A Robust Solution to Henry's Problem

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August 13, 2012

Abstract

Henry's Semi-Analytic Solution to Seawater Intrusion has been used as a benchmark for numerical solutions for many years. However, its ability to use parameters that are more representative of narrow zones of dispersion has been limited. The method used by Henry to evaluate his analytic solution has been tempered by the limited ability to increase the number of Fourier Coefficients used, and this may possibly lead to the inability of those methods to solve Henry's Problem using parameters that would simulate narrow zones of dispersion. Newton's Method however, is a more robust numerical method than those previously used, and therefore will not limit the number of coefficients used. The robust nature of Newton's Method leads to the ability to use an increased number of Fourier coefficients, and therefore will allow for parameters that can simulate narrower zones of dispersion.

1 Introduction

Harold R. Henry [2] [3] considered the problem of seawater intrusion into coastal aquifers for the case of an isotropic homogeneous confined aquifer, in which there is a steady seaward flow of freshwater. A constant flux of freshwater is applied to the inland boundary, while a body of higher density seawater is on the seaward side. Seawater therefore, intrudes from the seaward boundary towards the inland boundary until an equilibrium is reached between the heavier seawater and the lighter freshwater recharge. The domain described above is depicted in Figure 1.

Recently seawater intrusion has become a significant problem in coastal regions. As populations increase in these regions, natural recharge no longer counteracts the effects of withdrawals, and so it is important to have a tool that can estimate the amount of seawater intrusion that may occur with the continued withdrawal from coastal aquifers. Several numerical models have been developed to address seawater intrusion in coastal aquifers, and so it is important to have an analytic solution to use as a benchmark. The benchmarking of numerical code against analytic solutions is a necessary step in verifying the correctness of the numerical approximations [5].

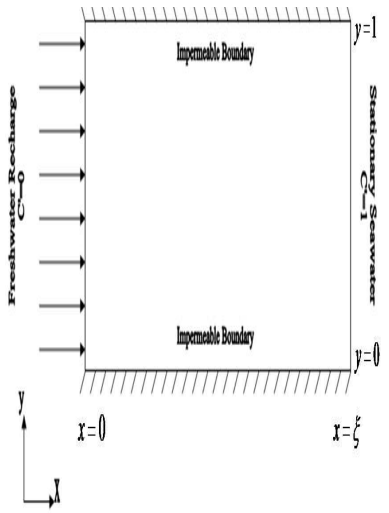

Figure 1: Depiction of 2D problem domain

Henry's Problem has been widely used as a benchmark case for numerical models of seawater intrusion. Henry's Semi-Analytic Solution to Seawater Intrusion has been tempered by both the inability to use, powerful computing technologies, which has become readily available to researchers, and a lack of a sufficient number of Fourier coefficients, so as to ensure a smooth convergence. In fact, Ségol [4] suggested that discrepancies between Henry and her own solutions may have been a result of a lack of convergence in Henry's solution, due to a lack of powerful computing of the time, and what Ségol suggests was a poor initial guess.

The method used by Henry to evaluate his Analytic solution has been plagued by the inability to use a large number of Fourier coefficients, and therefore does not allow for realistic parameter values Which have also prevented Henry's Solution from simulating narrow zones of dispersion. This is due to Henry's method having a substantially low convergences rate with respect to the important parameters a and b. Since Newton's method has a quadratic rate of convergence for well behaved problems, one might expect greater ease in calculation for lower values of a and b. Therefore, Newton's Method would address the convergence issues seen by Henry and others when trying to simulate narrow zones of dispersion.

Newton's Method not only provides for a previously unused method for solving Henry's Problem, but will also allow for more Fourier coefficients to be used in the solution, therefore allowing for a more realistic solution sets. As more terms are used in a Fourier series the closer the numerical solution comes to the true solution. Using Newton's method allows for the use of any number of Fourier coefficients, and so it is now possible to evaluate Henry's Semi-Analytic

Solution for narrow zones of dispersion. The improved accuracy and the ability to simulate narrow zones of dispersion will therefore improve Henry's Problem as a benchmark for comparison to numerical solutions.

In this paper, we calculate the Fourier coefficients for the Henry problem by using Newton's method to solve the nonlinear system of equations. Various truncations of the Fourier series are tested and results are compared against the available result using Henry's method. Finally, a and b are decreased to more realistic values and results are compared against numerical results from SUTRA with similar parameter values.

