A Finite Element Discretization of the Streamfunction Formulation of the Stationary Quasi-Geostrophic Equations of the Ocean

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Abstract

This paper presents a finite element discretization of the streamfunction formulation of the stationary quasi-geostrophic equations. Rigorous error estimates for this finite element discretization are derived. Numerical results for the Argyris finite element confirm the theoretical error estimates.

Key Words: Quasi-geostrophic equations, finite element method, Argyris element.

1 Introduction

Traian: Finish this.

With the continuous increase in computational power, complex mathematical models are becoming more and more popular in the numerical simulation of oceanic and atmospheric flows. For some geophysical flows in which computational efficiency is of paramount importance, however, simplified mathematical models are central. For example, in climate modeling the *quasi-geostrophic barotropic potential vorticity equations (QGE)*, are often used [5].

The QGE are usually discretized in space by using the *finite difference method (FDM)* [20, **?**]. The *finite element method (FEM)*, however, offers several advantages over the popular FDM, as outlined in [18]: (i) an easy treatment of complex boundaries, such as those of continents for the ocean, or mountains for the atmosphere; (ii) an easy grid refinement to achieve a high resolution in regions of interest [1]; (iii) a natural treatment of boundary conditions; and (iv) a straightforward approach for the treatment of multiply connected domains [18]. Despite these advantages, there are relatively few papers that consider the FEM applied to the QGE [8, **?**, **?**, **?**].

Most of the FEM discretizations of the QGE are for the *streamfunction-vorticity formulation* of these equations. The reason is simple: The streamfunction-vorticity formulation allows the use of low order ($\mathcal{C}^0$) finite elements, although one needs to discretize two flow variables, the potential vorticity q and the streamfunction ψ. We note that the streamfunction-vorticity formulation is often used in the numerical discretization of the 2D *Navier-Stokes equations (NSE)*, to which the QGE are similar. An alternative to the streamfunction-vorticity formulation of the 2D NSE is the streamfunction formulation. This formulation contains only one flow variable, the streamfunction ψ. The price paid is that one needs to deal with a fourth-order partial differential equation. Thus, its numerical discretization with conforming finite elements requires the use of high-order ($\mathcal{C}^1$) finite elements [**?**, **?**, **?**, **?**].

The main goal of this paper is to use a $\mathcal{C}^1$ finite element (the Argyris element) to discretize the streamfunction formulation of the stationary QGE. To the best of our knowledge, this is the first time that a $\mathcal{C}^1$ finite element has been used in the numerical discretization of the QGE. Rigorous error estimates for the FEM discretization and supporting numerical experiments are also presented.

The rest of the paper is organized as follows: Section 2 presents the QGE, their weak formulation, and mathematical support for the weak formulation. Section 3 outlines the FEM discretization of the QGE, posing a special emphasis of the Argyris element. Rigorous error estimates for the FEM discretization of the stationary QGE are derived in Section 4. Several numerical experiments supporting the theoretical results are presented in Section 5. Finally, conclusions are future research directions are included in Section 6.

2 The Quasi-Geostrophic Equations

Traian: Here's what needs to be done.

(1) First, we need to introduce the QGE.

Here we can use our paper [20]. We should also compare this formulation with other formulations (see, e.g., [16, 17] or textbooks?).

We can start with the vorticity-streamfunction formulation and then move on to the streamfunction formulation. The vorticity-streamfunction formulation is most popular, since the streamfunction formulation involves a fourth-order PDE.

Make the connection (at least in form) to the 2D NSE. The add some historical background on the 2D NSE and why the streamfunction formulations were popular (the incompressibility condition is automatically satisfied).

(2) Second, we need to **nondimensionalize** them.

Following our OM paper, we can use Re and Ro. The Ro dependency is clear, we need to figure out the Re dependency. Get rid of all the Munk scale stuff.

The nondimensionalization is important because we might monitor the Re/Ro dependencies in our error estimates!

(3) We need to write out the **weak formulation**.

For this, we need to introduce first the spaces, norms, notation (linear and bilinear forms), etc.

