A Framework for the Riemann Hypothesis via Symbolic Curvature Dynamics

Classified

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Abstract

We propose an analytic framework to prove the Riemann Hypothesis, using a corrected phase function $\vartheta(t) = \arg\zeta\left(\frac{1}{2} + it\right) - \theta(t)$, which isolates oscillations from nontrivial zeros via the functional equation. The second derivative exhibits negative averages on zero-free mesoscopic intervals. Off-line zeros produce positive curvature, contradicting the negative bound, proving all nontrivial zeros lie on $\mathrm{Re}(s) = \frac{1}{2}$. This first-principles approach contrasts with zero-density or spectral methods by leveraging phase curvature directly from the functional equation and Hadamard factorization.

1. Introduction

The Riemann Hypothesis asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Despite extensive studies using complex analysis, zero-density estimates, spectral interpretations, and random matrix analogies, the conjecture remains unproven [1, 2, 3].

This paper presents a curvature-based analytic framework to prove the Riemann Hypothesis, focusing on the corrected phase function

$$\vartheta(t) := \arg\zeta\left(\frac{1}{2} + it\right) - \theta(t), \tag{1.1}$$

where $\theta(t) = \mathrm{Im}\log\Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2}\log\pi$ isolates oscillations from nontrivial zeros, derived from the functional equation. The second derivative $\vartheta''(t)$ exhibits negative mesoscopic averages for $t \geq t_0$, ensuring zero spacing consistent with the global logarithmic law, derived analytically using the Hadamard product and Stirling's approximation. This approach differs from zero-density estimates [3] or spectral methods [5] by constructing a phase curvature model from first principles.

The threshold t_0 and mesoscopic length $L \asymp 2\pi/\log t$ are justified by curvature and spacing analyses in subsequent sections. Off-line zeros produce positive curvature, contradicting the negative bound, leading to the proof in Section 5 that all nontrivial zeros lie on $\mathrm{Re}(s) = \frac{1}{2}$.

Structure of the Paper. Section 2 reviews classical background. Section 3 defines the corrected phase function and its derivatives. Section 4 develops the symbolic energy model and recurrence law. Section 5 establishes the collapse of curvature structure off the critical line. Section 6 presents the final synthesis and states the Riemann Hypothesis theorem.

2. Classical Background

The Riemann zeta function is defined for $\mathrm{Re}(s) > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ [1, 2]. The functional equation and the completed zeta function are introduced in Section 3.1, where we define the corrected phase function central to our proof. Trivial zeros lie at the negative even integers, while the nontrivial zeros lie in the critical strip $0 < \mathrm{Re}(s) < 1$. The Riemann Hypothesis asserts that all nontrivial zeros satisfy $\mathrm{Re}(s) = \frac{1}{2}$ [4].

3. The Corrected Phase Function

We define the corrected phase function $\vartheta(t)$ as a real-valued function isolating the oscillatory structure of $\arg \zeta(s)$ along the critical line $s = \frac{1}{2} + it$, addressing a contradiction between its negative curvature and the increasing slope of $\vartheta'(t)$. We derive its derivatives, characterize its jump behavior at zeros, and establish curvature laws governing its global dynamics, including averaged negativity for $t \geq t_0$. In Subsection 3.6 we strengthen the averaged result via a bandlimited kernel that yields a strict uniform negativity floor on zero-free windows and a strict positive lower bound in the presence of any off-line zero. These results interface with the symbolic energy framework (Section 4) and enable contradictions against off-line zeros in Section 5. We use the principal branch of $\arg \zeta(s)$, continuous except at nontrivial zeros

$s = \rho$, where it exhibits jumps determined by analytic properties of $\zeta(s)$.

3.1 Definition from Principal Argument

Let

$$s = \frac{1}{2} + it.$$

Our objective is to define a real-valued corrected phase $\vartheta(t)$ that isolates the oscillatory structure of $\arg \zeta(s)$ due to nontrivial zeros, removing the smooth drift from the gamma factor, while accounting for the curvature's role in slope dynamics.

Step 1: Functional equation and completed zeta function.

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{3.1}$$

[1, Chap. II, §2.1, eq. (2.1.9)]

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s), \tag{3.2}$$

[1, Chap. II, §2.1, eq. (2.1.12)]

$$\xi(s) = \xi(1-s). \tag{3.3}$$

[1, Chap. II, §2.1, eq. (2.1.13)]

Step 2: Argument relations on the critical line and the corrected phase. From (3.2) and (3.3), for

$$s = \frac{1}{2} + it$$

we have

$$\xi\left(\frac{1}{2} + it\right) = \xi\left(\frac{1}{2} - it\right) \in \mathbb{R},$$

and by rearranging (3.2),

$$\xi\left(\frac{1}{2} + it\right) = \pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\zeta\left(\frac{1}{2} + it\right).$$

Hence

$$\arg\left[\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\zeta\left(\frac{1}{2} + it\right)\right] \equiv 0 \pmod{\pi}, \tag{3.4}$$

which expands to

$$-\frac{t}{2}\log\pi + \operatorname{Im}\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) + \arg\zeta\left(\frac{1}{2}+it\right) \equiv 0 \pmod{\pi}. \tag{3.5}$$

For

$$s = \frac{1}{2}+it,$$

the prefactor in (3.2) is

$$\frac{1}{2}s(s-1) = \frac{1}{2}\left(\frac{1}{2}+it\right)\left(-\frac{1}{2}+it\right) = \frac{1}{2}\left(t^2+\frac{1}{4}\right) \in \mathbb{R}_{\geq 0}. \tag{3.6}$$

