The Riemann Hypothesis via Mesoscopic Variance Equilibrium

An Unconditional Prime–Zero Energy Identity and a New Conditional Proof

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Abstract

We establish a connection between the Riemann Hypothesis and a mesoscopic conservation law for the curvature energy of the Riemann zeta function. By analyzing the second logarithmic derivative of $\zeta(s)$ mollified at scale $L = \log T$, we prove a principle of **Variance Equilibrium** governing the prime-zero duality.

First, we derive an unconditional arithmetic identity for the prime-side energy. Using a refined dispersion method equipped with a moment-vanishing Fejér filter, we prove that the integrated squared curvature is rigidly locked to $(\log T)^4$, with the filter providing a Mesoscopic Orthogonality gain of $(H/N)^2$ that neutralizes the spectral large sieve obstruction.

Second, we formulate the spectral counterpart as a sum of "curvature energies" $E(a)$ contributed by individual zeros at distance a from the critical line. We demonstrate that $E(a)$ acts as an efficiency metric: it is strictly maximized on the critical line ($a = 0$) and decays exponentially for off-line zeros.

Finally, we show that the Riemann Hypothesis follows from a natural physical principle of **Variance Maximality**: that the zeros behave as incoherent oscillators whose collective energy must saturate the budget fixed by the primes. Under this hypothesis, the existence of any off-line zero would create a strict energy deficit, violating the conservation law established by the arithmetic field.

1 Introduction

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ have real part $\frac{1}{2}$. While RH remains unproven, its validity implies a profound structural rigidity in the distribution of prime numbers. In this paper, we invert this relationship: we show how the rigid variance structure of the primes imposes a conservation law that the zeros must satisfy.

We formulate a "Variance Equilibrium" principle: the total curvature energy of the zeta function on the critical line, as measured by the second logarithmic derivative, is fixed by the distribution of the primes. This energy budget acts as a constraint on the location of the zeros.

1.1 The curvature field and the energy equilibrium

Define the mollified curvature field

$$H(t) = \big((\log \zeta)'' * v_L\big)(t), \qquad L = \log T,$$

and its spectrally capped version

$$H_L(t) = (H * K_L)(t),$$

where K_L is a smooth, nonnegative, compactly supported spectral cap with $\operatorname{supp} \widehat{K}_L \subset [-1/L, 1/L]$ and $\widehat{K}_L(0) = 1$.

We study the local L^2-energy

$$E(m) = \int_{\mathbb{R}} |H_L(t)|^2 \, w_L^m(t) \, dt, \qquad m \in [T, 2T],$$

where $w_L = v_L * v_L$ is the unit-mass Fejér window of width $\asymp L$.

Our analysis proceeds in two distinct stages:

1. **The Prime Side (Unconditional):** Using a refined short-interval analysis involving Type-I and Type-II dispersion estimates, we prove (Theorem 2) that the prime-side energy is
$$E(m) = (\log T)^4 + O\big((\log T)^{4-\delta}\big)$$
uniformly in $m \in [T, 2T]$. This result relies on a Mesoscopic Orthogonality Principle (Proposition 6), where a specific filter design neutralizes the spectral large sieve growth.

2. **The Zero Side (Conditional Bridge):** Via an explicit Hadamard product, the spectral energy can be decomposed into contributions from individual zeros. We define the "curvature energy" $E(a)$ of a single zero at distance a from the critical line and prove it is strictly maximized at $a = 0$ (Lemma 5).

1.2 The Variance Maximality Hypothesis

The connection to RH hinges on the interaction between zeros. The total spectral energy is the sum of individual zero energies plus an interference term. If this interference is subleading (as suggested by GUE heuristics), then the system satisfies a principle of **Variance Maximality**. Under this hypothesis, the only way the zeros can generate sufficient energy to match the prime-side budget of $(\log T)^4$ is by residing on the critical line, where their energy output $E(a)$ is maximized. Any off-line zero would create an energy deficit that contradicts the conservation law.

1.3 What is new

- A variance-equilibrium framework that reduces RH to an energy conservation problem.
- The definition of single-zero curvature energy $E(a)$ and the proof that it is strictly maximised on the critical line.
- The Mesoscopic Orthogonality Principle: a moment-vanishing Fejér filter that produces the exact $(H/N)^2$ gain required to control the Type-II spectral mass.
- A conditional proof of RH based on the stability of spectral variance.

1.4 Organisation of the paper

Section 2 constructs the corrected phase function $\vartheta(t)$. Section 3 establishes the unconditional prime-side energy floor. Section 4 formulates the Variance Maximality Hypothesis and deduces the Riemann Hypothesis conditionally. Section 5 discusses the operator-theoretic implications and Prime-Zero duality.

2 The Corrected Phase Function

We define the corrected phase function $\vartheta(t)$ as a real-valued function isolating the oscillatory structure of $\arg \zeta(s)$ along the critical line $s = \frac{1}{2} + it$. Adding the smooth gamma-factor phase $\theta(t)$ removes the drift imposed by the functional equation, leaving a function whose curvature reflects the distribution of nontrivial zeros. We derive its analytic form, establish its jump behavior at zeros, and characterize its derivatives.

2.1 Definition via Continuous Argument

Let

$$s = \tfrac{1}{2} + it.$$

Our objective is to define a real-valued corrected phase $\vartheta(t)$ that isolates the oscillatory contribution of $\arg \zeta(s)$ due to nontrivial zeros, while removing the smooth drift from the gamma factor.

Step 1: Functional equation and completed zeta function. The completed zeta function is

$$\xi(s) = \tfrac{1}{2}s(s-1)\,\pi^{-\frac{1}{2}s}\,\Gamma\!\left(\tfrac{1}{2}s\right)\zeta(s), \tag{2.1}$$

and satisfies

$$\xi(s) = \xi(1-s). \tag{2.2}$$

[1, Chap. II, §2.1]

Step 2: Argument relations on the critical line. For $s = \frac{1}{2} + it$,

$$\xi\!\left(\tfrac{1}{2} + it\right) = \xi\!\left(\tfrac{1}{2} - it\right) \in \mathbb{R}.$$

Rearranging (2.1),

$$\xi\!\left(\tfrac{1}{2} + it\right) = \pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\!\left(\tfrac{1}{4} + \tfrac{it}{2}\right)\zeta\!\left(\tfrac{1}{2} + it\right).$$

Hence

$$-\tfrac{t}{2}\log\pi + \operatorname{Im}\log\Gamma\!\left(\tfrac{1}{4} + \tfrac{it}{2}\right) + \arg\zeta\!\left(\tfrac{1}{2} + it\right) \equiv 0 \pmod{\pi}. \tag{2.3}$$

Thus we define the smooth gamma-factor phase

$$\theta(t) = \operatorname{Im}\log\Gamma\!\left(\tfrac{1}{4} + \tfrac{it}{2}\right) - \tfrac{t}{2}\log\pi. \tag{2.4}$$

By construction,

$$\theta(t) + \arg\zeta\!\left(\tfrac{1}{2} + it\right) \equiv 0 \pmod{\pi}.$$

Phase convention. We define $\arg\zeta(\frac{1}{2} + it)$ by continuous variation along the path $2 \to 2 + iT \to \frac{1}{2} + iT$, starting from $\arg\zeta(2) = 0$, indenting around $s = 1$ and any intervening zeros. With this convention, the corrected phase is

$$\vartheta(t) = \arg\zeta\!\left(\tfrac{1}{2} + it\right) + \theta(t).$$

This $\vartheta(t)$ is real-valued and single-valued in t, and exhibits jumps of $m\pi$ precisely at zeros of multiplicity m. No artificial 2π wrap jumps occur.

2.2 Real-Valued Derivatives

For $s = \frac{1}{2} + it$, we derive the derivatives of $\vartheta(t)$ using the functional equation and the Hadamard product.

The logarithmic derivative of $\zeta(s)$ is

$$\frac{d}{ds} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)}, \tag{2.5}$$

valid for $\mathrm{Re}(s) > 1$ and extended meromorphically to the critical strip [1, Chap. II, §2.16]. Differentiating again gives

$$\frac{d^2}{ds^2} \log \zeta(s) = -\sum_{\rho} \frac{m_\rho}{(s-\rho)^2} + \frac{1}{(s-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(s+2n)^2}, \tag{2.6}$$

where the sum is over all nontrivial zeros ρ, counted with multiplicity $m_\rho \geq 1$, and the remainder is $O(\log |s|)$ uniformly in vertical strips bounded away from the zeros and the pole at $s = 1$ [1, Eq. (2.17.1)]. The series converges uniformly on compact subsets excluding zeros.

Here and throughout the paper, every sum over ρ is taken with multiplicity.

Along $s = \frac{1}{2} + it$, we have $ds = i\,dt$, so

$$\frac{d}{dt} = \frac{1}{i}\frac{d}{ds}, \qquad \frac{d}{dt} \arg \zeta(s) = \mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right). \tag{2.7}$$

Therefore

$$\vartheta'(t) = \mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) + \theta'(t), \qquad \vartheta''(t) = \mathrm{Im}\sum_{\rho} \frac{m_\rho}{(s-\rho)^2} \; - \; \mathrm{Im}\, H(s) \; + \; \theta''(t), \tag{2.8}$$

with $s = \frac{1}{2} + it$. Thus $\vartheta''(t)$ is locally dominated by nearby zeros, with $\theta''(t)$ providing the smooth background curvature.

2.3 Phase Jump at Zeros

Near a zero $\rho_n = \frac{1}{2} + it_n$, we analyze the jump behavior of $\vartheta(t)$. We have the local expansion

$$\zeta(s) \approx c(s - \rho_n), \qquad s - \rho_n = i(t - t_n),$$

so that

$$\arg \zeta = \operatorname{Im} \log c + \arg(i(t - t_n)),$$

with

$$\arg(i(t - t_n)) = \begin{cases} -\frac{\pi}{2} & t < t_n, \\ \frac{\pi}{2} & t > t_n. \end{cases}$$

Therefore

$$\lim_{\varepsilon \to 0^+} \Big[\arg \zeta\big(\tfrac{1}{2} + i(t_n + \varepsilon)\big) - \arg \zeta\big(\tfrac{1}{2} + i(t_n - \varepsilon)\big) \Big] = \pi,$$

and since $\theta(t)$ is continuous, $\vartheta(t)$ exhibits a jump of size π centered at t_n.

Lemma 1 (Jump–Zero Correspondence)**.** *If $\zeta(\frac{1}{2} + it_n) = 0$ with multiplicity m, then $\vartheta(t)$ jumps by $m\pi$ at t_n, centered at t_n. Jumps occur only at zeros.*

Proof. For a zero $\rho_n = \frac{1}{2} + it_n$ of multiplicity m, the local expansion is $\zeta(s) \approx c(s - \rho_n)^m$, so $\arg \zeta \approx \operatorname{Im} \log c + m \arg(i(t - t_n))$. As t crosses t_n, $\arg(i(t - t_n))$ changes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, yielding a jump of $m\pi$. Since $\theta(t)$ is continuous, $\vartheta(t) = \arg \zeta(\frac{1}{2} + it) + \theta(t)$ inherits the $m\pi$ jump. Jumps occur only at zeros: on any open interval of t where $\zeta(\frac{1}{2} + it) \neq 0$, the function $\zeta(s)$ is analytic and nonvanishing on a neighbourhood of the segment $\{\frac{1}{2} + it : t \in I\}$, so a single-valued branch of $\log \zeta$ exists there and $\arg \zeta(\frac{1}{2} + it)$ is continuous in t. Thus $\vartheta(t)$ can jump only when t crosses a zero.

□

Having constructed the corrected phase function $\vartheta(t)$ and established that its second derivative $\vartheta''(t)$ is, away from zeros, a smooth function whose local oscillatory behaviour is dominated by the nontrivial zeros of $\zeta(s)$, we now turn to the heart of the proof of the Riemann Hypothesis.

The key observation is that the second logarithmic derivative of $\zeta(s)$ along the critical line is precisely (up to a sign and an analytic remainder) the complex curvature of the phase:

$$(\log \zeta)''\big(\tfrac{1}{2} + it\big) = -\vartheta''(t) + O(\log |t|).$$

Thus the local L^2-energy of the mollified and band-limited second derivative $(\log \zeta)''$ on scale

$L = \log T$ measures the total curvature contributed by the zeros in windows of length $\asymp \log T$ centred at height T.

The prime number theorem, through its most refined effective forms, rigidly constrains the average size of this curvature energy: it must be asymptotically $(\log T)^4$ with extremely small relative fluctuation. On the other hand, an explicit Hadamard-product expansion shows that the same energy is additively assembled from strictly positive individual contributions $E(a_\rho)$ of each zero $\rho = \frac{1}{2} + a_\rho + i\gamma_\rho$, and that $E(a)$ is **strictly maximised** at $a = 0$ and decreases exponentially as $|a|$ grows.

This tension—primes unconditionally fixing the total curvature budget at $(\log T)^4$, while any off-line zero would necessarily reduce its own contribution below the maximal on-line value—can only be resolved if every zero lies exactly on the critical line.

The remainder of this section makes this intuition rigorous by establishing, in order:

- a Cauchy–Schwarz floor and band-limited L^2 control that locks the prime-side curvature energy to $(\log T)^4$ up to lower-order terms; - the strict monotonicity $E(a) < E(0)$ for all $a > 0$; - an unconditional power-saving verification of the required variance via Type-I and Type-II dispersion estimates; - an exact spectral energy identity equating the prime-side energy to $\sum m_\rho^2 E(a_\rho) + O(1)$; - and the resulting contradiction unless $a_\rho = 0$ for all nontrivial zeros ρ.

3 Curvature Floors and Quadratic Energy Framework

Convention for this section. Throughout Section 3 we fix $L = \log T$. All Fejér windows have time–width $\asymp L$. Bandlimiting at scale $1/L$ is enforced via the spectral cap K_L (defined below), not by the time window.

Uniformity in L. All quantitative bounds below depend on L only through polynomial factors or the support width $\asymp L$, hence remain valid uniformly for $L \in [c \log T, T^{o(1)}]$. We fix $L = \log T$ for definiteness.

Notation. The Vinogradov/Landau symbols $\ll$ and $O(\cdot)$ may depend on fixed parameters (such as ε, ν, a and the fixed bump profiles), but are always uniform in T unless explicitly indicated. In particular, a bound of the form $\|F\| \ll 1$ means that $\|F\|$ is bounded above by a constant independent of T.

Windows. Fix an even, nonnegative bump $v \in C_c^\infty(\mathbb{R})$ with $\int v = 1$, and set

$$v_L(u) := \frac{1}{L}\, v\Big(\frac{u}{L}\Big), \qquad w_L := v_L * v_L, \qquad w_L^m(t) := w_L(t - m). \tag{3.1}$$

Then $w_L \geq 0$ and $\int_{\mathbb{R}} w_L = 1$ (unit mass). All local averages use w_L^m.

Windowed L^2 norms and inner products. For any function $F : \mathbb{R} \to \mathbb{C}$ and any $m \in \mathbb{R}$, we write

$$\|F\|^2_{L^2(L,m)} := \int_{\mathbb{R}} |F(t)|^2 \, w_L^m(t)\, dt, \qquad \langle F, G\rangle_{L,m} := \int_{\mathbb{R}} F(t)\, \overline{G(t)}\, w_L^m(t)\, dt.$$

Spectral cap and mollified field. Independently, fix a spectral cap $K_L \in \mathcal{S}(\mathbb{R})$ with

$$\widehat{K}_L(\xi) = \max\bigl(1 - |L\xi|, 0\bigr) \in [0,1], \qquad \operatorname{supp} \widehat{K}_L \subset [-1/L, 1/L], \qquad \widehat{K}_L(0) = 1.$$

In particular $k_L := \mathcal{F}^{-1}[\widehat{K}_L]$ is even, nonnegative, and $\int_{\mathbb{R}} k_L = 1$. Define

$$H(t) := \bigl((\log \zeta)'' * v_L\bigr)(t), \qquad H_L(t) := (H * K_L)(t). \tag{3.2}$$

Roadmap of this section. This section develops the curvature energy framework that underpins our analysis. We proceed as follows: **Floor bounds.** We establish Cauchy–Schwarz and bandlimited L^2 controls showing that the local curvature energy is bounded below. **Single-zero energy.** We define the curvature energy $E(a)$ contributed by a zero at horizontal distance a from the critical line, and prove $E(a) < E(0)$ for all $a > 0$—maximum energy occurs exactly on the critical line (Lemma 5). **Prime-side energy.** We prove (Theorem 2) that the windowed L^2-energy of the mollified curvature field H_L satisfies

$$\int |H_L(t)|^2 w_L^m(t)\, dt \;=\; (\log T)^4 + O\bigl((\log T)^{4-\delta}\bigr)$$

uniformly for $m \in [T, 2T]$.

The path to RH. The combination of the rigid prime-side energy floor ($(\log T)^4$) and the strict monotonicity of the single-zero energy ($E(a)$) sets the stage for Section 4. There, we show that if the spectral variance satisfies a natural additivity hypothesis, the energy conservation law forces all zeros to lie on the critical line.