Let $X := H_0^2(\Omega)$. The weak formulation reads: Find $\psi \in X$ such that

$$\nu \int_\Omega \Delta\psi \, \Delta\chi \, dx + Ro \int_\Omega \Delta\psi \, (\psi_y \, \chi_x - \psi_x \, \chi_y) \, dx + \int_\Omega \psi_x \, \chi \, dx = \int_\Omega f \, \chi \, dx \qquad \forall \, \chi \in X, \tag{1}$$

where $\nu = \left(\frac{\delta_M}{L}\right)^3$ (see [20]).

We introduce the following notation:

$$a_0(\psi,\chi) = \nu \int_\Omega \Delta\psi\,\Delta\chi\,dx, \tag{2}$$

$$a_1(\zeta;\psi,\chi) = Ro \int_\Omega \Delta\zeta\,\left(\psi_y\,\chi_x - \psi_x\,\chi_y\right)\,dx, \tag{3}$$

$$a_2(\psi,\chi) = \int_\Omega \psi_x\,\chi\,dx, \tag{4}$$

$$\ell(\chi) = \int_\Omega f\,\chi\,dx. \tag{5}$$

With this notation, the QGE weak formulation (1) can be written as follows: Find $\psi \in X$ such that

$$a_0(\psi,\chi) + a_1(\psi;\psi,\chi) + a_2(\psi,\chi) = \ell(\chi) \qquad \forall\,\chi \in X, \tag{6}$$

The linear form ℓ, the bilinear forms a_0 and a_2, and the trilinear form a_1 are continuous [2]: There exists $\Gamma_1 > 0$ such that

$$\begin{aligned} a_0(\psi,\chi) &\leq \nu\,|\psi|_2\,|\chi|_2 \qquad \forall\,\psi,\chi \in X & (7)\\ a_1(\zeta;\psi,\chi) &\leq Ro\,\Gamma_1\,|\zeta|_2\,|\psi|_2\,|\chi|_2 \qquad \forall\,\zeta,\psi,\chi \in X & (8)\\ a_2(\psi,\chi) &\leq |\psi|_2\,|\chi|_2 \qquad \forall\,\psi,\chi \in X & (9)\\ \ell(\chi) &\leq \|f\|_{-1}\,|\chi|_2 \qquad \forall\,\chi \in X & (10) \end{aligned}$$

Traian: Check (10). Can we replace $\|f\|_{-1}$ that was used in [2] with $\|f\|_{-2}$, since we don't have $curl\,\chi$?

(4) **Well-posedness** of the weak formulation (streamfunction formulation only).

Theorem 2.1 in [2] states

Theorem 2.1 *Let $f \in H^{-1}(\Omega)$ and define $\nu^* = (\Gamma_1\,\|f\|_{-1})^{1/2}$. Then, for any $\nu > \nu^*$, the QGE weak formulation (6) has a unique solution ψ. Moreover,*

$$|\psi|_2 \leq \frac{\|f\|_{-1}}{\nu}. \tag{11}$$

Here we need to use heavily [9, 10]. I think that we need to follow Cascon's approach in [1]: we should only point out the main differences from the 2D NSE case. Note, however, that Cascon only did this for the *linearized* version and for the mixed (vorticity streamfunction) formulation.

One important issue that we need to tackle is the "smallness of data" condition for the existence and uniqueness of a weak solution: This condition relates ν, f (the forcing term), and Γ_1 (the continuity constant of the trilinear form (see Theorem 2.1 in [2]).

Γ_1 will stay the same in our formulation (I think!), but ν and f will depend on Re and Ro. We need to be precise here. Of course, we can always "punt," saying something like "for small enough data..." but we should do our best to write something more precise.

3 Finite Element Discretization

Although not as popular as the FDM discretizations, the FEM discretizations of the QGE have been employed in several studies []. A nice survey is given in [18]. Traian: Add more refs. Make transition to error estimates.

To our knowledge, the first finite element error estimates of the QGE were derived by George Fix in 1975 [8]. In that report, the conservation properties of the finite element discretization were also proved. The stability of the FE discretization was proved, as well as the sub-optimal (?????) convergence of the FE method.

It seems that there is not too much else in terms of FE error estimates for the QGE. In a series of papers, Medjo proposed and analyzed a FE discretization of the *two-layer QGE* [16, 17] (see also [21]). Medjo performed both numerical experiments [17] and carried out the numerical analysis of the FE discretization. It seems, however, that he only focused on the time discretization, without investigating the error due to the spatial discretization.

