Homework 1

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Course: *Network dynamics and learning* – Professor: *Fagnani fabio*
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The homework has been carried out alone but then it has been checked and discussed with course students: Davide Bussone, Pedro Ramirez Hernandez, Antonio Dimitris Defonte.

Exercise 1

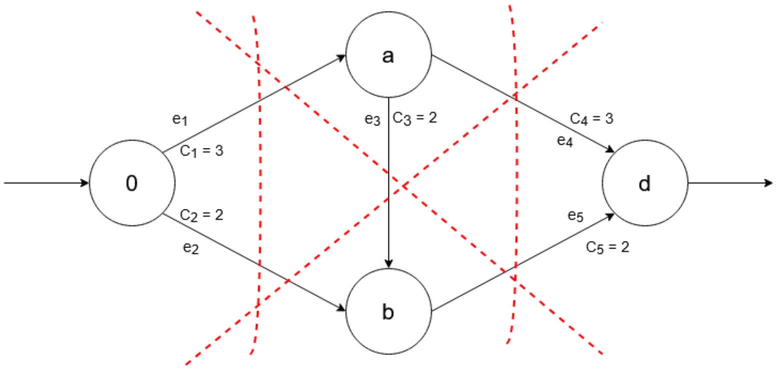

Figure 1: *graph cuts*

a-b). The four *o*-*d* cuts $U_1 = \{o\}$, $U_2 = \{o, a\}$, $U_3 = \{o, b\}$ and $U_4 = \{o, a, b\}$ have capacities:

$$C_{U_1} = 5, C_{U_2} = 7, C_{U_3} = 5 \text{ and } C_{U_4} = 5$$

According to the max-flow min-cut theorem, the max possible flow is the minimum between C_{U_n} which is $\tau_{o,d} = 5$ for C_{U_1},C_{U_3} and C_{U_4}. The infimum that if removed does not allow an unitary flow is $4 + \epsilon$. To maximize the throughput τ by adding 2 additional capacities points, they should be apportioned between C_2 and C_5 making $\tau^*_{o,d} = C_{U_1} = C_{U_3} = C_{U_4} = 6$.

c). Wardrop equilibrium. The paths colored in the picture above are p^1 ,p^2 , p^3 (red, green, blue), with respective flows z_1,z_2 and z_3. So it turns out that

- $x_1 = z_1 + z_2 \Rightarrow d_1(x) = (z_1 + z_2) + 1$
- $x_2 = z_3 \Rightarrow d_2(x) = 5(z_3) + 1$
- $x_3 = z_3 \Rightarrow d_3(x) = 1$

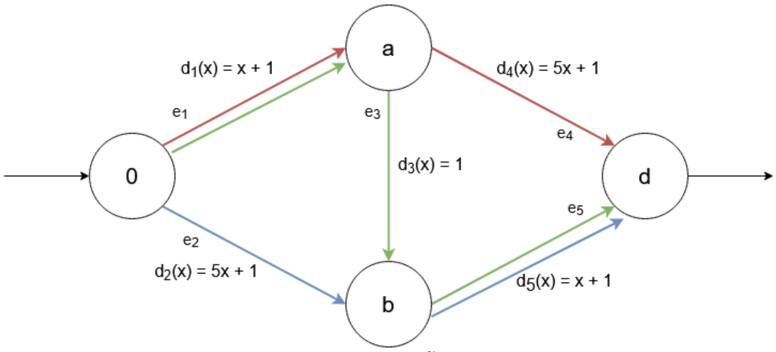

Figure 2: *flows*

- $x_4 = z_1 \Rightarrow d_4(x) = 5(z_1) + 1$
- $x_5 = z_2 + z_3 \Rightarrow d_5(x) = (z_2 + z_3) + 1$

and for each path the total delay is given by:

- $d_{p^1} = d_1 + d_4 = 6z_1 + z_2 + 2$
- $d_{p^2} = d_1 + d_3 + d_4 = z_1 + 2z_2 + z_3 + 3$
- $d_{p^3} = d_2 + d_5 = 6z_3 + z_2 + 2$

As it could be noticed in the figure 2 there is an horizontal symmetry of delays that could lead to have $d_1 = d_2$ given $z_1 = z_3$.