Hence (3.2) yields

$$\xi\left(\frac{1}{2}+it\right) = \frac{1}{2}\left(t^2+\frac{1}{4}\right)\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\zeta\left(\frac{1}{2}+it\right). \tag{3.7}$$

By (3.3),

$$\xi\left(\frac{1}{2}+it\right) = \xi\left(\frac{1}{2}-it\right) \in \mathbb{R}, \tag{3.8}$$

so

$$\arg\left[\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\zeta\left(\frac{1}{2}+it\right)\right] \equiv 0 \pmod{\pi}. \tag{3.9}$$

Expanding the argument gives

$$-\frac{t}{2}\log\pi + \operatorname{Im}\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) + \arg\zeta\left(\frac{1}{2}+it\right) \equiv 0 \pmod{\pi}. \tag{3.10}$$

We therefore conclude

$$\theta(t) = \operatorname{Im}\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) - \frac{t}{2}\log\pi. \tag{3.11}$$

From (3.5) and the definition (3.11) we have the congruence

$$\arg\zeta\left(\frac{1}{2}+it\right) + \theta(t) \equiv 0 \pmod{\pi}, \tag{3.12}$$

and hence

$$\arg\zeta\left(\frac{1}{2}+it\right) - \theta(t) \equiv 0 \pmod{\pi}. \tag{3.13}$$

This identity motivates the corrected phase (principal branch on each zero-free interval)

$$\vartheta(t) = \arg\zeta\left(\frac{1}{2}+it\right) - \theta(t), \tag{3.14}$$

which isolates the oscillatory component of $\arg \zeta(\frac{1}{2} + it)$.

3.2 Real-Valued Derivatives

For $s = \frac{1}{2} + it$, we derive the derivatives of $\vartheta(t)$ directly from the functional equation and the Hadamard product.

The logarithmic derivative of $\zeta(s)$ is

$$\frac{d}{ds} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)}, \tag{3.15}$$

valid for $\mathrm{Re}(s) > 1$, and extended meromorphically to the critical strip by analytic continuation [1, Chap. II, §2.16]. Differentiating again, the Hadamard product gives

$$\frac{d^2}{ds^2} \log \zeta(s) = -\sum_{\rho} \frac{m_\rho}{(s-\rho)^2} + \mathrm{regular}(s), \tag{3.16}$$

where ρ runs over nontrivial zeros with multiplicity m_ρ, and the regular term is holomorphic near the critical line [1, Chap. II, Eq. (2.17.1)]. The series converges uniformly on compact subsets of the critical strip excluding zeros.

Along the critical line $s = \frac{1}{2} + it$, we have $ds = i\,dt$, so the chain rule yields

$$\frac{d}{dt} = \frac{1}{i}\frac{d}{ds}, \qquad \frac{d}{dt} \arg \zeta(s) = \mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right). \tag{3.17}$$

Therefore

$$\vartheta'(t) = \mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) - \theta'(t), \qquad \vartheta''(t) = \frac{d}{dt}\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) - \theta''(t). \tag{3.18}$$

Applying (3.17) to $f(s) = \frac{\zeta'(s)}{\zeta(s)}$, one finds

$$\frac{d}{dt}\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = -\,\mathrm{Im}\left(\frac{d^2}{ds^2}\log\zeta(s)\right),$$

and substituting from (3.16) yields

$$\vartheta''(t) = \mathrm{Im}\sum_{\rho} \frac{m_\rho}{(s-\rho)^2} - \mathrm{Im}\left(\mathrm{regular}(s)\right) - \theta''(t), \tag{3.19}$$

with $s = \frac{1}{2} + it$. On zero-free intervals, the Hadamard product term $\mathrm{Im}\sum_\rho \frac{m_\rho}{(s-\rho)^2}$ is replaced by the Dirichlet polynomial side of the approximate functional equation [1, Chap. IV, §4.17], with the remainder contributing $O(1/\log t)$ after bandlimited averaging, as detailed in Subsection 3.6.

Remark 1 (On growth of $\vartheta'(t)$). On zero-free intervals, $\vartheta'(t)$ is oscillatory with bounded growth, while $\vartheta'_+(t_n) \approx \frac{1}{2}\log t_n$ at zeros due to jumps (Subsection 4.2).

3.3 Phase Jump at Zeros

Near a zero $\rho_n = \frac{1}{2} + it_n$, we analyze the jump behavior of $\vartheta(t)$. We have the local expansion

$$\zeta(s) \approx c(s - \rho_n), \qquad s - \rho_n = i(t - t_n),$$

so that

$$\arg\zeta = \operatorname{Im}\log c + \arg\left(i(t - t_n)\right),$$

where

$$\arg\left(i(t - t_n)\right) = \begin{cases} -\frac{\pi}{2} & t < t_n, \\ \frac{\pi}{2} & t > t_n. \end{cases}$$

Therefore

$$\lim_{\varepsilon \to 0^+} \left[\arg\zeta\left(\frac{1}{2} + i(t_n + \varepsilon)\right) - \arg\zeta\left(\frac{1}{2} + i(t_n - \varepsilon)\right)\right] = \pi,$$

and since $\theta(t)$ is continuous, $\vartheta(t)$ exhibits a jump of size π centered at t_n [1, Chap. IX, §9.3].

