2D Fourier Transform

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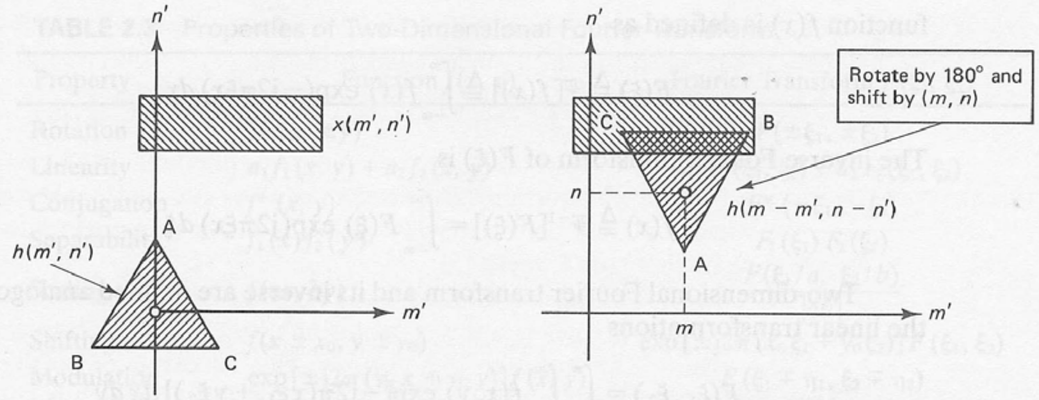
12 – 14 July 2006

2D Convolution

From: Fundamentals of Digitial Image Processing, AK Jain.



(a) impulse response

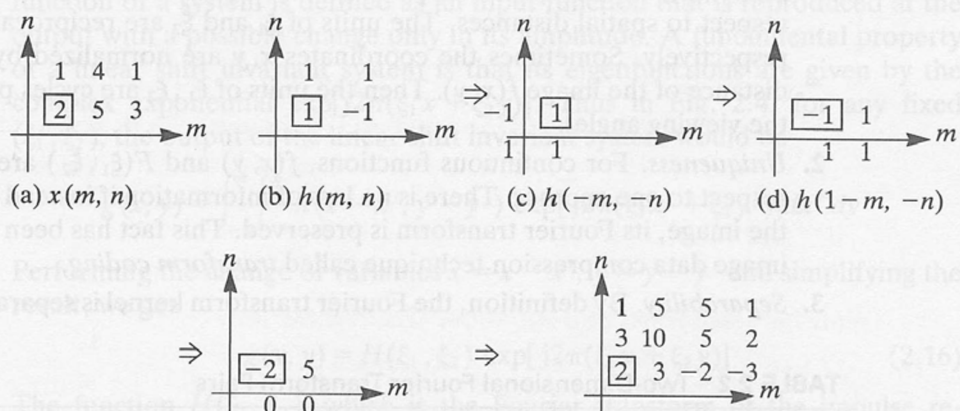
(b) output at location (m, n) is the sum of product of quantities in the area of overlap.

Figure 2.3 Discrete convolution in two dimensions

The convolution operation has several interesting properties, which are explored in Problems 2.2 and 2.3.

Example 2.1 (Discrete convolution)

