ERIC RAMOS RESEARCH STATEMENT

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My primary research focus is in algebraic combinatorics and its applications to topology and representation theory. In particular, I am interested in applying the techniques of the representation theory of categories in an attempt to understand the mechanisms underlying various well known asymptotic stability phenomena from combinatorics, topology, and other related fields. In the following research statement, I will outline four major projects that currently drive my present research program. These projects aim to further the state of the art in the representation theory of certain combinatorially flavored categories, and then use this advancement to solve concrete problems naturally arising from algebraic and topological combinatorics. As many of the relevant subject areas are fairly new, there are a plethora directions that are still unexplored. I will therefore outline smaller projects that would also be interesting to pursue beyond the major ones. To conclude, I will spend some time outlining my older work, as well as various problems that remain unsolved regarding those works.

Before we provide formal definitions, we begin with an example for motivation. For any finite set T, and any topological space X, we write

$$\mathrm{Conf}_T(X) := \{(x_t)_{t\in T} \mid x_t \neq x_{t'}\},$$

to denote the topological space of injections from T to X. If T' is any other finite set, and there is an injection $T \hookrightarrow T'$, then precomposition naturally defines a map of topological spaces $\mathrm{Conf}_{T'}(X) \to \mathrm{Conf}_T(X)$. For any fixed i, the cohomology groups $H_i(\mathrm{Conf}_\bullet(X); k)$ therefore inherit an interesting structure. Namely, for each n $H_i(\mathrm{Conf}_n(X); k)$ is a $k[\mathfrak{S}_n]$-module, and between any two values of n the corresponding modules are in some sense compatible. This can be thought of as the motivating philosophy for the study of what we will call FI-modules: a means of encoding an infinite collection of compatible symmetric group representations into a single object. Using this structure, Church, Ellenberg, and Farb were able to prove many non-trivial facts about the configuration spaces of orientable manifolds [CEF].

Write FI for the category of finite sets and injections. Note that we will usually think of this category equivalently as that whose objects are the sets $[n] = \{1, \ldots, n\}$ and whose morphisms are injections. An **FI-module** over a commutative ring k is a functor from FI to the category of k-modules. These modules, as defined here, were first introduced by Church, Ellenberg, and Farb in their seminal paper [CEF]. Since their inception in the work of Church, Ellenberg, and Farb, FI-modules, and modules over other similar combinatorially flavored categories, have seen an enormous amount of use in topology, representation theory, and other fields. This has been the subject of a recent AIM workshop, an AMS special session, an Oberwolfach workshop, an MSRI summer school, and numerous other conferences.

The category of FI-modules over k, FI-Mod, is an abelian category with abelian operations defined point-wise. We say that V is **finitely generated** if there is a finite set $\{v_i\} \subseteq \bigsqcup_T V(T)$, that is contained in no proper submodule of V. This notion was first explored by Church, Ellenberg, and Farb in [CEF], although it was not fully explored until the follow-up work of Church, Ellenberg, Farb, and Nagpal [CEFN, N]. In the paper [CEFN], it is shown that the category of finitely generated FI-modules over a Noetherian ring have a Noetherian property. That is, the category FI-mod of finitely generated FI-modules over a Noetherian ring is abelian. This fact was proven when k is a field of characteristic 0 by Church, Ellenberg, and Farb in [CEF] and by Snowden in [Sn]. It was proven in total generality by Church, Ellenberg, Farb, and Nagpal in [CEFN].

FI is the first, and perhaps most natural, example of what we will henceforth refer to as a **combinatorial category** - a category whose foundational structure is based in the combinatorics of some finite objects. Two of the projects proposed below involve applying our wealth of knowledge about FI-modules in new directions related to combinatorics and group theory. Not everything we discuss will be related with FI however, as another one of the objectives in my current research program is to push the boundaries of our understanding into the representation theory of combinatorial categories that are richer and less well understood than FI. We will encounter the primary such category in the next section - the category of graphs and contractions.

1. The contraction category of graphs

1.1. **Background.** Throughout this project description, A **graph** will always refer to an at most 1-dimensional CW-complex that is both connected and finite. Given a graph G we can write $V(G)$ for its set of **vertices** - or 0-cells - and $E(G)$ for its set of **edges** - or 1-cells. A **contraction** from a graph G to a graph G' is a topological map defined by selecting a sub-forest within the graph G, and contracting each component tree to obtain a new graph G'. Topology informs us that a contraction induces a homotopy equivalence between G and G', and thus necessarily $H_1(G) \cong H_1(G')$. We call the rank of $H_1(G)$ the **genus** of G. Finally, if $g \geq 0$ is an integer, we write $\mathcal{G}_g$ for the category whose objects are graphs with genus g, and whose morphisms are contractions. For instance, $\mathcal{G}_0$ is the category of trees and contractions. The primary object of study in what follows is the opposite category $\mathcal{G}_g^{op}$. In particular, we consider modules over the category $\mathcal{G}_g^{op}$.

Just as with our previous discussion of FI-modules, a $\mathcal{G}_g^{op}$-module over a Noetherian ring k is a functor from $\mathcal{G}_g^{op}$ to the category of k-modules. We say that a $\mathcal{G}_g^{op}$-module M is **finitely generated** if there is some finite list of genus g graphs $G_1, \ldots, G_N$ such that for any graph G of genus g, $M(G)$ is spanned by the images of $M(G_i)$ under the various transition maps induced from contractions. We usually refer to the graphs $\{G_i\}$ as being the **generators** of the module M. The main technical theorem, that informs everything in the sequel, is the following.

Theorem 1.1 (Proudfoot & Ramos [PR, PR2])**.** *Let k be a Noetherian ring, $g \geq 0$ be an integer, and let M be a finitely generated $\mathcal{G}_g^{op}$-module. Then all submodules of M are also finitely generated.*

One should note that Theorem 1.1 generalizes the aforementioned Noetherianity for FI-modules, as one can realize FI as the full subcategory of $\mathcal{G}_0^{op}$ of **star trees** - trees with a single vertex of degree ≥ 3. One may think about the relationship between these two statements as being analogous to the relationship between **Higman's Lemma** - that sequences within well-quasi-ordered posets are once again well-quasi-ordered - and **Kruskal's Tree Theorem** - that rooted trees are well-quasi-ordered by contractions.

1.2. **Applications of Noetherianity.** Just as was the case with FI-modules, Noetherianity of $\mathcal{G}_g^{op}$-modules can be leveraged to prove surprising facts in topology, as well as other fields. To illustrate these examples we will consider the configuration spaces of graphs. Configuration spaces of graphs, while not appearing much in classical literature, have recently seen a boom in their study. A later section of this statement will be dedicated to discussing the known theory of these spaces, and my contributions towards answering the natural question of, "what, if any, stable behaviors appear when one varies the number of points being configured n?" As these spaces relate to the category $\mathcal{G}_g^{op}$, however, we will instead be interested in the orthogonal question of, "what, if any, stable behaviors appear when one varies the graph and fixes the number of points?"

In their work [ADK], An, Drummond-Cole, and Knudsen show that if $\phi : G \to G'$ is a contraction of graphs, then there is an induced map $\phi^* : H_i(\mathrm{Conf}_n(G')) \to H_i(\mathrm{Conf}_n(G))$, for any fixed integers $i, n \geq 0$. This observation is made somewhat more surprising by the fact that the map ϕ^* does not exist at the level of topological spaces. Equipped with the existence of the maps ϕ^*, as well as our main Noetherianity theorem, we prove the following.

Theorem 1.2 (Proudfoot & Ramos [PR, PR2])**.** *For all $i, n, g \geq 0$, the assignment*
$$G \mapsto H_i(\mathrm{Conf}_n(G))$$
can be extended to a finitely generated $\mathcal{G}_g^{op}$-module over $\mathbb{Z}$.