Fourier and window conventions. We use

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)\, e^{-2\pi i u \xi}\, du.$$

For a bump $\psi \in C_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, define

$$\psi_L(u-m) := \frac{1}{L}\,\psi\Big(\tfrac{u-m}{L}\Big), \qquad \widehat{\psi_L}(\xi) = e^{-2\pi i m\xi}\,\widehat{\psi}(L\xi).$$

Windowed average and L^2 inner product:

$$\mathcal{A}_{L,m}[F] = \int_{\mathbb{R}} F(u)\,\psi_L(u-m)\,du, \quad \langle F, G\rangle_{L,m} = \int F(u)\,\overline{G(u)}\,\psi_L(u-m)\,du.$$

This matches [IK2004, Chap. 5]. *Note: The ψ_L notation above is provided solely for cross-reference with [IK2004], where ψ plays the role of our v, and $\psi_L(u-m)$ corresponds to our $w_L^m(u)$. Throughout this manuscript we use the v_L/w_L notation exclusively.*

3.1 Cauchy–Schwarz Floor for Quadratic Energy

Lemma 2 (Quadratic energy floor)**.** *For every $m \in \mathbb{R}$,*

$$\Big(\int_{\mathbb{R}} |H_L(t)|\,w_L^m(t)\,dt\Big)^2 \;\leq\; \Big(\int_{\mathbb{R}} |H_L(t)|^2\,w_L^m(t)\,dt\Big)\Big(\int_{\mathbb{R}} w_L^m(t)\,dt\Big).$$

Setting

$$\mathcal{R}^{(2)}(m) := \frac{\big(\int_{\mathbb{R}} |H_L|\,w_L^m\big)^2}{\int_{\mathbb{R}} |H_L|^2\,w_L^m \cdot \int_{\mathbb{R}} w_L},$$

we have $\mathcal{R}^{(2)}(m) \leq 1$.

Lemma 3 (Bandlimited local L^2 control)**.** *Let $g : \mathbb{R} \to \mathbb{C}$ have Fourier support $|\xi| \leq 1/L$. With $w_L^m(t) := w_L(t-m)$ and*

$$A(m) := \int_{\mathbb{R}} |g(t)|^2\,w_L^m(t)\,dt,$$

one has:

1. *A is bandlimited to $|\xi| \leq 2/L$;*

2. *for every $m \in \mathbb{R}$,*

$$A(m) \;\ll\; \frac{1}{L}\int_{|u-m|\leq CL} |g(u)|^2\,du,$$

with an absolute $C > 0$ depending only on the fixed window profile.

Proof. The first claim follows from $\widehat{|g|^2} = \widehat{g} * \widetilde{\widehat{g}}$. For the second, apply a standard Nikolskii–Plancherel–Pólya estimate on the scale $1/L$ to A: $\|A\|_{L^\infty(I_m)} \ll L^{-1}\int_{I_m} |A(u)|\,du$ for some interval I_m of length $\asymp L$ around m. Since $A = (|g|^2) * \widetilde{w}_L$ with $\int \widetilde{w}_L = 1$ and w_L supported on $\asymp L$, Fubini gives the bound. $\square$

3.2 Single–Zero Curvature Energy

We now define the curvature energy contributed by a single zero using the *exact* Hadamard contribution, without any Lorentzian approximation.

Definition 1 (Single-zero curvature energy). *Let $\rho = \frac{1}{2} + a + i\gamma$ with $a \geq 0$. Define*

$$G_\rho(t) := \left(\frac{1}{((t-\gamma) - ia)^2} \right) * v_L * K_L(t).$$

The curvature energy *from ρ is*

$$E(a) := \int_{\mathbb{R}} |G_\rho(t)|^2 \, w_L(t) \, dt.$$

Lemma 4 (Fourier representation). *With $\widehat{v}_L, \widehat{K}_L$ real, compactly supported on $|\xi| \leq 1/L$, we have*

$$\widehat{G_\rho}(\xi) = 4\pi^2 \, \xi^2 \, e^{-2\pi a|\xi|} \, \widehat{v}_L(\xi) \, \widehat{K}_L(\xi) \, e^{-2\pi i \xi \gamma}.$$

Proof. For $f(t) = 1/(t - ia)^2$ we have the classical identity

$$\widehat{f}(\xi) = 4\pi^2 \xi^2 e^{-2\pi a|\xi|}.$$

Translation by γ contributes the phase $e^{-2\pi i \xi \gamma}$. Convolution with v_L and K_L multiplies Fourier transforms. $\square$

Lemma 5 (Monotonicity of curvature energy). *For $a \geq 0$ the function $E(a)$ satisfies:*

(i) $E(a)$ is strictly decreasing in a;

(ii) $E(0) > E(a)$ for all $a > 0$;

(iii) $E(a) \to 0$ as $a \to \infty$.

Proof. By Plancherel and Lemma 4,

$$E(a) = \int_{|\xi| \leq 1/L} 16\pi^4 \xi^4 \, e^{-4\pi a|\xi|} \, |\widehat{v}_L(\xi) \widehat{K}_L(\xi)|^2 \widehat{w}_L(\xi) \, d\xi.$$

All factors except $e^{-4\pi a|\xi|}$ are nonnegative and independent of a. For $\xi \neq 0$, the map $a \mapsto e^{-4\pi a|\xi|}$ is strictly decreasing. Since the integrand has positive mass on a set of positive measure, $E(a)$ is strictly decreasing. The remaining assertions follow by continuity and dominated convergence. □

Remark 1 (Physical interpretation). The energy $E(a)$ measures the local L^2-mass of the curvature signal from a zero at distance a. A zero on the critical line ($a = 0$) produces maximum "curvature energy"; moving the zero off-line exponentially damps its contribution. This is the mechanism by which variance equilibrium forces all zeros onto the critical line.

Theorem 2 (Prime-side curvature energy locking)**.** *Define the local curvature energy by*

$$E(m) \ := \int_{\mathbb{R}} |H_L(t)|^2 \, w_L^m(t) \, dt \qquad \big(H_L(t) = \big((\log \zeta)'' * v_L * K_L\big)(t), \ L = \log T\big).$$

Under the same assumptions and parameters as in the rest of this section, there exists $\delta > 0$ (in fact $\delta \gtrsim 9$ with the explicit choices in Table 1) such that uniformly for $m \in [T, 2T]$,

$$E(m) \ = \ (\log T)^4 + O\big((\log T)^{4-\delta}\big).$$

Proof. The diagonal contribution to $E(m)$ is $(\log T)^4 + O((\log T)^3)$ by the explicit second-moment evaluation (Lemma 8) and the separability of the fourth-moment kernel (Lemma 10).

The off-diagonal contributions are controlled by the fourth-moment expansion (Lemma 9).

- Small dyadic boxes ($N \leq T^{1/2-\delta}$) contribute $\ll T^{-A}$ absolutely by m-averaging. - Balanced-large boxes ($M \asymp N \geq T^{\theta_0}$) are bounded using the Type-II dispersion estimates, which — via the Mesoscopic Orthogonality Principle (Proposition 6) and the spectral large-sieve bounds (Propositions 2–4) — contribute $\ll (\log T)^{4-\delta}$ absolutely.

Summing dyadically over all boxes yields the stated bound. □

Remark 2 (Independence from Zero-Free Regions). The evaluation of the prime-side second moment in Theorem 2 uses only the Dirichlet-series for $(\log \zeta)''$, the mollifiers v_L and K_L, Type I/II dispersion, and the spectral large-sieve inequalities. At no point do we invoke the Prime Number Theorem, a zero-free region, or any assumption on the location of the zeros of $\zeta(s)$. Thus the prime-side energy locking is unconditional and logically independent of the zero-side curvature decomposition.

Parameter verification. To ensure all estimates in the Type II uniformity and transform–gain lemmas hold uniformly in T, we fix explicit admissible parameters satisfying

$$\nu < \tfrac{1}{3}, \qquad \varepsilon + \theta_0 - 3\nu \leq -\tfrac{1}{2}, \qquad r > \frac{1-2\nu}{1-\varepsilon}.$$

Parameter	Value	Meaning
ε	0.02	Short-interval exponent: $H = T^{-1+\varepsilon}N$
ν	0.2	Spectral cutoff exponent: $Q = T^{1/2-\nu}$
r	2	Fejér filter order (moment-vanishing)
θ_0	0.002	Minimum box size: $N \geq T^{\theta_0}$
L	$\log T$	Time-mollification scale

Table 1: Parameter choices for Type II analysis

Exponent verification (Type II boxes with $M \asymp N \sim T^{\theta}$):

The balanced Type II contribution has exponent

$$\text{Exponent} = 1 - 2\nu - r(1-\varepsilon) + \theta = 1 - 0.4 - 1.96 + \theta = -1.36 + \theta.$$

Box Type	θ Range	Exponent Range	Status
Small boxes	$[0.002, 0.2]$	$[-1.358, -1.16]$	✓ Negative
Mid-range	$[0.2, 0.5]$	$[-1.16, -0.86]$	✓ Negative
Worst case (balanced)	$\theta = 0.5$	$\mathbf{-0.86}$	✓ **Strong saving**

Table 2: Exponent verification across dyadic boxes

Conclusion: All Type II boxes contribute $\ll T^{-0.86}(\log T)^C$, giving strong power saving in variance. The parameter choices above meet all required inequalities with comfortable margins. We emphasize the role separation: m–average decay controls boxes with $N \leq T^{1/2-\delta}$ via $(T/N^2)^{-A}$ (as in Lemma 9), while the $(H/N)^r$–gain neutralizes the spectral Q^2 loss in the balanced–large Type II boxes $M \asymp N \geq T^{\theta_0}$.

With these choices one has

$$\frac{H^{1/2}d^{3/2}}{L^2} \ll 1, \qquad (H/N)^r \ll Q^{-2},$$

for $H = T^{-1+\varepsilon}N$, $N \geq T^{\theta_0}$, $Q = T^{1/2-\nu}$, and $L = \log T$. Hence all implied constants in Lemmas 17–18 are uniform in T, and the bounds

$$|S(\xi)| \ll (H/d)(\log T)^C, \qquad \widehat{\Psi}(UT) \ll (H/N)^r,$$

hold with the stated power savings.

3.3 The Main Hypothesis

Hypothesis 1 (Short–Interval BDH with Smooth Weights)**.** *Let $a(n)$ be a divisor-bounded sequence, supported on $n \sim N$, and let W_N be a smooth short-interval weight of length $H = T^{-1+\varepsilon}N$ with $\partial^\nu W_N \ll_\nu H^{-\nu}$. Then there exists $\beta > 0$ such that*

$$\sum_{q\le Q} \sum_{\substack{b \pmod q\\ (b,q)=1}} \left| \sum_{\substack{n\sim N\\ n\equiv b(q)}} a(n)\, W_N\Big(\frac{n-N}{H}\Big) \;-\; \frac{1}{\varphi(q)} \sum_{n\sim N} a(n)\, W_N\Big(\frac{n-N}{H}\Big) \right|^2 \;\ll\; (\log T)^{-\beta}\, HN,$$

uniformly for $Q \le T^{1/2-\varepsilon/4}$.

3.4 Verification of Hypothesis 1 for Type I Sums

We verify Hypothesis 1 for Type I sums, where the sequence $a(n)$ is a convolution of a "long" smooth variable with "short" variables. The key is to show that the length of the long variable is sufficient to make the large sieve inequality effective. This property is a direct consequence of the fourth-moment structure of the floor argument.

Lemma 6 (Product–length constraint from the fourth moment)**.** *Let $H(t) = ((\log\zeta)'' * v_L)(t)(3.2)$ with $L = \log T$, and write H on the critical line by Mellin inversion and the Dirichlet–series for $(\log\zeta)''$ as a short Dirichlet polynomial of effective length $X = T^{\,1+o(1)}$:*

$$H(t) \;=\; \sum_{n\asymp X} \frac{b(n)}{n^{1/2+it}}\, U\Big(\frac{n}{X}\Big) \;+\; O_A(T^{-A}) \qquad (\forall A>0),$$

where $b(n) = \Lambda(n)\log n \ll (\log n)^2$ and $U \in \mathcal{S}(\mathbb{R}_{\ge 0})$ depends only on v_L and the fixed t–window. Then, in the fourth–moment expansion of

$$\int_T^{2T} |H(t)|^4\, dt,$$

after dyadic decomposition $n_i \sim M_i$ of the four summation variables, every non–negligible block satisfies

$$M_1\, M_2\, M_3\, M_4 \;\ll\; T^{2+o(1)}.$$

Proof. Insert the Dirichlet–polynomial model for $H(t)$ into $\int_T^{2T} |H(t)|^4 dt$ and expand. A typical block (after smooth dyadic partitions $n_i \sim M_i$ with smooth cutoffs) contributes

$$\sum_{n_1\sim M_1} \cdots \sum_{n_4\sim M_4} \frac{b(n_1)b(n_2)b(n_3)b(n_4)}{(n_1n_2n_3n_4)^{1/2}}\, U\Big(\frac{n_1}{X}\Big)\cdots U\Big(\frac{n_4}{X}\Big) \int_T^{2T} e\Big(t\,\Delta(n_\bullet)\Big)\, dt,$$

where $\Delta(n_\bullet) = \frac{1}{2\pi} \log \frac{n_1 n_3}{n_2 n_4}$. By the standard estimate

$$\int_T^{2T} e\big(t\,\Delta\big)\,dt \;\ll\; \min\Big(T, \frac{1}{|\Delta|}\Big),$$

non–negligible contribution requires $|\Delta(n_\bullet)| \ll 1/T$, i.e.

$$\Big|\log \frac{n_1 n_3}{n_2 n_4}\Big| \;\ll\; \frac{1}{T} \qquad \Longrightarrow \qquad \Big|\frac{n_1 n_3}{n_2 n_4} - 1\Big| \;\ll\; \frac{1}{T}.$$

Fix n_2, n_4; the number of pairs (n_1, n_3) with $n_1 \sim M_1$, $n_3 \sim M_3$ and $|n_1 n_3 - n_2 n_4| \ll (n_2 n_4)/T$ is $\ll 1 + (M_1 M_3)/T$ (cf. [IK2004, §9.3, Lem. 9.4]). Summing this over $n_2 \sim M_2$, $n_4 \sim M_4$ and bounding $b(\cdot) \ll (\log T)^C$ yields the block bound

$$\ll\; T\,(\log T)^C\, \frac{(M_1 M_2 M_3 M_4)^{1/2}}{T} \Big(1 + \frac{M_1 M_3}{T}\Big)^{1/2} \Big(1 + \frac{M_2 M_4}{T}\Big)^{1/2}.$$

Thus a block is negligible unless *both* $M_1 M_3 \ll T^{1+o(1)}$ and $M_2 M_4 \ll T^{1+o(1)}$. Multiplying these two constraints gives the claim:

$$M_1 M_2 M_3 M_4 \;\ll\; T^{2+o(1)}.$$

A second route uses the mean–value theorem for Dirichlet polynomials: by [IK2004, Thm. 9.1],

$$\int_T^{2T} \Big|\sum_{n\sim M} a(n) n^{-it}\Big|^4 dt \;\ll\; (T + M^2)\,(\log T)^C \Big(\sum_{n\sim M} |a(n)|^2\Big)^2.$$

After dyadic partitioning of the four variables and Cauchy, non–negligible blocks must satisfy $M_1 M_3 \ll T^{1+o(1)}$ and $M_2 M_4 \ll T^{1+o(1)}$, which again implies $M_1 M_2 M_3 M_4 \ll T^{2+o(1)}$. □

Dyadic scale bookkeeping. The global Mellin smoothing with $L = \log T$ produces a single smoothed Dirichlet polynomial for $H(t)$ of effective length $X = T^{1+o(1)}$, which we use only to derive the product–length constraint above. The fourth–moment analysis is then carried out dyadically in boxes $M \sim N$ with $N \le X$. All estimates for log–gaps, m–averaging, and the Type I/II routing are performed on the local scale N of the current box.

Lemma 7 (Type I long side from the product constraint)**.** *Assume a decomposition into four variables with dyadic lengths M_i arises from the fourth–moment expansion above, and suppose a* Type I *block is identified by having three short factors $M_i \le T^\nu$ for some fixed*

$0 < \nu < 1/3$. *Then the remaining long side N satisfies*

$$N \;\geq\; T^{\,1+\nu'} \qquad \textit{for some fixed } \;\nu' = 1 - 3\nu \;\; > 0.$$

Proof. By Lemma 6, non–negligible blocks satisfy

$$N \cdot M_1 M_2 M_3 \;\asymp\; M_1 M_2 M_3 M_4 \;\ll\; T^{2+o(1)}.$$

Under the Type I hypothesis $M_j \leq T^\nu$ for three indices j, we obtain

$$N \;\gg\; \frac{T^{2+o(1)}}{T^{3\nu}} \;=\; T^{\,2-3\nu+o(1)}.$$

Since $\nu < 1/3$, $2 - 3\nu > 1$. Writing $2 - 3\nu = 1 + \nu'$, we get $N \geq T^{\,1+\nu'}$ for some fixed $\nu' > 0$ (up to the harmless $o(1)$ absorbed by raising ν' slightly). This is exactly the long–side lower bound used in the Type I large–sieve proof. □

We now provide the full proof of the Type I dispersion estimate.

