ERIC RAMOS RESEARCH STATEMENT

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My primary research focus is in combinatorics and its interactions with representation theory, topology, and algebraic geometry. I am particularly interested in using the representation theories of certain combinatorially flavored categories to exhibit and explain certain stability phenomena throughout topology, algebraic geometry, and other fields. In the following research statement, I will outline four major projects that drive my present research program. These projects aim to further the state of the art in the representation theory of certain combinatorially flavored categories, and then use this advancement to solve concrete problems naturally arising from algebraic and topological combinatorics. As many of the relevant subject areas are fairly new, there are a plethora of directions that are still unexplored. I will therefore outline smaller projects that would also be interesting to pursue beyond the major ones. To conclude, I will spend some time outlining my older work, as well as various problems that remain unsolved regarding those works.

Before we provide formal definitions, we begin with an example for motivation. For any finite set T, and any topological space X, we write

$$\mathrm{Conf}_T(X) := \{(x_t)_{t \in T} \mid x_t \neq x_{t'}\},$$

to denote the topological space of injections from T to X. If T' is any other finite set, and there is an injection $T \hookrightarrow T'$, then precomposition naturally defines a map of topological spaces $\mathrm{Conf}_{T'}(X) \to \mathrm{Conf}_T(X)$. For any fixed i, the cohomology groups $H_i(\mathrm{Conf}_\bullet(X); k)$ therefore inherit an interesting structure. Namely, for each n $H_i(\mathrm{Conf}_n(X); k)$ is a $k[\mathfrak{S}_n]$-module, and between any two values of n the corresponding modules are in some sense compatible. This can be thought of as the motivating philosophy for the study of what we will call FI-modules: a means of encoding an infinite collection of compatible symmetric group representations into a single object. Using this structure, Church, Ellenberg, and Farb were able to prove many non-trivial facts about the configuration spaces of orientable manifolds [CEF].

Write FI for the category of finite sets and injections. Note that we will usually think of this category equivalently as that whose objects are the sets $[n] = \{1, \ldots, n\}$ and whose morphisms are injections. An **FI-module** over a commutative ring k is a functor from FI to the category of k-modules. These modules, as defined here, were first introduced by Church, Ellenberg, and Farb in their seminal paper [CEF]. Since their inception in the work of Church, Ellenberg, and Farb, FI-modules, and modules over other similar combinatorially flavored categories, have seen an enormous amount of use in topology, representation theory, and other fields. This has been the subject of a recent AIM workshop, an AMS special session, an Oberwolfach workshop, an MSRI summer school, and numerous other conferences.

The category of FI-modules over k, FI-Mod, is an abelian category with abelian operations defined point-wise. We say that V is **finitely generated** if there is a finite set $\{v_i\} \subseteq \bigsqcup_T V(T)$, that is contained in no proper submodule of V. This notion was first explored by Church, Ellenberg, and Farb in [CEF], although it was not fully explored until the follow-up work of Church, Ellenberg, Farb, and Nagpal [CEFN, Nagp]. In the paper [CEFN], it is shown that the category of finitely generated FI-modules over a Noetherian ring have a Noetherian property. That is, the category FI-mod of finitely generated FI-modules over a Noetherian ring is abelian. This fact was proven when k is a field of characteristic 0 by Church, Ellenberg, and Farb in [CEF] and by Snowden in [Sn]. It was proven in total generality by Church, Ellenberg, Farb, and Nagpal in [CEFN].

FI is the first, and perhaps most natural, example of what we will henceforth refer to as a **combinatorial category** - a category whose foundational structure is based in the combinatorics of some finite objects. Two of the projects proposed below involve applying our wealth of knowledge about FI-modules in new directions related to combinatorics and group theory. Not everything we discuss will be related with FI however, as another one of the objectives in my current research program is to push the boundaries of our understanding into the representation theory of combinatorial categories that are richer and less well understood than FI. We will encounter the primary such category in the next section - the category of graphs and contractions.

1. Categories of graphs

1.1. **Background.** Throughout this project description, A **graph** will always refer to a CW-complex of dimension at most one that is both connected and finite. Given a graph G we can write $V(G)$ for its set of **vertices** - or 0-cells - and $E(G)$ for its set of **edges** - or 1-cells. A **contraction** from a graph G to a graph G' is a topological map defined by selecting a sub-forest within the graph G, and contracting each component tree to obtain a new graph G'. Topology informs us that a contraction induces a homotopy equivalence between G and G', and thus necessarily $H_1(G) \cong H_1(G')$. We call the rank of $H_1(G)$ the **genus** of G. Finally, if $g \geq 0$ is an integer, we write $\mathcal{G}_g$ for the category whose objects are graphs with genus g, and whose morphisms are contractions. For instance, $\mathcal{G}_0$ is the category of trees and contractions. The primary object of study in what follows is the opposite category $\mathcal{G}_g^{op}$. In particular, we consider modules over the category $\mathcal{G}_g^{op}$.

Just as with our previous discussion of FI-modules, a $\mathcal{G}_g^{op}$-module over a Noetherian ring k is a functor from $\mathcal{G}_g^{op}$ to the category of k-modules. We say that a $\mathcal{G}_g^{op}$-module M is **finitely generated** if there is some finite list of genus g graphs $G_1, \ldots, G_N$ such that for any graph G of genus g, $M(G)$ is spanned by the images of $M(G_i)$ under various transition maps induced from contractions. We usually refer to the graphs $\{G_i\}$ as being the **generators** of the module M. The main technical theorem, that informs everything in the sequel, is the following.

Theorem 1.1 (Proudfoot & Ramos [PR, PR2])**.** *Let k be a Noetherian ring, $g \geq 0$ be an integer, and let M be a finitely generated $\mathcal{G}_g^{op}$-module. Then all submodules of M are also finitely generated.*

One should note that Theorem 1.1 generalizes the aforementioned Noetherianity for FI-modules, as one can realize FI as the full subcategory of $\mathcal{G}_0^{op}$ of **star trees** - trees with a single vertex of degree ≥ 3. One may think about the relationship between these two statements as being analogous to the relationship between **Higman's Lemma** - that sequences within well-quasi-ordered posets are once again well-quasi-ordered - and **Kruskal's Tree Theorem** - that rooted trees are well-quasi-ordered by contractions.

1.2. **The graph minor category and applications of Noetherianity.** At the end of the previous section, we described how the relationship between Noetherianity of FI and $\mathcal{G}_g^{op}$ modules is analogous to the relationship between Higman's Lemma and Kruskal's Tree Theorem. It is then natural to ask whether there exists some master category $\mathcal{G}$, generalizing both $\mathcal{G}_g$ and FI, whose Noetherianity reflects the celebrated **Graph Minor Theorem** of Robertson and Seymour [RS]. Given two graphs G, G', we say that G is a **minor** of G' if G can be obtained from G' via a sequence of edge contractions and deletions. The Graph Minor Theorem states that the graph minor relation is actually a well-quasi-order.

In [MPR], Miyata, Proudfoot, and myself constructed the category $\mathcal{G}$, which we call the **graph minor category**. Objects in this category are graphs, while the morphisms are what we call **minor morphisms**. The formal definition of minor morphism is a bit technical, but one can think of them as being built out of edge deletions, contractions, and graph automorphisms. For what follows we will write $\mathcal{G}_{\leq g}$ for the full subcategory of $\mathcal{G}$ whose objects are graphs with genus at most g. We have the following extension of Theorem 1.1.

Theorem 1.2 (Miyata, Proudfoot, & Ramos [MPR]). *If k is a Noetherian ring, and M is a finitely generated $\mathcal{G}^{op}_{\leq g}$-module, then all submodules of M are also finitely generated.*

Miyata, Proudfoot, and myself entered this project hoping to prove the obvious Noetherianity statement for the whole of $\mathcal{G}$, but have thusfar only been able to prove the weaker statement above. The desired statement could be viewed as a kind of categorification of the famous **Graph Minor Theorem**. The Graph Minor Theorem was proven by Robertson and Seymour in 20 papers over the course of 20 years. One can show that the aforementioned strengthening of Theorem 1.2 would imply the graph minor theorem. As is often the case with categorification, one may think of the categorical Graph Minor Theorem as providing an alternative yet equivalent formulation of the Graph Minor Theorem, that is perhaps more suited for certain topologically flavored applications. All this being said, however, we will see below that there are a large number of applications of even the weaker Theorem 1.2 that vastly generalize well known facts in topological combinatorics and other fields. It is obviously tantalizing to think that each of these applications can be pushed even further when the Strong version of Theorem 1.2 is eventually proven.

To illustrate these examples we will consider the configuration spaces of graphs. Configuration spaces of graphs, while not appearing much in classical literature, have recently seen a boom in their study. A later section of this statement will be dedicated to discussing the known theory of these spaces, and my contributions towards answering the natural question of, "what, if any, stable behaviors appear when one varies the number of points being configured n?" As these spaces relate to the category $\mathcal{G}^{op}_{\leq g}$, however, we will instead be interested in the orthogonal question of, "what, if any, stable behaviors appear when one varies the graph and fixes the number of points?"

In their work [ADK], An, Drummond-Cole, and Knudsen show that if $\phi : G \to G'$ is a contraction of graphs, then there is an induced map $\phi^* : H_i(\mathrm{Conf}_n(G')) \to H_i(\mathrm{Conf}_n(G))$, for any fixed integers $i, n \geq 0$. Only recently have we begun to understand where these maps are coming from at the level of topological space [ABGM]. Equipped with the existence of the maps ϕ^*, as well as our main Noetherianity theorem, we prove the following.

Theorem 1.3 (Miyata, Proudfoot & Ramos [PR, PR2, MR]). *For all $i, g, n \geq 0$, the assignment*

$$G \mapsto H_i(\mathrm{Conf}_n(G))$$

can be extended to a finitely generated $\mathcal{G}^{op}_{\leq g}$-module over $\mathbb{Z}$.

According to the definition of finite generation, the above theorem tells us that for any fixed $i, n \geq 0$, there is some finite list of generating graphs $G_1, \ldots, G_N$, such that for any graph G, every homology class in $H_i(\mathrm{Conf}_n(G))$ can be viewed as some combination of homology classes coming from $H_i(\mathrm{Conf}_n(G_j))$. One immediate consequence of this is the following.

Corollary 1.4 (Miyata, Proudfoot & Ramos [PR, PR2, MR]). *For any fixed $i, n \geq 0$, there exists an integer $d_{i,g,n}$ such that for any graph G, the exponent of the abelian group $H_i(\mathrm{Conf}_n(G))$ is at most $d_{i,g,n}$.*

Shades of this boundedness of torsion had been observed in, for instance, works of Ko and Park [KP] and Chettih and Lütgehetmann [CL]. The above result of Miyata, Proudfoot and I is the first time it was made precise. While the study of torsion in the homology of topological spaces has its own intrinsic value, it appears that the torsion in graph configuration spaces is especially compelling. Indeed, it is known [KP] that torsion in the first homology groups of **unordered configuration spaces of graphs** - i.e. the quotient of configuration space by the natural symmetric group action - exactly detects planarity of the graph.

