Compressible Flow Review

Lecture 3

Information propagation in a fluid

- Disturbances are transmitted through a gas as a result of collisions between randomly moving molecules.
- Transmission of a <u>small disturbance</u> through a gas can be treated as rev & adiabatic, i.e., isentropic.
- Conditions in the gas vary infinitesimally before and after the disturbance passes through.
- <u>Speed of Sound</u>: defined as the speed at which small (reversible) disturbances travel into a quiescent gas.

Speed of Sound

By reducing the N-S equations into the wave equation using the assumption of small disturbances, the speed of sound can be calculated as

$$a = \left(\frac{du}{d\rho / \rho}\right)_{isentropic} = \sqrt{\frac{dp}{d\rho}}_{isentropic} = \sqrt{\gamma R T}$$

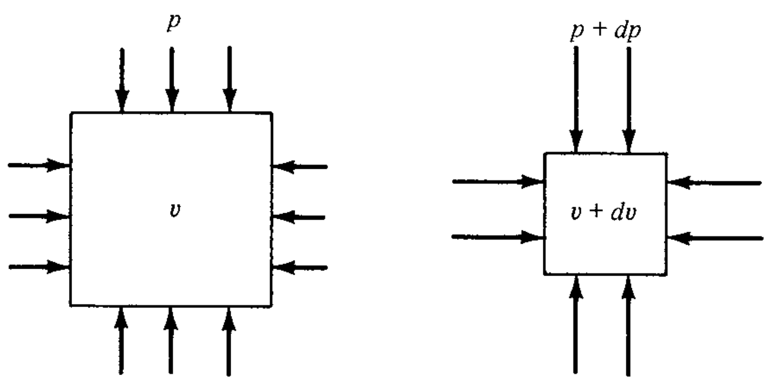
Since:

- *R* is constant
- γ is only a function of T
- speed of sound is only a function of temperature

AIR	
200 K	284 m/s
300K	347 m/s
1000K	634 m/s

For air γ = 1.4

What is compressibility



$$\tau = -\frac{1}{v}\frac{dv}{dp}$$

- All materials undergo a change in volume when a pressure is applied on them
  - Typically gases are more compressible than liquids and liquids are more compressible than solids
- Compressibility is a measure of the fractional change in volume for a given incremental pressure change
- The negative sign indicates that increasing the pressure, decreases the volume
- Also, since $v = 1/\rho$,

$$\tau = \frac{1}{\rho}\frac{d\rho}{dp}$$

Compressibility and Mach No.

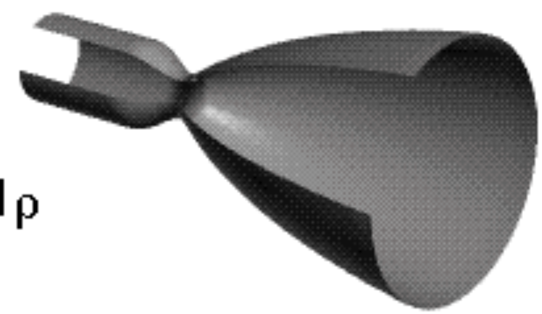
V = velocity
ρ = density

p = pressure
M = Mach

T = temperature
R = gas constant

a = speed of sound
γ = specific heat ratio

Conservation of Momentum: $\rho\, V\, dV = -\, dp$



Isentropic Flow: $\dfrac{dp}{p} = \gamma \dfrac{d\rho}{\rho}$

$$dp = \gamma \frac{p}{\rho}\, d\rho = \gamma R T\, d\rho$$

$$dp = a^2\, d\rho$$

Combine with Momentum: $\rho\, V\, dV = -\, a^2\, d\rho$

$$\boxed{-M^2 \frac{dV}{V} = \frac{d\rho}{\rho}}$$

- Air is a compressible fluid.
- Its density is dependent on velocity/pressure variations imposed by an external body
- However, if M << 1, then the density is essentially invariant with velocity

https://www.grc.nasa.gov/www/k-12/airplane/shortp.html

Stagnation Properties

- For high-speed flows, enthalpy, and kinetic energy are combined into a **stagnation enthalpy, *h_0***

$$h_0 = h + \frac{V^2}{2}$$

- h_0 represents the enthalpy of a fluid when it is brought to rest reversibly and adiabatically.

