1.

(a) Evaluate $\lim_{x\to 2} \dfrac{x-2}{\sqrt{3-x}-1}$.

(b) Is there any number a such that

$$\lim_{x\to -2} \frac{3x^2+ax+a+3}{x^2+x-2}$$

exists? If so, find the value a and the value of the limit.

(a)

$$\begin{aligned}
\lim_{x\to 2} \frac{x-2}{\sqrt{3-x}-1} &= \lim_{x\to 2} \frac{x-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} \\
&= \lim_{x\to 2} \frac{(x-2)\left(\sqrt{3-x}+1\right)}{3-x-1} \\
&= \lim_{x\to 2} \frac{(x-2)\left(\sqrt{3-x}+1\right)}{-(x-2)} \\
&= -\lim_{x\to 2} \sqrt{3-x}+1 \\
&= -\sqrt{3-2}+1 \\
&= \boxed{-2.}
\end{aligned}$$

■

(b) For this limit to exist we must require the limit be finite, i.e.

$$\lim_{x\to -2} \frac{3x^2+ax+a+3}{x^2+x-2} < \infty$$

The easiest way to insure this is to make sure the factor $x+2$ in the denominator is canceled by a similar factor in the numerator. If we let the second factor in the numerator be $x+b$ then

$$\begin{aligned}
3\left(x^2+\frac{a}{3}x+\frac{a}{3}+1\right) &= 3(x+2)(x+b) \\
&= 3(x^2+(2+b)x+2b)
\end{aligned}$$

and therefore we must have

$$\begin{aligned}
(2+b) &= \frac{a}{3} \\
2b &= \frac{a}{3}+1.
\end{aligned}$$

Fom here we can solve for b and a

$$\begin{aligned}
b &= \frac{a+3}{6} \\
\Rightarrow 2+\frac{a+3}{6} &= \frac{a}{3} \\
\frac{a+3}{6}-\frac{a}{3} &= -2 \\
a+3-2a &= -12 \\
a &= \boxed{15.} \Rightarrow b = 3
\end{aligned}$$

Now we just find the limit

$$\begin{aligned}\lim_{x\to-2}\frac{3(x+2)(x+3)}{(x+2)(x-1)} &= \lim_{x\to-2}\frac{3(x+3)}{x-1}\\ &= \frac{3(-2+3)}{-2-1}\\ &= \boxed{1.}\end{aligned}$$

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2. §2.3: 42.

$$\begin{aligned}f(x) &= \begin{cases} x^2, & x \neq 1\\ 2, & x = 1\end{cases}\\ L &= 2\\ x_0 &= -2\end{aligned}$$

Substituting these into the limit definition and reducing to $a < x < b$

$$\begin{aligned}\left|x^2-4\right| &< \epsilon\\ -\epsilon < x^2 &- 4 < \epsilon\\ 4-\epsilon < x^2 &< \epsilon+4\\ &\text{Assuming } \epsilon < 4 \text{ and using the negative square root}\\ -\sqrt{4-\epsilon} > x &> -\sqrt{\epsilon+4}\end{aligned}$$

Additionally,

$$\begin{aligned}|x+2| &< \delta\\ -\delta < x &+ 2 < \delta\\ -2-\delta < x &< -2+\delta\end{aligned}$$

From this we see that

$$\begin{aligned}-2-\delta &= -\sqrt{4+\epsilon}\\ \Rightarrow \delta &= \sqrt{4+\epsilon}-2\\ -2+\delta &= 2-\sqrt{4-\epsilon}\\ \Rightarrow \delta &= 2-\sqrt{4-\epsilon}\\ \delta &= \min(\sqrt{4+\epsilon}-2, 2-\sqrt{4-\epsilon})\end{aligned}$$