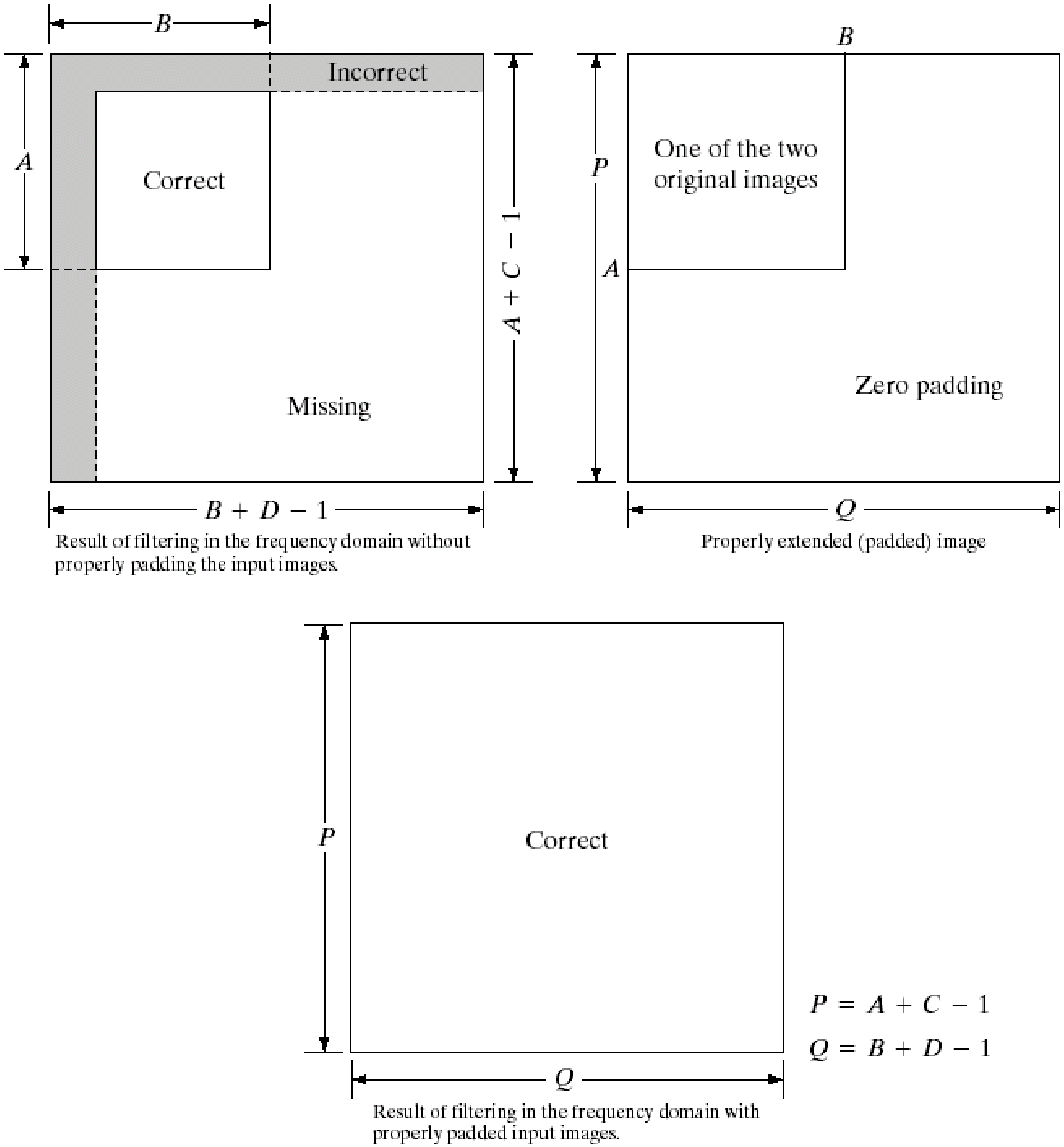
Consider the 2×2 and 3×2 arrays $h(m, n)$ and $x(m, n)$ shown next, where the boxed element is at the origin. Also shown are the various steps for obtaining the convolution of these two arrays. The result $y(m, n)$ is a 4×3 array. In general, the convolution of two arrays of sizes $(M_1 \times N_1)$ and $(M_2 \times N_2)$ yields an array of size $[(M_1 + M_2 - 1) \times (N_1 + N_2 - 1)]$ (Problem 2.5).



(e) $y(1, 0) = -2 + 5 = 3$

(f) $y(m, n)$

Padding



a b
c

FIGURE 4.38 Illustration of the need for function padding. (a) Result of performing 2-D convolution without padding. (b) Proper function padding. (c) Correct convolution result.

2D DFT

The two dimensional DFT and its inverse are:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M + vy/N)]$$

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux/M + vy/N)]$$

where *N* and *M* are the dimensions of the 2D matrix

$$\Delta u = 1/(M\Delta x); \Delta v = 1/(N\Delta y)$$

$$P(u,v) = \left|F(u,v)\right|^2$$

Separability

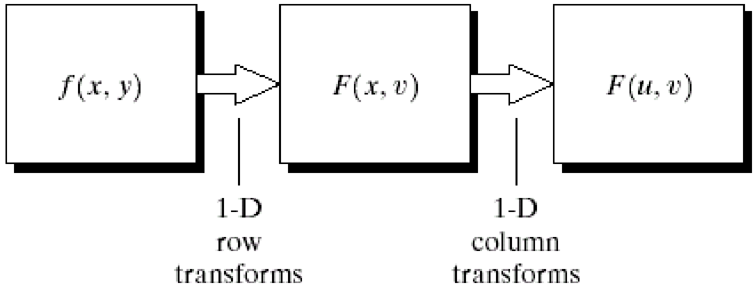
The 2D DFT is separable:

$$F(u,v) = \left[\frac{1}{M}\sum_{x=0}^{M-1}\exp[-j2\pi ux/M]\right]\left[\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N]\right]$$

$$F(u,v) = \frac{1}{M}\sum_{x=0}^{M-1} F(x,v)\exp[-j2\pi ux/M]$$

where

$$F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N]$$



Properties

Translation :

