COMP9417 - Machine Learning
Tutorial: Regression II

Question 1. Maximum Likelihood Estimation (MLE)

In this question we will first review and then work through a few examples of parameter estimation using the MLE technique. The following introduction can be skipped if you are comfortable with the MLE concept already.

The setting is as follows: we sample n observations (data), which we denote by $X_1, X_2, \ldots, X_n$, and we assume that the data is independently drawn from some probability distribution P. The shorthand for this is:

$$X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} P,$$

where i.i.d. stands for independent and identically distributed. In practice, we never have access to P, we are just able to observe samples from P (namely $X_1, \ldots, X_n$), which we will use to learn something about P. In the simplest case, we assume that P belongs to a parametric family. For example, if we assume that P belongs to the family of normal distributions, then we are assuming that P has a probability density function (pdf) of the form

$$p_\theta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad \theta = (\mu, \sigma^2), \qquad \mu \in \mathbb{R}, \sigma > 0,$$

where we will usually refer to μ as the mean, and σ^2 as the variance, and we combine all unkown parameters into a single parameter vector θ that lives in some parameter space Θ. In this particular example, $\Theta = \mathbb{R} \times [0, \infty)$. Under this assumption, if we knew θ, then we would know P, and so the learning problem reduces to learning the best possible parameter θ^*, hence the name *parametric*.

Continuing with this example, we need a way of quantifying how good a particular choice of θ is. To do this, we first recall the fact that for independent sets A, B, C, it holds that $P(A \text{ and } B \text{ and } C) = P(A)P(B)P(C)$. Therefore, we have:

$$\begin{aligned}
\text{Prob of observing } X_1, \ldots, X_n &= \text{Prob of observing } X_1 \times \cdots \times \text{Prob of observing } X_n \\
&= p_\theta(X_1) \times \cdots \times p_\theta(X_n) \\
&= \prod_{i=1}^{n} p_\theta(X_i) \\
&=: L(\theta).
\end{aligned}$$

We call $L(\theta)$ the *likelihood*, and it is a function of the parameter vector θ. We interpret this quantity as the probability of observing the data when using a particular choice of parameter. Obviously, we want to

choose the parameter θ that gives us the highest possible likelihood, i.e. we wish to find the *maximum likelihood estimator*

$$\hat{\theta}_{\text{MLE}} := \arg\max_{\theta \in \Theta} L(\theta).$$

Since this is just an optimization problem, we can rely on what we know about calculus to solve for the MLE estimator.

(a) Assume that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, 1)$, that is, we already know that the underlying distribution is Normal with a population variance of 1, but the population mean is unkown. Compute $\hat{\mu}_{\text{MLE}}$.
Hint: it is often much easier to work with the log-likelihood, i.e. to solve the optimisation:

$$\hat{\theta}_{\text{MLE}} := \arg\max_{\theta \in \Theta} \log L(\theta),$$

which gives exactly the same answer as solving the original problem (why?).

Solution:
The log-likelihood here is

$$\begin{aligned}\log L(m) &= \log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(X_i - m)^2\right)\right) \\ &= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(X_i - m)^2.\end{aligned}$$

Differentiating with respect to m and setting equal to zero yields:

$$\frac{\partial}{\partial m}\log L(m) = \sum_{i=1}^{n}(X_i - m) = 0 \implies \hat{\mu}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}.$$

To see that this is indeed a maximum, we should perform a second derivative test, which yields:

$$\frac{\partial^2}{\partial m^2}\log L(m) = -n < 0.$$

(b) Assume that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim}$ Bernoulli(p), compute $\hat{p}_{\text{MLE}}$. Recall that the Bernoulli distribution is discrete and has probability mass function:

$$\mathbb{P}(X = k) = p^k(1-p)^{1-k}, \qquad k = 0, 1 \quad p \in [0, 1].$$

Solution:
Note here that $\theta = p$ and the parameter space is $\Theta = [0, 1]$. We construct the log-likelihood in

the usual way:

$$\begin{aligned}\log L(p) &= \log\left(\prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i}\right) \\ &= n\overline{X}\log p + n(1-\overline{X})\log(1-p)\end{aligned}$$

Then, differentiating and setting to zero yields:

$$\frac{\partial}{\partial p}\log L(p) = 0 \implies \hat{p}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}.$$

(c) **optional:** Assume that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$. Compute $(\hat{\mu}_{\text{MLE}}, \hat{\sigma}^2_{\text{MLE}})$.