Another application of our machinery can be seen in the realms of topological combinatorics. If G is any graph, then an ***i*-matching** on G is a collection of edges $\{e_0, \ldots, e_i\}$ with no overlapping endpoints. The collection of i-matchings, for each i, can be pieced together to form a simplicial complex $\mathcal{M}_G$. One observes that if G is a minor of G', then its edges can be found in G' in such a way that if they had non-overlapping endpoints to in G, then the endpoints will remain non-overlapping in G'. In particular, the assignment $G \mapsto \mathcal{M}_G$ defines a functor from $\mathcal{G}_{\leq g}^{op}$ to topological spaces. Composing with homology, one can prove the following.

Theorem 1.5 (Miyata & Ramos, [MR]). *For all $i, g \geq 0$, the assignment*

$$G \mapsto H_i(\mathcal{M}_G)$$

can be extended to a finitely generated $\mathcal{G}_{\leq g}^{op}$-module over $\mathbb{Z}$. In particular, for any fixed $i, g \geq 0$, there exists an integer $d_{i,g}$ such that for any graph G, the exponent of the abelian group $H_i(\mathcal{M}_G)$ is at most $d_{i,g}$.

The presence of torsion in the homology of the matching complex has been a question of interest for many years. The current state of the art suggests that these groups have torsion, but there is also a conspicuous lack of large torsion in the smaller homological degrees. It is possible that the above theorem is a reflection of this apparent boundedness of the types of torsion that appear in low degree.

Outside of this uniformness in torsion, one may ask what exactly one can leverage from finite generation. In later sections, we will see that if V is an FI-module over a field k, then for all $n \gg 0$, $\dim_k V([n])$ agrees with a polynomial, where $[n] = \{1, \ldots, n\}$ [CEF, CEFN]. From a combinatorial perspective, one can think of this result as asserting the rationality of a particular **Hilbert Series**. This perspective is the one we will take going forward.

Let M be a finitely generated $\mathcal{G}_0^{op}$-module over a field k, the category of trees with contractions. It is a well known fact in combinatorics that to every **Dyck path** w, one may associated a tree $T(w)$, although this association is not one to one unless T is equipped with a choice of root and embedding into the plane. We define the Hilbert-Dyck series of the module M to be the formal power series

$$HD_M(t) := \sum_w \dim_k(M(T(w))t^{|T(w)|},$$

where the sum is over all Dyck paths, and $|T(w)|$ is the number of edges of $T(w)$ or, equivalently, half the length of the word w. For instance, if M is the module which assigns k to every tree, then $HD_M(t)$ is the generating function for the number of Dyck paths. That is, the generating function of the **Catalan Numbers**. Our main result generalizes one of the most famous results about this generating function.

Theorem 1.6 (Ramos [R8]). *For any finitely generated $\mathcal{G}_0^{op}$-module M over a field k, the Hilbert-Dyck series $HD_M(t)$ is algebraic.*

The combinatorics of algebraic generating functions implies a range of facts about the constituent pieces. In particular, one obtains an exponential bound on how these graded pieces can grow.

Our second application considers the **cohomology of the category** $\mathcal{G}_g^{op}$. We recall for any small category $\mathcal{C}$, its cohomology is defined by

$$H^i(\mathcal{C}) := \mathrm{Ext}^i_{\mathcal{C}-\mathrm{Mod}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}})$$

where $\underline{\mathbb{Z}}$ is the $\mathcal{C}$-module that takes the value $\mathbb{Z}$ at each object, and whose every induced map is the identity. It is classically known that the cohomology of any small category can be naturally identified with the singular cohomology of a topological space associated to the category; the **nerve** of the category.

While my work with Proudfoot does not completely resolve the issue of computing the cohomology of $\mathcal{G}_g^{op}$, we do show the following.

Theorem 1.7 (Proudfoot & Ramos [PR2]). *We say that a graph is reduced if it contains no vertices of degree 2, nor any edges whose deletion disconnects the graph. Write $\mathcal{G}^{op}_{g,red}$ for the full subcategory of $\mathcal{G}^{op}_g$ whose objects are reduced graphs of genus g. Then one has,*

$$H^i(\mathcal{G}^{op}_{g,red}) \cong H^i(\mathrm{Out}(F_g))$$

where $\mathrm{Out}(F_g)$ is the outer automorphism group of the free group F_g.

The above theorem is related to, and indeed depends on, Culler and Vogtmann's seminal work on **Outer Space**. Roughly speaking, the nerve of the category $\mathcal{G}^{op}_{g,red}$ can be seen to be homotopy equivalent to the quotient of Outer Space by the $\mathrm{Out}(F_g)$-action, where a collection of infinite-dimensional pockets are glued in at various points. While this space therefore becomes a considerably more cumbersome geometrically, the upshot is that its cohomology agrees with that of $\mathrm{Out}(F_g)$ with integral coefficients, whereas the Culler-Vogtmann Outer Space can only be used to compute rational cohomology of these groups.

1.3. **Future directions.** The first and most obvious question that I would like to resolve is the question of whether or not our Noetherianity result can be extended to the full category $\mathcal{G}^{op}$.

Question 1.8. Is it the case that modules over $\mathcal{G}^{op}$ satisfy a Noetherian property generalizing Theorem 1.2?

We have already shown [MPR], that all of the aforementioned finite generation results of $\mathcal{G}^{op}_{\leq g}$-modules will directly lift to results about the analogous $\mathcal{G}^{op}$-modules assuming that this question is shown in the positive.

The study of modules over the category $\mathcal{G}^{op}_g$, and more generally $\mathcal{G}^{op}$, is very new, and therefore lends itself well to a variety of interesting open questions. These questions range from intrinsic structural concerns, to questions about the aforementioned applications. We therefore begin this discussion by addressing questions of the first type.

While there are a nearly limitless number of directions one can take with regards to structural concerns, there is one particular direction which I am currently most interested in pursuing. We have already seen results related with a kind of Hilbert series for modules over the contraction category of trees. Similar statements can be shown for finitely generated modules over a large number of related combinatorial categories such as FS^{op} - of finite sets and surjective maps - and FA - of finite sets and all set maps [SS2].

Question 1.9. What is the correct definition for the Hilbert series of a $\mathcal{G}^{op}_g$-module when $g > 0$? Recent work of Chan, Faber, Galatius, and Payne [CGP, CGP2, CFGP] has used a construction they call orbifold sums of graphs to great effect in studying moduli spaces of tropical curves. Can one modify this definition to fit our setting? If so, what kind of formal rationality statements can one prove about these Hilbert series?

I believe that understanding Hilbert series is an extremely important first step in understanding the overall structure of the representation theory of a category. Indeed, it both informs the best way forward with regard to other structural concerns, and is useful in leveraging the most that you can from a finite generation statement in applications.

Moving on to our applications, recall the theorem of Miyata, Proudfoot and I that the homology groups of the configuration spaces of graphs of fixed genus g can be encoded in a finitely generated $\mathcal{G}^{op}_{\leq g}$-module. The most natural followup question to this is whether or not one can actually provide the (finitely many) generators of this module. In the case of genus 0, i.e. trees, the answer is known due to results of Chettih and Lütgehetmann [CL]. Outside of this case, however, virtually nothing is known. In fact [CL] provides examples of graphs whose configuration space homology can become almost arbitrarily pathological.

Question 1.10. Can one provide the generators of the aforementioned $\mathcal{G}^{op}_{\leq g}$-modules, even in cases where the genus, the number of points being configured, and/or the homological index is kept small? Finding these generators has a similar flavor to constructing a finite set of **forbidden minors** for some minor-closed property - such as K_5 and $K_{3,3}$ as they relate to planarity. Could the techniques used in those types of problems be admissible here?

We have discussed how the full subcategory of $\mathcal{G}^{op}_g$ whose objects are reduced graphs could be used to compute the cohomology of $\mathrm{Out}(F_g)$. It is still an open question, however, whether the nerve of the entire $\mathcal{G}^{op}$ is topologically related to anything in the literature.

Question 1.11. Can we find subcategories of $\mathcal{G}^{op}$ whose nerve can be linked to any one of the various generalizations and compactifications of Culler-Vogtmann's Outer Space? For instance, the moduli space of tropical curves, as studied by Chan, Galatius, and Payne [CGP, CGP2], seems like it would quite naturally fit into such a framework. Moreover, having formed such a link, can our general categorical results be used to prove something new about these relatively mysterious spaces?

One notes that while the Graph Minor Theorem is one particularly well-known and well-studied well-quasi-order theorems, there are many others that are potentially ripe for possible categorifications. For instance, in [KR], Knudsen and myself consider categorifications of the topological minor relation as well as the induced subgraph relation on cographs with applications.

Question 1.12. What are some natural graph categories that can potentially be used to prove categorifications for the litany of well-quasi-order results on graphs?

One final direction that is of great interest for applications can be found by enhancing our graph categories with some form of extra data. For instance, in [MRT], Matherne, Tymoczko, and myself consider contraction categories of graphs that have been enhanced with the data of edge-labels. We prove a kind of Noetherian property for modules over this category, and use this to prove the existence of non-trivial and hitherto unobserved universality phenomena in the behavior of generalized splines [GTV].

Question 1.13. To what extent are the Noetherianity phenomena in this work amenable to the inclusion of extra data such as edge-labels, vertex-labels, or otherwise? In the cases where they are, what applications can one find beyond the aforementioned work on generalized splines?

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2. Families of highly symmetric graphs

2.1. **Background.** A **homomorphism** of graphs is a map between their vertex sets that preserves adjacency. An **FI-graph** is a functor from FI to the category of graphs and graph homomorphisms.

Just as with FI-modules, we begin by limiting our attention to a collection of FI-graphs that are well-behaved enough to warrant study. We say that an FI-graph $G_\bullet$ is **finitely generated** if, for all $n \gg 0$, every vertex of G_{n+1} appears in the image of some vertex of G_n under the action of FI. Examples of such families include the complete graphs, the Johnson and Kneser graphs, and the Crown graphs. The primary tool for working with finitely generated FI-graphs is the following.

Theorem 2.1 (Ramos & White [RW]). *Let $G_\bullet$ denote a finitely generated FI-graph. Then for $n \gg 0$:*

1. *if $f : [n] \hookrightarrow [n+1]$ is an injection of sets, then the induced map $G(f)$ is injective;*
2. *if $f : [n] \hookrightarrow [n+1]$ is an injection of sets, then the image of the induced map $G(f)$ is an induced subgraph of G_{n+1};*

3. for any $r \geq 0$ and any collection of vertices $\{v_1, \ldots, v_r\}$ of G_{n+1}, there exists a collection of vertices $\{w_1, \ldots, w_r\}$ of G_n as well as an injection $f : [n] \hookrightarrow [n+1]$, such that $G(f)(\{w_1, \ldots, w_r\}) = \{v_1, \ldots, v_r\}$.