- During a stagnation process, kinetic energy is converted to enthalpy.

- Stagnation enthalpy remains constant for steady, adiabatic flow, with no shaft/electrical work, and no change in potential energy, i.e., energy is conserved.

Stagnation Properties

- **For an ideal gas**, $\Delta h = \Delta(c_p T)$, which allows h_0 to be rewritten as

$$c_p T_0 = c_p T + \frac{u^2}{2} \quad \Rightarrow \quad T_0 = T + \frac{u^2}{2c_p}$$

  – T_0 is the stagnation temperature; it represents *the temperature an ideal gas attains when it is brought to rest adiabatically.*
  – $u^2/2c_p$ corresponds to the temperature rise during stagnation, and is called the **dynamic temperature.**

- For and ideal gas with constant specific heats, stagnation pressure and density can be expressed as

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)} \qquad \frac{v}{v_0} = \frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{1/(\gamma-1)}$$

Isentropic Property Relationships

- Relationships between static properties and stagnation properties in terms of Mach# are easy to obtain. For example, in the case of stagnation temperature for an ideal gas:

$$T_0 = T + \frac{u^2}{2c_p} = const \qquad \frac{T_0}{T} = 1 + \frac{u^2}{2c_p T}$$

- Using the expressions below,

$$\frac{R}{c_p} = \frac{\gamma - 1}{\gamma} \quad a^2 = \gamma R T \quad M = \frac{u}{a}$$

the static properties can be expressed in terms of *Mach #* and γ

$$\frac{T_0}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2 \qquad \frac{p_0}{p} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{\gamma/(\gamma-1)} \qquad \frac{\rho_0}{\rho} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{1/(\gamma-1)}$$

Area Velocity Relationships

Conservation of Mass (1)

$$\rho u A = \text{const} \quad \Rightarrow \quad d(\rho u A) = 0$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Euler's equation (2)

