ERIC RAMOS RESEARCH STATEMENT

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My primary research focus is in algebraic combinatorics and its applications to topology and representation theory. In particular, I am interested in applying the techniques of the representation theory of categories in an attempt to understand the mechanisms underlying various well known asymptotic stability phenomena from combinatorics, topology, and other related fields. In the following research statement, I will outline four major projects that currently drive my research program. These projects aim to further the state of the art in the representation theory of certain combinatorially flavored categories, and then use this advancement to solve concrete problems naturally arising from algebraic and topological combinatorics. One should note that the following statement is more a summary of my entire research program, and therefore does not include some of my older works, as well as a wealth of alternative smaller questions which I am currently working on. To see my unabridged research statement, please visit my webpage at **ADD LINK**.

Before we provide formal definitions, we begin with an example for motivation. For any finite set T, and any topological space X, we write

$$\mathrm{Conf}_T(X) := \{(x_t)_{t\in T} \mid x_t \neq x_{t'}\},$$

to denote the topological space of injections from T to X. If T' is any other finite set, and there is an injection $T \hookrightarrow T'$, then precomposition naturally defines a map of topological spaces $\mathrm{Conf}_{T'}(X) \to \mathrm{Conf}_T(X)$. For any fixed i, the cohomology groups $H_i(\mathrm{Conf}_\bullet(X); k)$ therefore inherit an interesting structure. Namely, for each n $H_i(\mathrm{Conf}_n(X); k)$ is a $k[\mathfrak{S}_n]$-module, and between any two values of n the corresponding modules are in some sense compatible. This can be thought of as the motivating philosophy for the study of what we will call FI-modules: a means of encoding an infinite collection of compatible symmetric group representations into a single object. Using this structure, Church, Ellenberg, and Farb were able to prove many non-trivial facts about the configuration spaces of orientable manifolds [CEF].

Write FI for the category of finite sets and injections. Note that we will usually think of this category equivalently as that whose objects are the sets $[n] = \{1, \ldots, n\}$ and whose morphisms are injections. An **FI-module** over a commutative ring k is a functor from FI to the category of k-modules. These modules, as defined here, were first introduced by Church, Ellenberg, and Farb in their seminal paper [CEF]. Since their inception in the work of Church, Ellenberg, and Farb, FI-modules, and modules over other similar combinatorially flavored categories, have seen an enormous amount of use in topology, representation theory, and other fields. This has been the subject of a recent AIM workshop, an AMS special session, an Oberwolfach workshop, an MSRI summer school, and numerous other conferences.

The category of FI-modules over k, FI-Mod, is an abelian category with abelian operations defined point-wise. We say that V is **finitely generated** if there is a finite set $\{v_i\} \subseteq \bigsqcup_T V(T)$, that is contained in no proper submodule of V. This notion was first explored by Church, Ellenberg, and Farb in [CEF], although it was not fully explored until the follow-up work of Church, Ellenberg, Farb, and Nagpal [CEFN, Nagp]. In the paper [CEFN], it is shown that the category of finitely generated FI-modules over a Noetherian ring have a Noetherian property. That is, the category FI-mod of finitely generated FI-modules over a Noetherian ring is abelian. This fact was proven when k is a field of characteristic 0 by Church, Ellenberg, and Farb in [CEF] and by Snowden in [Sn]. It was proven in total generality by Church, Ellenberg, Farb, and Nagpal in [CEFN].

FI is the first, and perhaps most natural, example of what we will henceforth refer to as a **combinatorial category** - a category whose foundational structure is based in the combinatorics of some finite objects. Two of the projects proposed below involve applying our wealth of knowledge about FI-modules in new directions related to combinatorics and group theory. Not everything we discuss will be related with FI however, as another one of the objectives in my current research program is to push the boundaries of our understanding into the representation theory of combinatorial categories that are richer and less well understood than FI. We will encounter the primary such category in the next section - the category of graphs and contractions.

1. Categories of graphs

1.1. **Background.** Throughout this project description, A **graph** will always refer to an at most 1-dimensional CW-complex that is both connected and finite. Given a graph G we can write $V(G)$ for its set of **vertices** - or 0-cells - and $E(G)$ for its set of **edges** - or 1-cells. A **contraction** from a graph G to a graph G' is a topological map defined by selecting a sub-forest within the graph G, and contracting each component tree to obtain a new graph G'. Topology informs us that a contraction induces a homotopy equivalence between G and G', and thus necessarily $H_1(G) \cong H_1(G')$. We call the rank of $H_1(G)$ the **genus** of G. Finally, if $g \geq 0$ is an integer, we write $\mathcal{G}_g$ for the category whose objects are graphs with genus g, and whose morphisms are contractions. For instance, $\mathcal{G}_0$ is the category of trees and contractions. The primary object of study in what follows is the opposite category $\mathcal{G}_g^{op}$. In particular, we consider modules over the category $\mathcal{G}_g^{op}$.

