Research Statement

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Overview

My research lies mainly in the field of number theory. Much of my research so far has specifically focused on modular forms, but I also have research interests in many other areas of number theory as well. I am also interested in several other fields of mathematics, including harmonic analysis, ergodic theory/dynamical systems, and theoretical computer science. To date, I have completed thirteen different research papers, and I also have several projects in preparation. I have found computational experimentation to be extremely useful in my research, both as a way to check small examples, and also as a way to observe interesting patterns. Much of the progress I have made in research has come from getting my hands dirty and carrying out the explicit calculations (on paper and by computer) to compute the relevant number-theoretic objects.

Of the thirteen papers I have completed, seven are related to the asymptotic behavior of the Hecke operators [1, 2, 3, 4, 5, 6, 7], two are related to dimensions of modular forms spaces [8, 9], two deal with other topics in modular forms [10, 11], and two deal with questions in other areas of number theory [12, 13]. In the following, we summarize some of these projects; the full papers can all be found on my website [14]. I will also mention here that I have had the opportunity to mentor several undergraduate research groups through summer REU's at Clemson University. Several of the research projects discussed below have involved undergraduate students from these REU's.

Asymptotic behavior of the Hecke operators

In this section, we let $S_k(\Gamma_0(N))$ denote the space of cuspidal modular forms of weight k and level N, and $T_m(N,k)\colon S_k(\Gamma_0(N)) \to S_k(\Gamma_0(N))$ denote the m-th Hecke operator on this space.[1] I have completed several projects investigating the asymptotic behavior of these operators. This asymptotic behavior includes both the vertical perspective (where m is fixed and $N+k \to \infty$), and the horizontal perspective (where N and k are fixed and $m \to \infty$).

Average size of the eigenvalues of the Hecke operators.

One interesting question is to determine the average size of the eigenvalues of the (normalized) Hecke operators $T'_m(N,k)$. From the vertical perspective, Serre [15] showed that for prime indexed Hecke operators, the eigenvalues of $T'_p(N,k)$ tend to the distribution $\mu_p = \frac{p+1}{\pi}\frac{(1-t^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2-t^2}\,dt$ as $N+k\to\infty$ (this can be viewed as a type of vertical Sato-Tate distribution). Using this distribution, one can easily compute that the average size of the eigenvalues of $T'_p(N,k)$ tends to $\sqrt{(p+1)/p}$ as $N+k\to\infty$. However, for general m, we had to use a completely different method. By analyzing the terms of the Eichler-Selberg trace formula, we managed to generalize the above result, showing that the average size of the eigenvalues of $T'_m(N,k)$ tends to $\sqrt{\sigma_1(m)/m}$ as $N+k\to\infty$ [3].

One can also compute the average eigenvalue size from the horizontal perspective. This turns out to correspond with the classical Sato-Tate distribution $\mu_{ST} = \frac{1}{\pi}\sqrt{1-t^2/4}\,dt$ (with one formulation of the question following from the Sato-Tate theorem [16]).

Coefficients of Hecke Polynomials.

Write the characteristic polynomial for $T_m(N,k)$ as $x^d + c_1(m,N,k)x^{d-1} + c_2(m,N,k)x^{d-2} + \ldots + c_d(m,N,k)$, where $d = \dim S_k(\Gamma_0(N))$. We note that the first coefficient $c_1(m,N,k) = -\operatorname{Tr} T_m(N,k)$

[1] The reader need not be concerned if they are unfamiliar with modular forms For purposes of this research statement, all one really needs to know is that $S_k(\Gamma_0(N))$ is a vector space (made up of certain functions called modular forms) that many number theorists care about. And the $T_m(N,k)$ form an important family of linear operators on this space.

has historically been of great importance in the theory of modular forms. Instead of the first coefficient, however, I have been interested in studying the second coefficient $c_2(m, N, k)$.

My coauthors and I were able to show many interesting results about this second coefficient c_2, including many properties that are not even known about the trace (which has been studied extensively in the past). This is useful because in certain arguments which have historically used the trace $-c_1$, we can let c_2 play the role of c_1 and obtain stronger results.

First, via a careful analysis of the terms of the Eichler-Selberg trace formula, we showed that the second coefficient is non-vanishing in almost every scenario [1]. One can compare this result with the generalized Lehmer conjecture proposed in 2006 by Rouse [17]: that $\operatorname{Tr} T_m(N, k) = -c_1(m, N, k)$ is nonvanishing for all N coprime to m and $k \geq 12$, $k \neq 14$. In particular, it turns out that one can actually obtain stonger results here for $c_2(m, N, k)$ than for $c_1(m, N, k)$. This result on the nonvanishing of second coefficient was also later extended to the newspace $S_k^{\text{new}}(\Gamma_0(N))$ in [2].

