Let us now briefly explain how to compute the constant e', and more generally, how to accurately estimate the coefficients $u_0, u_1, \ldots, u_k$ in the singular expansion of $U(z)$ to any fixed order k:

$$U(z) = u_0 + u_1 Z + \cdots + u_k Z^k + o_{z\to\rho}(Z^k),$$

with $Z = \sqrt{1 - z/\rho}$. The first step is to estimate the coefficients in the singular expansion of $S_X(z)$, of the form

$$S_X(z) = c_0 + c_1 Z + \cdots + c_k Z^k + o_{z\to\rho}(Z^k),$$

For any fixed m (with the notations $F^{[m]}(z, y), \rho^{[m]}, \tau^{[m]}$ introduced above), we let $y^{[m]} := \tau^{[m]} + c_1^{[m]} Z + c_2^{[m]} Z^2 + c_3^{[m]} Z^3 + \cdots + c_k^{[m]} Z^k$, and consider the equation

$$-y^{[m]} + F^{[m]}(\rho^{[m]} \cdot (1 - Z^2), y^{[m]}) = 0,$$

which we expand order by order in Z, each coefficient $[Z^i]$ in $H := -y^{[m]} + F^{[m]}(\rho^{[m]} \cdot (1 - Z^2), y^{[m]})$ being a certain polynomial expression in $c_1^{[m]}, \ldots, c_k^{[m]}$. As it turns out, the coefficient $[Z^0]H$ and $[Z^1]H$ are 0, the coefficient $[Z^2]H$ is of the form $\frac{1}{2}(c_1^{[m]})^2 - a$ with $a \approx 1.46797$, which gives $c_1^{[m]} = -\sqrt{2a} \approx -1.71346$, and then for $3 \le i \le k+1$ the coefficient $[Z^i]H$ is of the form $c_1^{[m]} c_{i-1}^{[m]} - P_i(c_1^{[m]}, \ldots, c_{i-2}^{[m]})$ for a certain explicit polynomial P_i. This allows us to solve iteratively for the constants $c_2^{[m]}, c_3^{[m]}, \ldots$; we find $c_2^{[m]} \approx 1.45297$, $c_3^{[m]} \approx -0.33156$, etc, and we observe exponentially fast convergence as m increases.

Then, to obtain the coefficients u_i, we simply use the explicit expression $U(z) = G(z, S_X(z))$, which ensures that the singular expansion of $U(z)$ is the same as the singular expansion of $S_X(z) \cdot \left(1 - \frac{z}{1-z} - \frac{1}{2} S_X(z)\right)$. Expanding order by order in Z, we find that each u_i is a polynomial expression in $c_1, \ldots, c_i$, which allows us to compute the u_i's from the c_i's, giving $e' = u_3 \approx 1.67688$.