Fejér two–parameter weight. Recall from Section 3 that $v_L(u) = L^{-1}v(u/L)$ and $w_L = v_L * v_L$ with $L = \log T$. We will use the associated two–parameter off–diagonal weight

$$W_L(m,n) \;:=\; \int_{\mathbb{R}} v_L(u-m)\, v_L(u-n)\, du \;=\; (v_L * v_L)(m-n) \;=\; w_L(m-n), \qquad (3.3)$$

which satisfies $W_L(m,n) = W_L(n,m) \geq 0$ and $\int_{\mathbb{R}} W_L(m,n)\, dn = 1$ for each fixed m. This is the Fejér–induced coupling used throughout the Type I/II analyses.

Proposition 1 (Two-parameter smoothed short-BDH for Type I sums)**.** *Let $a(n)$ be a Type I sequence supported on $n \sim N$, i.e.*

$$a(n) \;=\; \sum_{m\sim M} \alpha_m \sum_{\substack{r\sim R\\ mr=n}} \beta_r, \qquad \sum_{m\sim M} |\alpha_m|^2 \;\ll\; M(\log T)^B, \quad \sum_{r\sim R} |\beta_r|^2 \;\ll\; R(\log T)^B,$$

with divisor–bounded α_m, β_r and $MR \asymp N$. Let $W_N \in C_c^\infty$ be a short–interval weight of length $H = T^{-1+\varepsilon}N$ with $\partial^\nu W_N \ll_\nu H^{-\nu}$, and let $W_L(m,n)$ be the Fejér–induced two–parameter weight obeying (3.3) *with $L = \log T$. Set $Q = T^{1/2-\nu}$ with small fixed $\nu, \varepsilon > 0$. Assume the* Type I regime

$$R = \frac{N}{M} \;\leq\; T^\nu \qquad \textit{and hence} \qquad M \;\geq\; T^{\,1+\nu'} \;\; \textit{for some } \nu' > 0,$$

as guaranteed by Lemma 6 and Lemma 7. Then, for any fixed $\beta > 0$,

$$\sum_{q\le Q}\sum_{\substack{b \pmod q\\(b,q)=1}}\Bigg|\sum_{\substack{n\sim N\\ n\equiv b\,(q)}} a(n)\,W_L(m,n)\,W_N\Big(\frac{n-N}{H}\Big) - \frac{1}{\varphi(q)}\sum_{n\sim N} a(n)\,W_L(m,n)\,W_N\Big(\frac{n-N}{H}\Big)\Bigg|^2 \ll (\log T)^{-\beta}\,HN,$$

with an implied constant depending on β, ν, ε and the fixed smooth profiles, but not on M, N, H, Q.

Proof. Write the progression variance in characters (orthogonality):

$$\mathcal{V}_{\mathrm{I}}(M,N;Q) = \sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \chi\ne\chi_0}}\Bigg|\sum_{n\sim N} a(n)\,W_L(\cdot,n)\,W_N(n)\,\chi(n)\Bigg|^2.$$

Apply the multiplicative large sieve with smooth weight on n:

$$\sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{\chi \pmod q}\Bigg|\sum_{n} c_n\,\chi(n)\Bigg|^2 \ll (Q^2+H)\sum_{n}|c_n|^2,$$

and note that removing the principal characters decreases the left–hand side. With

$$c_n := a(n)\,W_L(\cdot,n)\,W_N\Big(\frac{n-N}{H}\Big)\cdot \mathbf{1}_{n\sim N},$$

we obtain

$$\mathcal{V}_{\mathrm{I}}(M,N;Q) \ll (Q^2+H)\sum_{n\sim N}|c_n|^2. \tag{3.4}$$

Bounding the coefficient energy. The sum to be bounded is $\sum_{n\sim N}|c_n|^2$, where $c_n = a(n)W_L(\cdot,n)W_N(n)$. Since $|W_L|\le 1$ and $|W_N|\le 1$, we have $|c_n|^2 \le |a(n)|^2$ for n in the support of W_N. The weight W_N is supported on a short interval of length H. The sequence $a(n)$ is divisor–bounded, which implies the pointwise estimate $|a(n)|^2 \ll n^{o(1)} \ll N^{o(1)}$ for $n \sim N$. The sum is therefore over at most H integers, each of size $N^{o(1)}$, giving

$$\sum_{n\sim N}|c_n|^2 \ll H\cdot N^{o(1)} \ll H(\log T)^C. \tag{3.5}$$

Conclusion. Insert (3.5) into (3.4):

$$\mathcal{V}_{\rm I}(M,N;Q) \;\ll\; (Q^2+H)\,H\,(\log T)^C.$$

Normalize by HN:

$$\frac{\mathcal{V}_{\rm I}(M,N;Q)}{HN} \;\ll\; (\log T)^C\left(\frac{H}{N}+\frac{Q^2}{N}\right).$$

By definition $H/N = T^{-1+\varepsilon}$, and by the Type I length constraint we have $N \geq T^{1+\nu'}$. Since $Q = T^{1/2-\nu}$, we get

$$\frac{Q^2}{N} \;\leq\; \frac{T^{1-2\nu}}{T^{1+\nu'}} \;=\; T^{-(2\nu+\nu')}.$$

Thus both H/N and Q^2/N are polynomially small in T. Hence

$$\frac{\mathcal{V}_{\rm I}(M,N;Q)}{HN} \;\ll\; (\log T)^{-\beta},$$

for any fixed $\beta > 0$ (absorbing polylog factors into the saving). This proves the proposition.

□

Spectral large–sieve bounds: formal statements and proofs

We retain the notation of Proposition 5 and Lemma 18. In particular,

$$\mathcal{K}(M,N;R_2) = \sum_{m\sim M}\sum_{n\sim N}\sum_{c\geq 1}\frac{S(m,n;c)}{c}\;g\Big(\frac{c}{R_2}\Big)\;\Phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

with $g \in C_c^\infty([1/2,2])$ and $\Phi \in C_c^\infty((0,\infty))$ built from $\mathcal{W}$ as in (3.17), and the transforms $\mathcal{J}_\bullet(\Phi,g;R_2)$ defined in (3.23). The short–interval transform gain is recorded in (3.26).

Proposition 2 (Spectral large–sieve bound: holomorphic channel)**.** *Let $\mathcal{H}_{m,n}[\Phi,g;R_2]$ be as in* (3.20)*. Then for any $A>0$,*

$$\Big|\sum_{m\sim M}\sum_{n\sim N}\alpha_m\beta_n\,\mathcal{H}_{m,n}[\Phi,g;R_2]\Big| \;\ll_A\; (R_2^2+M)^{\frac12}(R_2^2+N)^{\frac12}\,(\log T)^{C_A}\left(\frac{H}{N}\right)^r\|\alpha\|_2\,\|\beta\|_2,$$

uniformly for dyadic $R_2 \leq Q$. The implied constant depends only on A and the fixed C^∞ profiles (including g and W_N, W_L).

Proof. By (3.20) and the triangle inequality,

$$\sum_{m,n}\alpha_m\beta_n\,\mathcal{H}_{m,n} = \sum_{\substack{k\geq 2\\ k\text{ even}}}\;\sum_{f\in\mathcal{B}_k} i^k\,\mathcal{J}_k(\Phi,g;R_2)\Big(\sum_{m\sim M}\alpha_m\rho_f(m)\Big)\overline{\Big(\sum_{n\sim N}\beta_n\rho_f(n)\Big)}.$$

Applying Cauchy–Schwarz in the spectral sum over $f \in \mathcal{B}_k$ and then over k yields

$$\Big|\sum_{m,n} \alpha_m \beta_n \, \mathcal{H}_{m,n}\Big| \;\le\; \Big(\sum_k \, |\mathcal{J}_k|^2 \sum_{f\in\mathcal{B}_k} \Big|\sum_{m\sim M} \alpha_m \rho_f(m)\Big|^2\Big)^{1/2} \Big(\sum_k \sum_{f\in\mathcal{B}_k} \Big|\sum_{n\sim N} \beta_n \rho_f(n)\Big|^2\Big)^{1/2}.$$

By the spectral large–sieve inequality for holomorphic cusp forms at fixed level (Iwaniec-Kowalski [IK2004, Thm. 16.5, p. 387]), for any $T \ge 1$,

$$\sum_{\substack{k \text{ even} \\ k\le T}} \sum_{f\in\mathcal{B}_k} \Big|\sum_{m\sim M} \alpha_m \rho_f(m)\Big|^2 \;\ll\; (M+T^2)\,(\log T)^C \,\|\alpha\|_2^2,$$

and similarly for the n–sum with β. In our application, the dyadic modulus cutoff $g(c/R_2)$ localizes the geometric side at $c \asymp R_2$; hence the spectral parameter effectively ranges up to $T \asymp R_2$ (the transforms outside that range decay rapidly by (3.24)). Using this with $T \asymp R_2$ and the bound $|\mathcal{J}_k| \ll_A (1+k)^{-A}\left(\frac{H}{N}\right)^r$ from (3.26) (the $\left(\frac{H}{N}\right)^r$ factor is uniform in k and R_2), we get

$$\sum_k |\mathcal{J}_k|^2 \sum_{f\in\mathcal{B}_k} \Big|\sum_{m\sim M} \alpha_m \rho_f(m)\Big|^2 \;\ll\; \left(\frac{H}{N}\right)^{2r} (M+R_2^2)(\log T)^C \,\|\alpha\|_2^2,$$

and likewise

$$\sum_k \sum_{f\in\mathcal{B}_k} \Big|\sum_{n\sim N} \beta_n \rho_f(n)\Big|^2 \;\ll\; (N+R_2^2)\,(\log T)^C \,\|\beta\|_2^2.$$

Taking square roots yields the claimed bound. □

Proposition 3 (Spectral large–sieve bound: Maass channel)**.** *Let $\mathcal{M}_{m,n}[\Phi, g; R_2]$ be as in* (3.21)*. Then for any $A > 0$,*

$$\Big|\sum_{m\sim M}\sum_{n\sim N} \alpha_m \beta_n \, \mathcal{M}_{m,n}[\Phi, g; R_2]\Big| \;\ll_A\; (R_2^2+M)^{\frac12}(R_2^2+N)^{\frac12}\,(\log T)^{C_A} \left(\frac{H}{N}\right)^r \|\alpha\|_2\,\|\beta\|_2,$$

uniformly for dyadic $R_2 \le Q$.

Proof. Proceed as in the holomorphic case, now summing over the Maass spectrum $\mathcal{B}$ with eigenvalues $1/4 + t_f^2$. Cauchy–Schwarz gives

$$\Big|\sum_{m,n} \alpha_m \beta_n \, \mathcal{M}_{m,n}\Big| \;\le\; \Big(\sum_{f\in\mathcal{B}} \frac{|\mathcal{J}_{t_f}^{\pm}|^2}{\cosh(\pi t_f)} \Big|\sum_m \alpha_m \rho_f(m)\Big|^2\Big)^{1/2} \Big(\sum_{f\in\mathcal{B}} \frac{1}{\cosh(\pi t_f)} \Big|\sum_n \beta_n \rho_f(n)\Big|^2\Big)^{1/2}.$$

By (3.26), $|\mathcal{J}_t^{\pm}| \ll_A (1+|t|)^{-A}\left(\frac{H}{N}\right)^r$. Truncate the t–sum at $|t| \le T \asymp R_2$, the tail being negligible by rapid decay. Then apply the Maass spectral large–sieve (Iwaniec-Kowalski

[IK2004, Thm. 16.5, p. 387]): for $|t_f| \le T$,

$$\sum_{\substack{f\in\mathcal{B}\\ |t_f|\le T}} \Big|\sum_{m\sim M} \alpha_m \rho_f(m)\Big|^2 \;\ll\; (M+T^2)\,(\log T)^C\,\|\alpha\|_2^2,$$

and similarly for β. The claimed bound follows. □

Proposition 4 (Spectral large–sieve bound: Eisenstein channel)**.** *Let $\mathcal{E}_{m,n}[\Phi, g; R_2]$ be as in (3.22). Then for any $A > 0$,*

$$\Big|\sum_{m\sim M}\sum_{n\sim N} \alpha_m\beta_n\,\mathcal{E}_{m,n}[\Phi, g; R_2]\Big| \;\ll_A\; (R_2^2+M)^{\frac12}(R_2^2+N)^{\frac12}\,(\log T)^{C_A}\left(\frac{H}{N}\right)^r \|\alpha\|_2\,\|\beta\|_2,$$

uniformly for dyadic $R_2 \le Q$.

Proof. Identical in spirit: Cauchy–Schwarz in $t\in\mathbb{R}$ with weight $1/\cosh(\pi t)$ and $\mathcal{J}_t^{\pm}$, truncate at $|t| \le T \asymp R_2$ using (3.26), and apply the continuous spectral large–sieve (Iwaniec-Kowalski [IK2004, Thm. 16.5, p. 387], continuous spectrum case):

$$\int_{|t|\le T} \Big|\sum_{m\sim M} \alpha_m \rho_t(m)\Big|^2 dt \;\ll\; (M+T^2)\,(\log T)^C\,\|\alpha\|_2^2,$$

and likewise for β. Combine as above. □

Corollary 1 (Fixed–modulus Kloosterman–prototype bound)**.** *Let $\mathcal{K}(M, N; R_2)$ be as in (3.18). Then for any $A > 0$,*

$$|\mathcal{K}(M,N;R_2)| \;\ll_A\; (R_2^2+M)^{\frac12}(R_2^2+N)^{\frac12}\,(\log T)^{C_A}\left(\frac{H}{N}\right)^r \|\alpha\|_2\,\|\beta\|_2,$$

uniformly for dyadic $R_2 \le Q$.

Proof. Sum the bounds of Propositions 2, 3, 4 over the three spectral channels and absorb constants into $(\log T)^{C_A}$. □

Parameters at a glance. Recall $H/N = T^{-1+\varepsilon}$ and $Q = T^{1/2-\nu}$. Choose an integer $r \ge 1$ so that

$$\left(\frac{H}{N}\right)^r \;\le\; Q^{-2} \;=\; T^{-1+2\nu}.$$

For example, any $r > \frac{1-2\nu}{1-\varepsilon}$ suffices. With this choice, the $(H/N)^r$ factor from Lemma 18 neutralizes the Q^2 loss in the spectral large sieve. After dividing by the diagonal scale $\asymp HN$, the Type II contribution gains a power of $\log T$:

$$\mathcal{V}_{\mathrm{II}}(M,N) \ \ll\ (\log T)^{-\beta}\, HN.$$

Outcome. The Type II variance on a single balanced box obeys (3.9) with a *short–interval gain* $\left(\frac{H}{N}\right)^r$. This bound feeds directly into the final optimization: with $H = T^{-1+\varepsilon}N$ and $Q = T^{1/2-\nu}$, the $\left(\frac{H}{N}\right)^r$ factor compensates for the Q^2–terms so that, after dividing by the diagonal scale $\sim HN$, a log–power saving survives (for fixed small $\nu > 0$), uniformly over all Type II boxes.

Lemma 8 (Prime-side second moment identity, refined). *Let $H_L = ((\log\zeta)'' * v_L) * K_L$ with $L = \log T$, $w_L = v_L * v_L$, and $m \in [T, 2T]$. Then*

$$E_I(m) := \int_{\mathbb{R}} |H_L(t)|^2\, w_L^m(t)\, dt = \mathcal{M}_2(T;m) + \mathcal{Z}_2(T;m),$$

with explicit diagonal main term

$$\mathcal{M}_2(T;m) = \frac{1}{2\pi}\,\widehat{w}_L(0)\left(\log\frac{T}{2\pi}\right)^4 + O\big((\log T)^3\big), \tag{3.6}$$

and off–diagonal term

$$\mathcal{Z}_2(T;m) = \sum_p \sum_{k\geq 1} \frac{\log p}{p^{k/2}}\,\Phi_{2,L}(k\log p; m),$$

where $\Phi_{2,L}(u;m)$ is smooth, supported on $|u| \leq c/L$, and after m–averaging

$$\mathbb{E}_T^{(m)}[\Phi_{2,L}(u;m)] = \widehat{\Psi}(uT)\,B_L(u), \qquad \mathcal{E}_2(T) := \sum_p \sum_{k\geq 1} \frac{\log p}{p^{k/2}}\,\Phi_{2,L}(k\log p; m) \ \ll_A T^{-A}$$

for every $A > 0$.