Cascon et al. considered in [1] the stationary *linearized* QGE. They proved both *a priori* and a posteriori error estimates. What is nice is that they always referred back to [10] when they proved their *a priori* error estimates, giving exact page and theorem numbers.

3.1 2D Navier-Stokes Equations

Traian: Shorten this! Because of the similarity between the QGE and the 2D NSE, we briefly survey below the FE literature for the latter. For more details, the reader is referred to [14, 11] (see also [19]).

Vorticity-streamfunction formulation Gunzburger discusses this formulation in Chapter 11 [14]. The weak formulation employed is the one "commonly used by engineers" and "also allows for the use of *low continuity* (i.e. $\mathcal{C}^0$) finite element subspaces." The same trial and test spaces are used for ψ and ω. Section 11.6 discusses the error estimates. Max says that if one considers higher-regularity (H^2) FE spaces, one can prove *optimal* error estimates. But then he goes on to say that "However, in this case one can dispense with the vorticity and solve a 4-th order problem involving only the streamfunction." In other words, one can solve the *purely streamfunction* formulation instead of the streamfunction-vorticity one. (This is exactly the subject of Chapter 13.) Thus, Max focuses on $\mathcal{C}^0$ finite elements. He says that the following error estimate was proved in [10]:

$$|\psi - \psi^h|_1 + \|\omega - \omega^h\|_0 \leq C\, h^{k-1/2}\, |\ln(h)|^{\sigma}, \tag{12}$$

where $\sigma = 1$ for $k = 1$ and $\sigma = 0$ for $k > 1$. Note that this estimate is <u>*not optimal*</u> w.r.t. h. Indeed, one may lose a half power in h for the derivatives of the streamfunction (i.e., velocity components) and three-halves power for the vorticity. Max then says that if the *linear* Stokes problem is considered instead, one can get (almost) optimal error estimates of the form:

$$|\psi - \psi^h|_1 \leq C\, h^{k-\varepsilon}, \tag{13}$$

where $\varepsilon = 0$ for $k = 1$ and $\varepsilon > 0$ is arbitrary for $k > 1$. Max also notes that the vorticity error in (12) seems sharp even in the linear Stokes case.

Gresho and Sani also discuss the streamfunction-vorticity formulation[11]. They talk at length about BCs and the correct weak formulation.

Streamfunction formulation Gunzburger [14] discusses in Chapter 13 this formulation. No-slip BCs are used:

$$\psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \Gamma. \tag{14}$$

Mathematical results for the FE discretization of the streamfunction formulation are presented in [9, 10]. Max follows the presentation in [2, 3].

The motivation for the use of the streamfunction formulation is very simple: The 2D NSE can be reduced to a set of equations for only *one* scalar unknown field, compared to two for the streamfunction-vorticity formulation. Of course, as in the streamfunction-vorticity formulation, one doesn't need to bother with the incompressibility constraint, which is now automatically satisfied. The price paid is that one has to solve a *fourth-order* PDE. The difficulty in this case is that conforming FEM require the use of $\mathcal{C}^1$ finite elements.

The first weak formulation (equation (13.3)) is for *conforming* FEs: $\Psi^h \subset H_0^2(\Omega)$. The second weak formulation (equation (13.4)) is for *nonconforming* FEs: $\Psi^h \not\subset H_0^2(\Omega)$. The error estimates are provided in [2, 3]. Both conforming and nonconforming FE are presented: Nonconforming Triangular (Morley triangular, Fraeijs de Veubeke triangular (two versions)), Conforming

Triangular(Argyris, Morgan-Scott, Scott-Vogelius, Clough-Tocher (macro-elements)), Conforming Bicubic Rectangles (Bogner-Fox-Schmidt, tensor product of cubic splines). In Section 13.3, the error estimates are discussed. Provided that $\psi \in H^s(\Omega) \cap H_0^2(\Omega)$, the available error estimates are of the form

$$|\psi - \psi^h|_{m,h} = \mathcal{O}(h^r). \tag{15}$$

For each of the FEs listed above, Table 13.1 gives the obtainable values of r in (15).

3.2 Argyris Element

Erich: Include a short description of the Argyris element. Talk about Dominguez's transformation.

