1. Prove the following equations have exactly one real solution in the given interval (do not find the root)

(a) $\cos x = e^x$ in the interval $[-1, 1]$.

(b) $\ln x = \tan x$ in the interval $(3, 4.5)$

(a) **Proof:** Let $f(x) = \cos x - e^x$ then since both e^x and $\cos x$ are continuous and differential on the given interval then so is $f(x)$. Since $f(x)$ is continuous on $[-1, 1]$ by the IVT we know that $f(x)$ achieves every value between $f(-1) \approx 0.17$ and $f(1) \approx -2.2$. Since zero is between $f(-1)$ and $f(1)$ there must be at least one zero in the interval $[-1, 1]$.
On the other hand, assume that $f(x) = 0$ for two values of $x \in [-1, 1]$, i.e. $f(x_1) = f(x_2) = 0$ for $x_1, x_2 \in [-1, 1]$. Since $f(x)$ is continuous on $[-1, 1]$ and differential on $(-1, 1)$ and $f(x_1) = f(x_2)$ we know, by Rolle's Theorem, that there exists a $c \in (-1, 1)$ such hat $f'(c) = 0$. However,

$$f'(x) = \sin x - e^x < 0 \ \forall x \in (-1, 1).$$

a contradiction and so our initial assumption must have been incorrect. Therefore, we must have exactly one root to $f(x)$ in the interval $[-1, 1]$, which implies $e^x = \cos x$ for exactly one $x \in [-1, 1]$. ■

(b) **Proof:** Let $f(x) = \ln x - \tan x$ then since both $\ln x$ and $\tan x$ are continuous and differential on the given interval then so is $f(x)$. Since $f(x)$ is continuous on $[3, 4.5]$ by the IVT we know that $f(x)$ achieves every value between $f(3) \approx 1.24$ and $f(4.5) \approx -3.13$. Since zero is between $f(3)$ and $f(4.5)$ there must be at least one zero in the interval $[3, 4.5]$.
On the other hand, assume that $f(x) = 0$ for two values of $x \in [3, 4.5]$, i.e. $f(x_1) = f(x_2) = 0$ for $x_1, x_2 \in [3, 4.5]$. Since $f(x)$ is continuous on $[3, 4.5]$ and differential on $(3, 4.5)$ and $f(x_1) = f(x_2)$ we know, by Rolle's Theorem, that there exists a $c \in (3, 4.5)$ such hat $f'(c) = 0$. However,

$$f'(x) = \frac{1}{x} - \sec^2 x < 0 \ \forall x \in (3, 4.5).$$

a contradiction and so our initial assumption must have been incorrect. Therefore, we must have exactly one root to $f(x)$ in the interval $[-1, 1]$, which implies $\tan x = \ln x$ for exactly one $x \in [3, 4.5]$. ■

2. Given $f(x) = \sin 2x$ in $[0, \pi/2]$

(a) Verify f meets the conditions of the Mean Value Theorem.

(b) Find the value(s) of c that satisfies the conclusion of the Mean Value Theorem.

(a) Since $\sin x$ is continuous and differential on $(-\infty, \infty)$ then so is $\sin 2x$ and therefore $\sin 2x$ is continous on $[0, \pi/2]$ and differential on $(0, \pi/2$. ■

(b) Since $f(x)$ satisfies the hypotheses of the MVT there exists a $c \in (0, \pi/2$ such that

$$\begin{aligned} f'(c) &= \frac{f(\pi/2) - f(0)}{\pi/2 - 0} \\ 2 \cdot \cos 2c &= 0 \\ c &= \frac{\cos^{-1} 0}{2} \\ c &= \boxed{\frac{\pi}{4}.} \end{aligned}$$

■

3. For $f(x) = \left|x^3+1\right|$ defined on the entire real number line

(a) Find all Critical Points

(b) Find all Inflection Points

(c) Use the First Derivative to find any max/mins.

(d) Use the Second Derivative to find any max/mins.