The above theorem allows us to paint the following picture of what finitely generated FI-graphs look like: for $n \gg 0$ there is a chain of induced subgraphs $G_n \subseteq G_{n+1} \subseteq \ldots$ such that each G_n is equipped with an action of the symmetric group $\mathfrak{S}_n$, and these actions respect the inclusions in the obvious way. That is to say, a finitely generated FI-graph can be thought of as a family of nested graphs, equipped with compatible symmetric group actions.

2.2. **Applications of the theory.** In this section we more thoroughly detail the theory of FI-graphs.

From the relatively simple setup of the previous sections, one obtains a wealth of interesting consequences and applications. In the original papers in which the theory of FI-graphs were developed, these applications were broken into three categories: combinatorial, topological, and algebraic. We continue this trichotomy here.

To begin, we consider the case of the complete graphs, K_n. Note that if T is any fixed finite graph, then the number of distinct copies of T appearing as a subgraph of K_n can be easily counted, and it is seen to be a polynomial in n. This behavior is common among all finitely generated FI-graphs.

Theorem 2.2 (Ramos & White [RW])**.** *Let $G_\bullet$ be a finitely generated FI-graph, and let T be any fixed graph. Then for all $n \gg 0$, the following quantities each agree with a polynomial in n:*

1. the number of copies of T appearing as a subgraph of G_n;
2. the number of copies of T appearing as an induced subgraph of G_n.

As an immediate application of this theorem, we consider what it implies about the Hom-graph of an FI-graph. For any finitely generated FI-graph $G_\bullet$, and any fixed graph T, we define a new FI-graph whose vertices are labeled by graph homomorphisms from T to G_n, and whose edges indicate that two homomorphisms differ on precisely one vertex of T. It can be shown that this new FI-graph is finitely generated, granted that $G_\bullet$ was, and therefore the above theorem immediately implies the following.

Corollary 2.3 (Ramos & White [RW])**.** *Let $G_\bullet$ be a finitely generated FI-graph, and let T be any fixed graph. Then the number of graph homomorphisms from T to G_n agrees with a polynomial for all $n \gg 0$.*

It is classically known that the number of graph homomorphisms from T to the complete graph counts the number of proper vertex colorings of T. Therefore, the above corollary can be seen as a generalization of the existence of the **chromatic polynomial**.

If instead we counted homomorphisms from the path of length r to G_n, we find that the number of walks of length r inside G_n is in agreement with a polynomial, for all $n \gg 0$. Seeing this, one might immediately be tempted to ask whether anything can be said about how the statistics of random walks on G_n vary with n. This is indeed the case.

Fix $m \geq 0$. For any vertex x of G_m, and any $n \geq m$, we write $x(n)$ for the vertex $G(\iota)(x)$ of G_n, where ι is the standard inclusion $\iota : [m] \hookrightarrow [n]$. We also write $\tau_{x,y}(n)$ for the hitting time random variable between $x(n)$ and $y(n)$. That is, the random variable which marks the time the walk takes getting from $x(n)$ to $y(n)$.

Theorem 2.4 (Ramos & White [RW2])**.** *Let $G_\bullet$ be a finitely generated FI-graph, and fix $m \gg 0$, as well as a pair of vertices $x, y \in G_m$. We write $\tau_{x,y}(n)$ for the hitting time random variable of the simple random walk on G_n between the vertices $x(n)$ and $y(n)$. Then for all $n \gg 0$ the function*

$$n \mapsto \mu_i(\tau_{x,y}(n))$$

agrees with a rational function, where μ_i is any one of the i-th moment, the i-th central moment, or the i-th cumulant.

Moving on from these combinatorial results, we next turn our attention to how the theory of FI-graphs can be applied to topology. If G is a graph, recall that we may consider the configuration space, $\mathrm{Conf}_n(G)$. Configuration spaces of graphs are a recurring theme throughout my work, as they have proven to be a rich area for new applications. In a future section the I will go into much more detail about the existing theory of these spaces, and their contributions, but for now it suffices to say that they behave much differently from configuration spaces of manifolds, as considered by Church, Ellenberg, and Farb [CEF]. Unlike our discussion in the introduction for other configuration spaces, we proceed by considering how their behavior varies when the number of points is fixed, and we instead make our graph bigger in a natural way. The following result generalizes work of Lütgehetmann from [Lu].

Theorem 2.5 (Ramos & White [RW]). *Let $G_\bullet$ be a finitely generated FI-graph. Then for all fixed $m, q \geq 0$, the FI-module*

$$n \mapsto H_q(\mathrm{Conf}_m(G_n))$$

is finitely generated.

To conclude, we discuss certain algebraic properties of FI-graphs. Namely, we consider the spectra of these graphs, and how they can vary with n. Recall that if G is a graph, then the **adjacency matrix** A_G of G is the endomorphism of the $\mathbb{Q}$-linearization of the vertex set $V(G)$ of G that maps a vertex x to the sum of vertices it is adjacent to. The **spectrum** of the graph G is then defined to be the spectrum of A_G. Spectral analysis of graphs is in many ways the foundation of algebraic graph theory [Bi, CDS], and has found applications to the study of Markov chains [LPW], and graph limits [Lo], among many others.

Let $G_\bullet$ be a finitely generated FI-graph, and write A_n for the adjacency matrix A_{G_n}. While it is the case that the functor $n \mapsto \mathbb{Q}V(G_n)$ is a finitely generated FI-module, it is not the case that the collection $\{A_n\}$ can be extended to an endomorphism of this FI-module. Despite this fact, it is undeniable that these maps are strongly related to the FI-structure underlying $G_\bullet$. This intuition was made precise by Speyer, White, and I in [RSW]. In fact, [RSW] works within the much more general context of relations between FI-sets, that the edge relation of an FI-graph is an example of. One consequence of that work is the following.

Theorem 2.6 (Ramos, Speyer, & White [RSW]). *Let $G_\bullet$ be a finitely generated FI-graph. For each n, the matrix A_n is real and symmetric, whence its distinct eigenvalues can be written as*

$$\lambda_1(n) < \ldots < \lambda_{r(n)}(n)$$

for some function $r(n)$. Then for all $n \gg 0$:

1. *$r(n) = r$ is independent of n. In particular, the number of distinct eigenvalues of A_n is eventually independent of n;*
2. *for each i the function $n \mapsto \lambda_i(n)$ is algebraic over the field $\mathbb{Q}(n)$;*
3. *for each i, the multiplicity of $\lambda_i(n)$ agrees with a polynomial.*

2.3. **Future directions.** For our first project, we turn our attention to various extremal invariants of FI-graphs. For instance, For a general graph G, recall the **independence number**, $\alpha(G)$. This invariant is defined as the size of the largest collection of vertices that are pairwise not connected by an edge. We will consider the question of the asymptotic behavior of the function $n \mapsto \alpha(G_n)$ whenever $G_\bullet$ is a finitely generated FI-graph. In examples of FI-graphs where the independence is known, it is conspicuously the case that it agrees with a quasi-polynomial in n. This includes the case of the Kneser graphs, whose independence number is the subject of the famous Erdös–Ko–Rado Theorem. On the other hand, there are many important examples of finitely generated FI-graphs whose independence numbers are still not known. In fact, there are well known conjectures of Erdös on the extremal theory of finite sets whose statement can be rephrased as computing the independence number of some FI-graph.

Question 2.7. Let $G_\bullet$ be a finitely generated FI-graph. Then can anything be said about generating function

$$\sum_{n\geq 0} \alpha(G_n)t^n?$$

It is my belief that this power series will always be rational, but are other behaviors also possible?

That a generating function is rational implies quite a lot about the sequence $\alpha(G_n)$. For instance, this is the subject of the entire Chapter 4 of Stanley's classic text on enumerative combinatorics [St]. It is a famous fact that the dimensions of the graded pieces of a finitely generated FI-module obey a polynomial growth rule [CEF, CEFN]. We will also see throughout this proposal other invariants of FI-objects which behave similarly. It is therefore not a stretch to believe that the generating function of $\alpha(G_n)$ should be rational. However, having a rational generating function is a more general phenomenon than simply being polynomial. In particular, this project would be the first instance of higher-order stability phenomena in FI objects.

Given a graph G, we may define a polynomial ring A_G whose variables are indexed by the edges of G. The **edge ideal** of G is defined to be the ideal generated by the monomials x_ix_j, where $\{i, j\}$ is an edge of G. It is a straight forward computation to show that the codimension of the edge ideal is precisely $\alpha(G)$.

Moving back to the context of FI-graphs, one may associate to $G_\bullet$ an FI-algebra $A_{G_\bullet}$. The collection of edge ideals of A_{G_n}, with n varying, then becomes an ideal within this FI-algebra. Putting it all together, one may therefore reduce our question about the independence number of an FI-graph to the question of the behavior of codimensions of (square-free) monomial ideals in FI-algebras. Questions of this type are precisely the subject of [LNNR], where they prove that the behavior is polynomial under certain strong hypotheses. This implies the conclusion of our the above question in this circumstance, and provides more evidence that the relevant power series are indeed rational.

This being said, the techniques of [LNNR] cannot be directly generalized to work in our setting. The issue is that The FI-algebra $A_{G_\bullet}$ does not generally satisfy a Noetherian property. In [LNNR], a Gröbner basis approach is used that mixes the classical theory from commutative algebra with the combinatorics of FI-modules. Without a Noetherian property, however, this Gröbner theory approach is doomed to fail.

There is a second approach, however, that has been used by Sam and Snowden [SS2], Krone, Leykin, and Snowden [KLS], Nagel [Nage], and myself [R8]. This approach involves leveraging the symmetry of the ideals in question to encode their members as words in regular language. One of the foundational results in language theory is that normed regular languages have rational Hilbert series. Using the relationship between the codimension of an ideal, and the Hilbert series of its associated module, it is my belief that the rationality can be proven.

The above question is one example of a much more general phenomenon which we now formally state.

Question 2.8. How does the FI-formalism interact with questions about minimums, such as in the case of the chromatic number, and maximums, such as in the case of independence numbers? For instance, is it always the case that the generating functions of these invariants have rational Hilbert series?

The more general question posed here about minimums and maximums has received attention recently in various places throughout the literature. We will later see how it appears in the work of Bahran [B], and it has recently appeared in work of Le, Nagel, Nguyen, and Römer [LNNR] on the projective dimensions of ideals within a certain FI-algebra. The more specific question about the chromatic number, while less tractable than the previous question, would have widespread implications throughout graph theory. In examples of FI-graphs where the chromatic number is known, it is conspicuously the case that it agrees with a polynomial in n. This includes the case of the Kneser graphs, whose chromatic number was computed in famous work of Lovász [Lo2]. On the other hand, there are many important examples of finitely generated FI-graphs whose chromatic numbers are still not known. This includes, for instance, the Johnson graphs.

