WATER RESOURCES RESEARCH, VOL. 31, NO. 7, PAGES 1809–1814, JULY 1995

The Henry problem for saltwater intrusion

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Abstract. Previous solutions of the idealized saltwater intrusion problem known as the Henry problem are discussed, and possible reasons for the observed discrepancies between them are given. High-accuracy finite difference techniques are used to solve the nondimensionalized equations governing the problem, and a fine grid is used so that the solutions obtained contain only very small truncation errors. Such errors are investigated by means of grid refinement. Comparison of past results with the present solutions indicate, first, the presence of significant inaccuracy in certain earlier results and, second, the effects of numerical dispersion in other previous solutions calculated using relatively few grid points.

Introduction

The intrusion of salt water into coastal aquifers is a common problem in many parts of the world, and, over the last three decades, has been the subject of many numerical modeling investigations. The first such numerical modeling work to simulate the intrusion of salt water, including the effects of the density dependence on salt concentration and the dispersion of salt, was carried out by *Henry* [1964]. In this paper an idealized intrusion problem was presented, and it was solved for the steady state salt distribution using a semianalytical solution technique. The "Henry problem" has subsequently been widely used as a test case for the verification of saltwater intrusion simulation codes.

While *Henry* [1964] solved only for the steady state distribution of salt, *Pinder and Cooper* [1970] used the method of characteristics to obtain a solution of a transient version of the Henry problem and compared the solution they obtained after a long simulated time with the steady state solution given by *Henry* [1964]. *Lee and Cheng* [1974] produced another steady state solution of the problem using the finite element method. The agreement between these early solutions, though considered reasonable at the time, was in retrospect not good [*Frind*, 1982].

The first of a series of finite element solutions of the transient Henry problem was published by *Segol et al.* [1975]. This solution was in reasonable agreement with that of *Pinder and Cooper* [1970] and was followed by further transient finite element solutions developed by *Frind* [1982], *Huyakorn et al.* [1987], and *Voss and Souza* [1987], all of which agreed quite closely with that of *Segol et al.* [1975]. None, however, resembled the solutions given by *Henry* [1964] or by *Lee and Cheng* [1974] to any great extent.

The present paper outlines some possible reasons for the discrepancies between the previously published solutions of the Henry problem, and presents a set of new steady state solutions of the problem, generated by applying high-accuracy finite difference approximations to a nondimensionalized form of the governing equations. Efficient solution methods are employed, so that finite difference grids containing very large numbers of grid points may be used. In this way, it is intended that highly accurate solutions may be produced, which can be used as a basis for comparison with solutions generated by other simulators.

Paper number 95WR00431.
0043-1397/95/95WR-00431$02.00

Theory

In the Henry problem a vertical slice is taken through a homogeneous isotropic aquifer confined above and below by impermeable boundaries (see Figure 1). A constant flux of freshwater is applied to one end, while the other end is exposed to a stationary body of seawater. Salt water intrudes from the seaward end until an equilibrium is reached with the opposing freshwater inflow.

The relevant governing equations for the problem may be expressed as follows:

$$\mathbf{q} = \frac{-k}{\mu}(\nabla p - \rho \mathbf{g}) \tag{1}$$

$$\rho = \rho_0(1 + \varepsilon c) \tag{2}$$

$$\nabla \cdot (\rho \mathbf{q}) = 0 \tag{3}$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{v}) - \nabla \cdot (D \nabla c) = 0 \tag{4}$$

The notation used is listed in Table 1.

Darcy's law is expressed by (1), while (2) states that the saltwater density varies linearly with salt concentration. Conservation of mass is expressed by (3), and (4) is the transport equation for salt, including dispersion effects. (It is assumed that the dispersion coefficient D in (4) is a constant scalar.) These four equations may be combined and expressed in a nondimensional form, suitable for numerical solution.

