A Conforming Finite Elements for the Pure Streamfunction Form of the Quasi-Geostrophic Equations

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- In collaboration with: Traian Iliescu (VT), Zhu Wang (IMA), and Dave Wells (VT)
- National Science Foundation Grant DMS-1025314
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1. Large Scale Ocean Surface Currents

2. Quasi-Geostrophic Equations

3. Argyris Finite Element

4. Error Estimates

5. Time Dependence

6. Future Work

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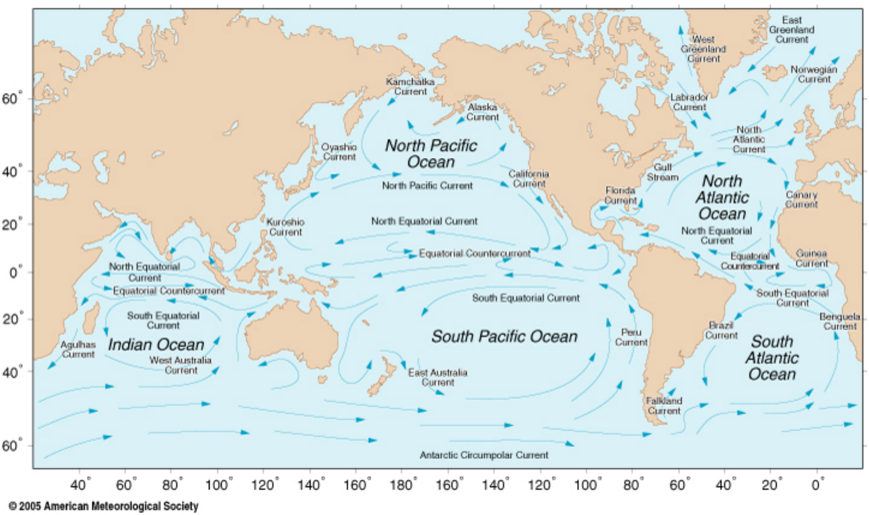

src:http://oceanmotion.org/html/background/wind-driven-surface.htm

Large scale surface currents of the oceans

Characteristics of large scale oceanic surface currents

- Driven by forces such as
  - Wind
  - Coriolis Force, i.e. the deflection of moving objects due to the rotation of the Earth.

Quasi-Geostrophic Equations

- The QGE are a simplified model for planet-scale flows.
- Streamfunction-Vorticity Formulation

$$\frac{\partial q}{\partial t} + J(\psi, q) = -Re^{-1} \Delta q + F \tag{1}$$

$$q = -Ro\, \Delta \psi + y \tag{2}$$

where Ro, Re are the Rossby and Reynolds numbers, and the Jacobian

$$J(\xi, \eta) = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}$$

- The Jacobian is associated with $(u \cdot \nabla)\, u$ in the Navier-Stokes Equations

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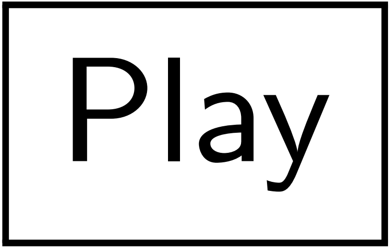

Demonstration of the Coriolis force.

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Further simplification by substituting (2) into the (1).

$$\frac{\partial q}{\partial t} + J(\psi, q) = -Re^{-1}\Delta q + F$$

$$q = -Ro\,\Delta\psi + y$$

Pure Streamfunction Formulation

$$-\frac{\partial\,[\Delta\psi]}{\partial t} + Re^{-1}\Delta^2\psi + J(\psi, \Delta\psi) - Ro^{-1}\frac{\partial\psi}{\partial x} = Ro^{-1}F \qquad (3)$$

Pro	Con
One Variable "Simpler"	Fourth-Order PDE Requires C^1 FE to be Conforming

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QGE is not Navier-Stokes

NSE

$$-\frac{\partial[\Delta\psi]}{\partial t} + Re^{-1}\,\Delta^2\psi + J(\psi,\Delta\psi) = F,$$

QGE

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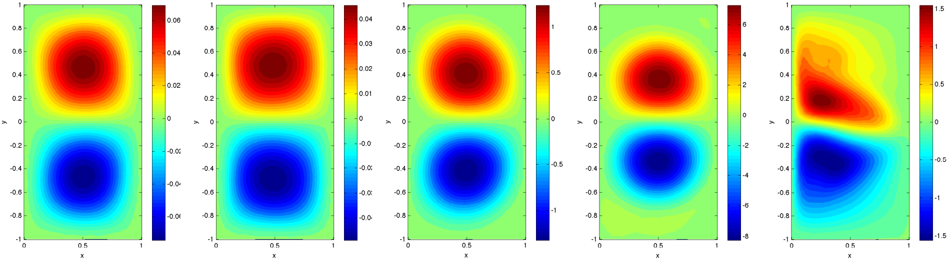

