1. Use Linearization to approximate the following values. Additionally, find the error and the relative error.

(a) $\sqrt[3]{1.1}$

(b) $\tan^{-1}(\sqrt{3}+0.15)$ (Convert your answer to degrees)

To use a linearization we must first have an x_0 for which we know the exact value of $f(x_0)$.

(a) Let $f(x) = \sqrt[3]{x}$ then the linearization of $f(x)$ about $x_0 = 1$ is

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= 1 + \frac{1}{3(1)^{2/3}}(x-1) \\ &= 1 + \frac{1}{3}(x-1) \end{aligned}$$

Thus, the approximate value to $\sqrt[3]{1.1}$ is

$$L(1.1) \approx 1 + 0.0333 = \boxed{1.0333}$$

The error is

$$\begin{aligned} \epsilon &= |f(1.1) - L(1.1)| \\ &= |1.0323 - 1.0333| = \boxed{1 \times 10^{3}} \end{aligned}$$

and the relative error is

$$\begin{aligned} \varepsilon &= \left|\frac{f(1.1) - L(1.1)}{f(1.1)}\right| \\ &\approx \left|\frac{1.0323 - 1.0333}{1.0323}\right| \approx \boxed{1 \times 10^{-3}} \end{aligned}$$

(b) Let $f(x) = \tan^{-1} x$ then the linearization of $f(x)$ about $x_0 = \sqrt{3}$ is

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= \frac{\pi}{3} + \frac{1}{1+(\sqrt{3})^2}(x - \sqrt{3}) \\ &= \frac{\pi}{3} + \frac{1}{4}(x - \sqrt{3}) \end{aligned}$$

Thus, the approximate value of $\tan^{-1}(\sqrt{3}+0.15)$ is

$$\begin{aligned} L(\sqrt{3}+0.15) &= \frac{\pi}{3} + \frac{1}{4}(\sqrt{3}+0.15-\sqrt{3}) \\ &= \frac{\pi}{3} + \frac{1}{4}0.15 \\ &\approx 1.085 = 1.085 \cdot \frac{180^\circ}{\pi} \approx \boxed{62.15^\circ} \end{aligned}$$

The error is

$$\begin{aligned}\epsilon &= |f(\sqrt{3}+0.15) - L(\sqrt{3}+0.15)| \\ &\approx |62.02^\circ - 62.15^\circ| = \boxed{0.13^\circ}\end{aligned}$$

and the relative error is

$$\begin{aligned}\varepsilon &= \left|\frac{f(1.1) - L(1.1)}{f(1.1)}\right| \\ &\approx \left|\frac{0.13^\circ}{62.02^\circ}\right| \approx \boxed{2.1\times 10^{-3}}\end{aligned}$$

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2. Using Newton's Method find the root to the following functions, to the nearest hundreth. (You must show each iteration to get full credit)

(a) $f(x) = x^5 + x + 1$

(b) $g(x) = \cos^{-1} x - e^x$

(a) Let $x_0 = 0$ and $'f(x) = 5x^4 + 1$ then we will use a table to find the solution

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0	1	1	-1
1	-1	-1	6	-0.833
2	-0.833	-0.235	3.411	-0.764
3	-0.764	-0.025	2.707	-0.755
4	-0.755	-0.000	2.625	-0.755

By the fifth iteration we see that difference between x_5 and x_4 is less than 0.01 and so our answer is $x \approx -0.755$. ■

(b) Let $x_0 = 0$ and $'f(x) = -\frac{1}{\sqrt{1-x^2}} - e^x < ++ >$ then we will use a table to find the solution

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0.000	0.571	-2.000	0.285
1	0.285	-0.049	-2.374	0.265
2	0.265	0.000	-2.340	0.265

By the fifth iteration we see that difference between x_2 and x_3 is less than 0.01 and so our answer is $x \approx 0.265$. ■

3. Using the Taylor series expansion about $\theta_0 = 0$ show that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $i^2 = -1$.