Just as with our previous discussion of FI-modules, a $\mathcal{G}_g^{op}$-module over a Noetherian ring k is a functor from $\mathcal{G}_g^{op}$ to the category of k-modules. We say that a $\mathcal{G}_g^{op}$-module M is **finitely generated** if there is some finite list of genus g graphs $G_1, \ldots, G_N$ such that for any graph G of genus g, $M(G)$ is spanned by the images of $M(G_i)$ under the various transition maps induced from contractions. We usually refer to the graphs $\{G_i\}$ as being the **generators** of the module M. The main technical theorem, that informs everything in the sequel, is the following.

Theorem 1.1 (Proudfoot & Ramos [PR, PR2])**.** *Let k be a Noetherian ring, $g \geq 0$ be an integer, and let M be a finitely generated $\mathcal{G}_g^{op}$-module. Then all submodules of M are also finitely generated.*

One should note that Theorem 1.1 generalizes the aforementioned Noetherianity for FI-modules, as one can realize FI as the full subcategory of $\mathcal{G}_0^{op}$ of **star trees** - trees with a single vertex of degree ≥ 3. One may think about the relationship between these two statements as being analogous to the relationship between **Higman's Lemma** - that sequences within well-quasi-ordered posets are once again well-quasi-ordered - and **Kruskal's Tree Theorem** - that rooted trees are well-quasi-ordered by contractions.

1.2. **The graph minor category and applications of Noetherianity.** At the end of the previous section, we described how the relationship between Noetherianity of FI and $\mathcal{G}_g^{op}$ modules is analogous to the relationship between Higman's Lemma and Kruskal's Tree Theorem. It is then natural to ask whether there exists some master category $\mathcal{G}$, generalizing both $\mathcal{G}_g$ and FI, whose Noetherianity reflects the celebrated **Graph Minor Theorem** of Robertson and Seymour [RS]. Given two graphs G, G', we say that G is a **minor** of G' if G can be obtained from G' via a sequence of edge contractions and deletions. The Graph Minor Theorem states that the graph minor relation is actually a well-quasi-order.

In [MPR], Miyata, Proudfoot, and myself constructed the category $\mathcal{G}$, which we call the **graph minor category**. Objects in this category are graphs, while the morphisms are what we call **minor morphisms**. The formal definition of minor morphism is a bit technical, but one can think of them as being built out of edge deletions, contractions, and graph automorphisms. We have the following extension of Theorem 1.1.

Theorem 1.2 (Miyata, Proudfoot, & Ramos [MPR])**.** *If k is a Noetherian ring, and M is a finitely generated $\mathcal{G}^{op}$-module, then all submodules of M are also finitely generated.*

The Graph Minor Theorem was proven by Robertson and Seymour in 20 papers over the course of 20 years. One can show that Theorem 1.2 implies the graph minor theorem. This is not to say that [MPR] provides a new proof of the Graph Minor Theorem, as the Theorem and its proof are both inputs into the proof of Theorem 1.2, however it does suggest that Theorem 1.2 is at least as deep as the Graph Minor Theorem.

Just as was the case with FI-modules, Noetherianity of $\mathcal{G}^{op}$-modules can be leveraged to prove surprising facts in topology, as well as other fields. To illustrate these examples we will consider the configuration spaces of graphs. Configuration spaces of graphs, while not appearing much in classical literature, have recently seen a boom in their study. A later section of this statement will be dedicated to discussing the known theory of these spaces, and my contributions towards answering the natural question of, "what, if any, stable behaviors appear when one varies the number of points being configured n?" As these spaces relate to the category $\mathcal{G}^{op}$, however, we will instead be interested in the orthogonal question of, "what, if any, stable behaviors appear when one varies the graph and fixes the number of points?"

Theorem 1.3 (Miyata, Proudfoot & Ramos [PR, PR2, MPR])**.** *For all $i, n \geq 0$, the assignment*

$$G \mapsto H_i(\mathrm{Conf}_n(G))$$

can be extended to a finitely generated $\mathcal{G}^{op}$-module over $\mathbb{Z}$.

According to the definition of finite generation, the above theorem tells us that for any fixed $i, n \geq 0$, there is some finite list of generating graphs $G_1, \ldots, G_N$, such that for any graph G, every homology class in $H_i(\mathrm{Conf}_n(G))$ can be viewed as some combination of homology classes coming from $H_i(\mathrm{Conf}_n(G_j))$. One immediate consequence of this is the following.