Note first that as the fluid density is nowhere greater than the density of seawater,

$$\frac{\rho - \rho_0}{\rho_0} \leq \frac{\rho_s - \rho_0}{\rho_0} \approx 0.027 \ll 1 \tag{5}$$

This means that following other authors, the Boussinesq approximation ($\rho \approx \rho_0$) may be applied to (3), giving

$$\nabla \cdot \mathbf{q} = 0 \tag{6}$$

A stream function can now be introduced [see *Henry*, 1964]. First, we introduce the nondimensional variables

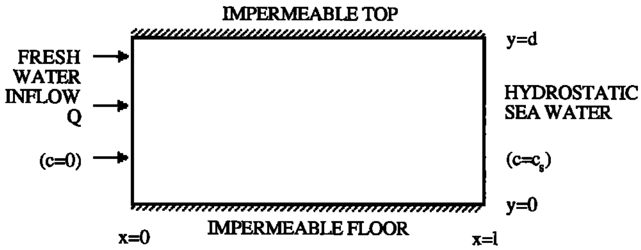

Figure 1. The Henry problem.

$$x' = \frac{x}{d} \qquad y' = \frac{y}{d} \qquad \mathbf{q}' = \frac{d}{Q}\mathbf{q} \qquad c' = \frac{c}{c_s}$$

$$\rho' = \frac{\rho - \rho_0}{\rho_s - \rho_0} = c'$$

Then the dimensionless stream function ψ may be defined by

$$q'_x = \frac{\partial \psi}{\partial y'} \qquad q'_y = -\frac{\partial \psi}{\partial x'} \tag{7}$$

The pressure variable p can be eliminated from (1) by differentiating q_x with respect to y, differentiating q_y with respect to x, and subtracting one result from the other:

$$\frac{\partial q_x}{\partial y} - \frac{\partial q_y}{\partial x} = \frac{kg}{\mu}\frac{\partial \rho}{\partial x} \tag{8}$$

and so

$$\frac{\partial q'_x}{\partial y'} - \frac{\partial q'_y}{\partial x'} = \frac{kgd(\rho_s - \rho_0)}{\mu Q}\frac{\partial c'}{\partial x'} \tag{9}$$

Hence, on substituting (7) into (9), we obtain

$$\nabla'^2 \psi = \frac{1}{a}\frac{\partial c'}{\partial x'} \tag{10}$$

where the dimensionless parameter a is given by

$$a = \frac{\mu Q}{kgd(\rho_s - \rho_0)} \tag{11}$$

Nondimensionalizing (4) and making use of (6) yields

$$\frac{\partial}{\partial t}(c_s c') = \frac{c_s D}{d^2}\nabla'^2 c' - \frac{Q}{nd^2}\mathbf{q}' \cdot \nabla' c' \tag{12}$$

so that

$$\frac{\partial c'}{\partial t} = \frac{D}{d^2}\left(\nabla'^2 c' - \frac{Q}{nD}\mathbf{q}' \cdot \nabla' c'\right) \tag{13}$$

and hence, on substituting (7) into (13), we obtain

$$\frac{\partial c'}{\partial t} = \frac{D}{d^2}\left(\nabla'^2 c' - \frac{1}{b}\left(\frac{\partial \psi}{\partial y'}\frac{\partial c'}{\partial x'} - \frac{\partial \psi}{\partial x'}\frac{\partial c'}{\partial y'}\right)\right) \tag{14}$$

where the dimensionless parameter b is given by

$$b = nD/Q \tag{15}$$

A nondimensional time variable may be introduced:

$$t' = (D/d^2)t \tag{16}$$

so that (14) becomes

$$\frac{\partial c'}{\partial t'} = \nabla'^2 c' - \frac{1}{b}J(c', \psi) \tag{17}$$

where

$$J(c', \psi) = \frac{\partial \psi}{\partial y'}\frac{\partial c'}{\partial x'} - \frac{\partial \psi}{\partial x'}\frac{\partial c'}{\partial y'} \tag{18}$$

For the steady state Henry problem, (17) becomes

$$J(c', \psi) - b\nabla'^2 c' = 0 \tag{19}$$

In summary, the nondimensionalized governing equations for the steady state Henry problem are given by (10) and (19), with the two dimensionless parameters a and b given by (11) and (15). These two parameters correspond to a specific permeability and dispersion coefficient, respectively. A third dimensionless parameter, the "aspect ratio" ξ, is also required to specify the domain over which the problem is to be solved:

$$\xi = l/d \tag{20}$$

The boundary conditions for the problem, in terms of the nondimensional variables, are given by

$$c'(0, y') = 0 \qquad c'(\xi, y') = 1 \tag{21a}$$

$$\frac{\partial c'(x', 0)}{\partial y'} = \frac{\partial c'(x', 1)}{\partial y'} = 0 \tag{21b}$$

$$\psi(x', 0) = 0 \qquad \psi(x', 1) = 1 \tag{21c}$$

$$\frac{\partial \psi(0, y')}{\partial x'} = \frac{\partial \psi(\xi, y')}{\partial x'} = 0 \tag{21d}$$

for $x' \in [0, \xi]$ and $y' \in [0, 1]$. Some past studies have used another boundary condition at the saltwater end, $x' = \xi$ (see below).