$$f(x,y)\exp[j2(u_0x+v_0y)/N] \Leftrightarrow F(u-u_0,v-v_0)$$

$$f(x-x_0,y-y_0) \Leftrightarrow F(u,v)\exp[-j2(u_0x+v_0y)/N]$$

Periodicity

$$F(u,v)=F(u+N,v)=F(u,v+N)=F(u+N,v+N)$$

if *f*(*x*,*y*) is real, we have conjugate symmetry

$$F(u,v)=F^*(-u,-v)$$

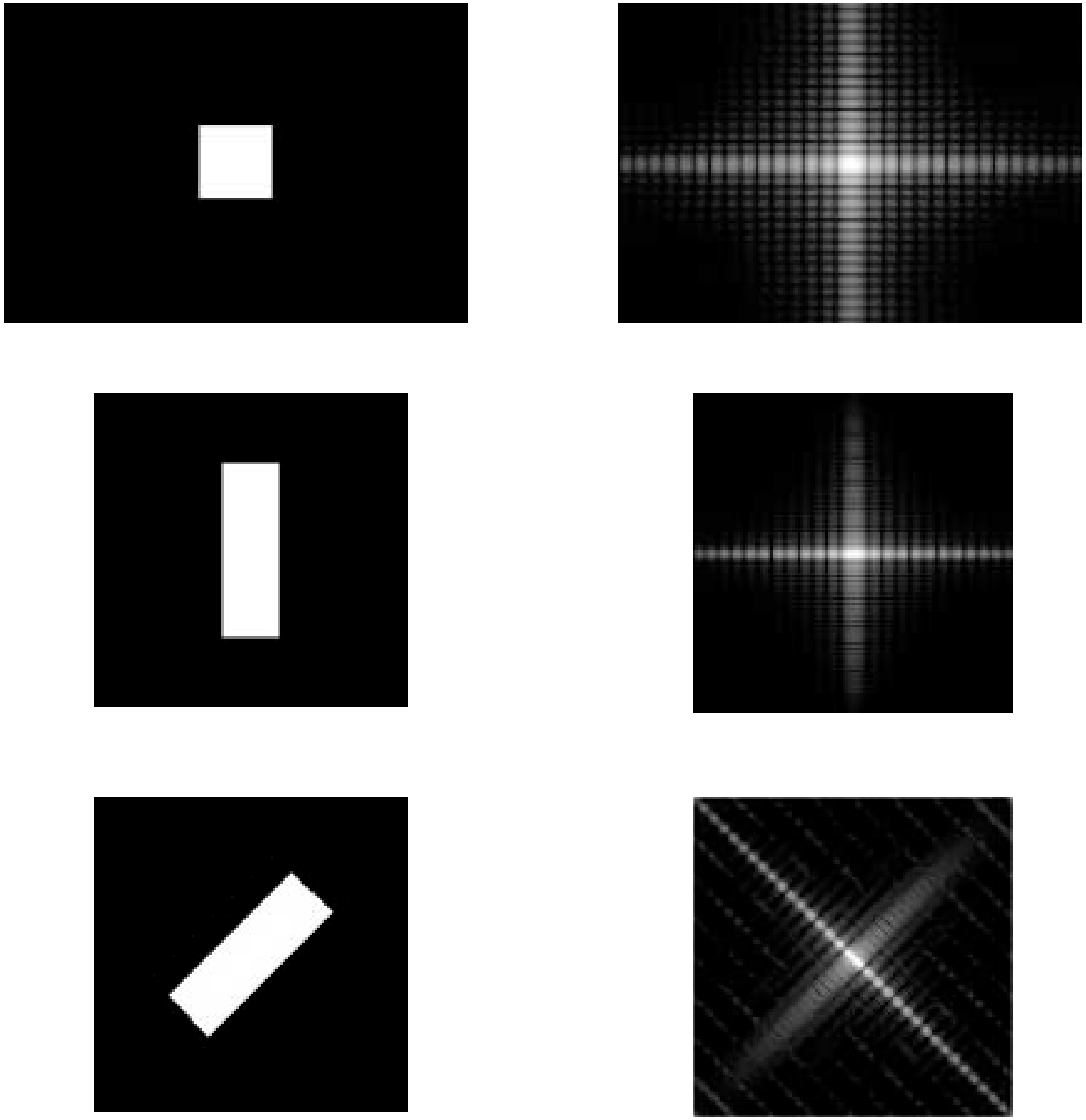
$$|F(u,v)|=|F(-u,-v)|$$

Rotation :

$$x=r\cos\theta, y=r\sin\theta, u=\omega\cos\phi, v=\omega\sin\phi$$

$$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\phi+\theta_0)$$

i.e. Rotating *f*(*x*,*y*) by θ_0 rotates *F*(*u*,*v*) by the same angle.