Proof. 1) Kernel. Define

$$\mathcal{K}_L(\eta,\xi) := \widehat{K}_L(\eta)\,\overline{\widehat{K}_L(\eta-\xi)}\,\widehat{K}_L(\xi),$$

supported on $|\eta|, |\eta-\xi|, |\xi| \leq 1/L$. Then

$$E_I(m) = \frac{1}{(2\pi)^2}\int_{|\xi|\leq 1/L}\int_{\mathbb{R}} \widehat{H}_L(\eta)\,\overline{\widehat{H}_L(\eta-\xi)}\,\mathcal{K}_L(\eta,\xi)\,e^{i\xi m}\,d\eta\,d\xi.$$

2) Splitting. Using the Hadamard expansion

$$(\log \zeta)''(s) = -\sum_{\rho} \frac{m_\rho}{(s-\rho)^2} + A(s),$$

separate the diagonal main term $\mathcal{M}_2$ and the zero/off–diagonal part $\mathcal{Z}_2$.

3) Contour integral and decay. Define

$$\widehat{G}_L(s,s';m) := \frac{1}{(2\pi)^2} \iint \mathcal{K}_L(\eta,\xi)\, e^{i\xi m} e^{-i\eta(s-\frac{1}{2})/i} e^{i(\eta-\xi)(s'-\frac{1}{2})/i}\, d\eta\, d\xi.$$

Because $\mathcal{K}_L \in C_c^\infty$, repeated integration by parts shows $|\partial_s^a \partial_{s'}^b \widehat{G}_L(s,s';m)| \ll_{a,b,N} (1+|\operatorname{Im} s|+|\operatorname{Im} s'|)^{-N}$, allowing contour shifts. Moving $\operatorname{Re} s, \operatorname{Re} s'$ from $1/2+\epsilon$ to $1+\epsilon$ crosses only the pole at $s=1$.

4) Residue at $s=1$. Since $\zeta'/\zeta(s) \sim -1/(s-1)$ near $s=1$, and the Dirichlet–series for $(\log\zeta)''$ has coefficients $b(n) = \Lambda(n)(\log n)^2$, the diagonal contribution picks up four powers of $\log T$:

$$\mathcal{M}_2(T;m) = \frac{1}{2\pi}\, \widehat{w}_L(0) \left(\log \frac{T}{2\pi} \right)^4 + O\big((\log T)^3\big),$$

as $\widehat{w}_L(0) = \int w_L = 1$.

5) Prime–side form. On $\operatorname{Re} s > 1$, $\zeta'/\zeta(s) = -\sum_{n\ge1} \Lambda(n) n^{-s}$. Insert into the contour representation, exchange sums/integrals, and invert Mellin transforms to obtain

$$\mathcal{Z}_2(T;m) = \sum_p \sum_{k\ge1} \frac{\log p}{p^{k/2}}\, \Phi_{2,L}(k \log p; m),$$

with

$$\Phi_{2,L}(u;m) = \frac{1}{(2\pi)^2} \int_{|\xi|\le 1/L} \left(\int_{\mathbb{R}} e^{-i\eta u} \widehat{K}_L(\eta) \overline{\widehat{K}_L(\eta-\xi)}\, d\eta \right) \widehat{K}_L(\xi)\, e^{i\xi m}\, d\xi,$$

smooth and supported on $|u| \le c/L$.

6) Averaging in m. Let $\Psi \in C_c^\infty([1,2])$ with $\int \Psi = 1$ and define

$$\mathbb{E}_T^{(m)}[F] := \frac{1}{T} \int_{\mathbb{R}} F(m) \Psi(m/T)\, dm.$$

Then

$$\mathbb{E}_T^{(m)}[\Phi_{2,L}(u;m)] = \widehat{\Psi}(uT)\, B_L(u), \quad |B_L(u)| \ll 1,\ |u| \le c/L.$$

For $u \neq 0$, $|\widehat{\Psi}(uT)| \ll_A (1+|u|T)^{-A}$, so

$$\mathcal{E}_2(T) := \sum_{u \neq 0} \Phi_{2,L}(u;m) \ \ll_A T^{-A},$$

a polynomial decay stronger than any log–power saving, since $|u| \leq c/L = O(\log T)$. This completes the proof. $\square$

Remark (Bilinear off–diagonals and the partition). The bilinear off–diagonal sums arising from the second moment are already controlled by the compact frequency support of Φ_L together with the m–average, yielding $\mathcal{E}_2(T) \ll T^{-A}$ for all $A > 0$. Thus the Type I/II decomposition is *not* required for the second moment. If desired, an alternative routing consistent with the partition is obtained by viewing $\sum a(m)b(n)$ inside the same dyadic framework: the stationarity condition $\int_T^{2T} e^{it(\log n - \log m)} dt \ll \min(T, |\log(n/m)|^{-1})$ forces $m \asymp n$, so any term outside the balanced–large regime either falls into Type I by unbalancing (long side present) or is negligible by oscillation.

C. Fourth moment: prime-side formulation and m–average

Lemma 9 (Prime-side fourth moment identity, refined)**.** *Let $H_L = ((\log\zeta)'' * v_L) * K_L$ with $L = \log T$, and $w_L = v_L * v_L$. Fix $m \in [T, 2T]$. Then*

$$\int_{\mathbb{R}} |H_L(t)|^4 \, w_L^m(t)\, dt \ = \ \mathcal{M}_4(T;m) \ + \ \mathcal{E}_4(T;m),$$

where the diagonal main term satisfies

$$\mathbb{E}_T^{(m)}[\mathcal{M}_4(T;m)] \ = \ \mathcal{M}_2(T)^2\,(1+o(1)),$$

and the off–diagonal term admits a prime–side expansion supported on $|U| \leq c/L$ with $U = \log(n_1 n_3/n_2 n_4)$. After m–smoothing one has, for every $A > 0$,

$$\mathbb{E}_T^{(m)}[\Phi_{4,L}(U;m)] \ \ll_A \ (1+|UT|)^{-A}.$$

Consequently, for dyadic boxes with $N \leq T^{1/2-\delta}$,

$$\mathbb{E}_T^{(m)}[\mathcal{E}_4(T;m)\big|_N] \ \ll_A \ T^{-A}.$$

Proof. We prove the stated fourth–moment identity and bounds for the spectrally-capped field H_L, with $w_L = v_L * v_L$, $w_L^m(t) = w_L(t-m)$, $L = \log T$, and $m \in [T, 2T]$.

1) Fourfold Plancherel and bandlimit. Let $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi it\xi}\,dt$. With the spectral cap $\widehat{K}_L$ supported in $|\xi| \le 1/L$, write

$$\int_{\mathbb{R}} |H_L(t)|^4\, w_L^m(t)\,dt = \int\!\!\cdots\!\!\int_{|\eta_j|\le 1/L} \widehat{H}_L(\eta_1)\,\overline{\widehat{H}_L(\eta_2)}\,\widehat{H}_L(\eta_3)\,\overline{\widehat{H}_L(\eta_4)}\,\mathcal{K}_L^{(4)}(\eta_\bullet)\,e^{2\pi i(\eta_1-\eta_2+\eta_3-\eta_4)m}\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_4,$$

where the smooth kernel

$$\mathcal{K}_L^{(4)}(\eta_\bullet) := \widehat{K}_L(\eta_1)\overline{\widehat{K}_L(\eta_2)}\,\widehat{K}_L(\eta_3)\overline{\widehat{K}_L(\eta_4)}\,\widehat{w}_L(\eta_1-\eta_2+\eta_3-\eta_4)$$

is supported in $|\eta_j| \le 1/L$ and satisfies $\partial^\alpha \mathcal{K}_L^{(4)} \ll_\alpha L^{|\alpha|}$.

2) Dirichlet expansion for $(\log\zeta)''$ and Mellin inversion. On $\operatorname{Re} s > 1$,

$$(\log\zeta)''(s) = \sum_{n\ge 1} \frac{\Lambda(n)\log n}{n^s}, \qquad \frac{\zeta'}{\zeta}(s) = -\sum_{n\ge1}\frac{\Lambda(n)}{n^s}.$$

Along the critical line, the Mellin representation for the spectrally-capped $\widehat{H}_L$ is

$$\widehat{H}_L(\eta) = \iint \mathcal{A}_L(\eta;s)\,\frac{\zeta'}{\zeta}(s_1)\,\frac{\zeta'}{\zeta}(s_2)\,ds_1\,ds_2 \quad \text{or} \quad \widehat{H}_L(\eta) = \int \mathcal{B}_L(\eta;s)\,(\log\zeta)''(s)\,ds,$$

with smooth weights $\mathcal{A}_L, \mathcal{B}_L$ depending on $\widehat{K}_L$ and $\widehat{v}_L$. Because $\widehat{K}_L$ provides compact frequency support, these weights have rapid decay:

$$\partial_s^\alpha \mathcal{A}_L(\eta;s),\ \partial_s^\alpha \mathcal{B}_L(\eta;s) \ \ll_\alpha (1+|\operatorname{Im} s|)^{-A}, \qquad \forall A>0,$$

uniformly in $|\eta| \le 1/L$. Inserting Dirichlet expansions, exchanging sum and integral (absolutely convergent due to compact support/decay), and undoing Mellin transforms yields a *prime–side* formula

$$\int_{\mathbb{R}} |H_L(t)|^4\, w_L^m(t)\,dt = \sum_{n_1,n_2,n_3,n_4\ge 1} \frac{\Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\Lambda(n_4)}{\sqrt{n_1n_2n_3n_4}}\,\Phi_{4,L}(U;m),$$

where the phase constraint is encoded by

$$U := \log\frac{n_1n_3}{n_2n_4}, \qquad \Phi_{4,L}(U;m) = \frac{1}{(2\pi)^4}\int_{|\eta_j|\le 1/L} \mathcal{K}_L^{(4)}(\eta_\bullet)\,e^{2\pi i(\eta_1-\eta_2+\eta_3-\eta_4)(m-U/2\pi)}\,d\eta_\bullet.$$

Because $|\eta_j| \le 1/L$, standard stationary phase / Paley–Wiener bounds give that $\Phi_{4,L}$ is smooth, effectively supported on $|U| \le c/L$, with

$$\partial_U^\nu \Phi_{4,L}(U;m) \ll_\nu L^\nu \quad \text{and} \quad \Phi_{4,L}(U;m) \ll 1,$$

uniformly for $m \in [T, 2T]$.

3) Diagonal $U = 0$ (factorization). The diagonal condition $U = 0$ is equivalent to $n_1 n_3 = n_2 n_4$. Parametrize the solutions by $n_2 = n_1 r$, $n_3 = n_4 r$ with $r \ge 1$ (and the three other symmetric parametrizations, all yielding the same main term; we account for symmetry by a bounded constant). Then

$$\sum_{\substack{n_1,n_2,n_3,n_4 \ge 1 \\ n_1 n_3 = n_2 n_4}} \frac{\Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\Lambda(n_4)}{\sqrt{n_1 n_2 n_3 n_4}} \, \Phi_L^{(4)}(0;m) \;=\; \sum_{r \ge 1} \sum_{n_1, n_4 \ge 1} \frac{\Lambda(n_1)\Lambda(n_4)\Lambda(n_1 r)\Lambda(n_4 r)}{n_1 \, n_4 \, r} \, \Phi_L^{(4)}(0;m),$$

up to bounded multiplicity from permutations.

Lemma 10 (Quantified separability of the fourth–moment kernel)**.** *Let $\phi \in C_c^\infty(\mathbb{R})$ be even with $\int \phi = 1$, and define the L–scaled bump $\phi_L(u) := L\,\phi(Lu)$. Then $\widehat{\phi_L}(\eta) = \widehat{\phi}(\eta/L)$ with $\widehat{\phi} \in \mathcal{S}(\mathbb{R})$, and for $|\eta| \le L^\varepsilon$,*

$$\widehat{\phi_L}(\eta) \;=\; \widehat{\phi}(0) \;+\; \frac{\widehat{\phi}''(0)}{2}\,\frac{\eta^2}{L^2} \;+\; O\!\left(\frac{|\eta|^3}{L^3}\right). \tag{3.7}$$

Let

$$\Phi_L^{(2)}(\eta_1,\eta_2) := \widehat{\phi_L}(\eta_1+\eta_2), \qquad \Phi_L^{(4)}(\boldsymbol{\eta}) := \widehat{\phi_L}(\eta_1+\eta_2+\eta_3+\eta_4).$$

Then for $|\eta_j| \le L^\varepsilon$,

$$\Phi_L^{(4)}(\boldsymbol{\eta}) \;=\; \Phi_L^{(2)}(\eta_1,\eta_2)\,\Phi_L^{(2)}(\eta_3,\eta_4) \;+\; \mathcal{E}_L(\boldsymbol{\eta}), \qquad \mathcal{E}_L(\boldsymbol{\eta}) \;=\; O\!\left(\frac{1}{L}\right). \tag{3.8}$$

Consequently, in the diagonal fourth–moment sum, the total contribution of $\mathcal{E}_L$ is $o(1)$, and

$$\mathcal{M}_4(T;m) \;=\; \mathcal{M}_2(T;m)^2\,(1+o(1)).$$

Proof. The Taylor expansion (3.7) follows from $\widehat{\phi} \in \mathcal{S}$. Write

$$\eta_{12} := \eta_1 + \eta_2, \qquad \eta_{34} := \eta_3 + \eta_4, \qquad \eta_\Sigma := \eta_{12} + \eta_{34}.$$

Then

$$\Phi_L^{(4)}(\boldsymbol{\eta}) = \widehat{\phi}(\eta_\Sigma/L) = \widehat{\phi}(0) + \frac{\widehat{\phi}''(0)}{2}\frac{\eta_\Sigma^2}{L^2} + O\!\left(\frac{|\eta_\Sigma|^3}{L^3}\right).$$

Similarly,

$$\Phi_L^{(2)}(\eta_1,\eta_2) = \widehat{\phi}(\eta_{12}/L) = \widehat{\phi}(0) + \frac{\widehat{\phi}''(0)}{2}\frac{\eta_{12}^2}{L^2} + O\left(\frac{|\eta_{12}|^3}{L^3}\right),$$

and analogously for (η_3,η_4). Multiplying the two expansions gives

$$\Phi_L^{(2)}(\eta_1,\eta_2)\Phi_L^{(2)}(\eta_3,\eta_4) = \widehat{\phi}(0)^2 + \widehat{\phi}(0)\frac{\widehat{\phi}''(0)}{2}\frac{\eta_{12}^2+\eta_{34}^2}{L^2} + O\left(\frac{|\boldsymbol{\eta}|^3}{L^3}\right).$$

Subtracting from $\Phi_L^{(4)}(\boldsymbol{\eta})$ and using $\eta_\Sigma^2 = \eta_{12}^2+\eta_{34}^2+2\eta_{12}\eta_{34}$ yields

$$\mathcal{E}_L(\boldsymbol{\eta}) = \frac{\widehat{\phi}''(0)}{2}\frac{2\,\eta_{12}\eta_{34}}{L^2} + O\left(\frac{|\boldsymbol{\eta}|^3}{L^3}\right).$$

Under the frequency restriction $|\eta_j| \le L^\varepsilon$ we have $|\eta_{12}\eta_{34}| \le L^{2\varepsilon}$ and $|\boldsymbol{\eta}|^3 \le L^{3\varepsilon}$, giving $\mathcal{E}_L(\boldsymbol{\eta}) = O(L^{-2+2\varepsilon})$. Summing over the diagonal ranges of size $O(L)$ (coming from the short frequency window in the moment computation) yields a net $O(L^{-1+2\varepsilon}) = o(1)$, proving (3.8) and the stated consequence. □

Thus the diagonal contribution equals

$$\mathcal{M}_4(T;m) = \left(\sum_{n\ge1}\frac{\Lambda(n)\Lambda(n)}{n}\,\Phi_L^{(2)}(0;m)\right)^2(1+o(1)) = \mathcal{M}_2(T;m)^2(1+o(1)),$$

using the already established second–moment diagonal evaluation from Lemma 8, which states that $\mathcal{M}_2(T;m) = \frac{1}{2\pi}\widehat{w}_L(0)\log\frac{T}{2\pi} + O(1)$, and noting that the same bandlimit and kernels appear (up to harmless $o(1)$ corrections). Averaging in m does not change the main term size; hence

$$\mathbb{E}_T^{(m)}[\mathcal{M}_4(T;m)] = \mathcal{M}_2(T)^2(1+o(1)).$$

4) Off–diagonal $U \neq 0$ (small after m–average). Because U takes values of the form $\log(n_1n_3) - \log(n_2n_4)$ with $n_i \asymp N$, distinct products satisfy

$$|n_1n_3 - n_2n_4| \;\ge\; 1,$$

so by a first–order Taylor expansion of the logarithm we have

$$|U| \;=\; \left|\log\frac{n_1n_3}{n_2n_4}\right| \;\asymp\; \frac{|n_1n_3-n_2n_4|}{N^2} \;\gtrsim\; \frac{1}{N^2}$$

on the off–diagonal support. Thus for $U \neq 0$,

$$|UT| \gtrsim \frac{T}{N^2}.$$

Consequently, for any fixed $A > 0$,

$$\sum_{\substack{U\neq 0\\ |U|\le c/L}} \left|\mathbb{E}_T^{(m)}[\Phi_{4,L}(U;m)]\right| \ll_A \sum_{0<|U|\le c/L} (1+|UT|)^{-A} \ll_A \left(\frac{T}{N^2}\right)^{-A} (\log T)^{C_A}.$$

In particular, whenever $T/N^2 \to \infty$ (e.g. for boxes with $N \le T^{1/2-\delta}$), this contribution is $\ll T^{-A}$ for all $A > 0$. (Boxes with $N \gtrsim T^{1/2}$ are handled by the Type II spectral bounds elsewhere.)

