Boussinesq systems in two space dimensions over a variable bottom for the generation and propagation of tsunami waves (Technical Report)

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1 Introduction

In applications, especially in the studies of tsunamis generation it is a common practice to assume that the tsunami wave pattern, i.e. the free surface deformation (due to the bottom deformation) is analogous to the bottom deformation. For this reason we will assume that the moving bottom consist of two parts $h(x,y,t) = D(x,y) + \zeta(x,y,t)$ where $\zeta(x,y,t)$ will be the same order of magnitude as the free surface elevation.

2 Derivation of the new set of Boussinesq-type equations

We denote by $(\tilde{x},\tilde{y},\tilde{z})$ a cartesian coordinate system. Let $\tilde{x}$, $\tilde{y}$ be the horizontal coordinates and $\tilde{z}$ measured upwards from the still water level. Consider a three-dimensional wave field with water-surface deviation propagating from its rest position, $\tilde{\eta}(\tilde{x},\tilde{y},\tilde{t})$, at time $\tilde{t}$, over a variable bottom bellow the undisturbed surface of the water $\tilde{h}(\tilde{x},\tilde{y},\tilde{t}) = \tilde{D}(\tilde{x},\tilde{y}) + \tilde{\zeta}(\tilde{x},\tilde{y},\tilde{t})$. The fluid is assumed to be inviscid and incompressible, and the flow is assumed to be irrotational. The fluid velocity is denoted by $\hat{\mathbf{u}} = (\tilde{u},\tilde{v},\tilde{w})^T$ in the $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ directions, respectively. The Euler equations which describe three dimensional wave propagation on the free surface are written in the form (cf. [W])

$$\hat{\mathbf{u}}_{\tilde{t}} + (\hat{\mathbf{u}}\cdot\widetilde{\nabla})\hat{\mathbf{u}} + \frac{1}{\rho}\widetilde{\nabla}\tilde{P} = -g\mathbf{j}, \tag{2.1}$$

$$\widetilde{\nabla}\cdot\hat{\mathbf{u}} = 0, \qquad \text{for } -\tilde{h} < \tilde{z} < \tilde{\eta}(\tilde{x},\tilde{y},\tilde{t}) \tag{2.2}$$

$$\widetilde{\nabla}\times\hat{\mathbf{u}} = 0, \tag{2.3}$$

where $\tilde{P}$ is the pressure field, ρ is the density and g is the acceleration due to gravity and $\mathbf{j} = (0,0,1)^T$. ($\widetilde{\nabla} = (\partial_{\tilde{x}},\partial_{\tilde{y}},\partial_{\tilde{z}})$.) The first equation expresses the conservation of momentum, where the other two equations express the conservation of mass and the irrotationality of the flow, respectively. The kinematic boundary condition at the free surface and seabed can be expressed as

$$\tilde{\eta}_{\tilde{t}} + \hat{\mathbf{u}}\cdot\widetilde{\nabla}\tilde{\eta} = 0, \ \text{ for } \tilde{z} = \tilde{\eta}(\tilde{x},\tilde{y},\tilde{t}), \tag{2.4}$$

and

$$\tilde{h}_{\tilde{t}} + \hat{\mathbf{u}}\cdot\widetilde{\nabla}\tilde{h} = 0, \ \text{ for } \tilde{z} = -\tilde{h}(\tilde{x},\tilde{y},\tilde{t}), \tag{2.5}$$

respectively. The fluid is assumed to satisfy the dynamic boundary condition $\tilde{P}(x,y,t) = \tilde{P}_0(x,y)$ at the free surface $\tilde{z} = \tilde{\eta}(\tilde{x},\tilde{y},\tilde{t})$.

Consider a characteristic water depth h_0, a typical wavelength λ_0 and a typical wave height a_0. A natural scaling of the independent and dependent variables for the purposes of long wave modelling is chosen, cf. [P], [BCS1], [C2], [N], so

$$x = \frac{\tilde{x}}{\lambda_0}, \;\; y = \frac{\tilde{y}}{\lambda_0}, \;\; z = \frac{\tilde{z}}{h_0}, \;\; t = \frac{c_0}{\lambda_0}\tilde{t},$$

and

$$u = \frac{h_0}{a_0 c_0}\tilde{u}, \;\; v = \frac{h_0}{a_0 c_0}\tilde{v}, \;\; w = \frac{\lambda_0}{a_0 c_0}\tilde{w}, \;\; \eta = \frac{\tilde{\eta}}{a_0}, \;\; h = \frac{\tilde{h}}{h_0}, \;\; D = \frac{\tilde{D}}{h_0}, \;\; \zeta = \frac{\tilde{\zeta}}{a_0},$$

where $c_0 = \sqrt{g h_0}$.

