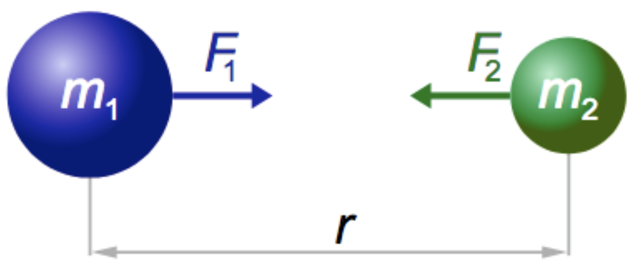
Lecture 36-37

Rocket Performance and Staging

Newton’s Laws of Gravitation

- Every particle in the universe attracts every other particle with the force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

- **The goal of rocket propulsion is to overcome the gravitational pull of the earth**

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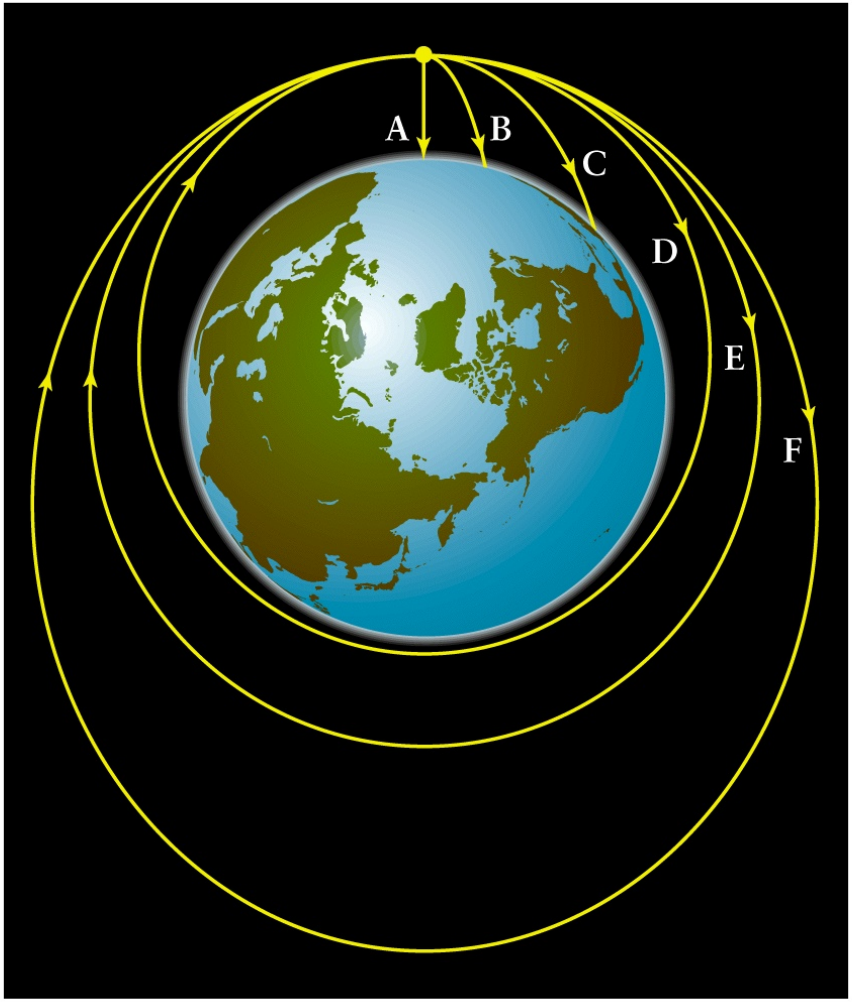
Orbital Mechanics

- Objects in orbit around Earth are constantly falling towards the Earth.
- They are acted upon by gravity, and are in free-fall towards Earth.
- They will not hit the Earth if their transverse speed is large enough.



Vis-Viva Equation

- The Vis-viva equation represents an energy balance between kinetic and potential energies of an orbiting system.

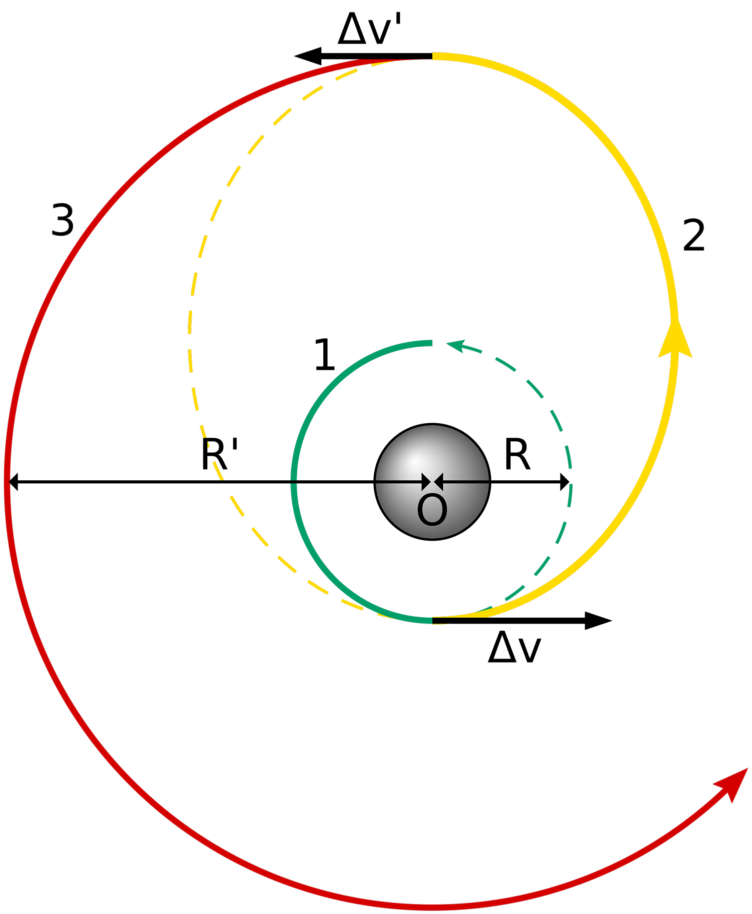
- Solving the equation for the velocity v gives:

$$v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

- where a is the semi-major axis of the orbit, r is any point in the orbit, and v is the speed in the orbit at r.
- Circular orbits apply when $r = a$ everywhere, *i.e.*:

$$v_{\text{circular}} = \sqrt{\frac{GM}{a}}$$

Delta-V and Delta-V budget



Simplest maneuver: Hohmann Orbit Transfer: Circular to Circular
Needs a total Delta-V budget of Δv + Δv’

- Delta-v is the difference in velocity from a satellite position in one orbit to a position in another orbit.
- Orbit-to-orbit maneuvers require that the vehicle be imparted a certain amount of increase or decrease in velocity
- A delta-v budget is an estimate of the total delta-v required for a space mission - calculated using orbital mechanics as the sum of the delta-v required for the propulsive maneuvers during the mission
- Delta-v determines how much propellant is required for a vehicle of given mass and propulsion system.

