1.

(a) §2.4 : 49.

(b) §2.4 : 50.

(c) §2.4 : 74.

(a)

$$\begin{aligned}
\lim_{x\to-\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}} &= \lim_{x\to-\infty}\frac{e^x\left(e^x-e^{-x}\right)}{e^x\left(e^x+e^{-x}\right)}\\
&= \lim_{x\to-\infty}\frac{e^{2x}-1}{e^{2x}+1}\\
&= \frac{0-1}{0+1}\\
&= \boxed{-1}.
\end{aligned}$$

■

(b)

$$\begin{aligned}
\lim_{x\to\infty}\frac{3x^2+e^{-x}}{\sin\frac{1}{x}-2x^2} &= \lim_{x\to\infty}\frac{\frac{1}{x^2}\left(3x^2+e^{-x}\right)}{\frac{1}{x^2}\left(\sin\frac{1}{x}-2x^2\right)}\\
&= \lim_{x\to\infty}\frac{3+\frac{e^{-x}}{x^2}}{\frac{\sin\frac{1}{x}}{x^2}-2}\\
&= \frac{3+0}{0-2}\\
&= \boxed{-\frac{3}{2}}
\end{aligned}$$

■

(c)

$$\begin{aligned}
\lim_{x\to\infty}\sqrt{x^2+x}-\sqrt{x^2-x} &= \lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\\
&= \lim_{x\to\infty}\frac{x^2+x-x^2+x}{\sqrt{x^2+x}+\sqrt{x^2-x}}\\
&= \lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}\\
&= \lim_{x\to\infty}\frac{\frac{1}{x}2x}{\frac{1}{x}\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)}\\
&= \lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}}\\
&= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}\\
&= \boxed{1}
\end{aligned}$$

■

2. §2.5 : 51.

(a) We say that $f(x)$ approaches **infinity** as x approaches x_0 from the **left**, and write

$$\lim_{x \to x_0^-} f(x) = \infty,$$

if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad f(x) > B.$$

■

(b) We say that $f(x)$ approaches **negative infinity** as x approaches x_0 from the right, and write

$$\lim_{x \to x_0^+} f(x) = -\infty,$$

if, for every **negative** real number B, there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad f(x) < B.$$

■

(c) We say that $f(x)$ approaches **negative infinity** as x approaches x_0 from the **left**, and write

$$\lim_{x \to x_0^-} f(x) = -\infty,$$

if, for every **negative** real number B, there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad f(x) < B.$$

■

3. Find any and all vertical and horizontal asymptotes of the following functions. Additionally, be sure to describe the behavior at the vertical asymptotes, i.e. evaluate the limits at those asymptotes.

(a) $\dfrac{x^2 - 3x + 2}{x^2 - 2x}$

(b) $\dfrac{3}{2}\left(\dfrac{x}{x-1}\right)^{2/3}$

(a) **Horizontal Asymptotes:**

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2-3x+2}{x^2-2x} &= \lim_{x\to\infty} \frac{\frac{1}{x^2}\left(x^2-3x+2\right)}{\frac{1}{x^2}\left(x^2-2x\right)} \\
&= \lim_{x\to\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{1-\frac{2}{x}} \\
&= 1 \\
\lim_{x\to-\infty} \frac{x^2-3x+2}{x^2-2x} &= \lim_{x\to-\infty} \frac{\frac{1}{x^2}\left(x^2-3x+2\right)}{\frac{1}{x^2}\left(x^2-2x\right)} \\
&= \lim_{x\to-\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{1-\frac{2}{x}} \\
&= 1
\end{aligned}$$

Therefore the horizontal asymptote is $\boxed{y=1}$.
Vertical Asymptotes: We are looking for x values which result in the function becoming undefined. In this case we are looking for x values that result in division by zero.

$$\begin{aligned}
\frac{x^2-3x+2}{x^2-2x} &= \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}x} \\
&= \frac{x-1}{x} \quad \text{for } x \neq 2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lim_{x\to 0^-} \frac{x^2-3x+2}{x^2-2x} &= \lim_{x\to 0^-} \frac{\overset{(-)}{x-1}}{\underset{(-)}{x}} \\
&= \infty
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lim_{x\to 0^+} \frac{x^2-3x+2}{x^2-2x} &= \lim_{x\to 0^+} \frac{\overset{(-)}{x-1}}{\underset{(+)}{x}} \\
&= -\infty
\end{aligned}$$

Therefore, the vertical asymptote is $\boxed{y=0}$. ■

(b) **Horizontal Asymptotes:**

$$\begin{aligned}
\lim_{x\to\infty}\frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3} &= \frac{3}{2}\left(\lim_{x\to\infty}\frac{\frac{1}{x}(x)}{\frac{1}{x}(x-1)}\right)^{2/3}\\
&= \frac{3}{2}\left(\lim_{x\to\infty}\frac{1}{1-\frac{1}{x}}\right)^{2/3}\\
&= \frac{3}{2}(1)^{2/3}\\
&= \frac{3}{2}\\
\lim_{x\to-\infty}\frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3} &= \frac{3}{2}\left(\lim_{x\to-\infty}\frac{\frac{1}{x}(x)}{\frac{1}{x}(x-1)}\right)^{2/3}\\
&= \frac{3}{2}\left(\lim_{x\to-\infty}\frac{1}{1-\frac{1}{x}}\right)^{2/3}\\
&= \frac{3}{2}(1)^{2/3}\\
&= \frac{3}{2}
\end{aligned}$$

Therefore the horizontal asymptote is $\boxed{y=\frac{3}{2}}$.

Vertical Asymptotes: We are looking for x values which result in the function becoming undefined. In this case we are looking for x values that result in division by zero.

$$\begin{aligned}
\Rightarrow \lim_{x\to1^-}\frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3} &= \frac{3}{2}\left(\lim_{x\to1^-}\frac{\overset{(+)}{x}}{\underset{(-)}{x-1}}\right)^{2/3}\\
&= \frac{3}{2}(-\infty)^{2/3}\\
&= \infty\\
\Rightarrow \lim_{x\to1^+}\frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3} &= \frac{3}{2}\left(\lim_{x\to1^+}\frac{\overset{(+)}{x}}{\underset{(+)}{x-1}}\right)^{2/3}\\
&= \frac{3}{2}(\infty)^{2/3}\\
&= \infty
\end{aligned}$$

Therefore, the vertical asymptote is $\boxed{y=1}$. ∎

4. For what value(s) of b is the function g continuous on $(-\infty,\infty)$?

$$g(x)=\begin{cases}-4x+1 & x\le b\\ x^2-2x+2 & x>b\end{cases}$$

For this function to be continuous we want the limit to exist and we want

$$\lim_{x \to b} f(x) = f(b).$$

Let's start with the existence of the limit. For a limit to exist we want

$$\begin{aligned}
\lim_{x \to b^-} f(x) &= \lim_{x \to b^+} f(x) \\
\lim_{x \to b^-} -4x + 1 &= \lim_{x \to b^+} x^2 - 2x + 2 \\
-4b + 1 &= b^2 - 2b + 2 \\
b^2 + 2b + 1 = 0 &\Rightarrow \boxed{b = -1}
\end{aligned}$$

Since, $-4x + 1 = x^2 - 2x + 2$ at $x = b = -1$ we also know that the limit is equal to the function value at $x = b = -1$. Therefore, the function is continuous when $b = -1$. ∎