Solution:
The log-likelihood here is

$$\begin{aligned}\log L(m, s^2) &= \log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{(X_i - m)^2}{2s^2}\right)\right) \\ &= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(s^2) - \frac{1}{2s^2}\sum_{i=1}^{n}(X_i - m)^2.\end{aligned}$$

To solve for the two MLE estimates simultaneously, we need to differentiate the log-likelihood with respect to each of the two parameters and setting each to zero, which will yield two equations (i.e. a system of equations). Solving these equations simultaneously yields the correct solution. Differentiating with respect to m first and setting equal to zero yields:

$$\frac{\partial}{\partial m}\log L(m, s^2) = \frac{1}{s^2}\sum_{i=1}^{n}(X_i - m) = 0 \implies \hat{\mu}_{\text{MLE}} = \overline{X}.$$

Note that in this case, the first equation does not depend on the second parameter, so we can solve it directly (this is not always the case). Next, differentiating with respect to s^2

$$\frac{\partial}{\partial s^2}\log L(m, s^2) = -\frac{n}{2s^2} - \frac{1}{2s^4}\sum_{i=1}^{n}(X_i - m)^2 = 0 \implies \hat{\sigma}^2_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n}(X_i - m)^2 = 0.$$

To solve this, we need to refer to the first of the two equations, which tells us that $m = \overline{X}$ is optimal, and so

$$(\hat{\mu}_{\text{MLE}}, \hat{\sigma}^2_{\text{MLE}}) = \left(\overline{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right).$$

Note, in order to be completely rigorous when finding a local maximum $(\hat{t}_1, \hat{t}_2)$ of a function $F(t1, t2)$, we need to check the following three condtions:

1. The first order partial derivatives at $(\hat{t}_1, \hat{t}_2)$ are zero.
2. At least one second-order partial derivative is negative
3. The determinant of the Hessian matrix is positive.

We will not verify these conditions here, and this is beyond the scope of the course, but it is important to note that we are leaving out some details.

Question 2. Bias and Variance of an Estimator

In the previous question, we discussed the MLE as a method of estimating a parameter. But there are an infinite number of ways to estimate a parameter. For example, one could choose to use the sample median instead of the MLE. It is useful to have a framework in which we can compare estimators in a systematic fashion, which brings us to two central concepts in machine learning: bias and variance. Assume that the true parameter is θ, and we have an estimate $\hat{\theta}$. Note that an estimator is just a function of the observed (random) data (i.e. we can always write $\hat{\theta} = \hat{\theta}(X)$) and so is itself a random variable! We can therefore define:

$$\text{bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta,$$
$$\text{var}(\hat{\theta}) = \mathbb{E}(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2.$$

The lab this week explores these concepts as well, and you are encouraged to do the lab exercise as you complete this question to get a full picture. A short summary of the lab in words:

- bias: tells us how far the expected value of our estimator is from the truth. Recall that an estimator is a function of the data sample we observe. The expectation of an estimator can be thought of in the following manner: imagine instad of having a single data sample, we have an infinite number of data samples. We compute the same estimator on each sample, and then take an average. This is the expected value of the estimator.
- varaiance: how variable our estimator is. Again, if we have an infinite number of data samples, we would be able to compute the estimator an infinite number of times, and check the variation in the estimator across all samples.

A good estimator should have low bias and low variance.

(a) Find the bias and variance of $\hat{\mu}_{\text{MLE}}$ where $X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, 1)$.

Solution:

We have already found that $\hat{\mu}_{\text{MLE}} = \overline{X}$. Therefore

$$\begin{aligned}
\text{bias}(\hat{\mu}_{\text{MLE}}) &= \text{bias}(\overline{X}) \\
&= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_i) - \mu \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu - \mu \\
&= \mu - \mu \\
&= 0,
\end{aligned}$$

and we say $\overline{X}$ is an unbiased estimator for μ. Next, we have

$$\begin{aligned}
\text{var}(\hat{\mu}_{\text{MLE}}) &= \text{var}(\overline{X}) \\
&= \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \text{var}(X_i) \\
&= \frac{1}{n},
\end{aligned}$$

where in the third equality we have used the independence of the X_i's, and in the final equality we have used the fact that $X_i \sim N(\mu, 1)$.