2 Governing Equations

Henry [2] assumed that a "representative average of the value of the diffusion coefficient, D, could be found and used as a constant scalar throughout the field of flow without distorting the essential features of the problem." Additionally, Henry assumed the Bousinesq approximation, which implies the existence of a stream function, defined as

$$u' = \frac{\partial \psi'}{\partial y'},\ v' = -\frac{\partial \psi'}{\partial x'}$$

Thus, using these assumptions allows one to write the steady state transport equations in the following dimensionless form

$$a\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\psi y^2}\right) = \frac{\partial C}{\partial x} + \frac{1}{\xi}, \tag{1}$$

$$b\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\psi y^2}\right) = \frac{\partial \psi}{\partial y}\frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial C}{\partial y} + \frac{1}{\xi}\frac{\partial \psi}{\partial y} + \frac{\partial C}{\partial x} + \frac{1}{\xi}, \tag{2}$$

where ψ is the dimensionless stream function, C is the dimensionless concentration, $\xi = \frac{L}{d}$ is the aspect ratio of the problem domain. Additionally, the dimensionless parameters a and b are given by

$$a = \frac{Q}{k_1 d} \text{ and } b = \frac{D}{Q} \tag{3}$$

where Q is the freshwater recharge, k_1 is related to the hydraulic conductivity, K, by $k_1 = K\left(\frac{\rho_s - \rho_0}{\rho_0}\right)$, with ρ_0 and ρ_s the freshwater and seawater densities, respectively.

Henry used the boundary conditions

$$\begin{array}{ll} \psi = 0, \frac{\partial c}{\partial y} = 0 \text{ at } y = 0 & \frac{\partial \psi}{\partial x} = 0, c = 0 \text{ at } x = 0 \\ \psi = 1, \frac{\partial c}{\partial y} = 0 \text{ at } y = 1 & \frac{\partial \psi}{\partial x} = 0, c = 1 \text{ at } x = \xi \end{array} \tag{4}$$

and what he termed as Fourier-Galerkin method, whereby one substitutes

$$\psi = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_{m,n} \sin m\pi y \cos n\pi \frac{x}{\xi} \tag{5}$$

$$C = \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} B_{r,s} \cos r\pi y \sin s\pi \frac{x}{\xi} \tag{6}$$

into (1) and (2), then multiplying (1) by $4\sin(g\pi y)\cdot\cos\left(h\pi\frac{x}{\xi}\right)$ and (2) by $4\cos(g\pi y)\cdot\sin\left(h\pi\frac{x}{\xi}\right)$ to obtain

$$\varepsilon(h)\cdot a\pi^2 A_{g,h}\left(g^2+\frac{h^2}{\xi^2}\right)\xi = \sum_{r=0}^{\infty} B_{r,h}\cdot h\cdot N(g,r) + \frac{4}{\pi}W(g,h) \tag{7}$$

$$\begin{aligned}\varepsilon(g)\cdot b\pi^2 B_{g,h}\left(g^2+\frac{h^2}{\xi^2}\right)\xi = {} & \frac{\pi}{4}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=1}^{\infty} A_{m,n}B_{r,s}(msLR-nrFG) \\ & + \sum_{n=0}^{\infty} A_{g,n}\cdot g\cdot N(h,n) + \varepsilon(g)\cdot\sum_{s=1}^{\infty} B_{g,s}\cdot s\cdot N(h,s) \\ & + \frac{4}{\pi}W(g,h)\end{aligned} \tag{8}$$

where

$$\varepsilon(g) = \begin{cases} 2 & \text{if } g = 0 \\ 1 & \text{if } g \neq 0 \end{cases}$$

$$N(h,n) = \frac{(-1)^{h+n}-1}{h+n} + \frac{(-1)^{h-n}-1}{h-n},$$

$$W(h,n) = \begin{cases} \frac{(-1)^h-1}{h} & \text{if } g = 0 \\ 0 & \text{if } g \neq 0 \end{cases}$$

$$F = \delta_{(m-r),g} + \delta_{(r-m),g} - \delta_{(m+r),q},$$
$$L = \delta_{(m-r),g} + \delta_{(r-m),g} + \delta_{(m+r),q},$$

$$G = \frac{(-1)^{h+n-s}-1}{h+n-s} + \frac{(-1)^{h-n+s}-1}{h-n+s} - \frac{(-1)^{h+n+s}-1}{h+n+s} + \frac{(-1)^{h-n-s}-1}{h-n-s},$$

$$R = \frac{(-1)^{h+n-s}-1}{h+n-s} + \frac{(-1)^{h-n+s}-1}{h-n+s} + \frac{(-1)^{h+n+s}-1}{h+n+s} + \frac{(-1)^{h-n-s}-1}{h-n-s},$$

and $\delta_{i,j}$ is the Kronecker delta such that

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

3 Newton-Raphson Formulation

To improve the results obtained by Henry's Analytic solution a Newton-Raphson Method was used. Originally Henry used a method simple method of linearization, which will be discussed in depth later. The method originally used by Henry will be discussed in depth later. For Newton's Method to make sense, in the contexts of Henry's Problem, it is necessary to define a vector X made up of vectors A and B, therefore let

$$X = \begin{bmatrix} A_{1,0} \\ A_{1,1} \\ \vdots \\ A_{i_a,j_a} \\ B_{0,1} \\ B_{0,2} \\ \vdots \\ B_{i_b,j_b} \end{bmatrix} \quad i_a,\ i_b,\ j_a,\ j_b \in \mathbb{Z}^+ \tag{9}$$

