THREE DENSITY-ONE FORMULATIONS OF CONVERGENCE

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Abstract. In this expository paper, we discuss three different density-one formulations of convergence. We give a proof showing the implications between all of these formulations. We also take note of two surprising corollaries of these implications.

1. Introduction

In number theory, if one cannot prove a conjecture for all natural numbers $n \in \mathbb{N}$, the next best thing might be to at least prove the conjecture for a density-one subset of $\mathbb{N}$. However, this brings up the question of how to formulate a density-one notion of convergence. In particular, given a sequence of real numbers $\{\alpha_n\}_{n\geq 1}$, what is the correct density-one analog of α_n converging to L? We list what are, in the authors' opinion, the three most natural possible formulations:

(A) For all $\varepsilon > 0$, $\{n : |\alpha_n - L| < \varepsilon\}$ is a density-one subset of $\mathbb{N}$.
(B) The average distance $\frac{1}{x}\sum_{n\leq x} |\alpha_n - L| \to 0$ as $x \to \infty$.
(C) There exists a density-one subset $S \subseteq \mathbb{N}$ such that $\alpha_n \to L$ as $n \to \infty$ along S.

We note here that formulation (A) is known as *statistical convergence*, formulation (B) is known as *strong Cesàro convergence*, and formulation (C) is known as s^**-convergence* [1, 2].

The question of which formulation to use recently came up in two unrelated works of the authors, [5] and [4].

In [5], the authors were interested in the sequence $\{\alpha_n\}_{n\geq 1}$, where α_n denotes the inverse of the degree of the coefficient field of the n-th weight k newform (with the set of weight k newforms ordered by level); see [5] for details. The weak level-N Maeda conjecture states that $\alpha_n \to 0$. However, we were only able to show the weaker result, formulation (A), for this sequence. So, the question came up if our result could truly be called a density-one version of the weak level-N Maeda conjecture, or if (C) was in fact the correct density-one formulation. A closely related question was also raised by Serre [6, Question, p. 89], where he proved formulation (A) for $\{\alpha_n\}_{n\geq 1}$, then asked if this result could be strengthened.

2020 *Mathematics Subject Classification.* 40A35 and 11B05.

Key words and phrases. density one, convergence.

In [4], the first author wanted to prove that the expected density $\mathbb{E}[d(T)] = \lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} \mathbb{P}[n \in T]$ of a certain random set $T \subseteq \mathbb{N}$ was equal to 1. One strategy to prove this fact would be to show that $\alpha_n := \mathbb{P}[n \in R]$ converges to 1 along a density-one subset $S \subseteq \mathbb{N}$. However, this brought up the question if the above strategy (i.e. formulation (C)) is actually necessary to show the desired result (i.e. formulation (B)), or if it is a strictly stronger statement.

In general, formulation (B) turns out to be strictly stronger than formulations (A) and (C). Surprisingly, for bounded sequences $\{\alpha_n\}_{n\ge 1}$, on the other hand, all three formulations turn out to be equivalent. These implications were originally shown in [3].

Theorem 1.1. *Let $\{\alpha_n\}_{n\ge1}$ denote a sequence of real numbers, and $L \in \mathbb{R}$. Then assuming the sequence $\{\alpha_n\}_{n\ge1}$ is bounded, the following are equivalent.*

(A) *For all $\varepsilon > 0$, $\{n : |\alpha_n - L| < \varepsilon\}$ is a density-one subset of $\mathbb{N}$.*

(B) *The average distance from L, $\displaystyle\frac{1}{x}\sum_{n\le x} |\alpha_n - L| \to 0$ as $x \to \infty$.*

(C) *There exists a density-one subset $S \subseteq \mathbb{N}$ such that $\alpha_n \to L$ as $n \to \infty$ along S.*

In general, (B) *implies* (A) *and* (C)*, which are both equivalent.*

This theorem has two surprising corollaries. Corollary 2.1 states that convergence along sets of density $1-\delta$ implies convergence along a set of density 1. Corollary 2.2 gives a set-theoretic interpretation of Theorem 1.1, not involving any sequences of real numbers.