Second, we were also able to show non-repetition of the second coefficient in various settings [5]. This is especially interesting because we used this non-repetition result to show that (conditional on Maeda's conjecture), level one Hecke eigenforms can be distinguished by their fourth Fourier coefficients.

Finally, we also studied the nonvanishing and asymptotic behavior of general Hecke polynomial coefficients $c_r(m, N, k)$ in [4]. This work verified [4, Conjecture 7.1] (a generalization of [17, Conjecture 1.5] and [18, Conjecture 5.1] predicting the non-vanishing of general Hecke polynomial coefficients) in all but finitely many cases.

Equidistribution of Eigenvalues.

As noted previously in this section, Serre [15] showed that the eigenvalues of $T'_m(N, k)$ are μ_p equidistributed as $N + k \to \infty$. He also generalized this result over $S_k(\Gamma_0(N))$ to the newspace $S_k^{\text{new}}(\Gamma_0(N))$ and to the space with character $S_k(\Gamma_0(N), \chi)$. However, the techniques of [15] do not apply directly to the newspace with chracter $S_k^{\text{new}}(\Gamma_0(N), \chi)$. Using the results of [8], I was was recently able to generalize the above equidistribution result to this final remaining case of newspaces with character $S_k^{\text{new}}(\Gamma_0(N), \chi)$ [6]. Later, my coauthors and I also gave a generalization of Serre's equidistribution result to the Atkin-Lehner sign pattern spaces $S_k^{\sigma}(N)$ and $S_k^{\text{new},\sigma}(N)$ [7].

Dimensions of modular forms spaces.

Note that much of the work mentioned so far implicitly depends on the spaces $S_k(\Gamma_0(N))$ existing asymptotically. For example, in order to talk about the asymptotic behavior of the eigenvalues of $T_m(N, k)$, we must have $\dim S_k(\Gamma_0(N)) > 0$ for $N + k$ sufficiently large.

With this motivation, I studied when exactly the newspaces with character $S_k^{\text{new}}(\Gamma_0(N), \chi)$ will exist (and more generally, will be small). Considering newspaces without character, $\dim S_k^{\text{new}}(\Gamma_0(N)) = 0$ for only finitely many pairs (N, k). And considering full spaces with character, it turns out that $\dim S_k(\Gamma_0(N), \chi) = 0$ for only finitely many triples (N, k, χ). But surprisingly, this property does not hold for newspaces with character; there exist infinitely many triples (N, k, χ) such that $\dim S_k^{\text{new}}(\Gamma_0(N), \chi) = 0$. However, I gave a complete classification of when this occurs [8].

In [8], I also settled (negatively) a 20-year-old conjecture from Martin [19], which posited that the dimension sequence $\{\dim S_2^{\text{new}}(\Gamma_0(N))\}_{N \geq 1}$ takes on every natural number. Later, I further investigated several variations of this conjecture in [9]. For each variation, I gave a complete classification of when such a property holds.

Two additional projects in modular forms

To any newform $f \in S_k(\Gamma_0(N))$, one can associate the (full) period polynomial $r_f(X)$, the even period polynomial $r_f^+(X)$, and the odd period polynomial $r_f^-(X)$, all defined in terms of special values of the L-function associated to f. It turns out that these three period polynomials all satisfy a certain functional equation, with circle of symmetry $|X| = \frac{1}{\sqrt{N}}$. In 2016, Jin-Ma-Ono-Soundararajan [20] showed what they called the "Riemann Hypothesis for period polynomials": that all the zeros of $r_f(X)$ lie on the circle of symmetry $|X| = \frac{1}{\sqrt{N}}$. In [10], my coauthors and I showed an analogous result for the even and odd period polynomials $r_f^+(X)$ and $r_f^-(X)$. For small levels N and weights k, there turn out to be many examples of $r_f^+(X)$ and $r_f^-(X)$ with zeros off the circle of symmetry. However, for sufficiently large level and weight, we proved that the zeros $r_f^+(X)$ and $r_f^-(X)$ all lie on the circle of symmetry.