(b) Find the bias and variance of $\hat{p}_{\text{MLE}}$ where $X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$.

Solution:
We have already found that $\hat{p}_{\text{MLE}} = \overline{X}$, and it can easily be shown that $\text{var}(X_i) = p(1-p)$. We therefore get that

$$\text{bias}(\hat{p}_{\text{MLE}}) = 0, \qquad \text{var}(\hat{p}_{\text{MLE}}) = \frac{p(1-p)}{n}.$$

(c) The mean squared error (MSE) is a metric that is widely used in statistics and machine learning. For an estimator $\hat{\theta}$ of the true parameter θ, we define its MSE by:

$$\text{MSE}(\hat{\theta}) := \mathbb{E}(\hat{\theta} - \theta)^2.$$

Show that the MSE obeys a bias-variance decomposition, i.e. we can write

$$\text{MSE}(\hat{\theta}) := \text{bias}(\hat{\theta})^2 + \text{var}(\hat{\theta}).$$

Solution:

$$\begin{aligned}
\text{MSE}(\hat{\theta}) &= \mathbb{E}(\hat{\theta} - \theta)^2 \\
&= \mathbb{E}[\hat{\theta} - \mathbb{E}(\hat{\theta}) + \mathbb{E}(\hat{\theta}) - \theta]^2 \\
&= \mathbb{E}[(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 + 2(\hat{\theta} - \mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta}) - \theta) + (\mathbb{E}(\hat{\theta}) - \theta)^2] \\
&= \mathbb{E}(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 + 2\mathbb{E}[(\hat{\theta} - \mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta}) - \theta)] + \mathbb{E}[\mathbb{E}(\hat{\theta}) - \theta]^2 \\
&= \mathbb{E}(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 + 2(\mathbb{E}(\hat{\theta}) - \mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta}) - \theta)] + [\mathbb{E}(\hat{\theta}) - \theta]^2 \\
&= \mathbb{E}(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 + 0 + [\mathbb{E}(\hat{\theta}) - \theta]^2 \\
&= \text{var}(\hat{\theta}) + \text{bias}(\hat{\theta})^2.
\end{aligned}$$

Question 3. Probabalistic View of Least-Squares regression

In the tutorial last week, we viewed the least-squares problem purely from an optimisation point of view. We specified the model we wanted to fit, namely:

$$\hat{y} = w^T x$$

as well as a loss function (MSE), and simply found the weight vector w that minimized the loss. We proved that when using MSE, the best possible weight vector was given by

$$\hat{w} = (X^T X)^{-1} X^T y.$$

In this question, we will explore a different point of view, which we can call the statistical view. At the heart of the statistical view is the data generatinc process (DGP), which assumes that there is some true underlying function that generates the data, which we call f, but we only have access to noisy observations of f. That is, we observe

$$y = f(x) + \epsilon, \qquad \epsilon \text{ is some random noise.}$$

For example, assume your y's represent the daily temperature in Kensington. Any thermometer - even the most expensive - is prone to measurement error, and so what we actually observe is the true temperature ($f(x)$) plus some random noise ϵ. Most commonly, we will assume that the noise is normally distributed with zero mean, and variance σ^2. Now, consider the (strong) assumption that $f(x)$ is linear, which means that there is some true β^* such that $f(x) = x^T\beta^*$. Therefore, we have that

$$y = x^T\beta^* + \epsilon, \qquad \epsilon \sim N(0, \sigma^2),$$

and therefore,

$$y|x \sim N(x^T\beta^*, \sigma^2).$$

What this says is that our response (conditional on knowing the feature value x) follows a normal distribution with mean $x^T\beta^*$ and variance σ^2. We can therefore think of our data as a random sample of observations coming from this distribution, which in turn allows us to estimate unknown parameters via maximum likelihood, just as we did in the previous questions.

(a) You are given a dataset $D = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ and you make the assumption that $y_i|x_i = x_i^T\beta^* + \epsilon_i$ for some unknown β^* and $\epsilon_i \sim N(0, \sigma^2)$, where all the ϵ_i's are independent of each other. Write down the log-likelihood for this problem as well as the maximum likelihood estimation objective and solve for the MLE estimator $\hat{\beta}_{\text{MLE}}$.