Corollary 1.4 (Proudfoot & Ramos [PR2])**.** *For any fixed $i, n \geq 0$, there exists an integer $d_{i,n}$ such that for any graph G, the exponent of the abelian group $H_i(\mathrm{Conf}_n(G))$ is at most $d_{i,n}$.*

Shades of this boundedness of torsion had been observed in, for instance, works of Ko and Park [KP] and Chettih and Lütgehetmann [CL]. The above result of Miyata, Proudfoot and I is the first time it was made precise. While the study of torsion in the homology of topological spaces has its own intrinsic value, it appears that the torsion in graph configuration spaces is especially compelling. Indeed, it is known [KP] that torsion in the first homology groups of **unordered configuration spaces of graphs** - i.e. the quotient of configuration space by the natural symmetric group action - exactly detects planarity of the graph.

1.3. **The current project.** .

While there are a nearly limitless number of directions one can take with regards to structural concerns, there is one particular direction which I am currently most interested in pursuing. We have already seen results related with a kind of Hilbert series for modules over the contraction category of trees. Similar statements can be shown for finitely generated modules over a large number of related combinatorial categories such as FS^{op} - of finite sets and surjective maps - and FA - of finite sets and all set maps [SS2].

Question 1.5. What is the correct definition for the Hilbert series of a $\mathcal{G}_g^{op}$-module when $g > 0$? More generally, what is the correct definition for the Hilbert series of a $\mathcal{G}$-module? Recent work of Chan, Faber, Galatius, and Payne [CGP, CGP2, CFGP] has used a construction they call orbifold sums of graphs to great effect in studying moduli spaces of tropical curves. Can one modify this definition to fit our setting? If so, what kind of formal rationality statements can one prove about these Hilbert series?

I believe that understanding Hilbert series is an extremely important first step in understanding the overall structure of the representation theory of a category. Indeed, it both informs the best way forward with regard to other structural concerns, and is useful in leveraging the most that you can from a finite generation statement in applications.

As a proof of concept of the above question, let M be a finitely generated $\mathcal{G}_0^{op}$-module over a field k, the category of trees with contractions. It is a well known fact in combinatorics that to every **Dyck path** w, one may associate a tree $T(w)$, although this association is not one to one unless T is equipped with a choice of root and embedding into the plane. We define the Hilbert-Dyck series of the module M to be the formal power series

$$HD_M(t) := \sum_w \dim_k(M(T(w))t^{|T(w)|},$$

where the sum is over all Dyck paths, and $|T(w)|$ is the number of edges of $T(w)$ or, equivalently, half the length of the word w. For instance, if M is the module which assigns k to every tree, then $HD_M(t)$ is the generating function for the number of Dyck paths. That is, the generating function of the **Catalan Numbers**. The main result of [R8] generalizes one of the most famous results about this generating function.

Theorem 1.6 (Ramos [R8])**.** *For any finitely generated $\mathcal{G}_0^{op}$-module M over a field k, the Hilbert-Dyck series $HD_M(t)$ is algebraic.*

The combinatorics of algebraic generating functions implies a range of facts about the constituent pieces. In particular, one obtains an exponential bound on how these graded pieces can grow.

2. Families of highly symmetric graphs

2.1. **Background.** A **homomorphism** of graphs is a map between their vertex sets that preserves adjacency. An **FI-graph** is a functor from FI to the category of graphs and graph homomorphisms.

We say that an FI-graph $G_\bullet$ is **finitely generated** if, for all $n \gg 0$, every vertex of G_{n+1} appears in the image of some vertex of G_n under the action of FI. Examples of such families include the complete graphs, the Johnson and Kneser graphs, and the Crown graphs. The primary tool for working with finitely generated FI-graphs is the following.

Theorem 2.1 (Ramos & White [RW])**.** *Let $G_\bullet$ denote a finitely generated FI-graph. Then for $n \gg 0$:*

1. *if $f : [n] \hookrightarrow [n+1]$ is an injection of sets, then the induced map $G(f)$ is injective;*
2. *if $f : [n] \hookrightarrow [n+1]$ is an injection of sets, then the image of the induced map $G(f)$ is an induced subgraph of G_{n+1};*
3. *for any $r \geq 0$ and any collection of vertices $\{v_1, \ldots, v_r\}$ of G_{n+1}, there exists a collection of vertices $\{w_1, \ldots, w_r\}$ of G_n as well as an injection $f : [n] \hookrightarrow [n+1]$, such that $G(f)(\{w_1, \ldots, w_r\}) = \{v_1, \ldots, v_r\}$.*

The above theorem allows us to paint the following picture of what finitely generated FI-graphs look like: for $n \gg 0$ there is a chain of induced subgraphs $G_n \subseteq G_{n+1} \subseteq \ldots$ such that each G_n is equipped with an action of the symmetric group $\mathfrak{S}_n$, and these actions respect the inclusions in the obvious way. That is to say, a finitely generated FI-graph can be thought of as a family of nested graphs, equipped with compatible symmetric group actions.