Frequency spectra of images at two resolutions

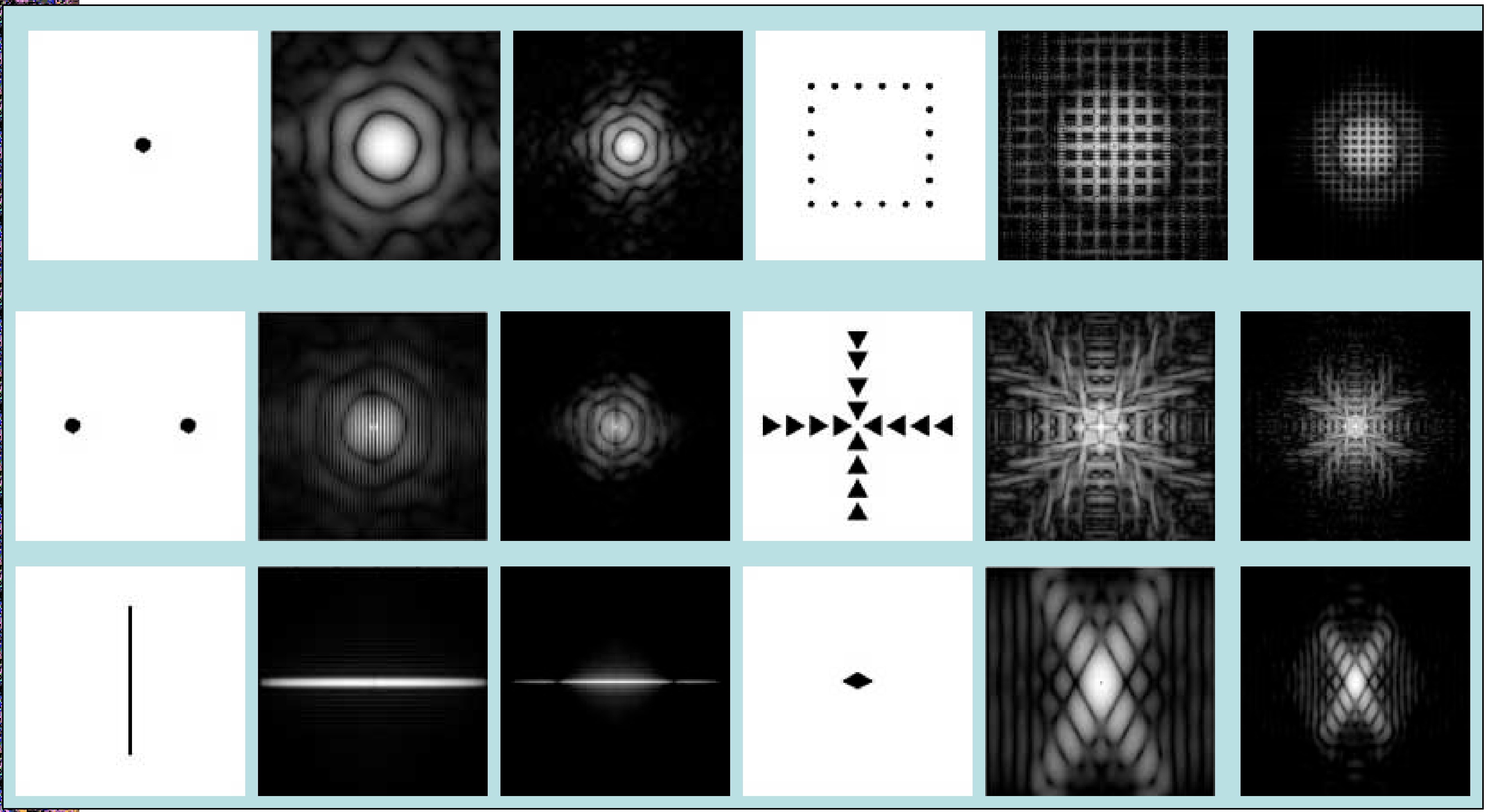


Image | low resolution | high resolution | Image | low resolution | high resolution