2. Proof of the main theorem

We now give a proof of Theorem 1.1. Throughout the proof, we will denote the density of a set $T \subseteq \mathbb{N}$ by $d(T) := \lim_{x\to\infty} \frac{1}{x}\#\{n \le x : n \in T\}$.

Proof.

Part 1: (B) implies (A)

For all $\varepsilon > 0$, observe that

$$d(\{n : |\alpha_n - L| \ge \varepsilon\}) = \lim_{x\to\infty} \frac{1}{x}\#\{n \le x : \frac{1}{\varepsilon}|\alpha_n - L| \ge 1\} \le \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\frac{1}{\varepsilon}|\alpha_n - L| = 0.$$

Hence $\{n : |\alpha_n - L| < \varepsilon\}$ is a density-one subset of $\mathbb{N}$, as desired.

Part 2: (C) implies (A)

Let $S \subseteq \mathbb{N}$ denote the density-one set from (C). Then since $\alpha_n \to L$ along S, we must have that for all $\varepsilon > 0$, $\{n \in S : |\alpha_n - L| \ge \varepsilon\}$ is finite. This then yields

$$d(\{n : |\alpha_n - L| \ge \varepsilon\}) \le d(\mathbb{N}\backslash S) + d(\{n \in S : |\alpha_n - L| \ge \varepsilon\}) = 0,$$

which means that $\{n : |\alpha_n - L| < \varepsilon\}$ is a density-one subset of $\mathbb{N}$, as desired.

Part 3: (A) implies (C)

For a set $T \subseteq \mathbb{N}$, let $d_x(T)$ denote the interval density

$$d_x(T) := \frac{1}{x}\#\{n \le x : n \in T\}.$$

Observe that for all $k \in \mathbb{N}, \varepsilon > 0$; since $\{n : |\alpha_n - L| \ge 1/k\}$ has density zero, there exists an $M \in \mathbb{N}$ such that

$$d_x(\{n : |\alpha_n - L| \ge 1/k\}) < \varepsilon \quad \text{for all } x \ge M.$$

Hence, we can construct a strictly increasing sequence $\{M_k\}_{k\ge1}$ such that $M_1 = 1$, and for each $k \ge 2$,

$$d_x\left(\{n : |\alpha_n - L| \ge 1/k\}\right) < 1/2^k \quad \text{for all } x \ge M_k. \tag{2.1}$$

We then define the set S as follows:

$$S := \bigcup_{k=1}^{\infty}\{n \in [M_k, M_{k+1}) : |\alpha_n - L| < 1/k\}.$$

Clearly, $\alpha_n \to L$ along S, since for all $k \in \mathbb{N}$, $|\alpha_n - L| < 1/k$ for all $n \ge M_k$ in S. It remains to show that $d(S) = 1$. For positive real numbers x, let

$$k(x) := \max\{k \in \mathbb{N} : M_k \le x\}.$$

Here, $k(x) \to \infty$ as $x \to \infty$ since $\{M_k\}_{k\ge1}$ is strictly increasing. Then for each $n \le x$, observe that $n \in [M_k, M_{k+1})$ for some $k \le k(x)$. This implies that $\{n \le x : n \notin S\} \subseteq \left\{n \le x : |\alpha_n - L| \ge \frac{1}{k(x)}\right\}$, and so

$$\begin{aligned}
d(\mathbb{N}\backslash S) &= \lim_{x\to\infty} \frac{1}{x}\#\{n \le x : n \notin S\} \\
&\le \lim_{x\to\infty} \frac{1}{x}\#\left\{n \le x : |\alpha_n - L| \ge \frac{1}{k(x)}\right\} \\
&= \lim_{x\to\infty} d_x\left(\left\{n : |\alpha_n - L| \ge \frac{1}{k(x)}\right\}\right) \\
&\le \lim_{x\to\infty} \frac{1}{2^{k(x)}} \qquad \text{(by (2.1))} \\
&= 0,
\end{aligned}$$

as desired.

