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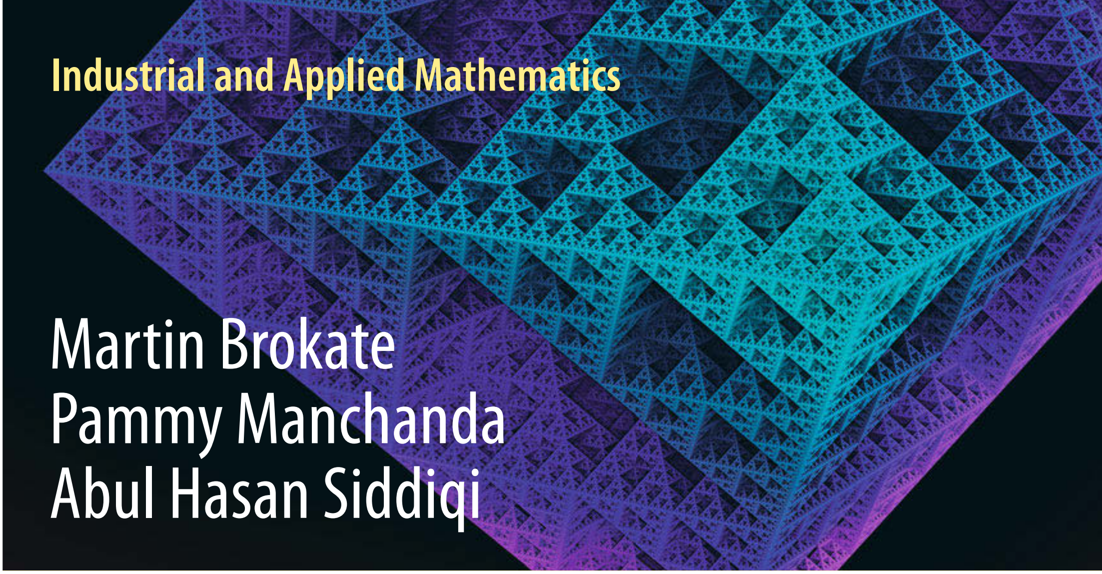

Martin Brokate
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Abul Hasan Siddiqi

Calculus for Scientists and Engineers

Springer

Industrial and Applied Mathematics

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Foreword

In my past position as Director of the Abdus Salam International Centre of Theoretical Physics in Trieste, Italy, I was deeply involved in strengthening advanced research in developing countries. Even though the Centre's mandate was postdoctoral research, it was clear that only by enhancing the quality of education at all levels can one enable advanced research at a sustainable level of excellence. The Centre had thus ventured, often and deliberately, into undergraduate education. One lacuna we had observed was the lack of good, affordable, and motivating textbooks in science and mathematics.

I was thus pleased when the authors of this book—all of whom are well-known researchers and teachers I know personally—approached me in the summer of 2006 with a proposal for writing a quality book on calculus for use in undergraduate education. Their goals were to make the book useful for instruction in any country but priced such that students in developing countries could afford it. With this understanding, I facilitated several visits of the authors to the Centre, during which they actively collaborated on the book. I am pleased that the book is now being published in its new edition by Springer Nature, and thank the authors for persisting with the collaboration despite geographical separation.

I am satisfied that the coverage and presentation in the book are at a high level and meet one of our principal requirements. The authors have endeavored to present the basic concepts clearly and point to their applications in diverse fields. I hope that many brilliant young students will benefit from this book, which is the result of a continuing collaboration among the authors, on which I congratulate them warmly.

New York, USA

K. R. Sreenivasan
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Preface

This book is meant to be used as a first course in calculus for students of science and engineering. It will also be useful for students of other disciplines who are interested in learning calculus.

We endeavored to explain the basic concepts of calculus hand in hand with their relevance to real-world problems. We have given special emphasis on applications without compromising rigorous analysis. Plenty of solved examples have been given to clarify techniques related to a particular theme. In appendices, we have discussed concepts and themes we regard as prerequisites, like the number system, trigonometric functions, and analytic geometry. Moreover, proofs of some of the theorems have been included there in order to not interrupt the flow of the argument in the main body of the text. Some references to other books on calculus that have motivated our presentation have been given in the bibliography.

The text is application oriented. Many interesting, relevant, and up-to-date applications have been drawn from the fields of business, economics, social and behavioral sciences, life sciences, physical sciences, and other fields of general interest. Applications are found in the main body of the text as well as in the exercise sets. In fact, one goal of the text is to include at least one real-life application in each section wherever possible.

The book comprises 12 chapters. Chapter 1 is devoted to an introduction of functions of one independent variable. Chapter 2 provides the concepts of limit and continuity along with their physical and geometrical interpretations. Chapter 3 deals with derivatives and the techniques of differentiation. Chapter 4 discusses the optimization of a function, that is, finding minima and maxima of a function over an interval. Moreover, applications of optimization to various real-world problems, including a fairly large number of solved examples from business and finance, are also studied in this chapter. Chapter 5 considers sequences and series, in particular Maclaurin and Taylor series. Chapters 6 and 7 are devoted to the process of integration and its applications in business and industry, engineering problems, and probability theory. We show with a lot of examples that the utility of integrals has expanded far from their original purpose, the computation of the area below a curve.

Chapter 8 introduces functions of several variables. Concepts of level curves (level sets) or contours, graphs of functions of two variables, and equipotential and isothermal surfaces are developed. Physical situations represented by functions of more than one variable are discussed. This chapter also deals with the extension of the concepts of limit, continuity, differentiability, optimization, and integration to functions of several variables. Physical situations, where such extensions are required, are discussed in detail. Often we have restricted our presentation to functions of two variables only, for the sake of clarity and easier understanding and because most results which hold true for two variables can be readily extended to functions of more than two variables.

Chapter 9 is devoted to the calculus of vector-valued functions (vector fields), that is, functions that are defined on a domain of dimension 1, 2, or 3 and take values in the plane or the space. Continuity, differentiability, and integration for vector-valued functions are introduced, and the theorems of Green, Gauss, and Stokes are discussed. Applications of vector calculus and of these theorems to problems of science and engineering are presented.

Chapter 10 deals with Fourier methods and their applications to real-world problems for readers who want to pursue this topic.

Chapter 11 is devoted to the introduction of ordinary and partial differential equations. Modeling of real-world problems with these equations is explained.

Chapter 12 shows how MATLAB can be used as an aid for teaching and learning concepts of calculus, in particular those we have discussed in this book. Teachers may use MATLAB as a tool for vivid and precise demonstrations, while students may use MATLAB as a tool for exploring by themselves various concepts of calculus. Indeed, MATLAB is being used nowadays practically in every branch of science and engineering.

* Chapter 12 is mainly written by Dr. A. K. Verma, Assistant Professor, Sharda Group of Institutions, Agra, and Dr. Jean-Marc Ginoux from France. Dr. Verma has also drawn all figures of this book using MATLAB.