According to the definition of finite generation, the above theorem tells us that for any fixed $i, n, g \geq 0$, there is some finite list of generating graphs of genus g $G_1, \ldots, G_N$, such that for any graph G of genus g, every homology class in $H_i(\mathrm{Conf}_n(G))$ can be viewed as some combination of homology classes coming from $H_i(\mathrm{Conf}_n(G_j))$. One immediate consequence of this is the following.

Corollary 1.3 (Proudfoot & Ramos [PR2])**.** *For any fixed $i, n, g \geq 0$, there exists an integer $d_{i,n,g}$ such that for any graph G of genus g, the exponent of the abelian group $H_i(\mathrm{Conf}_n(G))$ is at most $d_{i,n,g}$.*

Shades of this boundedness of torsion had been observed in, for instance, works of Ko and Park [KP] and Chettih and Lütgehetmann [CL]. The above result of Proudfoot and I is the first time it was made precise. While the study of torsion in the homology of topological spaces has its own intrinsic value, it appears that the torsion in graph configuration spaces is especially compelling. Indeed, it is known [KP] that torsion in the first homology groups of **unordered configuration spaces of graphs** - i.e. the quotient of configuration space by the natural symmetric group action - exactly detects planarity of the graph.

Our second application considers the **cohomology of the category** $\mathcal{G}_g^{op}$. We recall for any small category $\mathcal{C}$, its cohomology is defined by
$$H^i(\mathcal{C}) := \mathrm{Ext}^i_{\mathcal{C}-\mathrm{Mod}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}})$$
where $\underline{\mathbb{Z}}$ is the $\mathcal{C}$-module that takes the value $\mathbb{Z}$ at each object, and whose every induced map is the identity. It is classically known that the cohomology of any small category can be naturally identified with the singular cohomology of a topological space associated to the category; the **nerve** of the category.

While my work with Proudfoot does not completely resolve the issue of computing the cohomology of $\mathcal{G}_g^{op}$, we do show the following.

Theorem 1.4 (Proudfoot & Ramos [PR2])**.** *We say that a graph is reduced if it contains no vertices of degree 2, nor any edges whose deletion disconnects the graph. Write $\mathcal{G}_{g,red}^{op}$ for the full subcategory of $\mathcal{G}_g^{op}$ whose objects are reduced graphs of genus g. Then one has,*
$$H^i(\mathcal{G}_{g,red}^{op}) \cong H^i(\mathrm{Out}(F_g))$$
where $\mathrm{Out}(F_g)$ is the outer automorphism group of the free group F_g.

The above theorem is related to, and indeed depends on, Culler and Vogtmann's seminal work on **Outer Space**. Roughly speaking, the nerve of the category $\mathcal{G}_{g,red}^{op}$ can be seen to be homotopy equivalent to the quotient of Outer Space by the $\mathrm{Out}(F_g)$-action, where a collection of infinite-dimensional pockets are glued in at various points. While this space therefore becomes a considerably more cumbersome geometrically, the upshot is that its cohomology agrees with that of $\mathrm{Out}(F_g)$ with integral coefficients, whereas the Culler-Vogtmann Outer Space can only be used to compute rational cohomology of these groups.

1.3. **The major project.** In [PR2], Proudfoot and the I note it is very likely that the Noetherianity statement for $\mathcal{G}_g^{op}$-modules is actually a specific case of something much more general. We write $\mathcal{G}$ for the category whose objects are graphs, and whose morphisms are pairs $(G', \phi) : G \to G''$, where G' is a subgraph of G, and $\phi : G' \to G''$ is a contraction. In other words, there exists a morphism in $\mathcal{G}$ from G to G'' if and only if G'' is a **minor** of G. For any $g \geq 0$, $\mathcal{G}_g$ is clearly a subcategory of $\mathcal{G}$. Just as with FI and $\mathcal{G}_g^{op}$, we may make sense of finitely generated $\mathcal{G}^{op}$-modules over any commutative ring k.

Major Question A. Do $\mathcal{G}^{op}$-modules have a Noetherian property similar to that of FI-modules or $\mathcal{G}_g^{op}$-modules?

Just as the jump from the Noetherianity of FI-modules that that of $\mathcal{G}_g^{op}$-modules is morally similar to the jump from Higman's Lemma to Kruskal's Tree Theorem, I believe it will be the case that proving a Noetherianity statement for $\mathcal{G}^{op}$-modules would require one to wield the the strength of the **Graph Minor Theorem** [RS].

To make this a bit more precise, we briefly recall the proof of the classical Hilbert Basis Theorem, for modules over polynomial rings, via **Gröbner bases**. In this proof, one begins by applying elementary algebraic arguments to show that it suffices to prove that ideals of the polynomial ring are finitely generated. Using any well ordering of monomials, say the lexicographical order, one can then associate a monomial ideal to any given ideal, from which a generating set can be lifted. In particular, it then only suffices to prove that all monomial ideals of the polynomial ring are finitely generated. To conclude the proof, one then shows that this statement is equivalent to the poset $\mathbb{N}^r$ being Noetherian, where r is the number of variables of the polynomial ring.

In their seminal work [SS2], Sam and Snowden showed how the above argument could be generalized to prove Noetherianity statements for representations of combinatorial categories. In brief, they show that to prove a category $\mathcal{C}$ has Noetherian representation theory it suffices to construct a category $\mathcal{C}'$, which we call the **Gröbner cover** of $\mathcal{C}$, that admits a functor $\mathcal{C}' \to \mathcal{C}$ with certain easily verified technical properties. The morphisms of this Gröbner cover are then required to satisfy a similar role to the monomials in the aforementioned proof. In particular, one must show that certain families of morphisms can be encoded into a Noetherian poset, and that these same families can be equipped with a well ordering that mimics the role of the lexicographical order of monomials in the classical setting.

Returning to the context of the category $\mathcal{G}^{op}$, It is my belief that the correct choice of Gröbner cover is the category whose objects are graphs paired with a total ordering of their edges. The morphisms in this category will be those morphisms from $\mathcal{G}^{op}$ that respect the edge orderings. Objects of this kind have seen use in recent work of Chan, Galatius, and Payne [CGP, CGP2]. These works were, in many ways, my primary inspiration in considering this category.

This choice of Gröbner cover has the benefit that the necessary families of morphisms seem to be encoded into a poset related to the poset of the aforementioned graph minor theorem. On the other hand, constructing an appropriate analog of the lexicographical order seems to be much more complicated. Work of Barter [Ba] as well as Proudfoot and I [PR, PR2] have accomplished similar tasks in more limited contexts, but the techniques of these papers do not easily generalize to all graphs.

We note here that an affirmative answer to Question A would imply a finite generation statement about graph configuration spaces similar to Theorem 1.2. Such a statement would unify a number of results in the field, and be the ultimate culmination of the aforementioned question, "what, if any, stable behaviors appear when one varies the graph and fixes the number of points?"

1.4. **Other future directions.** The study of modules over the category $\mathcal{G}_g^{op}$ is very new, and therefore lends itself well to a variety of interesting open questions. These questions range from intrinsic structural concerns, to questions about the aforementioned applications. We therefore begin this discussion by addressing questions of the first type. Note that, any of the following questions can also be stated for the aforementioned $\mathcal{G}^{op}$-modules, although it seems likely that their proofs in that context would be considerably more difficult.

While there are a nearly limitless number of directions one can take with regards to structural concerns, there is one particular direction which I am currently most interested in pursuing. In later sections, we will see that if V is an FI-module over a field k, then for all $n \gg 0$, $\dim_k V([n])$ agrees with a polynomial, where $[n] = \{1, \ldots, n\}$. Put another way, the **Hilbert series** $H_V(t) := \sum_{n\geq 0} \dim_k V([n])$ is necessarily rational of a particular form. Similar statements can be shown for finitely generated modules over a large number of related combinatorial categories such as FS^{op} - of finite sets and surjective maps - and FA - of finite sets and all set maps [SS2]. In my work with Proudfoot, it is shown that for finitely generated $\mathcal{G}_g^{op}$-modules over a field, the dimension of $M(G)$ is *bounded* by a polynomial in the number of edges of G, although it does not necessarily exactly agree with a polynomial. While this statement is considerably weaker than what is known about FI modules, it is also natural to believe that one should lose such precision, as the number of edges in a graph is a far more coarse invariant of the graph than the number of elements in a set is to the set. Put another way, the most naive way one might think to define the notion of Hilbert series for a $\mathcal{G}_g^{op}$-module is perhaps not the most ideal.