To calculate the W.E. for an unitary flow we consider $z_2 = 1 - z_1 - z_3$.

- $d_{p^1} = 5z_1 - z_3 + 3$
- $d_{p^2} = -z_1 - z_3 + 5$
- $d_{p^3} = 5z_3 - z_1 + 3$

Then we suppose that $z_1 > 0 \Rightarrow d_{p^1} \leq d_{p^2}$, $d_{p^1} \leq d_{p^3}$ (the condition is based on the selfish behaviour of users).

- $5z_1 - z_3 + 3 \leq -z_1 - z_3 + 5 \Rightarrow z_1 \leq 1/3$
- $5z_1 - z_3 + 3 \leq 5z_3 - z_1 + 3 \Rightarrow z_1 \leq z_3$

But if $z_1 \leq z_3$ means that $z_3 > 0$ so $d_{p^1} = d_{p^3}$. As we supposed before the flows are symmetric $z_1 = z_3$, moreover $z_1 = z_3 = 1/3$, that means $z_2 = 1 - z_1 - z_3 = 1/3$

- $z_1 = 1/3 \Rightarrow d_{p^1} = 13/3$
- $z_2 = 1/3 \Rightarrow d_{p^2} = 13/3$
- $z_3 = 1/3 \Rightarrow d_{p^3} = 13/3$

And so the average delay is $13/3$.

d). The social optimum flow vector is the vector $\mathbf{z} = [z_1, z_2, z_3]$ that minimizes the total cost associated.

$C = 6z_1^2 + 2z_2^2 + 6z_3^2 + 2z_1z_2 + 2z_2z_3 + 2z_1 + 3z_2 + 2z_3$

That by substitution becomes:

$C = 6z_1^2 + 6z_3^2 - 3z_1 - 3z_3 + 5$

To find the minimum we consider the partial derivatives of C over z_1 and z_2.

$\frac{\partial C}{\partial z_1} = 12z_1 - 3$

$\frac{\partial C}{\partial z_3} = 12z_3 - 3$

In the end given that the cost function is an increasing function, being a set of sum/-multiplication of delays and flows that are non negative, we have that for $\partial C = 0$ there is a global minimum, condition satisfied by $z_1 = z_3 = 1/4 \Rightarrow z_2 = 1/2$. We could also state that it is the global minimum by checking the second derivative which is 12 for both variables, that means the function is convex over $z_1, z_3 \in [0, 1]$.

The min cost is: $\frac{17}{4}$ for $Z = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

e). The associated price of anarchy is: $6z_1 + 6z_3 + 2z_2 + 5$

$PoA = \frac{\frac{13}{3}}{\frac{17}{4}} = \frac{52}{51}$

f). The tolls must be set to reduce the PoA to 1 are:

$w^* = c'(f^*) - d_f^* = f^* d'(f^*)$

$d'(f^*) = \{1, 5, 0, 5, 1\}$

$f^* = \{3/4, 1/4, 1/2, 1/4, 3/4\}$

$\Rightarrow w^* = \{3/4, 5/4, 0, 5/4, 3/4\}$

Exercise 2

$w_e = 1 \ \forall e \in \Sigma \setminus \{w_0 = a\}$ The related condensation graph H_g is composed from a single node, that means it is strongly connected.

