1. Suppose f is defined on $[-1,3]$ and satisfies:

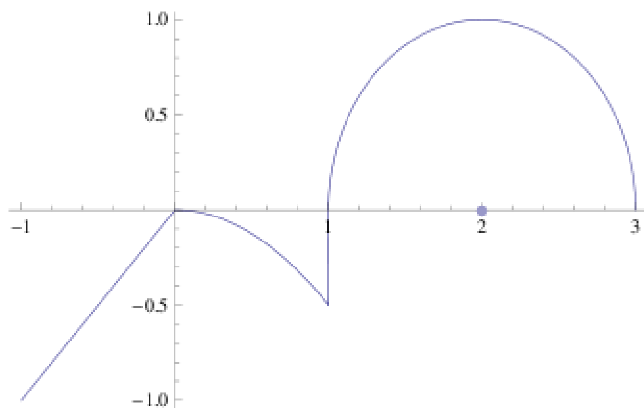
$$f(x) = \begin{cases} x, & -1 \leq x < 0 \\ -\frac{1}{2}x^2, & 0 \leq x < 1 \\ \sqrt{1-(x-2)^2}, & 1 \leq x \leq 3, x \neq 2 \\ 0, & x = 2 \end{cases}$$

(a) Sketch the graph of the function given above.

(b) Does $\lim_{x\to 2} f(x)$ exist? Justify your answer.

(c) Does $\lim_{x\to 1} f(x)$ exist? Justify your answer.

(d) Does $\lim_{x\to 4} f(x)$ exist? Justify your answer.



(a)

■

(b) Yes, the limit exists, because the function approaches the same value from the left and from the right, i.e 1. ■

(c) No, the limit doesn't exist because the function approaches two different values from either side, i.e $-\frac{1}{2}$ from the left and 0 from the right. ■

(d) No, the limit doesn't exist, since the function is not defined for values of x outside of $[-1,3]$. ■

2.

(a) Suppose that a function $f(x)$ is defined for all $x \in [-2,2]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

(b) Suppose that g is a function defined for all x. If $g(1) = 5$, must $\lim_{x\to 1} g(x)$ exist? If it does, then must $\lim_{x\to 1} g(x) = 5$? Can we conclude anything about $\lim_{x\to 1} g(x)$? Explain!

(a) No, we cannot say anything about the existence of $\lim_{x\to 0} f(x)$, because we don't know the behavior of the function near $x = 0$. ∎

(b) No, the limit does not need to exists, since the value of the function could approach one value from the left and a different value from the right. If the limit does exist it does not have to be equal to $g(1)$, since the function could be piecewise, e.g.

$$g(x) = \begin{cases} x^2 & x \neq 1 \\ 5 & x = 1 \end{cases}$$

The $lim_{x\to 1} g(x) = 1 \neq g(1) = 5$. We cannot conclude anything about $\lim_{x\to 0} g(x)$, since we do not have enough information about the function. ∎

3. Find the following limits.

(a) $\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$

(b) $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}$

(c) $\lim_{t\to 0} \frac{1+t+\sin(t)}{3\cos(t)}$

(d) $\lim_{u\to 1} \frac{u^6-1}{u^4-1}$

(a)

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h} &= \lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h} \cdot \frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2} \\
&= \lim_{h\to 0} \frac{5h+4-4}{h\cdot\sqrt{5h+4}+2} \\
&= \lim_{h\to 0} \frac{5h}{h\cdot\sqrt{5h+4}+2} \\
&= \lim_{h\to 0} \frac{5}{\sqrt{5h+4}+2} \\
&= \frac{5}{\lim_{h\to 0}\sqrt{5h+4}+2} \\
&= \frac{5}{\sqrt{\lim_{h\to 0}(5h+4)}+2} \\
&= \frac{5}{\sqrt{\lim_{h\to 0}(5h)+4}+2} \\
&= \frac{5}{\sqrt{4}+2} \\
&= \boxed{\frac{5}{4}.}
\end{aligned}$$

∎

(b)

$$
\begin{aligned}
\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3} &= \lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} \\
&= \lim_{x\to -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{x^2+5-9} \\
&= \lim_{x\to -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{x^2-4} \\
&= \lim_{x\to -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{(x-2)(x+2)} \\
&= \frac{\lim_{x\to -2}(\sqrt{x^2+5}+3)}{\lim_{x\to -2}(x-2)} \\
&= \frac{\sqrt{(-2)^2+5}+3)}{-2-2} \\
&= -\frac{\sqrt{9}+3)}{4} \\
&= -\frac{6}{4} \\
&= \boxed{-\frac{3}{2}.}
\end{aligned}
$$