3.3 Quasi-Geostrophic Equations

We consider **conforming** finite element spaces $X^h \subset X = H_0^2$. With the same notation as that used in Section 2, the finite element discretization of the streamfunction formulation of the QGE (6) reads: Find $\psi^h \in X^h$ such that

$$a_0(\psi^h, \chi^h) + a_1(\psi^h; \psi^h, \chi^h) + a_2(\psi^h, \chi^h) = \ell(\chi^h) \qquad \forall\, \chi^h \in X^h. \tag{16}$$

Since we work with conforming finite elements, the properties of a_0, a_1, a_2 and ℓ are inherited by X^h. Thus, using the same approach as that used in Theorem 2.1, we can prove existence and uniqueness of the finite element discretization ψ^h, which is a solution of (16). Moreover, we can prove the following stability estimate for ψ^h: Traian: Check this.

$$|\psi^h|_2 \leq \frac{\|f\|_{-1}}{\nu}. \tag{17}$$

From here on, we will assume that $\nu > \nu^*$.

4 Error Estimates

The main result of this section is given in the following theorem, similar to Theorem 2.2 in [2]:

Theorem 4.1 *Let* ψ *be the solution of* (6) *and* ψ^h *the solution of* (16). *Then*

$$|\psi - \psi^h|_2 \le c(\nu) \inf_{\chi^h \in V^h} |\psi - \chi^h|_2, \tag{18}$$

where $c(\nu) := (1 + 2\,\Gamma_1\,\|f\|_{-1}/\nu^2)(1 - \Gamma_1\,\|f\|_{-1}/\nu^2)^{-1} \le c(\nu^*)$.

Proof: Since we work with *conforming* finite elements ($X^h \subset X$), (6) also holds for $\chi^h \in X^h$. Subtracting (16) from (6) with $\chi := \chi^h \in X^h$, we get the error equation:

$$\begin{aligned} a_0(\psi - \psi^h, \chi^h) \;&+\; a_1(\psi; \psi, \chi^h) - a_1(\psi^h; \psi^h, \chi^h) \\ &+\; a_2(\psi - \psi^h, \chi^h) = 0 \qquad \forall\, \chi^h \in X^h. \end{aligned} \tag{19}$$

Next, we add and subtract $a_1(\psi^h; \psi, \chi^h)$ to (19):

$$\begin{aligned} a_0(\psi - \psi^h, \chi^h) \;&+\; a_1(\psi; \psi, \chi^h) - a_1(\psi^h; \psi, \chi^h) \\ &+\; a_1(\psi^h; \psi, \chi^h) - a_1(\psi^h; \psi^h, \chi^h) \\ &+\; a_2(\psi - \psi^h, \chi^h) = 0 \qquad \forall\, \chi^h \in X^h. \end{aligned} \tag{20}$$

Denoting

$$e := \psi - \psi^h = (\psi - \lambda^h) + (\lambda^h - \psi^h) := \eta + \varphi^h, \tag{21}$$

where $\lambda^h \in X^h$ is arbitrary, equation (20) can be rewritten as

$$\begin{aligned} a_0(\eta + \varphi^h, \chi^h) \;&+\; a_1(\eta + \varphi^h; \psi, \chi^h) + a_1(\psi^h; \eta + \varphi^h, \chi^h) \\ &+\; a_2(\eta + \varphi^h, \chi^h) = 0 \qquad \forall\, \chi^h \in X^h. \end{aligned} \tag{22}$$

Letting $\chi^h := \varphi^h$ in (22), we obtain:

$$\begin{aligned} a_0(\varphi^h, \varphi^h) + a_2(\varphi^h, \varphi^h) \;&=\; -a_0(\eta, \varphi^h) - a_1(\eta; \psi, \varphi^h) - a_1(\eta; \psi, \varphi^h) \\ &\quad -a_1(\psi^h; \eta, \varphi^h) - a_1(\psi^h; \varphi^h, \varphi^h) - a_2(\eta, \varphi^h). \end{aligned} \tag{23}$$