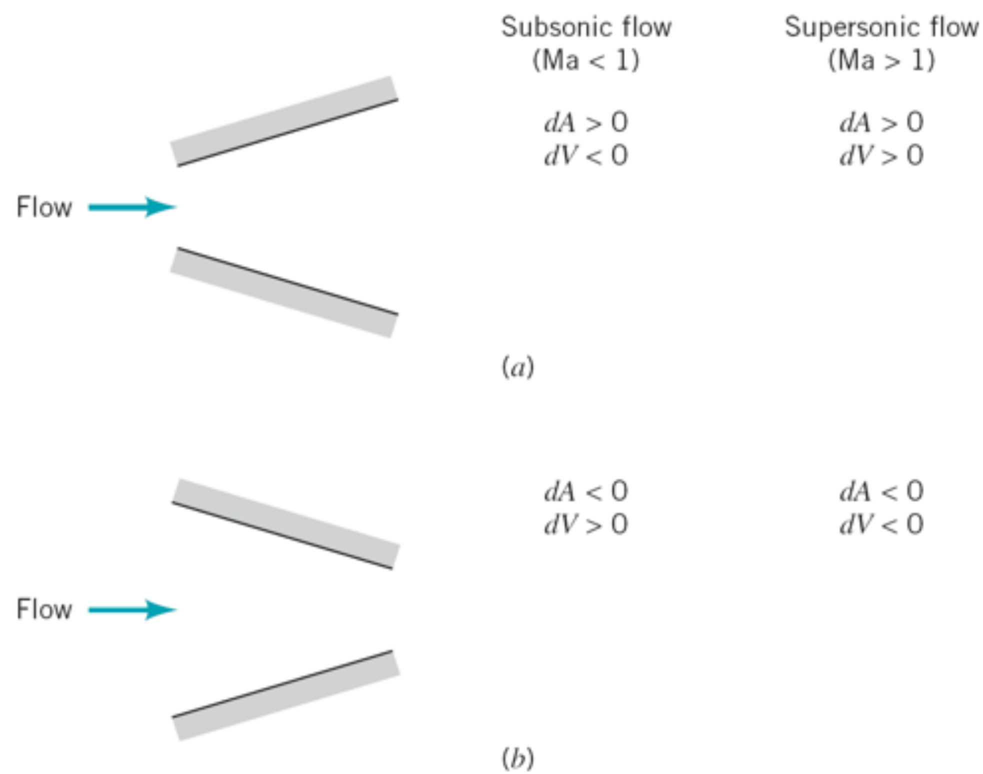
$$dp = -\rho u \, du$$

Speed of Sound (3)