While the my work with Speyer, and White has established the foundations for studying FI-graphs, there is still much left to be understood about these objects. To begin, we have seen that finitely generated FI-graphs can be associated to a large variety of polynomials arising from combinatorial questions. These include the polynomials that count the number of occurrences of certain fixed substructures, as well as the polynomials that count homomorphisms into the FI-graph. We have already seen that at least one of these polynomials has appeared in the literature as the chromatic polynomial. We are therefore left with the following question.

Question 2.9. What can be said about polynomials associated to FI-graphs that go beyond simple degree bounds? Work of Huh implies that the coefficients of the chromatic polynomial form a sign-alternating log-concave sequence [Hu]. Can statements of this form be proved for homomorphism numbers into other FI-graphs? The chromatic polynomial is known to have interesting counting properties at non-positive integer values, are similar statements true about homomorphism numbers into other FI graphs?

We have seen how the combinatorics of FI-graphs reveals interesting consequences about random walks on these graphs. In the paper [RW2], it is shown that the mixing time of such a random walk, as a function of n, can be understood in terms of the mixing time of an associated Markov process on a state space of fixed size. Aside from this, however, much is still not understood about mixing times of FI-graphs. Specifically I am interested in understanding when random walks on these objects exhibit the cutoff phenomenon. This property of families of Markov chains was formally introduced by Diaconis [D], building off of work of Diaconis and Aldous [AD], and has since been extensively studied [BHP, LS]. Interesting, despite a variety of heuristics predicting otherwise, It was proven by White and myself in [RW3] that FI-graphs will never exhibit cutoff. This leads to the following followup question.

Question 2.10. Despite the lack of total variation cutoff, are there weaker properties of the mixing time that can be still shown to appear? Moreover, what results can be proved about other Markov process one can perform on FI-graphs, such as those related to chip-firing and sand pile games?

Moving on to future directions in topology, we begin with questions about configuration spaces. Note that while we consider these configuration spaces specifically in how they relate to FI-graphs here, we dedicate the entire third section to other questions about these spaces, and related structures. Above, it was shown that if $G_\bullet$ is a finitely generated FI-graph, then the associated FI-modules $H_q(\mathrm{Conf}_m(G_\bullet))$ are finitely generated. The proof of this fact is non-constructive, and consequently does not provide bounds on the degree of generation. While work of Lütgehetmann [Lu] provides such bounds in a very particular circumstance, nothing is known otherwise.

Question 2.11. Can general bounds on generating degree be given in terms of the graphs G_n, the number of points m, and the homological degree q?

Another topological application of FI-graphs was recently discovered by Bahran [B]. For each n, let $G_n^{a,p}$ denote the graph whose vertices are labeled by elements of the symmetric group of prime order p with at most a cycles in their cycle decomposition, and whose edges indicate that the associated permutations commute. Then, for any fixed p, a, the family $G_\bullet^{p,a}$ form a finitely generated FI-graph. In his work [B], Bahran considers the clique complex of the graphs $G_n^{p,a}$; the abstract simplicial complex formed by the complete graphs inside of $G_n^{p,a}$. These spaces are of great interest in the study of finite groups [Ks, Ks2]. Using the framework of FI-graphs, Bahran is able to prove that these simplicial complexes are what he calls **highly acyclic**, i.e. that the index of their lowest non-vanishing homotopy group can be made arbitrarily big with n.

Question 2.12. Let G_n^p be the graph whose vertices are labeled by the elements of $\mathfrak{S}_n$ of order p, without any condition on the number of cycles, and whose edges are as with $G_n^{p,a}$. Bahran says that his study of $G_n^{p,a}$ was motivated by an attempt at understanding the more general G_n^p [B]. The FI-graph $G_\bullet^p$ is unfortunately not finitely generated, although it is very nearly so. Can the theory of finitely generated FI-graphs be expanded so as to prove similar results to Bahran? Otherwise, is there an action by a different category that makes the family $G_\bullet^p$ finitely generated in some other sense?

3. Matroidal representation stability

3.1. **Background.** Arguably one of the most influential trends in the past few years in algebraic combinatorics has been the influx of results tied to matroid geometry. For instance, one can point to the work of Adiprasito, Huh, and Katz [AHK] on the Hodge theory of combinatorial geometries, or the even newer work of Braden, Huh, Matherne, Proudfoot, and Wang [BHMPW] that considers the singular picture. In my own work, I have considered applications of the representation theory of categories to the various matroidal geometries.

The most classical and simple geometry that one might naturally associate with a realizable matroid M, or equivalently with a hyperplane arrangement $\mathcal{A}$, is the complement space $\mathcal{M}(\mathcal{A})$. For instance, in the case where $\mathcal{A}$ is the **braid matroid** - i.e. the cycle matroid associated to the complete graph - this complementary space is usual configuration space of the plane. Despite the simplicity of this particular construction, the study of these spaces often leads to surprising connections between their basic topological invariants, and combinatorial invariants of the associated matroid. For instance, the cohomology ring of this space is an object of classical significance known as the **Orlik-Solomon algebra** $OS_{\mathcal{A}}$. In their seminal work, Adiprasito, Huh, and Katz [AHK] were able to show that in a very general context that this algebra satisfies very strong combintorial conditions on the dimensions of its graded pieces. Namely, that they form a **log-concave** sequence. A log-concave sequence is a sequence of positive integers $\{a_n\}$ satisfying the condition $a_i^2 \geq a_{i-1}a_{i+1}$ for all i. In my own work I have focused on showing that these complement spaces, beyond the aforementioned configuration space, often times satisfy certain representation stability phenomena [PR3, MMPR, FRY].

The second major facet of my work in this direction are related with the **Kazhdan-Lusztig polynomials of a matroid**. For a matroid M, these objects were defined by Elias, Proudfoot, and Wakefield in [EPW]. In that work the authors use a purely combinatorial construction, based around the lattice of flats of M, to define these objects. This combinatorial formulation resembles similar constructions from the classical theory of Kazhdan-Lusztig polynomials. Importantly for this discussion, in the cases wherein M is realizable, there is a canonical way to realize the Kazhdan-Lusztig polynomial as arising from the geometry of the associated hyperplane arrangement.

To describe this perspective, let $\mathcal{A}$ denote the hyperplane in $\mathbb{C}^n$ associated with the matroid M. Then the complementary variety $\mathcal{M}(\mathcal{A})$ can be embedded into the torus $(\mathbb{C}^\times)^{\mathcal{A}}$ by setting the H-th coordinate to be the inner product $\langle a(H), \bullet\rangle$, where $a(H)$ is the normal vector to H. We define $\mathcal{M}^{-1}(\mathcal{A})$ to be the image of $\mathcal{M}(\mathcal{A})$ under the involution of the torus which inverts every coordinate. The **reciprocal plane** $X_{\mathcal{A}}$ is then defined to be the closure of $\mathcal{M}^{-1}(\mathcal{A})$ in the affine space $\mathbb{C}^{\mathcal{A}}$. As proven in [EPW], the aforementioned Kazhdan-Lusztig polynomial of the matroid M is equal to the Poincaré polynomial of the *intersection cohomology* groups of the reciprocal plane.

This more geometric perspective makes it obvious that the Kazhdan-Lusztig polynomials of realizable matroids satisfy a variety of desirable properties, such as coefficient positivity. It also opens the doors to an equivariant version of the theory, where one considers the formal Poincaré polynomial whose coefficients are the intersection cohomology groups, in cases where there is a natural group action on the matroid and associated hyperplane arrangement [GPY]. For instance, the aforementioned braid matroid associated with the complete graph carries an action by the symmetric group. In particular, it opens the field up to potential interaction with representation stability phenomena. In their work [PY] Proudfoot and Young prove the following.

Theorem 3.1 (Proudfoot and Young, [PY])**.** *Let for each $n \geq 0$, let M_n denote the braid matroid on n letters and let $\mathcal{A}_n$ denote the associated hyperplane arrangement. Then for any fixed $i \geq 0$, the i-th equivariant Kazhdan-Lusztig coefficients $IH^{2i}(X_{\mathcal{A}_\bullet})$ form a finitely generated module over the category FS^{op}, the opposite category of finite sets and surjective maps.*

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Modules over the opposite category of finite sets and surjections, while not as well understood as FI-modules, have been studied in a variety of contexts [PY, SS3, T]. For instance, in contrast to FI-modules, it is known that the dimensions of the pieces of a finitely generated FS^{op}-module are eventually sums of exponentials.

3.2. **Equivariant Log Concavity and Kazhdan-Lusztig Polynomials.** Returning to the context of the Orlik-Solomon algebra, we have discussed how in the realizable case this will be isomorphic to the cohomology ring of the complementary space. In particular, in cases wherein these spaces are equipped with the action of a group, the Orlik-Solomon algebra inherit the structure of a graded representation of this group. Keeping this in mind, we say that a graded representation V of a group G is **strongly equivariantly log-concave** if for each $i \leq j \leq k \leq l$ such that $i + l = j + k$, the representation $V_i \otimes V_l$ embeds into the representation $V_j \otimes V_k$. Observe that taking $i = n-1, j = k = n, l = n+1$, this definition resembles our previously mentioned definition of a log-concave sequence. Indeed, is the primary conjecture of my work with Matherne, Miyata, and Proudfoot that the theorem of Adiprasito, Huh, and Katz on log concavity of Orlik-Solomon algebras will remain true in this equivariant. To this end, we have proven the following.

Theorem 3.2 (Matherne, Miyata, Proudfoot, and Ramos, [MMPR])**.** *Write A_n for the Orlik-Solomon algebra of the braid matroid. Then for all $n \geq 0$, and all $i \leq j \leq k \leq l \leq 14$ such that $i + l = j + k$, the representation $(A_n)_i \otimes (A_n)_l$ embeds into $(A_n)_j \otimes (A_n)_k$.*

To prove this theorem, we used what we refer to as "quantitative representation stability." To see what is meant by this, note that for any fixed i, the representations $(A_\bullet)_i$ form an FI-module. This FI-module is precisely that which encodes the i-th cohomology group of configuration space in the plane. We have already discussed the fact that the core underlying philosophy of representation stability theory is that in a family of representation stable representations, the behaviors of some finite subfamily determine the behaviors in the entire family. In our setting, this guiding philosophy essentially tells us that to show $(A_n)_i \otimes (A_n)_l$ appears a subrepresentation of $(A_n)_j \otimes (A_n)_k$ for all n, we only need to show that this is the case for some finite collection of n. Homological theorems on FI-modules discussed below give us a bound on what finite means in this context, while computer calculation can verify that all members in this finite list satisfy the necessary property. Note that in the paper [MMPR], the case of the Orlik-Solomon algebra of the braid matroid is only one of four different matroidal spaces associated to the braid matroid that we apply this kind of reasoning to.