Let $g(\theta) = \cos\theta + i\sin\theta$ then

n	$g^{(n)}(\theta)$	$g^{(n)}(\theta_0)$
0	$\cos\theta + i\sin\theta$	1
1	$-\sin\theta + i\cos\theta$	i
2	$-\cos\theta - i\sin\theta$	-1
3	$\sin\theta - i\cos\theta$	$-i$
4	$\cos\theta + i\sin\theta$	1

Thus, we see that the Taylor series is

$$\begin{aligned} g(\theta) &\approx g(\theta_0) + g'(\theta_0)(\theta-\theta_0) + \frac{g''(\theta_0)}{2!}(\theta-\theta_0) + \frac{g'''(\theta_0)}{3!}(\theta-\theta_0)^3 + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \end{aligned}$$

On the other hand let $f(x) = e^{i\theta}$ then

n	$f^{(n)}(\theta)$	$f^{(n)}(\theta_0)$
0	$e^{i\theta}$	1
1	$ie^{i\theta}$	i
2	$-e^{i\theta}$	-1
3	$-ie^{i\theta}$	$-i$
4	$e^{i\theta}$	1

Thus, we see that the Taylor series is

$$\begin{aligned} f(\theta) &\approx f(\theta_0) + f'(\theta_0)(\theta-\theta_0) + \frac{f''(\theta_0)}{2!}(\theta-\theta_0) + \frac{f'''(\theta_0)}{3!}(\theta-\theta_0)^3 + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \end{aligned}$$

and we see that $e^{i\theta} = \cos\theta + i\sin\theta$. ■

4. Find the fourth-order Taylor series expansion of the following function about the given x_0.

(a) $f(x) = 2x^4 + 3x^2 - x + 4,\ x_0 = 0$ (Simplify)

(b) $g(x) = \ln x,\ x_0 = 1$

(a)

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$
0	$2x^4 + 3x^2 - x + 4$	4
1	$8x^3 + 6x - 1$	-1
2	$24x^2 + 6$	6
3	$48x$	0
4	48	48

And so the fourth-order taylor series is

$$\begin{aligned} p_4(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4 \\ &= 4 - x + \frac{6}{2}x^2 + \frac{48}{24}x^4 \\ &= \boxed{4 - x + 3x^2 + 2x^4} \end{aligned}$$

■

(b)

n	$g^{(n)}(x)$	$g^{(n)}(x_0)$
0	$\ln x$	0
1	$\frac{1}{x}$	1
2	$-\frac{1}{x^2}$	-1
3	$\frac{2}{x^3}$	2
4	$-\frac{6}{x^4}$	-6

And so the gourth-order taylor series is

$$\begin{aligned} p_4(x) &= g(x_0) + g'(x_0)(x-x_0) + \frac{g''(x_0)}{2!}(x-x_0)^2 + \frac{g'''(x_0)}{3!}(x-x_0)^3 + \frac{g^{(4)}(x_0)}{4!}(x-x_0)^4 \\ &= \boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4} \end{aligned}$$

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5. Find the absolute maximum and absolute minimum values of f on the given interval.

(a) $f(x) = x^3 - 12x + 1,\ [-3, 5]$

(b) $f(x) = x - 2\cos x,\ [-\pi, \pi]$

(a) First we must find all critical points (in the given interval)

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 0 \\ \Rightarrow x &= \{\pm 2\} \end{aligned}$$

Now, we check the value of the function at the end points and the critical points

$$\begin{aligned} f(-3) &= 10 \\ f(5) &= 66 \\ f(2) &= -15 \\ f(-2) &= 17 \end{aligned}$$

Therefore, we see that the **absolute max** is $\boxed{66 \text{ at } x = 5}$ and the **absolute min** is $\boxed{-15 \text{ at } x = 2.}$ ■

(b) First we must find all critical points (in the given interval)

$$\begin{aligned} f'(x) &= 1 + 2\sin x = 0 \\ \Rightarrow x &= \left\{-\frac{\pi}{6}, -\frac{5\pi}{6}\right\} \end{aligned}$$

Now, we check the value of the function at the end points and the critical points

$$\begin{aligned} f(-\pi) &= -\pi + 2 \\ f(\pi) &= \pi + 2 \\ f(-\pi/6) &= -\pi/6 - \sqrt{3} \\ f(-5\pi/6) &= -5\pi/6 + \sqrt{3}. \end{aligned}$$

Therefore, we see that the **absolute max** is $\boxed{\pi + 2 \text{ at } x = \pi}$ and the **absolute min** is $\boxed{-\pi/6 - \sqrt{3} \text{ at } x = -\pi/6.}$

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