Table 1. Notation

Symbol	Quantity	Value and/or Units
$\mathbf{q}$	specific discharge = $n\mathbf{v}$	m s^{-1}
$\mathbf{v}$	water velocity	m s^{-1}
k	aquifer permeability	m^2
n	aquifer porosity	dimensionless
μ	dynamic viscosity of salt water	kg m^{-1} s^{-1}
p	saltwater pressure	Pa
ρ	saltwater density	kg m^{-3}
$\mathbf{g}$	gravitational field vector	$(0, -9.8)^T$ m s^{-2}
ρ_0	freshwater density (20°C)	998.2 kg m^{-3}
ρ_s	seawater density (20°C)	1025 kg m^{-3}
ε	density dependence coefficient	7.3×10^{-4} m^3 kg^{-1}
c	concentration of salt in salt water	kg m^{-3}
c_S	concentration of salt in seawater	36.6 kg m^{-3}
D	salt dispersion coefficient	m^2 s^{-1}
d	height of the aquifer	m
l	length of the aquifer	m
Q	volumetric freshwater inflow rate	m^2 s^{-1}
t	time	s

Method of Solution

In the present work only the steady state problem is considered, and an iterative solution technique is used to deal with the nonlinear nature of the coupled governing equations (10) and (19). (In this section, primes will be omitted from nondimensionalized variables.) Given initial estimates of the steady state salt distribution and stream function, $c^{(0)}$ and $\psi^{(0)}$, a sequence of refined estimates $c^{(p)}$ and $\psi^{(p)}$ is calculated, for $p = 1, 2, 3, \cdots$, using the relations

$$J(c^{(p+1)}, \psi^{(p)}) - b\nabla^2 c^{(p+1)} = 0 \quad (22)$$

$$\nabla^2 \psi^{(p+1)} = \frac{1}{a}\frac{\partial c^{(p+1)}}{\partial x} \quad (23)$$

At each iteration, (22) is used to solve for $c^{(p+1)}$, and (23) is then solved for $\psi^{(p+1)}$. The iteration process is terminated when the average difference between $\psi^{(p+1)}$ and $\psi^{(p)}$ over the problem domain is less than a prescribed tolerance.

Equations (22) and (23) are solved approximately using finite difference methods. A regular grid of m by n points, with grid spacings Δx and Δy in the x and y directions respectively, is superimposed on the problem domain. As all previously published solutions of the Henry problem have used an aspect ratio of $\xi = 2$, a "square" finite difference grid in which $\Delta x = \Delta y$ may be employed, thus simplifying the finite difference analogues of (22) and (23).

The standard five-point second-order difference approximation is used for the Laplacian term in (22). The Jacobian term $J(c^{(p+1)}, \psi^{(p)})$ is more difficult to represent accurately in difference form. If, for example, simple centered differences are used to approximate the spatial derivatives in the term, numerical instability can occur as a result of "aliasing errors" [*Horne and O'Sullivan*, 1974]. However, *Arakawa* [1966] developed more sophisticated nine- and 13-point discrete Jacobians that avoid such problems by ensuring that the approximations conserve certain properties of the flow, such as the mean kinetic energy. The nine-point Arakawa approximation, which is of second order, is used here [see *Arakawa*, 1966]. The resulting difference equation corresponding to (22) is solved using the technique of successive overrelaxation.

Equation (23) is a Poisson problem in $\psi^{(p+1)}$. The right-hand side is evaluated using a centered difference to approximate the spatial derivative, and the problem is solved using the Buneman algorithm [*Buzbee et al.*, 1970], which is a direct (i.e., noniterative) method for solving a Poisson problem on a finite difference grid, again using the standard five-point difference approximation for the Laplacian operator. The method is very efficient, but because of its intricate nature, a detailed explanation of it is not appropriate here (refer to *Buzbee et al.* [1970]). A special requirement of the algorithm is that the number of grid points in the y direction be given by

$$n = 2^{k+1} + 1 \quad (24)$$

where k is a positive integer. As the aspect ratio $\xi = 2$, the number of grid points in the x direction must be given by $m = 2n - 1 = 2^{k+2} + 1$.

