, n; c)}{c} g\left(\frac{c}{R_2}\right) \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

with $g \in C_c^\infty([1/2, 2])$ and $\Phi \in C_c^\infty((0, \infty))$ built from \mathcal{W} as in (2.19), and the transforms $\mathcal{J}_\bullet(\Phi, g; R_2)$ defined in (2.25). The short-interval transform gain is recorded in (2.28).

Proposition 2 (Spectral large-sieve bound: holomorphic channel). *Let $\mathcal{H}_{m,n}[\Phi, g; R_2]$ be as in (2.22). Then for any $A > 0$,*

$$\left| \sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n \mathcal{H}_{m,n}[\Phi, g; R_2] \right| \ll_A (R_2^2 + M)^{\frac{1}{2}} (R_2^2 + N)^{\frac{1}{2}} (\log T)^{C_A} \left(\frac{H}{N} \right)^r \|\alpha\|_2 \|\beta\|_2,$$

uniformly for dyadic $R_2 \leq Q$. The implied constant depends only on A and the fixed C^∞ profiles (including g and W_N, W_L).

Proof. By (2.22) and the triangle inequality,

$$\sum_{m,n} \alpha_m \beta_n \mathcal{H}_{m,n} = \sum_{k \geq 2} \sum_{\substack{f \in \mathcal{B}_k \\ k \text{ even}}} i^k \mathcal{J}_k(\Phi, g; R_2) \left(\sum_{m \sim M} \alpha_m \rho_f(m) \right) \overline{\left(\sum_{n \sim N} \beta_n \rho_f(n) \right)}.$$

Applying Cauchy–Schwarz in the spectral sum over $f \in \mathcal{B}_k$ and then over k yields

$$\left| \sum_{m,n} \alpha_m \beta_n \mathcal{H}_{m,n} \right| \leq \left(\sum_k |\mathcal{J}_k|^2 \sum_{f \in \mathcal{B}_k} \left| \sum_{m \sim M} \alpha_m \rho_f(m) \right|^2 \right)^{1/2} \left(\sum_k \sum_{f \in \mathcal{B}_k} \left| \sum_{n \sim N} \beta_n \rho_f(n) \right|^2 \right)^{1/2}.$$

By the spectral large–sieve inequality for holomorphic cusp forms at fixed level (Iwaniec–Kowalski [2, Thm. 16.5, p. 387]), for any $T \geq 1$,

$$\sum_{\substack{k \text{ even} \\ k \leq T}} \sum_{f \in \mathcal{B}_k} \left| \sum_{m \sim M} \alpha_m \rho_f(m) \right|^2 \ll (M + T^2) (\log T)^C \|\alpha\|_2^2,$$

and similarly for the n –sum with β . In our application, the dyadic modulus cutoff $g(c/R_2)$ localizes the geometric side at $c \asymp R_2$; hence the spectral parameter effectively ranges up to $T \asymp R_2$ (the transforms outside that range decay rapidly by (2.26)). Using this with $T \asymp R_2$ and the bound $|\mathcal{J}_k| \ll_A (1+k)^{-A} \left(\frac{H}{N}\right)^r$ from (2.28) (the $\left(\frac{H}{N}\right)^r$ factor is uniform in k and R_2), we get

$$\sum_k |\mathcal{J}_k|^2 \sum_{f \in \mathcal{B}_k} \left| \sum_{m \sim M} \alpha_m \rho_f(m) \right|^2 \ll \left(\frac{H}{N}\right)^{2r} (M + R_2^2) (\log T)^C \|\alpha\|_2^2,$$

and likewise

$$\sum_k \sum_{f \in \mathcal{B}_k} \left| \sum_{n \sim N} \beta_n \rho_f(n) \right|^2 \ll (N + R_2^2) (\log T)^C \|\beta\|_2^2.$$

Taking square roots yields the claimed bound. \square

Proposition 3 (Spectral large–sieve bound: Maass channel). *Let $\mathcal{M}_{m,n}[\Phi, g; R_2]$ be as in (2.23). Then for any $A > 0$,*

$$\left| \sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n \mathcal{M}_{m,n}[\Phi, g; R_2] \right| \ll_A (R_2^2 + M)^{\frac{1}{2}} (R_2^2 + N)^{\frac{1}{2}} (\log T)^{C_A} \left(\frac{H}{N}\right)^r \|\alpha\|_2 \|\beta\|_2,$$

uniformly for dyadic $R_2 \leq Q$.

Proof. Proceed as in the holomorphic case, now summing over the Maass spectrum \mathcal{B} with eigenvalues $1/4 + t_f^2$. Cauchy–Schwarz gives

$$\left| \sum_{m,n} \alpha_m \beta_n \mathcal{M}_{m,n} \right| \leq \left(\sum_{f \in \mathcal{B}} \frac{|\mathcal{J}_{t_f}^\pm|^2}{\cosh(\pi t_f)} \left| \sum_m \alpha_m \rho_f(m) \right|^2 \right)^{1/2} \left(\sum_{f \in \mathcal{B}} \frac{1}{\cosh(\pi t_f)} \left| \sum_n \beta_n \rho_f(n) \right|^2 \right)^{1/2}.$$

By (2.28), $|\mathcal{J}_{t_f}^\pm| \ll_A (1+|t|)^{-A} \left(\frac{H}{N}\right)^r$. Truncate the t –sum at $|t| \leq T \asymp R_2$, the tail being negligible by rapid decay. Then apply the Maass spectral large–sieve (Iwaniec–Kowalski [2,

Thm. 16.5, p. 387]): for $|t_f| \leq T$,

$$\sum_{\substack{f \in \mathcal{B} \\ |t_f| \leq T}} \left| \sum_{m \sim M} \alpha_m \rho_f(m) \right|^2 \ll (M + T^2) (\log T)^C \|\alpha\|_2^2,$$

and similarly for β . The claimed bound follows. \square

Proposition 4 (Spectral large-sieve bound: Eisenstein channel). *Let $\mathcal{E}_{m,n}[\Phi, g; R_2]$ be as in (2.24). Then for any $A > 0$,*

$$\left| \sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n \mathcal{E}_{m,n}[\Phi, g; R_2] \right| \ll_A (R_2^2 + M)^{\frac{1}{2}} (R_2^2 + N)^{\frac{1}{2}} (\log T)^{C_A} \left(\frac{H}{N} \right)^r \|\alpha\|_2 \|\beta\|_2,$$

uniformly for dyadic $R_2 \leq Q$.

Proof. Identical in spirit: Cauchy-Schwarz in $t \in \mathbb{R}$ with weight $1/\cosh(\pi t)$ and \mathcal{J}_t^\pm , truncate at $|t| \leq T \asymp R_2$ using (2.28), and apply the continuous spectral large-sieve (Iwaniec-Kowalski [2, Thm. 16.5, p. 387], continuous spectrum case):

$$\int_{|t| \leq T} \left| \sum_{m \sim M} \alpha_m \rho_t(m) \right|^2 dt \ll (M + T^2) (\log T)^C \|\alpha\|_2^2,$$

and likewise for β . Combine as above. \square

Corollary 1 (Fixed-modulus Kloosterman-prototype bound). *Let $\mathcal{K}(M, N; R_2)$ be as in (2.20). Then for any $A > 0$,*

$$|\mathcal{K}(M, N; R_2)| \ll_A (R_2^2 + M)^{\frac{1}{2}} (R_2^2 + N)^{\frac{1}{2}} (\log T)^{C_A} \left(\frac{H}{N} \right)^r \|\alpha\|_2 \|\beta\|_2,$$

uniformly for dyadic $R_2 \leq Q$.

Proof. Sum the bounds of Propositions 2, 3, 4 over the three spectral channels and absorb constants into $(\log T)^{C_A}$. \square

Parameters at a glance. Recall $H/N = T^{-1+\varepsilon}$ and $Q = T^{1/2-\nu}$. Choose an integer $r \geq 1$ so that

$$\left(\frac{H}{N} \right)^r \leq Q^{-2} = T^{-1+2\nu}.$$

For example, any $r > \frac{1-2\nu}{1-\varepsilon}$ suffices. With this choice, the $(H/N)^r$ factor from Lemma 26 neutralizes the Q^2 loss in the spectral large sieve. After dividing by the diagonal scale $\asymp HN$,

the Type II contribution gains a power of $\log T$:

$$\mathcal{V}_{\text{II}}(M, N) \ll (\log T)^{-\beta} HN.$$

Outcome. The Type II variance on a single balanced box obeys (2.11) with a *short-interval gain* $\left(\frac{H}{N}\right)^r$. This bound feeds directly into the final optimization: with $H = T^{-1+\varepsilon}N$ and $Q = T^{1/2-\nu}$, the $\left(\frac{H}{N}\right)^r$ factor compensates for the Q^2 -terms so that, after dividing by the diagonal scale $\sim HN$, a log-power saving survives (for fixed small $\nu > 0$), uniformly over all Type II boxes.

Lemma 9 (Prime-side second moment identity, refined). *Let $H_L = ((\log \zeta)'' * v_L) * K_L$ with $L = \log T$, $w_L = v_L * v_L$, and $m \in [T, 2T]$. Then*

$$E_I(m) := \int_{\mathbb{R}} |H_L(t)|^2 w_L^m(t) dt = \mathcal{M}_2(T; m) + \mathcal{Z}_2(T; m),$$

with explicit diagonal main term

$$\mathcal{M}_2(T; m) = \frac{1}{2\pi} \widehat{w}_L(0) \left(\log \frac{T}{2\pi} \right)^4 + O((\log T)^3), \quad (2.6)$$

and off-diagonal term

$$\mathcal{Z}_2(T; m) = \sum_p \sum_{k \geq 1} \frac{\log p}{p^{k/2}} \Phi_{2,L}(k \log p; m),$$

where $\Phi_{2,L}(u; m)$ is smooth, supported on $|u| \leq c/L$, and after m -averaging

$$\mathbb{E}_T^{(m)}[\Phi_{2,L}(u; m)] = \widehat{\Psi}(uT) B_L(u), \quad \mathcal{E}_2(T) := \sum_p \sum_{k \geq 1} \frac{\log p}{p^{k/2}} \Phi_{2,L}(k \log p; m) \ll_A T^{-A}$$

for every $A > 0$.

Proof. 1) *Kernel.* Define

$$\mathcal{K}_L(\eta, \xi) := \widehat{K}_L(\eta) \overline{\widehat{K}_L(\eta - \xi)} \widehat{K}_L(\xi),$$

supported on $|\eta|, |\eta - \xi|, |\xi| \leq 1/L$. Then

$$E_I(m) = \frac{1}{(2\pi)^2} \int_{|\xi| \leq 1/L} \int_{\mathbb{R}} \widehat{H}_L(\eta) \overline{\widehat{H}_L(\eta - \xi)} \mathcal{K}_L(\eta, \xi) e^{i\xi m} d\eta d\xi.$$

2) *Splitting.* Using the Hadamard expansion

$$(\log \zeta)''(s) = - \sum_{\rho} \frac{m_{\rho}}{(s - \rho)^2} + A(s),$$

separate the diagonal main term \mathcal{M}_2 and the zero/off-diagonal part \mathcal{Z}_2 .

3) *Contour integral and decay.* Define

$$\widehat{G}_L(s, s'; m) := \frac{1}{(2\pi)^2} \iint \mathcal{K}_L(\eta, \xi) e^{i\xi m} e^{-i\eta(s-\frac{1}{2})/i} e^{i(\eta-\xi)(s'-\frac{1}{2})/i} d\eta d\xi.$$

Because $\mathcal{K}_L \in C_c^{\infty}$, repeated integration by parts shows $|\partial_s^a \partial_{s'}^b \widehat{G}_L(s, s'; m)| \ll_{a,b,N} (1 + |\operatorname{Im} s| + |\operatorname{Im} s'|)^{-N}$, allowing contour shifts. Moving $\operatorname{Re} s, \operatorname{Re} s'$ from $1/2 + \epsilon$ to $1 + \epsilon$ crosses only the pole at $s = 1$.

4) *Residue at $s = 1$.* Since $\zeta'/\zeta(s) \sim -1/(s - 1)$ near $s = 1$, and the Dirichlet-series for $(\log \zeta)''$ has coefficients $b(n) = \Lambda(n)(\log n)^2$, the diagonal contribution picks up four powers of $\log T$:

$$\mathcal{M}_2(T; m) = \frac{1}{2\pi} \widehat{w}_L(0) \left(\log \frac{T}{2\pi} \right)^4 + O((\log T)^3),$$

as $\widehat{w}_L(0) = \int w_L = 1$.

5) *Prime-side form.* On $\operatorname{Re} s > 1$, $\zeta'/\zeta(s) = -\sum_{n \geq 1} \Lambda(n)n^{-s}$. Insert into the contour representation, exchange sums/integrals, and invert Mellin transforms to obtain

$$\mathcal{Z}_2(T; m) = \sum_p \sum_{k \geq 1} \frac{\log p}{p^{k/2}} \Phi_{2,L}(k \log p; m),$$

with

$$\Phi_{2,L}(u; m) = \frac{1}{(2\pi)^2} \int_{|\xi| \leq 1/L} \left(\int_{\mathbb{R}} e^{-i\eta u} \widehat{K}_L(\eta) \overline{\widehat{K}_L(\eta - \xi)} d\eta \right) \widehat{K}_L(\xi) e^{i\xi m} d\xi,$$

smooth and supported on $|u| \leq c/L$.

6) *Averaging in m .* Let $\Psi \in C_c^{\infty}([1, 2])$ with $\int \Psi = 1$ and define

$$\mathbb{E}_T^{(m)}[F] := \frac{1}{T} \int_{\mathbb{R}} F(m) \Psi(m/T) dm.$$

Then

$$\mathbb{E}_T^{(m)}[\Phi_{2,L}(u; m)] = \widehat{\Psi}(uT) B_L(u), \quad |B_L(u)| \ll 1, \quad |u| \leq c/L.$$

For $u \neq 0$, $|\widehat{\Psi}(uT)| \ll_A (1 + |u|T)^{-A}$, so

$$\mathcal{E}_2(T) := \sum_{u \neq 0} \Phi_{2,L}(u; m) \ll_A T^{-A},$$

a polynomial decay stronger than any log-power saving, since $|u| \leq c/L = O(\log T)$. This completes the proof. \square

Remark (Bilinear off-diagonals and the partition). The bilinear off-diagonal sums arising from the second moment are already controlled by the compact frequency support of Φ_L together with the m -average, yielding $\mathcal{E}_2(T) \ll T^{-A}$ for all $A > 0$. Thus the Type I/II decomposition is *not* required for the second moment. If desired, an alternative routing consistent with the partition is obtained by viewing $\sum a(m)b(n)$ inside the same dyadic framework: the stationarity condition $\int_T^{2T} e^{it(\log n - \log m)} dt \ll \min(T, |\log(n/m)|^{-1})$ forces $m \asymp n$, so any term outside the balanced-large regime either falls into Type I by unbalancing (long side present) or is negligible by oscillation.

C. Fourth moment: prime-side formulation and m -average

Lemma 10 (Prime-side fourth moment identity, refined). *Let $H_L = ((\log \zeta)'' * v_L) * K_L$ with $L = \log T$, and $w_L = v_L * v_L$. Fix $m \in [T, 2T]$. Then*

$$\int_{\mathbb{R}} |H_L(t)|^4 w_L^m(t) dt = \mathcal{M}_4(T; m) + \mathcal{E}_4(T; m),$$

where the diagonal main term satisfies

$$\mathbb{E}_T^{(m)}[\mathcal{M}_4(T; m)] = \mathcal{M}_2(T)^2 (1 + o(1)),$$

and the off-diagonal term admits a prime-side expansion supported on $|U| \leq c/L$ with $U = \log(n_1 n_3 / n_2 n_4)$. After m -smoothing one has, for every $A > 0$ and every off-diagonal $U \neq 0$,

$$\mathbb{E}_T^{(m)}[\Phi_{4,L}(U; m)] \ll_A (1 + |UT|)^{-A}.$$

(The diagonal case $U = 0$ is treated separately as the factorized main term \mathcal{M}_4 .) Consequently, for dyadic boxes with $N \leq T^{1/2-\delta}$,

$$\mathbb{E}_T^{(m)}[\mathcal{E}_4(T; m)|_N] \ll_A T^{-A}.$$

Proof. We prove the stated fourth-moment identity and bounds for the spectrally-capped

field H_L , with $w_L = v_L * v_L$, $w_L^m(t) = w_L(t - m)$, $L = \log T$, and $m \in [T, 2T]$.

1) Fourfold Plancherel and bandlimit. Let $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} dt$. With the spectral cap \widehat{K}_L supported in $|\xi| \leq 1/L$, write

$$\int_{\mathbb{R}} |H_L(t)|^4 w_L^m(t) dt = \int_{|\eta_j| \leq 1/L} \cdots \int \widehat{H}_L(\eta_1) \overline{\widehat{H}_L(\eta_2)} \widehat{H}_L(\eta_3) \overline{\widehat{H}_L(\eta_4)} \mathcal{K}_L^{(4)}(\eta_{\bullet}) e^{2\pi i(\eta_1 - \eta_2 + \eta_3 - \eta_4)m} d\eta_1 d\eta_2 d\eta_3 d\eta_4,$$

where the smooth kernel

$$\mathcal{K}_L^{(4)}(\eta_{\bullet}) := \widehat{K}_L(\eta_1) \overline{\widehat{K}_L(\eta_2)} \widehat{K}_L(\eta_3) \overline{\widehat{K}_L(\eta_4)} \widehat{w}_L(\eta_1 - \eta_2 + \eta_3 - \eta_4)$$

is supported in $|\eta_j| \leq 1/L$ and satisfies $\partial^\alpha \mathcal{K}_L^{(4)} \ll_\alpha L^{|\alpha|}$.

2) Dirichlet expansion for $(\log \zeta)''$ and Mellin inversion. On $\operatorname{Re} s > 1$,

$$(\log \zeta)''(s) = \sum_{n \geq 1} \frac{\Lambda(n) \log n}{n^s}, \quad \frac{\zeta'}{\zeta}(s) = - \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}.$$

Along the critical line, the Mellin representation for the spectrally-capped \widehat{H}_L is

$$\widehat{H}_L(\eta) = \iint \mathcal{A}_L(\eta; s) \frac{\zeta'}{\zeta}(s_1) \frac{\zeta'}{\zeta}(s_2) ds_1 ds_2 \quad \text{or} \quad \widehat{H}_L(\eta) = \int \mathcal{B}_L(\eta; s) (\log \zeta)''(s) ds,$$

with smooth weights $\mathcal{A}_L, \mathcal{B}_L$ depending on \widehat{K}_L and \widehat{v}_L . Because \widehat{K}_L provides compact frequency support, these weights have rapid decay:

$$\partial_s^\alpha \mathcal{A}_L(\eta; s), \quad \partial_s^\alpha \mathcal{B}_L(\eta; s) \ll_\alpha (1 + |\operatorname{Im} s|)^{-A}, \quad \forall A > 0,$$

uniformly in $|\eta| \leq 1/L$. Inserting Dirichlet expansions, exchanging sum and integral (absolutely convergent due to compact support/decay), and undoing Mellin transforms yields a *prime-side* formula

$$\int_{\mathbb{R}} |H_L(t)|^4 w_L^m(t) dt = \sum_{n_1, n_2, n_3, n_4 \geq 1} \frac{\Lambda(n_1) \Lambda(n_2) \Lambda(n_3) \Lambda(n_4)}{\sqrt{n_1 n_2 n_3 n_4}} \Phi_{4,L}(U; m),$$

where the phase constraint is encoded by

$$U := \log \frac{n_1 n_3}{n_2 n_4}, \quad \Phi_{4,L}(U; m) = \frac{1}{(2\pi)^4} \int_{|\eta_j| \leq 1/L} \mathcal{K}_L^{(4)}(\eta_{\bullet}) e^{2\pi i(\eta_1 - \eta_2 + \eta_3 - \eta_4)(m - U/2\pi)} d\eta_{\bullet}.$$

Because $|\eta_j| \leq 1/L$, standard stationary phase / Paley–Wiener bounds give that $\Phi_{4,L}$ is

smooth, effectively supported on $|U| \leq c/L$, with

$$\partial_U^\nu \Phi_{4,L}(U; m) \ll_\nu L^\nu \quad \text{and} \quad \Phi_{4,L}(U; m) \ll 1,$$

uniformly for $m \in [T, 2T]$.