Lemma 1 (Jump–Zero Correspondence)**.** *If $\zeta(\frac{1}{2} + it_n) = 0$, then $\vartheta(t)$ jumps by π at t_n, centered at t_n. Jumps occur only at zeros.*

Proof. The jump arises from the argument's discontinuity at ρ_n. As t crosses t_n, $\arg\zeta$ changes by π, while $\theta(t)$ remains continuous. Thus, $\vartheta(t) = \arg\zeta(\frac{1}{2} + it) - \theta(t)$ inherits the π jump. □

3.4 Persistent Curvature Negativity

For any zero-free interval $I \subset (t_n, t_{n+1})$, identity (3.19) decomposes as

$$\frac{1}{L}\int_I \vartheta''(u)\,du = -\mathcal{D}(t; I) + \mathcal{R}_{\text{off}}(t; I) + \mathcal{G}(t; I),$$

where $\mathcal{D}$ is the diagonal variance term, $\mathcal{R}_{\text{off}}$ the off-diagonal remainder, and $\mathcal{G}$ the gamma contribution.

Lemma 2 (Off-diagonal suppression)**.** *Let $I = [u_0 - L/2, u_0 + L/2]$ be a symmetric zero-free interval with $L \asymp 1/\log t$. With coefficients*

$$A_n = W\left(\frac{n}{N}\right) n^{-1/2} P(\log n), \qquad N = \sqrt{\frac{t}{2\pi}},$$

for $W \in C_c^\infty([1-\eta, 1+\eta])$ with $W \equiv 1$ on $[1-\eta_0, 1+\eta_0] \subset (1-\eta, 1+\eta)$, $\eta_0 < \eta$, and P a polynomial not identically zero, the averaged off-diagonal term

$$\mathcal{R}_{\mathit{off}}(t; I) := \frac{1}{L} \int_I \sum_{m \neq n} A_m \overline{A_n} e^{-iu(\log m - \log n)}\, du = \sum_{m \neq n} A_m \overline{A_n} e^{-iu_0(\log m - \log n)} \widehat{\phi}\big(L(\log m - \log n)\big)$$

satisfies $\mathcal{R}_{\mathit{off}}(t; I) = O(1/\log t)$.

Proof. Since $\widehat{\phi}$ is supported on $[-1, 1]$, only pairs with $|\log(m/n)| \le 1/L$ contribute. Partition by $k = m - n$, with $|k| \le Cn/L$, so $|\log(m/n)| \asymp |k|/n$. The bilinear form is bounded by

$$\frac{1}{L} \sum_n |A_n|^2 + \frac{1}{L} \sum_{|k| \le Cn/L} \sum_n |A_{n+k} - A_n|\, |A_n|,$$

where the second term is $O(1/L) \sum_n |A_n|^2$ by the smoothness of A_n, yielding $O(1/L) = O(1/\log t)$. □

Lemma 3 (Variance lower bound)**.** *With $S_k(u) = \sum_{n \ge 1} A_n (\log n)^k e^{-iu \log n}$ for $k = 0, 1, 2$,*

$$\mathcal{D}(t; I) := \frac{1}{L} \int_I \left(\frac{S_2}{S_0} - \left(\frac{S_1}{S_0} \right)^2 \right) du = c_0 + O\left(\frac{1}{\log t} \right),$$

where

$$c_0 = \kappa_\phi c_*, \qquad \kappa_\phi = \frac{1}{2\pi} \int_{-1}^{1} (\widehat{\phi}(\xi))^2 d\xi > 0,$$

and

$$c_* \ge \frac{c_1}{c_2} \cdot \frac{(\log 2)^2}{12} > 0,$$

uniformly in t, with constants $0 < c_1 \le |W|^2 |P|^2 \le c_2 < \infty$ on a fixed subinterval of $[1-\eta_0, 1+\eta_0]$.

Proof. For $n \sim N = \sqrt{t/(2\pi)}$, the weights are

$$w_n = \frac{|A_n|^2}{\sum_{m \sim N} |A_m|^2}, \qquad A_n = W(n/N)\, n^{-1/2} P(\log n).$$

Define $\mu = \sum w_n \log n$. Then

$$\frac{S_2}{S_0} - \left(\frac{S_1}{S_0}\right)^2 = \sum_{n \sim N} w_n (\log n - \mu)^2,$$

the variance of $\log n$ under the weights w_n.

For $P \equiv 1$, $W \equiv 1$ on $[1, 2]$, this reduces to the continuous distribution of $\log n$ on $[\log N, \log 2N]$. Its variance is exactly

$$\frac{1}{\log 2} \int_{\log N}^{\log 2N} \left(x - \log N - \tfrac{\log 2}{2}\right)^2 dx = \frac{(\log 2)^2}{12}.$$

For general W, P not identically zero, there exists a subinterval $J \subset [1 - \eta_0, 1 + \eta_0]$ where

$$c_1 \leq |W(e^y)|^2 |P(\log N + y)|^2 \leq c_2,$$

so the weighted variance is comparable to Lebesgue measure on J. Hence

$$c_* \geq \frac{c_1}{c_2} \cdot \frac{(\log 2)^2}{12} > 0,$$

uniformly in t. Edge effects from W contribute only $O(1/\log t)$ [1, Chap. II, §2.17.1]. $\square$

Lemma 4 (Gamma contribution). *For a symmetric zero-free I with $L \asymp 1/\log t$,*

$$\mathcal{G}(t; I) := \frac{1}{L} \int_I -\theta''(u)\, du = -\frac{1}{2t} + O\left(\frac{1}{t^2}\right).$$

In particular, since $\theta''(t) = \frac{1}{2t}$, we have $-\theta''(t) = -\frac{1}{2t} < 0$. Thus $\mathcal{G}(t; I)$ contributes strictly negatively for all sufficiently large t.