The boundary conditions on ψ and c are incorporated by direct substitution of prescribed boundary values and by using symmetry on boundaries where the normal derivative of ψ or c is specified as zero. Hence ψ must be symmetrical about the vertical boundaries, and c must be symmetrical about the horizontal boundaries. These symmetries are used to evaluate the discrete Laplacian and Jacobian terms on the boundaries. The evaluation of the discrete Jacobian term on the horizontal boundaries also requires extrapolation of ψ to points outside the problem domain, and simple linear extrapolation is used for this. The evaluation of the right-hand side of (23) on the vertical boundaries is carried out in a similar way.

Previous Solutions of the Henry Problem

Since the work of *Henry* [1964] appeared, at least six other authors have published solutions of the Henry problem [*Pinder and Cooper*, 1970; *Lee and Cheng*, 1974; *Segol et al.*, 1975; *Frind*, 1982; *Huyakorn et al.*, 1987; *Voss and Souza*, 1987]. All have attempted to duplicate as closely as possible the problem solved by Henry, so that comparisons between solutions can be made. However, while the more recent solutions are similar, they do not agree well with earlier results; in particular, the semianalytical steady state solution originally given by *Henry* [1964] does not, except in a qualitative way, resemble any of the subsequent solutions. Nor is the finite element steady state solution of *Lee and Cheng* [1974] in close agreement with any of the other results. There are several possible reasons for the discrepancies between the published solutions.

It should be noted that the solutions of *Henry* [1964] and *Lee and Cheng* [1974] are steady state, while the others are the results of long-term transient simulations. Most authors considered their transient results to be approximately steady after 100 min of simulated time, starting from hydrostatic freshwater conditions throughout an aquifer of height $d = 1$ m, and using a dispersion coefficient $D = 6.6 \times 10^{-6}$ m^2 s^{-1}. This corresponds to a nondimensional time, as given by (16), of $t' = 0.0396$. However, *Frind* [1982] reported that the equilibrium salt distribution (calculated from a simulation up to the time $t' = 2.28$) was attained only at about $t' = 0.103$ and that this equilibrium distribution was noticeably different from the solution at $t' = 0.0396$. Therefore the results given by various authors for $t' = 0.0396$ [*Pinder and Cooper*, 1970; *Segol et al.*, 1975; *Voss and Souza*, 1987] may be expected to differ slightly from the steady state salt distribution, as they may not have reached a true equilibrium. Note that the results given by *Huyakorn et al.* [1987] were obtained using different distance scales and timescales, so that while their steady state results are comparable with others, their transient results are not. The 100-day simulated time at which their transient results are given corresponds to $t' = 6.6 \times 10^{-4}$.

In addition, the original boundary conditions for the problem were altered slightly by *Segol et al.* [1975], as the abrupt front between seawater and freshwater flowing out of the top portion of the sea boundary causes numerical difficulties when a transient finite element solution is attempted. To overcome this, the new boundary condition

$$\frac{\partial c(\xi, y)}{\partial x} = 0 \quad (25)$$

was imposed on the upper section of the sea boundary where fluid leaves the system (*Huyakorn et al.* [1987] determined that this occurs approximately over the upper 20% of the boundary). As before, the condition $c = 1$ was enforced on the remainder of the sea boundary. *Frind* [1982], *Huyakorn et al.* [1987], and *Voss and Souza* [1987] have also used these mixed sea boundary conditions in their simulations. Therefore these solutions cannot be compared near the upper part of the sea boundary with solutions calculated using Henry's original

Table 2. Comparison of Parameters and Boundary Conditions Used

Author	b Value	Seaward Boundary Condition
Henry [1964]	0.1	$c = 1$
Pinder and Cooper [1970]	0.035 ?	$c = 1$
Lee and Cheng [1974]	0.1	$c = 1$
Segol et al. [1975]	0.035	mixed $c = 1$ and no flux
Frind [1982]	0.035	mixed $c = 1$ and no flux
Huyakorn et al. [1987]	0.035	mixed $c = 1$ and no flux
Voss and Souza [1987]	0.1 and 0.035	mixed $c = 1$ and no flux
Present study	0.1 and 0.035	$c = 1$

boundary conditions. Throughout the remainder of the aquifer, however, such comparisons may still be made.