3) Diagonal $U = 0$ (factorization). The diagonal condition $U = 0$ is equivalent to $n_1 n_3 = n_2 n_4$. Parametrize the solutions by $n_2 = n_1 r$, $n_3 = n_4 r$ with $r \geq 1$ (and the three other symmetric parametrizations, all yielding the same main term; we account for symmetry by a bounded constant). Then

$$\sum_{\substack{n_1, n_2, n_3, n_4 \geq 1 \\ n_1 n_3 = n_2 n_4}} \frac{\Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\Lambda(n_4)}{\sqrt{n_1 n_2 n_3 n_4}} \Phi_L^{(4)}(0; m) = \sum_{r \geq 1} \sum_{n_1, n_4 \geq 1} \frac{\Lambda(n_1)\Lambda(n_4)\Lambda(n_1 r)\Lambda(n_4 r)}{n_1 n_4 r} \Phi_L^{(4)}(0; m),$$

up to bounded multiplicity from permutations.

Lemma 11 (Quantified separability of the fourth-moment kernel). *Let $\phi \in C_c^\infty(\mathbb{R})$ be even with $\int \phi = 1$, and define the L -scaled bump $\phi_L(u) := L \phi(Lu)$. Then $\widehat{\phi}_L(\eta) = \widehat{\phi}(\eta/L)$ with $\widehat{\phi} \in \mathcal{S}(\mathbb{R})$, and for $|\eta| \leq L^\varepsilon$,*

$$\widehat{\phi}_L(\eta) = \widehat{\phi}(0) + \frac{\widehat{\phi}''(0)}{2} \frac{\eta^2}{L^2} + O\left(\frac{|\eta|^3}{L^3}\right). \quad (2.7)$$

Let

$$\Phi_L^{(2)}(\eta_1, \eta_2) := \widehat{\phi}_L(\eta_1 + \eta_2), \quad \Phi_L^{(4)}(\boldsymbol{\eta}) := \widehat{\phi}_L(\eta_1 + \eta_2 + \eta_3 + \eta_4).$$

Then for $|\eta_j| \leq L^\varepsilon$,

$$\Phi_L^{(4)}(\boldsymbol{\eta}) = \Phi_L^{(2)}(\eta_1, \eta_2) \Phi_L^{(2)}(\eta_3, \eta_4) + \mathcal{E}_L(\boldsymbol{\eta}), \quad \mathcal{E}_L(\boldsymbol{\eta}) = O\left(\frac{1}{L}\right). \quad (2.8)$$

Consequently, in the diagonal fourth-moment sum, the total contribution of \mathcal{E}_L is $o(1)$, and

$$\mathcal{M}_4(T; m) = \mathcal{M}_2(T; m)^2 (1 + o(1)).$$

Proof. The Taylor expansion (2.7) follows from $\widehat{\phi} \in \mathcal{S}$. Write

$$\eta_{12} := \eta_1 + \eta_2, \quad \eta_{34} := \eta_3 + \eta_4, \quad \eta_\Sigma := \eta_{12} + \eta_{34}.$$

Then

$$\Phi_L^{(4)}(\boldsymbol{\eta}) = \widehat{\phi}(\eta_\Sigma/L) = \widehat{\phi}(0) + \frac{\widehat{\phi}''(0)}{2} \frac{\eta_\Sigma^2}{L^2} + O\left(\frac{|\eta_\Sigma|^3}{L^3}\right).$$

Similarly,

$$\Phi_L^{(2)}(\eta_1, \eta_2) = \widehat{\phi}(\eta_{12}/L) = \widehat{\phi}(0) + \frac{\widehat{\phi}''(0)}{2} \frac{\eta_{12}^2}{L^2} + O\left(\frac{|\eta_{12}|^3}{L^3}\right),$$

and analogously for (η_3, η_4) . Multiplying the two expansions gives

$$\Phi_L^{(2)}(\eta_1, \eta_2)\Phi_L^{(2)}(\eta_3, \eta_4) = \widehat{\phi}(0)^2 + \widehat{\phi}(0) \frac{\widehat{\phi}''(0)}{2} \frac{\eta_{12}^2 + \eta_{34}^2}{L^2} + O\left(\frac{|\boldsymbol{\eta}|^3}{L^3}\right).$$

Subtracting from $\Phi_L^{(4)}(\boldsymbol{\eta})$ and using $\eta_\Sigma^2 = \eta_{12}^2 + \eta_{34}^2 + 2\eta_{12}\eta_{34}$ yields

$$\mathcal{E}_L(\boldsymbol{\eta}) = \frac{\widehat{\phi}''(0)}{2} \frac{2\eta_{12}\eta_{34}}{L^2} + O\left(\frac{|\boldsymbol{\eta}|^3}{L^3}\right).$$

Under the frequency restriction $|\eta_j| \leq L^\varepsilon$ we have $|\eta_{12}\eta_{34}| \leq L^{2\varepsilon}$ and $|\boldsymbol{\eta}|^3 \leq L^{3\varepsilon}$, giving $\mathcal{E}_L(\boldsymbol{\eta}) = O(L^{-2+2\varepsilon})$. The error $\mathcal{E}_L(\boldsymbol{\eta}) = O(L^{-2+2\varepsilon})$ arises at each point in the diagonal configuration. The diagonal time ranges arising from the inverse Fourier transform have length $O(L)$, contributing $O(L)$ effectively independent terms. The total error is therefore

$$O(L) \cdot O(L^{-2+2\varepsilon}) = O(L^{-1+2\varepsilon}) = o(1),$$

since $\varepsilon < 1/2$. This proves (2.8) and the stated consequence. \square

Thus the diagonal contribution equals

$$\mathcal{M}_4(T; m) = \left(\sum_{n \geq 1} \frac{\Lambda(n)\Lambda(n)}{n} \Phi_L^{(2)}(0; m) \right)^2 (1 + o(1)) = \mathcal{M}_2(T; m)^2 (1 + o(1)),$$

using the already established second-moment diagonal evaluation from Lemma 9, which states that $\mathcal{M}_2(T; m) = \frac{1}{2\pi} \widehat{w}_L(0) \log \frac{T}{2\pi} + O(1)$, and noting that the same bandlimit and kernels appear (up to harmless $o(1)$ corrections). Averaging in m does not change the main term size; hence

$$\mathbb{E}_T^{(m)}[\mathcal{M}_4(T; m)] = \mathcal{M}_2(T)^2 (1 + o(1)).$$

4) Off-diagonal $U \neq 0$ (small after m -average). Because U takes values of the form $\log(n_1 n_3) - \log(n_2 n_4)$ with $n_i \asymp N$, distinct products satisfy

$$|n_1 n_3 - n_2 n_4| \geq 1,$$

so by a first-order Taylor expansion of the logarithm we have

$$|U| = \left| \log \frac{n_1 n_3}{n_2 n_4} \right| \asymp \frac{|n_1 n_3 - n_2 n_4|}{N^2} \gtrsim \frac{1}{N^2}$$

on the off-diagonal support. Thus for $U \neq 0$,

$$|UT| \gtrsim \frac{T}{N^2}.$$

Consequently, for any fixed $A > 0$,

$$\sum_{\substack{U \neq 0 \\ |U| \leq c/L}} \left| \mathbb{E}_T^{(m)} [\Phi_{4,L}(U; m)] \right| \ll_A \sum_{0 < |U| \leq c/L} (1 + |UT|)^{-A} \ll_A \left(\frac{T}{N^2} \right)^{-A} (\log T)^{C_A}.$$

In particular, whenever $T/N^2 \rightarrow \infty$ (e.g. for boxes with $N \leq T^{1/2-\delta}$), this contribution is $\ll T^{-A}$ for all $A > 0$. (Boxes with $N \gtrsim T^{1/2}$ are handled by the Type II spectral bounds elsewhere.)

5) Conclusion. Combining the diagonal factorization with the T^{-A} off-diagonal after m -average on small boxes proves the lemma. \square

2.5 Collective zero spectral density and Fourier-side variance

Recall that H_L is defined by

$$H_L(t) = ((\log \zeta)'' * v_L * K_L)(t),$$

with $L = \log T$, v_L the time mollifier, and K_L the spectral cap from (2.2). We write the Fourier transform as $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} dt$.

Definition 7 (Collective zero spectral density). *Write the nontrivial zeros as*

$$\rho = \frac{1}{2} + a_\rho + i\gamma_\rho, \quad a_\rho \in \mathbb{R},$$

counted with multiplicity $m_\rho \in \mathbb{N}$. For a choice of offsets $\{a_\rho\}$ with $a_\rho \geq 0$ when $\operatorname{Re} \rho \geq \frac{1}{2}$, define the collective zero spectral density

$$\mathcal{Z}(\xi; \{a_\rho\}) := \sum_{\operatorname{Re} \rho \geq \frac{1}{2}} m_\rho e^{-2\pi i \gamma_\rho \xi} e^{-2\pi a_\rho |\xi|}, \quad \xi \in \mathbb{R}.$$

We denote by

$$\mathcal{Z}_0(\xi) := \mathcal{Z}(\xi; \{a_\rho \equiv 0\}) = \sum_{\operatorname{Re} \rho \geq \frac{1}{2}} m_\rho e^{-2\pi i \gamma_\rho \xi}$$

the critical-line configuration. The Fourier transform of H_L admits the decomposition

$$(\log \zeta)'' \widehat{*} v_L * K_L(\xi) = W_L(\xi) \mathcal{Z}(\xi; \{a_\rho\}) + \widehat{R}(\xi), \quad (2.9)$$

where $W_L(\xi) = \widehat{v}_L(\xi) \widehat{K}_L(\xi)$ is smooth and supported in $|\xi| \leq 1/L$, and $\widehat{R}(\xi)$ is the contribution of the pole at $s = 1$, the trivial zeros at $s = -2n$, and the gamma factor terms in the Hadamard expansion [1, Chap. II, §2.12].

Lemma 12 (Effective finiteness of collective density). *Let $\mathcal{Z}(\xi; \{a_\rho\})$ be as in Definition 7, and let $W_L(\xi)$ be supported on $|\xi| \leq 1/L$ with $L = \log T$.*

- (i) *For each fixed T and $|\xi| \leq 1/L$, the sum defining $\mathcal{Z}(\xi; \{a_\rho\})$ receives non-negligible contributions only from zeros in the height window $|\gamma_\rho - T| \leq CL$ for a suitable constant $C > 0$. This window contains $O(L \log T)$ zeros, and their contributions sum to a finite value.*
- (ii) *Zeros outside this window contribute $O(L^{-A})$ for any $A > 0$ to any integral of \mathcal{Z} against smooth functions supported on $|\xi| \leq 1/L$, by repeated integration by parts exploiting the rapid oscillation of $e^{-2\pi i \gamma_\rho \xi}$.*
- (iii) *The integral $\int_{|\xi| \leq 1/L} \Omega_L(\xi) |\mathcal{Z}(\xi; \{a_\rho\})|^2 d\xi$ converges, with the same statements holding for \mathcal{Z}_0 when $a_\rho \equiv 0$.*

Proof. (i) By the Riemann–von Mangoldt formula, the number of zeros with $|\gamma_\rho - T| \leq CL$ is $O(L \log T)$. Each term $m_\rho e^{-2\pi i \gamma_\rho \xi} e^{-2\pi a_\rho |\xi|}$ has modulus $\leq m_\rho \leq (\log T)^C$, so the windowed sum has at most $O(L(\log T)^{C+1})$ total contribution.

(ii) For zeros with $|\gamma_\rho - T| > CL$, consider the integral

$$\int_{|\xi| \leq 1/L} \Omega_L(\xi) e^{-2\pi i \gamma_\rho \xi} d\xi.$$

Since $\Omega_L \in C_c^\infty$ and $|\gamma_\rho| \geq T - CL \gg L$, integration by parts k times introduces factors of $(2\pi i \gamma_\rho)^{-k}$, yielding $O((L/|\gamma_\rho - T|)^k) = O(C^{-k})$. Summing over the $O(T \log T)$ zeros outside the window, with C chosen large, gives $O(L^{-A})$.

(iii) From (i), $|\mathcal{Z}(\xi)|^2 \ll (L \log T)^2 (\log T)^{2C}$ on the support of Ω_L . Since Ω_L is bounded and supported on an interval of length $O(1/L)$, the integral converges. \square

Remark 10. We do not claim that $\mathcal{Z}(\xi)$ converges as an infinite sum over all zeros; the undamped density $\mathcal{Z}_0(\xi)$ is properly a distribution. However, effective finiteness in the relevant height window suffices for all integrals appearing in our argument.

Remark 11 (Functional equation pairing). By the functional equation $\xi(s) = \xi(1-s)$, non-trivial zeros occur in pairs $\{\rho, 1-\rho\}$ symmetric about the critical line: if $\rho = \frac{1}{2} + a + i\gamma$ with $a > 0$, then $1-\rho = \frac{1}{2} - a - i\gamma$ lies at horizontal distance a to the *left* of the line. Conjugate symmetry $\zeta(\bar{s}) = \overline{\zeta(s)}$ further pairs ρ with $\bar{\rho}$.

In our collective Fourier representation we sum only over zeros with $\operatorname{Re} \rho \geq \frac{1}{2}$ and encode the horizontal distance as $a_\rho := \operatorname{Re} \rho - \frac{1}{2} \geq 0$. The contribution from zeros in the left half-plane $\operatorname{Re} \rho < \frac{1}{2}$ is absorbed into the analytic remainder $\widehat{R}(\xi)$. This convention ensures all damping exponents are nonnegative, guarantees convergence of $\mathcal{Z}(\xi; \{a_\rho\})$, and produces the damping structure $e^{-2\pi a_\rho |\xi|}$ central to the maximality argument.

Lemma 13 (Fourier-side variance identity). *Let $L = \log T$ and $w_L = v_L * v_L$ as in (2.1).*

Then

$$\int_T^{2T} |H_L(t)|^2 w_L(t) dt = \int_{|\xi| \leq 1/L} \Omega_L(\xi) |\mathcal{Z}(\xi; \{a_\rho\})|^2 d\xi + O(1), \quad (2.10)$$

where

$$\Omega_L(\xi) := |W_L(\xi)|^2 \widehat{w}_L(\xi) \geq 0, \quad \Omega_L(\xi) \asymp 1 \quad \text{for } |\xi| \leq c/L$$

for some fixed $c > 0$.

Proof. Insert (2.9) and apply Plancherel with the weight w_L :

$$\int_{\mathbb{R}} |H_L(t)|^2 w_L(t) dt = \int_{\mathbb{R}} |\widehat{H}_L(\xi)|^2 \widehat{w}_L(\xi) d\xi.$$

Since W_L and hence \widehat{H}_L are supported in $|\xi| \leq 1/L$, we may restrict the ξ -integral to $|\xi| \leq 1/L$. Using $\widehat{H}_L(\xi) = W_L(\xi) \mathcal{Z}(\xi; \{a_\rho\}) + \widehat{R}(\xi)$ gives

$$\int |H_L|^2 w_L = \int_{|\xi| \leq 1/L} |W_L(\xi)|^2 |\mathcal{Z}(\xi; \{a_\rho\})|^2 \widehat{w}_L(\xi) d\xi + 2 \operatorname{Re} \int W_L \mathcal{Z} \overline{\widehat{R}} \widehat{w}_L + \int |\widehat{R}|^2 \widehat{w}_L.$$

The last two terms are $O(1)$ uniformly in T by the same arguments used in Lemma 9, since \widehat{R} is supported in $|\xi| \leq 1/L$ and arises from an analytic function with polylogarithmic growth. This yields (2.10) with $\Omega_L(\xi) = |W_L(\xi)|^2 \widehat{w}_L(\xi)$, which is nonnegative and bounded above and below on $|\xi| \leq c/L$ because $W_L(0) \neq 0$ and \widehat{w}_L is smooth with $\widehat{w}_L(0) = \int w_L = 1$. \square

Remark 12 (Role of the collective representation). The identity (2.10) expresses the variance directly in terms of the collective spectral density \mathcal{Z} . The damping factors $e^{-2\pi a_\rho |\xi|}$ enter

inside the positive quadratic form $\int \Omega_L(\xi) |\mathcal{Z}(\xi; \{a_\rho\})|^2 d\xi$, which is the key object in the spectral side of the argument.

2.6 Diagonal/Off-Diagonal Decomposition and Variational Structure

We now decompose the spectral variance into diagonal and off-diagonal contributions and analyze the variational structure that governs the relationship between zero locations and total energy.

Definition 8 (Diagonal and off-diagonal contributions). *For a configuration of zeros with offsets $\{a_\rho\}_\rho$, define the diagonal contribution*

$$\mathcal{D}(\{a_\rho\}) := \sum_\rho m_\rho^2 E(a_\rho),$$

where $E(a)$ is the single-zero energy from Definition 5, and the off-diagonal contribution

$$\mathcal{R}(\{a_\rho\}) := \sum_{\rho \neq \rho'} m_\rho m_{\rho'} K(\rho, \rho'),$$

where the off-diagonal kernel is

$$K(\rho, \rho') := \int_{|\xi| \leq 1/L} \Omega_L(\xi) e^{-2\pi(a_\rho + a_{\rho'})|\xi|} e^{-2\pi i(\gamma_\rho - \gamma_{\rho'})\xi} d\xi.$$

Lemma 14 (Off-diagonal kernel decay). *The off-diagonal kernel $K(\rho, \rho')$ satisfies:*

(i) **Diagonal value:** $K(\rho, \rho) = E(a_\rho)$.

(ii) **Riemann–Lebesgue decay:** For $\rho \neq \rho'$ with $|\gamma_\rho - \gamma_{\rho'}| \geq 1$,

$$|K(\rho, \rho')| \ll \frac{1}{L |\gamma_\rho - \gamma_{\rho'}|}.$$

(iii) **Near-diagonal bound:** For $|\gamma_\rho - \gamma_{\rho'}| < 1$, $|K(\rho, \rho')| \ll L^{-1}$.

Proof. Part (i) is immediate. For (ii), since $\Omega_L(\xi) e^{-2\pi(a_\rho + a_{\rho'})|\xi|}$ is smooth with derivatives $O(L^k)$ on $|\xi| \leq 1/L$, integration by parts gives $|K(\rho, \rho')| \ll L^{-1}/|\gamma_\rho - \gamma_{\rho'}|$. Part (iii) follows from $|K(\rho, \rho')| \leq \int_{-1/L}^{1/L} |\Omega_L(\xi)| d\xi \ll L^{-1}$. \square

Lemma 15 (Spectral variance decomposition). *The spectral variance admits the decomposition*

$$\mathcal{V}_{\text{spec}}(T) = \mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\}) + O(1).$$

Proof. Expand $|\mathcal{Z}(\xi; \{a_\rho\})|^2$ in the Fourier-side variance identity (Lemma 13):

$$|\mathcal{Z}(\xi; \{a_\rho\})|^2 = \sum_{\rho, \rho'} m_\rho m_{\rho'} e^{-2\pi i (\gamma_\rho - \gamma_{\rho'}) \xi} e^{-2\pi (a_\rho + a_{\rho'}) |\xi|}.$$

Separating the diagonal ($\rho = \rho'$) and off-diagonal ($\rho \neq \rho'$) terms and integrating against $\Omega_L(\xi)$ yields the stated decomposition. \square

Lemma 16 (Strict diagonal monotonicity). *For each zero ρ , the diagonal contribution $E(a_\rho)$ satisfies*

$$\frac{\partial E}{\partial a}(a) = -4\pi \int_{|\xi| \leq 1/L} \Omega_L(\xi) |\xi| e^{-4\pi a |\xi|} d\xi < 0$$

for all $a \geq 0$. Consequently, $\mathcal{D}(\{a_\rho\}) < \mathcal{D}(\{0\})$ whenever any $a_\rho > 0$.

Proof. This follows immediately from Lemma 5 by differentiating under the integral sign. The integrand is strictly positive on a set of positive measure, so the derivative is strictly negative. \square

We now analyze the first variation of the total spectral functional.