It is also necessary to define a vector function based upon (**??**) and (**??**), where

$$\phi_{g,h}(X) = \varepsilon(h)\, a\pi^2 \cdot A_{g,h} \cdot \left(g^2 + \frac{h^2}{\xi^2}\right)\xi - \sum_{r=0}^{i_b} B_{r,h} \cdot h \cdot N(g,r) - \frac{4}{\pi} W(g,h) \tag{10}$$

and

$$\begin{aligned} &\gamma_{g,h}(X) = \varepsilon(g)\, b\pi^2 \cdot B_{g,h} \cdot \left(g^2 + \tfrac{h^2}{\xi^2}\right)\xi - \textstyle\sum_{n=0}^{i_a} A_{g,n} \cdot g \cdot N(h,n) \\ &-\varepsilon(g) \textstyle\sum_{s=1}^{i_b} B_{g,s} \cdot s \cdot N(h,s) - \frac{\pi}{4} \sum_{m=1}^{i_a} \sum_{n=0}^{j_a} \sum_{r=0}^{i_b} \sum_{s=1}^{j_b} A_{m,n} B_{r,s} (msLR - nrFG) - \tfrac{4}{\pi} W(h,g) \end{aligned} \tag{11}$$

Newton's Method however, takes only one function and therefore these two vector functions must be combined to make one new vector function. We solve it as a system and combine both of the above functions in one vector:

$$\Phi(X) = \begin{bmatrix} \phi(X) \\ \gamma(X) \end{bmatrix} \tag{12}$$

Let $X^{(n+1)}$be the value of X at the $n+1^{th}$ iteration of the Multi-Variate Newton-Raphson's Method. We then can write Newton's Method as

$$X^{(n+1)} = X^{(n)} - D\Phi^{-1}\left(X^{(n)}\right) \cdot \Phi\left(X^{(n)}\right), \qquad n \in \mathbb{N} \tag{13}$$

where $D\Phi\left(X^{(n)}\right)$ is the Jacobian matrix evaluated at $X^{(n)}$ and $D\Phi^{-1}\left(X^{(n)}\right)$ is the inverse to $D\Phi\left(X^{(n)}\right)$.

For simplicity let $\Phi^{(n)}$ denote $\Phi\left(X^{(n)}\right)$, $\phi^{(n)}$ denote $\phi\left(X^{(n)}\right)$, $\gamma^{(n)}$ denote $\gamma\left(X^{(n)}\right)$, $D\Phi^{(n)}$ denote $D\Phi\left(X^{(n)}\right)$, and $D\Phi^{-1}_{(n)}$ denote $D\Phi^{-1}\left(X^{(n)}\right)$, therefore one arrives at

$$D\Phi^{(n)} = \begin{bmatrix} \partial\phi_{1,0}^{(n)}/\partial X_1^{(n)} & \partial\phi_{1,0}^{(n)}/\partial X_2^{(n)} & \cdots & \partial\phi_{1,0}^{(n)}/\partial X_{i_a(j_a+1)+(i_b+1)j_b}^{(n)} \\ \partial\phi_{1,1}^{(n)}/\partial X_1^{(n)} & \partial\phi_{1,1}^{(n)}/\partial X_2^{(n)} & \cdots & \partial\phi_{1,1}^{(n)}/\partial X_{i_a(j_a+1)+(i_b+1)j_b}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \partial\phi_{i_a,j_a}^{(n)}/\partial X_1^{(n)} & \partial\phi_{i_a,j_a}^{(n)}/\partial X_2^{(n)} & \cdots & \partial\phi_{i_a,j_a}^{(n)}/\partial X_{i_a(j_a+1)+(i_b+1)j_b}^{(n)} \\ \partial\gamma_{0,1}^{(n)}/\partial X_1^{(n)} & \partial\gamma_{0,1}^{(n)}/\partial X_2^{(n)} & \cdots & \partial\gamma_{0,1}^{(n)}/\partial X_{i_a(j_a+1)+(i_b+1)j_b}^{(n)} \\ \partial\gamma_{0,2}^{(n)}/\partial X_1^{(n)} & \partial\gamma_{0,2}^{(n)}/\partial X_2^{(n)} & \cdots & \partial\gamma_{0,2}^{(n)}/\partial X_{i_a(j_a+1)+(i_b+1)j_b}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \partial\gamma_{i_b,j_b}^{(n)}/\partial X_1^{(n)} & \partial\gamma_{i_b,j_b}^{(n)}/\partial X_2^{(n)} & \cdots & \partial\gamma_{i_b,j_b}^{(n)}/\partial X_{i_a(j_a+1)+(i_b+1)j_b}^{(n)} \end{bmatrix} \tag{14}$$

Where i_a is the number of rows of A, j_a is the number of columns of A, i_b is the number of rows of B, and j_b is the number of columns of B, therefore

$$\begin{aligned} |\phi| &= |A| = i_a\left(j_a+1\right) \\ |\gamma| &= |B| = j_b\left(i_b+1\right) \end{aligned} \tag{15}$$

To understand how to write the algorithm for this specific use of Newton's Method, one must first understand the structure of Φ and $D\Phi$. The vector function Φ has two parts as was shown in (12). The first portion of the vector function (12) is that portion which represents the equations for the A's. This is the portion of Φ given by $\phi(X)$. The second portion of the vector function (12) is that portion for which the equations for the B's are represented, and is the portion of Φ given by $\gamma(X)$.