(a) To find all critical points we must determine when the derivative is zero and determine where the derivative is undefined. First let's find where the derivative is undefined since this will also be required for determining the derivative. The derivative will be undefined when $f(x) = x^3+1 = 0$, i.e. $x = -1$ since there will be a sharp corner at this point. The function $f(x)$ can therefore be written as

$$f(x) = \begin{cases} x^3+1 & x \geq -1 \\ -x^3-1 & x < -1 \end{cases}$$

Thus the derivative is

$$f'(x) = \begin{cases} 3x^2 & x > -1 \\ -3x^2 & x < -1 \end{cases}$$

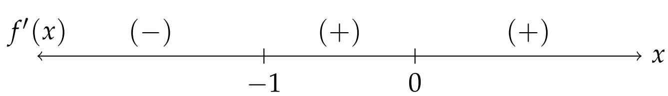
From this we see that the critical points are $x = \{-1, 0\}$. ■

(b) To find inflection points we must determine where $f''(x) = 0$ and where $f''(x)$ is undefined.

$$f''(x) = \begin{cases} 6x & x > -1 \\ -6x & x < -1 \end{cases}$$

From this we see that $f''(x) = 0$ when $x = 0$ and $f''(x)$ is undefined when $x = -1$. However, we must check that $f''(x)$ actually changes sign at $x = 0$. Since it changes from negative to positive as we travel from right to left across $x = 0$ there is indeed an inflection point when $x = 0$. On the other hand when we travel from left to right across $x = -1$ we see the value of $f''(x)$ go from positive to negative and so $x = -1$ would also be an inflection point. ■

(c) Using the first derivative



we see that $f(-1) = 0$ is a minimum and $f(0) = 1$ is a saddle-point. ■

(d) The Second derivative tells us nothing at $x = -1$ since it is undefined there. However, since $f''(0) = 0$ we see that the point $(0, 1)$ is a saddle-point. ■

4. Graph $f(x) = \dfrac{4x}{x^2+1}$. Don't use a table of values and show all work.

To draw the graph we first find all critical points.

$$\begin{aligned} f'(x) &= 4(x^2+1)^{-1} - 8x^2(x^2+1)^{-2} = 0 \\ \Rightarrow & 4(x^2+1) - 8x^2 = 0 \\ x &= \pm 1 \end{aligned}$$

and $f'(x)$ is undefined when

$$x^2 + 1 = 0.$$

But, this doesn't have a solution for real numbers and therefore $f'(x)$ is always defined. Thus, our critical points are $x = \pm 1$.
We also must determine when the function is increasing and decreasing, so let's use a number line of $f'(x)$

$f'(x)$	$(-)$		$(+)$		$(-)$	x
		-1		1		

Thus, $f(x)$ is increasing when $x \in [-1, 1]$ and $f(x)$ is decreasing when $x \in (-\infty, -1] \bigcup [1, \infty)$. Now, we must find the inflection points.

$$\begin{aligned} f''(x) &= -8x(x^2+1)^{-2} - 16x(x^2+1)^{-2} + 16x^3(x^2+1)^{-3} = 0 \\ \Rightarrow & -8x(x^2+1) - 16x(x^2+1) + 16x^3 = 0 \\ & -x^3 - x - 2x^3 - 2x + 2x^3 = 0 \\ & -x^3 - 3x = 0 \\ x &= 0 \end{aligned}$$

And therefore our inflection point is $x = 0$. Next, let's determine which critical points are min/max by using the second derivative test.

$$f''(-1) = 4 \qquad f''(1) = -4$$

Thus, we see that $f(-1) = -2$ is a minimum and $f(1) = 2$ is a maximum.
We also need to determine the concavity of our function. From when we used the second derivative test we see that $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$ and therefore $f(x)$ is concave up when $x > 0$ and concave down when $x < 0$. Finally, we must determine the behavior of $f(x)$ as $x \to \pm\infty$.