NSE $\qquad Ro = 1 \qquad Ro = 0.1 \qquad Ro = 0.01 \qquad Ro = 0.001$

Time Averaged, $t = [0, 10]$, $dt = 1\times 10^{-3}$, $Re = 200$, $F = \sin \pi y$

Stationary Quasi-Geostrophic Equations

- Pure Streamfunction Form of Stationary QGE

$$
\begin{gathered}
Re^{-1}\Delta^2\psi + J(\psi, \Delta\psi) - Ro^{-1}\frac{\partial\psi}{\partial x} = Ro^{-1}F \\
\psi = 0,\ \frac{\partial\psi}{\partial\vec{n}} = 0 \text{ on } \partial\Omega
\end{gathered} \tag{4}
$$

- Weak Form

$$
\begin{gathered}
\textit{Find } \psi \in X \textit{ such that } \forall\chi \in X := H_0^2(\Omega) \\
Re^{-1}(\Delta\psi,\Delta\chi) + b(\psi;\psi,\chi) - Ro^{-1}(\psi_x,\chi) = Ro^{-1}(F,\chi),
\end{gathered} \tag{5}
$$

where

$$
b(\psi;\psi,\chi) = [(\Delta\psi\cdot\psi_y,\chi_x) - (\Delta\psi\cdot\psi_x,\chi_y)]
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- Conforming Finite Element Formulation

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\begin{gathered}
\textit{Find } \psi^h \in X^h \textit{ such that } \forall\chi^h \in X^h \subset X \\
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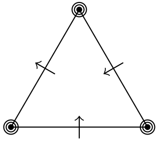

Argyris Triangle

- The Argyris Finite Element is C^1
  - Conforming Finite Element for fourth-order problems
- Fifth-order basis functions
  - Interpolation Error Bounds for Argyris

$$\|u - \Pi_h u\|_{2-s} \leq C h^{4+s} |u|_6 \text{ for } s = 0, 1, 2 \tag{7}$$

- 21 degrees of freedom
  - Function values at each vertex (3 values total)
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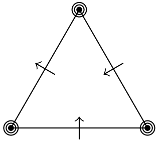
- https://github.com/VT-ICAM/ArgyrisPack



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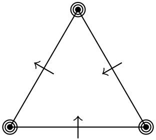

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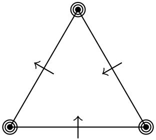

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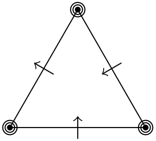

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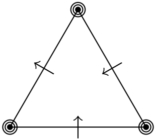

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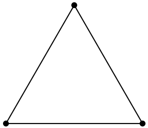

Node Values of Argyris Triangle

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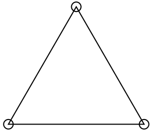

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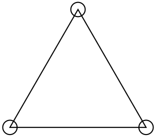

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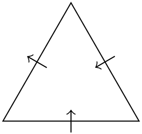

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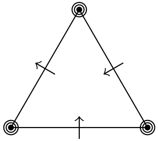

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Theorem (Argyris Error Estimates)

Let ψ *be the solution of* (5) *and* ψ^h *be the solution of* (6) *and assume the small data condition*

$$Re^{-2}\,Ro \geq \Gamma_1 \|F\|_{-2}.$$

Furthermore, assume that $\psi \in H^6(\Omega) \cap H_0^2(\Omega)$ *Then there exists positive constants* C_0, C_1, *and* C_2 *that depend on* $Re, Ro, \Gamma_1, \Gamma_2, F$ *but not* h *such that*

$$|\psi - \psi^h|_2 \leq C_2 \cdot h^4 \tag{8}$$

$$|\psi - \psi^h|_1 \leq C_1 \cdot h^5 \tag{9}$$

$$\|\psi - \psi^h\|_0 \leq C_0 \cdot h^6 \tag{10}$$

E. L. Foster, T. Iliescu, and Z. Wang, Submitted 2012, http://arxiv.org/abs/1210.3630

For the H^2 error estimate the proof follows in the standard way. To determine the H^1 and L^2 error estimates one uses the duality argument (Aubin-Nitsche) to bootstrap from H^2 norm into the appropriate norm.