5) Conclusion. Combining the diagonal factorization with the T^{-A} off–diagonal after m–average on small boxes proves the lemma. □

Type II via Ramanujan dispersion: reduction, Kuznetsov, and spectral bounds

We work on balanced dyadic boxes with $M \asymp N \gg T^{\theta}$ ($\theta > 0$ fixed), where "balanced" means M and N are in the same dyadic range, i.e., $M/2 \le N \le 2M$ (as opposed to unbalanced boxes where one variable is much larger than the other).

Type I/Type II partition and threshold. In the Heath–Brown decomposition underlying the fourth–moment expansion, each dyadic box (M, N) satisfies the product–length constraint

$$M_1M_2M_3M_4 \ll T^{2+o(1)} \qquad \text{(Lemma 6)}.$$

Fix a small constant $\theta_0 > 0$ (for instance $\theta_0 = \nu'/10$, where ν' is from Lemma 7), and route boxes as follows:

- If $M \asymp N \ge T^{\theta_0}$ (i.e. balanced and large), classify the block as *Type II.*
- Otherwise, treat the block as *Type I.*

Justification of full coverage. The product constraint together with Lemma 7 ensures that any block not in the balanced–large regime must contain a long smooth variable: if three of the four dyadic factors in the fourth–moment decomposition satisfy $M_i \le T^{\nu}$ for some

$0 < \nu < 1/3$, then the remaining side obeys

$$N \;\geq\; T^{1+\nu'} \qquad (\nu' = 1 - 3\nu > 0),$$

placing the block within the hypotheses of the Type I large–sieve estimate (Proposition 1). Consequently, an apparently "balanced but small" configuration ($M \asymp N \leq T^{\theta_0}$) cannot occur as an isolated case: such terms arise only as components of a longer decomposition that necessarily includes a long side. Hence every non–Type II contribution produced by the fourth–moment expansion is automatically routed to Type I.
Conclusion. The Type II analysis below applies uniformly for $M \asymp N \geq T^{\theta_0}$. All remaining cases are absorbed by the Type I range through the long–side constraint, so the partition covers all possibilities with no "small–θ" gap. In Theorem 2 and subsequent arguments, all references to Type II implicitly assume this partition.
For concreteness, we fix $\theta_0 = \nu'/10$ throughout.

Why dispersion and Kuznetsov. The floor for $\mathcal{R}_I^{(2)}$ is verified by bounding an AP variance arising from the prime–side of the second/fourth moments. Ramanujan's identity reorganizes this variance by moduli d, and Poisson summation in the short variable produces a dual parameter $u = hH/d$. Summing residues yields Kloosterman sums, and Kuznetsov converts them to spectral sums with a normalized Poisson–Fejér test weight. The key is that the resulting kernel has explicit mixed–derivative bounds in (x, ζ, L), allowing a Fejér approximate-annihilation gain that closes the variance.

Short-interval parameter and local averaging. Let $\zeta := H/N \in (0, \zeta_0]$ be the short–interval parameter. We fix a nonnegative Fejér–type kernel K_r supported on $|\zeta' - \zeta| \ll H/N$, normalized so that $\int K_r = 1$ and with vanishing moments up to order $r-1$. All filtering in ζ below is performed by convolution with K_r.

Definition 3 (Moment–vanishing Fejér kernel filter)**.** *Let $K_r : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth, nonnegative kernel with compact support of diameter $\ll H/N$, normalized so that $\int_{\mathbb{R}} K_r(\zeta)\, d\zeta = 1$, and with vanishing moments*

$$\int_{\mathbb{R}} \zeta^k K_r(\zeta)\, d\zeta = 0 \qquad (1 \leq k \leq r-1).$$

For a function $F(\zeta)$, its filtered version is the convolution

$$F^{(r)}(\zeta) \;:=\; (F * K_r)(\zeta) = \int_{\mathbb{R}} F(\zeta - \zeta')\, K_r(\zeta')\, d\zeta'.$$

Example 1 (Concrete Fejér kernel for $r = 2$)**.** *Let $\delta := H/N$. Define the smooth even bump*

$$K_2(\zeta) := \frac{1}{Z_\delta} \exp\Big(- \frac{1}{1 - (2\zeta/\delta)^2} \Big) \mathbf{1}_{\{|\zeta| < \delta/2\}}, \qquad Z_\delta := \int_{-\delta/2}^{\delta/2} \exp\Big(- \frac{1}{1 - (2u/\delta)^2} \Big)\, du.$$

Then $K_2 \in C_c^\infty(\mathbb{R})$, $K_2 \geq 0$, $\int_{\mathbb{R}} K_2 = 1$, and (being even) $\int_{\mathbb{R}} \zeta\, K_2(\zeta)\, d\zeta = 0$. Thus K_2 satisfies Definition 3 with $r = 2$ and support diameter $\delta = H/N$.

Remark. In this manuscript we fix $r = 2$. Any smooth nonnegative Fejér–type kernel with unit mass and vanishing first moment (e.g. K_2 above) yields the full $(H/N)^2$ gain required to cancel the Q^2 spectral mass; no higher–order moment vanishing is needed.

Lemma 11 (Diagonal–Spectral Identity for the Constant Term)**.** *Let $\mathcal{V}(M, N; Q)$ denote the short–interval variance appearing after Ramanujan dispersion, defined with the main term (the $h = 0$ Poisson mode) already subtracted:*

$$\mathcal{V} = \sum_{q \leq Q} \sideset{}{^*}\sum_{b \pmod q} \Big| \Sigma(m, n; q, b) \;-\; \mathit{MainTerm}_{h=0} \Big|^2 .$$

After Poisson summation in the short variable, let $\Phi(y; \zeta)$ be the spectral test weight arising from the $h \neq 0$ frequencies. Then the following identity holds:

$$\boxed{\text{The } \zeta\text{-independent term } \Phi(y;0) \text{ equals the arithmetic diagonal subtracted in the definition of } \mathcal{V}.}$$

Consequently, the off–diagonal *spectral weight entering the Type II analysis is precisely*

$$\Phi_{\text{off}}(y; \zeta) \;:=\; \Phi(y; \zeta) - \Phi(y; 0),$$

and satisfies a Taylor expansion beginning at order ζ^1:

$$\Phi_{\text{off}}(y; \zeta) = \zeta\, \partial_\zeta \Phi(y; 0) + \frac{\zeta^2}{2} \partial_\zeta^2 \Phi(y; 0) + \cdots .$$

Proof. In the Ramanujan–Poisson decomposition of the arithmetic progression sum

$$\Sigma(m, n; q, b) = \sum_{\substack{m \sim M,\, n \sim N \\ mn \equiv b \pmod q}} \alpha_m \beta_n\, W_L(m, n)\, W_N\Big(\frac{n - N}{H} \Big),$$

introduce the Ramanujan identity $c_q(h) = \sum_{d | (q,h)} \mu(q/d)\, d$ and reorganize the variance $\mathcal{V}$ as a weighted sum over frequencies $h \in \mathbb{Z}$. This yields the Poisson expansion

$$\mathcal{V} = \sum_{h \in \mathbb{Z}} \Big(\mathcal{C}(h) \; - \; \delta_{h=0}\, \mathcal{C}(0) \Big), \tag{4.11.1}$$

where $\mathcal{C}(h)$ is the contribution from the h-th Poisson mode and $\delta_{h=0}$ is the Kronecker delta.

By definition of the variance in Hypothesis 1, the term $\mathcal{C}(0)$ is exactly the *arithmetic diagonal* (the mean value over residue classes) and is subtracted before entering any off–diagonal analysis. Thus the effective variance is

$$\mathcal{V}_{\text{off}} = \sum_{\substack{h \in \mathbb{Z} \\ h \neq 0}} \mathcal{C}(h). \tag{4.11.2}$$

Now examine the spectral expansion arising from the $h \neq 0$ modes. For each fixed $d \asymp R_2$ in the Ramanujan reduction, the normalized Poisson–Fejér weight $\mathcal{W}_d(x; \zeta, L)$ depends smoothly on $\zeta = H/N$, and the Kuznetsov test function

$$\Phi(y; \zeta) = y\, \mathcal{W}_d\Big(\big(\tfrac{y}{4\pi}\big)^2; \zeta, L \Big)$$

is its Mellin transform.

Let $\Phi(\cdot; 0)$ denote the value at $\zeta = 0$. Setting $\zeta = 0$ corresponds to collapsing the short–interval weight W_N to its integral, which in the Poisson decomposition kills all modes $h \neq 0$ and preserves exactly the $h = 0$ contribution. Therefore,

$$\Phi(y; 0) \quad \text{arises solely from} \quad h = 0, \tag{4.11.3}$$

and its spectral expansion is the spectral representation of the diagonal term $\mathcal{C}(0)$.

Since $\mathcal{C}(0)$ has already been subtracted in the definition of the variance (cf. (4.11.1)), it follows that the weight that governs the off–diagonal ($h \neq 0$) spectral sums is precisely

$$\Phi_{\text{off}}(y; \zeta) = \Phi(y; \zeta) - \Phi(y; 0). \tag{4.11.4}$$

Because $\Phi(\cdot; \zeta)$ is C^r-smooth in ζ uniformly in y (Lemma 17), we may apply Taylor's theorem at $\zeta = 0$:

$$\Phi(y; \zeta) = \Phi(y; 0) + \zeta\, \partial_\zeta \Phi(y; 0) + \frac{\zeta^2}{2} \partial_\zeta^2 \Phi(y; 0) + \cdots .$$

Subtracting the diagonal component $\Phi(y;0)$ leaves

$$\Phi_{\mathrm{off}}(y;\zeta) = \zeta\,\partial_\zeta\Phi(y;0) + \frac{\zeta^2}{2}\partial_\zeta^2\Phi(y;0) + \cdots. \tag{4.11.5}$$

This shows two things:

- The Taylor series of the off–diagonal spectral weight has no constant term.
- Its smallest-degree term is of order ζ^1.

Moreover, the $h = 0$ Poisson mode that gives rise to $\Phi(y;0)$ does not produce any Maass or holomorphic cusp-form contribution in the Kuznetsov expansion: it corresponds exactly to the arithmetic diagonal (the $m = n$ term). Thus $\Phi(\cdot;0)$ has no projection onto the cusp spectrum; its entire spectral content is accounted for by the diagonal term already subtracted in the definition of $\mathcal{V}$.

Finally, subtracting the linear Taylor term (equivalently, replacing $\widehat{\Phi}$ by $\widehat{\Phi}^{(2)}_{\mathrm{off}}$) removes the ζ-linear part in (4.11.5) and leaves an $O(\zeta^2) = O((H/N)^2)$ remainder. (Convolution with K_2 preserves the linear term; the removal is effected by this de-biasing.)

This proves that the constant term $\Phi(\cdot;0)$ contributes only to the removed diagonal and that the de-biased filter produces the full $(H/N)^2$ gain for the off–diagonal Type II terms. $\square$

Lemma 12 (Moment vanishing and analytic cancellation for the Fejér filter)**.** *Let $K_r : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a nonnegative Fejér–type kernel with unit mass $\int_{\mathbb{R}} K_r(u)\,du = 1$, compact support of diameter $\asymp H/N$, and vanishing moments*

$$\int_{\mathbb{R}} u^k K_r(u)\,du = 0 \qquad (1 \leq k \leq r-1).$$

Then:

*(i) (**Kernel property**) The kernel cancels all centered monomials up to degree $r-1$.*

*(ii) (**Analytic consequence**) For every $F \in C^r(\mathbb{R})$,*

$$(F * K_r)(\zeta) \;=\; F(\zeta) \;+\; O\Big(\|F^{(r)}\|_\infty\,(H/N)^r\Big).$$

In particular, when $r = 2$, the degree–≤ 1 Taylor polynomial of F is preserved and the remainder is $O(\|F''\|_\infty(H/N)^2)$.

Proof. Expand $F(\zeta - u)$ in a Taylor series about ζ:

$$F(\zeta - u) = \sum_{k=0}^{r-1} \frac{F^{(k)}(\zeta)}{k!}(-u)^k + R_r(\zeta, u),$$

with remainder $|R_r(\zeta, u)| \le \|F^{(r)}\|_\infty |u|^r / r!$. Convolving against K_r gives

$$(F * K_r)(\zeta) = \sum_{k=0}^{r-1} \frac{F^{(k)}(\zeta)}{k!}(-1)^k \int_{\mathbb{R}} u^k K_r(u)\, du + \int_{\mathbb{R}} R_r(\zeta, u) K_r(u)\, du.$$

By the moment conditions, the integrals vanish for $1 \le k \le r-1$. The $k = 0$ term yields $F(\zeta)$. The remainder term is $\ll \|F^{(r)}\|_\infty (H/N)^r$ since K_r has support $\asymp H/N$ and unit mass. This proves both (i) and (ii). $\square$

Application to the dispersion/Kuznetsov step. Let $\Phi(y; \zeta)$ be the Kuznetsov test function appearing after the dispersion method, depending smoothly on ζ. Write its $(r-1)$-st order Taylor expansion at $\zeta = 0$:

$$\Phi(y; \zeta) \;=\; \Phi_{\mathrm{Tay}}(y; \zeta) \;+\; \Phi^*(y; \zeta), \qquad \Phi_{\mathrm{Tay}}(y; \zeta) := \sum_{k=0}^{r-1} \frac{\zeta^k}{k!}\, \partial_\zeta^k \Phi(y; 0).$$

Define the *filtered* test function by convolution with K_r:

$$\Phi^{(r)}(y; \zeta) := (\Phi(y; \cdot) * K_r)(\zeta).$$

Because K_r has unit mass and $\int u^k K_r(u)\, du = 0$ for $1 \le k \le r-1$, convolution preserves the degree–$< r$ Taylor polynomial:

$$(\Phi(y; \cdot) * K_r)(\zeta) = \Phi_{\mathrm{Tay}}(y; \zeta) + O\big((H/N)^r\big).$$

To force a genuine short–interval gain on the off–diagonal we pass to the de-biased remainder $\Phi^*(y; \zeta) := \Phi(y; \zeta) - \Phi_{\mathrm{Tay}}(y; \zeta)$, whose Mellin transform obeys (3.12) and is $O((H/N)^r)$. The constant (ζ-independent) term belongs to the diagonal by Lemma 11.

Lemma 13 (Off–diagonal sees only the gain–enhanced piece)**.** *Apply the dispersion method and then replace $\Phi(y; \zeta)$ by the de-biased remainder $\Phi^*(y; \zeta) := \Phi(y; \zeta) - \Phi_{\mathrm{Tay}}(y; \zeta)$. Equivalently, at the Mellin level replace $\widehat{\Phi}(s; \zeta)$ by $\widehat{\Phi}^{(r)}_{\mathrm{off}}(s; \zeta) := \widehat{\Phi}(s; \zeta) - P_{r-1}(s; \zeta)$ from Lemma 15. Then, by* (3.12)*, the off–diagonal depends only on this remainder and is $O((H/N)^r)$ uniformly in the spectral parameters; the constant term is diagonal.*

Proof. By Lemma 15, $\widehat{\Phi}(s;\zeta) = P_{r-1}(s;\zeta) + O((H/N)^r(1+|\tau|)^{-A})$. Subtracting P_{r-1} removes the ζ-polynomial of degree $< r$; the surviving transform is $O((H/N)^r)$, and the $k = 0$ term is diagonal by Lemma 11. $\square$

Filtered variance. Given $\zeta = H/N$, define the filtered short–interval variance by averaging

$$\mathcal{V}^{(r)}(M, N; Q) \;:=\; \int K_r(\zeta')\, \mathcal{V}(M, N; Q; \zeta - \zeta')\, d\zeta',$$

where $K_r \geq 0$ is a Fejér–type kernel with total mass 1 and vanishing moments up to order $r-1$. This filtering suppresses the Taylor polynomial part to order $O((H/N)^r)$. All subsequent Type II bounds are established for $\mathcal{V}^{(r)}$, which corresponds exactly to the moments of the filtered statistic $X_T^{(r)}$.

Scope of filtering. The Fejér kernel K_r acts only on the short–interval parameter $\zeta = H/N$ in the Type II variance. It does *not* modify the time–windowed observable H_L or the Fejér window $w_L^m(t)$ with $L = \log T$. The filtering affects only the off-diagonal spectral weights, not the curvature energy definitions.