It was independently discovered in [W-GWZ] and [TVY] that the braid matroid is only one in a family of matroids that are affiliated with the symmetric group. There is a natural way by which one can associate a realizable matroid to any irreducible representation of the symmetric group. This matroid is referred to as the **Specht matroid**, while the associated hyperplane arrangement is called the **intrinsic hyperplane arrangement**. One can think of this matroid as encoding the linearity relationships between the elements in the Young Tabloid spanning set of the Specht module. The associated hyperplane arrangement can then be realized as a hyperplane arrangement within this module. In the case of the standard irreducible representation, one recovers the braid matroid, while in all other cases these arrangements seem to be extremely rich and hard to understand. In [FRY], Flynn, Young, and myself tackle these arrangements and their various associated geometries from the perspective of representation stability, similarly to how the braid matroid is studied. In this direction we were able to prove the following.

Theorem 3.3 (Flynn, Ramos, and Young, [FRY])**.** *Let λ be a fixed partition of some integer m, and write $\lambda[n]$ for the* ***padded partition of*** *n $\lambda[n] = (n - |\lambda|, \lambda)$. Then for each fixed i the i-th cohomology group of the complementary space to the intrinsic arrangement associated to $\lambda[n]$ form a finitely generated* FI*-module.*

This theorem directly generalizes the corresponding statement for the braid matroid, as the braid matroid is precisely the case where λ is a single box. It also suggests that the question of whether the Orlik-Solomon algebras of these hyperplane arrangements are equivariant log-concave can be treated with similar quantitative means as the braid matroid.

Aside from the braid matroid and its generalizations, my work has also considered the complementary spaces of other hyperplane arrangements. One result of particular note relates with the **resonance arrangements**. Usually when one speaks of resonance arrangements, they are referring to the *real* hyperplane arrangement of all hyperplanes that at normal to some vector whose every coordinate is either a 0 or 1. Equivalently, it is the real hyperplane arrangement of hyperplanes whose every point has some partial coordinate sum is equal to 0. It is still an open question to determine a formula for the number of chambers of this arrangement as a function of n. While this remains an open and seemingly quite difficult question, there are ways that one can obtain a slightly weaker result as follows. Instead of thinking about the real hyperplane arrangement, one can consider the higher dimensional linear subspace arrangement of all n-tuples of vectors in $\mathbb{R}^3$ for which some partial sum is equal to zero. A theorem of Moseley [Mo] tells us that the number of chambers of the resonance arrangement will be equal to the **total cohomology** - i.e. the Poincaré polynomial evaluated at 1 - of the complementary space of this higher dimensional analog. In particular, the individual betti numbers of these higher dimensional complementary spaces provide a kind of partial count of the total number of chambers. To this end we find the following.

Theorem 3.4 (Proudfoot and Ramos, [PR3])**.** *Let $n \geq 0$, and write $\mathcal{M}(n)$ for the complementary space to the linear subspace arrangement in $(\mathbb{R}^3)^n$ of n-tuples of vectors for which some partial sum is zero. Then for any i the sequence of groups $H_i(\mathcal{M}(n))$ can be granted the structure of a finitely generated FS^{op}-module.*

One consequence of this finite generation result is that the Betti numbers $\dim(H_i(\mathcal{M}(n)))$ must grow like some sum of exponentials. In fact, this consequence, with a completely explicit description, had been obtained by Kühne in [Ku] earlier than our work. While our result is less explicit, it does also shed light on the representation theoretic structure of the groups $H_i(\mathcal{M}(n))$ that Kühne's work does not. The tools that we used in our work were also general enough to prove similar results about partial chamber counts for other real hyperplane arrangements such as the so-called **threshold arrangement** [GMP].

Moving on from the complementary space and the Orlik-Solomon algebra, our next avenue of interest are the in studying equivariant Kazhdan-Lusztig polynomials of various matroids. For our first application, we return to the categories $\mathcal{G}_g$ of genus g graphs with contractions. One of the great insights of the work [PY] is that any contraction of matroids induces maps in a contravariant fashion between their associated reciprocal planes. Moreover, these maps respect a particular spectral sequence that converges to the intersection cohomology groups. Being that edge-contractions of graphs induce contractions between the corresponding cycle matroids, we were able to prove the following.

Theorem 3.5 (Proudfoot and Ramos, [PR, PR2]). *Fix a natural number g and a positive integer i. The $\mathcal{G}_g^{op}$-module*

$$G \mapsto IH_{2i}(X_G)$$

is finitely generated, where X_G is the reciprocal plane of the cycle matroid of G.

One particular consequence of this theorem is that the coefficients of the Kazhdan-Lusztig polynomials of the cycle graphs are equal to polynomials in the number of vertices of the cycle. These polynomials have been computed and appear in [PR2]. In [PR], Proudfoot and myself prove a similar result for the family of graphs

Finally, we have seen Proudfoot and Young's result on the equivariant Kazhdan-Lusztig polynomials of braid matroids. Being that the aforementioned intrinsic arrangements directly generalize the braid arrangements, one might hope that a similar result is true in this context. This is precisely the content of the following theorem.

Theorem 3.6 (Flynn, Ramos, and Young, [FRY]). *Let for each $n \geq 0$, let $M_{\lambda,n}$ denote the Specht matroid of $\lambda[n]$, and let $\mathcal{A}_{\lambda,n}$ denote the associated hyperplane arrangement. Then for any fixed $i \geq 0$, the i-th equivariant Kazhdan-Lusztig coefficients $IH^{2i}(X_{\mathcal{A}_{\lambda,\bullet}})$ form a finitely generated module over the category FS^{op}, the opposite category of finite sets and surjective maps.*

3.3. **Future Directions.** Based on the explosion of interest in matroidal and other combinatorial geometries, it is perhaps unsurprising that there are a number of avenues for future research. For instance, the concept of equivariant log-concavity is so new that most of its statements are still conjectural!

Question 3.7. Is it the case that the Orlik-Solomon algebras of the braid group form an equivariant log-concave sequence? More generally, is it the case for the complementary spaces of the intrinsic hyperplane arrangements?

The aforementioned resonance arrangements and their higher dimensional analogs provide very natural families of arrangements whose complementary variety is being acted on by a category that is no FI. This is relevant because, unlike the previous examples, the theoretical underpinnings are far less developed. In particular, we are lead to the following pair of questions.

Question 3.8. In the case of the resonance arrangements, does one expect to find equivariant log concavity in the cohomology ring of their higher dimensional analogs? Moreover, can one develop strong homological methods for FS^{op}-modules so-as to replicate the quantitative methods mentioned above?

Moving on to graph categories and Kazhdan-Lusztig polynomials, one may have noticed that we were restricted to using the category of contractions because on the matroid side only contractions induce natural maps on the reciprocal plane. Indeed, while one does also obtain maps based on edge *deletions* these maps are known to be non-canonical and depend on the order in which the edges were deleted. Dove-tailing off a previous question on decorated graph categories that suggests the following question.

Question 3.9. Is there a finite generation result for equivariant Kazhdan-Lusztig coefficients that spans all graphical matroids? In particular, can one consider a modification of the graph minor category wherein the order of deletion and contraction is preserved, and use this modification to prove such a finite generation result?

4. Stable properties in families of groups and property (T)

4.1. **Background.** Having already discussed FI-graphs, FI-sets, and FI-modules, our next direction will be to consider **FI-groups**. An FI-group is a functor $\Gamma_\bullet$ from FI to the category of discrete groups. We say that an FI-group is **finitely generated** if there exists a finitely generated FI-subset $S_\bullet$ of $\Gamma_\bullet$, such that S_n is a generating set of Γ_n for all $n \gg 0$. As examples, one sees that the symmetric groups themselves form a finitely generated FI-group, where S_n is the set of transpositions and FI acts by conjugation. Free groups, automorphism groups of free groups, and $\mathrm{SL}_n(\mathbb{Z})$ are also interesting examples of these objects. One should keep the latter two examples in mind for what follows.

Much of The my work in the study of FI-groups has been in trying to understand how certain geometric-group-theoretic properties behave among the Γ_n as $n \gg 0$. Specifically, I have considered this question for **property (T) of Kazhdan**, **property FA of Serre**, as well as the **geometric Cayley graph property**. These three properties have been central to modern geometric group theory since their respective conceptions [BdLHV, Ka, KKN, M, O, Se, Sh, Z]. Famously, property (T) was used by Margulis to provide the first systematic approach to constructing **expander graphs** [M]. See [BdLHV] for a detailed exposition on all of these properties.

We say that a group Γ has property (T) if the trivial representation is isolated in the unitary dual of Γ. Roughly speaking, one can think of this as saying that there is an $\epsilon > 0$ and a compact subgroup $K \leq \Gamma$, such that any unitary representation that contains a vector that is only moved within an ϵ-ball of its starting position by K, must actually contain a vector that is fixed by Γ. Property FA, on the other hand, is a discrete version of (T) that requires that all actions of Γ on a tree admit a fixed point.

Finally, the Cayley graph property is a feature of a group Γ paired with a chosen (finite) generating set S, not containing the identity. Define the **level one Cayley graph** $C^{(1)}_{\Gamma,S}$ of (Γ, S) to be the induced subgraph of the usual Cayley graph generated by the elements of S. Then we say that (Γ, S) has the geometric Cayley graph property if $C^{(1)}_{\Gamma,S}$ is connected and its **Laplacian matrix** - the difference of the diagonal matrix of vertex degrees with the adjacency matrix of $C^{(1)}_{\Gamma,S}$ - has smallest non-zero eigenvalue strictly bigger than $1/2$.

In previous sections, we saw how finite generation conditions had drastic structural impacts on FI-graphs, sets, and modules. The first difficulty one faces with FI-groups, that is similarly found in the study of FI-algebras [NR], is that this is no longer necessarily the case. FI-groups come in a variety of different forms with the condition of finite generation having seemingly little impact on what kinds of group theoretic properties do and do not appear. We therefore shift our focus from questions of the form "what kinds of group theoretic properties are shared among finitely generated FI-groups," to questions of the form "what kinds of group theoretic properties are **stable** in finitely generated FI-groups?"

We say that a group property $\mathcal{P}$ is stable, if, for any finitely generated FI-group $\Gamma_\bullet$, there exists some $N \gg 0$ such that for all $n \geq N$, Γ_n has property $\mathcal{P}$ if and only if Γ_N does. As simple examples, one immediately sees that "being finite" is a stable property, whereas "having even rank" is not.

4.2. **Stability in FI-groups.** It is natural for one to ask whether there are any non-trivial group theoretic properties that are known to be stable. The proof of the following theorem uses the theory of FI-graphs in a non-trivial way.

Theorem 4.1 (Ellenberg and Ramos [ER])**.** *Let $(\Gamma_\bullet, S_\bullet)$ be a finitely generated FI-group with generating FI-set $S_\bullet$. Then property FA, as well as the geometric Cayley graph property, are stable properties of $(\Gamma_\bullet, S_\bullet)$.*

Note that, critical in the proof of this theorem, are the aforementioned stability phenomena which appear in the spectra of adjacency matrices associated to finitely generated FI-graphs.