It must also be remembered that all solutions of the problem have been calculated numerically and are therefore subject to discretization or truncation error [*Frind*, 1982]. Such errors may be investigated by comparison with results calculated using a finer grid; however, no such investigation has accompanied any of the published solutions. *Frind* [1982] did in fact publish results for a coarse (66-node) grid and for a finer (231-node) grid. These two results were significantly different, indicating some discretization error in the coarse-grid solution, but no conclusions can be drawn from this as to the accuracy of the fine-grid solution. The other published finite element solutions have used grids containing between 108 nodes [*Segol et al.*, 1975] and 231 nodes [*Voss and Souza*, 1987], and the possibility of discretization error in these solutions cannot be ruled out.

There is, however, a more fundamental inconsistency between the work of *Henry* [1964] and *Lee and Cheng* [1974], and that of the other authors. These two papers appear to employ a definition of the salt dispersion coefficient slightly different from that used by other authors. In the work by *Henry* [1964], the steady state salt transport equation is given in the form

$$\nabla \cdot (c\mathbf{q}) - \nabla \cdot (D_H \nabla c) = 0 \qquad (26)$$

where D_H is the dispersion coefficient as used by Henry. The appearance of the specific discharge $\mathbf{q}$, instead of the more usual velocity $\mathbf{v}$, means that (26) must be divided through by the porosity n in order to compare it with the transport equation (4) used here:

$$\nabla \cdot (c\mathbf{v}) - \nabla \cdot \left(\frac{D_H}{n} \nabla c\right) = 0 \qquad (27)$$

Comparing (27) with (4) (and letting $\partial c/\partial t = 0$ for steady state), it can be seen that the two equations are consistent only if

$$D_H = nD \qquad (28)$$

Henry's dimensionless parameter b, defined as D_H/Q, is then equivalent to the parameter b defined here by (15). *Lee and Cheng* [1974] used a formulation similar to that of *Henry* [1964].

Subsequent authors, however, have used the more common definition of dispersion coefficient embodied by (4). In seeking to duplicate the results of *Henry* [1964], which used the parameter values $a = 0.263$, $b = 0.1$, and $\xi = 2$, they have generally used physical parameters chosen so that $D/Q = 0.1$, rather than $D_H/Q = 0.1$, not noting the difference between D_H and D. In addition, an arbitrary porosity value of $n = 0.35$ has commonly been used, which effectively means that $b = 0.35 \times 0.1 = 0.035$ for these analyses. *Pinder and Cooper* [1970] also used the definition of the dispersion coefficient given by (4), but did not state the porosity value assumed in their calculations. Their solution does agree quite closely with the other $b = 0.035$ results. It is clear, however, that the two solutions calculated using $b = 0.1$ should not be compared with those that used $b = 0.035$. A summary of the values of b and the boundary conditions used by previous authors is given in Table 2.

Results and Discussion

Using the numerical method described earlier, the Henry problem was first solved using the parameters $\xi = 2$, $a = 0.263$, and $b = 0.1$, in order to compare results with those of *Henry* [1964] and *Lee and Cheng* [1974]. Four different finite difference grids were used, ranging from 17×9 nodes (corresponding to $k = 2$ in (24)) to 129×65 nodes ($k = 5$). Convergence of the solution was monitored as the grid was refined, in particular by observing the position of the $c' = 0.5$ salt contour where it intersects the base of the aquifer. Equations (22) and (23) were iterated until the average change in ψ over the grid was less than a tolerance of 10^{-6}. The successive overrelaxation algorithm, with a relaxation factor of 1.2, was used to solve for the salt concentration at each iteration. It was also terminated when the average change in c' over the grid was less than 10^{-6}.

The solution obtained using the coarse $k = 2$ (17×9) grid displayed oscillations near the sharp salt water–freshwater interface on the outflow section of the sea boundary, with the calculated salt concentration becoming negative at some points. As the grid was refined, these oscillations became progressively less severe. The toe of the $c' = 0.5$ salt contour advanced landward as the grid spacing was decreased, as shown in Figure 2, and approached a limiting value as k was increased beyond 4. This behavior is a result of numerical dispersion, which effectively increases artificially the dispersion coefficient D, approximately in proportion to the grid spacing (and to the local flow velocity). When the grid is refined, the effective dispersion is lowered, and the solution moves landward accordingly. As the solutions for $k = 4$ and $k = 5$ were very similar, it was concluded that the $k = 5$ grid was sufficiently fine to make truncation errors in the solution negligible. The $k = 5$ solution also displayed only very minor oscillations near the outflow face.