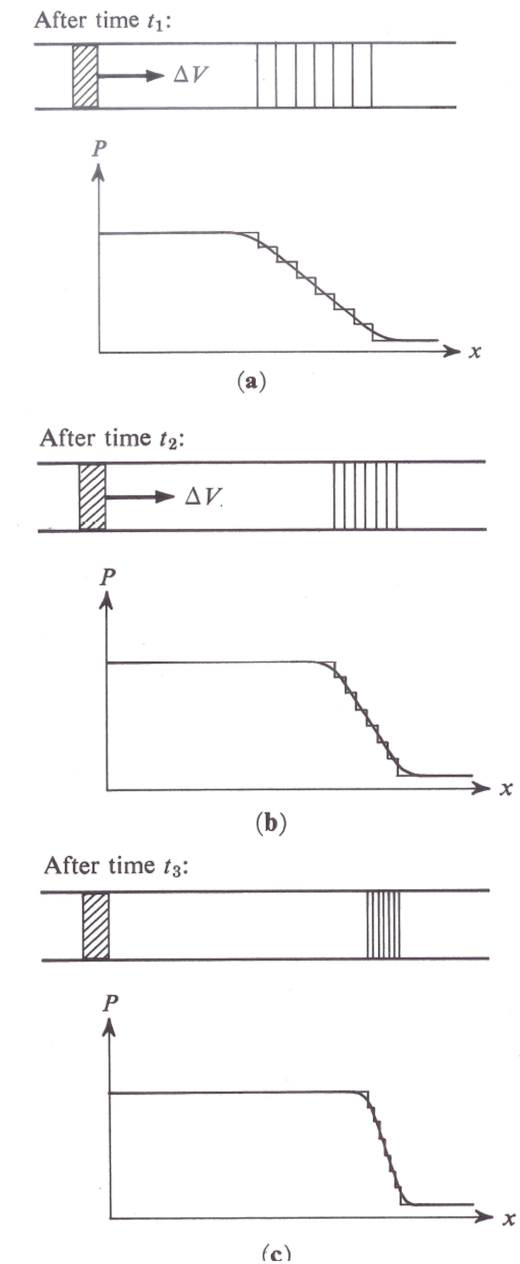
$$\frac{dp}{d\rho} = a^2$$

Using conservation of mass, Euler's equation, and speed of sound we get

$$\boxed{\frac{dA}{A} = (M^2 - 1)\frac{du}{u}}$$



Shock Waves

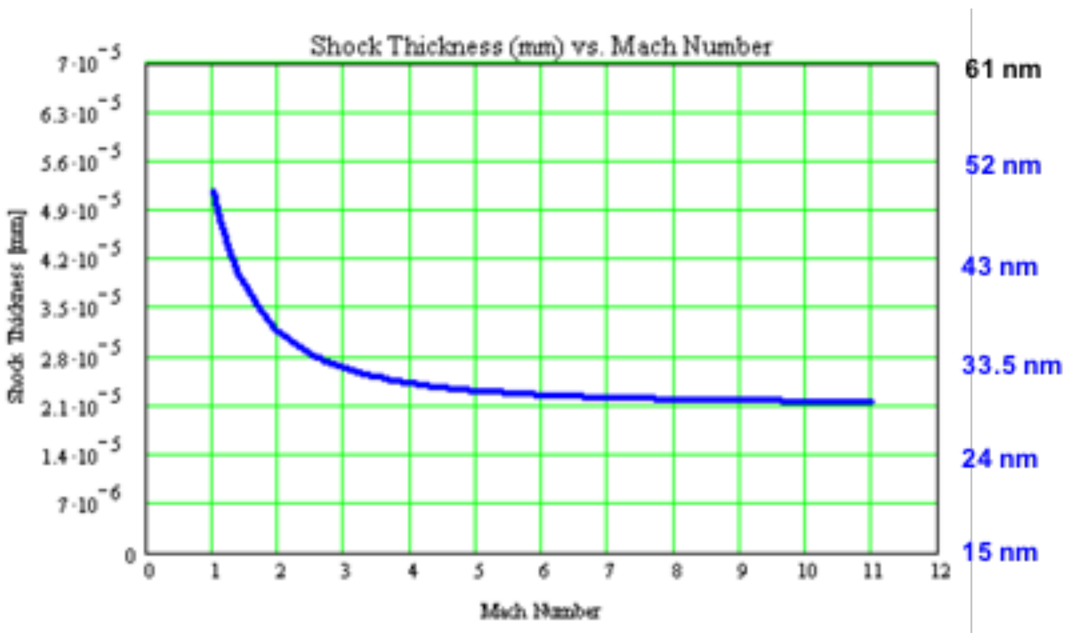


Shock Wave:

An abrupt change in fluid properties in a very small region of space.

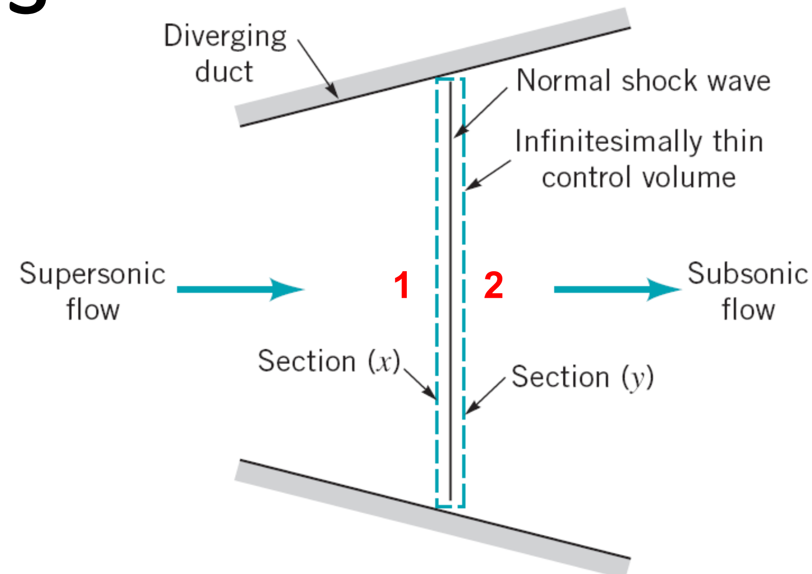
Types of Shock Waves:

1) normal (1-D)
2) oblique (2-D)
3) conical (3-D)



Governing Equations

Normal shock control volume.



Cons of Mass:

$$\rho_x u_x = \rho_y u_y = const$$

Cons of Momentum:

$$p_x + \rho_x u_x^2 = p_y + \rho_y u_y^2 = const$$

Cons of Energy:

$$h_x + u_x^2/2 = h_y + u_y^2/2 = const$$

2nd Law: condition of reversibility violated in a very narrow region

$$s_y - s_x = C_p \ln\frac{T_y}{T_x} - R\ln\frac{p_y}{p_x} > 0$$

Constitutive (Property) Relationships:

$\Delta h = \Delta(c_p T)$ $\quad$ $p = \rho RT$ $\quad$ $\gamma = c_p / c_v$ $\quad$ $M = u/a$ $\quad$ etc.