Static Thrust Equation for Rockets

We defined thrust for an an air-breathing engine as

$$\Im = \dot{m}_e u_e - \dot{m}_a u + (P_e - P_a) A_e$$

Since rockets don't have incoming airflow, this simplifies to

$$\mathcal{T} = \dot{m} u_e + (p_e - p_a) A_e$$

Where $\dot{m}$ is the flow of the propellant

This equation can be written as

$$\mathcal{T} = \dot{m} u_{\mathrm{eq}}$$

where

$$u_{\mathrm{eq}} = u_e + \left(\frac{p_e - p_a}{\dot{m}}\right) A_e$$

Force and Impulse

Newton's 2nd Law: The rate of change of momentum of a body is equal to the net force acting on it

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t}$$

The impulse acting on a body is defined as $\int_{t_1}^{t_2} \mathbf{F}\, dt$

Or, $I = (mv)_2 - (mv)_1$

- Impulse is the net change in momentum imparted to a body due to a force acting on it over a period of time.
- The same impulse can be produced with a small force acting over a long time or a large force acting over a small time

Specific Impulse in Rockets

Assuming that the propellant velocity is constant the impulse imparted by the rocket motor to the vehicle can then be calculated as

$$I = \int \mathcal{T}\, dt = \mathcal{M}_p u_{\mathrm{eq}}$$

where $\mathcal{M}_p$ is the total mass of expelled propellant.

The impulse per unit mass is then

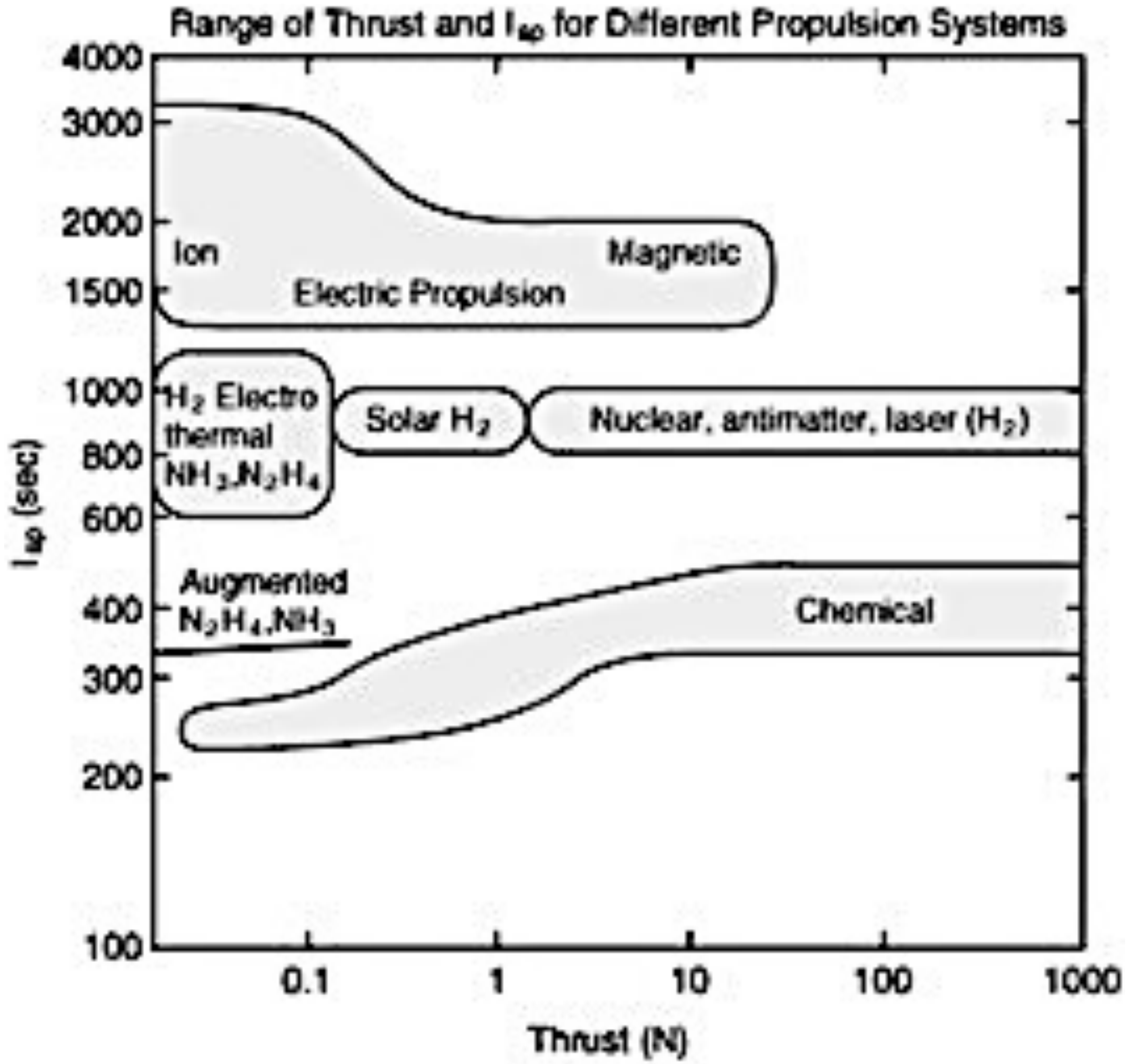
$$\frac{I}{\mathcal{M}_p} = \frac{\mathcal{T}}{\dot{m}} = u_{\mathrm{eq}}$$

Specific Impulse is defined as

$$\boxed{I_{\mathrm{sp}} = \frac{I}{\mathcal{M}_p g_e} = \frac{u_{\mathrm{eq}}}{g_e}}$$

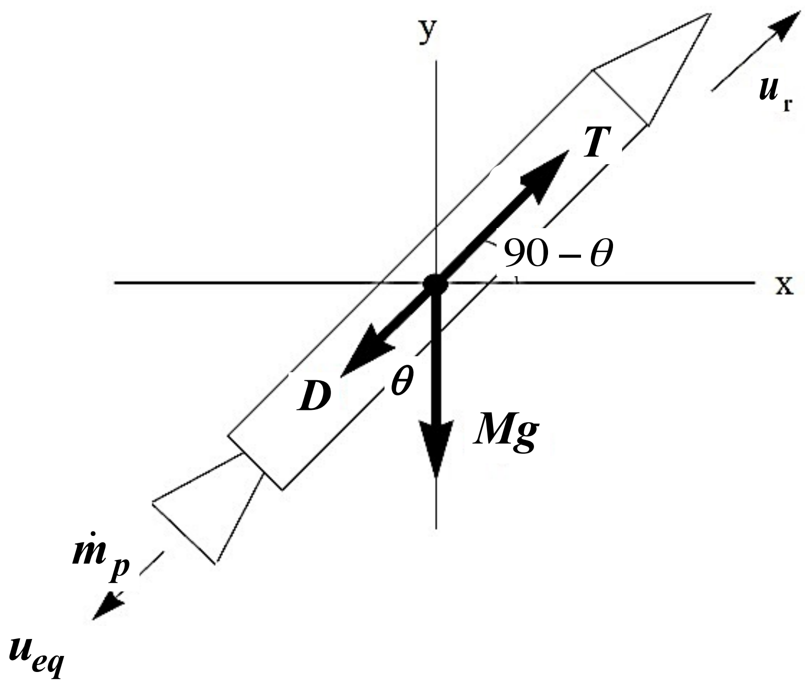
Isp of Different Space Propulsion Systems

- Chemical Propulsion systems can provide high thrust a low input power, but low Isp
- Electric Propulsion systems can provide High Isp, but have high power demands, and low Thrust