Frequency Domain Filtering

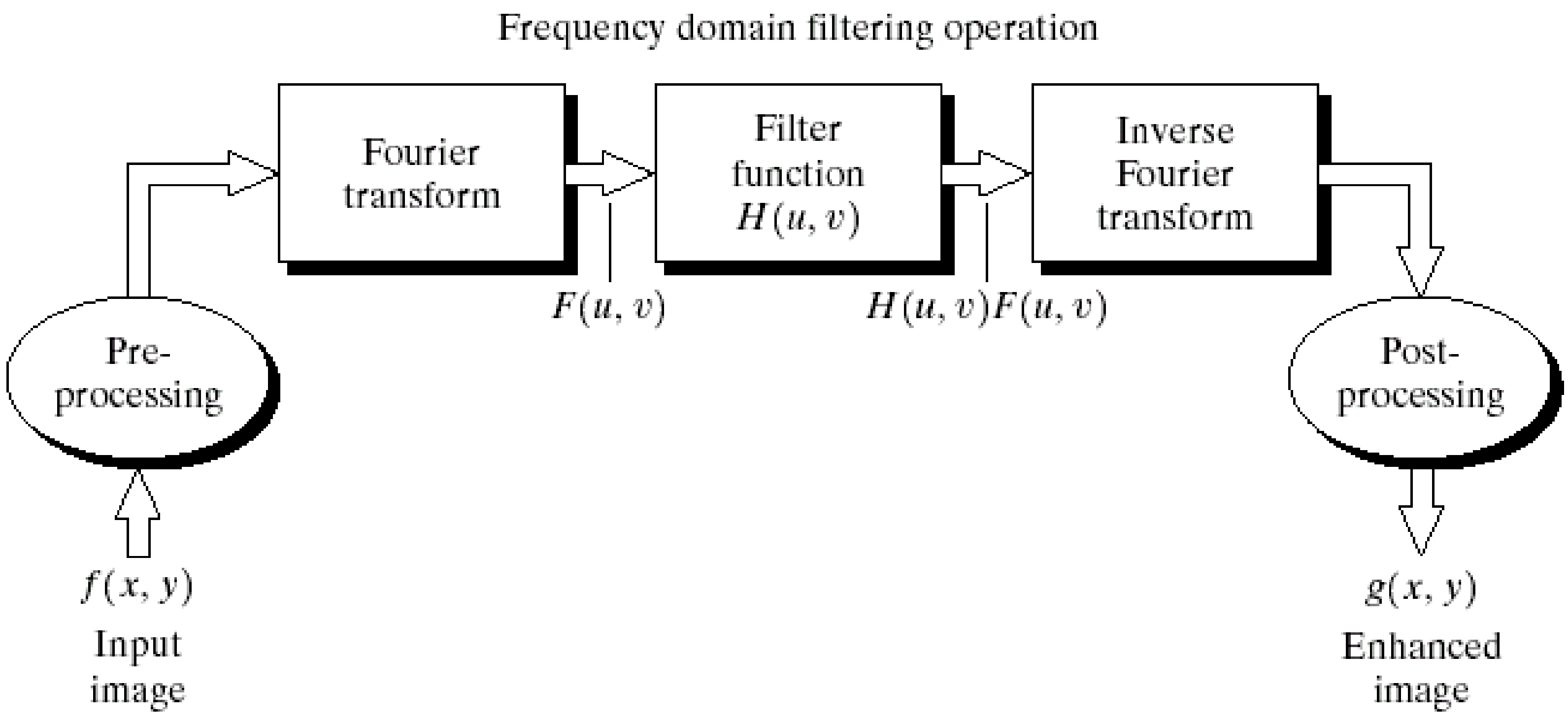


FIGURE 4.5 Basic steps for filtering in the frequency domain.