Governing Equations

$$M_y^2 = \frac{(\gamma-1)M_x^2+2}{2\gamma M_x^2-(\gamma-1)} = \frac{M_x^2+[2/(\gamma-1)]}{[2\gamma/(\gamma-1)]M_x^2-1}$$

$$\frac{p_y}{p_x} = \frac{2\gamma}{\gamma+1}M_x^2 - \frac{\gamma-1}{\gamma+1} = \frac{1+\gamma M_x^2}{1+\gamma M_y^2}$$

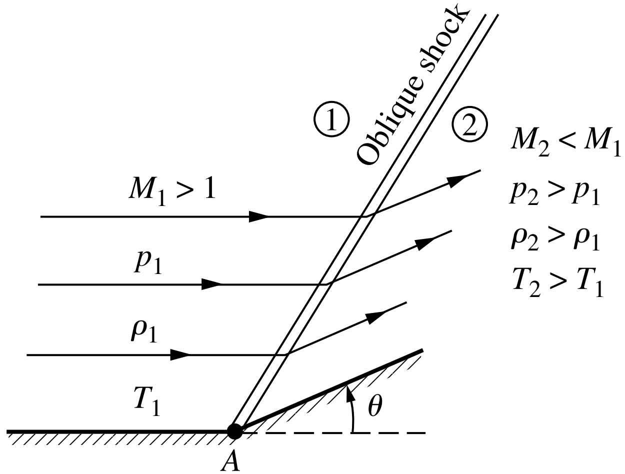
$$\frac{\rho_y}{\rho_x} = \frac{u_x}{u_y} = \frac{p_y}{p_x}\frac{T_x}{T_y} = \frac{(\gamma+1)M_x^2}{(\gamma-1)M_x^2+2}$$

$$\frac{T_y}{T_x} = \left(2+(\gamma-1)M_x^2\right)\frac{2\gamma M_x^2-(\gamma-1)}{(\gamma+1)^2 M_x^2} = \frac{1+[(\gamma-1)/2]M_x^2}{1+[(\gamma-1)/2]M_y^2}$$

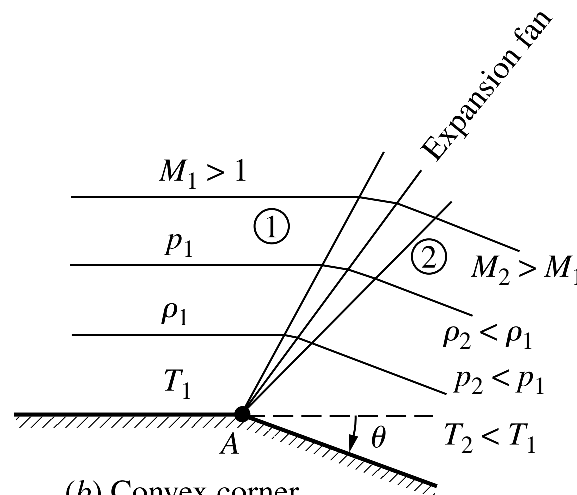
Summary of Normal Shock Wave Characteristics

Variable	Change Across Normal Shock Wave
Mach number	Decrease
Static pressure	Increase
Stagnation pressure	Decrease
Static temperature	Increase
Stagnation temperature	Constant
Density	Increase
Velocity	Decrease