Solution:
Under this assumption, we have that $y_i|x_i \sim N(x_i^T\beta^*), i = 1, \ldots, n$, or we can write this in matrix notation as:

$$y|X \sim N(X\beta^*, \sigma^2 I).$$

We can compute the likelihood of the data as

$$L(\beta) = P(y|X, \beta).$$

It is important to interpret this probability properly: it is the probabilty of seeing the responses y given that we have features X and we are assuming the underlying vector is β. $L(\beta)$ will give us a different value of the likelihood for different choices of β, and we want to choose the best β, i.e. the one that maximizes $L(\beta)$. Now, let's write out what $\log L(\beta)$ is in detail:

$$\begin{aligned}
\log L(\beta) &= \log P(y|X, \beta) \\
&= \log\left(\prod_{i=1}^{n} P(y_i|x_i, \beta)\right) \\
&= \sum_{i=1}^{n} \log P(y_i|x_i, \beta) \\
&= \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - x_i^T\beta)^2}{2\sigma^2}\right)\right) \\
&= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2 \\
&= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\|y - X\beta\|_2^2.
\end{aligned}$$

We therefore get that the MLE estimator of β is

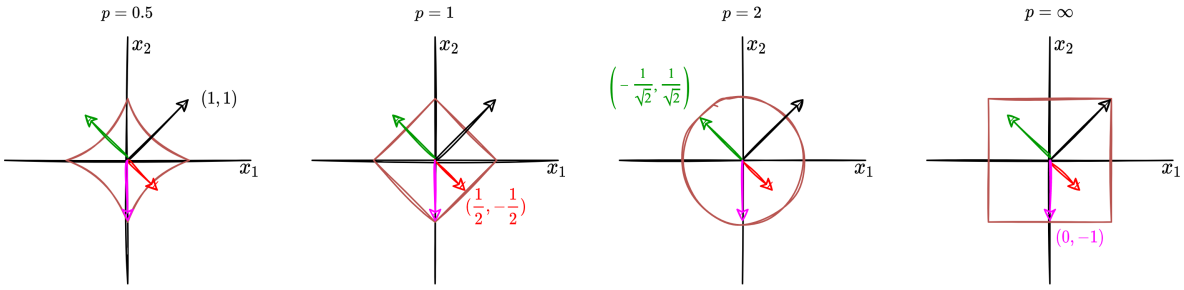
$$\begin{aligned}
\hat{\beta}_{\text{MLE}} &= \arg\max_{\beta} \log L(\beta) \\
&= \arg\max_{\beta}\left\{-\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\|y - X\beta\|_2^2\right\} \\
&= \arg\min_{\beta}\left\{\frac{n}{2}\log(2\pi\sigma^2) + \frac{1}{2\sigma^2}\|y - X\beta\|_2^2\right\} \\
&= \arg\min_{\beta} \|y - X\beta\|_2^2 \\
&= (X^TX)^{-1}X^Ty \\
&= \hat{\beta}_{\text{LS}}.
\end{aligned}$$

The second equality holds because maximising an objective is the same as minimizing the negative of the same objective, and the third equality holds since the minimizer is unaffected by the first term. Note that we have shown that the MLE in this case is exactly identical to the least squares estimator, this is not true in general. For example, had we used a different loss function (not MSE), or a different assumption about the noise, or a different distribution (other than the normal) then we would not get equivalence between LS and MLE. We can think of this as a probabalistic justification for doing least squares.

Question 4. Geometric Interpretations

In this question we will explore some geometric intuition for the least squares (LS), ridge and LASSO regression models.

(a) Consider the following diagram which represents the contour plot of the unit ball under various p-norms (see lab0 if you are unfamiliar with contour plots). Explain what is going on, and comment on the four vectors $((1,1), (1/2,-1/2), (-1/\sqrt{2}, 1/\sqrt{2}), (0,-1))$ that are represented in the plots. Further, what is the difference between the first plot ($p = 0.5$) and the others?



Solution:

First, a vector in $x \in \mathbb{R}^2$ is represented by two coordinates $x = (x_1, x_2)$. Next, recall that a p-norm of a vector in two dimensions is defined as

$$\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}, \qquad p \geq 1.$$

Note that this excludes $p = 0.5$ from being called a 'norm'. A norm is a notion of length and so any function g has to satisfy some conditions before we are allowed to call it a norm. These conditions are:

(i) Triangle inequality: $g(x + y) \leq g(x) + g(y)$ for any two vectors x, y. This just says that the length of the sum of two vectors is smaller than the sum of the lengths.