The International Centre for Theoretical Physics (ICTP), Trieste, Italy (a joint venture between UNESCO and the Italian government), has played a pivotal role in the creation of this book. Established in 1964 and renamed as the Abdus Salam ICTP in 1997, ICTP possesses a worldwide rather unique combination of features. It is a meeting point of scientists from developed and developing countries and, in particular, continues to provide yeoman's service in training bright scientists from developing countries. The authors of this book have been frequent visitors to this Centre since 1986 and have availed the Centre's hospitality to enhance their academic capabilities and cooperation. During one of their visits in 2006, while walking along the Adriatic Sea, they began to discuss the utility of writing a book on calculus for undergraduates and agreed to write such a book on calculus with special emphasis on the clarity of concepts and their applications in diverse fields. Our objective is to show that, throughout all of its contents, the mathematics of calculus is not just an abstract subject, but has relevance to many different fields of human knowledge. The next morning, Director of ICTP, Prof. K. R. Sreenivasan was approached with the request to support the writing of such a book. He was very

prompt in approving the idea and assured to provide all kinds of facilities in ICTP. We take this opportunity to thank him for the financial and infrastructural support without which this book could not have been completed. In particular, we have highly benefitted from the excellent library in ICTP.

We take this opportunity to thank Dr. Meenakshi, UGC Research Fellow, gold medalist, and now Lecturer at Dev Samaj College for Women, Ferozepur, who has gone through this book carefully and has given several valuable suggestions. Also, Sharda University deserves a special mention, as a major part of the technical work was carried out at this place.

Munich, Germany Martin Brokate
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Contents

About the Authors

Martin Brokate is Professor Emeritus of Applied Mathematics at the Technical University, Munich, Germany. He received his PhD in Mathematics at Freie Universität, Berlin, Germany, in 1980, and was appointed to the Chair of Numerical Analysis and Control Theory in 1999. He was the spokesman of Special Research Area 438 "Mathematical Modeling, Simulation and Verification in Material-Oriented Processes and Intelligent Systems" from 2001 to 2004. He was the Dean of the Department of Mathematics in 2003–2006. His interests lie in applied analysis and control theory, with a focus on the mathematical analysis of rate-independent evolutions and hysteresis operators.

Pammy Manchanda is Senior Professor in the Department of Mathematics at the Guru Nanak Dev University, Amritsar, India and Secretary of the Indian Society of Industrial and Applied Mathematics (ISIAM). She has published more than 50 research papers in several international journals of repute, edited 4 proceedings for international conferences of the ISIAM and co-authored 3 books. She has visited the International Centre for Theoretical Physics (ICTP) (a UNESCO institution) at Trieste, Italy, many times to carry out her research activities, attended and delivered talks and chaired sessions at several international conferences and workshops across the globe, including the International Council for Industrial and Applied Mathematics (ICIAM) during 1999–2015 and the International Congress of Mathematicians (ICM). She is the managing editor of the *Indian Journal of Industrial and Applied Mathematics* and a member of the editorial board of the Springer book series *Industrial and Applied Mathematics*.

Abul Hasan Siddiqi is a distinguished scientist and Adjunct Professor at the School of Basic Sciences and Research, and Coordinator at the Centre for Advanced Research in Applied Mathematics and Physics (CARAMP) at Sharda University, Greater Noida, India. He was a visiting consultant at ICTP; Sultan Qaboos University, Muscat, Oman; MIMOS, Kuala Lumpur, Malaysia; and a professor at several reputed universities including Aligarh Muslim University, Aligarh, India; and King Fahd University of Petroleum and Minerals, Dhahran,

Saudi Arabia. He has a long association with ICTP (a regular associate, guests of the director and senior associate). He was awarded the German Academic Exchange Fellowship thrice to carry out mathematical research in Germany. He has published more than 100 research papers jointly with his research collaborators, 13 books and edited proceedings of 17 international conferences, as well as supervised 29 PhD scholars. He is the founder secretary and the current President of the ISIAM, which celebrated its silver jubilee in January 2016. He is the editor-in-chief of the *Indian Journal of Industrial and Applied Mathematics* (published by ISIAM) and the Springer's book series *Industrial and Applied Mathematics*.

Chapter 1
Functions and Models

The concept of a function is of vital importance for the proper understanding of many phenomena occurring in different areas of human knowledge. The fundamental processes of calculus known as differentiation and integration are processes applied to functions. We discuss here the notion of a function, various forms of functions, and important classes of functions. We also discuss how functions express phenomena from other sciences, in particular, physics.

1.1 Function, Domain, and Range

Definition 1.1 (*Relation*) Let S be a set. A relation R on S is a set of ordered pairs (x, y), where $x, y \in S$. Moreover, if T is another set, a relation R on S and T is a set of ordered pairs (x, y), where $x \in S$ and $y \in T$.

The term "ordered pair" is used to emphasize that when we write (x, y), x is the first element of the pair, and y the second. This is to be distinguished from the notion of the set $\{x, y\}$, which consists of the elements x and y, but no order is implied. Thus, $\{x, y\}$ and $\{y, x\}$ denote the same set, but (x, y) and (y, x) are different ordered pairs. Having this explained once and for all, from now on we will just speak of a "pair" when we mean an ordered pair.

Example 1.1 Let

$$S = \{x \mid x \text{ is a natural number with } x < 6\}$$

and T be the set of natural numbers, which we denote by $\mathbb{N}$. We define a relation R as

$$R = \{(x, y) \mid y = 3x,\ x \in S\}\,. \tag{1.1}$$

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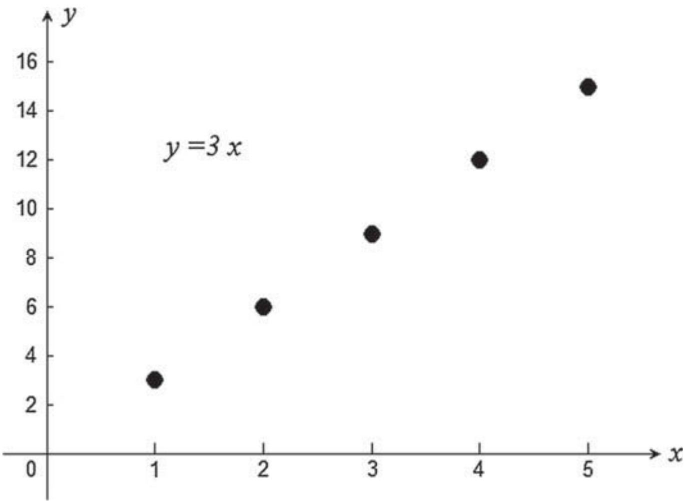


Fig. 1.1 Graph of the relation (1.1)

Since $S = \{1, 2, 3, 4, 5\}$, we can write R alternatively as the set

$$R = \{(1, 3), (2, 6), (3, 9), (4, 12), (5, 15)\}\,.$$

If the sets S and T consist of numbers, we can represent a relation graphically in the Cartesian plane, using the horizontal axis for S and the vertical axis for T. For the example above, this is given in Fig. 1.1. Such a pictorial representation of a relation is called a **graph**. Thus, a graph exhibits a pattern of points distributed over the plane, each of which represents a single pair (x, y). Most of the figures in this book are graphs of one form or another. The actual patterns formed by their points can have rather different shapes, and there may appear random collections of points or continuously filled areas, lines, or curves. All this depends on how the relationship between x and y is defined.