Question 1.5. What is the correct definition for the Hilbert series of a $\mathcal{G}_g^{op}$-module? Recent work of Chan, Faber, Galatius, and Payne [CGP, CGP2, CFGP] has used a construction they call orbifold sums of graphs to great effect in studying moduli spaces of tropical curves. Can one modify this definition to fit our setting? If so, what kind of formal rationality statements can one prove about these Hilbert series?

I believe that understanding Hilbert series is an extremely important first step in understanding the overall structure of the representation theory of a category. Indeed, it both informs the best way forward with regard to other structural concerns, and is useful in leveraging the most that you can from a finite generation statement in applications.

Moving on to our applications, recall the theorem of Proudfoot and I that the homology groups of the configuration spaces of graphs of fixed genus g can be encoded in a finitely generated $\mathcal{G}_g^{op}$-module. The most natural followup question to this is whether or not one can actually provide the (finitely many) generators of this module. In the case of genus 0, i.e. trees, the answer is known due to results of Chettih and Lütgehetmann [CL]. Outside of this case, however, virtually nothing is known. In fact [CL] provides examples of graphs whose configuration space homology can become almost arbitrarily pathological.

Question 1.6. Can one provide the generators of the aforementioned $\mathcal{G}_g^{op}$-modules, even in cases where the genus, the number of points being configured, and/or the homological index is kept small? Finding these generators has a similar flavor to constructing a finite set of **forbidden minors** for some minor-closed property - such as K_5 and $K_{3,3}$ as they relate to planarity. Could the techniques used in those types of problems be admissible here?

Finally, we have discussed how the full subcategory of $\mathcal{G}_g^{op}$ whose objects are reduced graphs could be used to compute the cohomology of $\mathrm{Out}(F_g)$. It is still an open question, however, whether the nerve of the entire $\mathcal{G}_g^{op}$ is topologically related to anything in the literature.

Question 1.7. Can we find subcategories of $\mathcal{G}_g^{op}$ whose nerve can be linked to any one of the various generalizations and compactifications of Culler-Vogtmann's Outer Space? For instance, the moduli space of tropical curves, as studied by Chan, Galatius, and Payne [CGP, CGP2], seems like it would quite naturally fit into such a framework. Moreover, having formed such a link, can our general categorical results be used to prove something new about these relatively mysterious spaces?

2. Families of highly symmetric graphs

2.1. **Background.** A **homomorphism** of graphs is a map between their vertex sets that preserves adjacency. An **FI-graph** is a functor from FI to the category of graphs and graph homomorphisms. More concretely, an FI-graph is a family of graphs $\{G_n\}$, indexed by the natural numbers, each carrying an action of the corresponding symmetric group $\mathfrak{S}_n$.

Just as with FI-modules, we begin by limiting our attention to a collection of FI-graphs that are well-behaved enough to warrant study. We say that an FI-graph $G_\bullet$ is **finitely generated** if, for all $n \gg 0$, every vertex of G_{n+1} appears in the image of some vertex of G_n under the action of FI. Examples of such families include the complete graphs, the Johnson and Kneser graphs, and the Crown graphs. The primary tool for working with finitely generated FI-graphs is the following.

Theorem 2.1 (Ramos & White [RW]). *Let $G_\bullet$ denote a finitely generated FI-graph. Then for $n \gg 0$:*

1. *if $f : [n] \hookrightarrow [n+1]$ is an injection of sets, then the induced map $G(f)$ is injective;*
2. *if $f : [n] \hookrightarrow [n+1]$ is an injection of sets, then the image of the induced map $G(f)$ is an induced subgraph of G_{n+1};*
3. *for any $r \geq 0$ and any collection of vertices $\{v_1, \ldots, v_r\}$ of G_{n+1}, there exists a collection of vertices $\{w_1, \ldots, w_r\}$ of G_n as well as an injection $f : [n] \hookrightarrow [n+1]$, such that $G(f)(\{w_1, \ldots, w_r\}) = \{v_1, \ldots, v_r\}$.*

The above theorem allows us to paint the following picture of what finitely generated FI-graphs look like: for $n \gg 0$ there is a chain of induced subgraphs $G_n \subseteq G_{n+1} \subseteq \ldots$ such that each G_n is equipped with an action of the symmetric group $\mathfrak{S}_n$, and these actions respect the inclusions in the obvious way. That is to say, a finitely generated FI-graph can be thought of as a family of nested graphs, equipped with compatible symmetric group actions.

2.2. **Applications of the theory.** From the relatively simple setup of the previous sections, one obtains a wealth of interesting consequences and applications. In the original papers in which the theory of FI-graphs were developed, these applications were broken into three categories: combinatorial, topological, and algebraic. We continue this trichotomy here.

To begin, we consider the case of the complete graphs, K_n. Note that if T is any fixed finite graph, then the number of distinct copies of T appearing as a subgraph of K_n can be easily counted, and it is seen to be a polynomial in n. This behavior is common among all finitely generated FI-graphs.

Theorem 2.2 (Ramos & White [RW]). *Let $G_\bullet$ be a finitely generated FI-graph, and let T be any fixed graph. Then for all $n \gg 0$, the following quantities each agree with a polynomial:*

1. *the number of copies of T appearing as a subgraph of G_n;*
2. *the number of copies of T appearing as an induced subgraph of G_n.*

As an immediate application of this theorem, we consider what it implies about the Hom-graph of an FI-graph. For any finitely generated FI-graph $G_\bullet$, and any fixed graph T, we define a new FI-graph whose vertices are labeled by graph homomorphisms from T to G_n, and whose edges indicate that two homomorphisms differ on precisely one vertex of T. It can be shown that this new FI-graph is finitely generated, granted that $G_\bullet$ was, and therefore the above theorem immediately implies the following.

Corollary 2.3 (Ramos & White [RW]). *Let $G_\bullet$ be a finitely generated FI-graph, and let T be any fixed graph. Then the number of graph homomorphisms from T to G_n agrees with a polynomial for all $n \gg 0$.*

It is classically known that the number of graph homomorphisms from T to the complete graph counts the number of proper vertex colorings of T. Therefore, the above corollary can be seen as a generalization of the existence of the **chromatic polynomial**.

If instead we counted homomorphisms from the path of length r to G_n, we find that the number of walks of length r inside G_n is in agreement with a polynomial, for all $n \gg 0$. Seeing this, one might immediately be tempted to ask whether anything can be said about how the statistics of random walks on G_n vary with n. This is indeed the case.

Fix $m \geq 0$. For any vertex x of G_m, and any $n \geq m$, we write $x(n)$ for the vertex $G(\iota)(x)$ of G_n, where ι is the standard inclusion $\iota : [m] \hookrightarrow [n]$.

Theorem 2.4 (Ramos & White [RW2]). *Let $G_\bullet$ be a finitely generated FI-graph, and fix $m \gg 0$, as well as a pair of vertices $x, y \in G_m$. We write $\tau_{x,y}(n)$ for the hitting time random variable of the simple random walk on G_n between the vertices $x(n)$ and $y(n)$. Then for all $n \gg 0$ the function*

$$n \mapsto \mu_i(\tau_{x,y}(n))$$

agrees with a rational function, where μ_i is any one of the i-th moment, the i-th central moment, or the i-th cumulant.