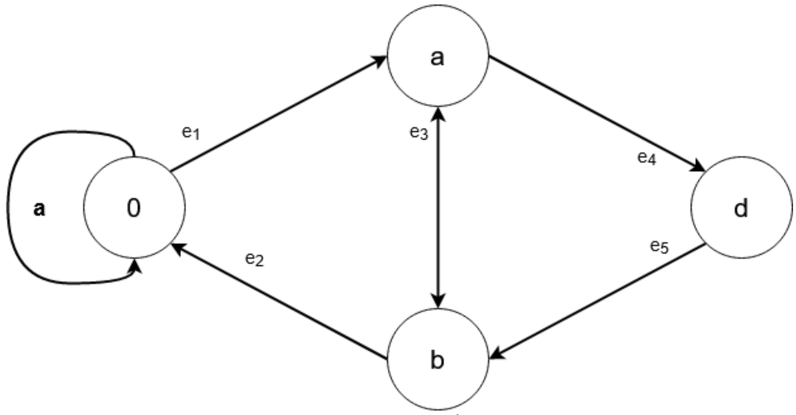

Figure 3: *graph cuts*

a). $W = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ $w = W\mathbb{1} = \begin{pmatrix} a+1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ $D = diag(w) = \begin{pmatrix} a+1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$$D^{-1} = \begin{pmatrix} \frac{a}{a+1} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} P = \begin{pmatrix} \frac{a}{a+1} & \frac{1}{a+1} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} L = D - W = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

b-c). As said at the beginning of this section, the graph is strongly connected and moreover the relative condensation graph is composed by a single node, that means a unique sink $s_g = 1$, that coincide with the number of consensus vectors possible. Moreover the graph is aperiodic $\forall a \geq 0$, it follows that

$$\lim_{t \to \infty} x(t)$$

converges $\forall a \geq 0$.

d). Given a=0 the graph is strongly connected and balanced, so for $x = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

$$\lim_{t \to \infty} x(t) = \mathbb{1}\pi' x(0)$$

$$\lim_{t \to \infty} x(t) = \mathbb{1}\alpha$$

$$\alpha = \pi' x(0)$$

$$\pi = \frac{w}{|w|} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$\alpha = -\frac{1}{6} + \frac{1}{3} - \frac{1}{6} + \frac{1}{3} = \frac{1}{3}$$
$$\Rightarrow \lim_{t\to\infty} x(t) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

e). For $a \geq 0$ the graph is still balanced,

$$\Rightarrow \pi = \begin{pmatrix} \frac{a+1}{a+6} & \frac{2}{a+6} & \frac{1}{a+6} & \frac{2}{a+6} \end{pmatrix}$$
$$\alpha = -\frac{a+1}{a+6} + \frac{2}{a+6} - \frac{1}{a+6} + \frac{2}{a+6} = \frac{2-a}{a+6}$$
$$a = \min_{a\geq 0} a \mid \lim_{t\to\infty} x_1(t) < 0 \iff a = 2$$

.

f).

$$\lim_{t\to\infty} x_1(t) = \alpha$$

Again for a generic $a \geq 0$,

$$\pi = \begin{pmatrix} \frac{a+1}{a+6} & \frac{2}{a+6} & \frac{1}{a+6} & \frac{2}{a+6} \end{pmatrix}$$

but alpha would be:

$$\alpha = x_1(0)\frac{a+1}{a+6} + x_2(0)\frac{2}{a+6} + x_3(0)\frac{1}{a+6} + x_4(0)\frac{2}{a+6} = \frac{x_1(0)(a+1) + 2x_2(0) + x_3(0) + 2x_4(0)}{a+6}$$

$$\mathrm{Var}(\lim_{t\to\infty} x_1(t)) = \mathrm{Var}(\alpha) = \mathbb{E}[\alpha^2] - \mathbb{E}[\alpha]^2$$
$$\mathbb{E}[x_i] = 0 \forall x_i \in x(0) \Rightarrow \mathbb{E}[\alpha]^2 = 0$$
$$\mathrm{Var}(\alpha) = \mathbb{E}[\alpha^2] =$$
$$\mathbb{E}[\frac{2(a+1)(2x_2x_1 + x_3x_1 + 2x_4x_1) + x_1^2(a+1)^2 + 4(x_2^2 + x_2x_3 + 2x_2x_4 + x_3x_4 + x_4^2) + x_3^2}{(a+6)^2}]$$
$$= \frac{(a+1)^2\mathbb{E}[x_1^2] + 4\mathbb{E}[x_2^2] + \mathbb{E}[x_3^2] + 4\mathbb{E}[x_4^2]}{(a+6)^2}$$
$$= \frac{a^2 + 2a + 10}{(a+6)^2}$$
$$\frac{\partial \mathrm{Var}}{\partial a} = \frac{2(5a-4)}{(6+a)^3}$$
$$\frac{\partial \mathrm{Var}}{\partial a} = 0 \iff a = \frac{4}{5}$$

So in the end, the variance would be minimized for a = 4/5.