The resulting salt distribution for the $k = 5$ case is represented by the $c' = 0.5$ contour given in Figure 3. The results of *Henry* [1964] and *Lee and Cheng* [1974], as well as the $b = 0.1$ results given by *Voss and Souza* [1987], are shown for comparison. The solution of *Voss and Souza* [1987] was obtained using the modified sea boundary conditions proposed by *Segol et al.* [1975] and therefore cannot be compared near the upper part of the sea boundary; however, the agreement with the present solution over the remainder of the contour is

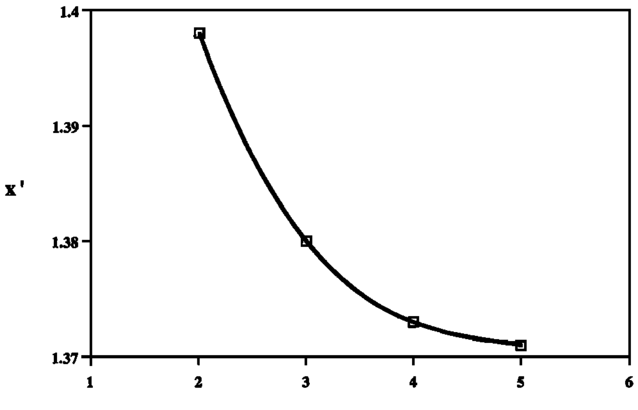

Figure 2. Position of the toe of the $c' = 0.5$ contour as the grid is refined ($b = 0.1$).

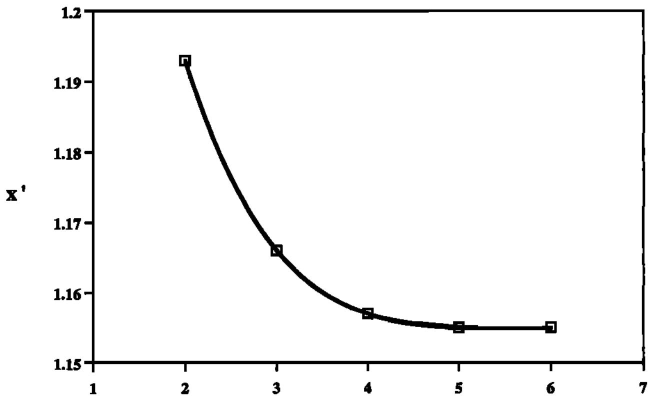

Figure 4. Position of the toe of the $c' = 0.5$ contour as the grid is refined ($b = 0.035$).

reasonable. The differences between the two curves may be a result of the somewhat coarser grid (21 × 11 nodes) employed by *Voss and Souza* [1987], which may have introduced some numerical dispersion into their solution.

It is clear from Figure 3 that the present solution is in disagreement with the results of *Henry* [1964] and *Lee and Cheng* [1974] over most of the $c' = 0.5$ contour, and that these two previous solutions are themselves not consistent. The possibility of some error in Henry's solution has been advanced by several authors [*Frind*, 1982; *Huyakorn et al.*, 1987; *Voss and Souza*, 1987]. Henry's solution was obtained by expanding the salt concentration and stream function in Fourier-Galerkin double series; these series were truncated, and the governing equations used to solve for the remaining series coefficients. *Voss and Souza* [1987] suggested that Henry's series may not have included enough higher-order terms to represent the solution accurately. The present results also confirm the existence of some form of error in Henry's solution. The results of *Lee and Cheng* [1974] are closer to the present results for $b = 0.1$ than are those of *Henry* [1964] but are still significantly different, with the toe of the $c' = 0.5$ contour at a distance approximately 0.12 units further inland. Lee and Cheng's solution resembles slightly more closely the $b = 0.035$ results of other authors, and has been compared with such results by *Frind* [1982] and *Huyakorn et al.* [1987], but the differences in the parameters used undermine the validity of such comparisons.

The Henry problem was also solved using the parameters $\xi = 2$, $a = 0.263$, and $b = 0.035$, using five different grids ranging from 17 × 9 nodes ($k = 2$) to 257 × 129 nodes ($k = 6$). While the two iteration tolerance levels were kept at 10^{-6}, as in the $b = 0.1$ case, more iterations both of the successive overrelaxation routine and of the generating equations (22) and (23) were required in order to attain these levels. The relaxation factor had to be decreased to 0.8 in order to obtain convergence when solving for the salt concentration. *Henry* [1964] and *Lee and Cheng* [1974] also reported numerical difficulties when using low values of the parameter b, caused by the dominance of the convective terms in the governing equations. The behavior of the solution as the grid was refined was similar to that displayed by the $b = 0.1$ solution, though the oscillations near the outflow boundary were more severe as a result of the higher salt gradients caused by the lower effective dispersion. As the grid spacing was decreased, these oscillations decreased steadily as before, and the toe of the $c' = 0.5$ salt contour advanced landward to a limiting value (see Figure 4).

