1. Find the equation of the tangent line and normal line to

(a) ~~$y^2 + \tan x^2 y = x$ at $(0,0)$.~~

(b) $x^2 + \left(y - \sqrt{|x|}\right)^2 = 3$ at $(1, 1-\sqrt{2})$.

(a)

(b) First notice that if $x > 0$ then $|x| = x$ and so we have

$$
\begin{aligned}
\frac{d}{dx}\left[x^2 + \left(y - \sqrt{x}\right)^2\right] &= \frac{d}{dx}[3] \\
2x + \frac{d\left[\left(y-\sqrt{x}\right)^2\right]}{d\left[y-\sqrt{x}\right]}\frac{d\left[y-\sqrt{x}\right]}{dx} \\
2x + 2\left(y-\sqrt{x}\right)\left[y' - \frac{1}{2\sqrt{x}}\right] &= 0 \\
2 + 2\left(1-\sqrt{2}-1\right)\left[y' - \frac{1}{2}\right] &= 0 \\
y' - \frac{1}{2} &= \frac{1}{\sqrt{2}} \\
y' &= \frac{1}{\sqrt{2}} + \frac{1}{2} \\
y' &= \frac{\sqrt{2}+1}{2} \Rightarrow y'_{\perp} = -\frac{2}{\sqrt{2}+1} = 2 - 2\sqrt{2}
\end{aligned}
$$

Tangent Line

$$
\begin{aligned}
y - y_1 &= y'(x - x_1) \\
y &= \frac{\sqrt{2}+1}{2}x + \frac{1}{2} \\
y - (1-\sqrt{2}) &= \frac{\sqrt{2}+1}{2}(x-1) \\
y &= \frac{\sqrt{2}+1}{2}x - \frac{\sqrt{2}+1}{2} + (1-\sqrt{2}) \\
&\boxed{y = \frac{\sqrt{2}+1}{2}x - \frac{3\sqrt{2}-1}{2}}.
\end{aligned}
$$

Normal Line

$$
\begin{aligned}
y - y_1 &= y'_{\perp}(x - x_1) \\
y - (1-\sqrt{2}) &= (2-2\sqrt{2})(x-1) \\
y &= (2-2\sqrt{2})x - (2-2\sqrt{2}) + (1-\sqrt{2}) \\
&\boxed{y = (2-\sqrt{2})x + \sqrt{2} - 1}
\end{aligned}
$$

■

2.

(a) If $y = A\sin(\ln x) + B\cos(\ln x)$, where A and B are constants, show that

$$x^2y'' + xy' + y = 0.$$

(b) Find the first and second derivative of

$$f(x) = \tan(\ln x^2).$$

(a)

$$
\begin{aligned}
y' &= \frac{d}{dx}\left[A\sin(\ln x) + B\cos(\ln x)\right] \\
&= A\frac{d\sin(\ln x)}{d\ln x}\frac{d\ln x}{dx} + B\frac{d\cos(\ln x)}{d\ln x}\frac{d\ln x}{dx} \\
&= A\cos(\ln x)\frac{1}{x} - B\sin(\ln x)\frac{1}{x} \\
&= \frac{1}{x}\left(A\cos(\ln x) - B\sin(\ln x)\right) \\
y'' &= \frac{d}{dx}\left[\frac{1}{x}\left(A\cos(\ln x) - B\sin(\ln x)\right)\right] \\
&= -\frac{1}{x^2}\left(A\cos(\ln x) - B\sin(\ln x)\right) + \frac{1}{x^2}\left(-A\sin(\ln x) - B\cos(\ln x)\right) \\
&= -\frac{1}{x^2}\left[A\left(\cos(\ln x) + \sin(\ln x)\right) - B\left(\sin(\ln x) - \cos(\ln x)\right)\right] \\
&\Rightarrow -A\cos(\ln x) - A\sin(\ln x) + B\sin(\ln x) - B\cos(\ln x) \\
&\quad + A\cos(\ln x) - B\sin(\ln x) + A\sin(\ln x) + B\cos(\ln x) = 0
\end{aligned}
$$

■

(b)

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\tan(\ln x^2)\right] \\
&= \frac{d\left[\tan(\ln x^2)\right]}{d\left[\ln x^2\right]}\frac{d\left[\ln x^2\right]}{d\left[x^2\right]}\frac{d\left[x^2\right]}{dx} \\
&= \sec^2(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x \\
&= \boxed{\frac{2\sec^2(\ln x^2)}{x}}
\end{aligned}
$$