Moving on from these combinatorial results, we next turn our attention to how the theory of FI-graphs can be applied to topology. If G is a graph, recall that we may consider the configuration space, $\mathrm{Conf}_n(G)$. Configuration spaces of graphs are a recurring theme throughout my work, as they have proven to be a rich area for new applications. In a future section the I will go into much more detail about the existing theory of these spaces, and their contributions, but for now it suffices to say that they behave much differently than configuration spaces of manifolds, as considered by Church, Ellenberg, and Farb [CEF]. As in previous sections, we proceed by considering how their behavior varies when the number of points is fixed, and we instead make our graph bigger in a natural way. The following result generalizes work of Lütgehetmann from [Lu].

Theorem 2.5 (Ramos & White [RW]). *Let $G_\bullet$ be a finitely generated FI-graph. Then for all fixed $m, q \geq 0$, the FI-module*

$$n \mapsto H_q(\mathrm{Conf}_m(G_n))$$

is finitely generated.

To conclude, we discuss certain algebraic properties of FI-graphs. Namely, we consider the spectra of these graphs, and how they can vary with n. Recall that if G is a graph, then the **adjacency matrix** A_G of G is the endomorphism of the $\mathbb{Q}$-linearization of the vertex set $V(G)$ of G that maps a vertex x to the sum of vertices it is adjacent to. The **spectrum** of the graph G is then defined to be the spectrum of A_G. Spectral analysis of graphs is in many ways the foundation of algebraic graph theory [Bi, CDS], and has found applications to the study of Markov chains [LPW], and graph limits [Lo], among many others.

Let $G_\bullet$ be a finitely generated FI-graph, and write A_n for the adjacency matrix A_{G_n}. While it is the case that the functor $n \mapsto \mathbb{Q}V(G_n)$ is a finitely generated FI-module, it is not the case that the collection $\{A_n\}$ can be extended to an endomorphism of this FI-module. Despite this fact, it is undefinable that these maps are strongly related to the FI-structure underlying $G_\bullet$. This intuition was made precise by Speyer, White, and myself in [RSW]. In fact, [RSW] works within the much more general context of relations between FI-sets, that the edge relation of an FI-graph is an example of. One consequence of that work is the following.

Theorem 2.6 (Ramos, Speyer, & White [RSW]). *Let $G_\bullet$ be a finitely generated FI-graph. For each n, the matrix A_n is real and symmetric, whence its distinct eigenvalues can be written as*

$$\lambda_1(n) < \ldots < \lambda_{r(n)}(n)$$

for some function $r(n)$. Then for all $n \gg 0$:

1. *$r(n) = r$ is independent of n. In particular, the number of distinct eigenvalues of A_n is eventually independent of n;*
2. *for each i the function $n \mapsto \lambda_i(n)$ is algebraic over the field $\mathbb{Q}(n)$;*
3. *for each i, the multiplicity of $\lambda_i(n)$ agrees with a polynomial.*

2.3. **The major project.** In this project, we turn our attention to various extremal invariants of FI-graphs. For instance, one such example of these invariants is the aforementioned chromatic number. In examples of FI-graphs where the chromatic number is known, it is conspicuously the case that it agrees with a polynomial in n. This includes the case of the Kneser graphs, whose chromatic number was computed in famous work of Lovász [Lo2]. On the other hand, there are many important examples of finitely generated FI-graphs whose chromatic numbers are still not known. This includes, for instance, the Johnson graphs.

Major Question B. Is it the case that the chromatic number of a finitely generated FI-graph must agree with a polynomial, or, perhaps, a quasi-polynomial? More generally, how does the FI-formalism interact with questions about minimums, such as in the case of the chromatic number, and maximums, such as in the case of **independence numbers**?

The more general question posed here about minimums and maximums has received attention recently in various places throughout the literature. We will later see how it appears in the work of Bahran [B], and it has recently appeared in work of Le, Nagel, Nguyen, and Römer [LNNR] on the projective dimensions of ideals within a certain FI-algebra.

The long history of graph theory would suggest that computing these chromatic numbers is extremely difficult, and so it is perhaps prudent to begin with the portion of this question that is more tractable. Recall that for a graph G, one defines its independence number, $\alpha(G)$, as the size of the largest collection of vertices that are pairwise not connected by an edge. We will consider the question of the asymptotic behavior of the function $n \mapsto \alpha(G_n)$ whenever $G_\bullet$ is a finitely generated FI-graph.

Given a graph G, we may define a polynomial ring A_G whose variables are indexed by the vertices of G. The **edge ideal** of G is defined to be the ideal generated by the monomials x_ix_j, where $\{i, j\}$ is an edge of G. It is a straight forward computation to show that the codimension of the edge ideal is precisely $\alpha(G)$.

Moving back to the context of FI-graphs, one may associate to $G_\bullet$ an FI-algebra $A_{G_\bullet}$. The collection of edge ideals of A_{G_n}, with n varying, then becomes an ideal within this FI-algebra. Putting it all together, one may therefore reduce our question about the independence number of an FI-graph to the question of the behavior of codimensions of monomial ideals in FI-algebras. Questions of this type are precisely the subject of [LNNR], where they prove that the behavior is polynomial under certain strong conditions. I believe that the techniques of [LNNR] can be directly generalized to work in our setting. This approach is also especially attractive myself, as it involves mixing tools from commutative algebra, combinatorics, and algebraic geometry.

2.4. **Other future directions.** While the my work with Speyer, and White has established the foundations for studying FI-graphs, there is still much left to be understood about these objects. To begin, we have seen that finitely generated FI-graphs can be associated to a large variety of polynomials arising from combinatorial questions. These include the polynomials that count the number of occurrences of certain fixed substructures, as well as the polynomials that count homomorphisms into the FI-graph. We have already seen that at least one of these polynomials has appeared in the literature as the chromatic polynomial. We are therefore left with the following question.

Question 2.7. What can be said about polynomials associated to FI-graphs that go beyond simple degree bounds? Work of Huh implies that the coefficients of the chromatic polynomial form a sign-alternating log-concave sequence [Hu]. Can statements of this form be proven for homomorphism numbers into other FI-graphs? The chromatic polynomial is known to have interesting counting properties at non-positive integer values, are similar statements true about homomorphism numbers into other FI graphs?

We have seen how the combinatorics of FI-graphs reveals interesting consequences about random walks on these graphs. In the paper [RW2], it is shown that the mixing time of such a random walk, as a function of n, can be understood in terms of the mixing time of an associated Markov process on a state space of fixed size. Aside from this, however, much is still not understood about mixing times of FI-graphs. Specifically I am interested in understanding when random walks on these objects exhibit the cutoff phenomenon. This property of families of Markov chains was formally introduced by Diaconis [D], building off of work of Diaconis and Aldous [AD], and has since been extensively studied [BHP, LS].

Question 2.8. It can be shown that in many of the most typical examples of finitely generated FI-graphs, the sequence of Markov chains obtained by considering random walks exhibits cutoff. However, one can also construct examples of FI-graphs that do not have this property, albeit by exploiting certain pathological constructions. Is there a natural condition that one can impose on a finitely generated FI-graph to assure random walks on it exhibit cutoff? Conversely, for those examples that do not exhibit cutoff, are there weaker properties of the mixing time that can be still shown to appear. Finally, what results can be proven about other Markov process one can perform on FI-graphs, such as those related to chip-firing and sand pile games?

Moving on to future directions in topology, we begin with questions about configuration spaces. Note that while we consider these configuration spaces specifically in how they relate to FI-graphs here, we dedicate an entire future section to other questions about these spaces, and related structures. Above, it was shown that if $G_\bullet$ is a finitely generated FI-graph, then the associated FI-modules $H_q(\mathrm{Conf}_m(G_\bullet))$ are finitely generated. The proof of this fact is non-constructive, and consequently does not provide bounds on the degree of generation. While work of Lütgehetmann [Lu] provides such bounds in a very particular circumstance, nothing is known otherwise.