The Tsiolkovsky Rocket Equation

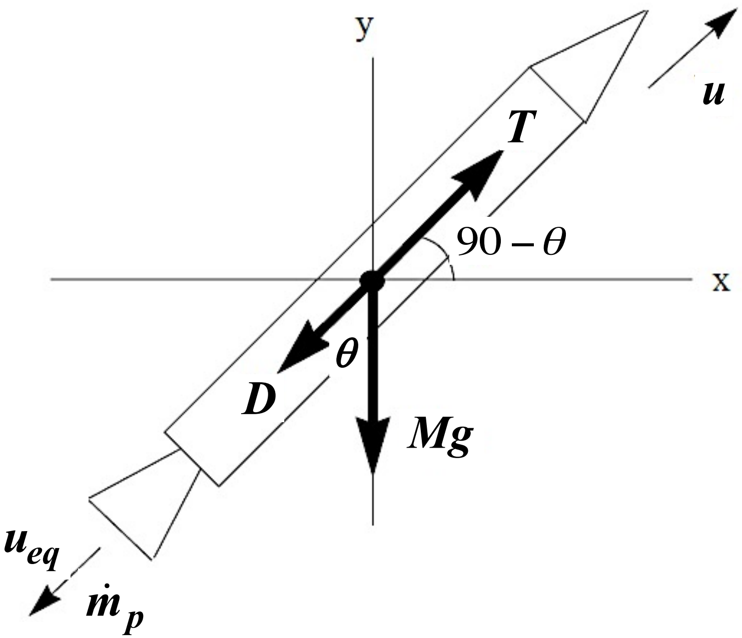
(fundamental equation of rocketry)



- T = vehicle thrust
- D = aerodynamic drag
- u = vehicle velocity
- θ = angle with respect to vertical
- M = vehicle mass
- g = local gravitational acceleration
- $\dot{m}_p$ = nozzle mass flow
- MR = mass ratio = M_o/M_b
- M_o = Initial Mass
- M_b = Final (Burnout Mass)
- $M_p = M_{propellant} = M_o - M_b$

The Tsiolkovsky Rocket Equation

(fundamental equation of rocketry)



$$m\bar{a} = \sum F_{axis}$$

$$M\frac{du}{dt} = T - D - Mg\cos\theta$$

$$\frac{du}{dt} = \frac{\dot{m}_p}{M}u_{eq} - \frac{D}{M} - g\cos\theta$$

$$\mathrm{u_{eq}} = const$$

integrating from initial to burnout

$$\Delta u = u_{eq}\ln\left(\frac{M_0}{M_b}\right) - \int_0^{t_b}\frac{D}{M}dt - \int_0^{t_b} g\cos\theta\, dt$$

Fundamental Equation of Rocketry

$$Since,\ I_{sp} = \frac{u_{eq}}{g_e}$$

$$\Delta u = g_e I_{sp} \ln(MR) - \int_0^{t_b} \frac{D}{M} dt - \int_0^{t_b} g \cos\theta \, dt$$

Vehicle *Δu* = Velocity imparted by Propulsion System - Drag Loss - Gravity (or "g-t") Loss

- (drag loss is absent in the case of space maneuvers)
- (propulsion forces >> gravity forces in many space maneuvers)
- Note: *Δu calculated here must be greater than the delta-V budget for the mission/maneuver*

Mission Delta-V Requirements

Mission (duration)	Delta-V (km/sec)
Earth surface to LEO	4-6
LEO to Mars Orbit (0.7 yr)	5.7
LEO to Neptune (5.0 yr)	11.7
LEO to alpha-Centauri (50 yr)	29.8

LEO = Low Earth orbit (approx. 274 km)

- For a typical launch vehicle headed to orbit: aerodynamic drag losses are small, on the order of 100 to 500 m/s.
- Gravitational losses are larger, generally ranging from 700 to 1200 m/s depending on the shape of the trajectory.

Propellant Calculation Exercise

- Determine the mass of propellant to send a 2500 kg spacecraft from LEO to Mars Orbit (0.7 yr mission).
  - Assume the 2500 kg includes the propellant on-board at the start of the burn.
  - Assume our engine has a specific impulse of 310 sec (typical of a small bipropellant engine).
  - **Use the *Rocket Equation:***

Ignore drag and gravity terms

$$\Delta u = g_e I_{sp} \ln(MR)$$

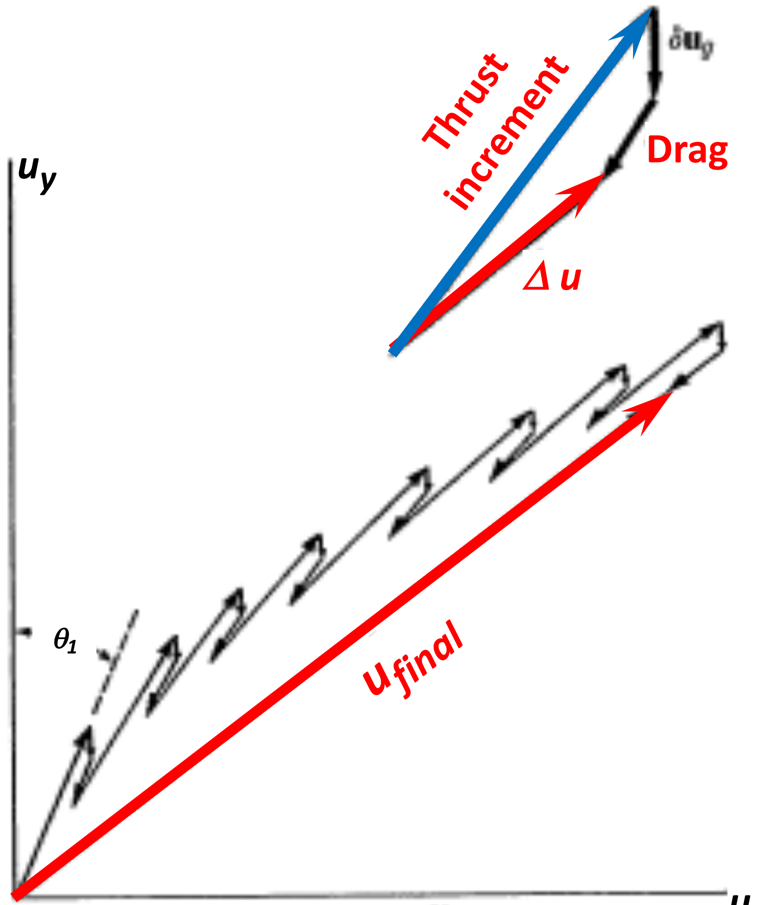
$$M_p = M_o\left[1 - exp\left(\frac{-\Delta u}{g_e I_{sp}}\right)\right]$$

$$= (2500\ kg)\left[1 - exp\left(\frac{-5700\ m/s}{(9.8\ m/s^2)(310\ s)}\right)\right] = 2117\ kg$$

Most of our spacecraft is propellant! Only 383 kg is left for structure and payload