Example 1.2 1. The graph of $y = x$ is given in Fig. 1.2.
2. The graph of $y = x^2$ is given in Fig. 1.3.

In Example 1.2, we have used the short notation "$y = x$", respectively, "$y = x^2$" for the relations

$$\{(x, y)\,|\, x \in \mathbb{R},\ y = x\}\,, \quad \text{resp.} \quad \{(x, y)\,|\, x \in \mathbb{R},\ y = x^2\}\,.$$

Here, $\mathbb{R}$ denotes the set of all real numbers.

Definition 1.2 (*Function*) A function f from a set S into a set T is a rule that assigns to each element $x \in S$ a unique element $y \in T$. The set S of elements, for which f is defined, is called the **domain** of f and usually denoted by $D(f)$, or simply D.

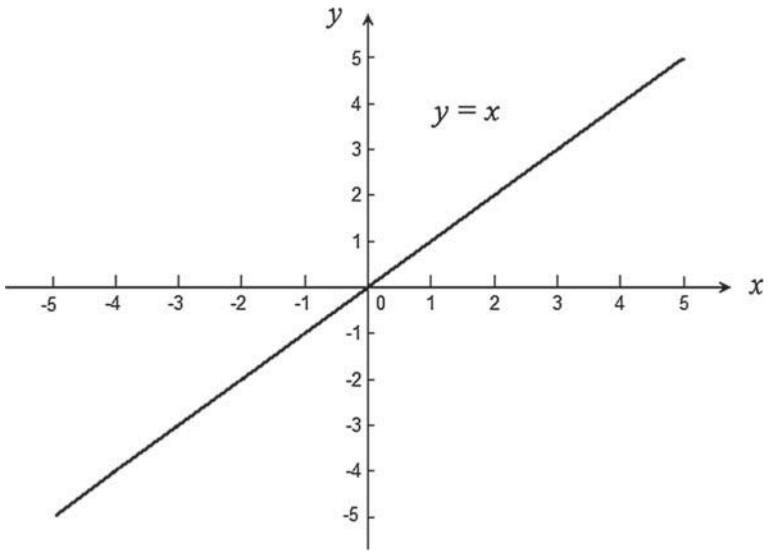
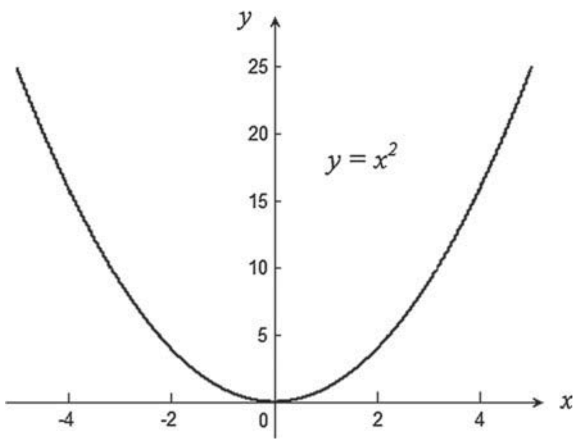


Fig. 1.2 Graph of the function $y = f(x) = x$

Fig. 1.3 Graph of the function $y = f(x) = x^2$



If f is a function from S into T, then the element $y \in T$, which is assigned to a specific element $x \in D$ by the function f, is denoted by $f(x)$ (read "f of x") and is called the **value** of f at x, while x is called the **argument** of f. The set of values of f,

$$R(f) = \{y|\, y \in T,\ y = f(x) \text{ for some } x \in D(f)\}$$

is called the **range** of f or the **image** of $D(f)$ under f. $\square$

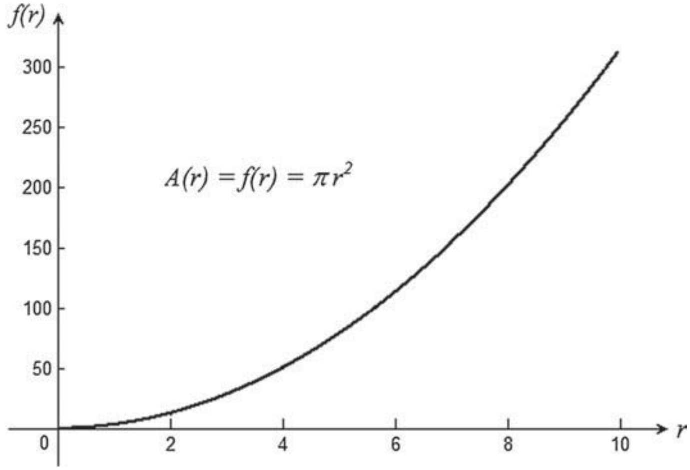


Fig. 1.4 Area of a circle as a function of its radius

Usually, a typical element of $D(f)$ is denoted by a specific letter, here x, and a typical element of $R(f)$ is denoted by a different letter, here y. In that case, x is called the **independent variable**, and y is called the **dependent variable**.

From Definition 1.2, we see that a function is a relation on S and T, where for each $x \in S = D(f)$ there is exactly one corresponding value $y \in T$. In other words, for each $x \in D(f)$ there is exactly one element y such that $y = f(x)$. Thus, the relation defined by f is

$$\{(x, y) | \, y = f(x)\} . \tag{1.2}$$

The study of calculus is based on functions, whose domain D is a set of real numbers, often an interval or a union of intervals, and whose range is a set of real numbers. In this context, a function f from D to R, where D and R are subsets of the set of real numbers, is called a real-valued function of a real variable. When we restrict ourselves to a single variable, as we do here, f is also called a function of one variable, or of a single variable. Later in Chap. 8, we will also encounter functions of two or more variables.

If $x \in D(f)$ is a real number, then the ordered pair $(x, f(x))$ with $x \in D(f)$ can be identified with a point in the xy-plane. The relation (1.2) formed by the set of all such points gives rise to a pictorial representation of f. Usually, both the picture and its formal description as the set (1.2) are called the **graph** of the function f. See, for example, Figs. 1.4 and 1.5.

There are four ways to represent a function: Verbally, numerically, visually (graphically), and algebraically.