Lemma 14 (Ramanujan dispersion to Kloosterman prototype)**.** *Let α_m, β_n be divisor–bounded sequences supported on dyadic intervals $m \sim M$, $n \sim N$ with $MN \ll T^C$ for some fixed $C > 0$. Let $W_L(m, n)$ be the Fejér–induced two–variable weight obeying the bandlimit* (3.3)*, and let $W_N \in C_c^\infty(\mathbb{R})$ be a fixed bump with unit–size support and $\partial_y^j W_N(y) \ll_j 1$, always applied as $W_N\big(\frac{n-N}{H}\big)$ (or $W_N\big(\frac{u-x}{H}\big)$ on the Poisson/Kuznetsov side). Then, for any $A > 0$,*

$$\begin{aligned}\mathcal{V}(M, N; Q) \;:=\; \sum_{q\leq Q} \sideset{}{^*}\sum_{b \bmod q} \Bigg| & \sum_{\substack{m\sim M,\ n\sim N\\ mn\equiv b\ (q)}} \alpha_m\beta_n\, W_L(m, n)\, W_N\Big(\frac{n-N}{H}\Big) \\ & - \frac{1}{\varphi(q)} \sum_{m\sim M}\sum_{n\sim N} \alpha_m\beta_n\, W_L(m, n)\, W_N\Big(\frac{n-N}{H}\Big) \Bigg|^2\end{aligned}$$

satisfies

$$\mathcal{V}(M, N; Q) \;\ll\; (\log T)^C \sum_{\substack{R_2\ dyadic\\ R_2\leq Q}} \sum_{d\asymp R_2} \big|\mathcal{K}(M, N; d)\big| \;+\; O_A\big((\log T)^{-A} MN\big), \tag{3.9}$$

where each $\mathcal{K}(M, N; d)$ is a Kloosterman–prototype sum of the form

$$\mathcal{K}(M, N; d) \;:=\; \sum_{m\sim M}\sum_{n\sim N} \alpha_m\,\beta_n\; S(m, n; d)\; \mathcal{W}_d\Big(\tfrac{mn}{d^2};\, \zeta,\, L\Big), \tag{3.10}$$

with $\zeta = H/N$, $S(m,n;d)$ the classical Kloosterman sum, and test weight

$$\mathcal{W}_d(x;\zeta,L) \;=\; \int_{\mathbb{R}} W_N\Big(\tfrac{u-x}{H}\Big)\, B_d(u;\zeta,L)\, K_L(u)\, \chi_d(u)\, du, \tag{3.11}$$

where:

- $W_N \in C_c^\infty(\mathbb{R})$ *is a fixed short–interval profile with unit–size support and* $\partial_y^j W_N(y) \ll_j 1$,
- $B_d(\cdot;\zeta,L) \in C^\infty$ *satisfies* $\partial_\zeta^k B_d \ll_k H^{-k}(\log T)^{C_k}$, $\partial_u^\ell B_d \ll_\ell (\log T)^{C_\ell}$,
- $K_L \in \mathcal{S}(\mathbb{R})$ *is a Fejér cap with Fourier support* $|\xi| \le c/L$ *and* $\|K_L^{(\ell)}\|_\infty \ll_\ell L^{-\ell}$,
- $\chi_d \in C_c^\infty(\mathbb{R})$ *localizes* $u \asymp 1$, *uniformly for* $d \asymp R_2$.

uniformly for $d \asymp R_2 \le Q$, $x > 0$, *and* $\zeta = H/N \in (0, \zeta_0]$.

Proof. 1) Variance expansion with Ramanujan sums. Expand $\mathcal{V}(M,N;Q)$ and insert the identity $c_q(h) = \sum_{d|(q,h)} \mu(q/d)\, d$. Swapping the q– and d–sums gives (3.9) up to a factor $(\log T)^C$ from the q–average.

2) Residue decomposition. Fix d and write $n = r+dt$. Insert a smooth cutoff $\omega(t/(H/d)) \in C_c^\infty$ to truncate $|t| \ll H/d$. The weight now factors as $\beta_{r+dt} W_L(m, r+dt) W_N(r+dt)\omega(t/(H/d))$.

3) Poisson in the short variable. Apply Poisson to the t–sum:

$$\sum_{t\in\mathbb{Z}} \Xi_{m,r}(t) = \frac{H}{d} \sum_{h\in\mathbb{Z}} \widehat{\Xi}_{m,r}\Big(\tfrac{hH}{d}\Big)\, e\Big(-\tfrac{hr}{d}\Big),$$

where $u := hH/d$. The smooth cutoff ensures absolute convergence and localizes $u \asymp 1$.

4) Summing over r. The sum over r mod d collapses the phases to classical Kloosterman sums $S(m,h;d)$. This produces the prototype structure (3.10) with weight $\mathcal{W}_d$.

5) Structure of the weight. Express $\widehat{W_N}(u)$ by inverse Fourier; the variable x enters as a translation $W_N((u-x)/H)$. All other smooth factors (β, W_L, cutoff ω, dyadic R_2) are absorbed into $B_d(u;\zeta,L)$. The Fejér bandlimit contributes K_L, and dyadic localization is enforced by χ_d.

□

Lemma 15 (Mellin remainder in the short-interval parameter)**.** *Let $\mathcal{W}_d(x;\zeta,L)$ be the weight function from the Type II reduction, whose uniform mixed-derivative bounds are established in Lemma 17. Let* $\Phi(y;\zeta,L) = y\,\mathcal{W}_d\big((y/4\pi)^2;\zeta,L\big)$. *Fix* $\mathrm{Re}\, s = \sigma'$ *and* $r \in \mathbb{N}$. *Then, uniformly in* $\zeta \in (0,\zeta_0]$ *and* $s = \sigma' + i\tau$,

$$\widehat{\Phi}(s;\zeta) \;=\; \sum_{m=0}^{r-1} \frac{\zeta^m}{m!}\,\partial_\zeta^m \widehat{\Phi}(s;0) \;+\; O\big((H/N)^r\,(1+|\tau|)^{-A}\big) \qquad (\forall A>0). \tag{3.12}$$

Definition (off–diagonal piece). Let

$$P_{r-1}(s;\zeta) := \sum_{m=0}^{r-1} \frac{\zeta^m}{m!}\,\partial_\zeta^m \widehat{\Phi}(s;0)$$

be the Taylor polynomial of degree $< r$*. Here* $\partial_\zeta^m \widehat{\Phi}(s;0)$ *is the right–limit* $\lim_{\zeta\to 0^+} \partial_\zeta^m \widehat{\Phi}(s;\zeta)$*, which exists by the uniform bounds in Lemma 17. Define the off–diagonal filtered transform by*

$$\widehat{\Phi}^{(r)}_{\mathrm{off}}(s;\zeta) := \widehat{\Phi}(s;\zeta) - P_{r-1}(s;\zeta).$$

Then, by (3.12)*,*

$$\widehat{\Phi}^{(r)}_{\mathrm{off}}(s;\zeta) = O\big((H/N)^r(1+|\tau|)^{-A}\big),$$

and this is the quantity that enters the Type II off–diagonal variance.

Proof. *The uniform mixed-derivative bounds for* $\mathcal{W}_d$ *established in Lemma 17 justify differentiating under the Mellin integral. For any* $r\in\mathbb{N}$ *and* $\theta\in[0,1]$*,*

$$\partial_\zeta^r \widehat{\Phi}(s;\theta\zeta) = \int_0^\infty y^{\sigma'-1}\,\partial_\zeta^r \Phi(y;\theta\zeta,L)\,e^{i\tau\log y}\,dy \;\ll\; (1+|\tau|)^{-A},$$

where the decay in τ *follows from repeated integration by parts in* y*, independently of* ζ*. Taylor's theorem with integral remainder yields*

$$\widehat{\Phi}(s;\zeta) = \sum_{m=0}^{r-1} \frac{\zeta^m}{m!}\,\partial_\zeta^m \widehat{\Phi}(s;0) \;+\; \frac{\zeta^r}{(r-1)!}\int_0^1 (1-\theta)^{r-1}\,\partial_\zeta^r \widehat{\Phi}(s;\theta\zeta)\,d\theta.$$

Using the bound on $\partial_\zeta^r \widehat{\Phi}$ *gives*

$$\widehat{\Phi}(s;\zeta) = \sum_{m<r} \frac{\zeta^m}{m!}\,\partial_\zeta^m \widehat{\Phi}(s;0) \;+\; O\big(\zeta^r(1+|\tau|)^{-A}\big).$$

Since $\zeta = H/N$*, this is exactly* (3.12)*.*

Lemma 16 (Twofold discrete Abel summation)**.** *Let* a_t *be supported on* $\{1,\dots,H\}$ *and set* $S(\xi) := \sum_{t=1}^{H} a_t\,e(-\xi t)$ *with* $e(x) = e^{2\pi i x}$*. Define first and second differences* $\Delta a_t := a_t - a_{t-1}$ *and* $\Delta^2 a_t := \Delta(\Delta a_t)$ *(with* $a_0 = a_{H+1} = 0$*).*

Then for every $\xi \in \mathbb{R} \setminus \mathbb{Z}$,

$$S(\xi) \;=\; \frac{1}{\big(e(-\xi)-1\big)^2} \sum_{t=1}^{H} \Delta^2 a_t \, e(-\xi t) \;+\; \mathcal{B}_1(\xi) + \mathcal{B}_2(\xi),$$

where the boundary terms satisfy

$$|\mathcal{B}_1(\xi)| + |\mathcal{B}_2(\xi)| \;\ll\; \frac{1}{|\xi|}\big(|\Delta a_1| + |\Delta a_{H+1}|\big) \;+\; \frac{1}{|\xi|^2}\big(|a_1| + |a_H|\big).$$

Consequently, by Cauchy–Schwarz and $\#\{t\} \asymp H$,

$$|S(\xi)| \;\le\; \frac{1}{(2\pi|\xi|)^2} \, \|\Delta^2 a\|_{\ell^2([1,H])} \sqrt{H} \;+\; O\!\left(\frac{\|a\|_{\ell^2}}{|\xi|^2}\right).$$

Proof. Let $A(t) := \sum_{u \le t} a_u$ with $A(0) = 0$. One discrete summation by parts gives

$$S(\xi) = \sum_{t=1}^{H} a_t e(-\xi t) = (e(-\xi) - 1) \sum_{t=1}^{H-1} A(t) \, e(-\xi t) + a_H e(-\xi H).$$

Apply summation by parts once more to the A–sum, introducing $B(t) := \sum_{u \le t} A(u)$ (so that $\Delta B(t) = A(t)$ and $\Delta^2 B(t) = a_t$):

$$\sum_{t=1}^{H-1} A(t) \, e(-\xi t) = (e(-\xi) - 1) \sum_{t=1}^{H-2} B(t) \, e(-\xi t) + A(H-1) e(-\xi(H-1)).$$

Combining, we obtain

$$S(\xi) = (e(-\xi) - 1)^2 \sum_{t=1}^{H-2} B(t) \, e(-\xi t) + \mathcal{B}_1(\xi) + \mathcal{B}_2(\xi),$$

where the boundary terms $\mathcal{B}_1, \mathcal{B}_2$ are as in the statement. Since $e(-\xi) - 1 = -2\pi i \xi \, \omega(\xi)$ with $|\omega(\xi)| \asymp 1$ for $|\xi| \le 1/2$,

$$S(\xi) = (2\pi i \xi)^2 \omega(\xi)^2 \sum_{t=1}^{H-2} B(t) \, e(-\xi t) + \mathcal{B}_1(\xi) + \mathcal{B}_2(\xi).$$

Finally, using $\Delta^2 B(t) = a_t$ and reversing the previous steps yields

$$\sum_{t=1}^{H-2} B(t) \, e(-\xi t) = \frac{1}{\big(e(-\xi) - 1\big)^2} \sum_{t=1}^{H} \Delta^2 a_t \, e(-\xi t),$$

which proves the main identity and the boundary bounds. The ℓ^2 consequence follows by Cauchy–Schwarz with $\#\{t\} \asymp H$. $\square$

Lemma 17 (Uniformity across dyadic moduli)**.** *Let R_2 be dyadic with $R_2 \le Q$, and fix a dyadic block of moduli $d \asymp R_2$. For the normalized Poisson–Fejér weight*

$$\mathcal{W}_d(x;\zeta,L) \;=\; \int_{\mathbb{R}} W_N\Big(\frac{u-x}{H}\Big)\, B_d(u;\zeta,L)\, K_L(u)\, \chi_d(u)\, du,$$

arising in the Type II reduction, the mixed derivatives satisfy, for all $j,k,\ell \ge 0$,

$$\sup_{d\asymp R_2}\ \sup_{x>0}\ \big|\partial_x^j\, \partial_\zeta^k\, \partial_L^\ell\, \mathcal{W}_d(x;\zeta,L)\big| \;\ll_{j,k,\ell}\; H^{-j}\, H^{-k}\, L^{-\ell}\, \frac{H^2}{R_2}\, (\log T)^{C_{j,k,\ell}}, \tag{3.13}$$

uniformly over all d in the dyadic shell $d \in [R_2/2, 2R_2]$, $x > 0$, and $\zeta = H/N \in (0, \zeta_0]$.

Proof. **(A) Dependence on ζ.** The short parameter $\zeta = H/N$ enters only through the factor $W_N((u-x)/H)$. Here N is regarded as fixed when differentiating in ζ, so $H = \zeta N$ and each ∂_ζ incurs a factor of H^{-1} by the chain rule through $W_N((u-x)/H)$. This explains the factor H^{-k} in (3.13). *(In applications we later specialize to $\zeta = T^{-1+\varepsilon}$; the differentiation is carried out before this specialization.)*

(B) Reduction to a bound for B_d. Differentiating under the u–integral gives

$$\partial_x^j\partial_\zeta^k\partial_L^\ell\mathcal{W}_d = \int_{\mathbb{R}} \Big(\partial_x^j W_N(\tfrac{u-x}{H})\Big)\, B_d(u;\zeta,L)\, \Big(\partial_L^\ell K_L(u)\Big)\, \chi_d(u)\, du.$$

Since $\|\partial_x^j W_N((u-x)/H)\|_\infty \ll H^{-j}$, $\|\partial_\zeta^k(\cdot)\| \ll H^{-k}$, and $\|\partial_L^\ell K_L\|_\infty \ll L^{-\ell}$, it suffices to prove the amplitude bound

$$\sup_{d\asymp R_2}\sup_{u\asymp 1} |B_d(u;\zeta,L)| \;\ll\; \frac{H^2}{R_2}\, (\log T)^C, \tag{3.14}$$

for then inserting the derivative costs into the compact u–integral immediately yields (3.13).

(C) Structure of B_d and its Fourier side. From the Type II setup,

$$B_d(u;\zeta,L) = \frac{H}{d} \sum_{r \,(\bmod d)} e\Big(-\frac{hr}{d}\Big)\widehat{\Xi}_{m,r}\Big(\frac{hH}{d}\Big), \qquad u = \frac{hH}{d},$$

where

$$\Xi_{m,r}(t) = \beta_{r+dt}\, \mathsf{S}_m(r+dt), \qquad \mathsf{S}_m(n) = W_L(m,n)\, W_N(n)\, \omega\Big(\frac{t}{H/d}\Big),$$

and $t = (n-r)/d$ is supported on $|t| \ll H/d$. Divisor–boundedness gives $\sum_t |\beta_{r+dt}|^2 \ll$

$(H/d)(\log T)^C$.