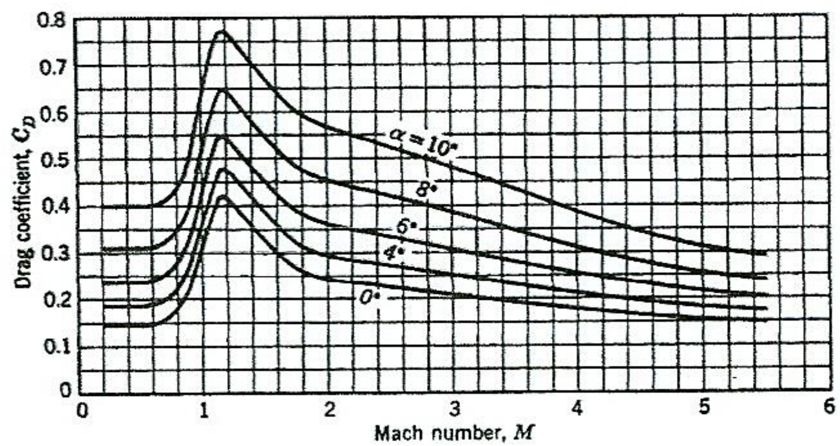
Effect of gravity and drag

H & P, Figure 10.5: calculation of a "gravity turn" from launch at θ_1



(a) Velocity coordinates

Drag ; nonlinear, transonic behavior



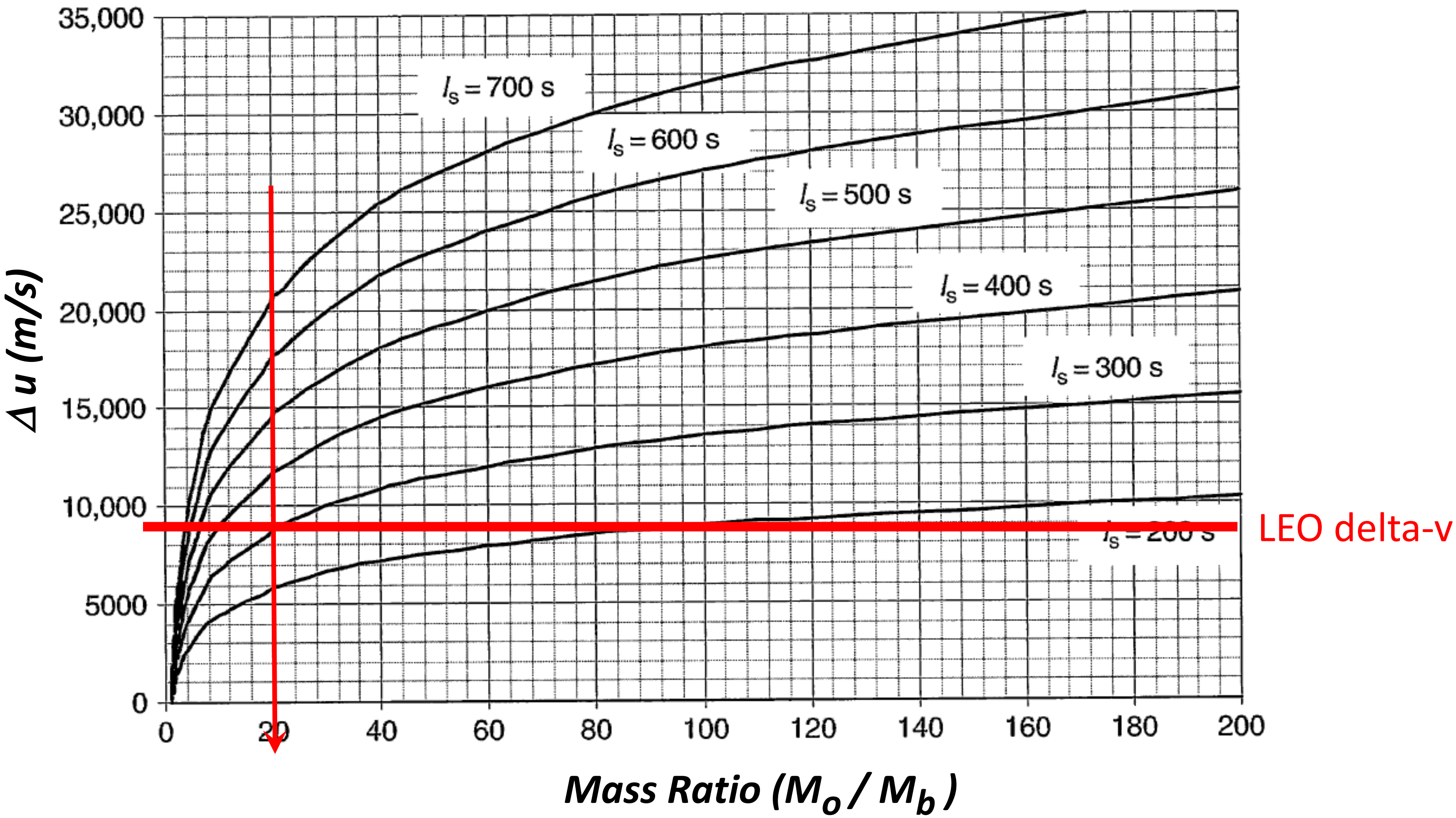
Launch vehicles must "throttle down" around maximum dynamic pressure.

Recall shuttle launch: "vehicle has throttled back to 67% of rated performance, reducing stress on shuttle as it breaks through the sound barrier".

$$\Delta u = g_e I_{sp} \ln(MR) - \int_0^{t_b} \frac{D}{M} dt - \int_0^{t_b} g \cos\theta \, dt$$

Rocket Velocity Increment vs. Mass Ratio

(as a function of specific impulse)



$M_o = 20 \times M_b = 20 \times (M_L + M_S) \Rightarrow$ **5 % payload and structure; 95% propellant**