2.2. **Applications of the theory.** To begin, we consider the case of the complete graphs, K_n. Note that if T is any fixed finite graph, then the number of distinct copies of T appearing as a subgraph of K_n can be easily counted, and it is seen to be a polynomial in n. This behavior is common among all finitely generated FI-graphs.

Theorem 2.2 (Ramos & White [RW]). *Let $G_\bullet$ be a finitely generated FI-graph, and let T be any fixed graph. Then for all $n \gg 0$, the following quantities each agree with a polynomial in n:*

1. *the number of copies of T appearing as a subgraph of G_n;*
2. *the number of copies of T appearing as an induced subgraph of G_n.*

As an immediate application of this theorem, we consider what it implies about the Hom-graph of an FI-graph.

Corollary 2.3 (Ramos & White [RW]). *Let $G_\bullet$ be a finitely generated FI-graph, and let T be any fixed graph. Then the number of graph homomorphisms from T to G_n agrees with a polynomial for all $n \gg 0$.*

It is classically known that the number of graph homomorphisms from T to the complete graph counts the number of proper vertex colorings of T. Therefore, the above corollary can be seen as a generalization of the existence of the **chromatic polynomial**.

If instead we counted homomorphisms from the path of length r to G_n, we find that the number of walks of length r inside G_n is in agreement with a polynomial, for all $n \gg 0$. Seeing this, one might immediately be tempted to ask whether anything can be said about how the statistics of random walks on G_n vary with n. This is indeed the case.

Fix $m \geq 0$. For any vertex x of G_m, and any $n \geq m$, we write $x(n)$ for the vertex $G(\iota)(x)$ of G_n, where ι is the standard inclusion $\iota : [m] \hookrightarrow [n]$. We also write $\tau_{x,y}(n)$ for the hitting time random variable between $x(n)$ and $y(n)$. That is, the random variable which marks the time the walk takes getting from $x(n)$ to $y(n)$.

Theorem 2.4 (Ramos & White [RW2]). *Let $G_\bullet$ be a finitely generated FI-graph, and fix $m \gg 0$, as well as a pair of vertices $x, y \in G_m$. We write $\tau_{x,y}(n)$ for the hitting time random variable of the simple random walk on G_n between the vertices $x(n)$ and $y(n)$. Then for all $n \gg 0$ the function*

$$n \mapsto \mu_i(\tau_{x,y}(n))$$

agrees with a rational function, where μ_i is any one of the i-th moment, the i-th central moment, or the i-th cumulant.

Moving on from these combinatorial results, we discuss certain algebraic properties of FI-graphs. Namely, we consider the spectra of these graphs, and how they can vary with n. Recall that if G is a graph, then the **adjacency matrix** A_G of G is the endomorphism of the $\mathbb{Q}$-linearization of the vertex set $V(G)$ of G that maps a vertex x to the sum of vertices it is adjacent to. The **spectrum** of the graph G is then defined to be the spectrum of A_G. Spectral analysis of graphs is in many ways the foundation of algebraic graph theory [Bi, CDS], and has found applications to the study of Markov chains [LPW], and graph limits [Lo], among many others.

Let $G_\bullet$ be a finitely generated FI-graph, and write A_n for the adjacency matrix A_{G_n}. While it is the case that the functor $n \mapsto \mathbb{Q}V(G_n)$ is a finitely generated FI-module, it is not the case that the collection $\{A_n\}$ can be extended to an endomorphism of this FI-module.

Theorem 2.5 (Ramos, Speyer, & White [RSW]). *Let $G_\bullet$ be a finitely generated FI-graph. For each n, the matrix A_n is real and symmetric, whence its distinct eigenvalues can be written as*

$$\lambda_1(n) < \ldots < \lambda_{r(n)}(n)$$

for some function $r(n)$. Then for all $n \gg 0$:

1. *$r(n) = r$ is independent of n. In particular, the number of distinct eigenvalues of A_n is eventually independent of n;*
2. *for each i the function $n \mapsto \lambda_i(n)$ is algebraic over the field $\mathbb{Q}(n)$;*
3. *for each i, the multiplicity of $\lambda_i(n)$ agrees with a polynomial.*

2.3. **The current project.** We turn our attention to various extremal invariants of FI-graphs. For instance, For a general graph G, recall the **independence number**, $\alpha(G)$. This invariant is defined as the size of the largest collection of vertices that are pairwise not connected by an edge. We will consider the question of the asymptotic behavior of the function $n \mapsto \alpha(G_n)$ whenever $G_\bullet$ is a finitely generated FI-graph. In examples of FI-graphs where the independence is known, it is conspicuously the case that it agrees with a quasi-polynomial in n. This includes the case of the Kneser graphs, whose independence number is the subject of the famous Erdös–Ko–Rado Theorem. On the other hand, there are many important examples of finitely generated FI-graphs whose independence numbers are still not known. In fact, there are well known conjectures of Erdös on the extremal theory of finite sets whose statement can be rephrased as computing the independence number of some FI-graph.