$$
\begin{aligned}
f''(x) &= \frac{d}{dx}\left[\frac{2\sec^2(\ln x^2)}{x}\right] \\
&= 2\frac{\frac{d\left[\sec^2(\ln x^2)\right]}{d\left[\sec(\ln x^2)\right]}\frac{d\left[\sec(\ln x^2)\right]}{d\left[\ln x^2\right]}\frac{d\left[\ln x^2\right]}{d\left[x^2\right]}\frac{d\left[x^2\right]}{dx}\cdot x - \sec^2(\ln x^2)}{x^2} \\
&= 2\frac{2\sec(\ln x^2)\cdot\sec(\ln x^2)\cdot\tan(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x\cdot x - \sec^2(\ln x^2)}{x^2} \\
&= 2\frac{4\sec^2(\ln x^2)\cdot\tan(\ln x^2) - \sec^2(\ln x^2)}{x^2} \\
&= \boxed{2\sec^2(\ln x^2)\frac{4\tan(\ln x^2) - 1}{x^2}}
\end{aligned}
$$

■

3.

(a) Find the first and second derivative of $\ln(y^2) + \sin(x+1) = e^x$.

(b) Find the first derivative of $y^x = \tan x$.
Hint: Notice $y^x = e^{x \ln y}$.

Be sure to substitute for y' when appropriate.

(a)

$$\begin{aligned}
&\frac{d}{dx}\left[\ln y^2 + \sin(x+1)\right] = \frac{d}{dx}\left[e^x\right] \\
&2\frac{d\left[\ln y\right]}{dy}\frac{dy}{dx} + \frac{d\left[\sin(x+1)\right]}{d\left[x+1\right]}\frac{d\left[x+1\right]}{dx} = e^x \\
&\frac{2}{y}y' + \cos(x+1) = e^x \\
&\frac{2}{y}y' = e^x - \cos(x+1) \\
&\boxed{y' = \frac{y}{2}\left(e^x - \cos(x+1)\right)}
\end{aligned}$$

$$\begin{aligned}
&y'' = \frac{d}{dx}\left[\frac{y}{2}\left(e^x - \cos(x+1)\right)\right] \\
&y'' = \frac{y'}{2}\left(e^x - \cos(x+1)\right) + \frac{y}{2}\left(e^x - \sin(x+1)\right) \\
&y'' = \frac{y}{4}\left(e^x - \cos(x+1)\right)\left(e^x - \cos(x+1)\right) + \frac{y}{2}\left(e^x - \sin(x+1)\right) \\
&\boxed{y'' = \frac{y}{2}\left[\frac{1}{2}\left(e^x - \cos(x+1)\right)^2 + e^x - \sin(x+1)\right]}
\end{aligned}$$

■

(b)

$$\begin{aligned}
&e^{x\ln y} = \tan x \\
&\frac{d}{dx}\left[e^{x\ln y}\right] = \frac{d}{dx}\left[\tan x\right] \\
&\frac{d\left[e^{x\ln y}\right]}{d\left[x\ln y\right]}\left(\ln y + x\frac{d\left[\ln y\right]}{dy}\frac{dy}{dx}\right) = \sec^2 x \\
&e^{x\ln y}\left(\ln y + \frac{x}{y}y'\right) = \sec^2 x \\
&\left(\ln y + \frac{x}{y}y'\right) = e^{-x\ln y}\sec^2 x \\
&\boxed{y' = \frac{y}{x}\left(e^{-x\ln y}\sec^2 x - \ln y\right)}
\end{aligned}$$

¡++¿ ■

¡++¿

4. Show the definitions of the following derivatives are correct

(a) $\frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1-x^2}}$ by using **Theorem 4** in the book.
Hint: Let $f^{-1}(x) = \cos^{-1} x$.

(b) $\frac{d}{dx}\left[a^x\right] = \ln a \cdot a^x$, where a is a constant.

(i) By using **Theorem 4** in the book.
Hint: Let $f^{-1}(x) = a^x$.

(ii) By using **Logarithmic Differentiation** .

(a) Let $f^{-1}(x) = \cos^{-1} x$ then $f(y) = \cos y$ and by **Theorem 4** in the book

$$\begin{aligned}\left[f^{-1}(x)\right]' &= \frac{1}{f'(f^{-1}(x))}\\ &= -\frac{1}{\sin(\cos^{-1} x)}\\ &= -\frac{1}{\sqrt{1-\cos^2(\cos^{-1} x)}}\\ &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

■

(b) (i) Let $f^{-1}(x) = a^x$ then $f(y) = \frac{\ln y}{\ln a}$ and by **Theorem 4** in the book

$$\begin{aligned}\left[f^{-1}(x)\right]' &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{\frac{1}{a^x \ln a}}\\ &= a^x \ln a.\end{aligned}$$

■

(ii)

$$\begin{aligned}y &= a^x\\ \ln y &= x \ln a\\ \frac{d}{dx}\left[\ln y\right] &= \frac{d}{dx}\left[x \ln a\right]\\ \frac{y'}{y} &= \ln a\\ y' &= y \ln a\\ y' &= a^x \ln a.\end{aligned}$$

■