Social Choice Functions

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Outline

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What are Boolean Functions

- $f : \{-1, 1\}^n \to \{-1, 1\}$
- They can be thought of as a voting rule or a social choice function for an election with 2 candidates and n voters.

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Some Social Choice Fns.

Suppose $x \in \{-1, 1\}^n$

- Majority : Returns the more frequent entry in x.
  $Maj_n(x) = sgn(x_1 + \ldots + x_n)$
- Dictator : Returns the i-th coordinate.
  $\chi_i(x) = x_i$

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- The tribes function of width w and size s,
  $Tribes_{w,s}(x^{(1)}, \ldots, x^{(s)}) =$
  $OR_s(AND_w(x^{(1)}), \ldots, AND_w(x^{(s)}))$

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Properties Desired in Voting Functions

We say that a function $f : \{-1, 1\}^n \to \{-1, 1\}$ is :

- monotone if $f(x) \leq f(y)$ whenever $x \leq y$ coordinate-wise.
- odd if $f(x) = -f(-x)$
- unanimous if $f(1, \ldots, 1) = 1$ and $f(-1, \ldots, -1) = -1$.
- transitive-symmetric if $\forall i, i' \in [n], \exists \pi \in S_n$ taking i to i' such that $f(x) = f(x^\pi)$ for all $x \in \{-1, 1\}^n$.
  Stronger Condition : A function is called symmetric if $f(x) = f(x^\pi)$ for all permutations $\pi \in S_n$

Properties Desired in Voting Functions

Some Remarks :

- Another naturally desirable property of a 2-candidate voting rule is that its unbiased i.e. "equally likely" to elect ± 1.
- We also might assume that voter preferences are independent and uniformly random.

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Influence

Definition

The coordinate $i \in [n]$ is pivotal for $f : \{-1,1\}^n \to \{-1,1\}$ on input x if $f(x) \neq f(x^{\oplus i})$ where $x^{\oplus i} = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$.

Definition

The influence of a coordinate i on $f : \{-1,1\}^n \to \{-1,1\}$ is defined as the probability that i is pivotal for a random input :

$$Inf_i[f] = Pr_{x \sim \{-1,1\}^n}[f(x) \neq f(x^{\oplus i})]$$

A toy example

For the i-th dictator function χ_i, we have that the coordinate i is pivotal for every input x; hence $Inf_i[\chi_i] = 1$. If $j \neq i$, then the coordinate j is never pivotal; hence $Inf_j[\chi_i] = 0$.

Derviatives

Definition

The *i*-th derivative operator D_i maps the function $f : \{-1, 1\}^n \to \mathbb{R}$ to the function $D_i f(x) : \{-1, 1\}^n \to \mathbb{R}$ and is defined by :

$$D_i f(x) = \frac{f(x^{(i\to 1)}) - f(x^{(i\to -1)})}{2}$$

.

Remark : D_i is a linear operator, so $D_i(f + g) = D_i f + D_i g$.

Derviatives

- Note that if $f : \{-1,1\}^n \to \{-1,1\}$, then:

$$D_i f(x) = \begin{cases} 0 & \textit{if coordinate } i \textit{ is not pivotal for } x \\ \pm 1 & \textit{if coordinate } i \textit{ is pivotal for } x \end{cases}$$

- Thus $D_i f(x)^2$ is the $0-1$ indicator for whether i is pivotal for x.
- $Inf_i[f] = E_x[D_i f(x)^2]$

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Fourier Expansion

- The Fourier expansion of a Boolean function $f : \{-1, 1\}^n \to \mathbb{R}$ is simply its representation as a real, multilinear polynomial.
- The multilinear polynomial for f may have upto 2^n terms corresponding to $S \subseteq [n]$. The monomial corresponding to S is written as

$$x^S = \Pi_{i \in S} x_i$$

  and $x^\varnothing = 1$.
- Every function can be uniquely expressed as a multilinear polynomial,

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) x^S$$

where $\hat{f}(S)$ is the fourier coefficient corresponding to $S \subseteq [n]$.

Fourier Expansion

Example

Consider the boolean function that returns the "maximum" of two bits : $max_2(+1, +1) = +1$, $max_2(+1, -1) = +1$, $max_2(-1, +1) = -1$, $max_2(-1, -1) = -1$.
It can also be represented as a multilinear polynomial, $max_2(x_1, x_2) = \frac{1}{2} + \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_1x_2$. Now, we can simply "read off" the fourier coefficients of max_2, as $max_2(\emptyset) = \frac{1}{2}$, $max_2(\{x_1\}) = \frac{1}{2}$, $max_2(\{x_2\}) = \frac{1}{2}$, $max_2(\{x_1, x_2\}) = -\frac{1}{2}$

A formula for fourier coefficients

- The functions of the form $f : \{-1,1\}^n \to \mathbb{R}$ make a vector space of dimension 2^n over R.
- This is a vector space of dimension 2^n and since the 2^n "functions" of the form x^S (for all $S \subseteq [n]$) span the vector space (as evidenced by the fourier expansion), they form a basis for the vector space.
- We can also define the inner product $\langle f, g\rangle = E_{x\in\{-1,1\}^n}[f(x)g(x)]$.
  This gives us a natural formulation for for

$$\hat{f}(S) = \langle f, x^S\rangle = E_{x\in\{-1,1\}^n}[f(x)x^S]$$

  which follows from the fact that the basis formed by x^S's is orthonormal.