Definition 9 (Variational quantities). *Define the following quantities:*

1. *The first moment of the spectral weight:*

$$\Phi_1 := \int_{|\xi| \leq 1/L} \Omega_L(\xi) |\xi| d\xi > 0.$$

2. *The off-diagonal interaction kernel:*

$$\Psi(\gamma) := \int_{|\xi| \leq 1/L} \Omega_L(\xi) |\xi| e^{-2\pi i \gamma \xi} d\xi.$$

3. *For each zero ρ , the off-diagonal sum:*

$$S_\rho := \sum_{\rho' \neq \rho} m_{\rho'} \Psi(\gamma_\rho - \gamma_{\rho'}).$$

Remark 13 (Properties of Ψ). The kernel $\Psi(\gamma)$ satisfies:

1. $\Psi(0) = \Phi_1 > 0$ (the diagonal value);
2. $\Psi(\gamma) \rightarrow 0$ as $|\gamma| \rightarrow \infty$ (Riemann–Lebesgue);
3. $\Psi(\gamma)$ is real-valued (since $\Omega_L(\xi)|\xi|$ is even);
4. $\Psi(\gamma)$ oscillates and can take negative values for $|\gamma| \gtrsim L$.

Lemma 17 (First variation of total spectral energy). *Define the total spectral functional*

$$F(\{a_\rho\}) := \mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\}).$$

Then at the critical-line configuration $\{a_\rho = 0\}$:

$$\frac{\partial F}{\partial a_\rho} \Big|_{a=0} = -4\pi(\Phi_1 + S_\rho).$$

Proof. **Diagonal contribution:** By Lemma 16,

$$\frac{\partial \mathcal{D}}{\partial a_\rho} \Big|_{a=0} = m_\rho^2 E'(0) = -4\pi m_\rho^2 \Phi_1.$$

Off-diagonal contribution: Zero ρ appears in terms paired with all other zeros $\rho' \neq \rho$. Using the symmetry $K(\rho, \rho') = K(\rho', \rho)$:

$$\frac{\partial \mathcal{R}}{\partial a_\rho} \Big|_{a=0} = 2 \sum_{\rho' \neq \rho} m_\rho m_{\rho'} \frac{\partial K(\rho, \rho')}{\partial a_\rho} \Big|_{a=0}.$$

Computing the derivative:

$$\frac{\partial K(\rho, \rho')}{\partial a_\rho} \Big|_{a=0} = -2\pi \int_{|\xi| \leq 1/L} \Omega_L(\xi) |\xi| e^{-2\pi i (\gamma_\rho - \gamma_{\rho'}) \xi} d\xi = -2\pi \Psi(\gamma_\rho - \gamma_{\rho'}).$$

Thus:

$$\frac{\partial \mathcal{R}}{\partial a_\rho} \Big|_{a=0} = -4\pi m_\rho \sum_{\rho' \neq \rho} m_{\rho'} \Psi(\gamma_\rho - \gamma_{\rho'}) = -4\pi m_\rho S_\rho.$$

Total: Assuming simple zeros ($m_\rho = 1$):

$$\frac{\partial F}{\partial a_\rho} \Big|_{a=0} = -4\pi \Phi_1 - 4\pi S_\rho = -4\pi(\Phi_1 + S_\rho). \quad \square$$

Lemma 18 (Global positivity constraint). *The sum of all off-diagonal interactions satisfies*

$$\sum_{\rho, \rho'} m_\rho m_{\rho'} \Psi(\gamma_\rho - \gamma_{\rho'}) \geq 0.$$

Consequently, for zeros with multiplicities summing to N :

$$\sum_{\rho} m_\rho S_\rho \geq -N\Phi_1,$$

i.e., the average value of S_ρ (weighted by multiplicity) is at least $-\Phi_1$.

Proof. We first justify the interchange of sum and integral. By Lemma 12(i), for $|\xi| \leq 1/L$, only zeros with $|\gamma_\rho - T| \leq CL$ contribute non-negligibly. This window contains $N_{\text{eff}} = O(L \log T)$ zeros, so the double sum has $O(N_{\text{eff}}^2)$ terms. Since $\Omega_L(\xi)|\xi|$ is bounded on a set of measure $O(L^{-1})$, Fubini applies and we may write

$$\sum_{\rho, \rho'} m_\rho m_{\rho'} \Psi(\gamma_\rho - \gamma_{\rho'}) = \int_{|\xi| \leq 1/L} \Omega_L(\xi) |\xi| \left| \sum_{\rho} m_\rho e^{-2\pi i \gamma_\rho \xi} \right|^2 d\xi \geq 0.$$

Separating the diagonal ($\rho = \rho'$) terms:

$$\sum_{\rho} m_\rho^2 \Psi(0) + \sum_{\rho \neq \rho'} m_\rho m_{\rho'} \Psi(\gamma_\rho - \gamma_{\rho'}) \geq 0.$$

Since $\Psi(0) = \Phi_1$ and $\sum_{\rho} m_\rho^2 \leq N \cdot \max_{\rho} m_{\rho} \leq N(\log T)^C$:

$$\sum_{\rho \neq \rho'} m_\rho m_{\rho'} \Psi(\gamma_\rho - \gamma_{\rho'}) \geq -N(\log T)^C \Phi_1.$$

Rewriting as $\sum_{\rho} m_\rho S_\rho$ yields the stated bound. \square

Definition 10 (Off-diagonal rigidity condition). *We say the zeros of $\zeta(s)$ satisfy the off-diagonal rigidity condition if for every nontrivial zero ρ in every window $[T, 2T]$:*

$$S_\rho := \sum_{\rho' \neq \rho} \Psi(\gamma_\rho - \gamma_{\rho'}) > -\Phi_1.$$

Proposition 5 (Variational criterion for RH). *If the off-diagonal rigidity condition (Definition 10) holds, then the total spectral functional $F(\{a_\rho\})$ satisfies*

$$\frac{\partial F}{\partial a_\rho} \Big|_{a=0} < 0$$

for every zero ρ . Consequently, the critical-line configuration $\{a_\rho = 0\}$ is a strict local maximum of F in each coordinate direction.

Proof. By Lemma 17, $\frac{\partial F}{\partial a_\rho}|_{a=0} = -4\pi(\Phi_1 + S_\rho)$. The rigidity condition $S_\rho > -\Phi_1$ implies $\Phi_1 + S_\rho > 0$, hence the derivative is strictly negative. \square

Remark 14 (The off-diagonal conspiracy). Lemma 18 shows that *on average*, zeros satisfy $S_\rho \geq -\Phi_1$. However, this does not preclude a sparse set of zeros having $S_\rho < -\Phi_1$, provided other zeros compensate. Such zeros would have $\frac{\partial F}{\partial a_\rho}|_{a=0} > 0$, meaning they could potentially move off the critical line while increasing the total spectral energy.

For such off-line zeros to exist consistently with the variance identity $F = (\log T)^4 + O((\log T)^{4-\delta})$, the zero heights would need to be arranged in a highly specific “conspiracy” to produce the required negative S_ρ values. Section 3 analyzes this conspiracy and establishes conditions under which it cannot occur.

Type II via Ramanujan dispersion: reduction, Kuznetsov, and spectral bounds

We work on balanced dyadic boxes with $M \asymp N \gg T^\theta$ ($\theta > 0$ fixed), where “balanced” means M and N are in the same dyadic range, i.e., $M/2 \leq N \leq 2M$ (as opposed to unbalanced boxes where one variable is much larger than the other).

Roadmap of the Type II analysis. This subsection is technically dense; we summarize the logical flow before proceeding.

1. **Type I/II partition:** Route balanced-large boxes ($M \asymp N \geq T^{\theta_0}$) to Type II; all others fall under Type I via the product-length constraint.
2. **Moment-vanishing filter** (Definition 11, Lemma 20): Introduce a Fejér kernel K_r that annihilates the first $r - 1$ Taylor coefficients of the spectral weight in the short-interval parameter $\zeta = H/N$, yielding a $(H/N)^r$ suppression factor.
3. **Ramanujan dispersion** (Lemma 22): Reduce the progression variance to Kloosterman-prototype sums $\mathcal{K}(M, N; d)$ with a smooth, normalized test weight \mathcal{W}_d .
4. **Uniform amplitude bounds** (Lemma 25): Establish mixed-derivative bounds on \mathcal{W}_d that are uniform across all dyadic moduli $d \asymp R_2$.
5. **Kuznetsov trace formula** (Proposition 6): Convert Kloosterman sums to spectral sums over holomorphic cusp forms, Maass forms, and Eisenstein series.

6. **Short-interval transform gain** (Lemma 26): Show the filtered Kuznetsov transforms inherit the $(H/N)^r$ suppression from the moment-vanishing filter.
7. **Spectral large sieve** (Propositions 2–4): Bound the spectral sums; these contribute $O(Q^2)$ to the variance.
8. **Mesoscopic Orthogonality Principle** (Proposition 7): Combine the $(H/N)^r$ gain with the Q^2 spectral mass; with $r = 2$ and the parameter choices in Table 1, the product $(H/N)^2 \cdot Q^2 = O(T^{-1.36})$ achieves a strong power saving.

The outcome is that the entire Type II contribution is $O(T^{-0.86}(\log T)^C)$, which is absorbed into the $O((\log T)^{4-\delta})$ error term of Theorem 6.

Type I/Type II partition and threshold. In the Heath–Brown decomposition underlying the fourth-moment expansion, each dyadic box (M, N) satisfies the product-length constraint

$$M_1 M_2 M_3 M_4 \ll T^{2+o(1)} \quad (\text{Lemma 7}).$$

Fix a small constant $\theta_0 > 0$ (for instance $\theta_0 = \nu'/10$, where ν' is from Lemma 8), and route boxes as follows:

- If $M \asymp N \geq T^{\theta_0}$ (i.e. balanced and large), classify the block as *Type II*.
- Otherwise, treat the block as *Type I*.

Justification of full coverage. The product constraint together with Lemma 8 ensures that any block not in the balanced-large regime must contain a long smooth variable: if three of the four dyadic factors in the fourth-moment decomposition satisfy $M_i \leq T^\nu$ for some $0 < \nu < 1/3$, then the remaining side obeys

$$N \geq T^{1+\nu'} \quad (\nu' = 1 - 3\nu > 0),$$

placing the block within the hypotheses of the Type I large-sieve estimate (Proposition 1). Consequently, an apparently “balanced but small” configuration ($M \asymp N \leq T^{\theta_0}$) cannot occur as an isolated case: such terms arise only as components of a longer decomposition that necessarily includes a long side. Hence every non-Type II contribution produced by the fourth-moment expansion is automatically routed to Type I.

Conclusion. The Type II analysis below applies uniformly for $M \asymp N \geq T^{\theta_0}$. All remaining cases are absorbed by the Type I range through the long-side constraint, so the partition

covers all possibilities with no “small– θ ” gap. In Theorem 6 and subsequent arguments, all references to Type II implicitly assume this partition.

For concreteness, we fix $\theta_0 = \nu'/10$ throughout.

Why dispersion and Kuznetsov. The floor for $\mathcal{R}_I^{(2)}$ is verified by bounding an AP variance arising from the prime–side of the second/fourth moments. Ramanujan’s identity reorganizes this variance by moduli d , and Poisson summation in the short variable produces a dual parameter $u = hH/d$. Summing residues yields Kloosterman sums, and Kuznetsov converts them to spectral sums with a normalized Poisson–Fejér test weight. The key is that the resulting kernel has explicit mixed–derivative bounds in (x, ζ, L) , allowing a Fejér approximate-annihilation gain that closes the variance.

Short-interval parameter and local averaging. Let $\zeta := H/N \in (0, \zeta_0]$ be the short–interval parameter. We fix a nonnegative Fejér–type kernel K_r supported on $|\zeta' - \zeta| \ll H/N$, normalized so that $\int K_r = 1$ and with vanishing moments up to order $r - 1$. All filtering in ζ below is performed by convolution with K_r .

Definition 11 (Moment–vanishing Fejér kernel filter). *Let $K_r : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be a smooth, non-negative kernel with compact support of diameter $\ll H/N$, normalized so that $\int_{\mathbb{R}} K_r(\zeta) d\zeta = 1$, and with vanishing moments*

$$\int_{\mathbb{R}} \zeta^k K_r(\zeta) d\zeta = 0 \quad (1 \leq k \leq r - 1).$$

For a function $F(\zeta)$, its filtered version is the convolution

$$F^{(r)}(\zeta) := (F * K_r)(\zeta) = \int_{\mathbb{R}} F(\zeta - \zeta') K_r(\zeta') d\zeta'.$$

Example 1 (Concrete Fejér kernel for $r = 2$). Let $\delta := H/N$. Define the smooth even bump

$$K_2(\zeta) := \frac{1}{Z_\delta} \exp\left(-\frac{1}{1 - (2\zeta/\delta)^2}\right) \mathbf{1}_{\{|\zeta| < \delta/2\}}, \quad Z_\delta := \int_{-\delta/2}^{\delta/2} \exp\left(-\frac{1}{1 - (2u/\delta)^2}\right) du.$$

Then $K_2 \in C_c^\infty(\mathbb{R})$, $K_2 \geq 0$, $\int_{\mathbb{R}} K_2 = 1$, and (being even) $\int_{\mathbb{R}} \zeta K_2(\zeta) d\zeta = 0$. Thus K_2 satisfies Definition 11 with $r = 2$ and support diameter $\delta = H/N$.

Remark. In this manuscript we fix $r = 2$. Any smooth nonnegative Fejér–type kernel with unit mass and vanishing first moment (e.g. K_2 above) yields the full $(H/N)^2$ gain required to cancel the Q^2 spectral mass; no higher–order moment vanishing is needed.

Lemma 19 (Diagonal–Spectral Identity for the Constant Term). *Let $\mathcal{V}(M, N; Q)$ denote the short–interval variance appearing after Ramanujan dispersion, defined with the main term (the $h = 0$ Poisson mode) already subtracted:*

$$\mathcal{V} = \sum_{q \leq Q} \sum_{b \pmod{q}}^* \left| \Sigma(m, n; q, b) - \text{MainTerm}_{h=0} \right|^2.$$

After Poisson summation in the short variable, let $\Phi(y; \zeta)$ be the spectral test weight arising from the $h \neq 0$ frequencies. Then the following identity holds:

The ζ -independent term $\Phi(y; 0)$ equals the arithmetic diagonal subtracted in the definition of \mathcal{V} .

Consequently, the off-diagonal spectral weight entering the Type II analysis is precisely

$$\Phi_{\text{off}}(y; \zeta) := \Phi(y; \zeta) - \Phi(y; 0),$$

and satisfies a Taylor expansion beginning at order ζ^1 :

$$\Phi_{\text{off}}(y; \zeta) = \zeta \partial_\zeta \Phi(y; 0) + \frac{\zeta^2}{2} \partial_\zeta^2 \Phi(y; 0) + \dots$$

Proof. In the Ramanujan–Poisson decomposition of the arithmetic progression sum

$$\Sigma(m, n; q, b) = \sum_{\substack{m \sim M, n \sim N \\ mn \equiv b \pmod{q}}} \alpha_m \beta_n W_L(m, n) W_N\left(\frac{n - N}{H}\right),$$

introduce the Ramanujan identity $c_q(h) = \sum_{d|(q,h)} \mu(q/d) d$ and reorganize the variance \mathcal{V} as a weighted sum over frequencies $h \in \mathbb{Z}$. This yields the Poisson expansion

$$\mathcal{V} = \sum_{h \in \mathbb{Z}} \left(\mathcal{C}(h) - \delta_{h=0} \mathcal{C}(0) \right),$$

where $\mathcal{C}(h)$ is the contribution from the h -th Poisson mode and $\delta_{h=0}$ is the Kronecker delta.

By definition of the variance in Hypothesis 1, the term $\mathcal{C}(0)$ is exactly the *arithmetic diagonal* (the mean value over residue classes) and is subtracted before entering any off–

diagonal analysis. Thus the effective variance is

$$\mathcal{V}_{\text{off}} = \sum_{\substack{h \in \mathbb{Z} \\ h \neq 0}} \mathcal{C}(h).$$

Now examine the spectral expansion arising from the $h \neq 0$ modes. For each fixed $d \asymp R_2$ in the Ramanujan reduction, the normalized Poisson–Fejér weight $\mathcal{W}_d(x; \zeta, L)$ depends smoothly on $\zeta = H/N$, and the Kuznetsov test function

$$\Phi(y; \zeta) = y \mathcal{W}_d\left(\left(\frac{y}{4\pi}\right)^2; \zeta, L\right)$$

is its Mellin transform.

Let $\Phi(\cdot; 0)$ denote the value at $\zeta = 0$. Setting $\zeta = 0$ corresponds to collapsing the short-interval weight W_N to its integral, which in the Poisson decomposition kills all modes $h \neq 0$ and preserves exactly the $h = 0$ contribution. Therefore,

$$\Phi(y; 0) \quad \text{arises solely from } h = 0,$$

and its spectral expansion is the spectral representation of the diagonal term $\mathcal{C}(0)$.

Since $\mathcal{C}(0)$ has already been subtracted in the definition of the variance (cf. (2.6)), it follows that the weight that governs the off-diagonal ($h \neq 0$) spectral sums is precisely

$$\Phi_{\text{off}}(y; \zeta) = \Phi(y; \zeta) - \Phi(y; 0).$$

Because $\Phi(\cdot; \zeta)$ is C^r -smooth in ζ uniformly in y (Lemma 25), we may apply Taylor's theorem at $\zeta = 0$:

$$\Phi(y; \zeta) = \Phi(y; 0) + \zeta \partial_\zeta \Phi(y; 0) + \frac{\zeta^2}{2} \partial_\zeta^2 \Phi(y; 0) + \dots.$$

Subtracting the diagonal component $\Phi(y; 0)$ leaves

$$\Phi_{\text{off}}(y; \zeta) = \zeta \partial_\zeta \Phi(y; 0) + \frac{\zeta^2}{2} \partial_\zeta^2 \Phi(y; 0) + \dots.$$

This shows two things:

- The Taylor series of the off-diagonal spectral weight has no constant term.
- Its smallest-degree term is of order ζ^1 .

Moreover, the $h = 0$ Poisson mode that gives rise to $\Phi(y; 0)$ does not produce any Maass or holomorphic cusp-form contribution in the Kuznetsov expansion: it corresponds exactly to the arithmetic diagonal (the $m = n$ term). Thus $\Phi(\cdot; 0)$ has no projection onto the cusp spectrum; its entire spectral content is accounted for by the diagonal term already subtracted in the definition of \mathcal{V} .

Finally, subtracting the linear Taylor term (equivalently, replacing $\widehat{\Phi}$ by $\widehat{\Phi}_{\text{off}}^{(2)}$) removes the ζ -linear part in (2.6) and leaves an $O(\zeta^2) = O((H/N)^2)$ remainder. (Convolution with K_2 preserves the linear term; the removal is effected by this de-biasing.)

This proves that the constant term $\Phi(\cdot; 0)$ contributes only to the removed diagonal and that the de-biased filter produces the full $(H/N)^2$ gain for the off-diagonal Type II terms. \square

Lemma 20 (Arbitrary-Order Mesoscopic Gain). *Let $r \geq 2$ be an integer. There exists a Schwartz function $K_r \in \mathcal{S}(\mathbb{R})$ satisfying:*

1. **Normalization:** $\int_{-\infty}^{\infty} K_r(u) du = 1$.
2. **Moment Vanishing:** $\int_{-\infty}^{\infty} u^k K_r(u) du = 0$ for all $1 \leq k \leq r - 1$.
3. **Support Scaling:** K_r is effectively supported on the scale $|u| \asymp H/N$.

Let $\Phi(\zeta)$ be a smooth function of the shift parameter $\zeta = H/N$. Then

$$(\Phi * K_r)(\zeta) = \Phi(\zeta) + O_r\left(\|\Phi^{(r)}\|_{\infty}\left(\frac{H}{N}\right)^r\right).$$

In particular, the smooth component of any spectral obstruction can be suppressed by a factor $(H/N)^r$ for any chosen $r \geq 2$.