Therefore, given $\epsilon > 0$ there exists a δ such that

$$0 < |x+2| < \epsilon \Rightarrow |f(x)-4| < \epsilon.$$

NOTE: If $\epsilon > 4$ then we would take δ to be the distance from x_0 to the nearer endpoint of the interval $(-\sqrt{4+\epsilon}, 0)$. Therefore,

$$\begin{aligned}
-2-\delta &= -\sqrt{4+\epsilon} \\
\Rightarrow \delta &= \sqrt{4+\epsilon}-2 \\
-2+\delta &= 0 \\
\Rightarrow \delta &= 2 \\
\delta = \min(2, \sqrt{4+\epsilon}-2)
\end{aligned}$$

■

3. §2.3: 55.

$$\begin{aligned}
f(x) &= \frac{\pi}{4}x^2 \\
L &= 9\text{in}^2 \\
x_0 &= 3.385\text{in} \\
\epsilon &= 0.01\text{in}^2
\end{aligned}$$

We are looking for a δ defined in the same way that we used in the limit definition, and so we will reduce $|f(x)-L| < \epsilon$ to $a < x < b$, so that we can determine the tolerance δ.

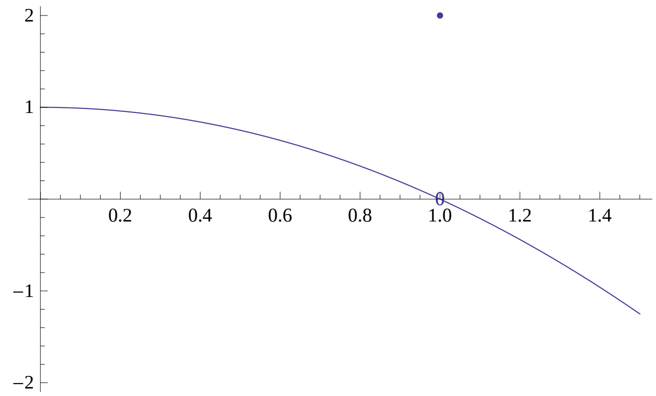
$$\begin{aligned}
\left|\frac{\pi}{4}x^2 - 9\right| &< 0.01 \\
-0.01 < \frac{\pi}{4}x^2 - 9 &< 0.01 \\
8.99 < \frac{\pi}{4}x^2 &< 9.01 \\
11.45 < x^2 &< 11.47 \\
\boxed{3.383 < x < 3.387}
\end{aligned}$$

So if we wanted to find the tolerance in x we would find that

$$\begin{aligned}
-\delta + 3.385 &= 3.383 \\
\Rightarrow \delta &= 3.385 - 3.383 = 0.002 \\
\delta + 3.385 &= 3.387 \\
\Rightarrow \delta &= 3.387 - 3.385 = 0.002 \\
\delta &= \min(0.002, 0.002) = 0.002
\end{aligned}$$

■

4. §2.4: 8.

(a) 

■

(b)

$$\lim_{x\to 1^+} f(x) = \boxed{0}$$
$$\lim_{x\to 1^-} f(x) = \boxed{0}$$

■

(c) Yes, the limit does exist and

$$\lim_{x\to 1} f(x) = \boxed{0,}$$

since the limit from the left is equal to the limit from the right. ■

5. §2.4: 18.

(a)

$$\begin{aligned}\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} &= \lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{x-1} \\ &= \lim_{x\to 1^+} \sqrt{2x} \\ &= \boxed{\sqrt{2}.}\end{aligned}$$

■

(b)

$$\begin{aligned}\lim_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} &= -\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{x-1} \\ &= -\lim_{x\to 1^+} \sqrt{2x} \\ &= \boxed{-\sqrt{2}.}\end{aligned}$$

■

6. Compute the following limits:

(a) $\lim_{t\to 0}\left(\dfrac{2t}{\tan(t)} - \dfrac{\sin(\sin(t))}{\sin(t)}\right)$

(b) $\lim_{y\to 0}\left(\dfrac{\sin(5y)}{\sin(4y)} + \dfrac{\sin(3y)\cot(5y)}{y\cot(4y)}\right)$

(a)