4.3. **Future directions.** Recall from our examples that $SL_\bullet(\mathbb{Z})$ and $\mathrm{Aut}(F_\bullet)$ are both examples of finitely generated FI-groups. It has been known since the original work of Kazhdan that $SL_n(\mathbb{Z})$ has property (T) for $n \geq 3$ [Ka]. Later, work of Shalom reproved this fact using more algebraic means [Sh]. What is relevant to us, however, is the very recent combinatorial approach of Kaluba, Kielak, and Nowak [KKN]. In this work, Kaluba, Kielak, and Nowak use a characterization of property (T) proposed by Ozawa [O] to show that both $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{Aut}(F_n)$ have property (T) for $n \geq 5$. Notable in this work is that their approach is fundamentally tied to the theory of FI-groups, though they do not use this language. We therefore have the following.

Major Question A. Is Property (T) a stable property of finitely generated FI-groups?

It is famously known that the geometric Cayley graph property implies property (T), which in turn implies property FA. The work of Ellenberg and I detailed above therefore seems to strongly suggest that the above question should have a positive answer. Moreover, work of Pak and Zuk has shown that the presence of property (T) seems to be heavily influenced by the existence of a generating set with a group action [PZ]. Of course, such a generating set is literally baked into the definition of an finitely generated FI-group! The approach to this question through the work of Kaluba, Kielak, and Nowak seems to be the best direction of attack for this problem. Let us make this a bit more concrete by exploring their work in a bit more detail.

Let G be a group, and assume we have a symmetric (finite) generating set S for G. Then we define Δ to be the element of the real group ring $\mathbb{R}[G]$ given by

$$\Delta = \sum_{s \in S} (1-s)(1-s)^*,$$

where $* : \mathbb{R}[G] \to \mathbb{R}[G]$ is the linear operator $g \mapsto g^{-1}$. Then Ozawa has shown [O] that G has property (T) if and only if there exists some $\lambda > 0$ such that $\Delta^2 - \lambda \cdot \Delta \in \mathbb{R}[G]$ can be expressed as a sum $\sum_i \zeta_i \zeta_i^*$, where $\zeta_i \in \mathbb{R}[G]$. This type of decomposition in $\mathbb{R}[G]$ is what is known as a **sum of squares** decomposition. Ozawa's theorem allows, among other things, for one to decide whether a group has (T) with semi-definite programming.

Coming back to the context of FI-groups, we have a family of group-ring elements Δ_n, which we are trying so show eventually satisfy the Ozawa condition with some sequence of positive constants, λ_n. In [KKN], Kaluba, Kielak, and Nowak make the observation that both Δ_{n+1} and Δ_{n+1}^2 can be expressed as a finite linear combination of group ring elements, that are each the images of Δ_n and Δ_n^2, respectively, under the FI-group's transition maps. They make this observation in the cases of the two group families they consider, but it is valid for any finitely generated FI-group. Following this key observation, they then use some clever algebra, as well as a clever choice of the constant λ_{n+1}, to write

$$\Delta_{n+1}^2 - \lambda_{n+1} \cdot \Delta_{n+1}$$

as a sum of things that look like $\Delta_n^2 - \lambda_n \cdot \Delta_n$, along with a "fudge-factor," which they show is tied to the $n = 5$ case. They then prove this fudge-factor has the necessary sum of squares decomposition using the aforementioned semi-definite program.

As outlined in the previous paragraph, most of the big observations made in [KKN] have obvious analogs for finitely generated FI-groups. Therefore the only difficulty with this approach is in understanding how their clever choice of λ_{n+1}, as well as their semi-definite programming approach to resolving the fudge-factor generalize. Both of these issues require a delicate understanding of the combinatorics of FI-groups, but ultimately seem tractable.

While this combinatorial approach of [KKN] seems the most promising, there are some other approaches, which we briefly outline now. There exists a cohomological characterization of property (T) which relies upon the vanishing of the first comology group with certain twisted coefficients. Work of Kassabov and Putman has shown that groups with presentations preserved by a group action can have extremely well controlled homology [KaPu]. Once again we see this possible connection between (T) and equivariant presentations. Using the techniques of Kassabov and Putman, I believe one can prove the necessary vanishing statements for cohomology. Another approach involves a relative version of (T). This **relative property (T)** of a pair of a group with a subgroup was introduced following Khazdan's original work, and has shown itself to be worthy of further study [CI, CWS, Fer]. Many theoretical results related to this relative version of (T) answer the question of when it can be leveraged to prove the full property (T). The inherit inductive structure of FI-groups therefore lends itself to an approach to this problem which uses the relative (T).

Implicit in any theorem about stable properties is the question of when this stable behavior begins. In our work, Ellenberg and the I have bounded stable ranges for property FA and the geometric Cayley graph property in terms of simple invariants of the FI-group. In the case where one can provide an affirmative answer to Question A, it is critical that we also be able to bound this stable range.

Question 4.2. Assuming that property (T) is a stable property of finitely generated FI-groups, can one provide bounds on when this stable behavior begins?

One might expect that the question of "what other group theoretic properties are stable" to be of interest. In fact, if one spends a considerable time working with examples, and examining some of the most common group theoretic properties, they will be struck with the realization that all of them are either provably stable, or otherwise do not seem to be unstable. For instance, solvability and nilpotency are both stable properties of FI-groups. This leads to the following question.

Question 4.3. What are some interesting group theoretic properties that are **not** stable? The conspicuous dearth of such examples suggests that finitely generated FI-groups satisfy an extremely rigid structure theory. Such rigidity has been proved for FI-sets [RSW], as well as FI-modules. What exactly is the correct structure theory for finitely generated FI-groups?

5. Graph configuration spaces

5.1. **Background.** The study of the configuration spaces of graphs has seen a recent surge in popularity due to their connection with robotics and physics [Far, G, MS, MS2]. This is also evidenced by a recent AIM workshop on the subject. In what follows, for any graph G, we will largely concern ourselves with unordered configurations, $\mathrm{UConf}_n(G)$, the quotient of the usual configuration space by the action of the symmetric group obtained by permuting coordinates in the obvious way.

It is a theorem of Abrams [A], that $\mathrm{UConf}_n(G)$ is a $K(\pi, 1)$ for all $n \geq 1$ and all graphs G. It follows from this that in order to understand the homotopy type of $\mathrm{UConf}_n(G)$, it largely suffices to understand the fundamental group $B_nG := \pi_1(\mathrm{UConf}_n(G))$. In the literature, the braid groups B_nG have largely been studied from the perspective of geometric group theory. For instance, it is now fairly well understood when these groups are right-angle Artin [KP, KKP].

For the purposes of this proposal, our primary interest is related to homology groups $H_i(B_nG)$. It follows from the discussion in the previous paragraph the singular homology of the space $\mathrm{UConf}_n(G)$ is canonically isomorphic to the group homology of B_nG. Using this isomorphism, authors such as Kim, Ko, and Park [KKP, KP], as well as Farley and Sabalka [FS], have been able to prove surprising facts about the homology groups $H_i(B_nG)$. For instance, Ko and Park have shown that $H_1(B_2G) \cong H_1(B_nG)$ for all $n \geq 2$ whenever G is biconnected [KP]. Farley has also used the isomorphism to provide a computational method for determining the groups $H_i(B_nG)$ whenever G is a tree [Fa]. All of these results use a discrete Morse structure on $\mathrm{UConf}_n(G)$, that was developed by Farley and Sabalka [FS]. In my work, I have also used this isomorphism to prove non-trivial facts about the homology of the unordered configuration spaces of trees.

5.2. **Stability phenomena in the homology of tree braid groups.** The philosophy behind my work on the configuration spaces of graphs is based in the philosophy of asymptotic algebra as a whole. Namely, whenever a family of algebraic objects exhibits asymptotic stability phenomena, it is often the case that they can be encoded in a single object, that is finitely generated in the appropriate sense. In their work on the configuration spaces of trees, I showed that the homology groups $H_i(B_nG)$ can be encoded in a finitely generated graded module over a polynomial ring.

For a graph G, an **essential vertex** is any vertex of degree at least 3. An **essential edge** is a connected component of the space obtained by removing all essential vertices of G. We also define the quantity Δ_G^i to be the maximum number of connected components that G can be broken into by removing exactly i vertices.

Theorem 5.1 (Ramos [R3])**.** *Let G be a tree, and fixed $i \geq 0$. Then there is a polynomial $P_i^G \in \mathbb{Q}[t]$ of degree $\Delta_G^i - 1$ such that for all $n \geq 0$*

$$P_i^G(n) = \dim_{\mathbb{Q}}(H_i(B_nG;\mathbb{Q}))$$

This was proved using the structure theorems of Farley and Sabalka [FS], as well as the computational theorems of Farley [Fa]. In fact, the I have computed the polynomials P_i^G explicitly in terms of certain invariants of the tree G in [R3]. It follows from this computation that the homology groups $H_i(B_nG)$ do not fully depend on the tree G.

Corollary 5.2 (Ramos [R3])**.** *Let G be a tree. Then the homology group $H_i(B_nG)$ depends only on i, n, and the degree sequence of G.*

The key insight to proving the above theorem, which is perhaps more significant than the result itself, is that the groups $H_i(B_nG)$ carry a natural action by a polynomial ring.

Theorem 5.3 (Ramos [R3])**.** *Let G be a tree, and let A_G denote the integral polynomial ring with variables indexed by the essential edges of G. Then for each $i \geq 0$, there is an action of A_G on the graded $\mathbb{Z}$-module $\mathcal{H}_i := \bigoplus_n H_i(B_nG)$, turning $\mathcal{H}_i$ into a finitely generated graded A_G-module. Moreover, $\mathcal{H}_i \otimes \mathbb{Q}$ decomposes as a direct sum of graded twists of squarefree monomial ideals of Krull dimension at most Δ_G^i.*

The first obvious question arising from my work is whether it can be applied to more general raphs. Indeed, in [R3] it is shown that the action of A_G on $\mathcal{H}_i$ will still be well defined, provided certain diagrams commute. This was proved by An, Drummond-Cole, and Knudsen in a recent paper [ADK]. In fact, in a follow up paper [ADK2] the following theorem was proved, resolving a conjecture of mine.

Theorem 5.4 (An, Drummond-Cole, and Knudsen)**.** *If G is a graph, that is neither a line segment nor a circle, then $n \mapsto \dim_{\mathbb{Q}}(H_i(B_nG;\mathbb{Q}))$ is eventually agrees with a polynomial of degree $\leq \Delta_G^i - 1$.*

5.3. **Future directions.** As our first question on graph configuration spaces, we will consider the problem of how much of a graph G can be recovered from the spaces $\mathrm{UConf}_n(G)$. It is not hard to show using the aforementioned discrete Morse theory that $\pi_1(B_2G)$ is determined by the degree sequence of G in this case, though not even uniquely so. Using Abrams' result that $\mathrm{UConf}_2(G)$ is a $K(\pi, 1)$, one concludes that the homotopy type of $\mathrm{UConf}_2(G)$ is determined only by a numerical invariant which is weaker than the degree sequence. Namely, what is called the star-norm of the degree sequence in [R8]. In particular, one cannot recover G from $\mathrm{UConf}_2(G)$ using traditional topological invariants.