Question 2.6. Let $G_\bullet$ be a finitely generated FI-graph. Then can anything be said about generating function

$$\sum_{n \geq 0} \alpha(G_n)t^n?$$

It is my belief that this power series will always be rational, but are other behaviors also possible?

It is a famous fact that the dimensions of the graded pieces of a finitely generated FI-module obey a polynomial growth rule [CEF, CEFN]. We have also seen throughout this proposal other invariants of FI-objects which behave similarly. It is therefore not a stretch to believe that the generating function of $\alpha(G_n)$ should be rational. However, having a rational generating function is a more general phenomenon than simply being polynomial. In particular, this project would be the first instance of higher-order stability phenomena in FI objects.

Given a graph G, we may define a polynomial ring A_G whose variables are indexed by the edges of G. The **edge ideal** of G is defined to be the ideal generated by the monomials $x_i x_j$, where $\{i, j\}$ is an edge of G. It is a straight forward computation to show that the codimension of the edge ideal is precisely $\alpha(G)$.

There is an approach that has been used by Sam and Snowden [SS2], Krone, Leykin, and Snowden [KLS], Nagel [Nage], and myself [R8]. This approach involves leveraging the symmetry of the ideals in question to encode their members as words in regular language, and gets at the combinatorial heart of the work [LNNR], where similar statements are proven using commutative algebra. One of the foundational results in language theory is that normed regular languages have rational Hilbert series. Using the relationship between the codimension of an ideal, and the Hilbert series of its associated module, it is my belief that the rationality can be proven.

3. Stable properties in families of groups and property (T)

3.1. **Background.** Having already discussed FI-graphs, FI-sets, and FI-modules, our next direction will be to consider **FI-groups**. An FI-group is a functor $\Gamma_\bullet$ from FI to the category of discrete groups. We say that an FI-group is **finitely generated** if there exists a finitely generated FI-subset $S_\bullet$ of $\Gamma_\bullet$, such that S_n is a generating set of Γ_n for all $n \gg 0$. As examples, one sees that the symmetric groups themselves form a finitely generated FI-group, where S_n is the set of transpositions and FI acts by conjugation. Free groups, automorphism groups of free groups, and $\mathrm{SL}_n(\mathbb{Z})$ are also interesting examples of these objects. One should keep the latter two examples in mind for what follows.

Much of The my work in the study of FI-groups has been in trying to understand how certain geometric-group-theoretic properties behave among the Γ_n as $n \gg 0$. Specifically, I have considered this question for **property (T) of Kazhdan**, **property FA of Serre**, as well as the **geometric Cayley graph property**. These three properties have been central to modern geometric group theory since their respective conceptions [BdLHV, Ka, KKN, M, O, Se, Sh, Z]. Famously, property

(T) was used by Margulis to provide the first systematic approach to constructing **expander graphs** [M]. See [BdLHV] for a detailed exposition on all of these properties.

We say that a group Γ has property (T) if the trivial representation is isolated in the unitary dual of Γ. Roughly speaking, one can think of this as saying that there is an $\epsilon > 0$ and a compact subgroup $K \leq \Gamma$, such that any unitary representation that contains a vector that is only moved within an ϵ-ball of its starting position by K, must actually contain a vector that is fixed by Γ. Property FA, on the other hand, is a discrete version of (T) that requires that all actions of Γ on a tree admit a fixed point.

Finally, the Cayley graph property is a feature of a group Γ paired with a chosen (finite) generating set S, not containing the identity. Define the **level one Cayley graph** $C^{(1)}_{\Gamma,S}$ of (Γ, S) to be the induced subgraph of the usual Cayley graph generated by the elements of S. Then we say that (Γ, S) has the geometric Cayley graph property if $C^{(1)}_{\Gamma,S}$ is connected and its **Laplacian matrix** - the difference of the diagonal matrix of vertex degrees with the adjacency matrix of $C^{(1)}_{\Gamma,S}$ - has smallest non-zero eigenvalue strictly bigger than $1/2$.

In previous sections, we saw how finite generation conditions had drastic structural impacts on FI-graphs, sets, and modules. The first difficulty one faces with FI-groups, that is similarly found in the study of FI-algebras [NR], is that this is no longer necessarily the case. FI-groups come in a variety of different forms with the condition of finite generation having seemingly little impact on what kinds of group theoretic properties do and do not appear. We therefore shift our focus from questions of the form "what kinds of group theoretic properties are shared among finitely generated FI-groups," to questions of the form "what kinds of group theoretic properties are **stable** in finitely generated FI-groups?"