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{2t}{\tan(t)} - \frac{\sin(\sin t)}{\sin t}\right) &= \lim_{t\to 0}\left(\frac{2t\cos t}{\sin t} - \frac{\sin(\sin t)}{\sin t}\right) \\
&= 2\left(\lim_{t\to 0}\cos t\right)\cdot\left(\lim_{t\to 0}\frac{\sin t}{t}\right)^{-1} - \left(\lim_{t\to 0}\frac{\sin(\sin(t))}{\sin(t)}\right) \\
&= 2(\cos 0)\cdot(1)^{-1} - \left(\lim_{t\to 0}\frac{\sin(\sin(t))}{\sin(t)}\right) \\
&= 2 - \left(\lim_{t\to 0}\frac{\sin(\sin t)}{\sin t}\right)
\end{aligned}$$

Now we will make the following substitution $u = \sin t$. If $t \to 0$ then $\sin t \to 0$ and so $u \to 0$. Putting this together gives

$$\lim_{t\to 0}\frac{\sin(\sin t)}{\sin t} = \lim_{u\to 0}\frac{\sin u}{u}.$$

Finally, substitute this into the our original problem and we have

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{2t}{\tan(t)} - \frac{\sin(\sin(t))}{\sin(t)}\right) &= 2 - \left(\lim_{t\to 0}\frac{\sin(\sin(t))}{\sin(t)}\right) \\
&= 2 - \left(\lim_{u\to 0}\frac{\sin u}{u}\right) \\
&= 2 - 1 \\
&= \boxed{1.}
\end{aligned}$$

■

(b)

$$
\begin{aligned}
\lim_{y\to 0}\left(\frac{\sin(5y)}{\sin(4y)}+\frac{\sin(3y)\cot(5y)}{y\cot(4y)}\right) &= \left(\lim_{y\to 0}\frac{\sin(5y)}{\sin(4y)}\right)+\left(\lim_{y\to 0}\frac{\sin(3y)\cot(5y)}{y\cot(4y)}\right)\\
&= \left(\lim_{y\to 0}\frac{4}{4}\frac{5}{5}\frac{y}{y}\frac{\sin(5y)}{\sin(4y)}\right)+\left(\lim_{y\to 0}\frac{\sin(3y)\cos(5y)\sin(4y)}{y\cos(4y)\sin(5y)}\right)\\
&= \frac{5}{4}\left(\lim_{y\to 0}\frac{4y}{\sin(4y)}\frac{\sin(5y)}{5y}\right)+\left(\lim_{y\to 0}\frac{3}{3}\frac{4}{4}\frac{5}{5}\frac{y}{y}\frac{\sin(3y)\cos(5y)\sin(4y)}{y\cos(4y)\sin(5y)}\right)\\
&= \frac{5}{4}\left(\lim_{y\to 0}\frac{\sin(4y)}{4y}\right)^{-1}\left(\lim_{y\to 0}\frac{\sin(5y)}{5y}\right)\\
&\qquad+\frac{3\cdot 4}{5}\left(\lim_{y\to 0}\frac{\cos(5y)}{\cos(4y)}\right)\cdot\left(\lim_{y\to 0}\frac{\sin(3y)\sin(4y)5y}{3y\cdot 4y\sin(5y)}\right)\\
&= \frac{5}{4}(1)^{-1}\cdot(1)+\frac{12}{5}\left(\frac{\cos(0)}{\cos(0)}\right)\cdot\left(\lim_{y\to 0}\frac{\sin(3y)\sin(4y)5y}{3y\cdot 4y\sin(5y)}\right)\\
&= \frac{5}{4}+\frac{12}{5}\cdot(1)\cdot\left(\lim_{y\to 0}\frac{\sin(3y)}{3y}\right)\cdot\left(\lim_{y\to 0}\frac{\sin(4y)}{4y}\right)\cdot\left(\lim_{y\to 0}\frac{\sin(5y)}{5y}\right)^{-1}\\
&= \frac{5}{4}+\frac{12}{5}\cdot(1)\cdot(1)\cdot(1)^{-1}\\
&= \frac{5}{4}+\frac{12}{5}\\
&= \frac{25+48}{20}\\
&= \boxed{\frac{73}{20}.}
\end{aligned}
$$

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