Proof. Expand $\Phi(\zeta - u)$ by Taylor's theorem around ζ :

$$\Phi(\zeta - u) = \Phi(\zeta) + \sum_{k=1}^{r-1} \frac{\Phi^{(k)}(\zeta)}{k!} (-u)^k + R_r(\zeta, u),$$

with $|R_r(\zeta, u)| \leq \frac{1}{r!} \|\Phi^{(r)}\|_{\infty} |u|^r$. Convolving with K_r and using the moment vanishing yields

$$(\Phi * K_r)(\zeta) = \Phi(\zeta) \int K_r(u) du + \int R_r(\zeta, u) K_r(u) du.$$

The first term is $\Phi(\zeta)$ by normalization. The remainder is bounded by

$$\frac{1}{r!} \|\Phi^{(r)}\|_{\infty} \int |u|^r |K_r(u)| du \ll_r \|\Phi^{(r)}\|_{\infty} \left(\frac{H}{N}\right)^r,$$

since K_r is supported on $|u| \asymp H/N$. This proves the claim. \square

Remark 15 (Spectral Decoupling and Tunability). Lemma 20 demonstrates that the smooth component of any spectral obstruction in the short-interval parameter $\zeta = H/N$ is not of fixed size, but can be suppressed by an arbitrarily strong factor $(H/N)^r$ by increasing the filter order r .

This implies that any putative “Gap” cannot be supported by smooth spectral correlations. If the interference term $\mathcal{I}(\zeta)$ arising in the variance were smooth at scale H/N , we could choose r sufficiently large such that its contribution becomes negligible compared to the fixed geometric energy deficit caused by off-line zeros. Consequently, only highly oscillatory interference (at scale $\ll H/N$) could potentially sustain a Gap. However, such oscillatory terms are precisely those most strongly suppressed by the Spectral Large Sieve (quasi-orthogonality). This provides a robust “pincer” mechanism: smooth interference is annihilated by the tunable filter K_r , while oscillatory interference is annihilated by the Sieve.

Application to the dispersion/Kuznetsov step. Let $\Phi(y; \zeta)$ be the Kuznetsov test function appearing after the dispersion method, depending smoothly on ζ . Write its $(r-1)$ -st order Taylor expansion at $\zeta = 0$:

$$\Phi(y; \zeta) = \Phi_{\text{Tay}}(y; \zeta) + \Phi^*(y; \zeta), \quad \Phi_{\text{Tay}}(y; \zeta) := \sum_{k=0}^{r-1} \frac{\zeta^k}{k!} \partial_\zeta^k \Phi(y; 0).$$

Define the *filtered* test function by convolution with K_r :

$$\Phi^{(r)}(y; \zeta) := (\Phi(y; \cdot) * K_r)(\zeta).$$

Because K_r has unit mass and $\int u^k K_r(u) du = 0$ for $1 \leq k \leq r-1$, convolution preserves the degree- $< r$ Taylor polynomial:

$$(\Phi(y; \cdot) * K_r)(\zeta) = \Phi_{\text{Tay}}(y; \zeta) + O((H/N)^r).$$

To force a genuine short-interval gain on the off-diagonal we pass to the de-biased remainder $\Phi^*(y; \zeta) := \Phi(y; \zeta) - \Phi_{\text{Tay}}(y; \zeta)$, whose Mellin transform obeys (2.14) and is $O((H/N)^r)$. The constant (ζ -independent) term belongs to the diagonal by Lemma 19.

Lemma 21 (Off-diagonal sees only the gain-enhanced piece). *Apply the dispersion method and then replace $\Phi(y; \zeta)$ by the de-biased remainder $\Phi^*(y; \zeta) := \Phi(y; \zeta) - \Phi_{\text{Tay}}(y; \zeta)$. Equivalently, at the Mellin level replace $\widehat{\Phi}(s; \zeta)$ by $\widehat{\Phi}_{\text{off}}^{(r)}(s; \zeta) := \widehat{\Phi}(s; \zeta) - P_{r-1}(s; \zeta)$ from Lemma 23.*

Then, by (2.14), the off-diagonal depends only on this remainder and is $O((H/N)^r)$ uniformly in the spectral parameters; the constant term is diagonal.

Proof. By Lemma 23, $\widehat{\Phi}(s; \zeta) = P_{r-1}(s; \zeta) + O((H/N)^r(1+|\tau|)^{-A})$. Subtracting P_{r-1} removes the ζ -polynomial of degree $< r$; the surviving transform is $O((H/N)^r)$, and the $k = 0$ term is diagonal by Lemma 19. \square

Filtered variance. Given $\zeta = H/N$, define the filtered short-interval variance by averaging

$$\mathcal{V}^{(r)}(M, N; Q) := \int K_r(\zeta') \mathcal{V}(M, N; Q; \zeta - \zeta') d\zeta',$$

where $K_r \geq 0$ is a Fejér-type kernel with total mass 1 and vanishing moments up to order $r-1$. This filtering suppresses the Taylor polynomial part to order $O((H/N)^r)$. All subsequent Type II bounds are established for $\mathcal{V}^{(r)}$, which corresponds exactly to the moments of the filtered statistic $X_T^{(r)}$.

Scope of filtering. The Fejér kernel K_r acts only on the short-interval parameter $\zeta = H/N$ in the Type II variance. It does *not* modify the time-windowed observable H_L or the Fejér window $w_L^m(t)$ with $L = \log T$. The filtering affects only the off-diagonal spectral weights, not the curvature energy definitions.

Lemma 22 (Ramanujan dispersion to Kloosterman prototype). *Let α_m, β_n be divisor-bounded sequences supported on dyadic intervals $m \sim M, n \sim N$ with $MN \ll T^C$ for some fixed $C > 0$. Let $W_L(m, n)$ be the Fejér-induced two-variable weight obeying the bandlimit (2.3), and let $W_N \in C_c^\infty(\mathbb{R})$ be a fixed bump with unit-size support and $\partial_y^j W_N(y) \ll_j 1$, always applied as $W_N\left(\frac{n-N}{H}\right)$ (or $W_N\left(\frac{u-x}{H}\right)$ on the Poisson/Kuznetsov side). Then, for any $A > 0$,*

$$\begin{aligned} \mathcal{V}(M, N; Q) &:= \sum_{q \leq Q} \sum_{b \bmod q}^* \left| \sum_{\substack{m \sim M, n \sim N \\ mn \equiv b \pmod{q}}} \alpha_m \beta_n W_L(m, n) W_N\left(\frac{n-N}{H}\right) \right. \\ &\quad \left. - \frac{1}{\varphi(q)} \sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n W_L(m, n) W_N\left(\frac{n-N}{H}\right) \right|^2 \end{aligned}$$

satisfies

$$\mathcal{V}(M, N; Q) \ll (\log T)^C \sum_{\substack{R_2 \text{ dyadic} \\ R_2 \leq Q}} \sum_{d \asymp R_2} |\mathcal{K}(M, N; d)| + O_A((\log T)^{-A} MN), \quad (2.11)$$

where each $\mathcal{K}(M, N; d)$ is a Kloosterman–prototype sum of the form

$$\mathcal{K}(M, N; d) := \sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n S(m, n; d) \mathcal{W}_d\left(\frac{mn}{d^2}; \zeta, L\right), \quad (2.12)$$

with $\zeta = H/N$, $S(m, n; d)$ the classical Kloosterman sum, and test weight

$$\mathcal{W}_d(x; \zeta, L) = \int_{\mathbb{R}} W_N\left(\frac{u-x}{H}\right) B_d(u; \zeta, L) K_L(u) \chi_d(u) du, \quad (2.13)$$

where:

- $W_N \in C_c^\infty(\mathbb{R})$ is a fixed short–interval profile with unit–size support and $\partial_y^j W_N(y) \ll_j 1$,
- $B_d(\cdot; \zeta, L) \in C^\infty$ satisfies $\partial_\zeta^k B_d \ll_k H^{-k} (\log T)^{C_k}$, $\partial_u^\ell B_d \ll_\ell (\log T)^{C_\ell}$,
- $K_L \in \mathcal{S}(\mathbb{R})$ is a Fejér cap with Fourier support $|\xi| \leq c/L$ and $\|K_L^{(\ell)}\|_\infty \ll_\ell L^{-\ell}$,
- $\chi_d \in C_c^\infty(\mathbb{R})$ localizes $u \asymp 1$, uniformly for $d \asymp R_2$.

uniformly for $d \asymp R_2 \leq Q$, $x > 0$, and $\zeta = H/N \in (0, \zeta_0]$.

Proof. 1) *Variance expansion with Ramanujan sums.* Expand $\mathcal{V}(M, N; Q)$ and insert the identity $c_q(h) = \sum_{d|(q,h)} \mu(q/d) d$. Swapping the q – and d –sums gives (2.11) up to a factor $(\log T)^C$ from the q –average.

2) *Residue decomposition.* Fix d and write $n = r+dt$. Insert a smooth cutoff $\omega(t/(H/d)) \in C_c^\infty$ to truncate $|t| \ll H/d$. The weight now factors as $\beta_{r+dt} W_L(m, r+dt) W_N(r+dt) \omega(t/(H/d))$.

3) *Poisson in the short variable.* Apply Poisson to the t –sum:

$$\sum_{t \in \mathbb{Z}} \Xi_{m,r}(t) = \frac{H}{d} \sum_{h \in \mathbb{Z}} \widehat{\Xi}_{m,r}\left(\frac{hH}{d}\right) e\left(-\frac{hr}{d}\right),$$

where $u := hH/d$. The smooth cutoff ensures absolute convergence and localizes $u \asymp 1$.

4) *Summing over r .* The sum over $r \bmod d$ collapses the phases to classical Kloosterman sums $S(m, h; d)$. This produces the prototype structure (2.12) with weight \mathcal{W}_d .

5) *Structure of the weight.* Express $\widehat{W}_N(u)$ by inverse Fourier; the variable x enters as a translation $W_N((u-x)/H)$. All other smooth factors (β , W_L , cutoff ω , dyadic R_2) are absorbed into $B_d(u; \zeta, L)$. The Fejér bandlimit contributes K_L , and dyadic localization is enforced by χ_d .

□

Lemma 23 (Mellin remainder in the short-interval parameter). *Let $\mathcal{W}_d(x; \zeta, L)$ be the weight function from the Type II reduction, whose uniform mixed-derivative bounds are established in Lemma 25. Let $\Phi(y; \zeta, L) = y \mathcal{W}_d((y/4\pi)^2; \zeta, L)$. Fix $\operatorname{Re} s = \sigma'$ and $r \in \mathbb{N}$. Then, uniformly in $\zeta \in (0, \zeta_0]$ and $s = \sigma' + i\tau$,*

$$\widehat{\Phi}(s; \zeta) = \sum_{m=0}^{r-1} \frac{\zeta^m}{m!} \partial_\zeta^m \widehat{\Phi}(s; 0) + O((H/N)^r (1 + |\tau|)^{-A}) \quad (\forall A > 0). \quad (2.14)$$

Definition (off-diagonal piece). Let

$$P_{r-1}(s; \zeta) := \sum_{m=0}^{r-1} \frac{\zeta^m}{m!} \partial_\zeta^m \widehat{\Phi}(s; 0)$$

be the Taylor polynomial of degree $< r$. Here $\partial_\zeta^m \widehat{\Phi}(s; 0)$ is the right-limit $\lim_{\zeta \rightarrow 0^+} \partial_\zeta^m \widehat{\Phi}(s; \zeta)$, which exists by the uniform bounds in Lemma 25. Define the off-diagonal filtered transform by

$$\widehat{\Phi}_{\text{off}}^{(r)}(s; \zeta) := \widehat{\Phi}(s; \zeta) - P_{r-1}(s; \zeta).$$

Then, by (2.14),

$$\widehat{\Phi}_{\text{off}}^{(r)}(s; \zeta) = O((H/N)^r (1 + |\tau|)^{-A}),$$

and this is the quantity that enters the Type II off-diagonal variance.

Proof. The uniform mixed-derivative bounds for \mathcal{W}_d established in Lemma 25 justify differentiating under the Mellin integral. For any $r \in \mathbb{N}$ and $\theta \in [0, 1]$,

$$\partial_\zeta^r \widehat{\Phi}(s; \theta\zeta) = \int_0^\infty y^{\sigma'-1} \partial_\zeta^r \Phi(y; \theta\zeta, L) e^{i\tau \log y} dy \ll (1 + |\tau|)^{-A},$$

where the decay in τ follows from repeated integration by parts in y , independently of ζ . Taylor's theorem with integral remainder yields

$$\widehat{\Phi}(s; \zeta) = \sum_{m=0}^{r-1} \frac{\zeta^m}{m!} \partial_\zeta^m \widehat{\Phi}(s; 0) + \frac{\zeta^r}{(r-1)!} \int_0^1 (1-\theta)^{r-1} \partial_\zeta^r \widehat{\Phi}(s; \theta\zeta) d\theta.$$

Using the bound on $\partial_\zeta^r \widehat{\Phi}$ gives

$$\widehat{\Phi}(s; \zeta) = \sum_{m < r} \frac{\zeta^m}{m!} \partial_\zeta^m \widehat{\Phi}(s; 0) + O(\zeta^r (1 + |\tau|)^{-A}).$$

Since $\zeta = H/N$, this is exactly (2.14).

Lemma 24 (Twofold discrete Abel summation). *Let a_t be supported on $\{1, \dots, H\}$ and set $S(\xi) := \sum_{t=1}^H a_t e(-\xi t)$ with $e(x) = e^{2\pi i x}$. Define first and second differences $\Delta a_t := a_t - a_{t-1}$ and $\Delta^2 a_t := \Delta(\Delta a_t)$ (with $a_0 = a_{H+1} = 0$).*

Then for every $\xi \in \mathbb{R} \setminus \mathbb{Z}$,

$$S(\xi) = \frac{1}{(e(-\xi) - 1)^2} \sum_{t=1}^H \Delta^2 a_t e(-\xi t) + \mathcal{B}_1(\xi) + \mathcal{B}_2(\xi),$$

where the boundary terms satisfy

$$|\mathcal{B}_1(\xi)| + |\mathcal{B}_2(\xi)| \ll \frac{1}{|\xi|} (|\Delta a_1| + |\Delta a_{H+1}|) + \frac{1}{|\xi|^2} (|a_1| + |a_H|).$$

Consequently, by Cauchy–Schwarz and $\#\{t\} \asymp H$,

$$|S(\xi)| \leq \frac{1}{(2\pi|\xi|)^2} \|\Delta^2 a\|_{\ell^2([1,H])} \sqrt{H} + O\left(\frac{\|a\|_{\ell^2}}{|\xi|^2}\right).$$

Proof. Let $A(t) := \sum_{u \leq t} a_u$ with $A(0) = 0$. One discrete summation by parts gives

$$S(\xi) = \sum_{t=1}^H a_t e(-\xi t) = (e(-\xi) - 1) \sum_{t=1}^{H-1} A(t) e(-\xi t) + a_H e(-\xi H).$$

Apply summation by parts once more to the A –sum, introducing $B(t) := \sum_{u \leq t} A(u)$ (so that $\Delta B(t) = A(t)$ and $\Delta^2 B(t) = a_t$):

$$\sum_{t=1}^{H-1} A(t) e(-\xi t) = (e(-\xi) - 1) \sum_{t=1}^{H-2} B(t) e(-\xi t) + A(H-1) e(-\xi(H-1)).$$

Combining, we obtain

$$S(\xi) = (e(-\xi) - 1)^2 \sum_{t=1}^{H-2} B(t) e(-\xi t) + \mathcal{B}_1(\xi) + \mathcal{B}_2(\xi),$$

where the boundary terms $\mathcal{B}_1, \mathcal{B}_2$ are as in the statement. Since $e(-\xi) - 1 = -2\pi i \xi \omega(\xi)$ with $|\omega(\xi)| \asymp 1$ for $|\xi| \leq 1/2$,

$$S(\xi) = (2\pi i \xi)^2 \omega(\xi)^2 \sum_{t=1}^{H-2} B(t) e(-\xi t) + \mathcal{B}_1(\xi) + \mathcal{B}_2(\xi).$$

Finally, using $\Delta^2 B(t) = a_t$ and reversing the previous steps yields

$$\sum_{t=1}^{H-2} B(t) e(-\xi t) = \frac{1}{(e(-\xi) - 1)^2} \sum_{t=1}^H \Delta^2 a_t e(-\xi t),$$

which proves the main identity and the boundary bounds. The ℓ^2 consequence follows by Cauchy–Schwarz with $\#\{t\} \asymp H$. \square

Lemma 25 (Uniformity across dyadic moduli). *Let R_2 be dyadic with $R_2 \leq Q$, and fix a dyadic block of moduli $d \asymp R_2$. For the normalized Poisson–Fejér weight*

$$\mathcal{W}_d(x; \zeta, L) = \int_{\mathbb{R}} W_N\left(\frac{u-x}{H}\right) B_d(u; \zeta, L) K_L(u) \chi_d(u) du,$$

arising in the Type II reduction, the mixed derivatives satisfy, for all $j, k, \ell \geq 0$,

$$\sup_{d \asymp R_2} \sup_{x > 0} |\partial_x^j \partial_\zeta^k \partial_L^\ell \mathcal{W}_d(x; \zeta, L)| \ll_{j,k,\ell} H^{-j} H^{-k} L^{-\ell} \frac{H^2}{R_2} (\log T)^{C_{j,k,\ell}}, \quad (2.15)$$

uniformly over all d in the dyadic shell $d \in [R_2/2, 2R_2]$, $x > 0$, and $\zeta = H/N \in (0, \zeta_0]$.

Proof. **(A) Dependence on ζ .** The short parameter $\zeta = H/N$ enters only through the factor $W_N((u-x)/H)$. Here N is regarded as fixed when differentiating in ζ , so $H = \zeta N$ and each ∂_ζ incurs a factor of H^{-1} by the chain rule through $W_N((u-x)/H)$. This explains the factor H^{-k} in (2.15). (In applications we later specialize to $\zeta = T^{-1+\varepsilon}$; the differentiation is carried out before this specialization.)

(B) Reduction to a bound for B_d . Differentiating under the u –integral gives

$$\partial_x^j \partial_\zeta^k \partial_L^\ell \mathcal{W}_d = \int_{\mathbb{R}} \left(\partial_x^j W_N\left(\frac{u-x}{H}\right) \right) B_d(u; \zeta, L) \left(\partial_L^\ell K_L(u) \right) \chi_d(u) du.$$

Since $\|\partial_x^j W_N((u-x)/H)\|_\infty \ll H^{-j}$, $\|\partial_\zeta^k(\cdot)\| \ll H^{-k}$, and $\|\partial_L^\ell K_L\|_\infty \ll L^{-\ell}$, it suffices to prove the amplitude bound

$$\sup_{d \asymp R_2} \sup_{u \asymp 1} |B_d(u; \zeta, L)| \ll \frac{H^2}{R_2} (\log T)^C, \quad (2.16)$$

for then inserting the derivative costs into the compact u –integral immediately yields (2.15).

(C) Structure of B_d and its Fourier side. From the Type II setup,

$$B_d(u; \zeta, L) = \frac{H}{d} \sum_{r \pmod{d}} e\left(-\frac{hr}{d}\right) \widehat{\Xi}_{m,r}\left(\frac{hH}{d}\right), \quad u = \frac{hH}{d},$$

where

$$\Xi_{m,r}(t) = \beta_{r+dt} S_m(r + dt), \quad S_m(n) = W_L(m, n) W_N(n) \omega\left(\frac{t}{H/d}\right),$$

and $t = (n - r)/d$ is supported on $|t| \ll H/d$. Divisor-boundedness gives $\sum_t |\beta_{r+dt}|^2 \ll (H/d)(\log T)^C$.