The Jacobian matrix $D\Phi\left(X\right)$ has a similar split along its columns. Where the first portion is given by the derivatives of the function$\phi\left(X\right)$, and the second portion is given by the derivatives of the function$\gamma\left(X\right)$. But, now there is an additional split along the rows; portion one being the derivatives of the vector function $\Phi\left(X\right)$ with respect to the A's, and portion two being the derivatives of the vector function $\Phi\left(X\right)$ with respect to the B's. Therefore, the matrix $D\Phi$ is actually split into four quadrants. The matrix therefore can be broken down into its component sections as follows

$$D\Phi = \begin{bmatrix} I & II \\ III & IV \end{bmatrix}$$

Where section I, II, III, and IV represent $\partial\phi/\partial A$, $\partial\phi/\partial B$, $\partial\gamma/\partial A$, and $\partial\gamma/\partial A$ respectively, and can be written

$$D\Phi = \begin{bmatrix} \partial\phi/\partial A & \partial\phi/\partial B \\ \partial\gamma/\partial A & \partial\gamma/\partial B \end{bmatrix} \tag{16}$$

Where

$$dim\left(I\right) = i_a\left(j_a+1\right) \times i_a\left(j_a+1\right)$$

$$dim\,(II) = i_a\,(j_a + 1) \times j_b\,(i_b + 1)$$

$$dim\,(III) = j_b\,(i_b + 1) \times i_a\,(j_a + 1)$$

$$dim\,(IV) = j_b\,(i_b + 1) \times j_b\,(i_b + 1)$$

and therefore

$$dim(D\Phi) = [i_a\,(j_a + 1) + (i_b + 1)\,j_b] \times [i_a\,(j_a + 1) + (i_b + 1)\,j_b] \tag{17}$$

The size of each quadrant is dependent on the size of the vectorsϕ andγ, defined in (15), which in turn is determined by the size of the vectors A and B.

The number of rows depends on which function (ϕ or γ) is being used, and the number of columns depends on whether the quadrant takes the derivative with respect to A or B. To encode this matrix, one must translate a loop counter into the indices for A and B. Since quadrants I and III are derivatives with respect to A, they have one pattern, and quadrants II and IV are derivatives with respect to B, they have another pattern.

The matrix $D\Phi$ is a square matrix; therefore a main diagonal is defined by the derivatives of $\phi$and γ, with respect to$A_{g,h}$ and the $B_{g,h}$ terms, respectively. For these terms, one only needs to determine a pattern that translates the row number in the matrix $D\Phi$ into g and h, and then place the diagonal derivatives (e.g. $\partial\phi_{g,h}/\partial A_{g,h}$ and $\partial\gamma_{g,h}/\partial B_{g,h}$) into the main diagonal of $D\Phi$. For the rest of the matrix $D\Phi$ the pattern will begiven by the column. For quadrants I and II the pattern is given by the function ϕ, and for quadrants III and IV the pattern is given by that of the function γ. If the column counter is less than or equal to the total number of A's then the pattern for A is used. If the counter for the columns is greater than that of the total number of A's the pattern for B is used. If the counter for the rows is less than or equal to the total number of ϕ functions then the equation for the function ϕ is used, while if the counter for the rows is greater than the total number of ϕ functions then the equations for the function γ are used.

Let i be the counter for the rows, then the pattern that translates the row number i into g and h can be discovered by writing out a table.

Table 1

i	g	h
1	1	0
2	1	1
3	1	2
$\vdots$	$\vdots$	$\vdots$
$j_a + 1$	1	j_a
$j_a + 2$	2	0
$(j_a + 2) + 1$	2	1
$\vdots$	$\vdots$	$\vdots$
$(j_a + 2) + j_a$	2	j_a
$2j_a + 3$	3	0
$(2j_a + 3) + 1$	3	1
$\vdots$	$\vdots$	$\vdots$
$((i_a - 1)\, j_a + i_a) + j_a$	i_a	j_a
$i_a\,(j_a + 1) + 1$	0	1
$i_a\,(j_a + 1) + 2$	0	2
$\vdots$	$\vdots$	$\vdots$
$i_a\,(j_a + 1) + j_b$	0	j_b
$i_a\,(j_a + 1) + j_b + 1$	1	1
$i_a\,(j_a + 1) + j_b + 2$	1	2
$\vdots$	$\vdots$	$\vdots$
$i_a\,(j_a + 1) + j_b + j_b$	1	j_b
$i_a\,(j_a + 1) + 2j_b + 1$	2	1
$\vdots$	$\vdots$	$\vdots$
$i_a\,(j_a + 1) + j_b\,(i_b + 1)$	i_b	j_b