To get around this obstacle, Levin, Young, and I have considered the **finite exclusion process** as a stochastic process on the configuration space [LRY]. The finite exclusion process is a Markov process on (unordered) pairs of vertices of G defined by first flipping a coin to see that vertex in the pair will change, and then uniformly at random choosing an edge adjacent to this chosen vertex to move along. If the two original vertices were connected by an edge, and this is the edge chosen to move along, the process stalls. This finite exclusion process has been a topic of continuing interest in Markov chain theory for many years (see [PP, NN], and the references therein). Calling the transition matrix of this Markov process P_G, we have shown that G is uniquely recoverable from P_G, whenever G is a tree. However, we also believe that something stronger is true.

Assume for simplicity that we have chosen an embedding of our tree G into the plane, and that we have designated a leaf of G to be the root. Then one obtains a well ordering on vertices via a depth-first numbering originating from the root. from any starting location, running the Markov process for some fixed number of steps defines a path in the configuration space $\mathrm{UConf}_2(G)$. Levin, Young, and I then define the **closure** of this path to be the loop obtained from the path by allowing the two points to flow back towards their starting positions, one at a time, where the first point to move is that that is currently sitting on the smaller vertex. From the closure operation, one obtains an element of $H_1(B_2G)$. Assuming that G is a tree, $H_1(B_2G)$ is a free group, whence one can study this closure as a kind of (multivariate) statistic associated to the walk. We call this statistic the **winding statistic of the process**. Note that, while we call it *the* winding statistic, in truth there is a winding statistic for each choice of the number of steps taken before we stop and close the path. The main theorem of [LRY] is that the winding statistic asymptotically satisfies a central limiting theorem. That is, up to a factor of the square root of the number of steps taken, the winding statistic is converging in distribution to a centered multi-variate normal distribution with some covariance matrix $\mathbf{\Sigma}_G$. Critical in the proof of this fact was the previously mentioned discrete Morse structure of Farley and Sabalka [FS]

Question 5.5. If G and G' are two trees with no vertices of degree 2 such that $\mathbf{\Sigma}_G = \mathbf{\Sigma}_{G'}$, is it the case that G must be isomorphic to G'?

We note that in [LRY], this conjecture is verified both theoretically and experimentally for all trees of the required form with up to 10 vertices. Questions of this form have precedence in the study of random processes on topological spaces. For instance classic, work of Spitzer [Sp] considered the winding behavior of two particles in the plane, each performing independent Brownian walks. Spitzer's work lead to a plethora of similar works related to winding of randomly moving particles on surfaces [RH, WT]. One may think of this work of Levin, Young, and myself as trying to understand winding of a random process within a 1-dimensional topological space.

One to resolving this question begins with a slight simplification of the problem. While the exclusion process on general trees still evades complete characterization, there seems to be more hope if one limits themselves to **regular trees**. That is, trees whose every non-leaf vertex has the same degree. The exclusion process for trees of this form were one topic of study in the paper [NN], where the eigenspectrum is considered.

In [LRY], the covariance matrix $\mathbf{\Sigma}_G$ is proved to arise from a quadratic form associated to the **discrete Green's function** associated to the exclusion process. That is to say, the quasi-inverse of the difference $I - P$, where I is the identity matrix and P is the transition matrix associated to the exclusion process. Classically speaking, these discrete Green's functions arise due to their relation to expected hitting times [CY]. They are also, unfortunately, notorious difficult to explicitly compute in most cases. This is why the work of Nestoridi and Nguyen [NN] is critical. Using their computations related to the eigenvalues and eigenbasis of P, it should be possible to determine the aforementioned quadratic form.

Moving on from the regular case, things get considerably more difficult. One possible approach to this more general case once again relies on the discrete Green's function formulation mentioned above. We have already discussed that these Green's functions can be written in terms of expected hitting times of the exclusion process. While these hitting times have not yet been computed in the case of trees, they have been implicitly computed for the exclusion process on various other structures. For instance, they are computed for the exclusion process on the complete graph in [LL]. It is my belief that these computations can be adapted to our setting, at least for certain families of trees. Having computed these mixing times, one would gain access to the discrete Green's functions and therefore also our covariance matrices.

The second perspective on randomness in graph configuration spaces relates with the question of torsion in $H_i(\mathrm{UConf}_n(G))$. One of the greatest concerns in the study of graph configuration spaces is whether or not their homologies can contain odd torsion. An affirmative answer to this question would have implications even in physics [MS], where it would imply the existence of non-abelian quantum statistics. Inspired by work of Kahle, Lutz, Newman, and Parsons [**?**], on torsion in random simplicial complexes, I believe that the correct approach to finding exotic torsion is by randomizing the space in the following way: One considers the random process that begins at the empty graph on n vertices, and (randomly) adds an edge with some probability. This is then repeated until terminating at the complete graph. Writing $G(n,e)$ for the random graph on n vertices with e edges, one then charts how the homology groups $H_i(\mathrm{UConf}_{2n}(G(n,e)))$ change during this process. It is my interest to consider the homology groups of $G(n,e)$.

Work of Kahle, Lutz, Newman, and Parsons [**?**] has shown that in similar processes on random simplicial complexes one observes a massive **torsion burst** once the correct number of faces have been added. It is my belief something similar will happen in our context as well.

Question 5.6. Will the homology group $H_{n-1}(\mathrm{UConf}_{2n}(G(n,e)))$ experience a torsion burst for some n,e? If this would be the case, it greatly suggests that the homology of graph configuration spaces will experience odd torsion in their penultimate homology group.

Note that there are theoretical reasons why exotic torsion is more likely to appear in the penultimate homology group than any other. One evidence for this is the theorem of Miyata Proudfoot and myself [MPR], which states that torsion in $H_i(\mathrm{UConf}_n(G))$ is bounded independently of G and n for any fixed i. Therefore, the only way a torsion explosion is even possible in this context is if one considers homology in degrees that change with the size of the graph. The penultimate homological degree is essentially equal to one less than the number of vertices, so this makes sense.

The original work [**?**] outlines a general experimental method for detecting such phenomena, provided your topological space is a finite simplicial complex. The aforementioned discrete Morse theory of graph configuration space therefore provides a potentially computationally feasible model to apply the necessary techniques. In fact, this has been implemented in the work [GG]. For this particular problem we are most interested in showing that odd torsion exists somewhere, and so experimental methods are more than capable of telling us what we need, though it would also be nice to develop a solid theoretical foundation for what these torsion classes represent. Moreover, given the abundance of literature that suggests that planar graphs lack torsion in the homologies of their configuration spaces [KP, ADK3], it is sensible to change the model of our random graph to (almost surely) include certain minors. These kinds of modifications are well established in the literature.

Returning to questions more in the realm of topology and algebra, the aforementioned theorems of An, Drummond-Cole, and Knudsen are some of the few results that one has about the A_G-module structure of the homology groups of B_nG, whenever G is a general graph. While it might be too much to ask to be able to prove a structure theorem similar to Theorem 5.3 for general graphs, there seems to be hope in studying these modules within certain families of graphs. For instance, I have recently shown that certain families of FI-graphs exhibit very restrictive behavior in the Betti numbers, in the commutative algebra sense, of their unordered configuration spaces [R4]. This result was recently generalized using a categorical version of the graph minor theorem in [MPR].

Question 5.7. Let $\mathcal{P}$ denote the class of graphs with some particularly nice property. For instance, $\mathcal{P}$ might denote the class of regular graphs, planar graphs, or graphs of some restricted maximal vertex degree. What results can one prove about the structure of $H_i(B_nG)$ as a module over A_G for $G \in \mathcal{P}$?

The theorem of An, Drummond-Cole, and Knudsen is also striking in that it provides an explicit connection between the modules $H_i(B_nG)$ and connectivity invariants of the graph G. This leads to the following question.

Question 5.8. What graph theoretic invariants are encoded in the commutative algebra of $H_i(B_nG)$? For instance, what does the Hilbert Polynomial of this module tell us about G?

Tying the previous two questions together, An, Drummond-Cole, and Knudsen recently proved that the commutative algebra of $H_2(\mathrm{UConf}_n(G))$ does indeed encode other connectivity invariants, at least when G is planar [ADK3]. Another question that one might ask is whether anything can be said about the usual configuration spaces $\mathrm{Conf}_n(G)$. The current consensus in the literature is that these spaces are considerably more complicated than the unordered configuration spaces, and very little is known about their behavior [BF, CL]. The aforementioned work of White and the myself [RW], Miyata, Proudfoot and myself [MPR, PR, PR2], as well as Lütgehetmann [Lu], has provided evidence that it is perhaps easier to think about these spaces by not allowing n to vary. That being said, just as in the unordered case, one would like to understand these spaces without this restriction.

Question 5.9. What is the correct way to approach configuration spaces of graphs, while allowing the number of points being configured to vary? Considering the polynomial ring action described above in the unordered case, should one expect the correct action here to be by a **twisted commutative algebra** as described by Sam and Snowden [SS]?

All of these questions form a tiny sample of the various interesting open problems in the study of graph configuration spaces. These questions seem to require techniques from graph theory, commutative algebra, topology, and even computational mathematics to solve. See the introduction of [ADK3] for more on this.

6. Homological invariants of FI-modules

6.1. **Background.** Having reviewed my most recent work in algebraic combinatorics, we now step back to my earlier works, which are more traditionally homological in nature. While many of my initial conjectures in these directions have been proven (see [NSS], for instance), there is still much that we do not understand.

The Noetherian property is the first step in treating finitely generated FI-modules from a homological perspective. Prior to my own work, this philosophy is most apparent in works of Church and Ellenberg [CE], Sam and Snowden [SS], and Gan and Li [GL]. In [CE], Church and Ellenberg consider **homology functors** H_i for FI-modules. If V is an FI-module, then $H_0(V)$ is defined to be the FI-module with points $H_0(V)(T) = V(T)/\cup_{T' \subsetneq T} V(T')$. The homology functors H_i are defined to be the (left) derived functors of H_0. It follows directly from the definition that the module $H_0(V)$ is **finitely supported**, i.e. $V(T) = 0$ for $|T| \gg 0$, whenever V is finitely generated. Remarkably, it can be shown that for any finitely generated module V , the quantity $\max_n\{H_i(V)(n) \neq 0\} - i$ is bounded independently of i. This was first proven for certain choices of k by Sam and Snowden in [SS], and expanded to allow for any commutative ring k by Church and Ellenberg [CE]. This bound, which we shall henceforth refer to as the **regularity** of V, was used by Church and Ellenberg to generalize certain stability results of Putman on congruence subgroups [Pu]. Note that the regularity in this context is exactly analogous to the notion of Castelnuovo-Mumford regularity, a concept of foundational importance to commutative algebra [E]. Understanding the mechanisms underlying the finiteness of regularity therefore becomes critical in explaining certain concrete phenomena in the study of congruence subgroups.