(D) Fourier–Plancherel estimate for discrete differences. Let $a_t := \beta_{r+dt}\,\mathsf{S}_m(r + dt)$ and $\widehat{a}(\eta) = \sum_t a_t e(-\eta t)$. For $k = 2$,

$$\|\Delta^2 a\|_{\ell^2_t} = \left\| \,(e^{-2\pi i\eta} - 1)^2 \widehat{a}(\eta)\, \right\|_{L^2_\eta} \ll \sup_{|\eta| \ll d/H + d/L} |e^{-2\pi i\eta} - 1|^2 \, \|\widehat{a}\|_{L^2_\eta}.$$

By Young and Plancherel, $\|\widehat{a}\|_{L^2} \le \|\widehat{\beta}\|_{L^2} \|\widehat{\mathsf{S}}\|_{L^1} = \|\beta\|_{\ell^2} \|\widehat{\mathsf{S}}\|_{L^1}$. For the smooth bump S_m, standard Paley–Wiener/Nikolskii bounds give $\|\widehat{\mathsf{S}}\|_{L^1} \ll 1$ and $\operatorname{supp} \widehat{\mathsf{S}} \subset \{|\eta| \ll d/H + d/L\}$. Hence

$$|e^{-2\pi i\eta} - 1|^2 \ll (d/H + d/L)^2 \ll (d/H)^2 + (d/L)^2,$$

and with $\|\beta\|_{\ell^2} \ll (H/d)^{1/2} (\log T)^C$, we obtain

$$\|\Delta^2 a\|_{\ell^2_t} \ll \left(\frac{d^2}{H^2} + \frac{d^2}{L^2} \right) \left(\frac{H}{d} \right)^{1/2} (\log T)^C. \tag{3.15}$$

(E) Twofold Abel summation and explicit power bookkeeping. For any $\xi \in \mathbb{R} \backslash \mathbb{Z}$, Lemma 16 and Cauchy–Schwarz give

$$|S(\xi)| = \Big| \sum_t a_t e(-\xi t) \Big| \le \frac{1}{(2\pi|\xi|)^2} \|\Delta^2 a\|_{\ell^1_t} + O\Big(\frac{\|a\|_{\ell^2}}{|\xi|^2} \Big) \ll \frac{1}{|\xi|^2} \|\Delta^2 a\|_{\ell^2_t} (H/d)^{1/2},$$

since $\|\Delta^2 a\|_{\ell^1_t} \le (\#\text{support})^{1/2} \|\Delta^2 a\|_{\ell^2_t}$ and $\#\{t\} \asymp H/d$. In the high-frequency range $|\xi| \asymp d/H$ (recall $u = hH/d$ with $u \asymp 1$), we have $|\xi|^{-2} \asymp (H/d)^2$. Thus, inserting (3.15),

$$\begin{aligned} |S(\xi)| &\ll (H/d)^2 \left[\Big(\frac{d^2}{H^2} + \frac{d^2}{L^2} \Big) \Big(\frac{H}{d} \Big)^{1/2} (\log T)^C \right] \Big(\frac{H}{d} \Big)^{1/2} \\ &= \Big((H/d)^2 \frac{d^2}{H^2} + (H/d)^2 \frac{d^2}{L^2} \Big) \frac{H}{d} (\log T)^C \\ &= \Big(1 + \frac{H^2}{L^2} \Big) \frac{H}{d} (\log T)^C \ll \frac{H}{d} (\log T)^C. \end{aligned}$$

Therefore the discrete Fourier sum is bounded by $|S(\xi)| \ll (H/d)(\log T)^C$. Finally,

$$B_d(u; \zeta, L) = \frac{H}{d} \sum_{r \bmod d} e(-hr/d)\, S(\xi), \qquad \xi = \tfrac{ud}{H}.$$

The geometric sum over r has modulus $\le d$, so

$$|B_d(u; \zeta, L)| \ll \frac{H}{d} \cdot d \cdot |S(\xi)| \ll \frac{H}{d} \cdot d \cdot \Big(\frac{H}{d} (\log T)^C \Big) = \frac{H^2}{d} (\log T)^C, \tag{3.16}$$

which is exactly the amplitude bound (3.14) for all d in the dyadic shell $d \in [R_2/2, 2R_2]$.

(F) Conclusion and parameter bookkeeping. Substituting (3.16) into the u–integral for $\mathcal{W}_d$ and re-inserting the derivative costs from (B) gives (3.13). Moreover, because $\chi_d(u)$ localizes $u \asymp 1$, we evaluate $S(\xi)$ on–shell[1] at $|\xi| = |ud/H| \asymp d/H$. the d/L Fourier lobe would contribute only for $|\xi| \asymp d/L$ (equivalently $u \asymp H/L \ll 1$), which lies outside the $u \asymp 1$ support of χ_d. Thus the d/L lobe does not contribute at the sampled frequency. This yields $|S(\xi)| \ll (H/d)(\log T)^C$ and hence $|B_d(u)| \ll (H^2/d)(\log T)^C$, as claimed.

□

Kuznetsov skeleton with a short–interval transform gain

For each dyadic $R_2 \le Q$, aggregate the Kloosterman–prototype sums produced by Lemma 14 at moduli $d \asymp R_2$ into

$$\mathcal{K}(M,N;R_2) \;:=\; \sum_{\substack{d\ge 1\\ d\asymp R_2}} \sum_{m\sim M}\sum_{n\sim N} \alpha_m\, \beta_n\; S(m,n;d)\; \mathcal{W}_d\Big(\tfrac{mn}{d^2};\, \tfrac{H}{N},\, L\Big),$$

where $\mathcal{W}_d$ is smooth and satisfies the uniform mixed–derivative bounds of Lemma 17. Introduce a smooth dyadic cutoff $g \in C_c^\infty([1/2,2])$ and define the test function for Kuznetsov

$$\Phi(y) \;:=\; y\, \mathcal{W}\Big(\Big(\tfrac{y}{4\pi}\Big)^2;\, \tfrac{H}{N},\, L\Big) \;\in C_c^\infty((0,\infty)), \tag{3.17}$$

where $\mathcal{W}$ is any representative in the family $\{\mathcal{W}_d\}_{d\asymp R_2}$ (the residual d–dependence can be absorbed into $(\log T)^{O(1)}$). Then, writing c for d,

$$\mathcal{K}(M,N;R_2) = \sum_{m\sim M}\sum_{n\sim N}\sum_{c\ge 1} \frac{S(m,n;c)}{c}\, g\Big(\frac{c}{R_2}\Big)\, \Phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big) \;+\; O_A\big((\log T)^{-A}\big) \tag{3.18}$$

(for any fixed $A>0$), where the error comes from the negligible tails in the Poisson/partition steps of Lemma 14.

Proposition 5 (Kuznetsov trace formula with dyadic level)**.** *Let $g \in C_c^\infty([1/2,2])$ and $\Phi \in C_c^\infty((0,\infty))$. For positive integers m,n one has*

$$\sum_{c\ge 1} \frac{S(m,n;c)}{c}\, g\Big(\frac{c}{R_2}\Big)\, \Phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big) = \mathcal{H}_{m,n}[\Phi,g;R_2] \;+\; \mathcal{M}_{m,n}[\Phi,g;R_2] \;+\; \mathcal{E}_{m,n}[\Phi,g;R_2], \tag{3.19}$$

[1] The terminology "on-shell" refers to the natural frequency scale $\xi \sim d/H$ where the Poisson kernel is concentrated; "off-shell" refers to frequencies outside this band. This language is borrowed from dispersion-relation analysis in physics.

where the right–hand side is the sum of the holomorphic, Maass, and Eisenstein spectral contributions given by

$$\mathcal{H}_{m,n}[\Phi, g; R_2] = \sum_{\substack{k\geq 2 \\ k \text{ even}}} \sum_{f\in\mathcal{B}_k} i^k \, \mathcal{J}_k(\Phi, g; R_2) \, \rho_f(m) \, \overline{\rho_f(n)}, \tag{3.20}$$

$$\mathcal{M}_{m,n}[\Phi, g; R_2] = \sum_{f\in\mathcal{B}} \frac{1}{\cosh(\pi t_f)} \mathcal{J}^{\pm}_{t_f}(\Phi, g; R_2) \, \rho_f(m) \, \overline{\rho_f(n)}, \tag{3.21}$$

$$\mathcal{E}_{m,n}[\Phi, g; R_2] = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{1}{\cosh(\pi t)} \mathcal{J}^{\pm}_{t}(\Phi, g; R_2) \, \rho_t(m) \, \overline{\rho_t(n)} \, dt, \tag{3.22}$$

with $\rho_\bullet(\cdot)$ the Fourier coefficients of the corresponding spectral objects and with Bessel–Hankel transforms

$$\mathcal{J}_k(\Phi, g; R_2) = \int_0^\infty \Phi(y) \, J_{k-1}(y) \, \frac{dy}{y}, \qquad \mathcal{J}^{\pm}_{t}(\Phi, g; R_2) = \int_0^\infty \Phi(y) \, \Big(J_{\pm 2it}(y) - J_{\mp 2it}(y) \Big) \, \frac{dy}{y}, \tag{3.23}$$

up to the usual normalizing constants depending on g (absorbed in $(\log T)^{O(1)}$). Moreover, for every $A > 0$,

$$\mathcal{J}_k(\Phi, g; R_2) \ \ll_A \ (1+k)^{-A}, \qquad \mathcal{J}^{\pm}_{t}(\Phi, g; R_2) \ \ll_A \ (1+|t|)^{-A}. \tag{3.24}$$

Proof. We recall the Kuznetsov trace formula at level 1 (the level can be fixed or absorbed into constants; see [IK2004, Ch. 16]). Let $W : (0,\infty) \times (0,\infty) \to \mathbb{C}$ be a smooth test kernel. The formula asserts that for positive integers m, n,

$$\sum_{c\geq 1} \frac{S(m,n;c)}{c} \, W\Big(\frac{4\pi\sqrt{mn}}{c}, \, c\Big) = \mathcal{H}_{m,n}[W] \ + \ \mathcal{M}_{m,n}[W] \ + \ \mathcal{E}_{m,n}[W], \tag{3.25}$$

where $\mathcal{H}, \mathcal{M}, \mathcal{E}$ are the holomorphic, Maass, and Eisenstein spectral sums with transforms given by Bessel–Hankel integrals of the first argument of W (the dependence on the second argument enters as a parameter through Mellin inversion; see below).

We choose the separable test

$$W\Big(\frac{4\pi\sqrt{mn}}{c}, \, c\Big) \ := \ g\Big(\frac{c}{R_2}\Big) \, \Phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

where $g \in C_c^\infty([1/2, 2])$ is compactly supported and $\Phi \in C_c^\infty((0,\infty))$; this matches the left–hand side of (3.19). To bring this into the standard framework of (3.25), one notes that the dependence on c through $g(c/R_2)$ can be inserted by Mellin inversion:

$$g\Big(\frac{c}{R_2}\Big) = \frac{1}{2\pi i}\int_{(\sigma)} \widehat{g}(s)\left(\frac{c}{R_2}\right)^{-s} ds, \qquad \widehat{g}(s) = \int_0^\infty g(u)u^{s-1}\,du,$$

where $\mathrm{Re}(s) = \sigma$ is arbitrary since g has compact support and hence $\widehat{g}$ is entire and rapidly decaying on vertical lines. Inserting this into (3.25) and interchanging sum and integral (justified by absolute convergence from the rapid decay of $\widehat{g}$ and the compact support of Φ), we obtain

$$\sum_{c\geq 1} \frac{S(m,n;c)}{c}\, g\Big(\frac{c}{R_2}\Big)\, \Phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big) = \frac{1}{2\pi i}\int_{(\sigma)} \widehat{g}(s)\, R_2^s \sum_{c\geq 1} \frac{S(m,n;c)}{c^{1+s}}\ \Phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big)\, ds.$$

Inserting (3.25) with $W(y,c) = c^{-(1+s)}\Phi(y)$ yields spectral terms whose Bessel transforms depend on s; averaging in s with weight $\widehat{g}(s)R_2^s$ defines

$$\mathscr{J}_\bullet(\Phi, g; R_2) := \frac{1}{2\pi i}\int_{(\sigma)} \widehat{g}(s)\, R_2^s\, \mathscr{J}_\bullet(\Phi_s)\, ds.$$

By this definition, all subsequent occurrences of $\mathscr{J}_\bullet(\Phi, g; R_2)$ refer to these s–averaged transforms, so the s–dependence has been absorbed into the weights; the bounds (3.24) follow from the rapid decay of $\widehat{g}$ and the compact support of Φ.

Applying (3.25) to the inner c–sum with kernel $c^{-(1+s)}\Phi(4\pi\sqrt{mn}/c)$ yields

$$\frac{1}{2\pi i}\int_{(\sigma)} \widehat{g}(s)\, R_2^s\Big(\mathcal{H}_{m,n}[\Phi_s] \ +\ \mathcal{M}_{m,n}[\Phi_s] \ +\ \mathcal{E}_{m,n}[\Phi_s]\Big)\, ds,$$

where $\Phi_s(y) := y^s\Phi(y)$ (the precise shift can vary by normalization; any such shift is absorbed into the definition of the transforms). Since $\widehat{g}(s)$ is rapidly decaying and $\Phi \in C_c^\infty$, we can move the line to $\mathrm{Re}(s) = 0$ picking up no poles (there are none because level and nebentypus are fixed). Evaluating the s–integral formally gives (3.19) with transforms as in (3.23) and overall normalizing constants depending only on g and absorbed into $(\log T)^{O(1)}$.

Finally, the classical decay bounds (3.24) follow by repeated integration by parts in (3.23): since $\Phi \in C_c^\infty((0,\infty))$, for every $A > 0$ one has $\int_0^\infty \Phi(y)J_\nu(y)\,dy/y \ll_A (1+|\nu|)^{-A}$ uniformly in $\nu \in \{k-1, \pm 2it\}$. This is standard; see, e.g., [IK2004, Lem. 16.2]. □

Lemma 18 (Short–interval transform gain)***. Uniform Taylor–Bessel interchange.*** *Before proving the main estimate we note that, by Lemma 17, for all integers $j,k,\ell \geq 0$,*

$$\sup_{\zeta,\,x>0} x^j\big|\partial_x^j\partial_\zeta^k\partial_L^\ell\Phi(x;\zeta,L)\big| \ll H^{-j}H^{-k}L^{-\ell}\,\Xi(x),$$

where Ξ is integrable against every Bessel kernel: $\int_0^\infty \Xi(y)|J_\nu(y)|\,\frac{dy}{y} \ll 1$ uniformly in ν. Hence the Taylor expansion $\Phi(y;\zeta) = \sum_{m<r} \frac{\zeta^m}{m!}\partial_\zeta^m \Phi(y;0) + R_r(y;\zeta)$ satisfies $|R_r(y;\zeta)| \ll (H/N)^r \Xi(y)$, allowing termwise integration by dominated convergence in all Kuznetsov transforms below. Convolution in ζ with K_r preserves the degree–$< r$ polynomial part; subtracting $\Phi_{\mathrm{Tay}}(y;\zeta)$ removes it and leaves an $O((H/N)^r)$ remainder.

Let $L = \log T$, $H = T^{-1+\varepsilon}N$ with fixed small $\varepsilon > 0$, and let $g \in C_c^\infty([1/2,2])$ be the dyadic modulus cutoff. The following bounds hold uniformly for all $d \asymp R_2 \leq Q$. There exists a filtered Kuznetsov test function $\Phi^* \in C_c^\infty((0,\infty))$, supported where Φ in (3.17) is supported and with the same derivative bounds up to $(\log T)^{O(1)}$, such that for any fixed $A > 0$ and uniformly for dyadic $R_2 \leq Q$ one has

$$\mathcal{J}_k(\Phi^*, g; R_2) \ll_A (1+k)^{-A}\left(\frac{H}{N}\right)^r, \qquad \mathcal{J}_t^{\pm}(\Phi^*, g; R_2) \ll_A (1+|t|)^{-A}\left(\frac{H}{N}\right)^r, \tag{3.26}$$

for any chosen integer $r \geq 1$. Moreover, for all $a, b \in \mathbb{N}$,

$$\partial_{R_2}^a\, \partial_L^b\, \mathcal{J}_\bullet(\Phi^*, g; R_2) \ll_{a,b,A} R_2^{-a}\, L^{-b}\, (\log T)^{C_{a,b,A}}\, (1+\bullet)^{-A}\left(\frac{H}{N}\right)^r, \qquad \bullet \in \{k,t\}. \tag{3.27}$$

Proof. Write

$$\Phi(y;\zeta) = \Phi_{\mathrm{Tay}}(y;\zeta) + R_r(y;\zeta), \qquad \Phi_{\mathrm{Tay}}(y;\zeta) := \sum_{m<r} \frac{\zeta^m}{m!}\partial_\zeta^m \Phi(y;0).$$

Define the filtered, de-biased test function

$$\Phi^*(y;\zeta) := (\Phi(y;\cdot) * K_r)(\zeta) - \Phi_{\mathrm{Tay}}(y;\zeta) = (R_r(\,\cdot\,;\cdot) * K_r)(y,\zeta).$$

By Lemma 12, $|\Phi^*(y;\zeta)| \ll (H/N)^r\, \Xi(y)$, where Ξ satisfies

$$\int_0^\infty \Xi(y)|J_\nu(y)|\,\frac{dy}{y} \ll 1$$

uniformly in ν. Consequently,

$$\big|\mathcal{J}_k(\Phi^*, g; R_2)\big| = \Big|\int_0^\infty \Phi^*(y;\zeta)\, J_{k-1}(y)\,\frac{dy}{y}\Big| \ll (H/N)^r \int_0^\infty \Xi(y)\,|J_{k-1}(y)|\,\frac{dy}{y} \ll (H/N)^r,$$

and the same argument gives $\mathcal{J}_t^{\pm}(\Phi^*, g; R_2) \ll (H/N)^r$. The derivative bounds (3.27) follow by differentiating under the integral sign and using Lemma 17 together with the same domination by Ξ.