Proof. From $\theta(t) = \operatorname{Im} \log \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2} \log \pi$, we obtain $\theta''(t) = \frac{1}{2t} + O(t^{-2})$ (by Stirling's expansion; see [1, Chap. II, §2.15]), so the average is $-\frac{1}{2t} + O\left(\frac{1}{t^2}\right)$. $\square$

Theorem 1 (Averaged negativity on symmetric windows). *For any symmetric zero-free I of length $L \asymp 1/\log t$,*

$$\frac{1}{L} \int_I \vartheta''(u)\, du \leq -c_* + O\left(\frac{1}{\log t}\right) + O\left(\frac{1}{t}\right).$$

The bandlimited average $\mathcal{A}_{L,u_0}[\vartheta''] \leq -\frac{c_0}{2}$.

Proof. By Lemmas 2, 3, and 4, the contributions satisfy $-\mathcal{D}(t; I) = -(c_0 + O(1/\log t)) < 0$,

$\mathcal{R}_{\mathrm{off}}(t;I) = O(1/\log t)$, $\mathcal{G}(t;I) = -\frac{1}{2t} + O\left(\frac{1}{t^2}\right) < 0$. Thus, $\frac{1}{L}\int_I \vartheta''(u)\,du \le -c_* + O\left(\frac{1}{\log t}\right) + O\left(\frac{1}{t}\right)$, and the bandlimited average $\le -\frac{c_0}{2}$ follows from the kernel's stricter bound. $\square$

3.5 Bandlimited Curvature for Large t

Lemma 5. *For $t \ge t_0$, where $t_0 = \exp\left(\frac{16c_0}{c_2}\right)$ from Theorem 2, the bandlimited average $\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2} < 0$ on every zero-free subinterval (t_n, t_{n+1}).*

Proof. From Subsection 3.2, the pointwise expansion of $\vartheta''(t)$ is:

$$\vartheta''(t) = -\frac{t}{\left(\frac{1}{4}+t^2\right)^2} - \sum_{k=1}^{\infty} \frac{2t\left(\frac{1}{2}+2k\right)}{\left[\left(\frac{1}{2}+2k\right)^2+t^2\right]^2} - \theta''(t) + R(t),$$

where the first term is the pole contribution, the series arises from trivial zeros $\rho_k = -2k$, and $R(t)$ is the remainder from nontrivial zeros.

For $t \ge t_0$, the pole term is:

$$-\frac{t}{\left(\frac{1}{4}+t^2\right)^2} \approx -\frac{1}{t^3},$$

and the trivial zeros sum is bounded by $-c/t^3$ for some absolute $c > 0$, by comparing the $k = 1$ term and bounding the tail by a decreasing integral [1, Chap. II, §2.11]. On zero-free intervals, the Hadamard product term $R(t) = \mathrm{Im}\sum_\rho \frac{m_\rho}{(s-\rho)^2}$ is replaced by the Dirichlet polynomial side of the approximate functional equation [1, Chap. IV, §4.17], with the remainder contributing $O(1/\log t)$ after bandlimited averaging, as detailed in Subsection 3.6. Theorem 1 gives $\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2}$, with $c_0 = \kappa_\phi c_*$, $c_* \ge \frac{c_1}{c_2}\cdot\frac{|J|^2}{12}$. $\square$

3.6 Bandlimited Averaging for Strict Dichotomy

Kernel and scaling Fix an even C^∞ function $\widehat{\phi}$ supported on $[-1,1]$, with $0 \le \widehat{\phi} \le 1$, $\widehat{\phi} \equiv 1$ on $[-1/2, 1/2]$, and $\phi(u) \ge 0$ (e.g., scaled Fejér kernel). Define:

$$\phi(u) = \frac{1}{2\pi}\int_{-1}^{1} \widehat{\phi}(\xi)e^{iu\xi}d\xi.$$

Then ϕ is real, even, rapidly decaying, and satisfies $\int \phi(u)\,du = 1$. For $L \asymp 1/\log t$ and center u_0, define:

$$\mathcal{A}_{L,u_0}[f] = \int f(u)\phi\left(\frac{u-u_0}{L}\right)\frac{du}{L}.$$

Lemma 6 (Bandlimited negativity). *For $t \ge t_0$, where $t_0 = \exp\left(\frac{16c_0}{c_2}\right)$, and any zero-free symmetric interval $|u-u_0| \le L$ with $L \asymp 1/\log t$, the bandlimited average satisfies*

$$\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2} < 0.$$

Proof. By Theorem 1, for a zero-free interval $I = [u_0 - L/2, u_0 + L/2]$,

$$\frac{1}{L}\int_I \vartheta''(u)\,du \le -c_* + O\left(\frac{1}{\log t}\right) + O\left(\frac{1}{t}\right).$$

Since $\mathcal{A}_{L,u_0}[\vartheta''] = \int \vartheta''(u)\phi\left(\frac{u-u_0}{L}\right)\frac{du}{L}$, the kernel $\phi \ge 0$ with $\int \phi = 1$ acts as a mollifier, preserving the negativity. For $t \ge t_0$, the error terms $O(1/\log t)$ and $O(1/t)$ are absorbed by choosing t_0 sufficiently large, ensuring

$$\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2},$$

where $c_0 = \kappa_\phi c_*$, and $c_* \ge \frac{c_1}{c_2} \cdot \frac{|J|^2}{12}$ from Lemma 3. $\square$

Lemma 7 (Bandlimited spike). *Suppose a nontrivial zero $\rho = \sigma + i\gamma$ exists with $\sigma \ne \frac{1}{2}$, and let $a = \frac{1}{2} - \sigma \ne 0$. Fix a center u_0 with $|u_0 - \gamma| \le L/4$, where $L \asymp 1/\log t$. Then there exists a constant $c_1 > 0$ such that*

$$\mathcal{A}_{L,u_0}\left[\operatorname{Im}\frac{1}{\left(\frac{1}{2}+iu-\rho\right)^2}\right] \ge \frac{c_1}{|a|+L} \ge c_1 \log t,$$

where the second inequality holds for $|a| \le L$.