In view of the close agreement between the $k = 5$ and $k = 6$ results, it was concluded that in the $k = 6$ solution truncation errors had been reduced to negligible levels.

Figure 5 compares the present solution for $k = 6$ with the transient results at 100 min of simulated time given by three other authors, again with all results represented by $c' = 0.5$ salt contours. The agreement between the various curves is reasonable, although the previous results are displaced seaward slightly with respect to the present solution. This displacement may be the result of numerical dispersion in the

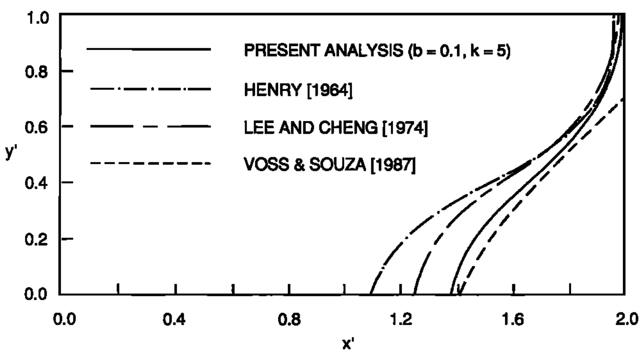

Figure 3. Position of the $c' = 0.5$ contour for $b = 0.1$, $k = 5$.

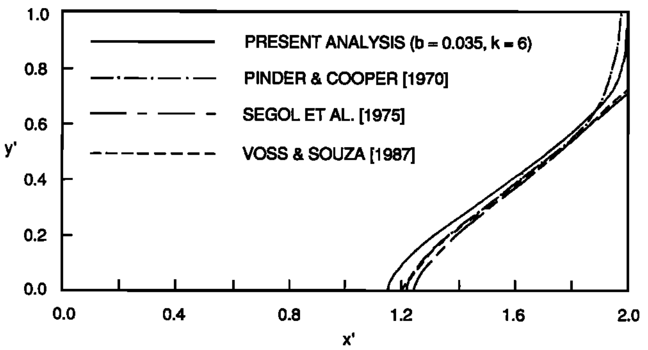

Figure 5. Position of the $c' = 0.5$ contour for $b = 0.035$, $k = 6$.

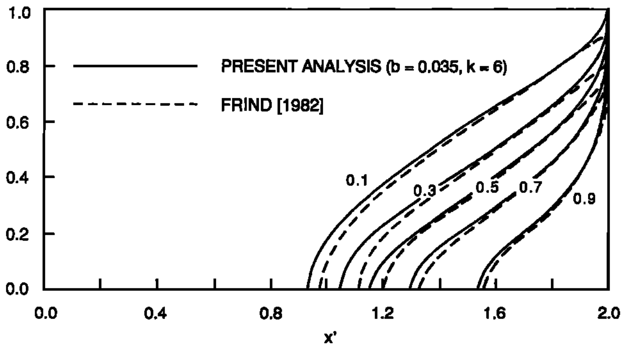

Figure 6. Comparison of steady state salt distribution with results of *Frind* [1982].

finite element solutions of *Segol et al.* [1975] and *Voss and Souza* [1987]. The solution of *Segol et al.* [1975], which used a 108-node grid, is displaced further seaward than is the solution of *Voss and Souza* [1987], which used a 231-node grid (while the $k = 6$ grid used here contained 33,153 nodes). *Pinder and Cooper* [1970] stated that their solution method did not introduce numerical dispersion.

Figure 6 compares the present $k = 6$ solution for $b = 0.035$ with the steady state results of *Frind* [1982]. As well as giving the $c' = 0.5$ contour, which shows the mean position of the salt water–freshwater transition zone, four other contours in the range $c' = 0.1$ to $c' = 0.9$ are given, in order to indicate the width of the zone. The solution of *Frind* [1982] is not comparable to the present solution near the outflow face, owing to the different sea boundary conditions used. Over the remainder of the solution, the curves are in reasonable agreement, although Frind's solution is again displaced somewhat toward the sea boundary with respect to the present results. As Frind's solution was calculated using a 21 × 11 node grid, this displacement may again be the result of numerical dispersion.

