1. Use the Intermediate Value Theorem to show that there exist a root of the given equation in the specified interval.

(a) $\cos(x) = x$ for $x \in (0, 1)$

(b) $\sqrt[3]{x} = 1 - x$ for $x \in (0, 1)$

(a) Let $f(x) = \cos x - x$ since $\cos x$ and x are both continuous, $f(x)$ is continuous.

$$f(0) = \cos 0 - 0 = 1$$
$$f(1) = \cos 1 - 1 \approx -0.46$$

Thus, since $f(x)$ changes sign in the interval $(0, 1)$ and is continuous in $(0, 1)$, by the IVT there exists a $c \in (0, 1)$ such that $f(c) = 0$. Therefore, there exists an $x \in (0, 1)$ such that $\cos x = x$. ∎

(b) Let $f(x) = \sqrt[3]{x} + x - 1$ since $\sqrt[3]{x}$ and $x - 1$ are both continuous, $f(x)$ is continuous.

$$f(0) = \sqrt[3]{0} + 0 - 1 = -1$$
$$f(1) = \sqrt[3]{1} + 1 - 1 = 1.$$

Thus, since $f(x)$ changes sign in the interval $(0, 1)$ and is continuous in $(0, 1)$, by the IVT there exists a $c \in (0, 1)$ such that $f(c) = 0$. Therefore, there exists an $x \in (0, 1)$ such that $\sqrt[3]{x} = 1 - x$. ∎

2. Is there a real number that is exactly 1 more that its cube? Why?

The question is asking if there is an x such that $x^3 + 1 = x$. Let $f(x) = x^3 - x + 1$, which is continuous since it is a polynomial. Looking at the interval $[-2, 1]$ we see that

$$f(-2) = -2^3 + 2 + 1 = -5$$
$$f(0) = 0^3 - 0 + 1 = 1.$$

Thus, since $f(x)$ changes sign in the interval $[-2, 0]$ and is continuous in $[-2, 0]$, by the IVT there exists a $c \in [-2, 0]$ such that $f(c) = 0$. Therefore, there exists an $x \in [-2, 0]$ such that $x^3 + 1 = x$. ∎

3. Find the slope of the tangent line to

$$f(x) = \sqrt{x}$$

at the point $x_0 = 1$ and then determine the equation of the tangent line.

Using the definition of the slope of a curve at a point we see

$$
\begin{aligned}
m &= \lim_{h\to 0} \frac{\sqrt{x_0+h}-\sqrt{x_0}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x_0+h}-\sqrt{x_0}}{h} \frac{\sqrt{x_0+h}+\sqrt{x_0}}{\sqrt{x_0+h}+\sqrt{x_0}} \\
&= \lim_{h\to 0} \frac{x_0+h-x_0}{h\left(\sqrt{x_0+h}+\sqrt{x_0}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x_0+h}+\sqrt{x_0}} \\
&= \frac{1}{2\sqrt{x_0}} \\
&= \frac{1}{2\sqrt{1}} \\
&= \boxed{\frac{1}{2}.}
\end{aligned}
$$

Now using the point-slope form at the point $(1, 1)$ we have

$$
\begin{aligned}
y - 1 &= \frac{1}{2}(x-1) \\
y &= \frac{1}{2}x - \frac{1}{2} + 1 \\
&\boxed{y = \frac{1}{2}x + \frac{1}{2}.}
\end{aligned}
$$

■

4. Using the definition of the derivative find the derivative of

$$f(x) = \sin x$$

Hint: Use the relation $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right) \cdot \sin\left(\frac{x-y}{2}\right)$

Using the definition of the derivative we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\cdot \sin\left(\frac{h}{2}\right)}{h} \\
&= \lim_{h\to 0} \cos\left(\frac{2x+h}{2}\right) \cdot \cancelto{1}{\lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}} \\
&= \boxed{\cos x.}
\end{aligned}
$$

■

5. Suppose $f(x)$ is given by
$$f(x) = \begin{cases} x^3 + x - 1 & x < 0 \\ x^2 - 1 & 0 \le x \le 1 \\ x^3 - 3x^2 + 3x & x > 1 \end{cases}.$$

(a) Does $f'(0)$ exist? If so, what is it? If not, why not? Explain!

(b) Does $f'(1)$ exist? If so, what is it? If not, why not? Explain!

(a) For the derivative to exist at $x = 0$ we must check if the right hand and left hand derivatives are equal at the point $x = 0$. Thus, the left hand derivative is

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(x_0+h) - f(x_0)}{h} &= \lim_{h\to 0^-} \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{h^3 + h - 1 - 0^2 + 1}{h} \\ &= \lim_{h\to 0^-} \frac{h^3 + h}{h} \\ &= \lim_{h\to 0^-} h^2 + 1 \\ &= 1 \end{aligned}$$

and the right hand derivative is

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h} &= \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{(0+h)^2 - 1 - 0^2 + 1}{h} \\ &= \lim_{h\to 0^+} \frac{h^2}{h} \\ &= \lim_{h\to 0^+} h \\ &= 0. \end{aligned}$$

Therefore, the left and right hand derivatives are not equal at $x = 0$ and so $\boxed{f'(0) \text{ does not exist.}}$ ■

(b) Again, for the derivative to exist at $x = 1$ we must check if the right hand and left hand derivatives are equal at the point $x = 1$. Thus, the left hand derivative is

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} &= \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0^-} \frac{(1+h)^2-1-1^2+1}{h} \\
&= \lim_{h\to 0^-} \frac{1+2h+h^2-1}{h} \\
&= \lim_{h\to 0^-} \frac{2h+h^2}{h} \\
&= \lim_{h\to 0^-} 2+h \\
&= 2
\end{aligned}$$

and the right hand derivative is

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} &= \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0^-} \frac{(1+h)^3-3(1+h)^2+3(1+h)-1^2+1}{h} \\
&= \lim_{h\to 0^+} \frac{1+3h+3h^2+h^3-3-6h-3h^2+3+3h}{h} \\
&= \lim_{h\to 0^+} \frac{1+h^3}{h} \\
&= \lim_{h\to 0^+} \frac{1}{h} + \lim_{h\to 0^+} h^2 \\
&= +\infty
\end{aligned}$$

Therefore, the left and right hand derivatives are not equal at $x = 0$ and, in fact the right hand derivative doesn't exist since it approaches $+\infty$ when we approach $x = 1$ from the right . Therefore, $\boxed{f'(0) \text{ does not exist.}}$ ■