(ii) Absolute homogeneity: $g(cx) = |c|g(x)$ for any vector x and scalar c. This says that if you multiple a vector by a constant, the length of the new vector is just the length of the old vector scaled by c.

(iii) Positive definiteness: $g(0) = 0$. The zero vector should have length zero.

You can check that these three are always satisfied for p-norms defined above. When $p = 0.5$ however, we can find a counter-example to the triangle inequality: Consider $x = (0, 9)$ and $y = (4, 0)$ then

$$\begin{aligned} \|x\|_{0.5} &= (\sqrt{0} + \sqrt{9})^2 = 3^2 = 9 \\ \|y\|_{0.5} &= (\sqrt{4})^2 = 4 \\ \|x\|_{0.5} + \|y\|_{0.5} &= 13 \\ \|x + y\|_{0.5} &= \|(4, 9)\|_{0.5} = (\sqrt{4} + \sqrt{9})^2 = 25 \end{aligned}$$

so we see that

$$\|x + y\|_{0.5} > \|x\|_{0.5} + \|y\|_{0.5}$$

and so the function $g(x) = \|x\|_{0.5}$ does not satisfy the triangle inequality and is not a norm, though it is often referrred to as a *pseudo-norm*. Now, let's try to figure out what is going on in the plots. In each plot, we have the *unit-circle* corresponding to the choice of p-norm for $p = 0.5, 1, 2, \infty$, which means that the shape outlined in brown in each plot represents the vectors x that satisfy $\|x\|_p = 1$. This is usually referred to as the *unit circle*, and is often written

$$\{x : \|x\|_p = 1\}.$$

This notation is to be interpreted in the following way: the set of all objects x such that x satisfies the requirement $\|x\|_p = 1$. What this tells us that depending on the choice of norm, the length of a vector can be different. Let's look at a concrete example: In black we have the vector $(1, 1)$, note that

$$\begin{aligned} \|(1, 1)\|_{0.5} &= (\sqrt{1} + \sqrt{1})^2 = 2^2 = 4 \\ \|(1, 1)\|_1 &= |1| + |1| = 2 \\ \|(1, 1)\|_2 &= \sqrt{|1|^2 + |1|^2} = \sqrt{2} \\ \|(1, 1)\|_\infty &= \max\{|1|, |1|\} = 1. \end{aligned}$$

So the vector is only on the ∞-norm circle, which means we only measure its length to be 1 when using the $p = \infty$-norm. In all other cases, the norm of $(1, 1)$ is larger than 1. Similar calculations can be done for the other 3 vectors. This gives us a general rule for p-norms:

$$\|x\|_p \geq \|x\|_q \quad \text{whenever } p < q.$$

Note that from a geometric point of view, the area enclosed by the unit circle (referred to as the unit ball) is a convex set whenever the $p \geq 1$, which can be seen in plots 2-4. Convexity does not hold when $p < 1$, as can be seen in the first plot. See the wiki page for more on convex sets.

(b) We previously saw that the ridge regression objective is defined by:

$$\hat{\beta}_{\text{ridge}} = \arg\min_{\beta} \left\{ \|y - X\beta\|_2^2 + \lambda \|\beta\|_2^2 \right\}.$$

Another (equivalent) way of defining the ridge objective is through a constrained optimisation:

$$\hat{\beta}_{\text{ridge}} = \arg\min_{\beta} \|y - X\beta\|_2^2 \quad \text{subject to} \quad \|\beta\|_2 \leq k.$$