Then, the governing equations for the fluid motion in the non-dimensional and scaled form take the following form:

$$\varepsilon \mathbf{u}_t + \varepsilon^2((\mathbf{u}\cdot\nabla)\mathbf{u} + w\mathbf{u}_z) + \frac{1}{\rho c_0^2}\nabla P = 0, \tag{2.6}$$

$$\varepsilon\sigma^2 w_t + \varepsilon^2\sigma^2(\mathbf{u}\cdot\nabla w) + w w_z) + \frac{1}{\rho c_0^2}P_z = -1, \tag{2.7}$$

for $-h < z < \varepsilon\eta$. The parameters $\varepsilon = \alpha_0/h_0$ and $\sigma = h_0/\lambda_0$ are assumed to be small. The conservation of mass is formulated as

$$\nabla\cdot\mathbf{u} + w_z = 0 \;\; \text{for} \;\; -h < z < \varepsilon\eta, \tag{2.8}$$

the irrotationality condition is given by the formulas

$$u_y - v_x = 0, \tag{2.9}$$

$$\mathbf{u}_z - \sigma^2\nabla w = 0, \tag{2.10}$$

for $-h < z < \varepsilon\eta$, while the boundary conditions take the form

$$\eta_t + \varepsilon(\mathbf{u}\cdot\nabla\eta) - w = 0 \;\; \text{on} \;\; z = \varepsilon\eta, \tag{2.11}$$

$$\zeta_t + \mathbf{u}\cdot\nabla h + w = 0 \;\; \text{on} \;\; z = -h, \tag{2.12}$$

where $\mathbf{u} = (u,v)^T$ and $\nabla = (\partial_x, \partial_y)$. We note that $h = D + \varepsilon\zeta$ and thus $h_t = O(\varepsilon)$. Integrating equation (2.8) with respect to z from $-h$ to z, and using (2.12) we have

$$w = -\mathbf{u}\cdot\nabla h - \int_{-h}^{z}\nabla\cdot\mathbf{u} - \zeta_t. \tag{2.13}$$

After integration of equation (2.10) and using (2.13), we observe that

$$\mathbf{u} = \mathbf{u}_b + O(\sigma^2), \tag{2.14}$$

where $\mathbf{u}_b$ is the horizontal velocity of the fluid at the bottom $z = -h$. Substitution of (2.13) into the irrotationality condition (2.10), and using (2.14), yields

$$\mathbf{u}_z = -\sigma^2\nabla(\nabla\cdot(h\mathbf{u}_b)) - \sigma^2 z\nabla(\nabla\cdot\mathbf{u}_b) - \sigma^2\nabla\zeta_t + O(\sigma^4, \varepsilon\sigma^2). \tag{2.15}$$

Integration of (2.15) with respect to z from $-h$ to z gives

$$\mathbf{u} = \mathbf{u}_b - \sigma^2(z+h)\nabla(\nabla\cdot(h\mathbf{u}_b)) - \sigma^2\frac{z^2-h^2}{2}\nabla(\nabla\cdot\mathbf{u}_b) - \sigma^2(z+h)\nabla\zeta_t + O(\sigma^4, \varepsilon\sigma^2). \tag{2.16}$$

We note that using (2.14), equation (2.13) takes the form

$$w = -\nabla\cdot(h\mathbf{u}_b) - z\nabla\cdot\mathbf{u}_b - \zeta_t + O(\sigma^2), \tag{2.17}$$

and thus,

$$w_t = -\nabla\cdot(h\mathbf{u}_b)_t - z\nabla\cdot\mathbf{u}_{bt} - \zeta_{tt} + O(\sigma^2). \tag{2.18}$$