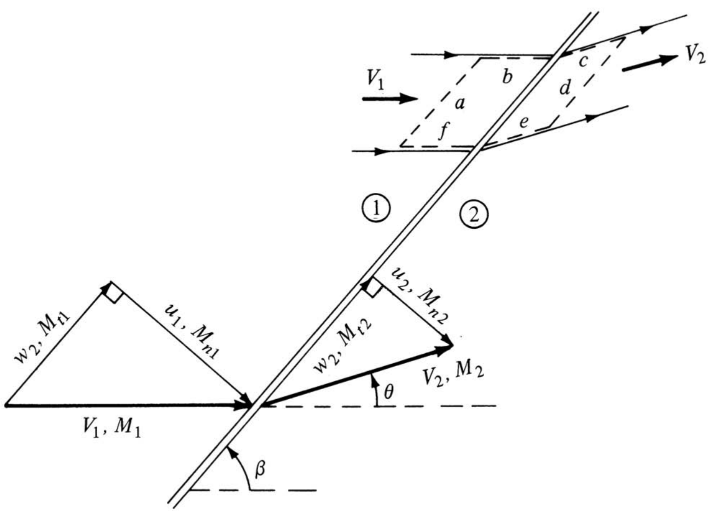
Oblique Shocks and Expansion Waves



(*a*) Concave corner



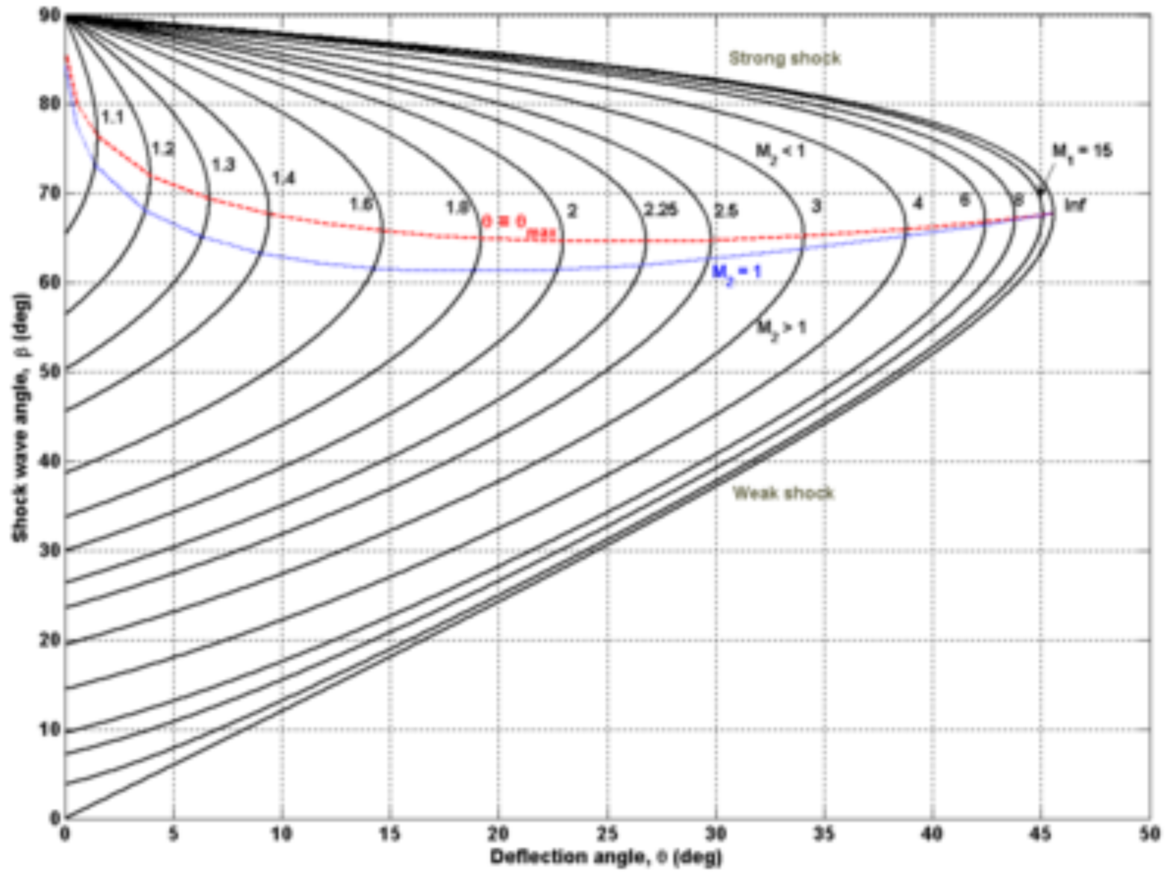
(*b*) Convex corner



$$\tan\theta = 2\cot\beta\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$$

Oblique Shock Solutions

$$\tan\theta = 2\cot\beta \frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$$



- Maximum deflection angle for a given incoming Mach No.
- Strong and Weak Shock Solutions