Question 2.9. Can bounds on generating degree be given in terms of the graphs G_n, the number of points m, and the homological degree q?

Another topological application of FI-graphs was recently discovered by Bahran [B]. For each n, let $G_n^{a,p}$ denote the graph whose vertices are labeled by elements of the symmetric group of prime order p with at most a cycles in their cycle decomposition, and whose edges indicate that the associated permutations commute. Then, for any fixed p, a, the family $G_\bullet^{p,a}$ form a finitely generated FI-graph. In his work [B], Bahran considers the clique complex of the graphs $G_n^{p,a}$; the abstract simplicial complex formed by the complete graphs inside of $G_n^{p,a}$. These spaces are of great interest in the study of finite groups [Ks, Ks2]. Using the framework of FI-graphs, Bahran is able to prove that these simplicial complexes are what he calls **highly acyclic**, i.e. that the index of their lowest non-vanishing homotopy group can be made arbitrarily big with n.

Question 2.10. Let G_n^p be the graph whose vertices are labeled by the elements of $\mathfrak{S}_n$ of order p, without any condition on the number of cycles, and whose edges are as with $G_n^{p,a}$. Bahran says that his study of $G_n^{p,a}$ was motivated by an attempt at understanding the more general G_n^p [B]. The FI-graph $G_\bullet^p$ is unfortunately not finitely generated, although it is very nearly so. Can the theory of finitely generated FI-graphs be expanded so as to prove similar results to Bahran? Otherwise, is there an action by a different category that makes the family $G_\bullet^p$ finitely generated in some other sense?

3. Stable properties in families of groups and property (T)

3.1. **Background.** Having already discussed FI-graphs, FI-sets, and FI-modules, our next direction will be to consider **FI-groups**. An FI-group is a functor $\Gamma_\bullet$ from FI to the category of discrete groups. We say that an FI-group is **finitely generated** if there exists a finitely generated FI-subset $S_\bullet$ of $\Gamma_\bullet$, such that S_n is a generating set of Γ_n for all $n \gg 0$. As examples, one sees that the symmetric groups themselves form a finitely generated FI-group, where S_n is the set of transpositions and FI acts by conjugation. Free groups, automorphism groups of free groups, and $\mathrm{SL}_n(\mathbb{Z})$ are also interesting examples of these objects. One should keep the latter two examples in mind for what follows.

Much of my work in the study of FI-groups has been in trying to understand how certain geometric-group-theoretic properties behave among the Γ_n as $n \gg 0$. Specifically, I have considered this question for **property (T) of Kazhdan**, **property FA of Serre**, as well as the **geometric Cayley graph property**. These three properties have been central to modern geometric group theory since their respective conceptions [BdLHV, Ka, KKN, M, O, Se, Sh, Z]. Famously, property (T) was used by Margulis to provide the first systematic approach to constructing **expander graphs** [M]. See [BdLHV] for a detailed exposition on all of these properties.

We say that a group Γ has property (T) if the trivial representation is isolated in the unitary dual of Γ. Roughly speaking, one can think of this as saying that any unitary representation that contains a vector that is only moved within an ϵ-ball of its starting position by a compact subgroup of Γ must actually contain a vector that is fixed by the action. Property FA, on the other hand, is a discrete version of (T) that requires that all actions of Γ on a tree admit a fixed point.

Finally, the Cayley graph property is a feature of a group Γ paired with a chosen (finite) generating set S, not containing the identity. Define the **level one Cayley graph** $C^{(1)}_{\Gamma,S}$ of (Γ, S) to be the induced subgraph of the usual Cayley graph generated by the elements of S. Then we say that (Γ, S) has the geometric Cayley graph property if $C^{(1)}_{\Gamma,S}$ is connected and its **Laplacian matrix** - the difference of the diagonal matrix of vertex degrees with the adjacency matrix of $C^{(1)}_{\Gamma,S}$ - has smallest non-zero eigenvalue strictly bigger than $1/2$.

3.2. **Stability in FI-groups.** In previous sections, we saw how finite generation conditions had drastic structural impacts on FI-graphs, sets, and modules. The first difficulty one faces with FI-groups, that is similarly found in the study of FI-algebras [NR], is that this is no longer necessarily the case. FI-groups come in a variety of different forms with the condition of finite generation having seemly little impact on what kinds of group theoretic properties do and do not appear. We therefore shift our focus from questions of the form "what kinds of group theoretic properties are shared among finitely generated FI-groups," to questions of the form "what kinds of group theoretic properties are **stable** in finitely generated FI-groups?"

We say that a group property $\mathcal{P}$ is stable, if, for any finitely generated FI-group $\Gamma_\bullet$, there exists some $N \gg 0$ such that for all $n \geq N$, Γ_n has property $\mathcal{P}$ if and only if Γ_N does. As simple examples, one immediately sees that "being finite" is a stable property, whereas "having even rank" is not. Our primary focus is in answering which of the properties discussed in the previous subsection are stable. The proof of the following theorem uses the aforementioned theory of FI-graphs in a non-trivial way.

Theorem 3.1 (Ellenberg and Ramos [ER])**.** *Let $(\Gamma_\bullet, S_\bullet)$ be a finitely generated FI-group with generating FI-set $S_\bullet$. Then property FA, as well as the geometric Cayley graph property, are stable properties of $(\Gamma_\bullet, S_\bullet)$.*

3.3. **The major project.** Recall from our examples that $SL_\bullet(\mathbb{Z})$ and $\mathrm{Aut}(F_\bullet)$ are both examples of finitely generated FI-groups. It has been known since the original work of Kazhdan that $SL_n(\mathbb{Z})$ has property (T) for $n \geq 3$ [Ka]. Later, work of Shalom reproved this fact using more algebraic means [Sh]. What is relevant to us, however, is the very recent combinatorial approach of Kaluba, Kielak, and Nowak [KKN]. In this work, Kaluba, Kielak, and Nowak use a characterization of property (T) proposed by Ozawa [O] to show that both $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{Aut}(F_n)$ have property (T) for $n \geq 5$. Notable in this work is that their approach is fundamentally tied to the theory of FI-groups, though they do not use this language. We therefore have the following.

Major Question C. Is property (T) a stable property of finitely generated FI-groups? The approach of Kaluba, Kielak, and Nowak can be translated into statements about the combinatorics of the FI-groups $\mathrm{SL}_\bullet(\mathbb{Z})$ and $\mathrm{Aut}(F_\bullet)$. Is it possible that their techniques can be generalized to statements about other FI-groups?

It is famously known that the geometric Cayley graph property implies property (T), which in turn implies property FA. The work of Ellenberg and I therefore seems to strongly suggest that the above question should have a positive answer. That being said, its proof has thus far evaded our various attempts. The approach to this question through the work of Kaluba, Kielak, and Nowak seems to be the best direction of attack for this problem.

3.4. **Other future directions.** Implicit in any theorem about stable properties is the question of when this stable behavior begins. In our work, Ellenberg and I have bounded stable ranges for property FA and the geometric Cayley graph property in terms of simple invariants of the FI-group. In the case where one can provide an affirmative answer to Question C, it is critical that we also be able to bound this stable range.

Question 3.2. Assuming that property (T) is a stable property of finitely generated FI-groups, can one provide bounds on when this stable behavior begins?

4. The configuration spaces of graphs

4.1. **Background.** The study of the configuration spaces of graphs has seen a recent surge in popularity due to their connection with robotics and physics [Far, G, MS, MS2]. This is also evidenced by the upcoming AIM workshop on the subject. In what follows, for any graph G, we will largely concern ourselves with unordered configurations, $\mathrm{UConf}_n(G)$, the quotient of the usual configuration space by the action of the symmetric group obtained by permuting coordinates in the obvious way.