Verbal representation: A function f is represented *verbally* if f is described in words. Examples are as follows:

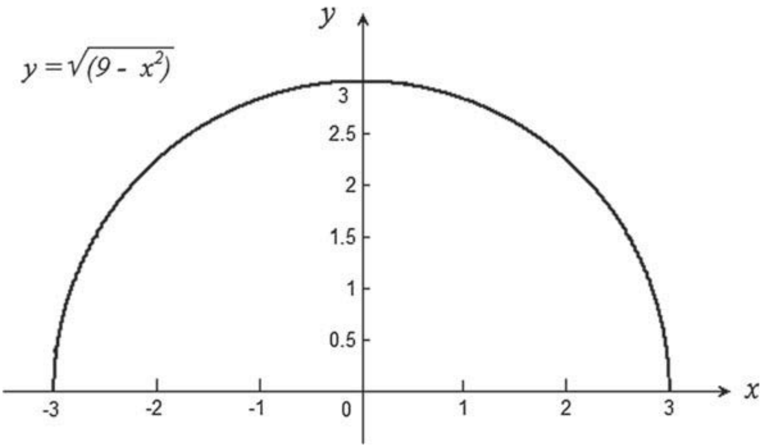


Fig. 1.5 Graph of the function $f(x) = \sqrt{9 - x^2}$

1. $A(r)$ is the area of a circle of radius r.
2. $T(t)$ is the temperature at time t.
3. $v(t)$ is the (instantaneous) velocity at time t.
4. $h(t)$ is the height at time t of a ball which has been thrown upward.

Tabular or numerical representation: A *numerical representation* of a function f is a table of its arguments and values. If done by rows, the upper row contains the arguments (the elements of the domain), and the lower row the values (the elements of the range). Example:

x	2	3	5	8	11
$f(x)$	11	9	12	−7	9

.

In this case,

$$D(f) = \{2, 3, 5, 8, 11\}, \quad R(f) = \{11, 9, 12, -7\}.$$

As a counterexample, the table of values

x	−2	1	3	5	3
$f(x)$	11	9	9	−7	−6

does not represent a function since there are two different values (9 and −6) associated to the number 3.

Graphical representation: A function f is represented *graphically* if there is a graph G in the coordinate plane such that the point (x, y) is on G if and only if $y = f(x)$, see Fig. 1.6. The domain of f consists of those points on the x-axis such that the vertical lines through these points meet the graph G. (Since f is a function, each vertical line may meet G at most once.) The range of f consists of those points

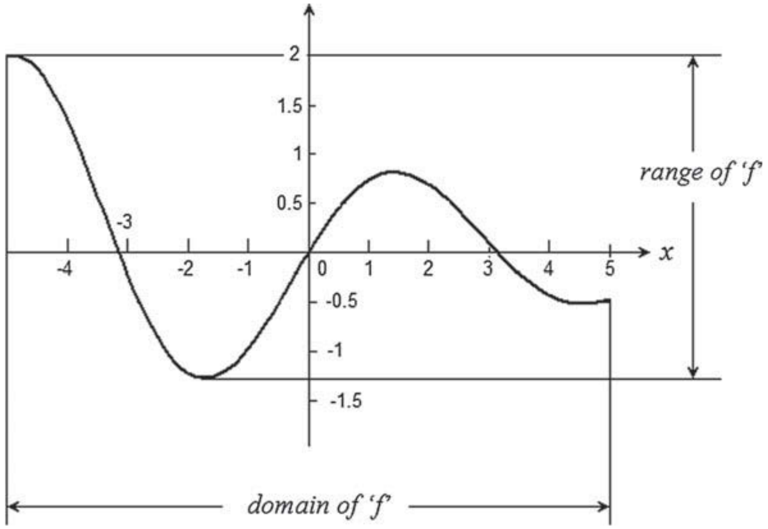


Fig. 1.6 Graph, domain, and range of a function of x

on the y-axis such that the horizontal line through these points meets the graph G. (It may happen that a horizontal line meets G more than once.)

Algebraic representation: A function f is said to be represented *algebraically* if, for each x in the domain of f, $f(x)$ is equal to some algebraic expression involving x. Examples are

1. $f(x) = x^2$,
2. $g(x) = x + 5$,
3. $h(x) = \sqrt{x}$.

The last example needs clarification, since the square root of a positive number is not unique. In this book, when we use the symbol $\sqrt{x}$, we always refer to the positive square root of x. Then h indeed defines a function.

A function may be thought of as an *input–output machine*, see Fig. 1.7. Given a particular input, say x, the function uses it as an argument and produces the value $f(x)$ as output.

$$\left\{\begin{array}{lcl} \text{input } t \rightarrow & \text{function } f & \rightarrow \text{output } f(t) \\ \text{input } 3 \rightarrow & \text{function } g & \rightarrow \text{output } g(3) \\ \text{input } 5 \rightarrow & f(t) = 2t - 6 \rightarrow & f(5) = 4 \end{array}\right\}$$

A function which describes the transmission of information is often called a **signal**.

Remark 1.1 There are many curves which we can draw in the plane, but which are not graphs of a function. In general, a curve is a graph of a function if and only if no vertical line intersects the curve more than once. In other words, if a vertical

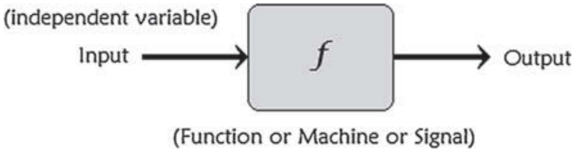


Fig. 1.7 Function as a machine or a signal

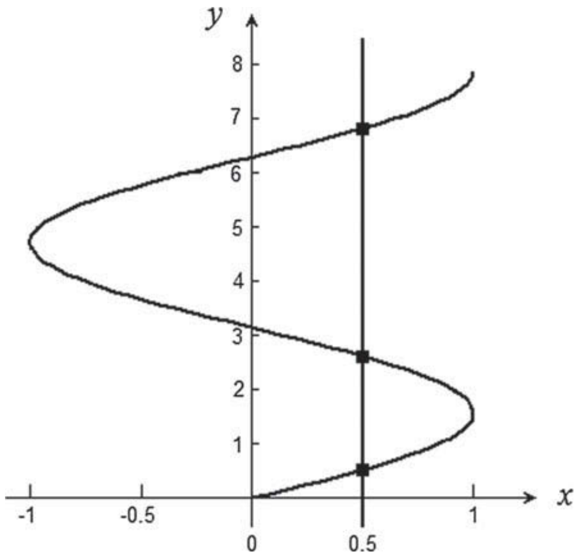


Fig. 1.8 Not a graph of a function of x

line intersects the curve at all, it does so only once. For example, the curve given in Fig. 1.8 is not a graph of a function of x as a vertical line intersects it more than once.

As a further example, let f be the absolute value function.