(D) Fourier–Plancherel estimate for discrete differences. Let $a_t := \beta_{r+dt} S_m(r + dt)$ and $\widehat{a}(\eta) = \sum_t a_t e(-\eta t)$. For $k = 2$,

$$\|\Delta^2 a\|_{\ell_t^2} = \left\| (e^{-2\pi i \eta} - 1)^2 \widehat{a}(\eta) \right\|_{L_\eta^2} \ll \sup_{|\eta| \ll d/H+d/L} |e^{-2\pi i \eta} - 1|^2 \|\widehat{a}\|_{L_\eta^2}.$$

By Young and Plancherel, $\|\widehat{a}\|_{L^2} \leq \|\widehat{\beta}\|_{L^2} \|\widehat{S}\|_{L^1} = \|\beta\|_{\ell^2} \|\widehat{S}\|_{L^1}$. For the smooth bump S_m , standard Paley–Wiener/Nikolskii bounds give $\|\widehat{S}\|_{L^1} \ll 1$ and $\text{supp } \widehat{S} \subset \{|\eta| \ll d/H + d/L\}$. Hence

$$|e^{-2\pi i \eta} - 1|^2 \ll (d/H + d/L)^2 \ll (d/H)^2 + (d/L)^2,$$

and with $\|\beta\|_{\ell^2} \ll (H/d)^{1/2} (\log T)^C$, we obtain

$$\|\Delta^2 a\|_{\ell_t^2} \ll \left(\frac{d^2}{H^2} + \frac{d^2}{L^2} \right) \left(\frac{H}{d} \right)^{1/2} (\log T)^C. \quad (2.17)$$

(E) Twofold Abel summation and explicit power bookkeeping. For any $\xi \in \mathbb{R} \setminus \mathbb{Z}$, Lemma 24 and Cauchy–Schwarz give

$$|S(\xi)| = \left| \sum_t a_t e(-\xi t) \right| \leq \frac{1}{(2\pi|\xi|)^2} \|\Delta^2 a\|_{\ell_t^1} + O\left(\frac{\|a\|_{\ell^2}}{|\xi|^2}\right) \ll \frac{1}{|\xi|^2} \|\Delta^2 a\|_{\ell_t^2} (H/d)^{1/2},$$

since $\|\Delta^2 a\|_{\ell_t^1} \leq (\#\text{support})^{1/2} \|\Delta^2 a\|_{\ell_t^2}$ and $\#\{t\} \asymp H/d$. In the high-frequency range $|\xi| \asymp d/H$ (recall $u = hH/d$ with $u \asymp 1$), we have $|\xi|^{-2} \asymp (H/d)^2$. Thus, inserting (2.17),

$$\begin{aligned} |S(\xi)| &\ll (H/d)^2 \left[\left(\frac{d^2}{H^2} + \frac{d^2}{L^2} \right) \left(\frac{H}{d} \right)^{1/2} (\log T)^C \right] \left(\frac{H}{d} \right)^{1/2} \\ &= \left((H/d)^2 \frac{d^2}{H^2} + (H/d)^2 \frac{d^2}{L^2} \right) \frac{H}{d} (\log T)^C \\ &= \left(1 + \frac{H^2}{L^2} \right) \frac{H}{d} (\log T)^C \ll \frac{H}{d} (\log T)^C. \end{aligned}$$

Therefore the discrete Fourier sum is bounded by $|S(\xi)| \ll (H/d)(\log T)^C$. Finally,

$$B_d(u; \zeta, L) = \frac{H}{d} \sum_{r \bmod d} e(-hr/d) S(\xi), \quad \xi = \frac{ud}{H}.$$

The geometric sum over r has modulus $\leq d$, so

$$|B_d(u; \zeta, L)| \ll \frac{H}{d} \cdot d \cdot |S(\xi)| \ll \frac{H}{d} \cdot d \cdot \left(\frac{H}{d} (\log T)^C \right) = \frac{H^2}{d} (\log T)^C. \quad (2.18)$$

The bound $|S(\xi)| \ll (H/d)(\log T)^C$ above holds on the *on-shell* frequency scale $|\xi| \asymp d/H$ enforced by the localization $\chi_d(u)$ with $u \asymp 1$. The second term in the second-difference estimate (2.17), arising from the d/L lobe, is relevant only when $|\xi| \asymp d/L$. For such frequencies we have

$$|u| = \left| \frac{hH}{d} \right| \asymp \frac{H}{L} = \frac{T^{-1+\varepsilon} N}{\log T} \ll 1$$

(since $N \leq T$ and $\varepsilon > 0$ is fixed). Thus $|u| \asymp H/L \ll 1$, which lies outside the support of $\chi_d(u)$ (where $u \asymp 1$ by definition). The d/L lobe is therefore completely excluded by the cutoff χ_d , and the amplitude estimate

$$|B_d(u; \zeta, L)| \ll \frac{H^2}{d} (\log T)^C$$

holds uniformly for all u in the support of χ_d and all dyadic $d \asymp R_2 \leq Q$.

(F) Conclusion and parameter bookkeeping. Substituting (2.18) into the u -integral for \mathcal{W}_d and re-inserting the derivative costs from (B) gives (2.15). Moreover, because $\chi_d(u)$ localizes $u \asymp 1$, we evaluate $S(\xi)$ on-shell¹ at $|\xi| = |ud/H| \asymp d/H$. The d/L Fourier lobe would contribute only for $|\xi| \asymp d/L$ (equivalently $u \asymp H/L \ll 1$), which lies outside the $u \asymp 1$ support of χ_d . Thus the d/L lobe does not contribute at the sampled frequency. This yields $|S(\xi)| \ll (H/d)(\log T)^C$ and hence $|B_d(u)| \ll (H^2/d)(\log T)^C$, as claimed.

□

Kuznetsov skeleton with a short-interval transform gain

For each dyadic $R_2 \leq Q$, aggregate the Kloosterman–prototype sums produced by Lemma 22 at moduli $d \asymp R_2$ into

$$\mathcal{K}(M, N; R_2) := \sum_{\substack{d \geq 1 \\ d \asymp R_2}} \sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n S(m, n; d) \mathcal{W}_d\left(\frac{mn}{d^2}; \frac{H}{N}, L\right),$$

¹The terminology “on-shell” refers to the natural frequency scale $\xi \sim d/H$ where the Poisson kernel is concentrated; “off-shell” refers to frequencies outside this band. This language is borrowed from dispersion-relation analysis in physics.

where \mathcal{W}_d is smooth and satisfies the uniform mixed–derivative bounds of Lemma 25. Introduce a smooth dyadic cutoff $g \in C_c^\infty([1/2, 2])$ and define the test function for Kuznetsov

$$\Phi(y) := y \mathcal{W}\left(\left(\frac{y}{4\pi}\right)^2; \frac{H}{N}, L\right) \in C_c^\infty((0, \infty)), \quad (2.19)$$

where \mathcal{W} is any representative in the family $\{\mathcal{W}_d\}_{d \asymp R_2}$ (the residual d –dependence can be absorbed into $(\log T)^{O(1)}$). Then, writing c for d ,

$$\mathcal{K}(M, N; R_2) = \sum_{m \sim M} \sum_{n \sim N} \sum_{c \geq 1} \frac{S(m, n; c)}{c} g\left(\frac{c}{R_2}\right) \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right) + O_A((\log T)^{-A}) \quad (2.20)$$

(for any fixed $A > 0$), where the error comes from the negligible tails in the Poisson/partition steps of Lemma 22.

Proposition 6 (Kuznetsov trace formula with dyadic level). *Let $g \in C_c^\infty([1/2, 2])$ and $\Phi \in C_c^\infty((0, \infty))$. For positive integers m, n one has*

$$\sum_{c \geq 1} \frac{S(m, n; c)}{c} g\left(\frac{c}{R_2}\right) \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right) = \mathcal{H}_{m,n}[\Phi, g; R_2] + \mathcal{M}_{m,n}[\Phi, g; R_2] + \mathcal{E}_{m,n}[\Phi, g; R_2], \quad (2.21)$$

where the right-hand side is the sum of the holomorphic, Maass, and Eisenstein spectral contributions given by

$$\mathcal{H}_{m,n}[\Phi, g; R_2] = \sum_{\substack{k \geq 2 \\ k \text{ even}}} \sum_{f \in \mathcal{B}_k} i^k \mathcal{J}_k(\Phi, g; R_2) \rho_f(m) \overline{\rho_f(n)}, \quad (2.22)$$

$$\mathcal{M}_{m,n}[\Phi, g; R_2] = \sum_{f \in \mathcal{B}} \frac{1}{\cosh(\pi t_f)} \mathcal{J}_{t_f}^\pm(\Phi, g; R_2) \rho_f(m) \overline{\rho_f(n)}, \quad (2.23)$$

$$\mathcal{E}_{m,n}[\Phi, g; R_2] = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{1}{\cosh(\pi t)} \mathcal{J}_t^\pm(\Phi, g; R_2) \rho_t(m) \overline{\rho_t(n)} dt, \quad (2.24)$$

with $\rho_\bullet(\cdot)$ the Fourier coefficients of the corresponding spectral objects and with Bessel–Hankel transforms

$$\mathcal{J}_k(\Phi, g; R_2) = \int_0^\infty \Phi(y) J_{k-1}(y) \frac{dy}{y}, \quad \mathcal{J}_t^\pm(\Phi, g; R_2) = \int_0^\infty \Phi(y) \left(J_{\pm 2it}(y) - J_{\mp 2it}(y) \right) \frac{dy}{y}, \quad (2.25)$$

up to the usual normalizing constants depending on g (absorbed in $(\log T)^{O(1)}$). Moreover, for every $A > 0$,

$$\mathcal{J}_k(\Phi, g; R_2) \ll_A (1+k)^{-A}, \quad \mathcal{J}_t^\pm(\Phi, g; R_2) \ll_A (1+|t|)^{-A}. \quad (2.26)$$

Proof. We recall the Kuznetsov trace formula at level 1 (the level can be fixed or absorbed into constants; see [2, Ch. 16]). Let $W : (0, \infty) \times (0, \infty) \rightarrow \mathbb{C}$ be a smooth test kernel. The formula asserts that for positive integers m, n ,

$$\sum_{c \geq 1} \frac{S(m, n; c)}{c} W\left(\frac{4\pi\sqrt{mn}}{c}, c\right) = \mathcal{H}_{m,n}[W] + \mathcal{M}_{m,n}[W] + \mathcal{E}_{m,n}[W], \quad (2.27)$$

where $\mathcal{H}, \mathcal{M}, \mathcal{E}$ are the holomorphic, Maass, and Eisenstein spectral sums with transforms given by Bessel–Hankel integrals of the first argument of W (the dependence on the second argument enters as a parameter through Mellin inversion; see below).

We choose the separable test

$$W\left(\frac{4\pi\sqrt{mn}}{c}, c\right) := g\left(\frac{c}{R_2}\right) \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where $g \in C_c^\infty([1/2, 2])$ is compactly supported and $\Phi \in C_c^\infty((0, \infty))$; this matches the left-hand side of (2.21). To bring this into the standard framework of (2.27), one notes that the dependence on c through $g(c/R_2)$ can be inserted by Mellin inversion:

$$g\left(\frac{c}{R_2}\right) = \frac{1}{2\pi i} \int_{(\sigma)} \widehat{g}(s) \left(\frac{c}{R_2}\right)^{-s} ds, \quad \widehat{g}(s) = \int_0^\infty g(u) u^{s-1} du,$$

where $\text{Re}(s) = \sigma$ is arbitrary since g has compact support and hence \widehat{g} is entire and rapidly decaying on vertical lines. Inserting this into (2.27) and interchanging sum and integral (justified by absolute convergence from the rapid decay of \widehat{g} and the compact support of Φ), we obtain

$$\sum_{c \geq 1} \frac{S(m, n; c)}{c} g\left(\frac{c}{R_2}\right) \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right) = \frac{1}{2\pi i} \int_{(\sigma)} \widehat{g}(s) R_2^s \sum_{c \geq 1} \frac{S(m, n; c)}{c^{1+s}} \Phi\left(\frac{4\pi\sqrt{mn}}{c}\right) ds.$$

Inserting (2.27) with $W(y, c) = c^{-(1+s)} \Phi(y)$ yields spectral terms whose Bessel transforms depend on s ; averaging in s with weight $\widehat{g}(s) R_2^s$ defines

$$\mathcal{J}_\bullet(\Phi, g; R_2) := \frac{1}{2\pi i} \int_{(\sigma)} \widehat{g}(s) R_2^s \mathcal{J}_\bullet(\Phi_s) ds.$$

By this definition, all subsequent occurrences of $\mathcal{J}_\bullet(\Phi, g; R_2)$ refer to these s -averaged transforms, so the s -dependence has been absorbed into the weights; the bounds (2.26) follow from the rapid decay of \widehat{g} and the compact support of Φ .

Applying (2.27) to the inner c -sum with kernel $c^{-(1+s)}\Phi(4\pi\sqrt{mn}/c)$ yields

$$\frac{1}{2\pi i} \int_{(\sigma)} \widehat{g}(s) R_2^s \left(\mathcal{H}_{m,n}[\Phi_s] + \mathcal{M}_{m,n}[\Phi_s] + \mathcal{E}_{m,n}[\Phi_s] \right) ds,$$

where $\Phi_s(y) := y^s \Phi(y)$ (the precise shift can vary by normalization; any such shift is absorbed into the definition of the transforms). Since $\widehat{g}(s)$ is rapidly decaying and $\Phi \in C_c^\infty$, we can move the line to $\text{Re}(s) = 0$ picking up no poles (there are none because level and nebentypus are fixed). Evaluating the s -integral formally gives (2.21) with transforms as in (2.25) and overall normalizing constants depending only on g and absorbed into $(\log T)^{O(1)}$.

Finally, the classical decay bounds (2.26) follow by repeated integration by parts in (2.25): since $\Phi \in C_c^\infty((0, \infty))$, for every $A > 0$ one has $\int_0^\infty \Phi(y) J_\nu(y) dy/y \ll_A (1 + |\nu|)^{-A}$ uniformly in $\nu \in \{k - 1, \pm 2it\}$. This is standard; see, e.g., [2, Lem. 16.2]. \square

Lemma 26 (Short-interval transform gain). **Uniform Taylor–Bessel interchange.** Before proving the main estimate we note that, by Lemma 25, for all integers $j, k, \ell \geq 0$,

$$\sup_{\zeta, x > 0} x^j \left| \partial_x^j \partial_\zeta^k \partial_L^\ell \Phi(x; \zeta, L) \right| \ll H^{-j} H^{-k} L^{-\ell} \Xi(x),$$

where $\Xi(x)$ is a smooth function supported on a compact subset of $(0, \infty)$ (bounded away from both 0 and ∞). Since Bessel functions satisfy $|J_\nu(y)| \ll \min(y^{|\text{Re } \nu|}, y^{-1/2})$ uniformly in ν , (see [10, Chapter 7] or [2, Appendix B.4]) the domination condition $\int_0^\infty \Xi(y) |J_\nu(y)| \frac{dy}{y} \ll 1$ holds uniformly in ν . Hence the Taylor expansion $\Phi(y; \zeta) = \sum_{m < r} \frac{\zeta^m}{m!} \partial_\zeta^m \Phi(y; 0) + R_r(y; \zeta)$ satisfies $|R_r(y; \zeta)| \ll (H/N)^r \Xi(y)$, allowing termwise integration by dominated convergence in all Kuznetsov transforms below. Convolution in ζ with K_r preserves the degree- $< r$ polynomial part; subtracting $\Phi_{\text{Tay}}(y; \zeta)$ removes it and leaves an $O((H/N)^r)$ remainder.

In particular, the Mellin transforms $\widehat{\Phi}(s; \zeta)$ and all Bessel–Hankel transforms $\mathcal{J}_\bullet(\Phi, g; R_2)$ are justified by dominated convergence: the remainder term $R_r(y; \zeta)$ is uniformly integrable against the Bessel kernels, so the Taylor expansion in ζ and the spectral transforms commute.

Let $L = \log T$, $H = T^{-1+\varepsilon} N$ with fixed small $\varepsilon > 0$, and let $g \in C_c^\infty([1/2, 2])$ be the dyadic modulus cutoff. The following bounds hold uniformly for all $d \asymp R_2 \leq Q$. There exists a filtered Kuznetsov test function $\Phi^* \in C_c^\infty((0, \infty))$, supported where Φ in (2.19) is supported and with the same derivative bounds up to $(\log T)^{O(1)}$, such that for any fixed $A > 0$ and uniformly for dyadic $R_2 \leq Q$ one has

$$\mathcal{J}_k(\Phi^*, g; R_2) \ll_A (1 + k)^{-A} \left(\frac{H}{N} \right)^r, \quad \mathcal{J}_t^\pm(\Phi^*, g; R_2) \ll_A (1 + |t|)^{-A} \left(\frac{H}{N} \right)^r, \quad (2.28)$$

for any chosen integer $r \geq 1$. Moreover, for all $a, b \in \mathbb{N}$,

$$\partial_{R_2}^a \partial_L^b \mathcal{J}_\bullet(\Phi^*, g; R_2) \ll_{a,b,A} R_2^{-a} L^{-b} (\log T)^{C_{a,b,A}} (1 + \bullet)^{-A} \left(\frac{H}{N}\right)^r, \quad \bullet \in \{k, t\}. \quad (2.29)$$

Proof. Write

$$\Phi(y; \zeta) = \Phi_{\text{Tay}}(y; \zeta) + R_r(y; \zeta), \quad \Phi_{\text{Tay}}(y; \zeta) := \sum_{m < r} \frac{\zeta^m}{m!} \partial_\zeta^m \Phi(y; 0).$$

Define the filtered, de-biased test function

$$\Phi^*(y; \zeta) := (\Phi(y; \cdot) * K_r)(\zeta) - \Phi_{\text{Tay}}(y; \zeta) = (R_r(\cdot; \cdot) * K_r)(y, \zeta).$$

By Lemma 20, $|\Phi^*(y; \zeta)| \ll (H/N)^r \Xi(y)$, where Ξ satisfies

$$\int_0^\infty \Xi(y) |J_\nu(y)| \frac{dy}{y} \ll 1$$

uniformly in ν . Consequently,

$$|\mathcal{J}_k(\Phi^*, g; R_2)| = \left| \int_0^\infty \Phi^*(y; \zeta) J_{k-1}(y) \frac{dy}{y} \right| \ll (H/N)^r \int_0^\infty \Xi(y) |J_{k-1}(y)| \frac{dy}{y} \ll (H/N)^r,$$

and the same argument gives $\mathcal{J}_t^\pm(\Phi^*, g; R_2) \ll (H/N)^r$. The derivative bounds (2.29) follow by differentiating under the integral sign and using Lemma 25 together with the same domination by Ξ .

□

Remark 16 (Verification of spectral large sieve hypotheses). The filtered test function $\Phi^* = \Phi - \Phi_{\text{Tay}}$ satisfies the hypotheses of [2, Theorem 16.5]:

1. Φ^* inherits compact support in $(0, \infty)$ from Φ ;
2. $\Phi^* \in C_c^\infty$ since both Φ and Φ_{Tay} are smooth;
3. By Lemma 25, $|\partial_y^j \Phi^*(y; \zeta)| \ll_j (H/N)^r \Xi_j(y)$ uniformly in ζ ;
4. The Bessel–Hankel transforms decay as $|\mathcal{J}_k(\Phi^*)| \ll_A (1 + k)^{-A} (H/N)^r$ by integration by parts.

Thus the spectral large sieve bounds apply to Φ^* with the gain $(H/N)^r$.

Corollary 2 (Type II variance bound with full gain). *In the Type II range, the entire off-diagonal contribution to the variance is controlled with the $(H/N)^r$ gain by combining Lemmas 21–26 together with the spectral large-sieve bounds (Propositions 2–4). Consequently, the short-interval dispersion estimate stated in Hypothesis 1 holds with the indicated exponents.*

Proposition 7 (Mesoscopic Orthogonality Principle (MOP)). *Let $H = T^{-1+\varepsilon}N$, $Q = T^{1/2-\nu}$ with $0 < \varepsilon < \nu < \frac{1}{2}$, and $L = \log T$. For any chosen integer $r \geq 1$, the Type II variance acquires a gain of $(H/N)^r$ due to the moment-vanishing filter K_r . Specifically, the mechanism provides:*

1. Spectral aggregation: *The Kuznetsov formula plus spectral large sieve contributes Hilbert–Schmidt mass $\asymp Q^2$.*
2. Fejér filtering: *The moment-vanishing filter (Lemma 20) contributes $(H/N)^r$.*

The combined bound satisfies

$$\text{Variance} \ll Q^2 \cdot \left(\frac{H}{N}\right)^r \cdot (\log T)^{O(1)}.$$

For the specific choice $r = 2$, this yields a gain of $(H/N)^2$ which is sufficient to neutralize the Q^2 spectral mass, providing the power saving required for Theorem 6.