Here one can see that when $i \leq i_a\,(j_a + 1)$ there is one pattern, and when $i > i_a\,(j_a + 1)$ there is another pattern. Therefore, when $i \leq i_a\,(j_a + 1)$ one arrives at the relationships

$$(p - 1)\, j_a + p \leq i \leq p(j_a + 1) \rightarrow g = p \quad p \in \mathbb{N}$$

$$h = \begin{cases} j_a & \text{if } mod(i, j_a + 1) = 0 \\ mod(i, j_a + 1) - 1 & \text{otherwise} \end{cases}$$

and when $i > i_a\left(j_a+1\right)$ one arrives at the similar relations

$$p \cdot j_b + 1 \leq i - i_a\left(j_a+1\right) \leq \left(p+1\right) j_b \rightarrow g = p \quad p \in \mathbb{Z}^+$$

$$h = \begin{cases} j_b & \text{if } mod(i - i_a\left(j_a+1\right), j_b) = 0 \\ mod(i - i_a\left(j_a+1\right), j_b) & \text{otherwise} \end{cases}$$

Now that the pattern has been established one can use this pattern inside a set of "Loops" to produce $D\Phi^{(n)}$ and $\Phi^{(n)}$.

To initialize $D\Phi$ one needs to translate the pattern as described in Table 1 into an easy way of taking those derivatives. The derivative will depend on which variable and which function one is working with. This will be determined by both the row and column of the matrix. Where the row indicates the function to be used and the column determines the which variables derivative to take. In general one has:

Section I

$$\frac{\partial \phi_{g,h}}{\partial A_{m,n}} = \begin{cases} \varepsilon\left(h\right) a\pi^2 \cdot \left(g^2 + \frac{h^2}{\xi^2}\right) \xi & \text{if } m = g \text{ and } n = h \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

Section II

$$\frac{\partial \phi_{g,h}}{\partial B_{r,s}} = \begin{cases} -h \cdot N\left(g,r\right) & \text{if } r \neq g \text{ and } s = h \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Section III

$$\frac{\partial \gamma_{g,h}}{\partial A_{m,n}} = \begin{cases} -g \cdot N\left(h,n\right) - \frac{\pi}{4} \sum_{r=0}^{i_b} \sum_{s=1}^{j_b} B_{r,s} \cdot \left(gsLR - nrFG\right) & \text{if } m = g \text{ and } n \neq h \\ -\frac{\pi}{4} \sum_{r=0}^{i_b} \sum_{s=1}^{j_b} B_{r,s} \cdot \left(msLR - nrFG\right) & \text{otherwise} \end{cases} \tag{20}$$

Section IV

$$\frac{\partial \gamma_{g,h}}{\partial B_{r,s}} = \begin{cases} \varepsilon\left(g\right) b\pi^2 \cdot \left(g^2 + \frac{h^2}{\xi^2}\right) \xi - \frac{\pi}{4} \sum_{m=1}^{i_a} \sum_{n=0}^{j_a} A_{m,n} \cdot \left(mhLR - ngFG\right) & \text{if } r = g \text{ and } s = h \\ -s \cdot N\left(h,s\right) - \frac{\pi}{4} \sum_{m=1}^{i_a} \sum_{n=0}^{j_a} A_{m,n} \cdot \left(msLR - ngFG\right) & \text{if } r = g \text{ and } s \neq h \\ -\frac{\pi}{4} \sum_{m=1}^{i_a} \sum_{n=0}^{j_a} A_{m,n} \cdot \left(mhLR - nrFG\right) & \text{if } r \neq g \text{ and } s = h \\ -\frac{\pi}{4} \sum_{m=1}^{i_a} \sum_{n=0}^{j_a} A_{m,n} \cdot \left(msLR - nrFG\right) \text{otherwise} & \end{cases} \tag{21}$$

4 Application

A Multi-Variate Newton-Raphson's Method was used to solve the system of non-linear equations. An initial guess is however required to begin Newton's

Method. It was noticed that a guess of 0 for A and B could be made and still have convergence at a value of $b = 0.1$. However, for lower values of b, Newton's method would no longer converge using an initial guess of 0, therefore for $b < 0.1$ an initial guess was obtained from the solution for $b = 0.1$. This method worked for values of $b \geq 6 \times 10^{-3}$.

To determine the appropriate number of Fourier coefficients to be used, a base case for $a = 0.263,\ b = 0.1,$ and $\xi = 2.0$ with $i_a = i_b = j_a = j_b = 15$ or 480 Fourier coefficients was compared to the solution for which $i_a = i_b = j_a = j_b =$ 20 or 840 Fourier coefficients. The i and j indices were again increased by five each for a total number of 1300 Fourier coefficients and compared to the solution for the previous solution for 840 coefficients. This process was repeated until little change was seen between the previous solution and the new solution. Then the final number of terms would be used as the preferred number of terms. This process resulted in a total number of 1860 Fourier coefficients being sufficient for proper convergence. The increase from one set of coefficients to another resulted mostly in a change in the upper 20% of the problem domain.