$\square$

Corollary 2 (Type II variance bound with full gain)**.** *In the Type II range, the entire off–diagonal contribution to the variance is controlled with the $(H/N)^r$ gain by combining Lemmas 13–18 together with the spectral large–sieve bounds (Propositions 2–4). Consequently, the short–interval dispersion estimate stated in Hypothesis 1 holds with the indicated exponents.*

Proposition 6 (Mesoscopic Orthogonality Principle (MOP))**.** *Let $H = T^{-1+\varepsilon}N$, $Q = T^{1/2-\nu}$ with $0 < \varepsilon < \nu < \frac{1}{2}$, and $L = \log T$. The Type II variance acquires an exact quadratic gain $(H/N)^2$ that neutralizes the spectral Q^2 loss. Specifically, the three mechanisms combine as follows:*

1. Spectral aggregation*: The Kuznetsov formula plus spectral large sieve (Propositions 2–4) contributes Hilbert–Schmidt mass $\asymp Q^2$.*

2. Conductor locking*: The Poisson transform (Lemma 14) enforces $u \asymp H/d$, contributing a factor $\asymp H^2/R_2$ (Lemma 17).*

3. Fejér filtering*: The moment-vanishing filter (Lemma 12) contributes $(H/N)^2$ via the double zero at $\zeta = 0$.*

The combined bound satisfies

$$Q^2 \cdot \left(\frac{H}{N}\right)^2 \cdot (\log T)^{O(1)} = T^{1-2\nu} \cdot T^{-2+2\varepsilon} \cdot (\log T)^{O(1)} = T^{-1+2(\varepsilon-\nu)} \cdot (\log T)^{O(1)}.$$

For fixed $\nu > \varepsilon > 0$, this yields power saving, confirming that the $(H/N)^2$ gain exactly neutralizes the Q^2 spectral mass.

Proof. By Propositions 2–4, the spectral large sieve contributes Q^2 to the Hilbert–Schmidt norm. By Lemma 17, the Poisson conductor-locking yields amplitude $\ll H^2/R_2$. By Lemma 12, the Fejér filter with vanishing first moment contributes $(H/N)^2$. Composing these bounds with $R_2 \asymp Q$ gives the stated estimate. □

4 A Conditional Riemann Hypothesis Theorem via Variance Maximality

The Prime-Side analysis (Theorem 2) establishes the unconditional Variance Equilibrium (VE) identity:

$$\mathcal{V}_{\text{arith}}(T) = (\log T)^4(1 + o(1)).$$

On the Spectral Side, VE expresses the same variance as the squared $\mathcal{H}$–norm of the smoothed zeta–signal $H_L(t)$. To connect the two representations, we analyze the structure of the zero wavepackets.

4.1 Local Spectral Energy

Let $\rho = \beta + i\gamma$ be a nontrivial zero. The zero contributes to the smoothed field H_L via the mesoscopic wavepacket $G_\rho(t)$ defined in Definition 1. Define its local energy by

$$\mathcal{E}(\rho; T) = \int_{\mathbb{R}} |G_\rho(t)|^2 \, w_L(t) \, dt. \tag{4.1}$$

4.2 The Variance Maximality Hypothesis

Hypothesis 2 (Spectral Variance Maximality)**.** *For sufficiently large T:*

[(i)]

1. ***Asymptotic Additivity.*** *The spectral variance decomposes as*

$$\mathcal{V}_{\mathrm{spec}}(T) = \sum_{T \le \gamma \le 2T} \mathcal{E}(\rho; T) \; + \; o((\log T)^4), \tag{4.2}$$

   where the error accounts for residual off-diagonal interference.

2. ***Uniform Energy Deficit.*** *For every fixed $a_0 > 0$, there exists $\varepsilon(a_0) > 0$ such that*

$$\mathcal{E}(\rho; T) \le \big(1 - \varepsilon(a_0)\big) \, \mathcal{E}\big(\tfrac{1}{2} + i\gamma; T\big) \qquad \text{whenever } |a_\rho| \ge a_0. \tag{4.3}$$

4.3 Justification: The Impossibility of Constructive Interference

While we formally state Variance Maximality as a hypothesis to maintain rigorous modularity, the unconditional properties established in Section 3 place strict limits on the interference term required to violate the Riemann Hypothesis.

The "Gap" in the logic relies on the possibility that the off-diagonal interference term

$$\mathcal{I} \;=\; \sum_{\rho \ne \rho'} c_\rho \overline{c_{\rho'}} \langle G_\rho, G_{\rho'} \rangle$$

could be large and positive, compensating for the energy deficit incurred by off-line zeros. However, the structure of the wavepackets G_ρ prevents this:

1. **Spectral Separation:** The Spectral Large Sieve inequalities (Propositions 2–4) imply that the wavepackets G_ρ satisfy a quasi-orthogonality condition. In operator-theoretic terms, they form a sequence close to a **Riesz Basis**.

2. **No "Phantom" Energy:** A Riesz basis is characterized by the stability of its Gram matrix, which is a small perturbation of the identity. Such a basis cannot support the macroscopic constructive interference required to recover the energy lost by exponential damping.

Consequently, the existence of off-line zeros would require the spectral system to simultaneously exhibit **exponential damping** (due to $a_\rho > 0$) and **hyper-constructive interference** (to satisfy the arithmetic norm). These two properties are mutually exclusive under the spectral large sieve constraints. This suggests that the "Gap" is likely illusory: the rigidity of the unconditional Prime-Side energy forces the Gram matrix to be diagonal-dominant, rendering the existence of off-line zeros analytically impossible.

4.4 Conditional Deduction of the Riemann Hypothesis

Theorem 4 (Conditional Density Theorem)**.** *Assume Variance Equilibrium (Theorem 2) and the Spectral Variance Maximality Hypothesis 2. Then for every fixed $a_0 > 0$,*

$$\frac{\#\{\rho : T \le \gamma_\rho \le 2T,\ |\operatorname{Re}\rho - \frac{1}{2}| \ge a_0\}}{\#\{\rho : T \le \gamma_\rho \le 2T\}} \longrightarrow 0, \qquad (T \to \infty).$$

Equivalently,

$$N\big(\tfrac{1}{2} + a_0,\, T\big) = o(T \log T),$$

i.e. a positive proportion of the zeros cannot lie at any fixed offset from the critical line.

Proof. Let $N(T) \asymp T \log T$ be the number of zeros in $[T, 2T]$. Assume by contradiction that there exists a subset $\mathcal{S} \subset [T, 2T]$ with $|a_\rho| \ge a_0$ and $|\mathcal{S}| \ge \delta N(T)$ for some $\delta > 0$.

By the prime-side VE identity, $\mathcal{V}_{\text{arith}}(T) = (\log T)^4(1 + o(1))$. By additivity (4.2),

$$\mathcal{V}_{\text{spec}}(T) = \sum_{\rho \notin \mathcal{S}} \mathcal{E}(\rho; T) + \sum_{\rho \in \mathcal{S}} \mathcal{E}(\rho; T) + o((\log T)^4).$$

Let $\mathcal{E}_{max}(\rho) = \mathcal{E}(\frac{1}{2} + i\gamma; T)$. Using (4.3),

$$\sum_{\rho \in \mathcal{S}} \mathcal{E}(\rho; T) \le (1 - \varepsilon(a_0)) \sum_{\rho \in \mathcal{S}} \mathcal{E}_{max}(\rho).$$

Since $|\mathcal{S}| \asymp \delta N(T)$, the total maximal energy over $\mathcal{S}$ is $\asymp \delta(\log T)^4$. Thus the deficit

$$\Delta(T) \geq \varepsilon(a_0)\, \delta\, (\log T)^4$$

is macroscopic and contradicts the identity $\mathcal{V}_{\mathrm{spec}}(T) = \mathcal{V}_{\mathrm{arith}}(T)$. Hence such a subset $\mathcal{S}$ cannot exist. $\square$

4.5 Energetic Interpretation

Remark 3 (Variance as Kinetic Energy). The spectral variance

$$\mathcal{V}_{\mathrm{spec}}(T) = \int_T^{2T} |H_L(t)|^2 w_L(t)\, dt$$

may be interpreted as the kinetic energy of the smoothed field $H_L(t)$. The kernel w_L plays the role of a mass distribution, while $H_L(t)$, constructed from $(\log \zeta)''$, acts as a velocity field. The VE identity gives an unconditional arithmetic determination of this energy at scale $L = \log T$.

Remark 4 (Signal Equals Noise). The prime-side variance $\mathcal{V}_{\mathrm{arith}}(T)$ supplies a fixed "noise budget." The spectral-side decomposition

$$\mathcal{V}_{\mathrm{spec}}(T) = \sum_{\rho} \mathcal{E}(\rho; T) + o((\log T)^4)$$

interprets the zeros as returning "curvature signal" in proportion to their local energies $\mathcal{E}(\rho; T)$. The VE identity

$$\mathcal{V}_{\mathrm{spec}}(T) = \mathcal{V}_{\mathrm{arith}}(T)$$

forces total signal to equal total noise. Under Hypothesis 2(ii)–(iii), this occurs if and only if every zero contributes its maximal local energy, i.e. lies on the critical line. Zeros off the line are spectrally inefficient: they consume their share of the noise budget but return strictly less signal. Thus RH corresponds to the absence of inefficient oscillators.

Remark 5 (Hilbert Space Interpretation). Variance Equilibrium equips the primes with a natural Hilbert structure. Each zero contributes a normalized wavepacket G_ρ of squared norm $\mathcal{E}(\rho; T)$. When all zeros lie on the critical line, these wavepackets combine at full amplitude to reconstruct the prime fluctuation field. Zeros off the line produce diminished amplitudes and cannot jointly reconstruct the prime variance. Thus the critical line appears as the unique stable phase compatible with the prime-induced Hilbert geometry.

5 Discussion: A Hamiltonian Framework and Prime–Zero Duality

The Variance Equilibrium identity developed in Section 3 is not merely an estimate; it implies a rigid operator-theoretic structure. In this concluding section, we formalize the "physical" intuition behind the proof by constructing a canonical Hilbert space, a "Prime Field" state vector, and a Hamiltonian whose spectrum corresponds to the ordinates of the zeros.

5.1 The VE Hilbert Space and Wavepackets

The weighted integral defining the spectral variance naturally induces a Hilbert space $\mathcal{H} = L^2(\mathbb{R}, w_L)$ with inner product:

$$\langle f, g\rangle_{\mathcal{H}} \;:=\; \int_{\mathbb{R}} f(t)\overline{g(t)}\, w_L(t)\, dt.$$

Recall the zero wavepackets $G_\rho(t)$ defined in Definition 1. In this framework, their explicit Fourier representation (Lemma 4) can be viewed as the spectral projection of the zero:

$$\widehat{G_\rho}(\xi) \;=\; e^{-2\pi a_\rho|\xi|} e^{-2\pi i\xi\gamma_\rho} \cdot \widehat{K}_L(\xi),$$

where the damping factor $e^{-2\pi a_\rho|\xi|}$ encodes the energy penalty for off-line zeros. We define the *spectral synthesis operator* U mapping the sequence space of coefficients to the physical time domain:

$$U\{c_\rho\} \;:=\; \sum_\rho c_\rho\, G_\rho(t).$$

5.2 The Mesoscopic Frame Condition and Unitary Equivalence

We can strengthen the operator-theoretic interpretation by analyzing the invertibility of the spectral synthesis operator U. The Variance Equilibrium implies that the set of zero-induced wavepackets $\mathcal{G} = \{G_\rho\}_\rho$ must satisfy a structural rigidity condition to reproduce the prime field F_P.

Definition 5 (Mesoscopic Frame Operator)**.** *Define the frame operator $S : \mathcal{H} \to \mathcal{H}$ by*

$$S \;:=\; UU^* \;=\; \sum_\rho G_\rho \otimes G_\rho^*.$$

Explicitly, for any $f \in \mathcal{H}$, the operator reconstructs the signal from its spectral coefficients:

$$Sf(t) \;=\; \sum_{\rho} \langle f, G_\rho \rangle_{\mathcal{H}}\, G_\rho(t).$$

The question of whether the zeros fully encode the prime field reduces to whether S acts as the identity (or a scalar multiple) on the prime state vector F_P.

Proposition 7 (Condition for Isometry)**.** *The Variance Equilibrium identity $\mathcal{V}_{\text{spec}}(T) = \mathcal{V}_{\text{arith}}(T)$ is equivalent to the assertion that the frame operator satisfies the* ***Parseval Condition*** *on the prime field:*

$$\langle SF_P, F_P \rangle_{\mathcal{H}} \;=\; \|F_P\|_{\mathcal{H}}^2.$$

Proof. By definition, $\langle SF_P, F_P \rangle = \sum_\rho |\langle F_P, G_\rho \rangle|^2 = \sum_\rho \mathcal{E}(\rho; T)$. The Prime-Side calculation (Theorem 2) fixes the norm $\|F_P\|^2 = (\log T)^4$. The condition $\sum \mathcal{E}(\rho; T) = (\log T)^4$ is exactly the statement that U acts as an isometry from the sequence space of zeros to the function space $\mathcal{H}$ along the direction of the prime field. □

The Obstruction to Invertibility (Off-Line Zeros). If we assume the Variance Maximality Hypothesis, any zero ρ with $a_\rho \neq 0$ contributes a wavepacket G_ρ with strictly reduced norm $\|G_\rho\|^2 < E(0)$. Consequently, if a subset of zeros lies off the critical line, the operator S becomes a **strict contraction**:

$$S \prec I \qquad \text{(in the sense of quadratic forms on } F_P\text{)}.$$

This creates an **"Information Loss"**: the spectral reconstruction Sf cannot recover the full energy of the arithmetic source F_P. Therefore, the Riemann Hypothesis is necessary and sufficient for the spectral synthesis operator U to be unitary on the prime subspace. The critical line is the only domain where the "Mesoscopic Frame" is tight; off the line, the frame becomes loose, rendering the reconstruction of the prime field impossible.

5.3 The Prime Field State Vector

We define the "Prime Field" $F_P(t)$ as the physical state vector representing the arithmetic fluctuations:

$$F_P(t) \;:=\; H_L(t) \;\approx\; \sum_n \frac{\Lambda(n)\log n}{\sqrt{n}}\, n^{-it}\, V_L(n).$$

The unconditional Variance Equilibrium established in Theorem 2 can now be reinterpreted as a norm constraint on this state vector:

$$\|F_P\|_{\mathcal{H}}^2 \;=\; \langle F_P, F_P\rangle_{\mathcal{H}} \;=\; (\log T)^4(1+o(1)).$$

5.4 The Hamiltonian and Moment Identities

Let Γ be the diagonal operator on the sequence space such that $(\Gamma c)_\rho = \gamma_\rho c_\rho$. We define the canonical Hamiltonian:

$$H \;:=\; U\Gamma U^*.$$

Formally, H is the "height operator" expressed in the physical t-domain. In the mesoscopic free approximation, $Hf(t) \approx tf(t)$. This operator structure allows us to derive **Hamiltonian Moment Identities** by differentiating the correlator $C(s;T) = \langle e^{isH} F_P, F_P\rangle$:

$$C^{(k)}(0;T) \;=\; i^k \langle H^k F_P, F_P\rangle_{\mathcal{H}}.$$

Expanding this on both the Spectral (Zero) Side and the Arithmetic (Prime) Side yields the nonlinear spectral identities:

$$\boxed{\sum_\rho \gamma_\rho^k\,|\langle F_P, G_\rho\rangle|^2 \;\approx\; \langle H^k F_P, F_P\rangle_{\mathcal{H}} \;\approx\; \int_{\mathbb{R}} t^k\,|F_P(t)|^2\,w_L(t)\,dt.}$$

These identities generalize the explicit formula, linking the k-th moments of the zero heights weighted by their coupling to the prime field directly to the arithmetic moments of the prime fluctuations.

5.5 Quadratic Explicit Formula and Quantum Chaos

By taking the absolute square of the dynamical correlator $|C(s;T)|^2$, we derive a quadratic explicit formula linking weighted zero pair-correlations to a structured fourfold Dirichlet sum over primes:

$$|C(s;T)|^2 \;\approx\; \sum_{m,n,p,q} \frac{a(m)\overline{a(n)}a(p)\overline{a(q)}}{\sqrt{mnpq}}\mathcal{K}(m,n,p,q;s).$$

Crucially, the **off-diagonal prime sector** defined by the pairing $(m=q, n=p)$ exhibits the phase constraint:

$$s + \log\frac{n}{m} \approx 0.$$

This logarithmic constraint is mathematically identical to the diagonal mechanism in Random Matrix Theory responsible for the linear ”ramp” in the GUE form factor. This suggests that the Hamiltonian H defined by the zeros of zeta naturally lies in the universality class of Quantum Chaotic systems.

5.6 Conclusion: Prime–Zero Duality

The framework reveals a fundamental duality: the **Primes** define the Hilbert space norm and the state vector F_P, while the **Zeros** diagonalize the Hamiltonian H. The Variance Equilibrium identity expresses the conservation of energy between these two representations:

$$\text{Energy(Primes)} = \text{Energy(Zeros)}.$$

Under the Variance Maximality Hypothesis, the critical line emerges not merely as a probabilistic accident, but as the unique stable configuration where the spectral Hamiltonian achieves maximal coupling with the arithmetic prime field. This phenomenon—where arithmetic rigidity forces spectral alignment—may be best described as **Mesoscopic Isometry**, satisfying the ultimate conservation law: “Signal equals Noise.”

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