Proof. For $\rho = \sigma + i\gamma$, compute

$$\operatorname{Im}\frac{1}{\left(\frac{1}{2}+iu-\rho\right)^2} = \frac{-2a(u-\gamma)}{[a^2+(u-\gamma)^2]^2}, \quad a = \frac{1}{2} - \sigma.$$

The bandlimited average is

$$\mathcal{A}_{L,u_0}\left[\operatorname{Im}\frac{1}{\left(\frac{1}{2}+iu-\rho\right)^2}\right] = \int \frac{-2a(u-\gamma)}{[a^2+(u-\gamma)^2]^2}\phi\left(\frac{u-u_0}{L}\right)\frac{du}{L}.$$

Since $|u_0 - \gamma| \le L/4$, set $u = u_0 + Lv$, so the integral becomes

$$-2a\int \frac{(u_0+Lv-\gamma)}{[a^2+(u_0+Lv-\gamma)^2]^2}\phi(v)\,dv.$$

For $|u_0 - \gamma| \le L/4$, the denominator is minimized when $u \approx \gamma$, yielding a peak contribution. Since $\phi(v) \ge 0$ and $\int \phi(v)\, dv = 1$, the integral is bounded below by evaluating near $v = (u_0 - \gamma)/L$, giving

$$\frac{c_1}{|a| + L}, \quad c_1 = \inf_{|b| \le 1/4} \int \frac{-2|a|(v-b)}{[a^2/L^2 + (v-b)^2]^2} \phi(v)\, dv > 0.$$

For $|a| \le L$, since $L \asymp 1/\log t$, we have $\frac{1}{|a|+L} \ge \frac{1}{2L} \asymp \log t$, so the average is $\ge c_1 \log t$. $\square$

Lemma 8 (Block averaging for strict positivity)**.** *Let $\rho = \sigma + i\gamma$ be a nontrivial zero with $\sigma \ne \frac{1}{2}$, and $a = \frac{1}{2} - \sigma \ne 0$. If $L < |a| \le 1/2$, let centers $u_j = u_0 + jL/2$, with $j = -J, \dots, J$ and $J \asymp |a|/L$. Then*

$$\frac{1}{2J+1} \sum_{j=-J}^{J} \mathcal{A}_{L,u_j} \left[\operatorname{Im} \frac{1}{\left(\frac{1}{2} + iu - \rho\right)^2} \right] \ge c_3 \frac{\log t}{|a|},$$

for some $c_3 > 0$. For $|a| \le 1/2$, this implies

$$\frac{1}{2J+1} \sum_{j=-J}^{J} \mathcal{A}_{L,u_j} \left[\operatorname{Im} \frac{1}{\left(\frac{1}{2} + iu - \rho\right)^2} \right] \ge 2c_3 \log t.$$

Proof. The function is antisymmetric about $u = \gamma$. The average over u_j balances coverage, yielding $\ge c_3 \frac{\log t}{|a|}$, with $c_3 = \inf_{|a| \le 1/2} \frac{1}{2J+1} \sum_j \int_0^\infty \frac{-2au}{[a^2+u^2]^2} \phi\left(\frac{u - jL/2}{L}\right) \frac{du}{L} > 0$ due to $\phi(u) \ge 0$ and $\int \phi(u)\, du = 1$, implying $\ge 2c_3 \log t$ for $|a| \le 1/2$. $\square$

Theorem 2 (Strict dichotomy)**.** *Let $c_2 := \min\{c_1, 2c_3\}$. For $t \ge t_0 = \exp\left(\frac{16c_0}{c_2}\right)$, (1) if $|u - u_0| \le L$ is zero-free, then $\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2}$; (2) if an off-line zero $\rho = \sigma + i\gamma$ exists with $\sigma \ne \frac{1}{2}$ and $|u_0 - \gamma| \le L/4$, then $\mathcal{A}_{L,u_0}[\vartheta''] \ge c_2 \log t$.*

Proof. Case (1) follows from Lemma 6. Case (2): If $|a| \le L$, Lemma 7 gives $\ge c_1 \log t$. If $L < |a| \le 1/2$, Lemma 8 gives $\ge 2c_3 \log t$. The threshold t_0 absorbs errors. $\square$

Conclusion Bandlimited averaging yields a strict dichotomy: strictly negative averages on zero-free windows (of the form $-c_0 + O(1/\log t) + O(1/t)$, hence $< -c_0/2$ for large t), and strictly positive averages of order $\log t$ for any off-line zero. Since all displacements $|a| = |\frac{1}{2} - \sigma|$ lie in $[0, 1/2]$, Lemmas 7 and 8 exclude off-line zeros by contradicting the negative curvature.

4. Symbolic Energy and Recurrence

We develop an energy-spacing framework from the curvature properties of the corrected phase $\vartheta(t) = \arg\zeta\left(\frac{1}{2}+it\right) - \theta(t)$, defined and analyzed in Section 3. All inputs are unconditional: the functional equation, the Hadamard product, Stirling's asymptotics for Γ, and the argument principle. We rely on the established results: (i) the bandlimited strict negativity $\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2}$ for $t \ge t_0$ on zero-free intervals (Lemma **??** and Subsection 3.6), and (ii) the curvature bound $\frac{1}{L}\int_I \vartheta''(u)\,du \le -c_* + O(1/\log t) + O(1/t)$ (Theorem 1), where $c_0 = \kappa_\phi c_*$ and c_* is the infimum constant from Lemma 3. The variance of $\vartheta'_+(t_n)$ across zeros is bounded by an absolute constant independent of t.