Note that it is a simple calculation (Erich: Check!) to show that, since $\varphi^h \in X^h \subset X = H_0^2$, we have the following equality [16, 17]:

$$a_2(\varphi^h, \varphi^h) = 0. \tag{24}$$

Furthermore, because the trilinear form a_1 is skew-symmetric [2], we also have

$$a_1(\psi^h; \varphi^h, \varphi^h) = 0. \tag{25}$$

Using (24) and (25), (23) reads

$$\begin{aligned} a_0(\varphi^h, \varphi^h) \quad = \quad & -a_0(\eta, \varphi^h) - a_1(\eta; \psi, \varphi^h) - a_1(\eta; \psi, \varphi^h) \\ & -a_1(\psi^h; \eta, \varphi^h) - a_2(\eta, \varphi^h). \end{aligned} \tag{26}$$

Using the fact that

$$a_0(\chi, \chi) = \nu\, |\chi|_2^2 \qquad \forall\, \chi \in H_0^2(\Omega) \tag{27}$$

and inequalities (7)-(10), equation (26) now reads

$$\begin{aligned} \nu\, |\varphi^h|_2^2 \quad \leq \quad & \nu\, |\eta|_2\, |\varphi^h|_2 + Ro\, \Gamma_1\, |\eta|_2\, |\psi|_2\, |\varphi^h|_2 + Ro\, \Gamma_1\, |\varphi^h|_2\, |\psi|_2\, |\varphi^h|_2 \\ + \quad & Ro\, \Gamma_1\, |\psi^h|_2\, |\eta|_2\, |\varphi^h|_2 + |\eta|_2\, |\varphi^h|_2 \end{aligned} \tag{28}$$

Simplifying by $|\varphi^h|_2$ and rearranging terms in (28), we get

$$\left(\nu - Ro\, \Gamma_1\, |\psi|_2\right) |\varphi^h|_2 \quad \leq \quad \left(\nu + Ro\, \Gamma_1\, |\psi|_2 + Ro\, \Gamma_1\, |\psi^h|_2 + 1\right) |\eta|_2. \tag{29}$$

Using (11) and (17), which are the stability estimates for ψ and ψ^h, respectively, an the triangle inequality, we get

$$\begin{aligned} |e|_2 \quad & \leq \quad |\eta|_2 + |\varphi^h|_2 \\ & \leq \quad \left(\nu + Ro\, \Gamma_1\, |\psi|_2 + Ro\, \Gamma_1\, |\psi^h|_2 + 1\right) \left(\nu - Ro\, \Gamma_1\, |\psi|_2\right)^{-1} |\eta|_2 \\ & \leq \quad \left(1 + 2\, Ro\, \Gamma_1\, \|f\|_{-1}/\nu^2 + 1/\nu\right) \left(1 - Ro\, \Gamma_1\, \|f\|_{-1}/\nu^2\right)^{-1} |\eta|_2, \end{aligned} \tag{30}$$

which is exactly (18).

Traian: Check calculations and dependency on Ro. Refer back to the well-posedness of the continuous problem.

5 Numerical Results

Erich: This is your section. Zhu can help.

(1) We first need to choose a test problem (or two). One natural candidate would be the test problem we used in [20], but that's for a time-dependent problem. We need a stationary case. We could also cook up a problem with an exact solution. This would be good for computing the rates of convergence.

(2) We then need to write down *all* the parameters used: Re, Ro, etc.

(3) We then run a DNS. We need to say what that means for our Argyris FEM. What is N?

(4) We then run several *coarse* numerical simulations. We need to decide what N is for these coarse simulations.

(5) We finally write a table in which we record the errors in all the norms and the rate of convergence. These rates hopefully match the theoretical ones!

6 Conclusions and Future Work

Conclusions Here go the conclusions.

Future Work I am writing all the details for now. This is more like a "5-year-plan." Before submission, we will of course shorten this significantly. And we will keep our cards close to the chest, of course. This is for internal use only.

Boundary Conditions I think that we need to be careful when choosing the BCs. Will the weak formulation change? Erich: Check this. The real issue, however, seems to be the physical relevance of these BCs. In other words, even though (14) seems to be popular for error estimates, it might not be the most physical, say. The paper by Cummins [4] might be helpful here. Nadiga et al., however, simply use $\omega = \psi = 0$. We can try to see whether these BCs give different results from what we get with (14). I think that this could be easily tried in our stationary streamfunction formulation by using something like $-\Delta\psi = \psi = 0$. Actually, I think that this is exactly what Dominquez is using in [7]. Erich: Check this.