Once the total number of Fourier coefficients has been determined, a simulation for $b = 0.05$ was performed at the specified number of Fourier coefficients. It was then desired to find out how low the Newton's Method could go with respect to b. An iterative process was used for $b < 0.05$, whereby every time Newton's Method converged for a value of b a new value of b smaller than the previous was used for attempting convergence. Using the initial guess obtained by the solution for $b = 0.1$, Newton's Method was able to converge down to a value of $b = 6 \times 10^{-3}$. Solutions for values of $b < 6 \times 10^{-3}$ were attempted using the solution obtained by $b = 6 \times 10^{-3}$, but results were difficult to obtain and were no longer consistent in terms of the expected results. It would most likely be possible to obtain more consistent solutions for values of $b < 6 \times 10^{-3}$, if one were to use more Fourier coefficients, but due to time constraints this was not tested. Therefore, only the results for $b = 0.1$, 0.05, and $6 \times 10^{-3}$are presented.

The results above were compared to that Henry's Method and SUTRA. Henry's Method requires one to assume a value for the non-linear terms in equation (**??**), i.e. $\frac{\pi}{4} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} A_{m,n} B_{r,s} \left(msLR - nrFG\right) = const.$ This essentially linearizes the system of equations. Initially the non-linear terms were assumed to be zero. After each iteration the non-linear terms are then updated using the new estimated values of A and B. The resulting linear equations are such that the equations for $B_{0,h}$ are linearly independent of $A_{g,h}$ and $B_{g>0,h}$. In addition to the non-linear terms being treated as constants, Henry also treated the linear terms involving only $A_{g,h}$ in equation (**??**) as a known value, i.e. $\sum_{n=0}^{\infty} A_{g,n} \cdot g \cdot N\left(h, n\right) = const.$ The value of these terms were obtained using previous iteration of $B_{g>0,h}$ (initially this was assumed to be zero), and the current iteration of $B_{0,h}$ substituted into equation (**??**). Once the non-linear and linear terms had been estimated the resulting system of linear equations involving $B_{g>0,h}$ were then solved. The newly obtained values for $B_{g>0,h}$ were then used to update the non-linear terms. When a complete set of B's and A's were evaluated, a new set of C's and ψ's are then evaluated. The process as described above was repeated until the error was less than some

epsilon. Where the error was defined to be

$$error = MAX\left(\left\|C_{new} - C_{old}\right\|_{\infty}, \left\|\psi_{new} - \psi_{old}\right\|_{\infty}\right) \tag{22}$$

and $epsilon = 10^{-4}$. In this analysis the Fourier series was truncated in the fashion described by Ségol [4];

$A_{1,0}$ through $A_{1,15}$ $A_{2,1}$ through $A_{2,10}$ $A_{3,0}$ through $A_{3,5}$ $A_{4,1}$ through $A_{4,3}$
$A_{5,0}$ through $A_{5,2}$ $B_{0,1}$ through $B_{0,20}$ $B_{1,1}$ through $B_{1,10}$ $B_{2,1}$ through $B_{2,5}$
$B_{3,1}$ through $B_{3,3}$ $B_{4,1}$ through $B_{4,2}$

This truncation results in a total of 78 Fourier coefficients.

The above method was used to solve for $b = 0.1$ with an initial guess, for the nonlinear terms, of zero. Once this solution was obtained the resulting values for the Fourier coefficients were used to obtain an initial guess, for $b = 0.09$. The same procedure was then used to obtain a solution for $b = 0.08$ using the solution for $b = 0.09$. This was repeated until a solution for $b = 0.05$ was obtained.

For SUTRA a scheme that would model Henry's Solution for $a = 0.263$ and $b =$ 0.1, 0.05, and 6×10^{-3} needed to be established, before comparing results. In previous papers such as Voss and Souza [6], comparisons using Henry's problem and SUTRA have never seemed to use consistent boundary conditions. That is, Henry's problem uses a fixed boundary condition for concentration on the seaward side, but many comparison of SUTRA to Henry's solution did not hold concentration constant on the seaward boundary. This results in both a difference in concentrations for the upper 20% of the solution domain and a differing position for the seawater lens. Thus, using such a comparison will not result in similar solution, and should not be used as a comparison, unless a comparison of shape is wanted for the bottom 80%. However, in the SUTRA code one can change the boundary condition, so that the seawater side has a constant concentration, and so this feature of SUTRA will be used in the comparison that follows.

Along with boundary condition, one must also match dispersion models. Henry used a simplified model of dispersion, where the total dispersion is not dependent on velocity, SUTRA, on the other hand, uses a total dispersion that is the sum of the velocity dependent dispersion and molecular diffusivity multiplied by porosity. To be able to compare a SUTRA model to Henry's model one would set the dispersivity to zero and use an artificially large molecular diffusivity.