Assuming that $P = 0$ at $z = \varepsilon\eta$, integrating (2.7) with respect to z from z to $\varepsilon\eta$, and using (2.18) we have

$$\frac{P}{\rho c_0^2} = \varepsilon\sigma^2(z\nabla\cdot(h\mathbf{u}_b)_t + \frac{z^2}{2}\nabla\cdot\mathbf{u}_{bt}) + \varepsilon\sigma^2 z\zeta_{tt} + \varepsilon\eta - z + O(\varepsilon\sigma^4, \varepsilon^2\sigma^2). \tag{2.19}$$

Using (2.16), (2.19), for $z = -h$, we have

$$\mathbf{u}_{bt} + \nabla\eta + \varepsilon(\mathbf{u}_b\cdot\nabla)\mathbf{u}_b - \sigma^2 h\nabla(\nabla\cdot(h\mathbf{u}_{bt})) + \sigma^2\frac{h^2}{2}\nabla(\nabla\cdot\mathbf{u}_{bt}) - \sigma^2 h\nabla\zeta_{tt} = O(\sigma^4, \varepsilon\sigma^2), \tag{2.20}$$

while we made use of the fact that $h_t = O(\varepsilon)$.

Integration of equation (2.8) with respect of z from $-h$ to $\varepsilon\eta$ yields

$$w(\varepsilon\eta) - w(-h) = -\int_{-h}^{\varepsilon\eta}\nabla\cdot\mathbf{u}dz, \tag{2.21}$$

and thus, adding the boundary conditions (2.11) and (2.12) we have

$$\eta_t + \nabla\cdot\int_{-h}^{\varepsilon\eta}\mathbf{u}dz + \zeta_t = 0. \tag{2.22}$$

Denote the depth-average horizontal velocity of the fluid by

$$\bar{\mathbf{u}} = \frac{1}{h+\varepsilon\eta}\int_{-h}^{\varepsilon\eta}\mathbf{u}dz. \tag{2.23}$$

then (2.22) becomes

$$\eta_t + \nabla\cdot[(h+\varepsilon\eta)\bar{\mathbf{u}}] + \zeta_t = 0. \tag{2.24}$$

Moreover, using (2.23), the equation (2.16) gives

$$\mathbf{u}_b = \bar{\mathbf{u}} + \sigma^2\frac{h}{2}\nabla(\nabla\cdot(h\bar{\mathbf{u}})) - \sigma^2\frac{h^2}{3}\nabla(\nabla\cdot\bar{\mathbf{u}}) + \sigma^2\frac{h}{2}\nabla\zeta_t + O(\sigma^4), \tag{2.25}$$

and thus, the equation (2.20) gives

$$\bar{\mathbf{u}}_t + \nabla\eta + \varepsilon(\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} - \sigma^2\frac{h}{2}\nabla(\nabla\cdot(h\bar{\mathbf{u}}_t)) + \sigma^2\frac{h^2}{6}\nabla(\nabla\cdot\bar{\mathbf{u}}_t) - \sigma^2\frac{h}{2}\nabla\zeta_{tt} = O(\varepsilon\sigma^2, \sigma^4). \tag{2.26}$$

Note that the system of equations (2.24), (2.26) is the Boussinesq system of equations derived by Peregrine, [P].

In the sequel we try to write system (2.24), (2.26) using the variables $H = h + \varepsilon\eta$ and $H\bar{u}$. Is is obvious that (2.24) can be written as

$$H_t + \varepsilon\nabla\cdot[H\bar{u}] = 0, \tag{2.27}$$

while equation (2.26) can be written as

$$(H\bar{\mathbf{u}})_t + \nabla\cdot\left(\varepsilon H\bar{\mathbf{u}}\otimes\bar{\mathbf{u}} + \frac{1}{2\varepsilon}H^2\mathbf{I}\right) - \sigma^2\frac{hH}{2}\nabla(\nabla\cdot(h\bar{\mathbf{u}})_t) + \sigma^2\frac{h^2H}{6}\nabla(\nabla\cdot\bar{\mathbf{u}}_t) = \frac{1}{\varepsilon}H\nabla h + \sigma^2\frac{hH}{2}\nabla\zeta_{tt} + O(\varepsilon\sigma^2, \sigma^4). \tag{2.28}$$