1. Verbal representation: $f(x)$ is the absolute value of x.
2. Tabular representation:

x	-2	-1	0	1	2
$f(x)$	2	1	0	1	2

3. Graphical representation: See Fig. 1.9.
4. Algebraic representation:

$$f(x) = |x| = \begin{cases} x\,, & x \geq 0\,, \\ -x\,, & x < 0\,. \end{cases}$$

Remark 1.2 Let us emphasize an important point concerning the notation of functions. In the majority of cases, people use the letter x to denote an argument of a function of a single variable. On the other hand, in particular, when the function

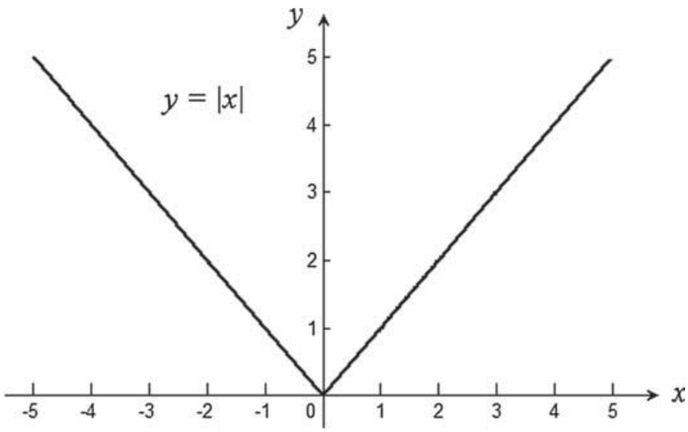


Fig. 1.9 Absolute value function

arises in the context of a real-world problem, other letters are chosen for the argument which reflects that context, like t for time. Now the point we want to make is that the definition $f(x) = x^2$ denotes **exactly the same function** as the definition $f(t) = t^2$. Indeed, according to Definition 1.2 a function f is specified completely by its domain and range (which are sets) and the rule by which elements of the range are associated to elements of the domain. Which letters or symbols we use to specify those elements are immaterial when we consider the function as a mathematical object. This observation may seem trivial to some. But it lies at the roots of the power of mathematics. Once you have a mathematical theory for the function $f(x) = x^2$, it applies no matter whether x stands for time, distance, price, signal intensity, or other quantities.

1.2 Various Types of Functions

Functions come in a large variety. Therefore, it makes sense not only to specify individual functions, like $f(x) = 3x^2 - 4$, but to consider certain classes of functions. The functions belonging to such a class are identified by certain features common to all functions in that class.

Definition 1.3 (*Constant function*) If $f(x) = c$ for all x in the domain of f, where c is a fixed number, then f is called a **constant function**.

The graph of a constant function is a horizontal line. For example, the function defined by $f(x) = 4$, $x \in [-4, 16]$, is a constant function. See Fig. 1.10.

Definition 1.4 (*Linear function*) A function of the form $y = f(x) = mx + b$, where m and b are given real numbers, is called a **linear function**.

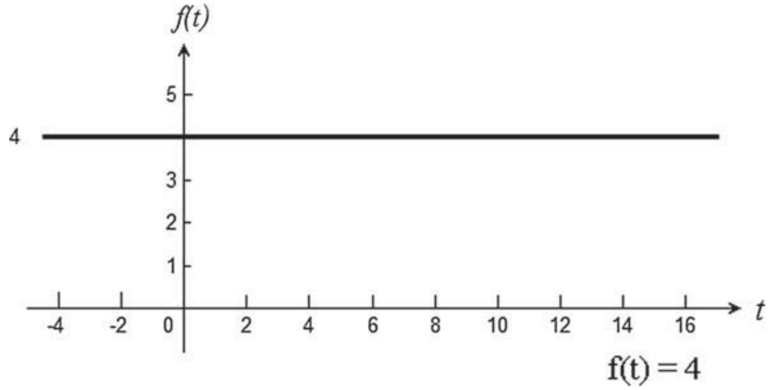


Fig. 1.10 Constant function

The graphs of linear functions are straight lines with slope m. Examples of linear functions are $f(x) = 9x + 2$ and $f(x) = \frac{1}{2}x - 3$.

Definition 1.5 (*Quadratic function*) A function of the form $y = f(x) = ax^2 + bx + c$, where a, b, and c are given real numbers and $a \neq 0$, is called a **quadratic function**.

The functions defined by $f(x) = 5x^2 + 9x + 4$ and $f(x) = -x^2 + 4x + 1$ are examples of quadratic functions.

Definition 1.6 (*Polynomial function*) A function of the form

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0\,,$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are given real numbers with $a_n \neq 0$, and $n \geq 0$ is a nonnegative integer, is called a **polynomial function** or simply a **polynomial** of degree n.

A linear function $f(x) = ax + b$ is a polynomial of degree 1 if $a \neq 1$, otherwise it reverts to the constant b, a polynomial of degree 0. A quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, is a polynomial of degree 2. As a further example, the function $f(x) = 4x^8 - 3x - 2$ is a polynomial of degree 8.

Definition 1.7 (*Rational function*) A function of the form

$$f(x) = \frac{P(x)}{Q(x)}\,,$$

where $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and $Q(x) = b_m x^m + \cdots + b_1 x + b_0$ are polynomials of degree n and m, respectively, is called a **rational function**.

For example, the function

$$f(x) = \frac{3x^3 - 4x^2 + 2}{5x^2 + x - 3}$$

is a rational function.

Definition 1.8 (*Power function*) A function of the form $y = f(x) = x^r$, where r is a given real number, is called a **power function**.

For power functions, one has to distinguish different cases. If r is a positive integer, then $f(x) = x^r$ is a polynomial of degree r, also called the **monomial** of degree r. Examples are $f(x) = x^2$ and $f(x) = x^7$. If r is a negative integer, then $-r$ is a positive integer and x^r is defined as

$$x^r = \frac{1}{x^{-r}}, \quad \text{if } x \neq 0.$$

For example,

$$x^{-2} = \frac{1}{x^2}.$$

If $r = 0$, then $x^0 = 1$ by convention (because it should hold that $x^0 = x^{r-r} = x^r x^{-r}$). If r is a rational number, then x^r may involve roots, for example

$$x^{\frac{1}{2}} = \sqrt{x}, \quad x^{\frac{1}{6}} = \sqrt[6]{x}, \quad x^{\frac{2}{5}} = \sqrt[5]{x^2}, \quad x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}.$$

In this case, x^r is defined only if x is nonnegative. The case when x is irrational is more complicated. We discuss it in the next section under the subheading "exponential function."

The classes of functions considered above have been specified by algebraic formulas. We now consider some classes which are specified by general properties.

Definition 1.9 (*Even and odd functions*) A function f having the property $f(-x) = f(x)$ for all $x \in D(f)$ is called an **even function**. An **odd function** is a function for which $f(-x) = -f(x)$ for all $x \in D(f)$. In both cases, we assume that $-x \in D(f)$ whenever $x \in D(f)$.

For example, $f(x) = x^4$ is an even function and $f(x) = x^3$ is an odd function. The trigonometric functions cos and sec are even functions, whereas sin, tan, cot, and csc are odd functions. The absolute value function $f(x) = |x|$ is an even function.

The graph of an odd function is symmetric with respect to the origin (Fig. 1.11), and the graph of an even function is symmetric w.r.t. the y-axis (Fig. 1.12). The zero function is the only function which is both even and odd, because the latter implies that $-f(x) = f(-x) = f(x)$, and hence $f(x) = 0$ for all x.