Numerical Test

$$Re^{-1}\Delta^2\psi + J(\psi, \Delta\psi) - Ro^{-1}\frac{\partial\psi}{\partial x} = Ro^{-1}F$$

$$\psi = 0,\ \frac{\partial\psi}{\partial\vec{n}} = 0 \text{ on } \partial\Omega,\ Re = Ro = 1$$

- $\Omega = [0,\,1] \times [0,\,1]$
- $\psi(x,y) = \sin^2 \pi x \cdot \sin^2 2\pi y$

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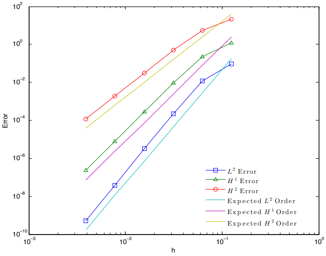

Numerical vs Theoretical Rates of Convergence

Time Dependence

- Recall the pure streamfunction form of QGE, i.e. Equation 3

$$-\frac{\partial\,[\Delta\psi]}{\partial t} + Re^{-1}\Delta^2\psi + J(\psi,\Delta\psi) - Ro^{-1}\frac{\partial\psi}{\partial x} = Ro^{-1}F$$

$$\psi(t;x,y) = \frac{\partial\psi}{\partial\vec{n}} = 0 \text{ on } \partial\Omega, \qquad \psi(0;x,y) = \psi_0(x,y)$$

- Full-discretization

$$\begin{aligned}[K + k\,(L-B)]\,\psi^h_{n+1} + b(\psi^h_{n+1};\psi^h_{n+1},\chi^h) \\ = K\,\psi^h_n + \ell(t_{n+1}), \quad \forall\chi^h \in X^h \subset X\end{aligned} \tag{11}$$

where

- K is the stiffness matrix associated with $(\nabla\psi^h,\nabla\chi^h)$
- L is the Laplace matrix associated with $Re^{-1}(\Delta\psi^h,\chi^h)$
- B is the β-plane matrix associated with $Ro^{-1}(\psi^h_x,\chi^h)$
- $\ell(t_{n+1})$ is the load vector associated with $Ro^{-1}(F(t_{n+1},\chi^h)$
- k is the time step

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Numerical Test

$$-\frac{\partial\left[\Delta\psi\right]}{\partial t} + Re^{-1}\Delta^2\psi + J(\psi,\Delta\psi) - Ro^{-1}\frac{\partial\psi}{\partial x} = Ro^{-1}F$$

$$\psi = 0,\ \frac{\partial\psi}{\partial\vec{n}} = 0 \text{ on } \partial\Omega,\ \psi(0;x,y) = \psi_0(x,y)$$

$$Re = Ro = 1$$

- $\Omega = [0,\,1]\times[0,\,1]$, $t = [0, 0.25]$
- $\psi(t;x,y) = \left[(1-x)\left(1-e^{-0.1\,x\,t}\right)\sin\pi y\right]^2$

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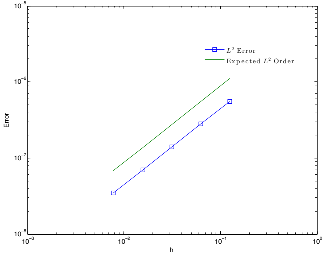
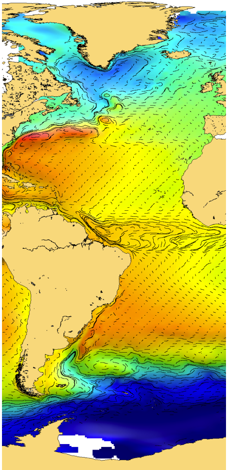
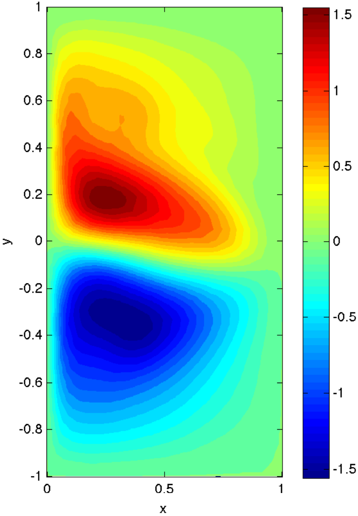
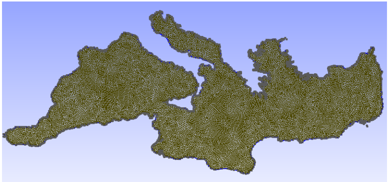


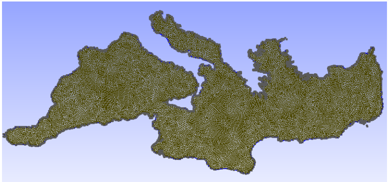
Notice anything different?







FE mesh for the Mediterranean Sea



FE mesh for the Mediterranean Sea

Play

QGE on the Mediteranean Sea

$F = sin(\pi y)$, $Re = 400$, $Ro = 1.6E - 4$, $DoFs = 46480$

Challenges and Future Work

- Realistic Parameters

Challenge	Possible Solution
Narrow Boundary Layer	Stabilization Methods
Dynamic Structures	Adaptive Mesh Refinement

- Ensemble Forecasting

Challenge	Possible Solutions
"Slow" Code	Parallel Processing Adaptive Mesh Refinement Proper Orthogonal Decomposition Large Eddy Simulation

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- Quasi-Geostrophic Equation
- Pure Stream Function form of QGE
- Optimal Error estimates
- Numerical Experiments
- Preliminary and Future Work

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Thank you!

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