Unfortunately, in the Boussinesq approximation the irrotationality condition is not fully satisfied, but only the relation $\bar{u}_y = \bar{u}_x + O(\varepsilon)$ and thus $\nabla(\nabla\cdot\bar{\mathbf{u}}) = \Delta\bar{\mathbf{u}}_t + O(\sigma^2)$. This is due to (2.9), (2.14) and (2.15). Using this relation, (2.28) can be written in the form:

$$(H\bar{\mathbf{u}})_t + \nabla\cdot\left(\varepsilon H\bar{\mathbf{u}}\otimes\bar{\mathbf{u}} + \frac{1}{2\varepsilon}H^2\mathbf{I}\right) - \sigma^2\frac{hH}{2}\nabla(\nabla\cdot(h\bar{\mathbf{u}})_t) + \sigma^2\frac{h^2H}{6}\Delta\bar{\mathbf{u}}_t = \frac{1}{\varepsilon}H\nabla h + \sigma^2\frac{hH}{2}\nabla\zeta_{tt} + O(\varepsilon\sigma^2, \sigma^4). \tag{2.29}$$

We observe now that $h = D + O(\varepsilon)(= D + \varepsilon\zeta)$ and thus $H = D + O(\varepsilon)$. Moreover, we observe that $h\Delta\bar{\mathbf{u}}_t = \nabla(\nabla\cdot(h\bar{\mathbf{u}}_t)) - \nabla(\nabla h\cdot\bar{\mathbf{u}}_t) - \nabla h\nabla\cdot\bar{\mathbf{u}}_t$, while for (2.26) we observe that $\bar{\mathbf{u}}_t = -\nabla\eta + O(\varepsilon,\sigma^2)$. Thus, we have

$$h\Delta\bar{\mathbf{u}}_t = \nabla(\nabla\cdot(h\bar{\mathbf{u}}_t)) + \nabla(\nabla h\cdot\nabla\eta) + \nabla h\nabla\cdot\nabla\eta + O(\varepsilon,\sigma^2). \tag{2.30}$$

Using (2.30) and (2.29) and using the above relations we have that

$$\begin{aligned}(H\bar{\mathbf{u}})_t - \sigma^2\frac{D^2}{3}\nabla(\nabla\cdot(H\bar{\mathbf{u}})_t) + \nabla\cdot\left(\varepsilon H\bar{\mathbf{u}}\otimes\bar{\mathbf{u}} + \frac{1}{2\varepsilon}H^2\mathbf{I}\right) + \sigma^2\frac{D^2}{6\varepsilon}(\nabla(\nabla D\cdot\nabla H) + \nabla D\Delta H)\\ = \sigma^2\frac{D^2}{6\varepsilon}(\nabla(\nabla D\cdot\nabla h) + \nabla D\Delta h) + \frac{1}{\varepsilon}H\nabla h + \sigma^2\frac{hH}{2}\nabla\zeta_{tt} + O(\varepsilon\sigma^2,\sigma^4)\end{aligned} \tag{2.31}$$

The new system consists of (2.27), (2.31). Further simplifications can be made when we consider the 1D case.

System (2.27), (2.28) can be written also in a conservative form. This form is convenient for the study of the long wave runup.

Using the fact that $H = D + O(\varepsilon)$ and denoting $Q = H\bar{\mathbf{u}}$ we have the system in dimensional variables:

$$H_t + \nabla\cdot Q = 0 \tag{2.32}$$

$$Q_t + \nabla\cdot\left(\frac{Q\otimes Q}{H} + \frac{g}{2}H^2\mathbf{I}\right) - P(H,Q) = gH\nabla h + \frac{D\,H}{2}\nabla\zeta_{tt}, \tag{2.33}$$

where

$$P(H,Q) = \frac{H^2}{2}\nabla(\nabla\cdot Q_t) - \frac{H^2}{6}\Delta Q_t - \left(\frac{|\nabla H|^2}{3} - \frac{H\,\Delta H}{6}\right)Q_t + \frac{1}{3}H\nabla H\cdot\nabla Q_t. \tag{2.34}$$

Remark 1: In (2.33) one may replace the source-term $\frac{D\,H}{2}\nabla\zeta_{tt}$ by the term $\frac{H^2}{2}\nabla\zeta_{tt}$

Remark 2: In (2.34) the term $\nabla(\nabla\cdot Q)$ cannot be simplified to ΔQ since the irrotationality condition applies only to u but not to Q.

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