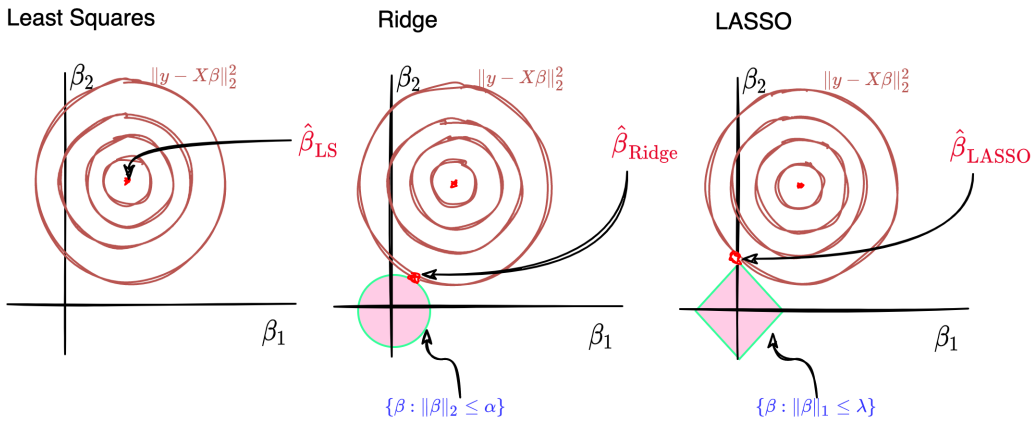
What this says is that we want to find β that minimizes the squared loss but the solution must also belong to the 2-norm ball of radius k. Note that in general, k and λ are not the same. The constrained optimisation statement gives us a nice geometric interpretation of the ridge solution which we will now explore. Before doing so, we also note that the LASSO has an unconstrained version:

$$\hat{\beta}_{\text{LASSO}} = \arg\min_{\beta} \left\{ \|y - X\beta\|_2^2 + \lambda \|\beta\|_1 \right\}.$$

and also a constrained version

$$\hat{\beta}_{\text{LASSO}} = \arg\min_{\beta} \|y - X\beta\|_2^2 \quad \text{subject to} \quad \|\beta\|_1 \leq k,$$

where λ, k for the Ridge and for the LASSO are different in general. These objectives are almost identical, they only differ by the choice of norm used for the penalty/constraint term. This actually leads to large and very important differences in practice. Now, with this in mind, interpret the following plots:



Discuss the differences in the Ridge and LASSO solutions explicitly.

Solution:
In these plots, we are graphically depicting least squares, ridge and LASSO optimisations in 2-dimensions, i.e. $\beta = (\beta_1, \beta_2)$. In the first plot, we are graphically depicting the least-squares optimization, which recall is:

$$\hat{\beta}_{\text{LS}} = \arg\min_{\beta} \|y - X\beta\|_2^2.$$

The brown circles represent the contours of the squared-error function $\|y - X\beta\|_2^2$. Again, if you are not familiar with this concept, see lab 0. We know that the smaller circles of the contour

represent smaller values of the function. The least-squares solution is therefore just the point directly in the middle, which gives us the smallest possible value of $\|y - X\beta\|_2^2$.

In the second plot, we are now doing ridge regression. Recall from the constrained optimisation view point, we still want to minimize squared error, so we still see the contours of the squared error. However, we also need to respect the constraint that the norm of our solution belongs to the norm-ball of radius α, i.e. our solution must be in the pink shaded region (this is called the set of feasible solutions). The solution of the ridge problem is then the point that is in the feasible solution region AND makes the term $\|y - X\beta\|_2^2$ as small as possible. Graphically, this is just the point of intersection highlighted in red. Finally, an identical interpretation holds for the LASSO problem, except now we are using the 1-norm instead of the 2-norm.

The LASSO is known to find *sparse* solutions. A vector is sparse if only a few of its elements are non-zero. For example, we would say the vector

$$[0, 0, 0, 1, 0, 0, 0.2, 0]$$

is sparse, whereas

$$[0, 2, 2, 1, 0, 3, 0.2, 1]$$

is not sparse. Note that the shape of the 1-norm ball gives us solutions that tend to be more sparse. The reason being that the ball has *corners* or *spikes* at the points $(1, 0), (0, 1), (-1, 0), (0, -1)$, and is flat between these points, so the squared error is likely to intersect with the constraint set at one of these points. This is important because these points correspond to setting one of the weights to zero. The 2-norm ball has round sides and so this is less likely in the ridge case.

(c) The LASSO is said to *induce sparsity*. What does this mean? Why might it be desirable to have a sparse solution?

Solution:

We have already seen in the previous question why the LASSO induces sparsity, but we have not discussed why we might want a sparse solution. Obtaining a sparse solution is often important in scientific applications: if we are trying to understand how something works, then having a sparse model means we can attribute the response to a few features, rather than having a model with a huge number of features which makes it impossible to do any reasonable interpretation. From another perspective, if gathering data is expensive, then narrowing down the effect to a few good features can make it much cheaper to operate your model.