A formula for fourier coefficients

Parseval's Theorem : For any $f : \{-1,1\}^n \to \mathbb{R}$,

$$\langle f, f \rangle = E_{x \in \{-1,1\}^n}[f(x)^2] = \sum_{S \subseteq [n]} \hat{f}(S)^2$$

.

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Influence in terms of Fourier Expansion

- D_i acts as a formal differentiation on the fourier expansion.

Theorem

Let $f : \{-1,1\}^n \to \{-1,1\}$, *have the fourier expansion* $f(x) = \sum_{S \subseteq [n]} \hat{f}(S) x^S$. *Then :*

$$D_i f(x) = \sum_{S \subseteq [n], S \ni i} \hat{f}(S) x^{S \setminus \{i\}}$$

.

- Now if we apply Parseval's Thm to the previous expression, we obtain that $Inf_i[f] = \sum_{S \ni i} \hat{f}(S)^2$.

Small Influences are "good"

Theorem

Let $f : \{-1,1\}^n \to \{-1,1\}$ *be transitive-symmetric and monotone. Then* $Inf_i[f] \leq 1/\sqrt{n}$ *for all* $i \in [n]$.

Remark : Both the majority function and the tribes function are monotone and transitive symmetric. For the majority function $Inf_i[Maj_n] \sim \frac{\sqrt{2/\pi}}{\sqrt{n}}$ for large n, whereas $Inf_i[Tribes_n] = \frac{ln(n)}{n}(1 \pm o(1))$.

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What do we want in a "good" voting function?

- We want that the function is monotone, odd, unanimous and symmetric. We might also want that it is unbiased.
- According to Rousseau, the ideal voting rule is one which maximizes the number of votes which agree with the outcome.

Theorem

Let $f : \{-1,1\}^n \to \{-1,1\}$ *be a voting rule for a 2-candidate election. Given votes* $x = (x_1, \ldots, x_n)$*, let* w *be the number of votes that agree with the outcome of the election,* $f(x)$*. Then:*

$$E[w] = \frac{n}{2} + \frac{1}{2}\sum_{i=1}^{n} \hat{f}(i)$$

Majority works in 2-party elections

- The only monotone, odd and symmetric boolean functions is the Majority function.
- The unique maximisers of $\sum_{i=1}^{n} \hat{f}(i)$ among all $f : \{-1, 1\}^n \to \{-1, 1\}$ are the majority functions. .

What if we have ≥ 3 parties?

- In his 1785 Essay on the Application of Analysis to the Probability of Majority Decisions, Condorcet suggested using the voters preferences to conduct the three possible pairwise elections, a vs. b, b vs. c, and c vs. a.
- Each individual election conducted through a 2-candidate voting rule. Condorcet suggested using Majority but we could technical use any suitable voting function.

Condorcet Election

What does it look like?

	Voters' Preferences #1	#2	#3	$\cdots$		Societal Aggregation
a (+1) vs. b (−1)	$+1$	$+1$	-1	$\cdots$	$= x$	$f(x)$
b (+1) vs. c (−1)	$+1$	-1	$+1$	$\cdots$	$= y$	$f(y)$
c (+1) vs. a (−1)	-1	-1	$+1$	$\cdots$	$= z$	$f(z)$

Condorcet Winner

- In an election employing Condorcets method with $f : \{-1, 1\}^n \to \{-1, 1\}$, we say that a candidate is a Condorcet winner if it wins all of the pairwise elections in which it participates.
- This lack of a Condorcet winner is termed Condorcets Paradox; it occurs when the outcome $(f(x), f(y), f(z))$ is one of the two all-equal triples $\{(-1, -1, -1), (1, 1, 1)\}$.

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Arrows'Theorem

- There might be some other function $f : \{-1,1\}^n \rightarrow \{-1,1\}$ than Majority that allows for the possibility of Condorcet Winner no matter what the "votes".
- Arrow's Theorem : Suppose $f : \{-1,1\}^n \rightarrow \{-1,1\}$ is a unanimous voting rule used in a 3-candidate Condorcet election. If there is always a Condorcet winner, then f must be a dictatorship.

Kalai's Proof of Arrow's Theorem

- Kalai's Theorem (?) : Consider a 3-candidate Condorcet election using an $f : \{-1,1\}^n \to \{-1,1\}$. Under the impartial culture assumption, the probability of a condorcet winner is precisely $\frac{3}{4} - \frac{3}{4}Stab_{-1/3}[f]$.
- Arrow's Theorem is a simple corollary. An advantage of Kalai's analytic proof of Arrow's Theorem is that we can deduce several more interesting results about the probability of a Condorcet winner :
    - Guilbaud's Formula: In a 3-candidate Condorcet election using Majority, the probability of a condorcet winner tends to 91.2% as $n \to \infty$.
    - Suppose that in a 3 - candidate Condorcet election using $f : \{-1,1\}^n \to \{-1,1\}$, the probability of a Condorcet winner is $1 - \varepsilon$. Then f is $O(\varepsilon)$ close to $\pm\chi_i$ for some $i \in [n]$.

Sources I

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Analysis of Boolean Functions.
Avaialble to Download Online for FREE!

Gil Kalai
A Fourier-Theoretic Perspective on the Condorcet Paradox and Arrows's Theorem.
http://www.cs.huji.ac.il/~noam/econcs/arr.pdf

Range Voting and Arrow's Theorem.
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