It is a theorem of Abrams [A], that $\mathrm{UConf}_n(G)$ is a $K(\pi, 1)$ for all $n \geq 1$ and all graphs G. It follows from this that in order to understand the homotopy type of $\mathrm{UConf}_n(G)$, it largely suffices to understand the fundamental group $B_nG := \pi_1(\mathrm{UConf}_n(G))$. In the literature, the braid groups B_nG have largely been studied from the perspective of geometric group theory. For instance, it is now fairly well understood when these groups are right-angle Artin [KP, KKP].

For the purposes of this proposal, our primary interest are related to homology groups $H_i(B_nG)$. It follows from the discussion in the previous paragraph the singular homology of the space $\mathrm{UConf}_n(G)$ is canonically isomorphic to the group homology of B_nG. Using this isomorphism, authors such as Kim, Ko, and Park [KKP, KP], as well as Farley and Sabalka [FS], have been able to prove surprising facts about the homology groups $H_i(B_nG)$. For instance, Ko and Park have shown that $H_1(B_2G) \cong H_1(B_nG)$ for all $n \geq 2$ whenever G is biconnected [KP]. Farley has also used the isomorphism to provide a computational method for determining the groups $H_i(B_nG)$ whenever G is a tree [Fa]. All of these results use a discrete Morse structure on $\mathrm{UConf}_n(G)$, that was developed by Farley and Sabalka [FS]. In my work, I have also used this isomorphism to prove non-trivial facts about the homology of the unordered configuration spaces of trees.

4.2. **Stability phenomena in the homology of tree braid groups.** The philosophy behind my work on the configuration spaces of graphs is based in the philosophy of asymptotic algebra as a whole. Namely, whenever a family of algebraic objects exhibits asymptotic stability phenomena, it is often the case that they can be encoded in a single object, that is finitely generated in the appropriate sense. In my work on the configuration spaces of trees, I showed that the homology groups $H_i(B_nG)$ can be encoded in a finitely generated graded module over a polynomial ring.

For a graph G, an **essential vertex** is any vertex of degree at least 3. An **essential edge** is a connected component of the space obtained by removing all essential vertices of G. We also define the quantity Δ_G^i to be the maximum number of connected components that G can be broken into by removing exactly i vertices.

Theorem 4.1 (Ramos [R3])**.** *Let G be a tree, and fixed $i \geq 0$. Then there is a polynomial $P_i^G \in \mathbb{Q}[t]$ of degree $\Delta_G^i - 1$ such that for all $n \geq 0$*

$$P_i^G(n) = \dim_{\mathbb{Q}}(H_i(B_nG; \mathbb{Q}))$$

This was proven using the structure theorems of Farley and Sabalka [FS], as well as the computational theorems of Farley [Fa]. In fact, I computed the polynomials P_i^G explicitly in terms of certain invariants of the tree G in [R3]. It follows from this computation that the homology groups $H_i(B_nG)$ do not fully depend on the tree G.

Corollary 4.2 (Ramos [R3])**.** *Let G be a tree. Then the homology group $H_i(B_nG)$ depends only on i, n, and the degree sequence of G.*

The key insight to proving the above theorem, that is perhaps more significant than the result itself, is that the groups $H_i(B_nG)$ carry a natural action by a polynomial ring.

Theorem 4.3 (Ramos [R3])**.** *Let G be a tree, and let S_G denote the integral polynomial ring with variables indexed by the essential edges of G. Then for each $i \geq 0$, there is an action of S_G on the graded $\mathbb{Z}$-module $\mathcal{H}_i := \bigoplus_n H_i(B_nG)$, turning $\mathcal{H}_i$ into a finitely generated graded S_G-module. Moreover, $\mathcal{H}_i \otimes \mathbb{Q}$ decomposes as a direct sum of graded twists of squarefree monomial ideals of Krull dimension at most Δ_G^i.*

The first obvious question arising from my work is whether it can be applied to more general graphs. Indeed, in [R3] it is shown that the action of S_G on $\mathcal{H}_i$ will still be well defined, provided certain diagrams commute. This was proven by An, Drummond-Cole, and Knudsen in a recent paper [ADK]. In fact, in a follow up paper [ADK2] the following theorem was proven, resolving a conjecture of myself.

Theorem 4.4 (An, Drummond-Cole, and Knudsen)**.** *If G is a graph, that is neither a line segment nor a circle, then $n \mapsto \dim_{\mathbb{Q}}(H_i(B_nG;\mathbb{Q}))$ is eventually agrees with a polynomial of degree $\leq \Delta_G^i - 1$.*

4.3. **The major project.** Throughout this proposal we have seen that there are two orthogonal ways to think about stability in unordered graph configuration spaces: varying the number of points, and varying the graph itself. These two approaches are most exemplified in Theorem 4.3 and Theorem 1.2, the latter of which can be proven in the unordered case using similar means to the original. It then becomes a natural question to ask whether these two different approaches to finite generation can be made compatible. This was partially accomplished by myself in [R7], although it is likely the case that something much more general is true.

Given a contraction of graphs $G \to G'$, there is a natural inclusion of the edges of G' into the edges of G. In particular, the above contraction induces a map of algebras $S_{G'} \to S_G$. This inspires the definition of the **canonical $\mathcal{G}_g^{op}$-algebra**, $S_\bullet$. Recalling that one may view FI as a subcategory of $\mathcal{G}_0^{op}$, the algebra $S_\bullet$ may be viewed as a generalization of the FI-algebra $X_\bullet$, where $X_n = \mathbb{Z}[x_1, \ldots, x_n]$. This particular FI-algebra has received a large amount of attention from the perspective of combinatorial commutative algebra [NR, LNNR].

One result of particular importance to us is that modules defined over the FI-algebra $X_\bullet$ - i.e. collections of X_n-modules M_n, which are compatible with the action of FI - satisfy a Noetherian property [NR]. This leads us into our main technical question

Major Question D. Does the canonical $\mathcal{G}_g^{op}$-algebra satisfy a Noetherian property generalizing that of $X_\bullet$?

The proof that $X_\bullet$ has such a property is performed via a Gröbner basis argument, making heavy use of the combinatorics of finite sets and Higman's lemma. It is therefore my belief that these arguments can be adapted to our setting, although certain difficulties may arise in the details. Specifically, one must work with the combinatorics of finite graphs and Kruskal's Tree Theorem.

It is notable that an affirmative answer to the above question could be used to strengthen many of the aforementioned theorems about graph configuration spaces. For instance, the version of Corollary 1.3 for unordered configuration spaces would no longer have a dependence on the number of points being configured. It is also notable that one could use Noetherianity to prove boundedness results about Betti-numbers in the study of edge-ideals. These bounds would directly generalize similar statements in [NR].

4.4. **Other future directions.** The aforementioned theorems of An, Drummond-Cole, and Knudsen are some of the few results that one has about the S_G-module structure of the homology groups of B_nG, whenever G is a general graph. While it might be too much to ask to be able to prove a structure theorem similar to Theorem 4.3 for general graphs, there seems to be hope in studying these modules within certain families of graphs. For instance, I have recently shown that certain families of FI-graphs exhibit very restrictive behavior in the Betti numbers, in the commutative algebra sense, of their unordered configuration spaces [R4].

Question 4.5. Let $\mathcal{P}$ denote the class of graphs with some particularly nice property. For instance, $\mathcal{P}$ might denote the class of regular graphs, planar graphs, or graphs of some restricted maximal vertex degree. What results can one prove about the structure of $H_i(B_nG)$ as a module over S_G for $G \in \mathcal{P}$?

The theorem of An, Drummond-Cole, and Knudsen is also striking in that it provides an explicit connection between the modules $H_i(B_nG)$ and connectivity invariants of the graph G. This leads to the following question.

Question 4.6. What graph theoretic invariants are encoded in the commutative algebra of $H_i(B_nG)$? For instance, what does the Hilbert Polynomial of this module tell us about G?