There are many functions which are neither even nor odd, for example

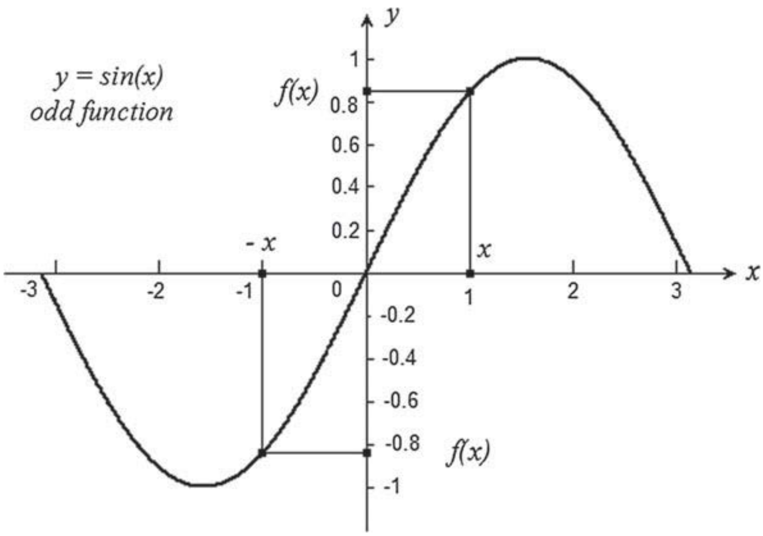


Fig. 1.11 The odd function $y = \sin x$

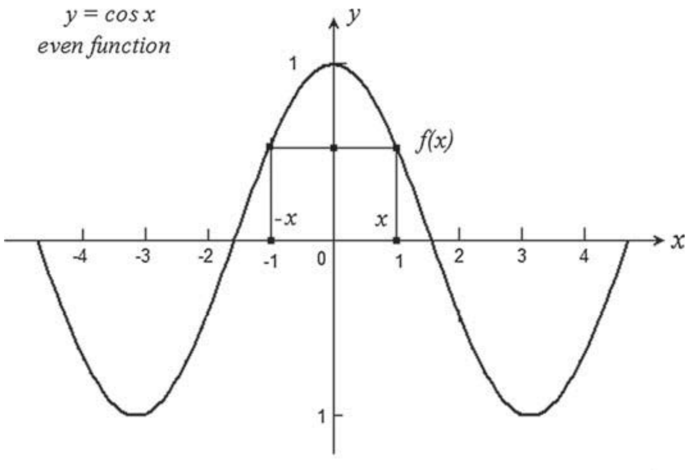


Fig. 1.12 The even function $y = \cos x$

$$f(x) = x + x^2 .$$

In fact, the sum of an even and an odd function is neither even nor odd, unless one of them is the zero function.

Theorem 1.1 *Let f and g be two real-valued functions defined on the same domain.*
(a) If f and g are even functions, then $f + g$ and $f \cdot g$ are even functions.

(b) If f and g are odd functions, then $f + g$ is an odd function, and $f \cdot g$ is an even function.
(c) If f is an even and g is an odd function, then $f \cdot g$ is an odd function.

For example,

$$\tan x = \frac{1}{\cos x} \cdot \sin x$$

is odd, by part (c) of the theorem above.

Definition 1.10 (*Periodic function*) A function f is said to be **periodic** with period p, if for each x in the domain of f, namely, $x \in D(f)$, the point $x + p$ also belongs to $D(f)$ and $f(x + p) = f(x)$. Such a number p is called a **period** of f. The smallest positive period of f is called the **fundamental period** of f.

The trigonometric functions $\sin x$ and $\cos x$ are periodic functions with the fundamental period 2π, as

$$\sin(x + 2\pi) = \sin x\,, \quad \cos(x + 2\pi) = \cos x\,,$$

and since there is no smaller positive period than 2π. On the other hand, all numbers of the form $2k\pi$, k being any integer, are periods of $\sin x$ and $\cos x$.

Definition 1.11 (*Increasing and decreasing functions*) A function f is said to be **increasing** on an interval I, if $f(x_2) > f(x_1)$ for every pair of points $x_1, x_2 \in I$ such that $x_2 > x_1$. Similarly, f is said to be **decreasing** on I if $f(x_2) < f(x_1)$ for every pair of points $x_1, x_2 \in I$ such that $x_2 > x_1$.

In addition, if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, f is called **nondecreasing**. If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$ then f is called **nonincreasing**.

It can be checked that the sum of two increasing functions is an increasing function. This statement remains true if we replace “increasing” by “decreasing”, “nonincreasing”, or “nondecreasing”.

There is a close relationship between increasing (decreasing, nonincreasing, nondecreasing) functions and the sign of their first derivative (see Sect. 4.3).

Example 1.3
1. The function $f(x) = x^2$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. On $(-\infty, \infty)$ it is neither decreasing nor increasing.
2. The function
$$f(x) = \begin{cases} 2, & x < 0 \\ x, & x \geq 0 \end{cases}$$
is constant on $(-\infty, 0)$ and increasing on $[0, \infty)$. (See Fig. 1.13.) It is neither decreasing nor increasing on $(-\infty, \infty)$.
3. The function $f(x) = x^3$ is increasing on $(-\infty, \infty)$. (See Fig. 1.14.)
4. The function defined by
$$f(x) = \begin{cases} 1\,, & x \text{ is rational,} \\ 0\,, & x \text{ is irrational,} \end{cases}$$

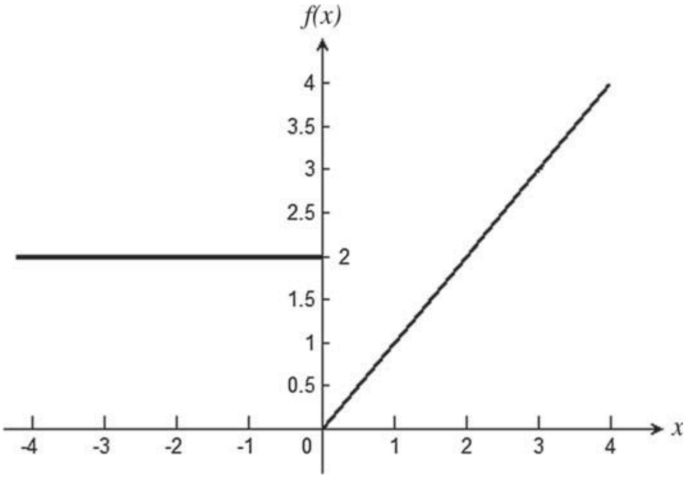


Fig. 1.13 A function which is neither decreasing nor increasing

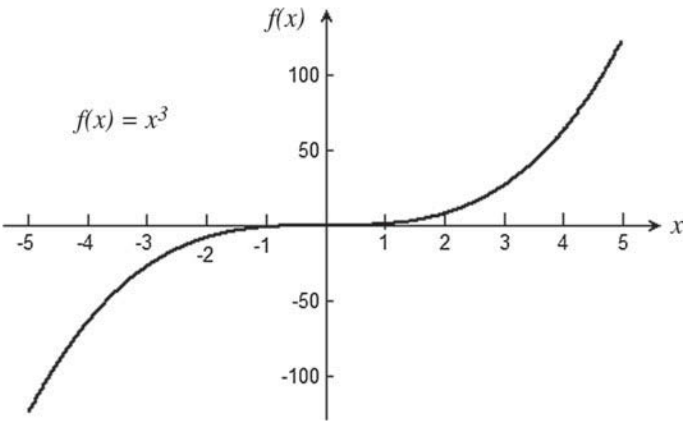


Fig. 1.14 An increasing function

is called the **Dirichlet function**. (see Fig. 1.15) There is no interval on which the function either increases or decreases. On every interval, however small, the function jumps back and forth between 0 and 1 an infinite number of times. Thus, there is no good way to draw the graph of this function, and Fig. 1.15 provides just a very rough approximation.