Another property connected to homological invariants is the existence of the **Hilbert Polynomial**. If V is a finitely generated FI-module over a field k, then there exists a polynomial

$P(X) \in \mathbb{Q}[X]$ such that for all finite sets T with $|T| \gg 0$,

$$\dim_k(V(T)) = P(|T|)$$

This fact was proven in the case where k is a field of characteristic 0 by Church, Ellenberg, and Farb in [CEF], and was expanded to all fields by Church, Ellenberg, Farb, and Nagpal in [CEFN]. Seeing a result like this, it therefore becomes quite natural to ask when this polynomial behavior begins. A bound is given by Church, Ellenberg, and Farb in the cases wherein k is a field of characteristic 0 in [CEF], although no bound is given in the cases in which k is a general field. My work reveals that this question, as well as that of the regularity of a module, are deeply connected to one another through the study of local cohomology.

6.2. **Local Cohomology.** Let V be an FI-module over a Noetherian ring k. Then we say that an element $v \in V(T)$ is **torsion** if there is some finite set T' and an injection $f : T \to T'$ such that v is in the kernel of the map induced by f.We define $H^0_{\mathfrak{m}}(V)$ to be the maximal torsion submodule of V. The assignment $V \mapsto H^0_{\mathfrak{m}}(V)$ defines a left exact functor, and we denote its derived functors by $H^i_{\mathfrak{m}}$. These are the **local cohomology functors**.

In the case where k is a field of characteristic 0, Sam and Snowden studied the local cohomology functors, and proved that many important invariants of FI-modules, such as regularity, were encoded by them [SS]. When k is a field of characteristic 0, however, Sam and Snowden show that FI-modules can be equivalently thought of as a subclass of modules over a polynomial ring in infinitely many variables [SS]. The first challenge to generalizing the results of Sam and Snowden to general coefficient rings was in finding a consistent language which would allow us to use the commutative algebra techniques of their work. This was accomplished in my work with Li. [LR].

To begin, we show that the modules $H^i_{\mathfrak{m}}(V)$ can always be computed within the category FI-mod of finitely generated FI-modules, so long as V is finitely generated.

Theorem 6.1 (Li and Ramos [LR]). *Let V be a finitely generated FI-module over a Noetherian ring k. Then for each i, the module $H^i_{\mathfrak{m}}(V)$ is finitely generated, and $H^i_{\mathfrak{m}}(V)(T) = 0$ for all T with $|T| \gg 0$. Moreover, $H^i_{\mathfrak{m}}(V) = 0$ for $i \gg 0$.*

Considering the theory of local cohomology in the context of local rings, the above might be quite surprising. Indeed, classically it is known that the top non-vanishing local cohomology module is never finitely generated [BS].

The above theorem implies that each finitely generated module V has a well defined smallest, and largest, non-vanishing local cohomology module. We define the **depth** of V to be the smallest index i such that $H^i_{\mathfrak{m}}(V) \neq 0$. In [R], I introduced a definition of depth which can be used for FI-modules over any ring. It is a theorem of Li and myself these two definitions are equivalent [LR].

Theorem 6.2 (Li and Ramos [LR]). *Let V be a finitely generated FI-module over a Noetherian ring k. Then the regularity of V is at most $\max_i\{\deg(H^i_{\mathfrak{m}}(V)) + i\}$.*

The quantity $\max_i\{\deg(H^i_{\mathfrak{m}}(V)) + i\}$ is very reminiscent of the Castelnuovo-Mumford regularity of a module over a polynomial ring [E]. In fact, It was proven by Nagpal, Sam, and Snowden that it is equal to the regularity of the FI-module V, just as is the case in the classical setting. [NSS]

Theorem 6.3 (Li and Ramos [LR]). *Let V be a finitely generated FI-module over a field k. Then the dimension $\dim_k(V(T))$ agrees with a polynomial for $|T|$ at least $\max_i\{\deg(H^i_{\mathfrak{m}}(V))\}$.*

Note that the above two theorems provide an explicit connection between the regularity of a module and the obstruction to its Hilbert polynomial. Once again, such a connection exists and is classically known about modules over a polynomial ring [E].

6.3. **Future Directions.** While my work with Li lays the groundwork for a "commutative algebra" approach to the study of FI-modules, there is still much which is not well understood. To begin, very little has been invested in trying to compute these local cohomology groups. Finite generation of these modules implies that such a theory is, at the very least, not completely hopeless. Assuming such tools exists, the theorem of Nagpal, Sam, and Snowden [NSS] would provide explicit means for computing the regularity of an FI-module. The work of Church and Ellenberg [CE] implies that these computations have real applications in the study of congruence subgroups.

Question 6.4. Can one develop a computational theory of the local cohomology of FI-modules, perhaps building on the computational theory of FI-modules developed by Wiltshire-Gordon [W-G]?

It is also an interesting question to ask how these homological invariants are interacting with the various topological examples of FI-modules. While the theorems of the previous section can be used to bound the obstruction to the Hilbert polynomial, for instance, it is often the case in practice that these bounds are non-optimal. This would suggest that "niceness" in the topology from which the example arises is being observed by the local cohomology modules. Can this be formally described? Conversely, can one deduce facts about the topology of a provided example, given its local cohomology? A new paper of Church, Miller, Nagpal, and Reinhold [CMNR] does some work in this direction, but there is apparently much more that one might be able to say.

Question 6.5. Let $X_\bullet$ be a functor from FI to the category of topological spaces, and assume, for all i, that $H_i(X_\bullet)$ is finitely generated as an FI-module over $\mathbb{Z}$. What does the topology of $X_\bullet$ tell us about the local cohomology groups of $H_i(X_\bullet)$? Conversely, what does the local cohomology of $H_i(X_\bullet)$ reveal about the topology of the spaces X_n?

Finally, while FI-modules are arguably the most well understood examples of representations of combinatorial categories, there is a large variety of other categories whose representation theory have become extremely relevant in the literature. For this reason it has become an interesting open question as to how much of the above formalism can be adapted to these different circumstances. For instance, work of the author and Li has illustrated similarities between the local cohomology theory of what are known as FI^m-modules, and the local cohomology of multi-graded modules over a polynomial ring [LR2], while work of Sam and Snowden has shown a deep connection between the local cohomology of what are known as FI_d-modules and the geometry of Grassmanians [SS2]. Moving forward I am interested in studying homological invariants of VI(q) and VIC(q)-modules. One may think of these categories as "q-versions" of FI, as their representation theory encodes families of $GL_n(q)$-modules [PS]. Work of Miller and Wilson [MW], Miller, Patzt, and Wilson [MPW], and Nagpal [N2, N3] have displayed a variety of applications one would unlock by developing a robust theory of the homological invariants of these categories.

Question 6.6. Building on recent work on Nagpal [N2, N3] and Harman [Ha], can one develop a satisfactory theory of homological invariants of VI(q) and VIC(q)-modules similar to the theory for FI-modules described above? In particular, can one prove strong finiteness results of regularity of VIC(q)-modules which improves upon the work of Miller and Wilson [MW]?

7. Coherence in the representation theory of categories

7.1. **Background.** In the previous sections, we saw that there is a large premium put on the condition that an FI-module be finitely generated. However, for one to have a Noetherian property, it is necessary that one work over a Noetherian ring. For some applications one would like to work with FI-modules over rings which are not Noetherian. Once again pulling from commutative algebra, perhaps the next best thing after Noetherianity is coherence. We say an FI-module over a ring k is **coherent** if it is generated by elements which only appear in finitely many degrees, and the module of relations between these generators is also generated in finitely many degrees. Note that a coherent module need not be finitely generated, but the (possibly infinitely many) generators must appear in only finitely many degrees.

7.2. **The coherence of FI-modules.** Much of the work described on the local cohomology of FI-modules was only doable because the Noetherian property allows us to treat FI-mod using the techniques of abelian categories. Our first goal will therefore be to prove something similar for coherent modules. The following theorem was independently proven by Li and myself.

Theorem 7.1 (Li [Li], Ramos [R2])**.** *The category of coherent* FI*-modules over a commutative ring k is abelian. That is, the kernel and cokernel of any morphism of coherent* FI*-modules are themselves coherent.*

Perhaps the most notable fact about the above theorem is that it does not have any conditions on the ring k. This implies that while FI-modules may not be Noetherian, they are always coherent. The regularity theorem of Church and Ellenberg, along with the above theorem, seems to imply that the work completed in Section 6.2 should still hold for coherent FI-modules. This is indeed the case.

Theorem 7.2 (Ramos [R2])**.** *The theorems of Section 6.2 continue to hold with finitely generated replaced in all places by coherent, and the Noetherian hypothesis removed from the ring k.*

7.3. **Future directions.** While many of the various well studied concrete examples of FI-modules often turn out to be finitely generated, more recent examples have shown that coherent modules also arise naturally. For instance, the work of Church and Ellenberg provides examples of this kind in the study of congruence subgroups for general rings [CE]. What remains interesting is the question of whether finite generation is something one should expect to find in more general and nuanced examples. For instance, if one were to change FI to some more general category, whose representations are not necessarily Noetherian, can something still be said about coherent representations?

As an example, consider the category $\mathrm{VI}(\mathbb{Z})$, of finite rank free $\mathbb{Z}$ modules, with split linear injections. Representations of this category were considered by Putman and Sam in [PS], and have more recently been used in the work of Patzt [Pa] and Miller, Nagpal, and Patzt [MNP]. Representations of similar categories were critical in the resolution of the Lannes-Schwartz Artinian conjecture [PS]. It was proven by Putman and Sam that representations of $\mathrm{VI}(\mathbb{Z})$ will not have the Noetherian property over any ring k. However, the work of Patzt [Pa] seems to imply that these representations appear in nature. In that work, Patzt specifically considers their applications to the study of Torelli groups.

I am interested in studying $\mathrm{VI}(\mathbb{Z})$ representations, and especially interested in understanding whether they satisfy any kind of coherence. A result in this direction would provide at least a framework for approaching the many natural examples of $\mathrm{VI}(\mathbb{Z})$ representations without needing to rely on any kind of Noetherian hypothesis. It would also expand upon results of Patzt [Pa].

Question 7.3. If k is a field of characteristic 0, can one prove a structure theorem similar to that of Nagpal [Nagp] for coherent $\mathrm{VI}(\mathbb{Z})$-modules?

The previously mentioned work of Miller, Nagpal, and Patzt [MNP] also considers a property that they call Koszulness. This is used to great affect in that paper, and suggests that there are notions which fall between finite generation and coherence which might be sufficient for many applications. This particular philosophy appears in work of Harman [Ha], where he considers a condition he calls rigidity.

Question 7.4. What finiteness properties of $\mathrm{VI}(\mathbb{Z})$-modules, and related objects, can one leverage to prove non-trivial facts about the growing list of examples of these objects?

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