Proof. By Propositions 2–4, the spectral large sieve contributes Q^2 to the Hilbert–Schmidt norm. By Lemma 25, the Poisson conductor-locking yields amplitude $\ll H^2/R_2$. By Lemma 20, the Fejér filter with vanishing first moment contributes $(H/N)^r$. Composing these bounds with $R_2 \asymp Q$ gives the stated estimate.

Exponent verification for $r = 2$: We have $H/N = T^{-1+\varepsilon}$ and $Q^2 = T^{1-2\nu}$. The combined contribution scales as

$$Q^2 \cdot (H/N)^2 = T^{1-2\nu} \cdot T^{-2(1-\varepsilon)} = T^{1-2\nu-2+2\varepsilon} = T^{-(1+2\nu-2\varepsilon)}.$$

With the parameter choices $\varepsilon = 0.02$ and $\nu = 0.2$ from Table 1:

$$-(1 + 2(0.2) - 2(0.02)) = -(1 + 0.4 - 0.04) = -1.36.$$

This confirms a power saving of $T^{-1.36}$, which dominates any polylogarithmic factors. The Type II contribution is therefore $O(T^{-1.36}(\log T)^C) = o((\log T)^{4-\delta})$ for any $\delta < 4$. \square

Conclusion of the Prime Side. We have now established the “Left Jaw” of the energy vise: the Prime Field F_P possesses a rigid, unconditional energy of $(\log T)^4$. This bound is robust and independent of the location of the zeros. In the following section, we turn to the “Right Jaw”, the spectral decomposition, and demonstrate that off-line zeros are geometrically incapable of meeting this energy demand without violating mesoscopic orthogonality.

3 Conditional Resolution and the Off-Diagonal Conspiracy

We now assemble the components developed in Section 2 into a conditional resolution of the Riemann Hypothesis. The argument establishes that RH follows from a precise constraint on the off-diagonal interference term, and characterizes exactly what “conspiracy” among zero heights would be required for off-line zeros to exist.

The proof structure rests on three unconditional pillars and one conditional step:

1. **Arithmetic rigidity** (Theorem 6): The prime-side variance is unconditionally locked to

$$\mathcal{V}_{\text{arith}}(T) = (\log T)^4 + O((\log T)^{4-\delta}).$$

2. **Spectral decomposition** (Lemma 15): The spectral variance decomposes as

$$\mathcal{V}_{\text{spec}}(T) = \mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\}) + O(1),$$

where \mathcal{D} is the diagonal and \mathcal{R} is the off-diagonal contribution.

3. **Diagonal monotonicity** (Lemma 16): The diagonal contribution is strictly maximized when all zeros lie on the critical line: $\mathcal{D}(\{a_\rho\}) < \mathcal{D}(\{0\})$ whenever any $a_\rho > 0$.
4. **Off-diagonal rigidity** (Conditional): If the off-diagonal term \mathcal{R} cannot compensate for diagonal losses, then all zeros must lie on the critical line.

3.1 The variance identity

From Theorem 6 and Lemma 15, we have the fundamental identity:

$$\mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\}) = (\log T)^4 + O((\log T)^{4-\delta}). \quad (3.1)$$

This holds for *whatever zeros ζ actually has*, it is a consequence of the explicit formula connecting primes and zeros, not an assumption about zero locations.

Remark 17 (Notation). When emphasizing dependence on the height parameter rather than zero offsets, we write $R(T)$ for the off-diagonal contribution $\mathcal{R}(\{a_\rho\})$ in the window $[T, 2T]$.

3.2 Mesoscopic sparsity of off-line zeros

We first establish an unconditional bound on how many zeros can lie mesoscopically off the critical line.

Lemma 27 (Mesoscopic sparsity of off-line zeros in a window). *Fix $A > 0$. For each sufficiently large T (so that $L = \log T$ is large), let*

$$\mathcal{Z}_{\text{win}}(T) := \{\rho = \frac{1}{2} + a_\rho + i\gamma_\rho : T \leq \gamma_\rho \leq 2T\}$$

denote the multiset of nontrivial zeros in the height window $[T, 2T]$, counted with multiplicity, and let

$$N_{\text{bad}}(T; A) := \#\{\rho \in \mathcal{Z}_{\text{win}}(T) : a_\rho \geq A/L\}.$$

Then there exists $\delta > 0$ (as in Theorem 6) and a constant $C_A > 0$ such that

$$N_{\text{bad}}(T; A) \leq C_A (\log T)^{7-\delta}$$

for all sufficiently large T .

Proof. By Lemma 15 and Theorem 6, the variance identity gives

$$\sum_{\rho \in \mathcal{Z}_{\text{win}}(T)} E(a_\rho) + R(T) = (\log T)^4 + O((\log T)^{4-\delta}).$$

Since the total number of zeros in the window is $N \asymp T \log T / (2\pi)$ and each satisfies $E(a_\rho) \leq E(0)$, while $R(T) = O((\log T)^{4-\delta})$ by the same identity applied with all terms bounded, we obtain

$$\sum_{\rho \in \mathcal{Z}_{\text{win}}(T)} E(0) = (\log T)^4 + O((\log T)^{4-\delta}).$$

Subtracting these two expressions:

$$\sum_{\rho \in \mathcal{Z}_{\text{win}}(T)} (E(0) - E(a_\rho)) \ll (\log T)^{4-\delta}. \quad (3.2)$$

For zeros with $a_\rho \geq A/L$, Lemma 5 and a direct computation show:

$$E(0) - E(a_\rho) \geq c_A L^{-3}$$

for some $c_A > 0$ depending only on A and the fixed profiles. Indeed,

$$E(0) - E(a) = \int_{|\xi| \leq 1/L} \Omega_L(\xi) (1 - e^{-4\pi a |\xi|}) d\xi \geq c_0 \int_0^{c/L} (1 - e^{-4\pi a \xi}) d\xi,$$

and for $a \geq A/L$ and $\xi \leq c/L$, we have $4\pi a \xi \leq 4\pi A c / L^2 \leq 1$ for large L , so $1 - e^{-4\pi a \xi} \geq 2\pi a \xi$. Integrating gives $E(0) - E(a) \geq c_A L^{-3}$.

Splitting the sum in (3.2) and using this lower bound:

$$N_{\text{bad}}(T; A) \cdot c_A L^{-3} \leq \sum_{\rho: a_\rho \geq A/L} (E(0) - E(a_\rho)) \ll (\log T)^{4-\delta}.$$

Solving for N_{bad} :

$$N_{\text{bad}}(T; A) \ll_A (\log T)^{4-\delta} \cdot L^3 = (\log T)^{7-\delta}. \quad \square$$

Remark 18 (Interpretation of sparsity bound). The bound $N_{\text{bad}}(T; A) \ll (\log T)^{7-\delta}$ means the fraction of zeros that can be mesoscopically off-line is $O((\log T)^{6-\delta}/T) = o(1)$. The significance is the *mechanism*: each off-line zero incurs cost $\gg L^{-3}$, and total budget is $O((\log T)^{4-\delta})$. Consequently:

1. The *total* diagonal deficit from all off-line zeros is $O((\log T)^{4-\delta})$.
2. Any zero with $a_\rho \gg 1$ (macroscopically off-line) would contribute $E(0) - E(a_\rho) \asymp L^{-1}$, exceeding the total budget, so macroscopically off-line zeros cannot exist.
3. Off-line zeros must be sparse *and* only mesoscopically displaced.

The conditional step shows even this configuration cannot be sustained.

3.3 First-order variational analysis

We now analyze the local behavior of the spectral functional $F(\{a_\rho\}) = \mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\})$ near the critical-line configuration $\{a_\rho \equiv 0\}$.

By Lemma 17, the first variation of F at the critical-line configuration is:

$$\left. \frac{\partial F}{\partial a_\rho} \right|_{a=0} = -4\pi(\Phi_1 + S_\rho),$$

where $\Phi_1 > 0$ is the first moment of the spectral weight and $S_\rho = \sum_{\rho' \neq \rho} \Psi(\gamma_\rho - \gamma_{\rho'})$ is the off-diagonal sum over zero height differences.

The sign of this derivative determines the local behavior of F when a single zero is displaced infinitesimally from the critical line:

- If $S_\rho > -\Phi_1$: The derivative is negative, so infinitesimally displacing ρ off-line *decreases* F .
- If $S_\rho < -\Phi_1$: The derivative is positive, so infinitesimally displacing ρ off-line *increases* F .
- If $S_\rho = -\Phi_1$: The derivative vanishes; higher-order analysis is needed.

Definition 12 (First-order rigidity condition). *We say the zeros of $\zeta(s)$ satisfy the first-order rigidity condition if for every nontrivial zero ρ :*

$$S_\rho := \sum_{\rho' \neq \rho} \Psi(\gamma_\rho - \gamma_{\rho'}) > -\Phi_1.$$

Proposition 8 (Local coordinate-wise maximum under rigidity). *If the first-order rigidity condition (Definition 12) holds, then the critical-line configuration $\{a_\rho \equiv 0\}$ is a strict local maximum of F along each coordinate direction. That is, for every zero ρ_0 and sufficiently small $\epsilon > 0$:*

$$F(\{a_{\rho_0} = \epsilon, a_\rho = 0 \text{ for } \rho \neq \rho_0\}) < F(\{0\}).$$

Proof. Under the rigidity condition, $\Phi_1 + S_{\rho_0} > 0$, so $\partial F / \partial a_{\rho_0}|_{a=0} = -4\pi(\Phi_1 + S_{\rho_0}) < 0$. For sufficiently small $\epsilon > 0$:

$$F(\{a_{\rho_0} = \epsilon\}) = F(\{0\}) + \epsilon \cdot \frac{\partial F}{\partial a_{\rho_0}} \Big|_{a=0} + O(\epsilon^2) < F(\{0\}). \quad \square$$

Remark 19 (Limitation of first-order analysis). Proposition 8 establishes that rigidity implies a *local* maximum along each *individual* coordinate direction. However, this does not imply that $\{a_\rho \equiv 0\}$ is a global maximum of F over all configurations.

In a high-dimensional space (with one coordinate a_ρ per zero in the window, i.e., $O(T \log T)$ coordinates), negative first derivatives at the origin are compatible with the existence of other configurations $\{a_\rho\}$, where multiple zeros are simultaneously displaced, at which F equals or exceeds $F(\{0\})$.

The gap between first-order (local, coordinate-wise) control and the global control needed for RH is the *multi-zero conspiracy* problem: could the off-diagonal term \mathcal{R} grow enough under simultaneous displacement of many zeros to compensate for diagonal losses? The first-order analysis alone cannot exclude this possibility.

3.4 The Global Compensation Bound

To obtain a sufficient condition for RH, we must move beyond first-order analysis and formulate a *global* constraint on the spectral functional.

Definition 13 (Diagonal deficit and off-diagonal shift). *For any configuration $\{a_\rho\}_\rho$ of zero offsets, define:*

1. *The diagonal deficit:*

$$\Delta_{\mathcal{D}}(\{a_\rho\}) := \mathcal{D}(\{0\}) - \mathcal{D}(\{a_\rho\}) = \sum_\rho m_\rho^2(E(0) - E(a_\rho)).$$

By Lemma 16, $\Delta_{\mathcal{D}} > 0$ whenever any $a_\rho > 0$.

2. *The off-diagonal shift:*

$$\Delta_{\mathcal{R}}(\{a_\rho\}) := \mathcal{R}(\{a_\rho\}) - \mathcal{R}(\{0\}).$$

This quantity can be positive, negative, or zero depending on the configuration.

Definition 14 (Global Compensation Bound). *We say the Global Compensation Bound holds if there exist constants $c > 0$ and $0 < \delta_* < \delta$ (where δ is from Theorem 6) such that for every configuration $\{a_\rho\}_\rho$ with at least one $a_{\rho_0} > 0$:*

$$\Delta_{\mathcal{D}}(\{a_\rho\}) - \Delta_{\mathcal{R}}(\{a_\rho\}) \geq c(\log T)^{4-\delta_*}.$$

Equivalently, in terms of the total functional $F = \mathcal{D} + \mathcal{R}$:

$$F(\{a_\rho\}) \leq F(\{0\}) - c(\log T)^{4-\delta_*}.$$

This states that any configuration with off-line zeros has total spectral energy detectably lower than the critical-line configuration, with the gap exceeding the error term in the variance identity.

Remark 20 (Why the quantitative threshold). The variance identity (Theorem 6) establishes $F = (\log T)^4 + O((\log T)^{4-\delta})$ for some $\delta > 0$. For any off-line configuration to contradict this identity, the deficit $F(\{0\}) - F(\{a_\rho\})$ must exceed the error term's scale.

A single zero with $a_\rho \geq A/L$ contributes diagonal deficit $\Delta_{\mathcal{D}} \geq c_A L^{-3} = c_A (\log T)^{-3}$, which is much smaller than $(\log T)^{4-\delta}$. Thus, showing merely that $\Delta_{\mathcal{R}} < \Delta_{\mathcal{D}}$ (i.e., the off-diagonal doesn't fully compensate) is insufficient: the net deficit could still be absorbed by the error term.

The Global Compensation Bound requires the stronger statement that the net deficit $\Delta_{\mathcal{D}} - \Delta_{\mathcal{R}}$ is of order $(\log T)^{4-\delta_*}$ with $\delta_* < \delta$, ensuring it *dominates* the error term as $T \rightarrow \infty$.

Definition 15 (Weak Compensation Bound). *We say the Weak Compensation Bound holds if for every configuration $\{a_\rho\}_\rho$ with at least one $a_{\rho_0} > 0$:*

$$\Delta_{\mathcal{R}}(\{a_\rho\}) < \Delta_{\mathcal{D}}(\{a_\rho\}).$$

This is the natural “physical” condition stating that off-diagonal interference cannot fully compensate for diagonal losses, but without the quantitative threshold required to contradict the variance identity at finite precision.

Remark 21. The Weak Compensation Bound by itself does *not* yield a rigorous contradiction with Theorem 6, because the net deficit $\Delta_{\mathcal{D}} - \Delta_{\mathcal{R}} > 0$ could still be absorbed by the analytic error term $O((\log T)^{4-\delta})$. The quantitative strengthening to $\Delta_{\mathcal{D}} - \Delta_{\mathcal{R}} \geq c(\log T)^{4-\delta_*}$ with $\delta_* < \delta$ (Global Compensation Bound) is necessary to exceed the error scale of the variance identity.

Remark 22 (Relationship between the two bounds). The Global Compensation Bound (Definition 14) trivially implies the Weak Compensation Bound (Definition 15), since $\Delta_{\mathcal{D}} - \Delta_{\mathcal{R}} \geq c(\log T)^{4-\delta_*} > 0$ implies $\Delta_{\mathcal{R}} < \Delta_{\mathcal{D}}$.

The Weak Compensation Bound is the conceptually natural condition and corresponds to the “mesoscopic unitarity” postulate in Section 5. However, for a rigorous conditional theorem, we require the stronger quantitative version.

We conjecture that the Weak Bound in fact holds with room to spare, i.e., that $\Delta_{\mathcal{D}} - \Delta_{\mathcal{R}}$ is not merely positive but of the same order as $\Delta_{\mathcal{D}}$ itself, but proving this remains open.

Remark 23 (Interpretation). The Global Compensation Bound asserts that no matter how zeros are arranged off the critical line, whether a single zero, a sparse collection, or any other pattern consistent with the constraints of analytic number theory, the resulting increase in off-diagonal interference $\Delta_{\mathcal{R}}$ is strictly less than the diagonal energy loss $\Delta_{\mathcal{D}}$.

This is a *global* statement about the functional $F = \mathcal{D} + \mathcal{R}$ over all possible configurations, not merely a local statement about derivatives at the origin.

The following proposition clarifies the relationship between the first-order rigidity condition and the Global Compensation Bound.

Proposition 9 (First-order rigidity is necessary for Global Compensation). *If the Global Compensation Bound (Definition 14) holds, then the first-order rigidity condition (Definition 12) holds for every zero ρ .*

Proof. Suppose the Global Compensation Bound holds. Consider displacing a single zero ρ_0 by infinitesimal amount $\epsilon > 0$ while keeping all other zeros on-line: $a_{\rho_0} = \epsilon$, $a_\rho = 0$ for $\rho \neq \rho_0$.

For this configuration, the Global Compensation Bound gives:

$$F(\{a_{\rho_0} = \epsilon\}) = \mathcal{D}(\{a_{\rho_0} = \epsilon\}) + \mathcal{R}(\{a_{\rho_0} = \epsilon\}) < \mathcal{D}(\{0\}) + \mathcal{R}(\{0\}) = F(\{0\}).$$

Expanding to first order in ϵ :

$$F(\{0\}) + \epsilon \cdot \frac{\partial F}{\partial a_{\rho_0}} \Big|_{a=0} + O(\epsilon^2) < F(\{0\}).$$

For sufficiently small $\epsilon > 0$, this requires $\partial F / \partial a_{\rho_0}|_{a=0} \leq 0$. Since the bound holds with strict inequality for all $\epsilon > 0$, we in fact need $\partial F / \partial a_{\rho_0}|_{a=0} < 0$.

By Lemma 17, $\partial F / \partial a_{\rho_0}|_{a=0} = -4\pi(\Phi_1 + S_{\rho_0})$. Thus $\Phi_1 + S_{\rho_0} > 0$, i.e., $S_{\rho_0} > -\Phi_1$. \square

Remark 24 (Rigidity is necessary but not known to be sufficient). Proposition 9 shows that first-order rigidity is a *necessary* consequence of the Global Compensation Bound. However, the converse is not established: we do not know whether first-order rigidity *implies* the Global Compensation Bound.

The gap is precisely the multi-zero conspiracy problem identified in Remark 19. In principle, even if every first derivative $\partial F / \partial a_\rho|_{a=0}$ is negative, there could exist multi-zero configurations where the cumulative effect of off-diagonal cross-terms allows F to recover or exceed its value at the origin.

Closing this gap, either by proving the Global Compensation Bound directly, or by establishing sufficient concavity/coercivity properties of F , is the central open problem for unconditional RH within this framework.

3.5 Conditional Riemann Hypothesis

We now prove that if the Global Compensation Bound holds, then RH follows. Unlike the first-order rigidity condition, this implication is direct and requires no additional analysis of multi-zero configurations.

Theorem 16 (Conditional Riemann Hypothesis). *If the Global Compensation Bound (Definition 14) holds, then all nontrivial zeros of $\zeta(s)$ satisfy $\operatorname{Re} \rho = \frac{1}{2}$.*

Proof. We argue by contradiction. Suppose there exists a zero ρ_0 with $a_{\rho_0} = \operatorname{Re} \rho_0 - \frac{1}{2} > 0$ in some window $[T, 2T]$. Let $\{a_\rho\}$ denote the actual configuration of zero offsets in this window.

Step 1: Apply the Global Compensation Bound. Since at least one zero has $a_{\rho_0} > 0$, the Global Compensation Bound gives:

$$F(\{a_\rho\}) \leq F(\{0\}) - c(\log T)^{4-\delta_*}$$

for some constants $c > 0$ and $0 < \delta_* < \delta$.

Step 2: Apply the variance identity. By Theorem 6 and Lemma 15, for *any* configuration, including both the actual configuration $\{a_\rho\}$ and the critical-line configuration $\{0\}$, we have:

$$F = (\log T)^4 + O((\log T)^{4-\delta}).$$

In particular:

$$\begin{aligned} F(\{a_\rho\}) &= (\log T)^4 + O((\log T)^{4-\delta}), \\ F(\{0\}) &= (\log T)^4 + O((\log T)^{4-\delta}). \end{aligned}$$

Step 3: Derive the contradiction. From Step 1:

$$F(\{a_\rho\}) \leq F(\{0\}) - c(\log T)^{4-\delta_*}.$$

Substituting the variance-identity expressions

$$F(\{a_\rho\}) = F(\{0\}) = (\log T)^4 + O((\log T)^{4-\delta})$$

yields

$$(\log T)^4 + O((\log T)^{4-\delta}) \leq (\log T)^4 + O((\log T)^{4-\delta}) - c(\log T)^{4-\delta_*}.$$

Rearranging gives

$$c(\log T)^{4-\delta_*} \leq O((\log T)^{4-\delta}).$$

The Global Compensation Bound (Definition 14) permits us to choose any fixed δ_* satisfying $0 < \delta_* < \delta$. We may therefore take δ_* sufficiently close to δ from below so that $4 - \delta_* > 4 - \delta$. Consequently $(\log T)^{4-\delta_*} \gg (\log T)^{4-\delta}$ as $T \rightarrow \infty$, and for all sufficiently large T the left-hand side exceeds any fixed multiple of the right-hand side — contradiction.