Low- and High-pass Filtering

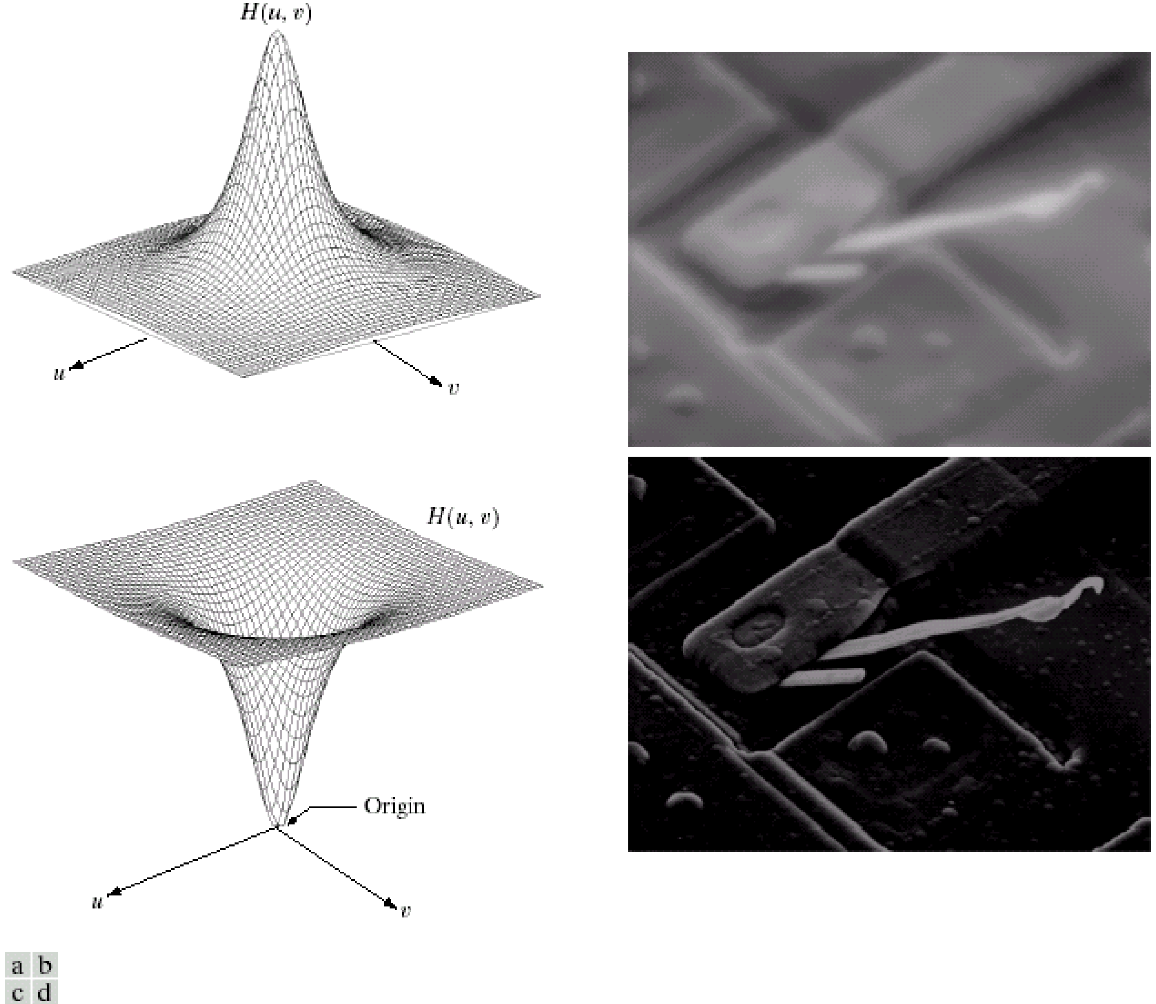
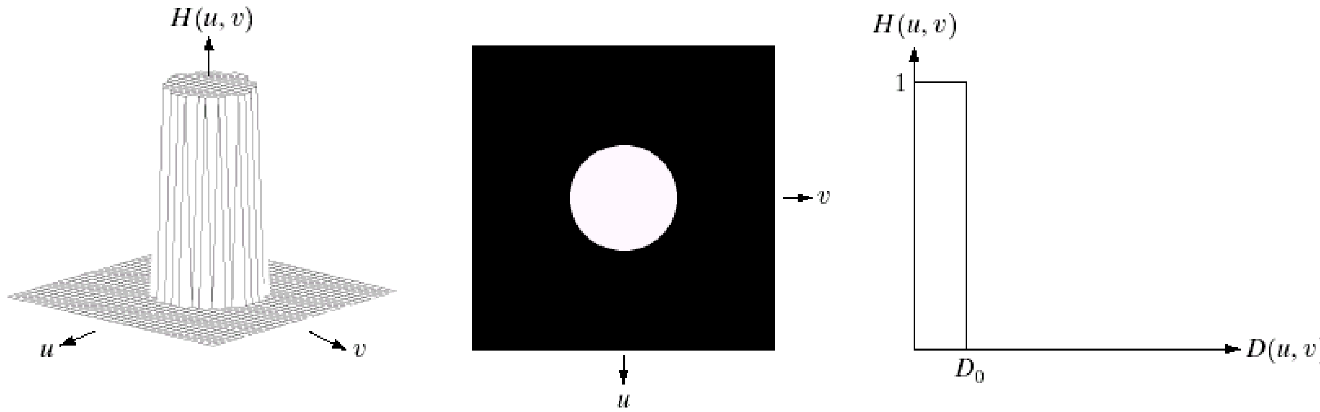


FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Ideal Low-pass Filter

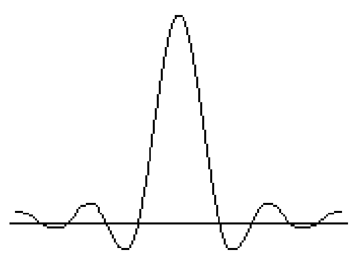
Freq Domain

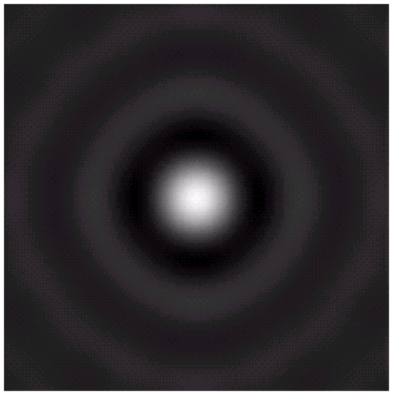


a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Spatial Domain





Example

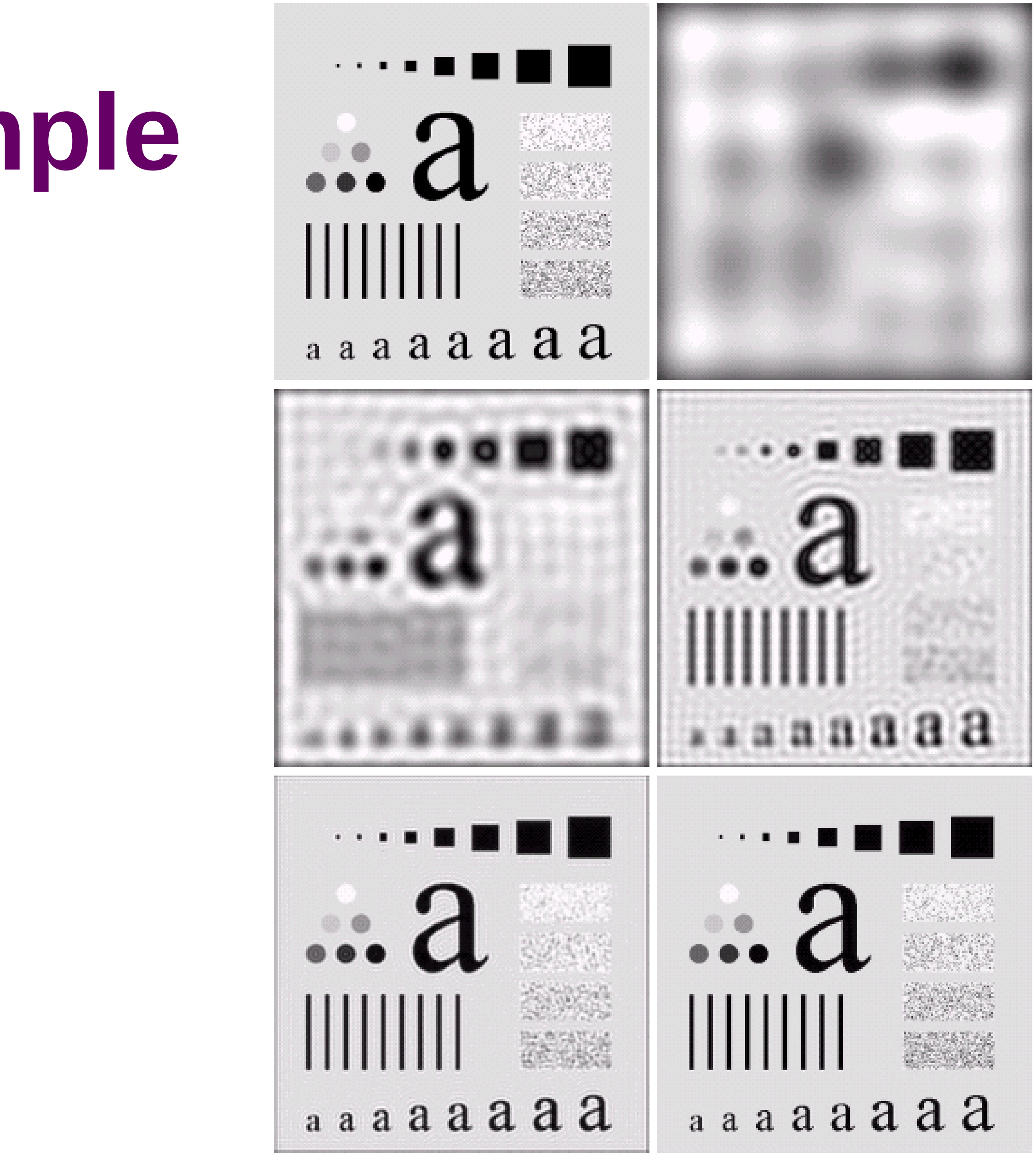
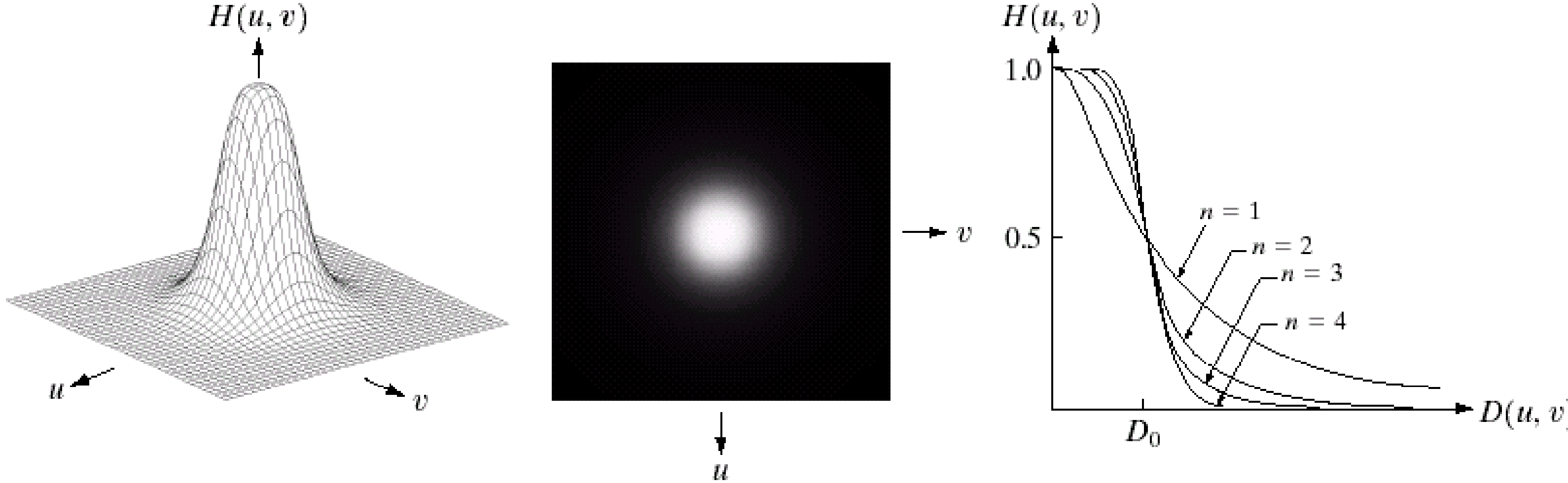


FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

Butterworth Filters

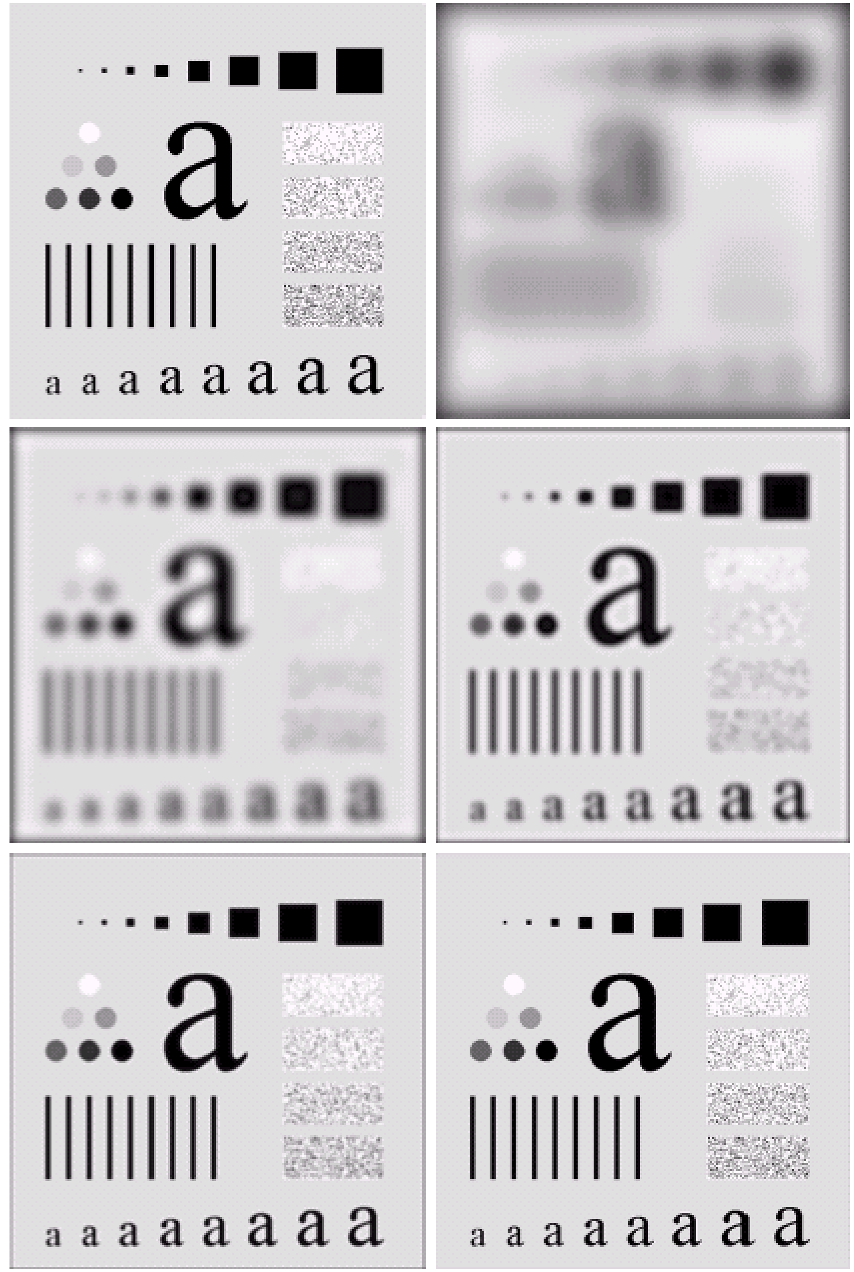


a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

$$H(u,v) = \frac{1}{1 + \left[D(u,v)/D_0\right]^{2n}}$$

Example



a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Summary

- The DFT (and FFT) is a powerful tool to analyze the frequency components in a signal.
- Convolution in the spatial domain = multiplication in the frequency domain.
  - This makes convolution fast!
- Filter design is a field of study by itself.
  - We barely scratched the surface here.