Part 4: (C) implies (B), assuming boundedness

Let $S \subseteq \mathbb{N}$ denote the density-one subset from (C). Since $\{\alpha_n\}_{n\geq 1}$ is bounded, let $M \in \mathbb{R}$ be such that $|\alpha_n - L| \leq M$ for all $n \geq 1$. Then

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} |\alpha_n - L| &= \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\leq x\\ n\in S}} |\alpha_n - L| + \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\leq x\\ n\notin S}} |\alpha_n - L| \\
&\leq \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\leq x\\ n\in S}} |\alpha_n - L| + M \cdot \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\leq x\\ n\notin S}} 1 \\
&= 0 + M \cdot d(\mathbb{N}\backslash S) \qquad \text{(since } |\alpha_n - L| \to 0 \text{ along } S\text{)} \\
&= 0,
\end{aligned}$$

completing the proof. □

In the following corollary, we show that if a sequence converges along density $1-\delta$ subsets of $\mathbb{N}$, then it also converges along a density 1 subset of $\mathbb{N}$.

Corollary 2.1. *Let $\{\alpha_n\}_{n\geq 1}$ be a sequence of real numbers, and $L \in \mathbb{R}$. Then the following are equivalent.*

- (A′) *For all $\delta > 0$, there exists a density $1-\delta$ subset $S_\delta \subseteq \mathbb{N}$ such that $\alpha_n \to L$ as $n \to \infty$ along S_δ.*
- (C′) *There exists a density-one subset $S \subseteq \mathbb{N}$ such that $\alpha_n \to L$ as $n \to \infty$ along S.*

Proof. The implication (C′) implies (A′) is trivial, and we will show (A′) implies (C′). By Theorem 1.1, it suffices to show that (A′) implies (A).

Fix $\varepsilon > 0$, and we will show that $d(\{n : |\alpha_n - L| \geq \varepsilon\}) = 0$. For arbitrary $\delta > 0$, let S_δ be the set from (A′) such that $d(S_\delta) = 1 - \delta$ and $\alpha_n \to L$ along S_δ. Then the set $\{n \in S_\delta : |\alpha_n - L| \geq \varepsilon\}$ is finite, which implies that

$$d(\{n : |\alpha_n - L| \geq \varepsilon\}) \leq d(\mathbb{N}\backslash S_\delta) + d(\{n \in S_\delta : |\alpha_n - L| \geq \varepsilon\}) < \delta + 0 = \delta.$$

Hence since δ was arbitrary, we have $d(\{n : |\alpha_n - L| \geq \varepsilon\}) = 0$, as desired. □

Finally, we give a set-theoretic interpretation of Theorem 1.1, not involving any sequences of real numbers.

Corollary 2.2. *Let $\{T_k\}_{k\geq 1}$ denote a collection of disjoint subsets of $\mathbb{N}$. Then the following are equivalent.*

- (A″) *Each T_k has density zero in $\mathbb{N}$.*
- (C″) *There exists a density-one subset $S \subseteq \mathbb{N}$ such that $T_k \cap S$ is finite for all $k \geq 1$.*

Proof. Let $\{\alpha_n\}_{n\geq 1}$ be the sequence given by

$$\alpha_n = \begin{cases} \frac{1}{k} & \text{if } n \in T_k \\ 0 & \text{if } n \notin T_k \text{ for all } k. \end{cases}$$

Note that for all $r \geq 1$, $\{n : \alpha_n \geq 1/r\} = \bigcup_{k=1}^{r} T_k$. Then considering the densities of both sides, one can immediately see that (A) and (A$''$) are equivalent. Similarly, it is immediate to see that (C) and (C$''$) are equivalent since $\alpha_n \to 0$ along S if and only if $T_k \cap S$ is finite for all $k \geq 1$. $\square$

Acknowledgements

The preparation of this paper was supported by NSF grant DMS-2349174. Hui Xue is supported by Simons Foundation Grant MPS-TSM-00007911.

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