■

(c)

$$
\begin{aligned}
\lim_{t\to 0} \frac{1+t+\sin(t)}{3\cos(t)} &= \frac{\lim_{t\to 0}(1+t+\sin(t))}{\lim_{t\to 0} 3\cos(t)} \\
&= \frac{1+0+\sin(0)}{3\cos(0)} \\
&= \boxed{\frac{1}{3}.}
\end{aligned}
$$

■

(d)

$$\begin{aligned}
\lim_{u\to 1}\frac{u^6-1}{u^4-1} &= \lim_{u\to 1}\frac{(u^3-1)(u^3+1)}{(u^2-1)(u^2+1)}\\
&= \lim_{u\to 1}\frac{(u^3-1)(u^3+1)}{(u^2-1)(u^2+1)}\\
&= \lim_{u\to 1}\frac{(u-1)(u^2+u+1)(u^3+1)}{(u+1)(u-1)(u^2+1)}\\
&= \lim_{u\to 1}\frac{(u^2+u+1)(u^3+1)}{(u+1)(u^2+1)}\\
&= \frac{(1^2+1+1)(1^3+1)}{(1+1)(1^2+1)}\\
&= \frac{3\cdot 2}{2\cdot 2}\\
&= \frac{6}{4}\\
&= \boxed{\frac{3}{2}.}
\end{aligned}$$

■

4. Use Sandwich Theorem and limit laws to show that

(a) $\lim_{t\to 0} t^2\cos(20\pi t) = 0$

(b) $\lim_{x\to 0}\sqrt{x^3+x^2}\sin(\frac{\pi}{x}) = 0$

(c) $\lim_{h\to 0}\Big(h^2\cos(\frac{2}{h})+1\Big)\Big(h^2\cos(\frac{2}{h})-1\Big) = -1$

(d) $\lim_{u\to 0} u^2 4^{\sin(\frac{\pi}{u})} = 0$

(a)

$$-1 \le \cos(20\pi t) \le 1 \Rightarrow -t^2 \le t^2\cos(20\pi t) \le t^2.$$

$$\begin{aligned}
\lim_{t\to 0} -t^2 &= 0\\
\lim_{t\to 0} t^2 &= 0
\end{aligned}$$

Thus, by the Squeeze Theorem we see that $\lim_{t\to 0} t^2\cos(20\pi t) = 0$. ■

(b)

$$-1 \le \sin(\frac{\pi}{x}) \le 1 \Rightarrow -\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin(\frac{\pi}{x}) \le \sqrt{x^3+x^2}.$$

$$\begin{aligned}
\lim_{x\to 0} -\sqrt{x^3+x^2} &= 0\\
\lim_{t\to 0} \sqrt{x^3+x^2} &= 0
\end{aligned}$$

Thus, by the Squeeze Theorem we see that $\lim_{x\to 0}\sqrt{x^3+x^2}\sin(\frac{\pi}{x}) = 0$. ■

(c)

$$\left(h^2\cos(\frac{2}{h}+1\right)\left(h^2\cos(\frac{2}{h}-1\right) = h^4\cos^2(\frac{2}{h}-1$$

$$0 \le \cos^2(\frac{2}{h}) \le 1$$

$$\Rightarrow 0 \le h^2\cos(\frac{2}{h}) \le h^2 \qquad \Rightarrow -1 \le h^2\cos(\frac{2}{h}) - 1 \le h^2 - 1$$

$$\lim_{x\to 0} -1 = -1$$
$$\lim_{x\to 0} h^2 - 1 = -1$$

Thus, by the Squeeze Theorem we see that $\lim_{h\to 0}\left(h^2\cos(\frac{2}{h})+1\right)\left(h^2\cos(\frac{2}{h})-1\right) = -1$ ■

(d)

$$-1 \le \sin(\frac{\pi}{u}) \le 1 \Rightarrow 4^{-1}u^2 \le u^2 4^{\sin(\frac{\pi}{u}} \le 4u^2.$$

$$\lim_{u\to 0} 4^{-1}u^2 = 0$$
$$\lim_{u\to 0} 4u^2 = 0$$

Thus, by the Squeeze Theorem we see that $\lim_{u\to 0} u^2 4^{\sin(\frac{\pi}{u})} = 0$. ■