With both the boundary conditions and dispersion models aligned correctly, one can now proceed to matching parameter distributions. Henry's problem uses parameters such as a and b, whereas SUTRA uses parameters such as freshwater recharge (Q), porosity (ε), hydraulic conductivity (k), seawater density (ρ_s), freshwater density (ρ_0), salt concentrations (C_s), and the dispersion coefficient (D), and so it is necessary to determine the parameter values that will represent the Henry's non-dimensional parameters a and b. The parameter values to be used, in the SUTRA model, will be similar to that of Voss and Souza [6], and are:

$\varepsilon = 0.35$
$C_s = 0.0357 \left[\frac{\text{kg(dissolved solids)}}{\text{kg(seawater)}}\right]$
$\rho_s = 1024.99 \text{ kg/m}^3$
$\frac{\partial \rho}{\partial C} = 700 \left[\frac{\text{kg(seawater)}^2}{\text{(kg(dissolved solids) m}^3\text{)}}\right]$
$\rho_0 = 1000 \text{ kg/m}^3$
$Q = 6.6 \times 10^{-2} \text{ kg/s}$
$k = 1.024698 \times 10^9$ (Voss and Souza used $k = 1.020408 \times 10^9$)
$|g| = 9.8 \text{ m/s}^2$
$\alpha_L = \alpha_T = 0$
$B = 1.0 \text{ m}$
$D_m = 18.85714 \times 10^{-6} \text{ m/s}^2$ for $b = 0.1$
$D_m = 9.428571 \times 10^{-6} \text{ m/s}^2$ for $b = 0.05$
$D_m = 1.131429 \times 10^{-6} \text{ m/s}^2$ for $b = 6 \times 10^{-3}$
$C_0 = 0$ (concentration of freshwater)

The hydraulic conductivity differs from Voss and Souza, because the values used by Voss and Souza ($k = 1.020408 \times 10^{-2}$ kg/s) did not seem to result in the appropriate values of a and b, for all other given parameter values. As one can see three different values of molecular diffusivity were used. These three different values were used to match the three different values of$b$that were used in Henry's problem. No other parameter values were changed during the simulation, since changing molecular diffusivity is sufficient to match the different values of b. Total simulated time and total number of nodes for the SUTRA simulation were the same as that used by Voss and Souza [6], 100 minutes and a total 231 nodes.

5 Results

All model comparisons used $a = 0.263$ and $\xi = 2$, while changing b. The non-dimensional variable$b$was varied over a, because "Henry's solution is very sensitive to changes in b" (Ségol 1994). The first comparison (Figure 2) was done for $b = 0.1$. As one can see the contours obtained by Henry's Method are not very smooth, and may be the reason why model comparisons using this method were unable to obtain a close match. However, a close match for the 0.5 isochlor is seen between Newton's Method and SUTRA. For the 0.9 isochlor, Newton's Method and SUTRA contours are almost indistinguishable. As one proceeds from the 0.9 isochlor to the 0.1 isochlor the contours of Newton's Method and SUTRA become further and further apart. This seems to be caused by the instabilities in the upper 20% of the solution domain for Newton's Method. These instabilities were also seen by Ségol [4], when using Henry's Method, so it appears that this is an inherent trait in Henry's problem. However, these instabilities seem to be reduced by increasing the total number of Fourier coefficients. It therefore may be possible to eliminate such instabilities with a greater number of Fourier coefficients, but due to time constraints a maximum of 1860 Fourier coefficients were used.

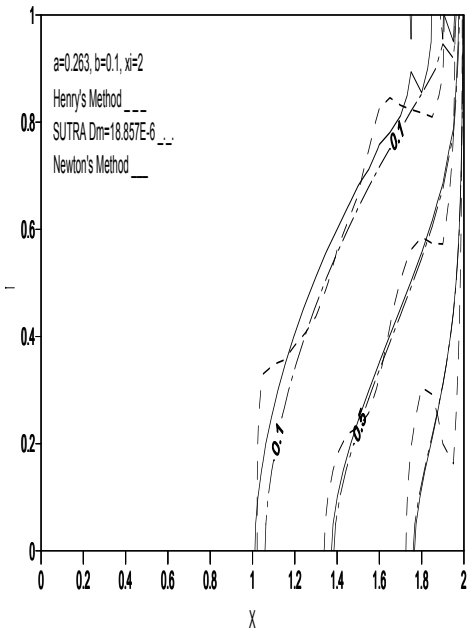

Figure 2: Isochlor concentrations for $a = 0.263$ and $b = 0.1$

The second comparison (Figure 3) used a value of 0.05 for b, and as one can see an increase in instabilities in the upper 20% of the solution domain resulted. As for Henry's Method, instabilities were so great that the 0.1 isochlor contour shows up near $x = 0.2$ m. Once again the 0.9 isochlor contours for Newton's Method and SUTRA match very well, and so does the 0.5 isochlors, but the 0.1 isochlor contours differ by a noticeable amount in their position. Again it appears that as the concentration decreases from 0.9 to 0.1 the isochlor contours begin to differ by greater amounts. As one can see from **??**, Newton's Method and SUTRA do not agree as well as that of the previous comparison for $b = 0.1$.