Excercise 3

a). The Florence's families graph is undirected, balanced and strongly-connected. This means that

$$\lim_{t\to\infty} x(t) = \mathbb{1}\pi' x(0)$$

. The initial opinion x(0) is 0 except for

- Medici : $1 \Rightarrow \pi_{Medici} = \frac{6}{38}$
- Strozzi : -1 $\Rightarrow \pi_{Strozzi} = \frac{4}{38}$

Every other product is 0, so

$$\alpha = \frac{1}{19}$$

and

$$\Rightarrow \lim_{t\to\infty} x(t) = \frac{1}{19}\mathbb{1}_{(15)}$$

A python version of this excercise is available at the endd of the homework.

c). In this part the graph will solved with the simplification of the electric networks. A variant is reported in the end.

In the final figure is shown the asymptotic opinion, with

- red : -1
- white : 0
- blue : 1

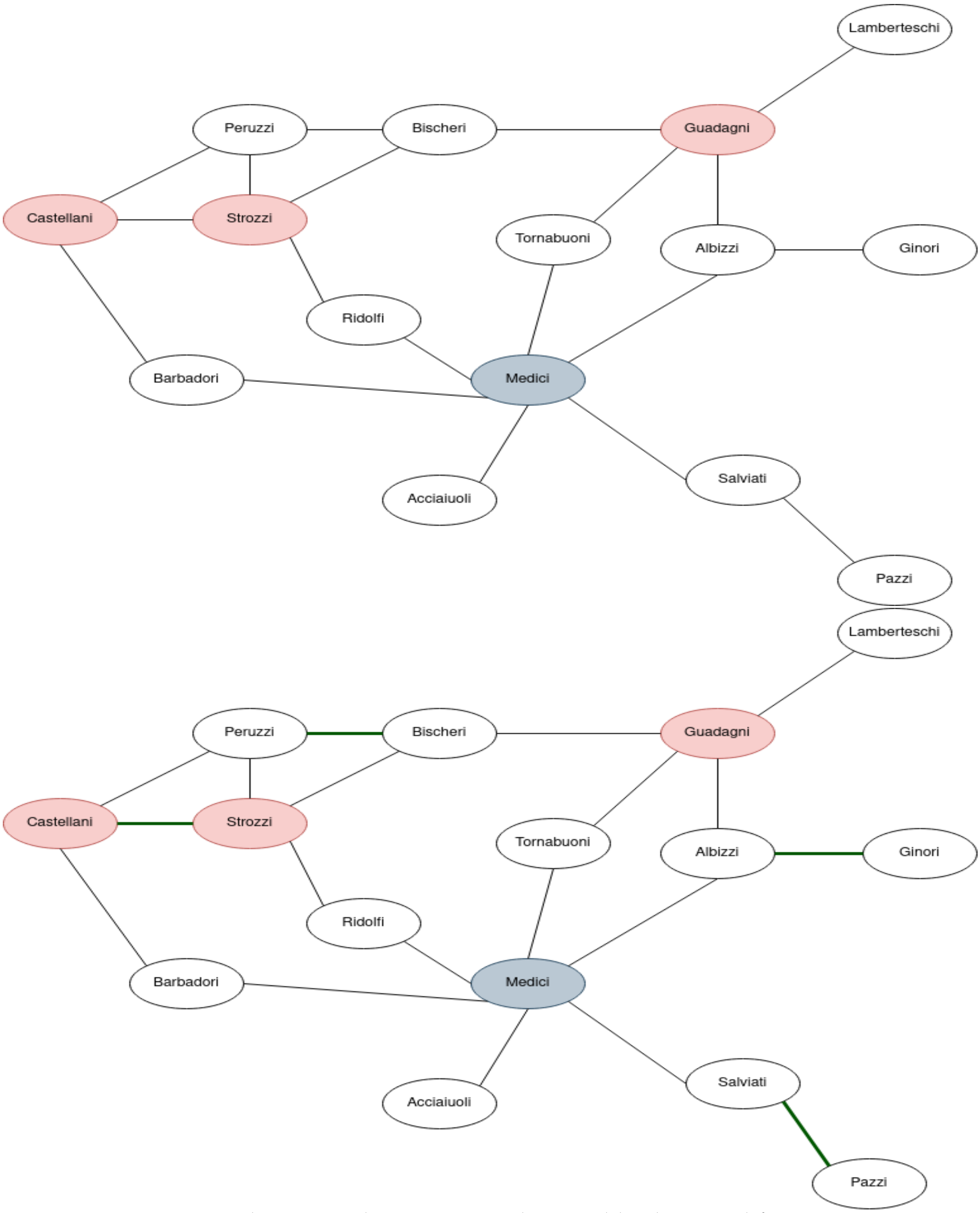

Figure 4: The green edges represent the possible glue simplifications

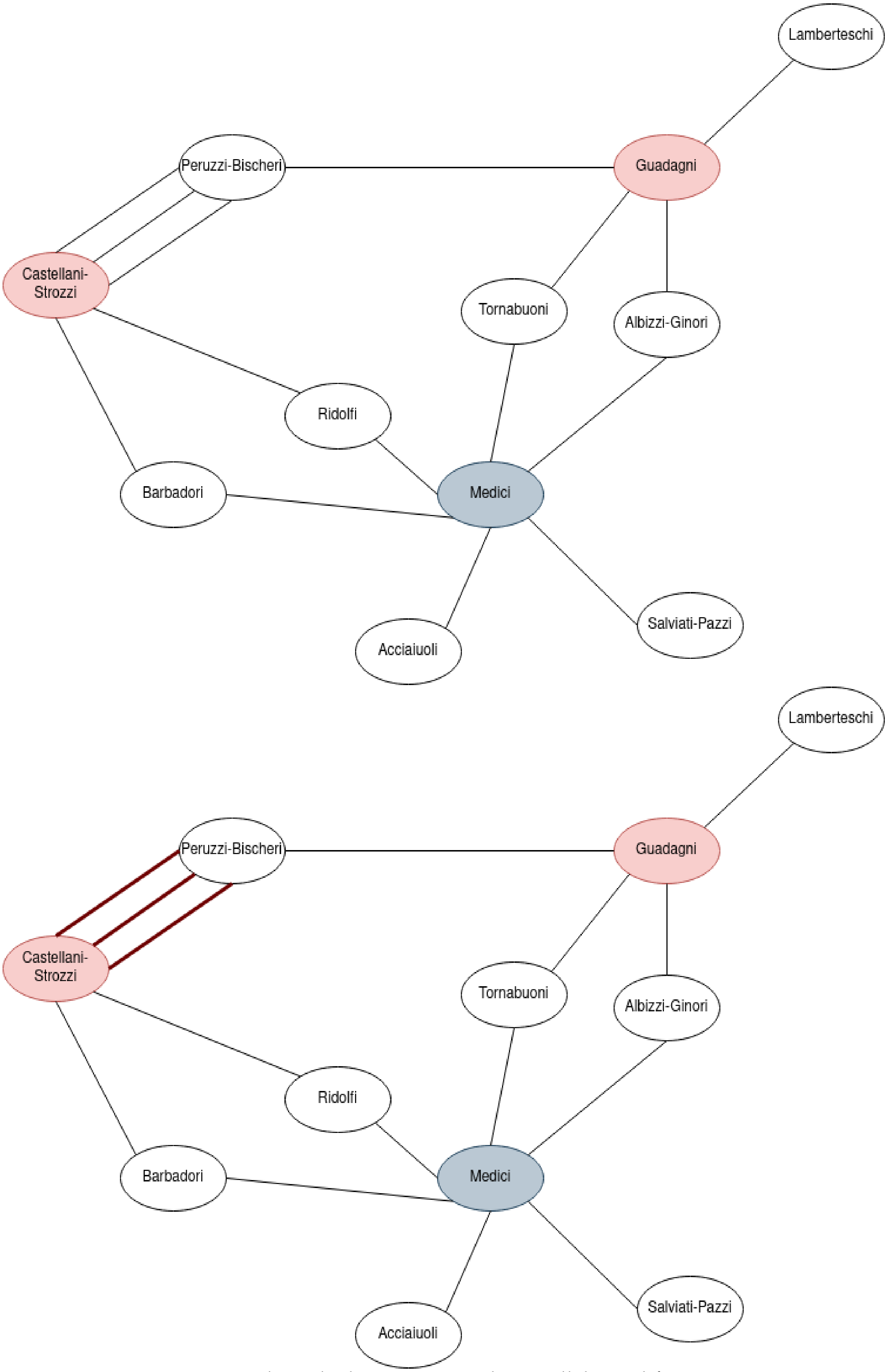

Figure 5: The red edges represent the parallel simplification

Figure 6: in this step intermediary nodes between concord stubborn nodes and dead end nodes are simplified (in an electric network no current flow would pass throught)