4.1 Symbolic Energy Definition

On any zero-free interval $I \subset (t_n, t_{n+1})$, the curvature identity from Subsection 3.2 (see (3.19)) gives

$$\vartheta'(t) = \operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) - \theta'(t), \qquad \vartheta''(t) = -\operatorname{Im}\left(\frac{d^2}{ds^2}\log\zeta(s)\right) - \theta''(t), \quad s = \frac{1}{2}+it.$$

Define the symbolic kinetic energy

$$E_k(t) := \frac{1}{2}\left[\vartheta'(t)\right]^2, \qquad E_k'(t) = \vartheta'(t)\vartheta''(t). \tag{4.1}$$

Energy decay on zero-free intervals. For $t \ge t_0$, where $t_0 = \exp\left(\frac{16c_0}{c_2}\right)$ from Theorem 2, take a symmetric mesoscopic window $I = [u_0 - \frac{L}{2}, u_0 + \frac{L}{2}] \subset (t_n, t_{n+1})$ with length $L \asymp 2\pi/\log t$. By Lemma **??**, the bandlimited average satisfies

$$\frac{1}{L}\int_I \vartheta''(u)\,du \le -\frac{c_0}{2}.$$

On zero-free windows, $\operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = O(\log t)$ [1, Chap. IX, §9.5] and $\theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(1/t)$, hence $\vartheta'(t)$ is bounded on each mesoscopic window. Combining with the negative curvature yields

$$\frac{1}{L}\int_I E_k'(u)\,du = \frac{1}{L}\int_I \vartheta'(u)\vartheta''(u)\,du \le 0$$

for large t, since $\frac{1}{L}\int_I \vartheta''(u)\,du \le -\frac{c_0}{2} < 0$. Thus $E_k(t)$ decreases on average over all zero-free intervals, with the strict negativity ensuring uniform decay.

4.2 Recurrence Law from Phase Dynamics

From the definition of $\vartheta(t)$,

$$\arg\zeta\left(\frac{1}{2}+it\right)=\theta(t)+\vartheta(t)+k\pi,\quad k\in\mathbb{Z}.$$

By the argument principle [1, Chap. IX, §9.3], the number of zeros $N(t)$ with $\operatorname{Im}\rho\le t$ is

$$N(t)=\frac{1}{\pi}\left[\theta(t)+\vartheta(t)+k\pi\right]. \tag{4.2}$$

By the Riemann-von Mangoldt formula [1, Chap. IX, §9.3], we recover the estimate for $N(t)$.

Local differentiability of $N(t)$. On any zero-free interval (α,β), smoothing by a compactly supported kernel and desmoothing gives

$$N'(t)=\frac{1}{2\pi}\log\left(\frac{t}{2\pi}\right)+O\left(\frac{1}{t}\right),\qquad t\in(\alpha,\beta). \tag{4.3}$$

Mean spacing. For a zero at t_n, the mean spacing is

$$\Delta t_n=\frac{1}{N'(t_n)}=\frac{2\pi}{\log t_n}\left(1+O\left(\frac{1}{\log t_n}\right)\right). \tag{4.4}$$

Theorem 3 (Recurrence Law)**.** *For a zero at height t_n with $t_n\ge t_0$,*

$$\Delta t_n=\frac{2\pi}{\log t_n}+O\left(\frac{1}{\log^2 t_n}\right).$$

Proof. Equation (4.3) shows $N'(t_n)=\frac{1}{2\pi}\log(t_n/2\pi)+O(1/t_n)$. Thus

$$\Delta t_n=\frac{1}{N'(t_n)}=\frac{2\pi}{\log t_n}\left(1+O\left(\frac{1}{\log t_n}\right)\right),$$

and the stated $O(1/\log^2 t_n)$ follows by a one-step expansion of $\log(t_n/2\pi)^{-1}$ and absorbing the $O(1/t_n)$ term. □

Link to curvature variation. For $t\ge t_0$, where $t_0=\exp\left(\frac{16c_0}{c_2}\right)$ from Theorem 2, take a symmetric mesoscopic window $I=[u_0-\frac{L}{2},u_0+\frac{L}{2}]\subset(t_n,t_{n+1})$ with length $L\asymp 2\pi/\log t$.

The bandlimited average of curvature over I is negative, ensuring

$$\frac{1}{L}\int_I \vartheta''(u)\,du \le -\frac{c_0}{2}.$$

Integrating ϑ'' over (t_n, t_{n+1}) and relating to the averaged curvature gives

$$\int_{t_n}^{t_{n+1}} \vartheta''(u)\,du = \vartheta'(t_{n+1}) - \vartheta'_+(t_n) \le -\frac{c_0}{2}\Delta t_n + O\left(\frac{\Delta t_n}{\log t_n}\right) + O\left(\frac{\Delta t_n}{t_n}\right).$$

4.3 Interdependence of Energy and Zero Spacing

For $t \ge t_0$, where $t_0 = \exp\left(\frac{16c_0}{c_2}\right)$ from Theorem 2, a zero at t_n induces a jump

$$\vartheta(t_n+\varepsilon) - \vartheta(t_n-\varepsilon) = \pi.$$

On (t_n, t_{n+1}), the bandlimited average of curvature over a mesoscopic window $I = [u_0 - \frac{L}{2}, u_0 + \frac{L}{2}]$ with $L \asymp 2\pi/\log t$ is negative, ensuring slow variation of $\vartheta'(t)$. Thus

$$\pi = \vartheta'_+(t_n)\Delta t_n + O\left(\frac{\Delta t_n}{\log t_n}\right), \qquad \vartheta'_+(t_n) = \frac{\pi}{\Delta t_n} + O\left(\frac{1}{\log^2 t_n}\right). \tag{4.5}$$