Another question that one might ask is whether anything can be said about the usual configuration spaces $\mathrm{Conf}_n(G)$. The current consensus in the literature is that these spaces are considerably more complicated than the unordered configuration spaces, and very little is known about their behavior [BF, CL]. The aforementioned work of White and I [RW], Proudfoot and I [PR, PR2], as well as Lütgehetmann [Lu], has provided evidence that it is perhaps easier to think about these spaces by not allowing n to vary. That being said, just as in the unordered case, one would like to understand these spaces without this restriction.

Question 4.7. What is the correct way to approach configuration spaces of graphs, while allowing the number of points being configured to vary? Considering the polynomial ring action described above in the unordered case, should one expect the correct action here to be by a **twisted commutative algebra** as described by Sam and Snowden [SS]?

As our final question on graph configuration spaces, we consider the problem of how much of a graph G can be recovered from the spaces $\mathrm{UConf}_n(G)$. One might we recall the statement that the homology groups $H_i(B_nG)$ are determined by i, n, and the degree sequence of G, whenever G is a tree. In fact, it is not hard to show that $\pi_1(B_2G)$ is uniquely determined by the degree sequence of G in this case. Using Abrams' result that $\mathrm{UConf}_2(G)$ is a $K(\pi, 1)$, one concludes that the homotopy type of $\mathrm{UConf}_2(G)$ is determined only by the degree sequence of the tree G. In particular, one cannot recover G from $\mathrm{UConf}_2(G)$ using traditional topological invariants.

To get around this obstacle, Levin, Young, and I have introduced a random process, that can recover G from $\mathrm{UConf}_2(G)$. We define a Markov process on (unordered) pairs of vertices of G by first flipping a coin to see that vertex in the pair will change, and then uniformly at random choosing an edge adjacent to this chosen vertex to move along. If the two original vertices were connected by an edge, and this is the edge chosen to move along, the process whiffs. Calling the transition matrix of this Markov process P_G, we have shown that G is uniquely recoverable from P_G, whenever G is a tree. However, we also believe that something stronger is true.

Assume for simplicity that we have chosen an embedding of our tree G into the plane, and that we have designated a leaf of G to be the root. Then one obtains a well ordering on vertices via a depth-first numbering originating from the root. Finally, assume that our two particles began the random process occupying the smallest two vertices in this order. Running the Markov process for some fixed number of steps defines a path in the configuration space $\mathrm{UConf}_2(G)$. Levin, Young, and I then define the **closure** of this path to be the loop obtained from the path by allowing the two points to flow back towards the root, one at a time, where the first point to move is that that is currently sitting on the smaller vertex. From the closure operation, one obtains an element of $H_1(B_2G)$. Assuming that G is a tree, $H_1(B_2G)$ is a free group, whence one can study this closure as a kind of (multivariate) statistic associated to the walk.

Question 4.8. Can one recover G from this statistic?

Questions of this form have precedence in the study of random processes on topological spaces. For instance classic, work of Spitzer [Sp] considered the winding behavior of two particles in the plane, each performing independent Brownian walks. Spitzer's work lead to a plethora of similar works related to winding of randomly moving particles on surfaces [RH, WT]. One may think of this work of Levin, Young, and I as trying to understand winding of a random process within a 1-dimensional topological space.

5. Homological invariants of FI-modules

5.1. **Background.** Having reviewed my most recent work in algebraic combinatorics, we now step back to my earlier works, which are more traditionally homological in nature. While many of my initial conjectures in these directions have been proven (see [NSS], for instance), there is still much that we do not understand.

The Noetherian property is the first step in treating finitely generated FI-modules from a homological perspective. Prior to my own work, this philosophy is most apparent in works of Church and Ellenberg [CE], Sam and Snowden [SS], and Gan and Li [GL]. In [CE], Church and Ellenberg consider **homology functors** H_i for FI-modules. If V is an FI-module, then $H_0(V)$ is defined to be the FI-module with points $H_0(V)(T) = V(T)/\cup_{T'\subsetneq T} V(T')$. The homology functors H_i are defined to be the (left) derived functors of H_0. It follows directly from the definition that the module $H_0(V)$ is **finitely supported**, i.e. $V(T) = 0$ for $|T| \gg 0$, whenever V is finitely generated. Remarkably, it can be shown that for any finitely generated module V , the quantity $\max_n\{H_i(V)(n) \neq 0\} - i$ is bounded independently of i. This was first proven for certain choices of k by Sam and Snowden in [SS], and expanded to allow for any commutative ring k by Church and Ellenberg [CE]. This bound, which we shall henceforth refer to as the **regularity** of V, was used by Church and Ellenberg to generalize certain stability results of Putman on congruence subgroups [Pu]. Note that the regularity in this context is exactly analogous to the notion of Castelnuovo-Mumford regularity, a concept of foundational importance to commutative algebra [E]. Understanding the mechanisms underlying the finiteness of regularity therefore becomes critical in explaining certain concrete phenomena in the study of congruence subgroups.

Another property connected to homological invariants is the existence of the **Hilbert Polynomial**. If V is a finitely generated FI-module over a field k, then there exists a polynomial $P(X) \in \mathbb{Q}[X]$ such that for all finite sets T with $|T| \gg 0$,

$$\dim_k(V(T)) = P(|T|)$$

This fact was proven in the case where k is a field of characteristic 0 by Church, Ellenberg, and Farb in [CEF], and was expanded to all fields by Church, Ellenberg, Farb, and Nagpal in [CEFN]. Seeing a result like this, it therefore becomes quite natural to ask when this polynomial behavior begins. A bound is given by Church, Ellenberg, and Farb in the cases wherein k is a field of characteristic 0 in [CEF], although no bound is given in the cases in which k is a general field. My work reveals that this question, as well as that of the regularity of a module, are deeply connected to one another through the study of local cohomology.

5.2. **Local Cohomology.** Let V be an FI-module over a Noetherian ring k. Then we say that an element $v \in V(T)$ is **torsion** if there is some finite set T' and an injection $f : T \to T'$ such that v is in the kernel of the map induced by f.We define $H^0_{\mathfrak{m}}(V)$ to be the maximal torsion submodule of V. The assignment $V \mapsto H^0_{\mathfrak{m}}(V)$ defines a left exact functor, and we denote its derived functors by $H^i_{\mathfrak{m}}$. These are the **local cohomology functors**.

In the case where k is a field of characteristic 0, Sam and Snowden studied the local cohomology functors, and proved that many important invariants of FI-modules, such as regularity, were encoded by them [SS]. When k is a field of characteristic 0, however, Sam and Snowden show that FI-modules can be equivalently thought of as a subclass of modules over a polynomial ring in infinitely many variables [SS]. The first challenge to generalizing the results of Sam and Snowden to general coefficient rings was in finding a consistent language which would allow us to use the commutative algebra techniques of their work. This was accomplished in my work with Li. [LR].

To begin, we show that the modules $H^i_{\mathfrak{m}}(V)$ can always be computed within the category FI-mod of finitely generated FI-modules, so long as V is finitely generated.

Theorem 5.1 (Li and Ramos [LR])**.** *Let V be a finitely generated* FI*-module over a Noetherian ring k. Then for each i, the module $H^i_{\mathfrak{m}}(V)$ is finitely generated, and $H^i_{\mathfrak{m}}(V)(T) = 0$ for all T with $|T| \gg 0$. Moreover, $H^i_{\mathfrak{m}}(V) = 0$ for $i \gg 0$.*

Considering the theory of local cohomology in the context of local rings, the above might be quite surprising. Indeed, classically it is known that the top non-vanishing local cohomology module is never finitely generated [BS].

The above theorem implies that each finitely generated module V has a well defined smallest, and largest, non-vanishing local cohomology module. We define the **depth** of V to be the smallest index i such that $H^i_{\mathfrak{m}}(V) \neq 0$. In [R], I introduced a definition of depth which can be used for FI-modules over any ring. It is a theorem of Li and myself these two definitions are equivalent [LR].