The third comparison (Figure 4), for which $b = 6 \times 10^{-3}$, resulted in instabilities that created negative and 0.1 isochlors across the entire upper 20% of the solution domain for Newton's Method. Similar instabilities can be seen in the SUTRA solution for 231 nodes. One can see in the upper right hand corner the similar pattern that is also present in the Newton's Method. The resulting 0.1 and 0.5 isochlors no longer match well. However the 0.9 isochlor is still very close. But, as one can see the lower values of b create greater and greater instabilities in the upper 20% of the solution domain. These instabilities then further degrade the accuracy of the isochlor contours in the lower 80% of the solution domain. It is likely that a larger number of Fourier coefficients could correct these problems, but due to a lack of computing power and time this was unable to be checked. Henry's Method is not represented in this comparison,

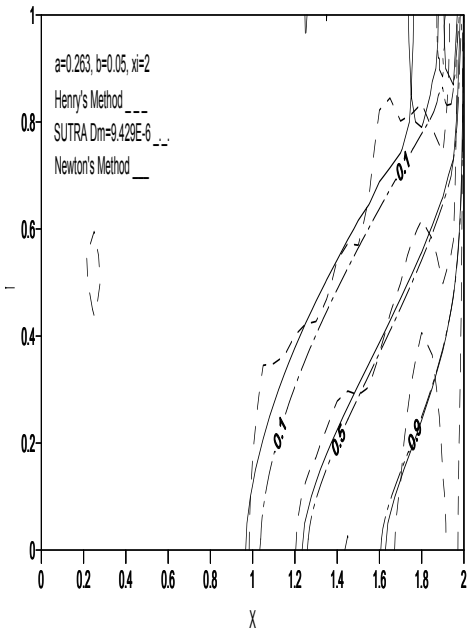

Figure 3: Isochlor concentrations fo $a = 0.263$ and $b = 0.05$

due to the inability of this method to converge for values of b lower than 0.05. It is believed that the use of more Fourier coefficients would result in better agreement between SUTRA and Henry's Semi-Analytic Solution.

6 Conclusions

Using Newton's Method to evaluate Henry's problem results in the ability to use a greater number of Fourier coefficients than that of Henry's Method. This allows for an increase in the accuracy of the solution. The increase in solution accuracy further allows for simulations of narrower zones of dispersion (small values of b). The lower values of b correspond to lower amounts of dispersion, and therefore address the concerns stated by Voss and Souza [6], in which they say, "because of the unrealistic large amount of dispersion introduced in the solution by the constant total dispersion coefficient, this test does not check whether a model is consistent or whether it accurately represents density driven flows, nor does it check whether a model can represent field situations with relatively narrow transition zones."

It appears that a further increase in the total number of Fourier coefficients may be necessary to address the issue of instabilities created when evaluating Henry's Problem for low values of b. Due to time constraints the number of Fourier coefficients for this analysis was limited to 1860 coefficients. It is believed

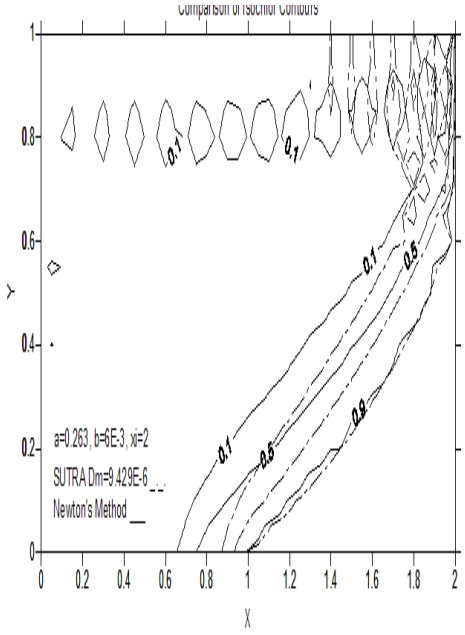

Figure 4: Isochlor concentrations for $a = 0.263$ and $b = 6 \times 10^{-3}$

that the number of Fourier coefficients to quell the instabilities may be in excess of 5100 coefficients. The majority of time needed for each iteration was in the evaluation of the Jacobian matrix $D\Phi$. A quicker algorithm for calculating the total derivative function may be able to be developed, and therefore it may be possible to evaluate Henry's Problem using a greater number of Fourier coefficients in a more timely fashion.

In addition to developing a more efficient algorithm for the total derivative function, it might be necessary to use a more robust numerical method than that of Newton's Method. If a more robust numerical method is used it may be able to address the instabilities in the upper 20% of the solution domain, without a further increase in the number of Fourier coefficients used in the evaluation of Henry's Problem. It is clear that using Newton's Method over that of Henry's Method increases the stability of Henry's Problem. This increase in stability resulted in the ability to use a larger number of Fourier coefficients, and therefore an increase in the accuracy of the solution. The increase in solution accuracy then resulted in the ability to evaluate Henry's Problem for values of $b < 0.05$. Using an even more robust numerical method may have similar results.

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