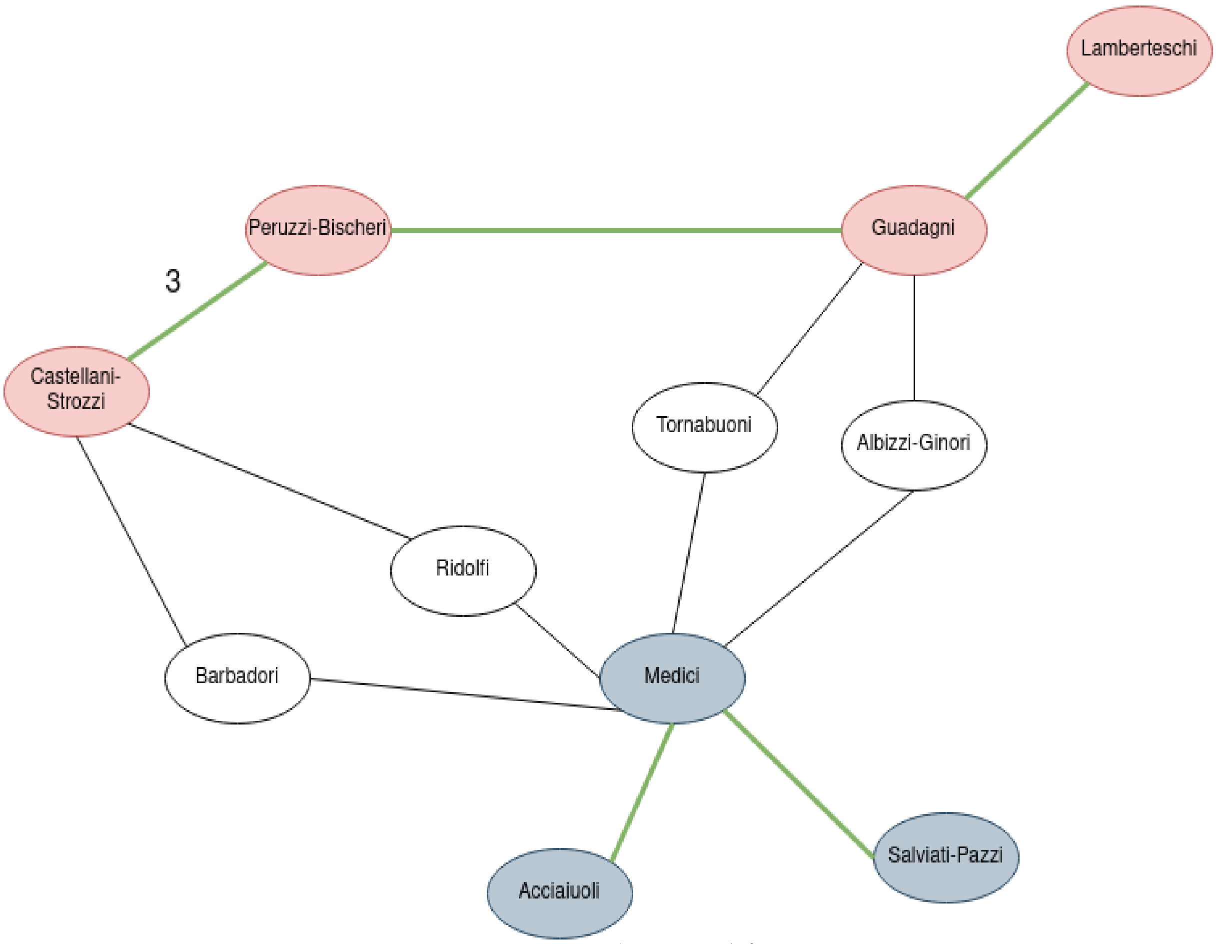

Figure 7: again a glue simplification

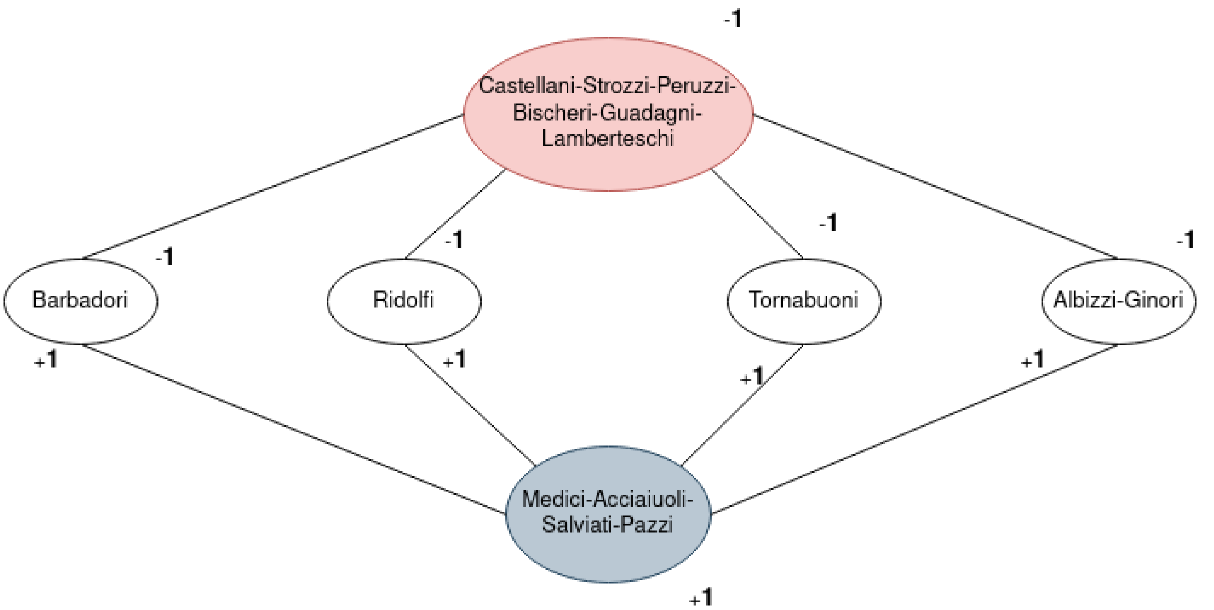

Figure 8: No more simplifications are required it is clear the result (red : -1, white : 0, blue : 1) the sum of currents for each white node is 0