Banach Algebras and their Spectral Properties

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December 2, 2025

Presentation Overview

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Definitions

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- A is called a *commutative algebra* if the multiplication is commutative (i.e. if $xy = yx$ for all $x, y \in A$).
- A is called an *algebra with identity* if there exists an element $e \in A$ such that

$$xe = ex = x \qquad \text{for all } x \in A$$

  [Blackboard]

Normed Algebras and Banach Algebras

- A *normed algebra* A is a normed space that is also an algebra satisfying:

$$\begin{aligned} \|xy\| &\leq \|x\| \, \|y\| && \text{for all } x, y \in A, \\ \|e\| &= 1 && \text{if } A \text{ has identity.} \end{aligned}$$

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- The space $C[0, 1]$ is a commutative Banach algebra with identity.
- The space $M_n(\mathbb{C})$ of $n \times n$ matrices is a Banach algebra with identity (non-commutative for $n > 1$).
- Let $X \neq \{0\}$ be a Banach space. Then $B(X, X)$ is a Banach algebra with identity (non-commutative for $\dim X > 1$).

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Resolvent Set / Spectrum

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- Note that this definition is equivalent to the classical definition of a resolvent set for linear operators $T \in B(X, X)$.
  ($A = B(X, X)$) is complex Banach algebra with identity for any Banach space X).

Resolvent Set / Spectrum

- We are losing some information by generalizing from $B(X,X)$ to general complex Banach algebras with identity (i.e. we no longer have that $\sigma(T) = \sigma_p(T) \dot\cup \sigma_r(T) \dot\cup \sigma_c(T)$).

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- Nevertheless;

Big Idea:

Many of the classical results about resolvent sets / spectra can be generalized to the Banach Algebra setting.

Geometric Series Inverses

Theorem 1

If $x \in A$ has norm $\|x\| < 1$, then $e - x$ is invertible and

$$(e - x)^{-1} = \sum_{j \geq 0} x^j.$$

(Note that $x^0 = e$.)
[Blackboard]

Properties of the set of invertible elements

Theorem 2

Let G denote the set of all invertible elements of A. Then

1. *G is a group,*
2. *G is an open subset of A.*

[Blackboard]

The spectrum

Define the *spectral radius* $r_\sigma(x)$ of an element $x \in A$ to be

$$r_\sigma(x) := \sup_{\lambda \in \sigma(x)} |\lambda| \,.$$

Theorem 3

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Theorem 4

For all $x \in A$, $\sigma(x) \neq \varnothing$ and $\rho(x) \neq \varnothing$.

The End