Staging

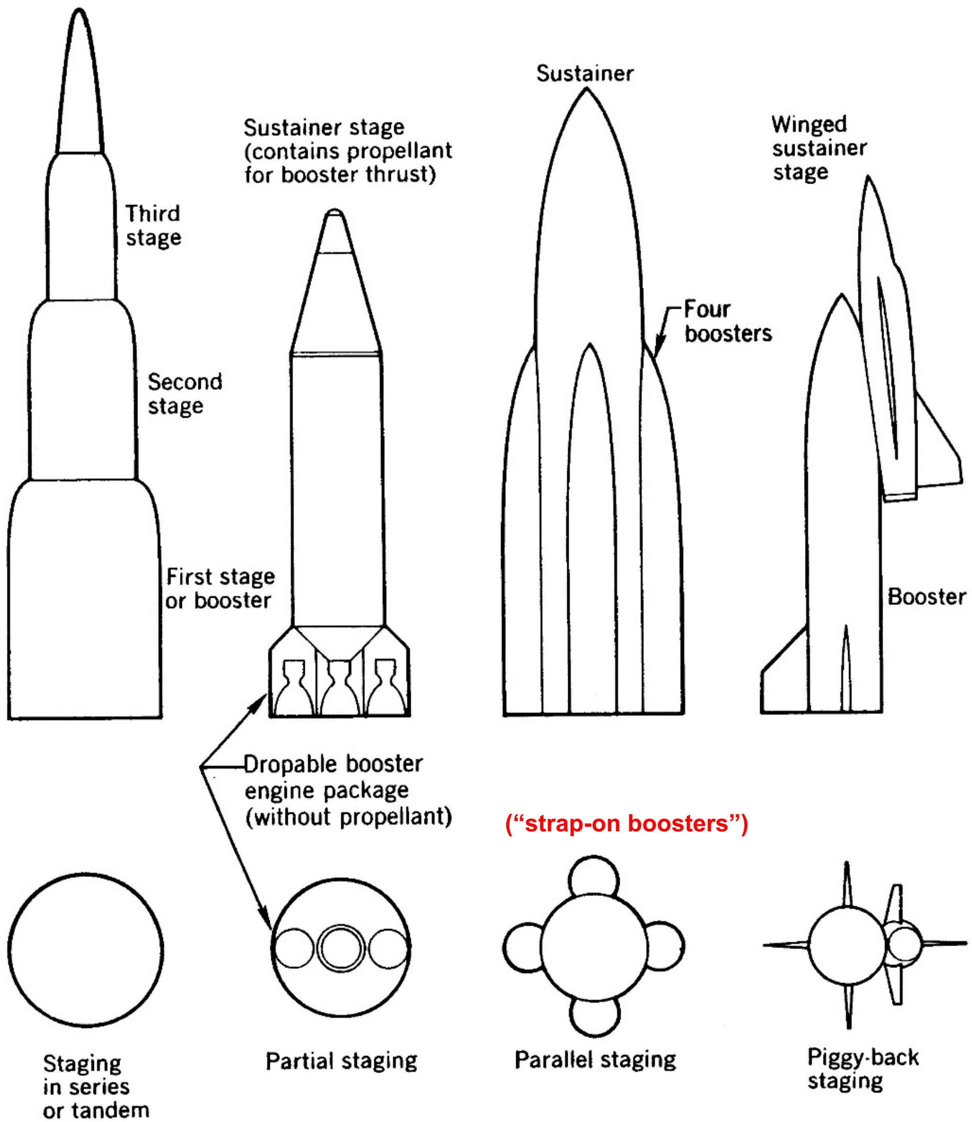
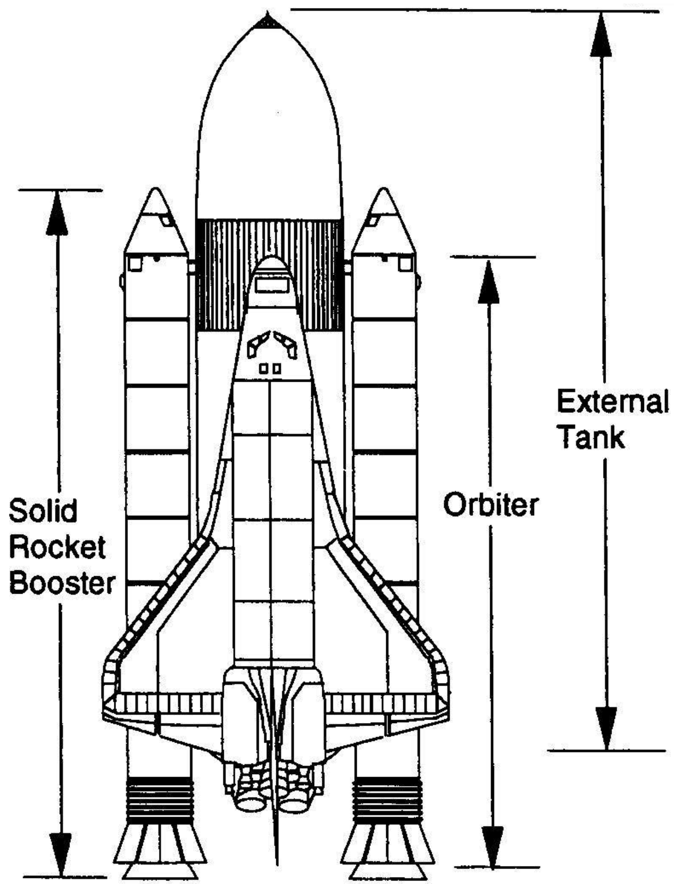
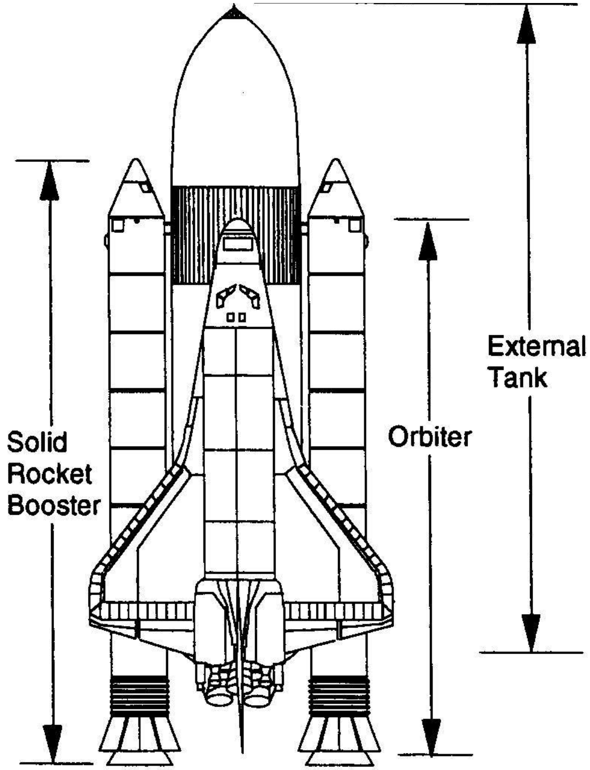


Fig. 5–6. Simplified sketches of several staging configurations.



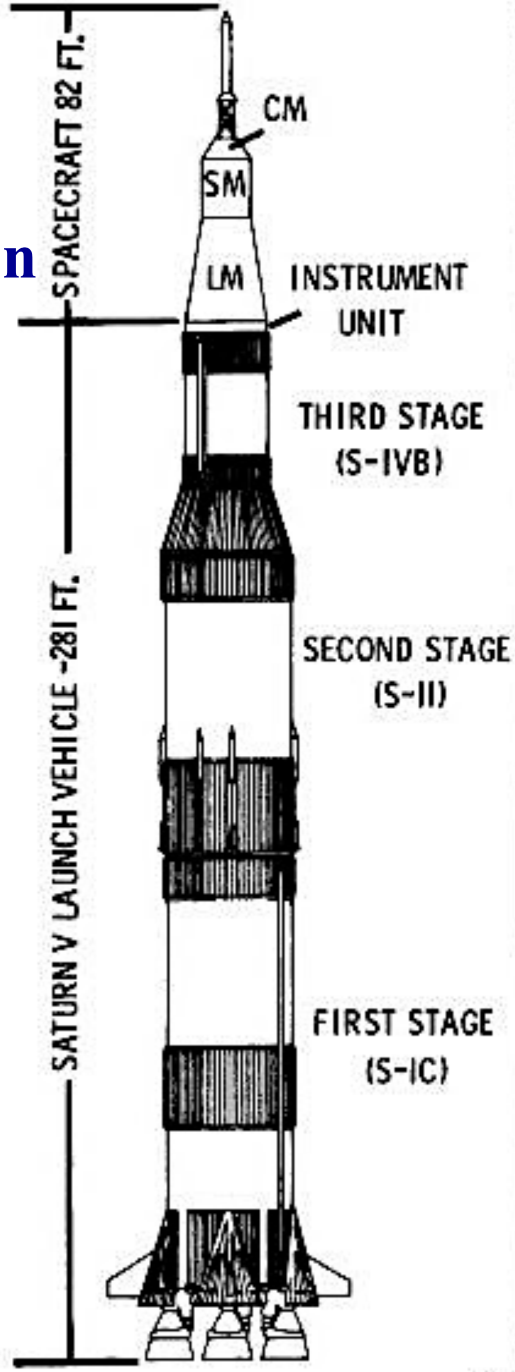
[Fig 11.18, Farokhi]

Staging Example: Space Shuttle



- Shuttle is launched vertically with all main engines firing.
- At an altitude of about 45 km, after some 2 minutes, the boosters separate for recovery and refurbishment.
- The Orbiter continues under SSME power until about 8 min 50 s after launch, when the external tank separates for destructive reentry.
- Earlier missions required two OMS burns to attain operational orbit but a direct ascent technique is now employed, omitting the OMS-1 burn and relying on the OMS-2 burn at apogee about 45 min after launch.