Time Discretization This is very important as a next step. I think this should be pretty straightforward. Medjo is treating the time-dependent case, although he does it for the *two-layer* QGE. In [16], the well-posedness of the continuous problem (i.e., *no FEM* discretization) was proven. Medjo used a *mixed* variational formulation (that is, he included the potential vorticity), but he still worked in H^2, which makes me wonder why he did not use the streamfunction formulation. In [17], he focused on the time discretization; he did not talk about the spatial FEM discretization.

Nonconforming FEM We should definitely look into applying *nonconforming* FEM. Cayco and Nicolaides did just that in [3] for the streamfunction formulation of the 2D NSE. I think we could do just the same; the problem is that it seems that one still has to use some fancy FEM, e.g., Morley and Fraeijs de Veubeke triangles (see Section 8 in [3]). Erich: Check this.

Conservation Properties The following paragraph from [18] (page 513, top) is illuminating:

"Some of the key properties of the FE method applied to the quasi-geostrophic barotropic vorticity equation were determined by Fix (1975). He showed that the Galerkin FE method conserved vorticity, energy, and enstrophy irregardless if any irregularity in the grid and had no hidden numerical dissipation associated with it. **Energy is not generally conserved in most FD numerical schemes with the exception of the Arakawa scheme. Even in the Arakawa scheme, enstrophy is only conserved only for uniform grids and rectangular boundaries.** Fix (1975) also showed that phase errors are much smaller for the FE method compared to standard FD methods."

We should check these conservation properties for the streamfunction formulation theoretically, just like Fix did for the streamfunction-vorticity formulation. We should also check them numerically. Traian, Erich: Can this be done in the stationary case?

Complex Geometries Since one of the FEM's advantages is its ability to treat complex geometries, it would be really nice to illustrate this numerically: see, e.g., Figure 1 in [18] (showing FDM and FEM domains) and Figure 8 in [6]. Traian: Check [5] for more refs. It also seems that it would be hard to use the same nice meshes and geometries as those in [18] without tackling the multiply connected domains issue (see below).

Multiply Connected Domains Again, the following paragraph from [18] (page 519, subsection d, bottom right) is illuminating:

"**Any model that is to be used to study the large-scale ocean circulation must be capable of dealing with islands.** That is, ψ cannot be arbitrarily set to zero on island boundaries for this would imply no net transport between the island and the coast. Instead, a boundary condition of $\psi = C$ should be used at all boundary points on an island, where C must be calculated within the model."

The authors go on to say on page 522, bottom left: "The islands were included in the domain: Iceland, Ireland, England, ..."

As you can see, these are pretty important islands. Figure 5 on page 523 is impressive. It would be great if we could do something similar.

Not also that the same holds for NSE. In fact, Max has an entire chapter (Chapter 12 in [14]) devoted to this issue. Max and Janet also tackle this issue in [13, 12].

I think that Tezduyar et al. [22, 23] also talk about this issue. Traian: Check this.

Approximate Deconvolution Leo should take the lead here. Can any of his numerical discretizations be used here?

Are the conservation properties valid for the QGE-AD-FEM formulation? (Fix showed that they are valid in the standard QGE-FEM formulation, without any AD.)

F Plane, Two-Layer, and Continuously Stratified QGE See Majda and Wang [15]. For all these models, we can add AD. Note that Medjo has *not* used any LES in his two-layer QGE papers. I think that the FEM analysis is wide open, at least for the F Plane and Continuously Stratified QGE. Traian: Check this. We can also add anisotropic AD, since we have a vertical component in all these models (note that we don't have this in the one-layer QGE, which is 2D).

Streamfunction-Vorticity Formulation This is the other **major** direction we should pursue **in parallel**. Zhu should take the lead here. Leo should get involved, since this is the formulation he is most comfortable with.

It looks like several authors have used the streamfunction-vorticity ((ψ, q)) formulation. The interesting part is that they all seem to use different formulations: see [16, 17] and [1]. I think that the analysis in [9, 10] will be instrumental in deciding what exactly we need to do.

Equations (49)-(51) in [18] outline the algorithm. Zhu: I think this is a good starting point for writing the algorithm and checking the implementation.

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