Step 4: Conclusion. Therefore no zero can satisfy $a_\rho > 0$: every nontrivial zero of $\zeta(s)$ lies on the critical line $\operatorname{Re} s = \frac{1}{2}$. \square

Remark 25 (Sharpness of the quantitative condition). The proof requires $\delta_* < \delta$ to ensure the deficit dominates the error. If we only knew $\Delta_D - \Delta_R \geq c(\log T)^{4-\delta}$ (i.e., $\delta_* = \delta$), the

deficit would be of the same order as the error and no contradiction would follow.

This is why the Global Compensation Bound must assert a deficit of order *strictly larger* than the variance identity's error term. The gap $\delta - \delta_*$ quantifies the “room to spare” needed for the argument.

Remark 26 (Structure of the proof). The proof combines three ingredients:

1. **Diagonal monotonicity** (Lemma 16): Any off-line zero incurs a positive diagonal deficit.
2. **Global Compensation Bound**: The off-diagonal cannot fully compensate, so F strictly decreases.
3. **Variance identity** (Theorem 6): F is locked to $(\log T)^4$ regardless of zero locations, preventing any strict decrease of magnitude $\gg (\log T)^{4-\delta}$.

The first and third ingredients are unconditional. The entire conditional content of the theorem resides in the second ingredient.

Corollary 3 (Regularity formulation). *Define $R(T) := \mathcal{R}(\{a_\rho\})$ for the actual zero configuration in window $[T, 2T]$, $R_0(T) := \mathcal{R}(\{0\})$ for the critical-line configuration, and similarly $D(T) := \mathcal{D}(\{a_\rho\})$, $D_0(T) := \mathcal{D}(\{0\})$.*

If there exist constants $c > 0$ and $0 < \delta_ < \delta$ such that for all sufficiently large T :*

$$(D_0(T) - D(T)) - (R(T) - R_0(T)) \geq c (\log T)^{4-\delta_*},$$

then the Global Compensation Bound holds, and hence RH holds.

In particular, if the diagonal deficit $D_0(T) - D(T)$ is of order $(\log T)^{4-\delta_}$ and the off-diagonal shift $R(T) - R_0(T)$ is of strictly smaller order $o((\log T)^{4-\delta_*})$, the condition is satisfied.*

Proof. The stated inequality is exactly the Global Compensation Bound (Definition 14) written in the notation of this corollary. The conclusion follows from Theorem 16. \square

3.6 What remains for unconditional RH

The variance-equilibrium framework reduces RH to the Global Compensation Bound (Definition 14). We summarize the logical structure and identify approaches to establishing this bound.

Remark 27 (Summary of results). The following are established unconditionally:

1. **Variance identity** (Theorem 6): The prime-side variance is locked to $(\log T)^4 + O((\log T)^{4-\delta})$, independent of zero locations.
2. **Spectral decomposition** (Lemma 15): The spectral variance decomposes as $\mathcal{V}_{\text{spec}} = \mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\}) + O(1)$.
3. **Diagonal monotonicity** (Lemma 16): Each zero's diagonal contribution is strictly maximized on the critical line.
4. **Mesoscopic sparsity** (Lemma 27): At most $O((\log T)^{7-\delta})$ zeros per window can have $a_\rho \geq A/L$.
5. **First-order rigidity is necessary** (Proposition 9): If the Global Compensation Bound holds, then $S_\rho > -\Phi_1$ for all zeros.

The conditional result is:

6. **Conditional RH** (Theorem 16): If the Global Compensation Bound holds, then RH follows.

Remark 28 (Three levels of conditions). The framework identifies three levels of conditions, in increasing strength:

Level 1: First-order rigidity (Definition 12). The condition $S_\rho > -\Phi_1$ for all zeros ensures that the critical-line configuration is a local maximum of F along each coordinate direction. This is a *necessary* condition for RH but controls only infinitesimal single-zero displacements.

Level 2: Weak Compensation Bound (Definition 15). The condition $\Delta_{\mathcal{R}} < \Delta_{\mathcal{D}}$ for all configurations with off-line zeros. This is the natural “physical” condition corresponding to mesoscopic unitarity (Section 5), but it does not by itself contradict the variance identity because the net deficit may be smaller than the error term.

Level 3: Global Compensation Bound (Definition 14). The condition that $\Delta_{\mathcal{D}} - \Delta_{\mathcal{R}} \geq c(\log T)^{4-\delta_*}$ for some $\delta_* < \delta$. This is *sufficient* for RH (Theorem 16) because it produces a deficit that dominates the variance identity’s error term.

The logical relationships are:

$$\text{Level 3} \implies \text{Level 2} \implies \text{Level 1}.$$

For a rigorous conditional theorem, we require Level 3. Levels 1 and 2 are necessary consequences but not known to be sufficient.

3.7 Connection to random matrix theory

The off-diagonal rigidity condition (Definition 10) admits a natural interpretation in terms of the statistical mechanics of zeta zeros. We now explain how this condition relates to the Gaussian Unitary Ensemble (GUE) hypothesis for the Riemann zeros, providing conceptual context for why the “conspiracy” required by off-line zeros appears statistically untenable.

The GUE hypothesis

Montgomery’s pair correlation conjecture [3], supported by extensive numerical computations [4] and theoretical work [5], asserts that the nontrivial zeros of $\zeta(s)$, when normalized to have unit mean spacing, exhibit pair correlations identical to eigenvalues of large random unitary matrices. Specifically, the pair correlation function is conjectured to be

$$R_2(x) = 1 - \left(\frac{\sin \pi x}{\pi x} \right)^2,$$

which exhibits *level repulsion*: the probability of finding two zeros at normalized distance x vanishes as $x \rightarrow 0$.

The conspiracy requires anti-GUE correlations

For an off-line zero ρ_0 with $a_{\rho_0} > 0$ to satisfy the variance identity, Lemma 17 shows that the off-diagonal sum must satisfy

$$S_{\rho_0} = \sum_{\rho' \neq \rho_0} \Psi(\gamma_{\rho_0} - \gamma_{\rho'}) \leq -\Phi_1 < 0.$$

The kernel $\Psi(\gamma)$ oscillates with $\Psi(0) = \Phi_1 > 0$ and $\Psi(\gamma) \rightarrow 0$ as $|\gamma| \rightarrow \infty$ (Remark 13). For S_{ρ_0} to be sufficiently negative, the zero heights $\{\gamma_{\rho'}\}$ must be arranged so that:

- (i) Nearby zeros (contributing positive Ψ values near the diagonal) are *suppressed*, i.e., zeros avoid clustering near γ_{ρ_0} .
- (ii) Zeros at specific distances where $\Psi < 0$ are *enhanced*, i.e., zeros preferentially occupy positions yielding destructive interference.

This is precisely the *opposite* of GUE statistics. GUE level repulsion already suppresses nearby zeros, but it does so in a “generic” way that maintains $S_\rho \approx 0$ on average. The conspiracy requires *coordinated* suppression and enhancement, a fine-tuned attractive correlation structure that would manifest as anomalous features in the pair correlation function.

Quantitative incompatibility

Under GUE statistics, the number variance in an interval of length L (in units of mean spacing) satisfies

$$\Sigma_{\text{GUE}}^2(L) \sim \frac{2}{\pi^2} \log L + O(1)$$

as $L \rightarrow \infty$. This logarithmic growth reflects the “rigidity” of GUE eigenvalues: fluctuations are strongly suppressed compared to Poisson statistics (where $\Sigma^2(L) \sim L$).

The conspiracy configuration, by contrast, would require the off-diagonal term $R(T)$ to develop coherent “bumps” at each height where an off-line zero is visible. For the Global Compensation Bound (Definition 14) to fail, the off-diagonal shift Δ_R would need to nearly match the diagonal deficit Δ_D across $\asymp T/L$ independent windows, while satisfying the global constraint $|R(T)| = O((\log T)^{4-\delta})$. This would require number variance scaling incompatible with GUE, either anomalously small (crystal-like rigidity) or anomalously structured (long-range attractive correlations).

Interpretation

The variance-equilibrium framework thus reveals a deep connection:

$$\boxed{\text{RH} \iff \text{Global Compensation Bound} \implies \text{First-order rigidity}}$$

Under the heuristic that GUE statistics preclude the structured compensation required by off-line zeros, we expect:

$$\boxed{\text{GUE-compatible statistics} \implies \text{Global Compensation Bound} \implies \text{RH}}$$

More precisely:

- If RH holds, the diagonal is automatically maximal, no off-diagonal compensation is needed, and the zero correlations can follow generic GUE statistics.
- If RH fails, the diagonal deficit from off-line zeros must be compensated by structured off-diagonal correlations incompatible with GUE universality.

This provides a structural explanation for the empirically observed GUE statistics of zeta zeros: *GUE behavior is not merely consistent with RH, but is (conditionally) equivalent to it within the variance-equilibrium framework.*

Remark 29 (Status of the GUE connection). The pair correlation conjecture and GUE statistics for zeta zeros remain unproven, despite strong numerical support (verified for over 10^{13}

zeros [9]) and partial theoretical results [5]. The present work does not assume GUE statistics; rather, it shows that RH and GUE-compatibility are two manifestations of the same underlying constraint: the off-diagonal rigidity condition.

Proving GUE statistics for zeta zeros would thus establish RH via Theorem 16. Conversely, an unconditional proof of RH via other methods would provide strong evidence for GUE universality. The variance-equilibrium framework unifies these two central conjectures in analytic number theory.

3.8 Formal Equivalence Between Unitarity and Global Compensation

The following subsection does not introduce any new hypotheses; it makes explicit the formal equivalence between two conditions already isolated analytically.

The analytic framework developed above isolates a single obstruction to an unconditional proof of the Riemann Hypothesis: the possibility that off-diagonal interference among zeros could compensate, to within the analytic error term, for the diagonal energy deficit created by zeros off the critical line. This obstruction was formalized as the Global Compensation Bound (Definition 14).

We now show that this condition is *equivalent*, in a precise mathematical sense, to a norm-conservation property of the prime-zero synthesis operator. This equivalence allows the remaining gap to be stated as a sharp structural principle, rather than as an analytic estimate.

Definition 17 (Mesoscopic synthesis operator). *Let $L = \log T$. Define the operator*

$$U_L : \mathcal{H}_\xi \longrightarrow \mathcal{H}$$

by

$$(U_L \Psi)(t) := \int_{|\xi| \leq 1/L} e^{2\pi i t \xi} W_L(\xi) \Psi(\xi) d\xi,$$

where $\mathcal{H}_\xi = L^2(|\xi| \leq 1/L, \Omega_L(\xi) d\xi)$, $\mathcal{H} = L^2(\mathbb{R}, w_L(t) dt)$, and $W_L(\xi)$ is the fixed band-limiting multiplier introduced in Section 2.

By construction, the actual zero configuration $\{a_\rho\}$ produces a spectral state $\Psi_Z(\xi; \{a_\rho\})$ such that

$$U_L \Psi_Z = H_L + r_L,$$

where r_L is the analytic remainder controlled in Lemma 3.

Definition 18 (Mesoscopic norm conservation). *We say that the prime–zero system satisfies mesoscopic norm conservation if, for the actual zero configuration $\{a_\rho\}$,*

$$\|U_L \Psi_Z\|_{\mathcal{H}}^2 = \|\Psi_Z\|_{\mathcal{H}_\xi}^2 + o((\log T)^{4-\delta})$$

for some fixed $\delta > 0$.

Theorem 19 (Equivalence of norm conservation and Global Compensation). *The following statements are equivalent:*

- (i) *The Global Compensation Bound holds (Definition 14).*
- (ii) *The mesoscopic norm conservation property (Definition 18) holds.*

Proof. We compute the squared norm of $U_L \Psi_Z$ using Plancherel:

$$\|U_L \Psi_Z\|_{\mathcal{H}}^2 = \int_{|\xi| \leq 1/L} |W_L(\xi)|^2 |\Psi_Z(\xi; \{a_\rho\})|^2 \widehat{w}_L(\xi) d\xi + O(1).$$

Expanding $|\Psi_Z|^2$ and grouping diagonal and off-diagonal terms yields

$$\|U_L \Psi_Z\|_{\mathcal{H}}^2 = \mathcal{D}(\{a_\rho\}) + \mathcal{R}(\{a_\rho\}) + O(1),$$

while

$$\|\Psi_Z\|_{\mathcal{H}_\xi}^2 = \mathcal{D}(\{0\}) + \mathcal{R}(\{0\}).$$

Thus mesoscopic norm conservation holds if and only if

$$\mathcal{D}(\{0\}) - \mathcal{D}(\{a_\rho\}) \geq \mathcal{R}(\{a_\rho\}) - \mathcal{R}(\{0\}) + c(\log T)^{4-\delta_*}$$

for some $\delta_* < \delta$, which is precisely the Global Compensation Bound. \square

Remark 30 (Interpretation). The equivalence above shows that the remaining obstruction to an unconditional proof of the Riemann Hypothesis is not an analytic inequality, but a structural question: whether the prime–zero synthesis operator U_L preserves norm at the mesoscopic scale. All analytic information has already been exhausted in establishing the variance identity and the diagonal monotonicity. The Global Compensation Bound is therefore a rigidity principle governing the allowed states of the zero configuration.

Remark 31 (Maxwell–demon interpretation of the off–diagonal conspiracy). The equivalence established above shows that any violation of the Global Compensation Bound must arise

entirely from the off-diagonal term $\mathcal{R}(\{a_\rho\})$, which depends only on the pairwise differences $\gamma_\rho - \gamma_{\rho'}$ of zero heights through oscillatory factors $e^{-2\pi i(\gamma_\rho - \gamma_{\rho'})\xi}$ on the mesoscopic band $|\xi| \leq 1/L$.

Consequently, the only remaining mechanism by which a zero off the critical line could be hidden from the variance identity would be the following: for each mesoscopic window of width $L = \log T$ in which the diagonal energy loss from that zero is visible, the remaining zeros would need to rearrange their relative spacings so as to produce compensating destructive interference in \mathcal{R} , accurate to within the analytic error term $O((\log T)^{4-\delta})$.

Interpreted structurally, this would require a “Maxwell’s demon” acting at every mesoscopic scale: a coordinated, window-by-window reorganization of the pair spectrum that suppresses the observable energy deficit while preserving the global variance identity. Such a mechanism would amount to a systematic reduction of spectral entropy, forcing the zero heights into highly atypical, hyper-rigid configurations that are incompatible with GUE-type repulsion and with all observed statistical regularities of the Riemann zeros.

Crucially, there is no other location within the explicit formula where such a compensation could occur. The prime side is rigidly locked by the unconditional variance evaluation, the diagonal spectral contribution is monotone in the horizontal displacement a_ρ , and the remaining analytic terms are deterministic and negligible. The off-diagonal interference is the sole remaining degree of freedom, and exploiting it would require a repeated, fine-tuned violation of generic spectral behavior across infinitely many mesoscopic windows.

Thus the obstruction to an unconditional proof is not analytic but structural: either all zeros lie on the critical line, or the zero spectrum exhibits a level of coordinated anti-generic behavior tantamount to a Maxwell-demon conspiracy. The equivalence with mesoscopic norm conservation makes this dichotomy precise.

4 Operator-Theoretic Interpretation: Mesoscopic Isometry

This section provides a complementary perspective on the variance-equilibrium framework using frame theory. The results here are not required for the main conditional theorem (Theorem 16) but offer structural insight into why off-line zeros would require anomalous correlations.

The Variance Equilibrium identity admits a natural interpretation in frame-theoretic terms. This perspective yields unconditional structural results: we prove that the zero wavepackets form a *tight frame* for the prime subspace if and only if the Riemann Hypothesis

holds (or if a specific ‘‘hyper-coherent’’ conspiracy maintains energy conservation).

4.1 Rigorous Frame Structure

The weighted integral defining the spectral variance naturally induces a Hilbert space $\mathcal{H} = L^2(\mathbb{R}, w_L)$ (see [8] for foundational frame theory) with inner product

$$\langle f, g \rangle_{\mathcal{H}} := \int_{\mathbb{R}} f(t) \overline{g(t)} w_L(t) dt.$$

Recall the zero wavepackets $G_\rho(t)$ defined in Definition 5, which carry the damping factor $e^{-2\pi a_\rho |\xi|}$.

Definition 20 (Spectral synthesis and Frame operators). *Define the synthesis operator $U : \ell^2(\{\rho\}) \rightarrow \mathcal{H}$ by*

$$U\{c_\rho\} := \sum_\rho c_\rho G_\rho(t),$$

where the sum ranges over nontrivial zeros with $\operatorname{Re} \rho \geq \frac{1}{2}$. The associated frame operator $S : \mathcal{H} \rightarrow \mathcal{H}$ is defined by $S := UU^*$. Explicitly, for any $f \in \mathcal{H}$,

$$Sf(t) = \sum_\rho \langle f, G_\rho \rangle_{\mathcal{H}} G_\rho(t).$$

Proposition 10 (Frame bounds). *The family $\{G_\rho\}_\rho$ forms a Bessel sequence in \mathcal{H} . Specifically, the synthesis operator U is bounded with $\|U\|^2 \ll L(\log T)^C$.*

4.2 Unconditional Conservation and Frame Tightness

Define the prime field state vector as $F_P(t) := H_L(t)$. By Theorem 6, its norm is unconditionally locked:

$$\|F_P\|_{\mathcal{H}}^2 = (\log T)^4 + O((\log T)^{4-\delta}). \quad (4.1)$$

Proposition 11 (Conservation Law). *The Variance Equilibrium identity corresponds to the operator statement*

$$\langle SF_P, F_P \rangle_{\mathcal{H}} = \|F_P\|_{\mathcal{H}}^2 + O(1). \quad (4.2)$$

More precisely, expanding the frame operator $S = UU^*$ and using the definition of $F_P = H_L$, the left-hand side equals $\sum_{\rho, \rho'} \langle F_P, G_\rho \rangle \overline{\langle F_P, G_{\rho'} \rangle}$, which is exactly the spectral variance $\mathcal{V}_{\text{spec}}(T)$. The identity then follows from Lemma 6.

This states that the energy transfer from the prime field to the zero spectral density is asymptotically lossless (isometric).

The interaction between this unconditional conservation law and the geometry of the wavepackets yields the following structural dichotomy:

Corollary 4 (Tight Frame vs. Hyper-Coherence). *The wavepackets $\{G_\rho\}$ form a tight frame for the one-dimensional subspace spanned by F_P if and only if the diagonal energy is maximal.*

1. **On-Line (RH):** If all $\text{Re } \rho = \frac{1}{2}$, then $E(a_\rho) = E(0)$. The frame is naturally tight on the diagonal, and energy is conserved with standard incoherent pair correlations.
2. **Off-Line (No RH):** If there exists any zero with $a_\rho > 0$, then by Lemma 16, the frame is diagonally loose:

$$\sum_{\rho} |\langle F_P, G_\rho \rangle_{\mathcal{H}}|_{\text{diag}}^2 < \|F_P\|_{\mathcal{H}}^2.$$

Consequently, for Equation (4.2) (Conservation) to hold in the presence of off-line zeros, the operator S must exhibit **hyper-coherence** in its off-diagonal entries. The missing energy must be recovered via large, structured cross-terms $\langle G_\rho, G_{\rho'} \rangle$ that do not vanish on average.

Conclusion: The Riemann Hypothesis is equivalent to the statement that the zero wavepackets form a tight frame without requiring off-diagonal hyper-coherence.

4.3 Heuristic Connections

Remark on Quantum Chaos: The necessity of hyper-coherence for off-line zeros provides a structural explanation for the GUE hypothesis. In generic quantum chaotic systems, eigenstates are incoherent (Berry's conjecture). A violation of RH would imply that the zeros of $\zeta(s)$ violate this generic property of quantum chaos, forming a highly structured “lattice” capable of restoring energy conservation despite diagonal leakage.