Saturn-V Rocket for Apollo 10 Moon Mission



FIRST STAGE (S-IC)	
DIAMETER	33 FEET
HEIGHT	138 FEET
WEIGHT	5,031,023 LBS. FUELED 294,200 LBS. DRY
ENGINES	FIVE F-1
PROPELLANTS	LIQUID OXYGEN (3,258,280 LBS.) RP-1 (KEROSENE) - (1,417,334 LBS.)
THRUST	7,680,982 LBS.

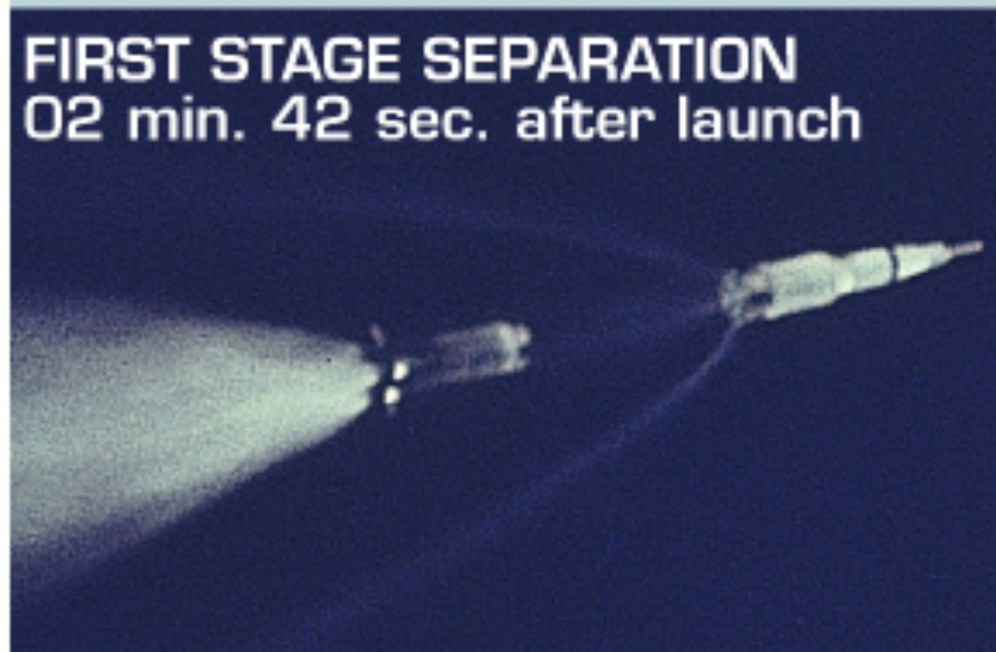
SECOND STAGE (S-II)	
DIAMETER	33 FEET
HEIGHT	81.5 FEET
WEIGHT	1,074,590 LBS. FUELED 84,367 LBS. DRY
ENGINES	FIVE J-2
PROPELLANTS	LIQUID OXYGEN (829,114 LBS.) LIQUID HYDROGEN (158,231 LBS.
THRUST	1,163,854 LBS.
INTERSTAGE	8,890 LBS.

THIRD STAGE (S-IVB)	
DIAMETER	21.7 FEET
HEIGHT	58.3 FEET.
WEIGHT	261,836 LBS. FUELED 25,750 LBS. DRY
ENGINES	ONE J-2
PROPELLANTS	LIQUID OXYGEN (190,785 LBS.) LIQUID HYDROGEN (43,452 LBS.)
THRUST	203,615 LBS.
INTERSTAGE	8,081 LBS.

INSTRUMENT UNIT	
DIAMETER	21.7 FEET
HEIGHT	3 FEET
WEIGHT	4,254 LBS.

Saturn-V Rocket for Apollo 10 Moon Mission

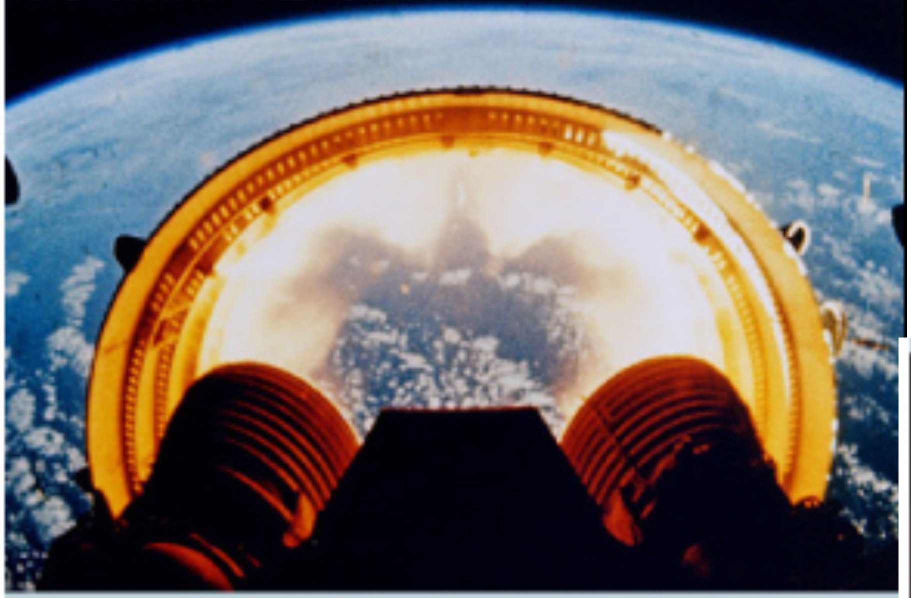
The first stage's five F-1 rocket engines ignite and produce 7.5 million pounds of thrust



At an altitude of 42 miles (67 kilometers), the F-1 engines shut down. Explosive bolts fire, and the severed first stage falls into the Atlantic Ocean

Saturn-V Rocket for Apollo 10 Moon Mission

INTERSTAGE SEPARATION
03 min. 12 sec.



The second stage's J-2 engines ignite, and the interstage skirt drops away

ESCAPE TOWER JETTISON
03 min. 18 sec.

Useful only up to about 19 miles altitude, the launch escape tower is now dead weight and is discarded

Saturn-V Rocket for Apollo 10 Moon Mission

The third stage's single J-2 engine ignites about ten seconds after the second stage is cut loose

At engine cutoff, the Apollo spacecraft has reached a speed of 17,432 miles per hour (28,000 kph) and is in orbit around Earth at an altitude of 118.8 miles (191.2 km)

Saturn-V Rocket for Apollo 10 Moon Mission

The third stage's J-2 engine is fired again to break out of Earth orbit and send the vehicle toward the moon

Now on course for the moon, Apollo separates, turns 180 degrees and pulls the lunar module from atop the third stage. Its job finished, the Saturn stage coasts away into space (some missions commanded the third stage to crash into the moon)

Rocket Staging

- Goal is to discard empty tanks and extra structure as rocket travels, so that this mass is not subjected to gravity losses
- Large engines used for initial high thrust phase, may produce excessive accelerations when propellant is nearly consumed
- Multistage rocket is a series of individual vehicles or stages, each with its own structure, tanks and engines
- Each stage accelerates payload before being detached