Theorem 5.2 (Li and Ramos [LR])**.** *Let V be a finitely generated* FI*-module over a Noetherian ring k. Then the regularity of V is at most* $\max_i\{\deg(H^i_{\mathfrak{m}}(V)) + i\}$.

The quantity $\max_i\{\deg(H^i_{\mathfrak{m}}(V)) + i\}$ is very reminiscent of the Castelnuovo-Mumford regularity of a module over a polynomial ring [E]. In fact, It was proven by Nagpal, Sam, and Snowden that it is equal to the regularity of the FI-module V, just as is the case in the classical setting. [NSS]

Theorem 5.3 (Li and Ramos [LR])**.** *Let V be a finitely generated* FI*-module over a field k. Then the dimension* $\dim_k(V(T))$ *agrees with a polynomial for $|T|$ at least* $\max_i\{\deg(H^i_{\mathfrak{m}}(V))\}$.

Note that the above two theorems provide an explicit connection between the regularity of a module and the obstruction to its Hilbert polynomial. Once again, such a connection exists and is classically known about modules over a polynomial ring [E].

5.3. **Future Directions.** While my work with Li lays the groundwork for a "commutative algebra" approach to the study of FI-modules, there is still much which is not well understood. To begin, very little has been invested in trying to compute these local cohomology groups. Finite generation of these modules implies that such a theory is, at the very least, not completely hopeless. Assuming such tools exists, the theorem of Nagpal, Sam, and Snowden [NSS] would provide explicit means for computing the regularity of an FI-module. The work of Church and Ellenberg [CE] implies that these computations have real applications in the study of congruence subgroups.

Question 5.4. Can one develop a computational theory of the local cohomology of FI-modules, perhaps building on the computational theory of FI-modules developed by Wiltshire-Gordon [W-G]?

It is also an interesting question to ask how these homological invariants are interacting with the various topological examples of FI-modules. While the theorems of the previous section can be used to bound the obstruction to the Hilbert polynomial, for instance, it is often the case in practice that these bounds are non-optimal. This would suggest that "niceness" in the topology from which the example arises is being observed by the local cohomology modules. Can this be formally described? Conversely, can one deduce facts about the topology of a provided example, given its local cohomology? A new paper of Church, Miller, Nagpal, and Reinhold [CMNR] does some work in this direction, but there is apparently much more that one might be able to say.

Question 5.5. Let $X_\bullet$ be a functor from FI to the category of topological spaces, and assume, for all i, that $H_i(X_\bullet)$ is finitely generated as an FI-module over $\mathbb{Z}$. What does the topology of $X_\bullet$ tell us about the local cohomology groups of $H_i(X_\bullet)$? Conversely, what does the local cohomology of $H_i(X_\bullet)$ reveal about the topology of the spaces X_n?

Finally, while FI-modules are arguably the most well understood examples of representations of combinatorial categories, there is a large variety of other categories whose representation theory have become extremely relevant in the literature. For this reason it has become an interesting open question as to how much of the above formalism can be adapted to these different circumstances. For instance, work of the author and Li has illustrated similarities between the local cohomology theory of what are known as FI^m-modules, and the local cohomology of multi-graded modules over a polynomial ring [LR2], while work of Sam and Snowden has shown a deep connection between the local cohomology of what are known as FI_d-modules and the geometry of Grassmanians [SS2]. Moving forward I am interested in studying homological invariants of VI(q) and VIC(q)-modules. One may think of these categories as "q-versions" of FI, as their representation theory encodes families of $GL_n(q)$-modules [PS]. Work of Miller and Wilson [MW], Miller, Patzt, and Wilson [MPW], and Nagpal [N2, N3] have displayed a variety of applications one would unlock by developing a robust theory of the homological invariants of these categories.

Question 5.6. Building on recent work on Nagpal [N2, N3] and Harman [Ha], can one develop a satisfactory theory of homological invariants of VI(q) and VIC(q)-modules similar to the theory for FI-modules described above? In particular, can one prove strong finiteness results of regularity of VIC(q)-modules which improves upon the work of Miller and Wilson [MW]?

6. Coherence in the representation theory of categories

6.1. **Background.** In the previous sections, we saw that there is a large premium put on the condition that an FI-module be finitely generated. However, for one to have a Noetherian property, it is necessary that one work over a Noetherian ring. For some applications one would like to work with FI-modules over rings which are not Noetherian. Once again pulling from commutative algebra, perhaps the next best thing after Noetherianity is coherence. We say an FI-module over a ring k is **coherent** if it is generated by elements which only appear in finitely many degrees, and the module of relations between these generators is also generated in finitely many degrees. Note that a coherent module need not be finitely generated, but the (possibly infinitely many) generators must appear in only finitely many degrees.

6.2. **The coherence of FI-modules.** Much of the work described on the local cohomology of FI-modules was only doable because the Noetherian property allows us to treat FI-mod using the techniques of abelian categories. Our first goal will therefore be to prove something similar for coherent modules. The following theorem was independently proven by Li and myself.

Theorem 6.1 (Li [Li], Ramos [R2]). *The category of coherent* FI*-modules over a commutative ring k is abelian. That is, the kernel and cokernel of any morphism of coherent* FI*-modules are themselves coherent.*

Perhaps the most notable fact about the above theorem is that it does not have any conditions on the ring k. This implies that while FI-modules may not be Noetherian, they are always coherent.

The regularity theorem of Church and Ellenberg, along with the above theorem, seems to imply that the work completed in Section 5.2 should still hold for coherent FI-modules. This is indeed the case.

Theorem 6.2 (Ramos [R2])**.** *The theorems of Section 5.2 continue to hold with finitely generated replaced in all places by coherent, and the Noetherian hypothesis removed from the ring k.*

6.3. **Future directions.** While many of the various well studied concrete examples of FI-modules often turn out to be finitely generated, more recent examples have shown that coherent modules also arise naturally. For instance, the work of Church and Ellenberg provides examples of this kind in the study of congruence subgroups for general rings [CE]. What remains interesting is the question of whether finite generation is something one should expect to find in more general and nuanced examples. For instance, if one were to change FI to some more general category, whose representations are not necessarily Noetherian, can something still be said about coherent representations?

As an example, consider the category $\mathrm{VI}(\mathbb{Z})$, of finite rank free $\mathbb{Z}$ modules, with split linear injections. Representations of this category were considered by Putman and Sam in [PS], and have more recently been used in the work of Patzt [Pa] and Miller, Nagpal, and Patzt [MNP]. Representations of similar categories were critical in the resolution of the Lannes-Schwartz Artinian conjecture [PS]. It was proven by Putman and Sam that representations of $\mathrm{VI}(\mathbb{Z})$ will not have the Noetherian property over any ring k. However, the work of Patzt [Pa] seems to imply that these representations appear in nature. In that work, Patzt specifically considers their applications to the study of Torelli groups.

I am interested in studying $\mathrm{VI}(\mathbb{Z})$ representations, and especially interested in understanding whether they satisfy any kind of coherence. A result in this direction would provide at least a framework for approaching the many natural examples of $\mathrm{VI}(\mathbb{Z})$ representations without needing to rely on any kind of Noetherian hypothesis. It would also expand upon results of Patzt [Pa].

Question 6.3. If k is a field of characteristic 0, can one prove a structure theorem similar to that of Nagpal [N] for coherent $\mathrm{VI}(\mathbb{Z})$-modules?

The previously mentioned work of Miller, Nagpal, and Patzt [MNP] also considers a property that they call Koszulness. This is used to great affect in that paper, and suggests that there are notions which fall between finite generation and coherence which might be sufficient for many applications. This particular philosophy appears in work of Harman [Ha], where he considers a condition he calls rigidity.

Question 6.4. What finiteness properties of $\mathrm{VI}(\mathbb{Z})$-modules, and related objects, can one leverage to prove non-trivial facts about the growing list of examples of these objects?

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