The stream function calculated using the $k = 6$ grid, with $b = 0.035$, is shown in Figure 7. The $\psi = 0$ streamline marks the leading edge of the region of saltwater recirculation, which contains only streamlines beginning and ending on the sea boundary. This dividing streamline meets the aquifer base, on which the boundary condition $\psi = 0$ is enforced, in a "stagnation point" at approximately $x' = 0.93$. The region of freshwater through-flow, containing only streamlines originating on the landward boundary, is characterized by ψ being positive. For this set of parameters, no other stream function results are currently available for comparison.

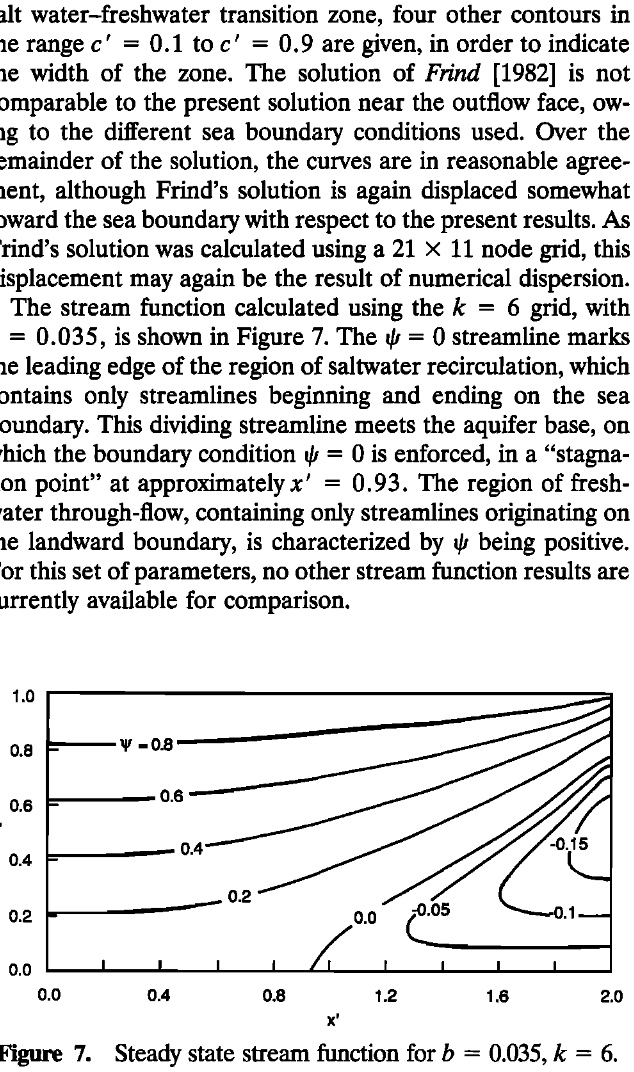

Figure 7. Steady state stream function for $b = 0.035$, $k = 6$.

Summary and Conclusions

The procedure of nondimensionalizing the governing equations reveals the underlying dimensionless parameters of the Henry problem. While a number of solutions of the problem are now available, the solutions of *Henry* [1964] and *Lee and Cheng* [1974] used parameter values different from those used by other authors, with the result that these two solutions should not be compared with the others. In addition, there is strong evidence for some form of inaccuracy in the original results of *Henry* [1964], and possibly in the solution given by *Lee and Cheng* [1974] also.

The numerical solution of the Henry problem presented here combines the highly accurate Arakawa difference approximation for the Jacobian term in the salt advection-diffusion equation with the Buneman algorithm for efficient solution of the stream function equation. In this way, accurate solutions of the problem can be calculated using very fine finite difference grids, while keeping computational costs at a reasonable level. To investigate truncation errors, and to ensure that such errors in the solution are reduced to negligible levels, comparisons between solutions calculated using progressively finer grids are necessary (although such a procedure is usually practicable only when solving an idealized problem). The steady state solutions presented here for two different sets of dimensionless parameters agree approximately with other published solutions calculated using transient simulators. In most cases the differences are probably a result of numerical dispersion introduced by the relatively coarse grids and simple differencing techniques employed in these transient solutions. The present solutions, because of their very low levels of numerical dispersion, may serve as a useful benchmark in the verification of saltwater intrusion simulators.

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(Received December 30, 1993; revised January 30, 1995; accepted February 2, 1995.)