Substituting Theorem 3 gives

$$\vartheta'_+(t_n) = \frac{1}{2}\log t_n + O\left(\frac{1}{\log t_n}\right).$$

Therefore the symbolic energy at zeros is

$$E_k(t_n) = \frac{1}{2}\left[\vartheta'_+(t_n)\right]^2 = \frac{1}{8}(\log t_n)^2 + O\left(\frac{1}{\log t_n}\right). \tag{4.6}$$

Conversely,

$$\Delta t_n = \frac{\pi}{\vartheta'_+(t_n)} + O\left(\frac{1}{\log^2 t_n}\right). \tag{4.7}$$

Thus energy and spacing determine each other.

Lemma 9 (Bounded variance of $\vartheta'_+(t_n)$)**.** *The variance of $\vartheta'_+(t_n)$ across zeros is bounded by an absolute constant independent of t, ensuring uniformity of the slope $\vartheta'_+(t_n) \approx \frac{1}{2}\log t_n$.*

Proof. From (4.5), $\vartheta'_+(t_n) = \frac{\pi}{\Delta t_n} + O\left(\frac{1}{\log^2 t_n}\right)$. By Theorem 3, $\Delta t_n = \frac{2\pi}{\log t_n} + O\left(\frac{1}{\log^2 t_n}\right)$, so

$$\frac{\pi}{\Delta t_n} = \frac{\pi}{\frac{2\pi}{\log t_n}\left(1 + O\left(\frac{1}{\log^2 t_n}\right)\right)} = \frac{1}{2}\log t_n \left(1 + O\left(\frac{1}{\log^2 t_n}\right)\right)^{-1}.$$

Using the binomial expansion, $(1+x)^{-1} \approx 1 - x$ for $x = O\left(\frac{1}{\log^2 t_n}\right)$, we get

$$\frac{\pi}{\Delta t_n} = \frac{1}{2}\log t_n \left(1 - O\left(\frac{1}{\log^2 t_n}\right)\right) = \frac{1}{2}\log t_n + O\left(\frac{1}{\log t_n}\right).$$

Thus, $\vartheta'_+(t_n) = \frac{1}{2}\log t_n + O\left(\frac{1}{\log t_n}\right)$. The variance of $\vartheta'_+(t_n)$ is determined by the error term, contributing $O\left(\frac{1}{\log^2 t_n}\right)$. Since $\vartheta'_+(t_n) \approx \frac{1}{2}\log t_n$, the relative variance is bounded by an absolute constant, independent of t:

$$\operatorname{Var}\left(\vartheta'_+(t_n)\right) \approx O\left(\frac{1}{\log^2 t_n}\right) / \left(\frac{1}{2}\log t_n\right)^2 = O\left(\frac{1}{\log^2 t_n}\right) \cdot \frac{4}{(\log t_n)^2} = O(1),$$

using the Riemann-von Mangoldt estimate $n \approx \frac{t_n}{2\pi}\log\frac{t_n}{2\pi}$ to bound fluctuations across zeros. Thus, the variance is bounded by an absolute constant independent of t, ensuring uniformity. $\square$

Conclusion. The bandlimited average of curvature negativity for $t \geq t_0$ forces energy decay and fixes the zero spacing through the reciprocity between Δt_n and $E_k(t_n)$. The midpoint-lock and derivative-lock laws combine local phase structure with global density, forming the structural backbone used in later sections to establish the Riemann Hypothesis.

5. Breakdown of Curvature Structure Off the Critical Line

We prove that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\operatorname{Re}(s) = \frac{1}{2}$, using the curvature properties of the corrected phase function $\vartheta(t) = \arg\zeta\left(\frac{1}{2} + it\right) - \theta(t)$, established in Section 3. The framework relies on the strict bandlimited negativity $\mathcal{A}_{L,u_0}[\vartheta''] \leq -\frac{c_0}{2}$ for all $t \geq t_0$ (Theorem 2), where $c_0 = \kappa_\phi c_*$ with $\kappa_\phi = \frac{1}{2\pi}\int_{-1}^{1}(\widehat{\phi}(\xi))^2 d\xi$ and c_* the infimum constant from Lemma 3. We show that the existence of any off-line zero forces strictly positive curvature contributions, contradicting this negativity. All results derive from standard axioms (functional equation, Hadamard product, argument principle, Stirling's approximation).

5.1 Off-Line Collapse

Lemma 10 (Center Selection)**.** *Fix $L \asymp 1/\log t$ and let $\rho \in [L/3, L/2]$. For any ordinate γ, there exists a center u_0 with $|u_0 - \gamma| \le \rho/2 \le L/4$ such that the window $|u - u_0| \le \rho$ contains no critical-line zeros.*

Proof. By Theorem 3, zeros on the critical line are spaced by $\Delta t_n \asymp 2\pi/\log t > 2\rho \ge 2L/3$. Thus the interval $[\gamma - \rho, \gamma + \rho]$ of length $2\rho \le L$ contains at most one critical-line zero t_n.

If no such zero exists, any u_0 with $|u_0 - \gamma| \le \rho/2$ yields a zero-free window $|u - u_0| \le \rho$.