We say that a group property $\mathcal{P}$ is stable, if, for any finitely generated FI-group $\Gamma_\bullet$, there exists some $N \gg 0$ such that for all $n \geq N$, Γ_n has property $\mathcal{P}$ if and only if Γ_N does. As simple examples, one immediately sees that "being finite" is a stable property, whereas "having even rank" is not.

3.2. **Stability in FI-groups.** It is natural for one to ask whether there are any non-trivial group theoretic properties that are known to be stable. The proof of the following theorem uses the theory of FI-graphs in a non-trivial way.

Theorem 3.1 (Ellenberg and Ramos [ER]). *Let $(\Gamma_\bullet, S_\bullet)$ be a finitely generated FI-group with generating FI-set $S_\bullet$. Then property FA, as well as the geometric Cayley graph property, are stable properties of $(\Gamma_\bullet, S_\bullet)$.*

Note that, critical in the proof of this theorem, are the aforementioned stability phenomena which appear in the spectra of adjacency matrices associated to finitely generated FI-graphs.

3.3. **The current project.** Recall from our examples that $SL_\bullet(\mathbb{Z})$ and $\mathrm{Aut}(F_\bullet)$ are both examples of finitely generated FI-groups. It has been known since the original work of Kazhdan that $SL_n(\mathbb{Z})$ has property (T) for $n \geq 3$ [Ka]. Later, work of Shalom reproved this fact using more algebraic means [Sh]. What is relevant to us, however, is the very recent combinatorial approach of Kaluba, Kielak, and Nowak [KKN]. In this work, Kaluba, Kielak, and Nowak use a characterization of property (T) proposed by Ozawa [O] to show that both $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{Aut}(F_n)$ have property (T) for $n \geq 5$. Notable in this work is that their approach is fundamentally tied to the theory of FI-groups, though they do not use this language. We therefore have the following.

Major Question A. Is Property (T) a stable property of finitely generated FI-groups?

It is famously known that the geometric Cayley graph property implies property (T), which in turn implies property FA. The work of Ellenberg and I detailed above therefore seems to strongly suggest that the above question should have a positive answer. Moreover, work of Pak and Zuk has shown that the presence of property (T) seems to be heavily influenced by the existence of a generating set with a group action [PZ]. Of course, such a generating set is literally baked into the definition of an finitely generated FI-group! The approach to this question through the work of Kaluba, Kielak, and Nowak seems to be the best direction of attack for this problem. Let us make this a bit more concrete by exploring their work in a bit more detail.

Let G be a group, and assume we have a symmetric (finite) generating set S for G. Then we define Δ to be the element of the real group ring $\mathbb{R}[G]$ given by

$$\Delta = \sum_{s\in S}(1-s)(1-s)^*,$$

where $* : \mathbb{R}[G] \to \mathbb{R}[G]$ is the linear operator $g \mapsto g^{-1}$. Then Ozawa has shown [O] that G has property (T) if and only if there exists some $\lambda > 0$ such that $\Delta^2 - \lambda \cdot \Delta \in$ $mathbbR[G]$ can be expressed as a sum $\sum_i \zeta_i\zeta_i^*$, where $\zeta_i \in \mathbb{R}[G]$. This type of decomposition in $\mathbb{R}[G]$ is what is known as a **sum of squares** decomposition. Ozawa's theorem allows, among other things, for one to decide whether a group has (T) with semi-definite programming.

Coming back to the context of FI-groups, we have a family of group-ring elements Δ_n, which we are trying so show eventually satisfy the Ozawa condition with some sequence of positive constants, λ_n. In [KKN], Kaluba, Kielak, and Nowak make the observation that both Δ_{n+1} and Δ^2_{n+1} can be expressed as a finite linear combination of group ring elements, that are each the images of Δ_n and Δ^2_n, respectively, under the FI-group's transition maps. They make this observation in the cases of the two group families they consider, but it is valid for any finitely generated FI-group. Following this key observation, they then use some clever algebra, as well as a clever choice of the constant λ_{n+1}, to write

$$\Delta^2_{n+1} - \lambda_{n+1} \cdot \Delta_{n+1}$$

as a sum of things that look like $\Delta^2_n - \lambda_n \cdot \Delta_n$, along with a "fudge-factor," which they show is tied to the $n = 5$ case. They then prove this fudge-factor has the necessary sum of squares decomposition using the aforementioned semi-definite program.

