1. Find the derivative and second derivative of the following

(a) $g(x) = \frac{1+\sec(x)}{1-\sec(x)}$

(b) $h(x) = \sin^2 x$

(a)

$$\begin{aligned}
g'(x) &= \frac{\frac{d}{dx}\left[1+\sec x\right](1-\sec x) - \frac{d}{dx}\left[1-\sec x\right](1+\sec x}{(1-\sec x)^2} \\
&= \frac{\sec x \tan x(1-\sec x) + \sec x \tan x(1+\sec x)}{(1-\sec x)^2} \\
&= \boxed{\frac{2\sec x \tan x}{(1-\sec x)^2}} \\
g''(x) &= \frac{\frac{d}{dx}\left[2\sec x \tan x\right](1-\sec x)^2 - \frac{d}{d(1-\sec x)}\left[(1-\sec x)^2\right]\frac{d}{dx}\left[1-\sec x\right](2\sec x\tan x)}{(1-\sec x)^4} \\
&= \frac{2\left(\frac{d}{dx}\left[\sec x\right]\tan x + \frac{d}{dx}\left[\tan x\right]\sec x\right)(1-\sec x)^2 - 2(1-\sec x)(\sec x\tan x)(2\sec x \tan x)}{(1-\sec x)^4} \\
&= \boxed{\frac{2\left(\sec x\tan^2 x + \sec^3 x\right)(1-\sec x)^2 - 2(1-\sec x)(\sec x\tan x)(2\sec x\tan x)}{(1-\sec x)^4}}
\end{aligned}$$

■

(b)

$$\begin{aligned}
h'(x) &= \frac{d}{d(\sin x)}\left[(\sin x)^2\right]\frac{d}{dx}\left[\sin x\right] \\
&= 2\sin x\cos x \\
&= \boxed{\sin 2x} \\
h''(x) &= \frac{d}{d(2x)}\left[\sin 2x\right]\frac{d}{dx}(2x) \\
&= (\cos 2x)2 \\
&= \boxed{2\cos 2x}
\end{aligned}$$

■

2. Suppose that an object moves back and forth according to the function

$$f(t) = t^3 + bt^2 + ct + d, \quad f(0) = 1,\ f'(0) = 0, \text{ and } f''(0) = 3.$$

(a) Using the information above find $f(t)$.

(b) When is the object at rest?

(c) When is the object moving forward? Moving backward?

(d) When is the object accelerating?

(e) How far did the object travel (counting retraces!) between $t = 0$ and $t = 8$?

(a)

$$
\begin{aligned}
&f(0) = d = 1 \\
&f'(t) = 3t^2 + 2bt + c \\
&f'(0) = c = 0 \\
&f''(t) = 6t + 2b \\
&f''(0) = 2b = 3 \Rightarrow b = \frac{3}{2} \\
\Rightarrow &\boxed{f(t) = t^3 + \frac{3}{2}t^2 + 1}
\end{aligned}
$$

$\blacksquare$

(b) The object is at rest when $f'(t)$ (velocity) is zero, i.e.

$$
\begin{aligned}
&f'(t) = 3t^2 + 3t = 0 \\
&3t(t+1) = 0 \\
&\boxed{t = 0, \quad t = -1}
\end{aligned}
$$

$\blacksquare$

(c) An object is moving forward when $f'(t) > 0$ and backward when $f'(t) < 0$, so give the zeros we found in (b) we can draw the following number line

$$(+) \quad -1 \quad (-) \quad 0 \quad (+) \qquad f'(t)$$

From the above number line we see that the particle is moving forward when $\boxed{t \in (-\infty, -1) \bigcup (0, \infty)}$ and moving backward when $\boxed{t \in (-1, 0)}$. $\blacksquare$

(d) The object is accelerating when $f''(t) > 0$, i.e.

$$
\begin{aligned}
f''(t) = 6t + 3 &> 0 \\
6t &> -3 \\
&\boxed{t > -\frac{1}{2}}
\end{aligned}
$$

Note: On a more complicated problem a number line could be used. $\blacksquare$

(e) Since the object is only moving forward in the given interval we can use the displacement formula

$$
\begin{aligned}
\Delta f &= f(8) - f(0) \\
&= 8^3 + \frac{3}{2}8^2 + 1 - 0^3 - \frac{3}{2}0^2 - 1 \\
&= 512 + \frac{3}{2} \cdot 64 \\
&= \boxed{608}
\end{aligned}
$$

$\blacksquare$

3. Suppose $f(u) = \cos(u)$ and $g(t) = 3t^4$. Using chain rule, compute:

(a) $(f \circ g)'(t)$

(b) $(g \circ f)'(u)$

(c) $(g \circ g)'(t)$

(d) $(f \circ f)'(u)$

(a)

$$\begin{aligned}(f \circ g)'(t) &= f'(g(t))g'(t)\\ &= \boxed{-\sin(3t^4) \cdot 12t^3}\end{aligned}$$

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(b)

$$\begin{aligned}(g \circ f)'(u) &= g'(f(u))f'(u)\\ &= \boxed{-12\cos^3 u \sin u}\end{aligned}$$

■

(c)

$$\begin{aligned}(g \circ g)'(t) &= g'(g(t))g'(t)\\ &= 12(3t^4)^3 12t^3\\ &= \boxed{3888t^{15}}\end{aligned}$$

■

(d)

$$\begin{aligned}(f \circ f)'(u) &= f'(f(u))f'(u)\\ &= \boxed{\sin(\cos u)\sin u}\end{aligned}$$

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4. Given that f and g are both differential functions find the derivative of the following

(a) $((f \cdot g) \circ f)(t)$

(b) $\left(f \circ \left(\frac{f}{g}\right)\right)(x)$

(a)

$$\begin{aligned}\frac{d}{dt}[((f \cdot g) \circ f)(t)] &= \frac{d}{dt}[f(f(t)) \cdot g(f(t))\\ &= \frac{d}{dt}[f(f(t))]g(f(t)) + \frac{d}{dt}[g(f(t))]f(f(t))\\ &= \boxed{f'(f(t))f'(t)g(f(t)) + g'(f(t))f'(t)f(f(t))}\end{aligned}$$

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(b)

$$\begin{aligned}
\frac{d}{dx}\left[\left(f \circ \left(\frac{f}{g}\right)\right)(x)\right] &= \frac{d}{dx}\left[f\left(\frac{f(x)}{g(x)}\right)\right] \\
&= f'\left(\frac{f(x)}{g(x)}\right)\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \\
&= \boxed{f'\left(\frac{f(x)}{g(x)}\right)\left[\frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}\right]}
\end{aligned}$$

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