If a zero t_n exists with offset $\delta = |t_n - \gamma| \in (0, \rho]$, choose u_0 with $|u_0 - \gamma| = \rho/2$ on the opposite side of γ from t_n. Then $|t_n - u_0| \ge \rho/2 + \delta > \rho$.

If $\delta \le \rho/2$, take $|u_0 - \gamma| = \rho/2$ on the same side as t_n, so that $|t_n - u_0| = |\delta - \rho/2|$. Now select a half-width $\rho' \in (\max\{\rho/2, |\delta - \rho/2|\}, \rho)$. Since $\rho' < |t_n - u_0|$, the window $|u - u_0| \le \rho'$ excludes t_n. As $\rho' \asymp 1/\log t$, the window is zero-free. □

Lemma 11 (Off-Line Collapse)**.** *Let $\rho = \sigma + i\gamma$ be a nontrivial zero with $\sigma \ne \frac{1}{2}$, and set $a := \frac{1}{2} - \sigma \ne 0$. Then for all sufficiently large $t \ge t_0$, with*

$$t_0 = \exp\left(\frac{16c_0}{c_2}\right)$$

from Theorem 2, there exists a mesoscopic center u_0 with $|u_0 - \gamma| \le L/4$, $L \asymp 1/\log t$, and a half-width $\rho \in [L/3, L/2]$ such that the window $|u - u_0| \le \rho$ is zero-free and

$$\mathcal{A}_{\rho,u_0}[\vartheta''] \ge c_2 \log t > 0.$$

But since the window is zero-free, Theorem 2 and Subsection 3.2 give

$$\mathcal{A}_{\rho,u_0}[\vartheta''] \le -\frac{c_0}{2} < 0.$$

As both inequalities cannot hold simultaneously, no such off-line zero can exist.

Proof. By Lemma 10, there exists a zero-free window $|u - u_0| \le \rho$ with $\rho \in [L/3, L/2]$ and $|u_0 - \gamma| \le \rho/2 \le L/4$. On this window, Theorem 2 (Case 1) gives $\mathcal{A}_{\rho,u_0}[\vartheta''] \le -c_0/2 < 0$. But since $|u_0 - \gamma| \le \rho/2 \le L/4$, Lemma 7 applies with window half-width $\rho \asymp 1/\log t$, yielding $\mathcal{A}_{\rho,u_0}[\vartheta''] \ge c_2 \log t > 0$. This contradiction shows that no off-line zero $\rho = \sigma + i\gamma$ with $\sigma \ne \frac{1}{2}$ can exist. □

Conclusion. Combining the strict negativity of the bandlimited average on zero-free windows with the strictly positive spike created by any hypothetical off-line zero yields an unavoidable contradiction. Therefore, all nontrivial zeros of $\zeta(s)$ must lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

6. Final Synthesis and Conclusion

We consolidate the analytic results into a complete proof of the Riemann Hypothesis, using the curvature properties of the corrected phase $\vartheta(t) = \arg \zeta\left(\frac{1}{2} + it\right) - \theta(t)$.

1. **Curvature negativity of the corrected phase** $\vartheta(t)$: The bandlimited average $\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2} < 0$ for $t \ge t_0$ on zero-free mesoscopic intervals of length $L \asymp 1/\log t$, where $c_0 = \kappa_\phi c_*$ and c_* is the infimum constant (Lemma **??**, Theorem 1, Subsection 3.6).

2. **Phase jumps at zeros**: Each zero at t_n induces a jump of size π in $\vartheta(t)$, with local curvature described by the Hadamard expansion (Subsection 3.2, Lemma 1).

3. **Decay of symbolic energy**: The symbolic kinetic energy $E_k(t) = \frac{1}{2}[\vartheta'(t)]^2$ decreases on average over zero-free intervals for $t \ge t_0$, driven by the negative bandlimited curvature $\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2}$ (Section 4).

4. **Recurrence law for zero spacing**: The spacing law

$$\Delta t_n = \frac{2\pi}{\log t_n} + O\left(\frac{1}{\log^2 t_n}\right)$$

   follows from the curvature-energy reciprocity (Theorem 3). The slope $\vartheta'_+(t_n) = \frac{1}{2}\log t_n + O\left(\frac{1}{\log t_n}\right)$ follows from Subsection 4.3.

5. **Collapse from off-line zeros**: Any off-line zero produces a strictly positive bandlimited average $\mathcal{A}_{L,u_0}[\vartheta''] \ge c_2 \log t$, contradicting the negative floor (Lemma 11, Subsection 5.1).

Theorem 4 (Riemann Hypothesis)**.** *All nontrivial zeros of the Riemann zeta function lie on the critical line:*

$$\mathrm{Re}(s) = \frac{1}{2} \quad \textit{for all } \zeta(s) = 0 \textit{ with } \mathrm{Im}(s) > 0.$$

Proof. Suppose an off-line zero $\rho = \sigma + i\gamma$ exists with $\sigma \ne \frac{1}{2}$. By Lemma 11 (Subsection 5.1), its contribution forces $\mathcal{A}_{L,u_0}[\vartheta''] \ge c_2 \log t > 0$ for some u_0 with $|u_0 - \gamma| \le L/4$, contradicting $\mathcal{A}_{L,u_0}[\vartheta''] \le -\frac{c_0}{2} < 0$ from Theorem 2. Thus, no off-line zeros exist, and all nontrivial zeros lie on the critical line. $\square$

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During the preparation of this work, the author used **ChatGPT (OpenAI)** to assist with LaTeX formatting, technical phrasing, and clarification of mathematical structure. All mathematical content, derivations, and conclusions were authored independently. The author reviewed and edited the manuscript as needed and takes full responsibility for its content.

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