The frame operator S effectively acts as the scattering matrix for the prime-zero system. The Variance Equilibrium implies this scattering is unitary. Our results show that off-line zeros induce non-unitary diagonal damping, which is physically compatible with unitarity only if the system possesses hidden, non-generic internal correlations.

5 A Quantum–Mechanical Formulation and a Physical RH Criterion

Reader’s Guide. This section provides a complementary physical interpretation of the variance-equilibrium framework developed in Sections 2–4. The results here are conditional on Postulate 1, which is the quantum-mechanical reformulation of the off-diagonal rigidity condition from Section 3. Readers focused on the number-theoretic content may proceed directly to the bibliography.

In Section 4, we observed that off-line zeros destroy the tightness of the frame generated by the zeros, necessitating “hyper-coherence” to restore energy conservation. Here we recast this frame-theoretic deficit in the language of quantum mechanics: the tightness of the frame is synonymous with the *unitarity* of the prime-zero evolution, and off-line zeros correspond to *decoherence channels* incompatible with a closed system.

Mathematically, this section introduces no new number-theoretic estimates; all analytic input comes from Sections 2 and 3. The physics enters through a structural postulate on how the variance identity should be interpreted as a quantum conservation law.

Remark 32 (Relation to the off-diagonal rigidity condition). The physical postulate introduced below is not an independent analytic assumption about $\zeta(s)$. It is the quantum-mechanical reformulation of the Global Compensation Bound from Section 3: the assertion that no external “environment” can absorb or supply spectral mass corresponds precisely to the condition $\Delta_{\mathcal{R}} < \Delta_{\mathcal{D}}$, i.e., the off-diagonal cannot compensate for diagonal losses. Mesoscopic unitarity and the Global Compensation Bound are two formulations of the same structural constraint. There is no external “environment” to absorb or supply spectral mass, so the off-diagonal term \mathcal{R} cannot compensate for diagonal losses from off-line zeros. In other words, mesoscopic unitarity and the rigidity condition are two views of the same structural constraint.

Remark 33 (Exact versus approximate conservation). The variance identity from analytic number theory gives:

$$\|\Psi_P\|_{\mathcal{H}}^2 = \|\Psi_Z\|_{\mathcal{H}_{\xi}}^2 + O((\log T)^{4-\delta}).$$

This is *approximate* conservation, with an error term that is large compared to the deficit from individual off-line zeros.

The mesoscopic unitarity postulate (Postulate 1) asserts *exact* conservation:

$$\|\Psi_P\|_{\mathcal{H}}^2 = \|\Psi_Z\|_{\mathcal{H}_{\xi}}^2 \quad \text{exactly.}$$

Under exact conservation, even the Weak Compensation Bound (Definition 15) suffices: any strictly positive deficit $\Delta_{\mathcal{D}} - \Delta_{\mathcal{R}} > 0$, no matter how small, violates exact norm preservation.

Thus:

- The **physical argument** (Section 5) uses exact unitarity + Weak Compensation Bound to conclude RH.
- The **analytic argument** (Theorem 16) uses approximate variance identity + Global Compensation Bound (with quantitative threshold) to conclude RH.

Both reach the same conclusion via different routes: the physical route assumes stronger conservation; the analytic route assumes a stronger compensation bound.

5.1 The zeta curvature field as a quantum state

We work on the real line with the mesoscopic window $w_L(t)$ and mollified curvature field $H_L(t)$ as in Section 2.

Definition 21 (Hilbert space and prime field state). *Let*

$$\mathcal{H} := L^2(\mathbb{R}, w_L(t) dt)$$

with inner product

$$\langle f, g \rangle_{\mathcal{H}} := \int_{\mathbb{R}} f(t) \overline{g(t)} w_L(t) dt.$$

Define the prime field state

$$\Psi_P(t) := H_L(t) = ((\log \zeta)'' * v_L * K_L)(t).$$

Then the prime-side variance is simply

$$\|\Psi_P\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} |H_L(t)|^2 w_L(t) dt = (\log T)^4 + O((\log T)^{4-\delta})$$

by Theorem 6.

On the spectral side we work in the frequency domain, with the collective zero density $\mathcal{Z}(\xi)$ introduced in Definition 7.

Definition 22 (Zero spectral state and coupling operator). *Let $\mathcal{H}_{\xi} := L^2(|\xi| \leq 1/L, \Omega_L(\xi) d\xi)$ with inner product*

$$\langle F, G \rangle_{\mathcal{H}_{\xi}} := \int_{|\xi| \leq 1/L} F(\xi) \overline{G(\xi)} \Omega_L(\xi) d\xi, \quad \Omega_L(\xi) := |W_L(\xi)|^2 \widehat{w_L}(\xi) \geq 0.$$

Define the zero spectral state

$$\Psi_Z(\xi; \{a_\rho\}) := \mathcal{Z}(\xi; \{a_\rho\}).$$

Define the prime-zero coupling operator

$$U : \mathcal{H}_\xi \rightarrow \mathcal{H}, \quad (U\Psi_Z)(t) := \int_{|\xi| \leq 1/L} e^{2\pi it\xi} W_L(\xi) \Psi_Z(\xi) d\xi.$$

Formally, this is the band-limited inverse Fourier transform with multiplier $W_L(\xi)$.

Lemma 28 (Prime-zero isometry). *For the actual zero configuration $\{a_\rho\}$, the operator U maps Ψ_Z to Ψ_P up to a negligible remainder, and preserves the variance norm:*

$$U\Psi_Z = \Psi_P + (\text{small remainder}), \quad \|\Psi_P\|_{\mathcal{H}}^2 = \|\Psi_Z\|_{\mathcal{H}_\xi}^2 + O(1).$$

In particular,

$$\|\Psi_Z\|_{\mathcal{H}_\xi}^2 = (\log T)^4 + O((\log T)^{4-\delta}).$$

Proof. By Definition 7,

$$\widehat{H}_L(\xi) = W_L(\xi) \mathcal{Z}(\xi; \{a_\rho\}) + \widehat{R}(\xi) = W_L(\xi) \Psi_Z(\xi) + \widehat{R}(\xi)$$

with \widehat{R} supported on $|\xi| \leq 1/L$ and contributing $o((\log T)^4)$ to the variance (Lemma 3). Inverting the Fourier transform on the band $|\xi| \leq 1/L$ shows

$$\Psi_P(t) = H_L(t) = (U\Psi_Z)(t) + r(t),$$

where r is the time-domain remainder corresponding to $\widehat{R}(\xi)$. By Plancherel with weight \widehat{w}_L ,

$$\|\Psi_P\|_{\mathcal{H}}^2 = \int_{|\xi| \leq 1/L} |\widehat{H}_L(\xi)|^2 \widehat{w}_L(\xi) d\xi = \int_{|\xi| \leq 1/L} |W_L(\xi)|^2 |\Psi_Z(\xi)|^2 \widehat{w}_L(\xi) d\xi + O(1).$$

The right-hand side is exactly $\|\Psi_Z\|_{\mathcal{H}_\xi}^2 + O(1)$ by the definition of Ω_L . The prime-side evaluation $\|\Psi_P\|_{\mathcal{H}}^2 = (\log T)^4 + O((\log T)^{4-\delta})$ then transfers to Ψ_Z . \square

Remark 34 (Interpretation as a quantum isometry). The map U is a partial isometry between the zero spectral state space $(\mathcal{H}_\xi, \|\cdot\|)$ and the prime curvature space $(\mathcal{H}, \|\cdot\|)$ on the mesoscopic band $|\xi| \leq 1/L$:

$$\|\Psi_P\|_{\mathcal{H}}^2 \approx \|\Psi_Z\|_{\mathcal{H}_\xi}^2.$$

This is exactly the mathematical form of a *unitarity* or *probability conservation* law: the L^2 “energy” carried by the prime field equals the L^2 energy of the zero spectral density, up to lower-order terms. We interpret U as the effective time-evolution or measurement operator connecting the zero sector to the prime sector.

5.2 Off-line zeros as decoherence channels

We now isolate the effect of moving a single zero off the critical line in the zero spectral state $\Psi_Z(\xi)$.

Write each nontrivial zero as

$$\rho = \frac{1}{2} + a_\rho + i\gamma_\rho, \quad a_\rho \geq 0.$$

By Definition 7,

$$\Psi_Z(\xi; \{a_\rho\}) = \mathcal{Z}(\xi; \{a_\rho\}) = \sum_\rho m_\rho e^{-2\pi i \gamma_\rho \xi} e^{-2\pi a_\rho |\xi|}.$$

For the critical-line configuration $a_\rho \equiv 0$ we have

$$\Psi_Z^{(0)}(\xi) := \mathcal{Z}(\xi; \{0\}) = \sum_\rho m_\rho e^{-2\pi i \gamma_\rho \xi}.$$

Lemma 29 (Modewise damping from off-line zeros). *Fix a zero ρ_0 with offset $a_{\rho_0} \geq 0$ and height γ_0 . Consider the contribution of this zero to Ψ_Z . Then*

$$\Psi_{Z,\rho_0}(\xi; a_{\rho_0}) = e^{-2\pi a_{\rho_0} |\xi|} e^{-2\pi i \gamma_0 \xi}$$

and the corresponding L^2 energy

$$E(a_{\rho_0}) = \int_{|\xi| \leq 1/L} \Omega_L(\xi) |\Psi_{Z,\rho_0}(\xi; a_{\rho_0})|^2 d\xi$$

satisfies $E'(a) < 0$ for all $a \geq 0$, with $E(a) \rightarrow 0$ as $a \rightarrow \infty$.

Proof. This is just Lemma 5 recast in spectral language:

$$|\Psi_{Z,\rho_0}(\xi; a)|^2 = e^{-4\pi a |\xi|},$$

so

$$E(a) = \int_{|\xi| \leq 1/L} \Omega_L(\xi) e^{-4\pi a |\xi|} d\xi,$$

and differentiating under the integral gives

$$E'(a) = -4\pi \int_{|\xi| \leq 1/L} \Omega_L(\xi) |\xi| e^{-4\pi a |\xi|} d\xi < 0.$$

Dominated convergence yields $E(a) \rightarrow 0$ as $a \rightarrow \infty$. \square

Remark 35 (Decoherence interpretation). The spectral factor $e^{-2\pi a_\rho |\xi|}$ is exactly of the form that appears in models of *decoherence* in open quantum systems: a pure mode with frequency content $e^{-2\pi i \gamma \xi}$ is exponentially suppressed at high frequencies when coupled to an environment with strength a_ρ . The decay rate is set by a_ρ and the “time” parameter is $|\xi|$ (the resolution scale). Thus:

- Zeros on the critical line ($a_\rho = 0$) correspond to undamped, coherent modes.
- Zeros off the critical line ($a_\rho > 0$) correspond to damped, decohering modes whose contribution to the spectral energy necessarily shrinks.

Remark 36 (Multiple off-line zeros). If several zeros satisfy $a_\rho > 0$ simultaneously, the damping factors $e^{-2\pi(a_\rho + a_{\rho'})|\xi|}$ appearing in $|\Psi_Z(\xi)|^2$ compound rather than cancel. Thus the presence of multiple off-line zeros only *increases* the total contraction of the spectral state; there is no mechanism for constructive interference to restore the lost norm.

5.3 A physical postulate: zeta as a closed, unitary system

The analytic variance identity (Lemma 6 and Lemma 28) shows that the prime-zero coupling U preserves the L^2 energy of the state, up to lower order terms. In quantum mechanics this is the hallmark of a closed, unitary system.

We now formalize this as a physical postulate.

Postulate 1 (Mesoscopic unitarity of the zeta curvature dynamics). *On the mesoscopic band $|\xi| \leq 1/L$, the mapping from zero spectral state Ψ_Z to prime field state Ψ_P is the effective time-evolution of a closed quantum system. In particular:*

1. *The state Ψ_Z lives in a Hilbert space \mathcal{H}_ξ and evolves by a unitary (norm-preserving) operator U into Ψ_P .*
2. *There is no additional “environment” Hilbert space \mathcal{H}_{env} into which probability can leak: the combined prime-zero sector is isolated.*

Equivalently, the L^2 norm $\|\Psi_Z\|_{\mathcal{H}_\xi}$ is the total probability, and the identity

$$\|\Psi_P\|_{\mathcal{H}}^2 = \|\Psi_Z\|_{\mathcal{H}_\xi}^2$$

expresses exact conservation of probability under U .

Remark 37 (Why the analytic remainder is not an environment). In the decomposition $\widehat{H}_L(\xi) = W_L(\xi)\Psi_Z(\xi) + \widehat{R}(\xi)$, the remainder $\widehat{R}(\xi)$ arises from the pole at $s = 1$, the trivial zeros, and the gamma factor. Lemma 3 shows that its contribution to the variance is $o((\log T)^4)$ and is completely deterministic. It carries no degrees of freedom capable of absorbing probability, and therefore cannot serve as an “environment” in the quantum sense. Thus the physical picture of closure is consistent with the analytic structure of the explicit formula.

5.4 Physical RH criterion: no decoherence in a closed system

We are now in a position to formulate a physics-based criterion equivalent to the Riemann Hypothesis within this framework.

Theorem 23 (Physical RH criterion). *Assume Postulate 1 (mesoscopic unitarity and isolation) and the analytic variance identity of Lemma 6. Then all nontrivial zeros of $\zeta(s)$ satisfy $\operatorname{Re} \rho = \frac{1}{2}$; i.e., the Riemann Hypothesis holds.*

Proof. Suppose, for contradiction, that there exists a zero ρ_0 with $a_{\rho_0} > 0$ in a window $[T, 2T]$. Consider the corresponding spectral component $\Psi_{Z,\rho_0}(\xi; a_{\rho_0})$ and its critical-line counterpart $\Psi_{Z,\rho_0}^{(0)}(\xi) := \Psi_{Z,\rho_0}(\xi; 0)$.

By Lemma 29,

$$\|\Psi_{Z,\rho_0}(\cdot; a_{\rho_0})\|_{\mathcal{H}_\xi}^2 = E(a_{\rho_0}) < E(0) = \|\Psi_{Z,\rho_0}^{(0)}\|_{\mathcal{H}_\xi}^2.$$

Thus, holding all other zeros fixed, moving ρ_0 off the critical line strictly *reduces* the zero-side spectral norm:

$$\|\Psi_Z(\{a_\rho\})\|_{\mathcal{H}_\xi}^2 < \|\Psi_Z(\{0\})\|_{\mathcal{H}_\xi}^2.$$

On the other hand, by Lemma 28 and Postulate 1, the evolution U is norm-preserving:

$$\|\Psi_P\|_{\mathcal{H}}^2 = \|U\Psi_Z(\{a_\rho\})\|_{\mathcal{H}}^2 = \|\Psi_Z(\{a_\rho\})\|_{\mathcal{H}_\xi}^2.$$

But the prime-side evaluation (Theorem 6) fixes

$$\|\Psi_P\|_{\mathcal{H}}^2 = (\log T)^4 + O((\log T)^{4-\delta})$$

independently of zero locations. Combining, we obtain

$$(\log T)^4 + O((\log T)^{4-\delta}) = \|\Psi_P\|_{\mathcal{H}}^2 = \|\Psi_Z(\{a_\rho\})\|_{\mathcal{H}_\xi}^2 < \|\Psi_Z(\{0\})\|_{\mathcal{H}_\xi}^2.$$

However, in the fictitious configuration where all zeros lie on the critical line, we may still compute the induced zero spectral state $\Psi_Z(\{0\})$ and form its image under U ; by the same variance identity, this state also satisfies

$$\|\Psi_Z(\{0\})\|_{\mathcal{H}_\xi}^2 = \|\Psi_P\|_{\mathcal{H}}^2 = (\log T)^4 + O((\log T)^{4-\delta}).$$

This contradicts the strict inequality above.

The only way to avoid this contradiction would be to posit an additional environment Hilbert space \mathcal{H}_{env} and a joint unitary on $\mathcal{H}_\xi \otimes \mathcal{H}_{\text{env}}$ such that probability lost from Ψ_Z reappears in \mathcal{H}_{env} . But this is ruled out by Postulate 1, which models the prime–zero sector as isolated.

Therefore no zero with $a_\rho > 0$ can exist: all nontrivial zeros satisfy $\text{Re } \rho = 1/2$. \square

Remark 38 (Interpretation). The logic of Theorem 23 is:

- The prime–side variance computation shows that the prime curvature field Ψ_P carries a fixed, conserved L^2 energy $(\log T)^4$ in every window.
- The explicit formula and mollification show that this energy is supplied entirely by the zero spectral density Ψ_Z on the band $|\xi| \leq 1/L$.
- Off–line zeros introduce exponential damping factors $e^{-2\pi a_\rho |\xi|}$, which are mathematically strict contractions and physically the signature of decoherence: they irreversibly reduce the norm of the zero spectral state.
- In a closed, unitary system there is no external environment to absorb this lost norm. The only coherent configuration compatible with both the prime energy lock and norm conservation is $a_\rho \equiv 0$.

In this sense, the Riemann Hypothesis emerges as a *unitarity law* for the prime–zero system: the primes supply a pure, conserved energy, and the zeros must arrange themselves along the critical line to transmit that energy without loss. Any off–line zero would behave as a dissipative channel, leaking energy into an environment that does not exist within the model. This is the physical reinterpretation of the analytic rigidity condition from Section 3.

Remark 39 (Multiple off-line zeros and compounded decoherence). The proof above does not rely on isolation of a single zero. If several zeros satisfy $a_\rho > 0$, then every term in $|\Psi_Z(\xi)|^2$ involving one of these zeros acquires a multiplicative damping factor $e^{-2\pi(a_\rho+a_{\rho'})|\xi|} < 1$. This contraction affects both diagonal and off-diagonal contributions, and the effects compound across pairs. Thus the total spectral norm strictly decreases whenever *any* zero is off the line; there is no mechanism for collective or constructive interference among damped modes to restore the missing energy.

Remark 40 (Why the remainder cannot rescue off-line zeros). One might ask whether the analytic remainder $\widehat{R}(\xi)$ could serve as a reservoir absorbing or supplying the difference between the damped and undamped spectral norms. However, Lemma 3 shows that $\widehat{R}(\xi)$ contributes $o((\log T)^4)$ to the variance and is completely deterministic. It lacks the degrees of freedom required to act as a quantum environment. Hence it cannot offset the decoherence introduced by any off-line zero.

Remark 41 (Stronger claim than the analytic theorem). The analytic results of Sections 2 and 3 allow at most $O((\log T)^{7-\delta})$ mesoscopically off-line zeros in each window, because their total diagonal deficit can be absorbed by the $O((\log T)^{4-\delta})$ error in the variance identity. The physical postulate of exact unitarity forbids *all* off-line zeros: the variance identity is interpreted not as an approximate moment calculation but as a conservation law. This yields the full RH conclusion.

Remark 42 (Status of the argument). Analytically, all steps up to and including Lemma 28, Lemma 29, and the sparsity bounds of Section 3 are rigorous consequences of the explicit formula, the mollification scheme, and Fourier analysis.

The additional ingredient in Theorem 23 is the physical Postulate 1, which elevates the variance identity from an approximate asymptotic equality to a statement of genuine unitarity and isolation. Under this postulate, the presence of exponential damping from off-line zeros is incompatible with conservation of L^2 norm, and RH follows.

Thus, within a hybrid analytic–physical framework where the zeta prime–zero dynamics are modeled as a closed quantum system, the Riemann Hypothesis appears as a necessary condition for unitarity. Whether such a physical model is the correct interpretation of $\zeta(s)$ lies at the interface of analytic number theory, quantum information, and spectral dynamics.

Remark 43 (Connection to random matrix theory). The physical interpretation aligns naturally with the GUE conjecture. In random matrix theory, the Gaussian Unitary Ensemble describes spectra of closed, Hermitian quantum systems exhibiting level repulsion [6]. Under GUE statistics, the pair-correlation structure enforces precisely the rigidity condition that in the analytic setting guarantees RH (Definition 10).

From the physical viewpoint, this is because GUE eigenstates represent coherent, undamped modes of a unitary system. Off-line zeros would correspond to non-Hermitian perturbations introducing decay, incompatible with GUE structure. Thus, the appearance of GUE statistics in the zeros of ζ is not merely numerical coincidence: it is the spectral fingerprint of the same unitarity principle underlying Theorem 23.

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