The next class of functions we consider is the class of **invertible functions**. It consists of all those functions for which we can find an inverse function according to the following definition.

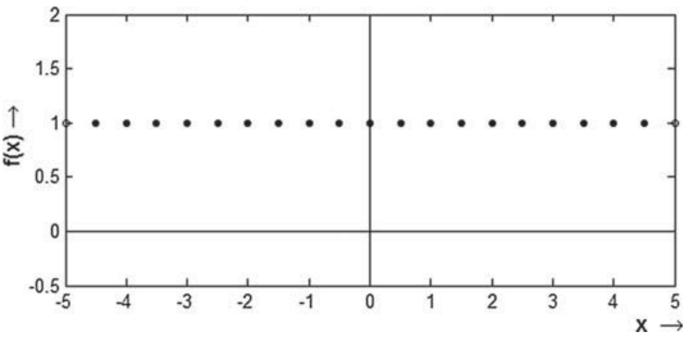


Fig. 1.15 A very rough approximation of the graph of the Dirichlet function

Definition 1.12 (*Inverse function*)

(a) A function f is said to be **one-to-one** or **injective** if any two different numbers x_1, x_2 in its domain yield different function values, that is, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. An equivalent formulation of this condition would be that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

(b) Let f be one-to-one. The **inverse** of f, denoted by f^{-1}, is the function whose domain $D(f^{-1})$ equals the range of f and is defined by

$$f^{-1}(y) = x\,, \tag{1.3}$$

for all y in the range of f, where x is the unique element in $D(f)$ with $f(x) = y$.

Note that the inverse f^{-1} indeed is a function according to Definition 1.2, since when f is one-to-one, for any given $y \in R(f)$, there is exactly one $x \in D(f)$ with $y = f(x)$. Inserting $y = f(x)$ into (1.3), we see that

$$f^{-1}(f(x)) = x\,, \quad \text{for all } x \in D(f), \tag{1.4}$$

furnishes an equivalent definition of the inverse function.

If we follow the convention that x denotes an argument of f and y denotes a value $y = f(x)$ of f, it is natural to use the letter y to denote the argument of f^{-1}. So, for example, the inverse function f^{-1} of $f(x) = x^3$ is given by $f^{-1}(y) = y^{1/3} = \sqrt[3]{y}$. Indeed, (1.4) holds since

$$f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x\,.$$

As another example, the inverse f^{-1} of $f(x) = 3x - 5$ is given by $f^{-1}(y) = \frac{1}{3}y + \frac{5}{3}$.

Theorem 1.2 *Let the function f, defined on an interval I, be either increasing or decreasing. Then f has an inverse.*

Proof Increasing and decreasing functions are one-to-one; therefore, they have an inverse.

Remark 1.3 1. If f has an inverse f^{-1}, then

$$f(f^{-1}(y)) = y\,, \quad \text{for all } y \text{ in the range of } f.$$

2. If f is a one-to-one function, then there is one and only one function g with domain equal to the range of f such that

$$g(f(x)) = x\,, \quad f(g(y)) = y\,,$$

for all $x \in D(f)$ and all $y \in R(f)$. Namely, $g = f^{-1}$.
3. There is no special meaning attached to the letter y as an argument of f^{-1}. We could equally well use the letter x, so that, for example, the inverse of $f(x) = x^3$ is given by $f^{-1}(x) = x^{1/3}$.
4. If f is a one-to-one function, every horizontal line in the coordinate plane meets the graph of f at most once.
5. When we reflect the graph of f in the coordinate plane across the straight line $y = x$, we obtain the graph of f^{-1}.

Definition 1.13 (*Bounded function*) A function f is said to be **bounded** on an interval I if there exists a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$. Any such number is called a **bound** for f on I.

For example, $f(x) = \sqrt{9 - x^2}$ is bounded on its domain, the interval $I = [-3, 3]$, because $0 \leq f(x) \leq 3$ for all $x \in I$, and $M = 3$ (or any larger number) is a bound for f. On the other hand, the function $g(x) = \pi x^2$ is unbounded on its domain of definition $(-\infty, \infty)$, but it is bounded if we restrict it to the interval $I = [0, 1]$, where it has the bound π.

1.3 Important Examples of Functions

Trigonometric functions. The trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are discussed in Appendix C.

Absolute value function. This function is denoted as $|x|$ and is defined by

$$f(x) = |x| = \begin{cases} x\,, & x \geq 0\,, \\ -x\,, & x < 0\,. \end{cases}$$

Its domain is $(-\infty, \infty)$ or $\mathbb{R}$, its range is $[0, \infty)$. If it appears as part of another function, one has to take care what becomes of the case distinction. For example, $g(x) = |2x - 3|$ can be expressed as

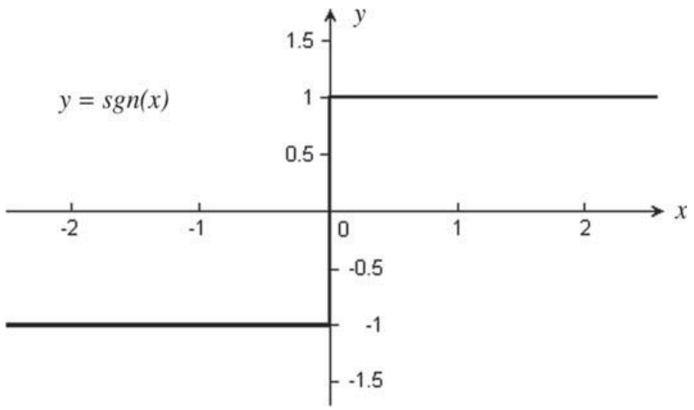


Fig. 1.16 Signum or sign function

$$g(x) = \begin{cases} 2x - 3\,, & x \geq \frac{3}{2}\,, \\ -(2x - 3)\,, & x < \frac{3}{2}\,. \end{cases}$$

Signum function. This function, also called **sign function**, is denoted by $\operatorname{sgn}(x)$ and defined by

$$\operatorname{sgn}(x) = \begin{cases} \dfrac{x}{|x|}\,, & x \neq 0\,, \\ 0\,, & x = 0\,. \end{cases}$$

The signum function has the values 1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$. Its domain is $\mathbb{R}$ or $(-\infty, \infty)$, and its range is the set $\{-1, 0, 1\}$ consisting of three elements only. (See Fig. 1.16.)