As outlined in the previous paragraph, most of the big observations made in [KKN] have obvious analogs for finitely generated FI-groups. Therefore the only difficulty with this approach is in understanding how their clever choice of λ_{n+1}, as well as their semi-definite programming approach to resolving the fudge-factor generalize. Both of these issues require a delicate understanding of the combinatorics of FI-groups, but ultimately seem tractable.

4. Graph configuration spaces

4.1. **Background.** The study of the configuration spaces of graphs has seen a recent surge in popularity due to their connection with robotics and physics [Far, G, MS, MS2]. This is also evidenced by a recent AIM workshop on the subject. In what follows, for any graph G, we will largely concern ourselves with unordered configurations, $\mathrm{UConf}_n(G)$, the quotient of the usual configuration space by the action of the symmetric group obtained by permuting coordinates in the obvious way.

It is a theorem of Abrams [A], that $\mathrm{UConf}_n(G)$ is a $K(\pi, 1)$ for all $n \geq 1$ and all graphs G. It follows from this that in order to understand the homotopy type of $\mathrm{UConf}_n(G)$, it largely suffices to understand the fundamental group $B_nG := \pi_1(\mathrm{UConf}_n(G))$. In the literature, the braid groups B_nG have largely been studied from the perspective of geometric group theory. For instance, it is now fairly well understood when these groups are right-angle Artin [KP, KKP].

For the purposes of this proposal, our primary interest is related to homology groups $H_i(B_nG)$. It follows from the discussion in the previous paragraph the singular homology of the space $\mathrm{UConf}_n(G)$ is canonically isomorphic to the group homology of B_nG. Using this isomorphism, authors such as Kim, Ko, and Park [KKP, KP], as well as Farley and Sabalka [FS], have been able to prove surprising facts about the homology groups $H_i(B_nG)$. For instance, Ko and Park have shown that $H_1(B_2G) \cong H_1(B_nG)$ for all $n \geq 2$ whenever G is biconnected [KP]. Farley has also used the isomorphism to provide a computational method for determining the groups $H_i(B_nG)$ whenever G is a tree [Fa]. All of these results use a discrete Morse structure on $\mathrm{UConf}_n(G)$, that was developed by Farley and Sabalka [FS].

4.2. **Stability phenomena in the homology of tree braid groups.** The philosophy behind my work on the configuration spaces of graphs is based in the philosophy of asymptotic algebra as a whole. Namely, whenever a family of algebraic objects exhibits asymptotic stability phenomena, it is often the case that they can be encoded in a single object, that is finitely generated in the appropriate sense. In their work on the configuration spaces of trees, I showed that the homology groups $H_i(B_nG)$ can be encoded in a finitely generated graded module over a polynomial ring.

For a graph G, an **essential vertex** is any vertex of degree at least 3. An **essential edge** is a connected component of the space obtained by removing all essential vertices of G. We also define the quantity Δ_G^i to be the maximum number of connected components that G can be broken into by removing exactly i vertices.

Theorem 4.1 (Ramos [R3])**.** *Let G be a tree, and fixed $i \geq 0$. Then there is a polynomial $P_i^G \in \mathbb{Q}[t]$ of degree $\Delta_G^i - 1$ such that for all $n \geq 0$*

$$P_i^G(n) = \dim_{\mathbb{Q}}(H_i(B_nG; \mathbb{Q}))$$

This was proved using the structure theorems of Farley and Sabalka [FS], as well as the computational theorems of Farley [Fa]. In fact, I have computed the polynomials P_i^G explicitly in terms of certain invariants of the tree G in [R3].

The key insight to proving the above theorem, which is perhaps more significant than the result itself, is that the groups $H_i(B_nG)$ carry a natural action by a polynomial ring.

Theorem 4.2 (Ramos [R3])**.** *Let G be a tree, and let A_G denote the integral polynomial ring with variables indexed by the essential edges of G. Then for each $i \geq 0$, there is an action of A_G on the graded $\mathbb{Z}$-module $\mathcal{H}_i := \bigoplus_n H_i(B_nG)$, turning $\mathcal{H}_i$ into a finitely generated graded A_G-module. Moreover, $\mathcal{H}_i \otimes \mathbb{Q}$ decomposes as a direct sum of graded twists of squarefree monomial ideals of Krull dimension at most Δ_G^i.*

The first obvious question arising from my work is whether it can be applied to more general graphs. Indeed, in [R3] it is shown that the action of A_G on $\mathcal{H}_i$ will still be well defined, provided certain diagrams commute. This was proved by An, Drummond-Cole, and Knudsen in a recent paper [ADK]. In fact, in a follow up paper [ADK2] the following theorem was proved, resolving a conjecture of mine.