Rocket Mass Definitions

- Total mass of rocket, M_o, may be written as sum of 3 primary components:
  - Payload mass, M_L
  - Propellant mass, M_P
  - Structural mass, M_S
    - Includes everything but payload and propellant
    - Engines, tanks, controls, etc.
- If rocket consumes all its propellant during firing, burnout mass consists of structure and payload:

$$\text{Initial Mass}, M_o = M_L + M_P + M_S$$

$$\text{Final (Burnout) Mass}, M_b = M_L + M_S$$

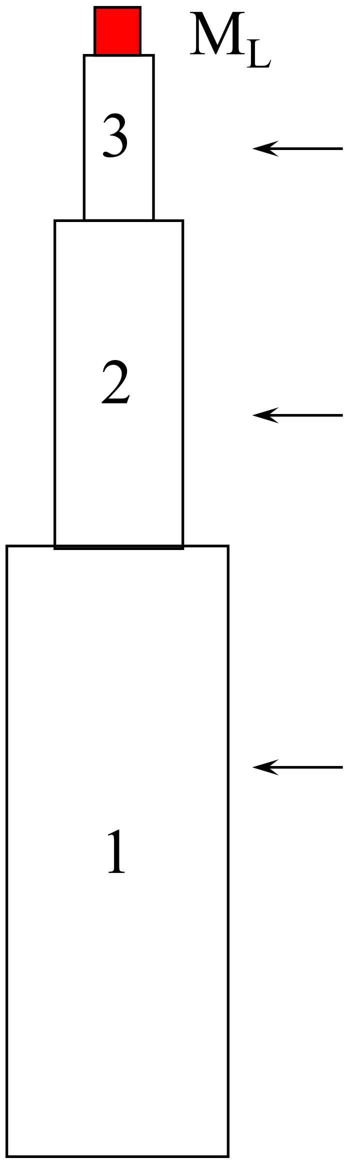
Mass Ratio Definitions

Symbol	Ratio	Description
MR	$MR = \frac{M_o}{M_b} = \frac{M_o}{M_L + M_S}$	**Mass Ratio**: initial mass / mass at the end of the thrust period. Want this ratio large.
λ	$\lambda = \frac{M_L}{M_o - M_L} = \frac{M_L}{M_P + M_S}$	**Payload Ratio**: ratio of payload to everything but payload. Want this large, but larger the payload, the lower maximum attainable velocity.
ε	$\varepsilon = \frac{M_S}{M_P + M_S} = \frac{M_b - M_L}{M_o - M_L}$	**Structural Coefficient**: ratio of the structural weight to everything but the payload. Want this small.
ζ	$\zeta = \frac{M_P}{M_P + M_S} = \frac{M_P}{M_o - M_L}$	**Propellant Ratio**: Ratio of propellant to everything but the payload.

Additional relationships

$$MR = \frac{1+\lambda}{\varepsilon+\lambda} \qquad \zeta = 1 - \varepsilon$$

Multistage Rocket Analysis



Total Mass 3: $M_{o3}=M_{P3}+M_{S3}+M_L$

Payload for Stage 3: $M_{L3}=M_L$

Total Mass 2: $M_{o2}=M_{P2}+M_{S2}+M_{o3}$

Payload for Stage 2: $M_{L2}=M_{o3}$

Total Mass 1: $M_{o1}=M_{P1}+M_{S1}+M_{o2}$

Payload for Stage 1: $M_{L1}=M_{o2}$

Total Mass i: $M_{oi}=M_{Pi}+M_{Si}+M_{o(i+1)}$

Payload for Stage i: $M_{Li}=M_{o(i+1)}$

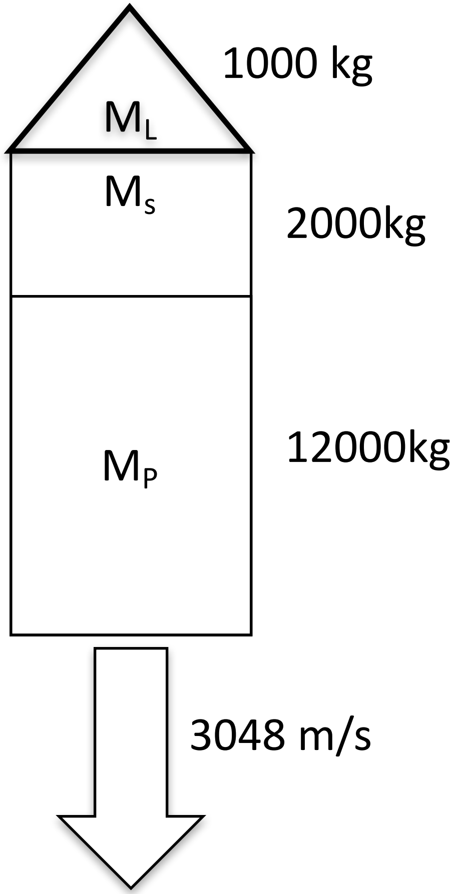
- Stages are numbered in order of firing
- Payload for an particular stage is the mass of all subsequent stages

Single vs Two Stage Rocket Example

- A single stage rocket with an exhaust velocity of 3048 m/s.
- M_L = 1000 kg, M_O = 15000kg, M_S = 2000kg.
- M_P = 15000-2000-1000 = 12000kg
- Everything but payload, $M_O - M_L$ = 14000kg
- Payload ratio, λ = 1000/14000 = 0.0714
- Structural coefficient, ε = 2000/14000 = 0.143
- Mass ratio, MR = 15000/3000 = 5
- Or we can use the ratios defined in Slide 12

$$MR = \frac{1+\lambda}{\varepsilon+\lambda} = \frac{1+0.714}{0.143+0.0714} = \frac{1.0714}{0.2144} \approx 5$$

Δu = Ue * ln (MR) = 3048 * ln(5) = 4905 m/s



Single vs Two Stage Rocket Example

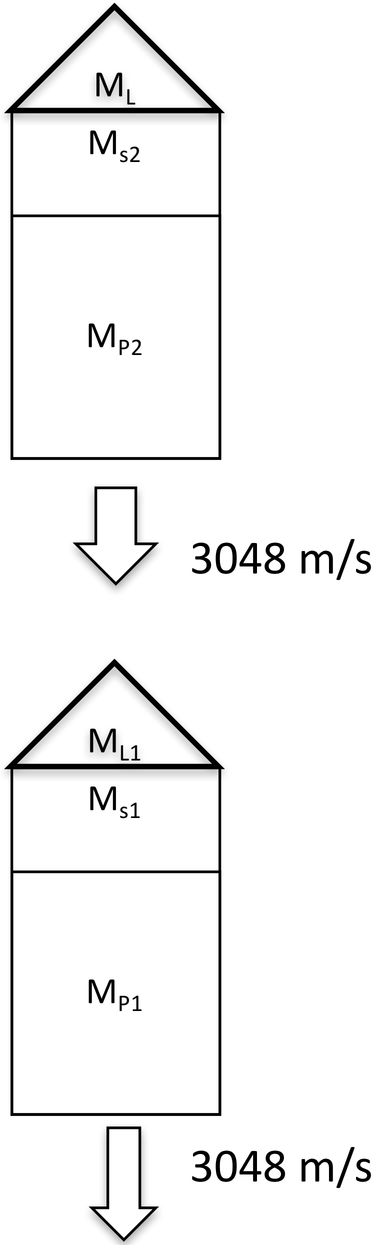
- A two stage rocket with an exhaust velocity of 3048 m/s and same initial mass, total structural mass, payload as the single stage rocket
- $M_{O1} = M_{P1} + M_{S1} + M_{L1} = 15000\text{kg}$ (total rocket weight is the same)
- $M_{O2} = M_{L1} = M_{P2} + M_{S2} + M_L$ (Second stage is the payload for stage 1)
- $M_L = 1000\text{kg}$ (desired payload to deliver)
- $M_{S1} + M_{S2} = 2000\text{kg}$ (same total structural mass)
- Assume payload ratio is the same, i.e,