I have also done some work studying the transcendence of zeros of modular forms. In [11], my coauthors and I were able to show that aside from a finite number of exceptions, the zeros of modular forms in several different families are all transcendental. These families included, for example, the Eisenstein series for several different $\Gamma_0^+(p)$ and $\Gamma_0(p)$, cuspidal projections of products of Eisenstein series, and many forms in the Miller basis.

Two additional projects in number theory

I recently completed a project in probabilistic number theory. Taking inspiration from the Prime Number Theorem, one can model the actual primes as a set of random natural numbers, where N is chosen with probability $1/\log N$. This is known as the Cramér random prime model. In some sense, the defining characteristic of the actual primes is the Fundamental Theorem of Arithmetic. So in [12], I investigated the natural question of how close the Cramér random primes come to satisfying the Fundamental Theorem of Arithmetic. I also addressed some related variations, slightly modifying the probability with which a random prime is chosen.

I also recently completed a project about CM points in the upper half-plane [13]. Let $\mathcal{F}$ denote the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane. William Duke [21] showed that as fundamental discriminants $D \to -\infty$, the sets CM_D (CM points of discriminant D) are equidistributed over $\mathcal{F}$. My coauthors and I investigated a slight variation: the behavior of CM points on the *boundary* of $\mathcal{F}$. In particular, we proved that such CM points are equidistributed on the boundary, and also gave a complete characterization of when every CM_D point lies on the boundary. This characterization turns out to be closely related to the structure of class groups. So to complete this project, we also (conditionally) gave a complete classification of negative discriminants with class group of small exponent. Additionally, we note here that the above characterization was used in [11] to prove certain results concerning the transcendence of zeros of modular forms.

Future Work

I am currently working on a project concerning the distribution of CM points. Generalizing my work in [13], I have already managed to show that CM points are equidistributed along every closed geodesic in the upper half-plane. (This can potentially be viewed as dual to Duke's equidistribution result [21].) However, there are still several closely related open questions. that I would like to resolve.

I am also working on investigating a project in algebraic number theory: giving a complete classification of imaginary quadratic order of small elasticity. I hope to accomplish this by generalizing some other aspects of my work in [13], using the recent result of Fan-Pollack [22]. This project will also involve a fair amount of computational work.

Another question I would like to answer comes from [12, Conjecture 7.3], concerning multiplicatively closed sets of natural numbers. Although an arbitrary multiplicatively closed set is not guaranteed to have a natural density (see [12, Proposition 7.2]), [12, Conjecture 7.3] posits that every multiplicatively closed set will have a logarithmic density. This is a generalization of the Davenport-Erdős Theorem [23], which states that every set closed under multiplication by the natural numbers is guaranteed to have a logarithmic density. The proof of this generalization would likely involve estimating the growth rate of certain partial summations by applying a Tauberian theorem to the Dirichlet series associated to the multiplicatively closed set.

Yet another question I would like to answer is why there exists an infinite family of triples (N, k, χ) such that $\dim S_k^{\mathrm{new}}(\Gamma_0(N), \chi) = 0$. I showed in [8] that this infinite family is precisely the (N, k, χ)

such that $2 \mid \mathfrak{f}(\chi)$ and $2 \parallel N/\mathfrak{f}(\chi)$. However, the proof does not shed any insight on why newforms cannot exist when $2 \mid \mathfrak{f}(\chi)$ and $2 \parallel N/\mathfrak{f}(\chi)$. I proved that in this case, a specific decomposition of forms in $S_k(\Gamma_0(N), \chi)$ must necessarily exist [8, Proposition 9.1]. But I would like to determine how to find this decomposition explicitly, and understand why it exists.

Lastly, there are still many unresolved questions about Hecke polynomial coefficients that I would like to investigate. For example, for $m = 2, 4$, we used the non-repetition of $c_2(m, 1, k)$ to verify that level one Hecke eigenforms can be distinguished by their m-th Fourier coefficient (assuming Maeda's conjecture) [5, Theorem 1.5]. It would be quite interesting to generalize the techniques of [5] in order to show this result for general m. For another example, I hope to use some of the strategies developed to study $c_2(m, N, k)$ in order to attack problems about the trace. Specifically, I believe that [2, Conjecture 7.1] could be tractable, which posits that for any fixed non-square $m \geq 2$, there exist N_1 and N_2 coprime to m such that $\operatorname{Tr} T_m^{\text{new}}(N_1, k)$ vanishes for all weights k, and $\operatorname{Tr} T_m^{\text{new}}(N_2, k)$ is non-vanishing for all weights k.

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