$$\lim_{x \to \pm\infty} \frac{4x}{x^2+1} = \lim_{x \to \pm\infty} \frac{4/x}{1 + 1/x^2} = 0$$

And the graph of $f(x)$ is ■

5. Given f is a continuous function and $f(0) = 0$ use the following table to graph the function f

x	$-\infty$		1		4		6		$+\infty$
$f'(x)$		$+$		$-$		$-$		$+$	
$f''(x)$		$-$		$-$		$+$		$+$	

The graph should have be shaped like the one below and should go through the point $(0, 0)$. ■

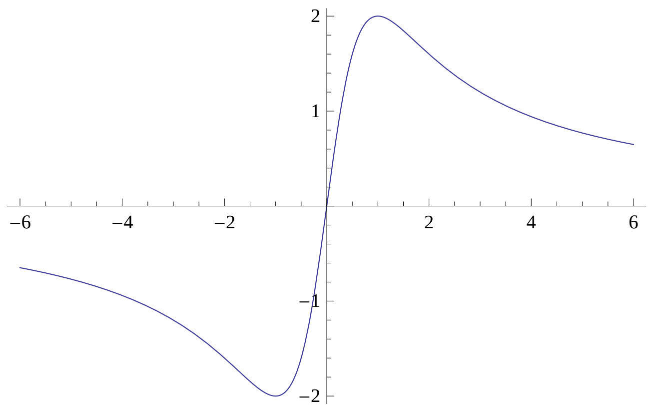


Figure 1: Graph for Problem 4

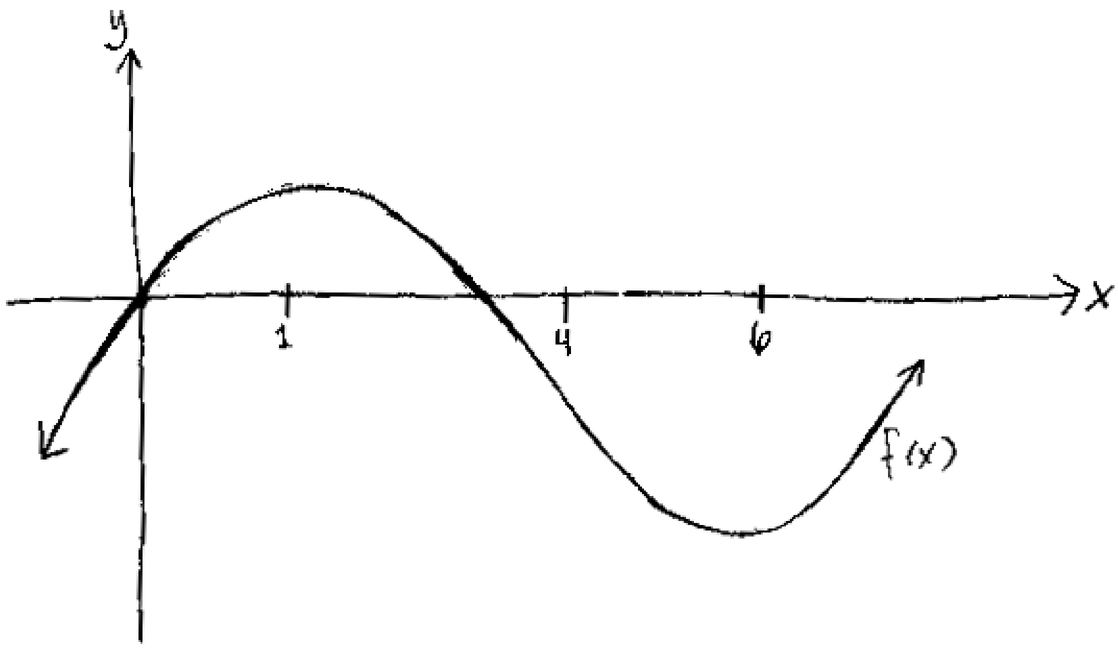


Figure 2: Graph for Problem 5