Theorem 4.3 (An, Drummond-Cole, and Knudsen [ADK2])**.** *If G is a graph, that is neither a line segment nor a circle, then $n \mapsto \dim_{\mathbb{Q}}(H_i(B_nG; \mathbb{Q}))$ is eventually agrees with a polynomial of degree $\leq \Delta_G^i - 1$.*

An, Drummond-Cole, and Knudsen have also recently proven that the commutative algebra of $H_2(\mathrm{UConf}_n(G))$ does indeed encode other connectivity invariants, at least when G is planar [ADK3].

4.3. **Future directions.** As our major question on graph configuration spaces, we will consider the problem of how much of a graph G can be recovered from the spaces $\mathrm{UConf}_n(G)$. It is not hard to show using the aforementioned discrete Morse theory that $\pi_1(B_2G)$ is determined by the degree sequence of G in this case, though not even uniquely so. Using Abrams' result that $\mathrm{UConf}_2(G)$ is a $K(\pi, 1)$, one concludes that the homotopy type of $\mathrm{UConf}_2(G)$ is determined only by a numerical invariant which is weaker than the degree sequence. Namely, what is called the star-norm of the degree sequence in [R8]. In particular, one cannot recover G from $\mathrm{UConf}_2(G)$ using traditional topological invariants.

To get around this obstacle, Levin, Young, and I have considered the **finite exclusion process** as a stochastic process on the configuration space [LRY]. The finite exclusion process is a Markov process on (unordered) pairs of vertices of G defined by first flipping a coin to see that vertex in the pair will change, and then uniformly at random choosing an edge adjacent to this chosen vertex to move along. If the two original vertices were connected by an edge, and this is the edge chosen to move along, the process stalls. This finite exclusion process has been a topic of continuing interest in Markov chain theory for many years (see [PP, NN], and the references therein).

Assume for simplicity that we have chosen an embedding of our tree G into the plane, and that we have designated a leaf of G to be the root. Then one obtains a well ordering on vertices via a depth-first numbering originating from the root. from any starting location, running the Markov process for some fixed number of steps defines a path in the configuration space $\mathrm{UConf}_2(G)$. Levin, Young, and I then define the **closure** of this path to be the loop obtained from the path by allowing the two points to flow back towards their starting positions, one at a time, where the first point to move is that that is currently sitting on the smaller vertex. From the closure operation, one obtains an element of $H_1(B_2G)$. Assuming that G is a tree, $H_1(B_2G)$ is a free group, whence one can study this closure as a kind of (multivariate) statistic associated to the walk. We call this statistic the **winding statistic of the process**. The main theorem of [LRY] is that the winding statistic asymptotically satisfies a central limiting theorem.

Theorem 4.4 (Levin, Ramos, and Young [LRY])**.** *Let G be a tree, and let $W(t)$ denote the winding of the finite exclusion process performed on G after t steps. Then there exists a matrix $\mathbf{\Sigma}_G$ such that,*

$$W(t)/\sqrt{t} \overset{\mathcal{D}}{\to} Norm(0, \mathbf{\Sigma_G})$$

where $Norm$ is the standard normal distribution with mean 0 and covariance matrix $\mathbf{\Sigma}_G$, and the above indicates convergence in distribution.

This leads to the following.

Question 4.5. If G and G' are two trees with no vertices of degree 2 such that $\mathbf{\Sigma}_G = \mathbf{\Sigma}_{G'}$, is it the case that G must be isomorphic to G'?

We note that in [LRY], this conjecture is verified both theoretically and experimentally for all trees of the required form with up to 10 vertices. Questions of this form have a history in the study of random processes on topological spaces. For instance classic, work of Spitzer [Sp] considered the winding behavior of two particles in the plane, each performing independent Brownian walks. Spitzer's work lead to a plethora of similar works related to winding of randomly moving particles on surfaces [RH, WT].

In [LRY], the covariance matrix $\mathbf{\Sigma}_G$ is proved to arise from a quadratic form associated to the **discrete Green's function** associated to the exclusion process. That is to say, the quasi-inverse of the difference $I - P$, where I is the identity matrix and P is the transition matrix associated to the exclusion process. Classically speaking, these discrete Green's functions arise due to their relation to expected hitting times [CY]. They are also, unfortunately, notorious difficult to explicitly compute in most cases. This is why the work of Nestoridi and Nguyen [NN] is critical. Using their computations related to the eigenvalues and eigenbasis of P, it should be possible to determine the aforementioned quadratic form, at least in the case of regular trees.

Moving on from the regular case, things get considerably more difficult. One possible approach to this more general case once again relies on the discrete Green's function formulation mentioned above. We have already discussed that these Green's functions can be written in terms of expected hitting times of the exclusion process. These hitting times have been implicitly computed for the exclusion process on various other structures. For instance, they are computed for the exclusion process on the complete graph in [LL]. It is my belief that these computations can be adapted to our setting, at least for certain families of trees.

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