$$\lambda_1 = \lambda_2 = \frac{M_{02}}{M_{01} - M_{02}} = \frac{M_L}{M_{02} - M_L}$$

$$\frac{M_{02}}{15000 - M_{02}} = \frac{1000}{M_{02} - 1000}$$

$$Or, M_{02} = 3872kg$$

$$And \ \lambda_1 = \lambda_2 = 0.348$$



Single vs Two Stage Rocket Example

- Assume structural ratio is the same, i.e,

$$\varepsilon_1 = \varepsilon_2 = \frac{M_{S1}}{M_{01} - M_{02}} = \frac{M_{S2}}{M_{02} - M_L}$$

$$M_{S1} = \frac{M_{S2}}{M_{02} - M_L}(M_{01} - M_{02})$$

$$But, M_{S1} = 2000 - M_{S2}$$

$$(2000 - M_{S2})(3872 - 1000) = M_{S2}(15000 - 3872)$$

$$M_{S2} = 411kg, M_{S1} = 1589kg$$

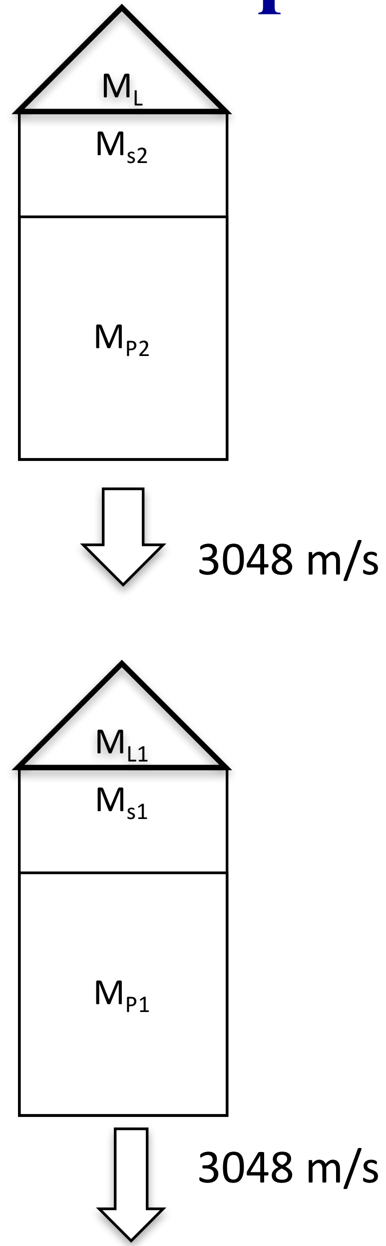
$$\text{And } \varepsilon_1 = \varepsilon_2 = 0.143$$

- Since the structural and payload ratios are assumed to be same, the mass ratios are the same. Therefore,

$$MR_1 = MR_2 = \frac{1+\lambda}{\varepsilon+\lambda} = \frac{1+0.348}{0.143+0.348} = 2.75$$

$$\Delta u_1 = \Delta u_2 = 3048\ln(2.75) = 3080\text{m/s}$$

$$\text{Total } \Delta u = \Delta u_1 + \Delta u_2 = 6160\text{m/s}$$



General n-stage rocket

- Terminal velocity increment $$u_n = \sum_{i=1}^{n} u_{ei} \ln \mathcal{R}_i$$

- Assuming all exhaust velocities are same $$u_n = u_e \ln\left(\prod_{1}^{n} \mathcal{R}_i\right)$$

- Assuming identical coefficients $$u_n = n u_e \ln \mathcal{R}$$

- Stage payload ratios can be related as

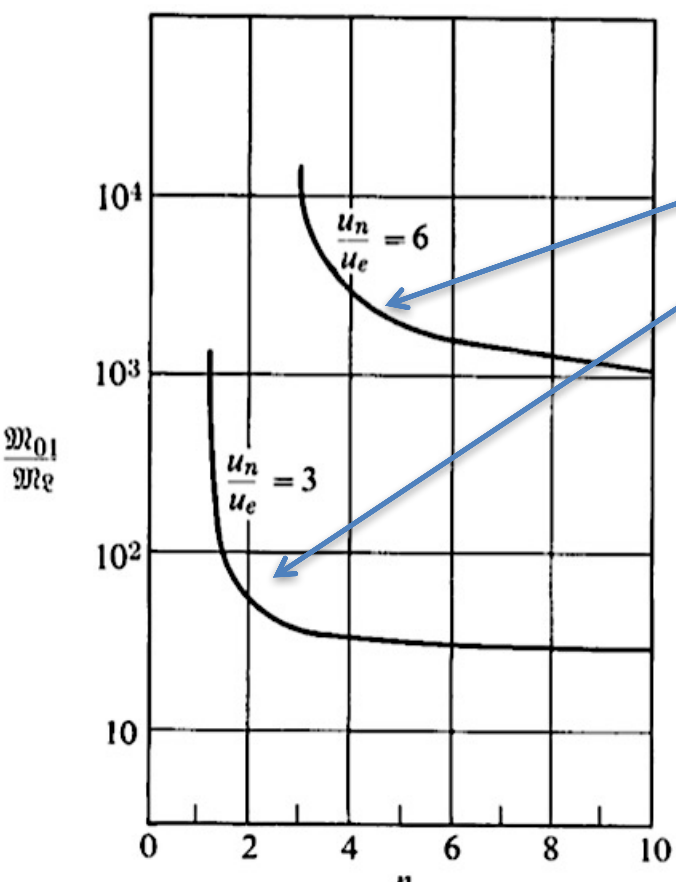
$$\frac{\mathcal{M}_{01}}{\mathcal{M}_{02}} = \frac{1+\lambda_1}{\lambda_1},$$
$$\frac{\mathcal{M}_{02}}{\mathcal{M}_{03}} = \frac{1+\lambda_2}{\lambda_2},$$
$$\vdots \qquad \vdots$$
$$\frac{\mathcal{M}_{0(n-1)}}{\mathcal{M}_{0n}} = \frac{1+\lambda_{n-1}}{\lambda_{n-1}},$$
$$\frac{\mathcal{M}_{0n}}{\mathcal{M}_{\mathcal{L}}} = \frac{1+\lambda_n}{\lambda_n}.$$

$$\Rightarrow \quad \frac{\mathcal{M}_{01}}{\mathcal{M}_{\mathcal{L}}} = \prod_{i=1}^{n}\left(\frac{1+\lambda_i}{\lambda_i}\right) \quad \Rightarrow \quad \frac{\mathcal{M}_{01}}{\mathcal{M}_{\mathcal{L}}} = \left(\frac{1+\lambda}{\lambda}\right)^n.$$

General n-stage rocket

Combining $\boxed{\frac{\mathcal{M}_{01}}{\mathcal{M}_{\mathcal{L}}} = \left(\frac{1+\lambda}{\lambda}\right)^n}$ $\boxed{u_n = n u_e \ln \mathcal{R}}$

$$\frac{u_n}{u_e} = n \ln\left\{\frac{(\mathcal{M}_{01}/\mathcal{M}_{\mathcal{L}})^{1/n}}{\epsilon[(\mathcal{M}_{01}/\mathcal{M}_{\mathcal{L}})^{1/n} - 1] + 1}\right\}$$



- Need to make M_{O1}/M_L as small as possible
- For a given terminal velocity, u_n, you reach a point of diminishing returns beyond a certain number of stages
- For most practical applications, cost is a driving factor, so typically #stages is kept limited
- Higher terminal velocities (i.e., Delta-v) requires more stages