Greatest Integer function. This function is denoted by $[x]$ and defined by

$$f(x) = [x] = \text{the largest integer } n \text{ such that } n \leq x.$$

See Fig. 1.17. It is also called the **integer part function**, and the number $[x]$ is called the **integer part** of x. The domain of this function is $\mathbb{R}$ or $(-\infty, \infty)$, and its range is the set of all integers. It is constant on each interval of the form $[n, n+1)$ where n is an integer. At each integer point $x = n$ the value of the function $[x]$ changes from $n - 1$ to n; the function is said to have a jump or a step of unit magnitude at those points. For example, we have $[2.1] = 2$, $[2] = 2$, $[1.9] = 1$, $[-2.1] = -3$, $[-2] = -2$.

Step function. Let a closed interval $[a, b]$ be divided into subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ by the points $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. A **step function** $s(x)$ on $[a, b]$ is a function that is constant on open intervals (x_{j-1}, x_j), $j = 1, 2, 3, \ldots, n$. The values $s(x_j)$ at the partition points need not to be related to

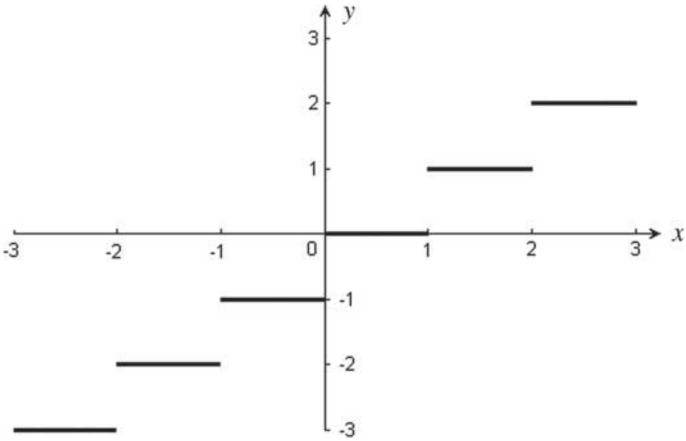
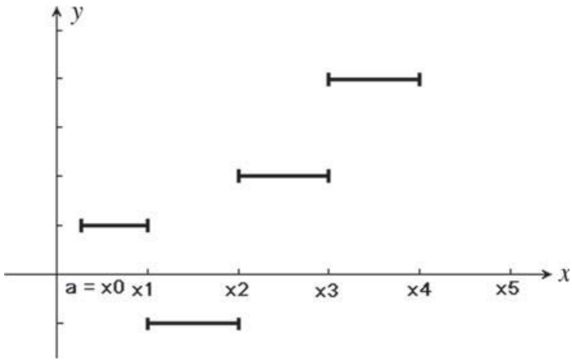


Fig. 1.17 Integer part function

Fig. 1.18 A step function



the values $s(x)$ on either adjoining subintervals. An example of a step function is shown in Fig. 1.18. The greatest integer function (if restricted to an interval $[a, b]$) also furnishes an example of a step function.

Factorial function. The factorial function or **factorial of n** is denoted by $n!$ and defined by

$$f(n) = n! = 1 \cdot 2 \cdot 3 \cdots n \,.$$

Its domain is the set of nonnegative integers including 0, and we have $0! = 1$ by convention (Fig. 1.19). Its range is a subset of the positive integers. Its graph consists of isolated points (0, 1), (1, 1), (2, 2), (3, 6), (4, 24) ….

Haar function. The function f defined by

$$f(x) = \begin{cases} 1\,, & 0 < x \le \frac{1}{2}\,, \\ -1\,, & \frac{1}{2} < x \le 1\,, \end{cases}$$

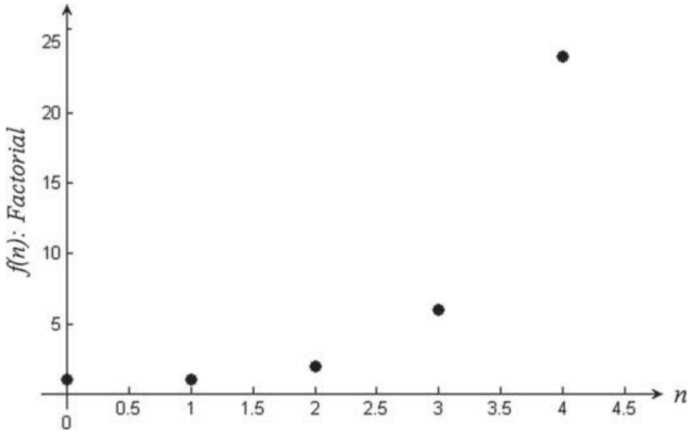
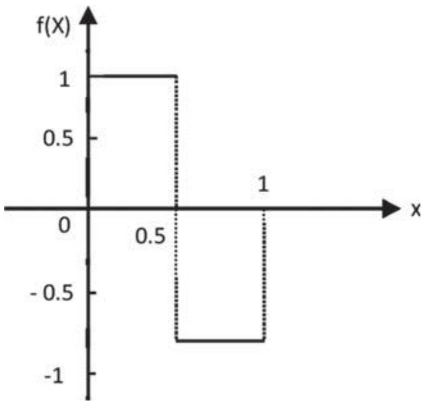


Fig. 1.19 Factorial function

Fig. 1.20 Haar function



is called the **Haar function**, see Fig. 1.20. The Haar function appears in approximation theory, in particular, in the construction of wavelets.

Heaviside function. (See Fig. 1.21) The Heaviside function $H(x)$ is defined as

$$H(x) = \begin{cases} 1\,, & x \geq 0\,, \\ 0\,, & x < 0\,. \end{cases}$$

Shannon sampling function. The function f defined by

$$f(x) = \frac{\sin \omega_0 x}{\pi x}\,,$$

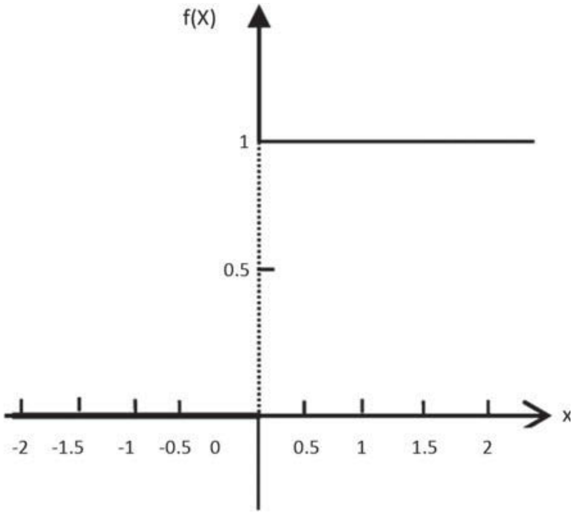


Fig. 1.21 Heaviside function

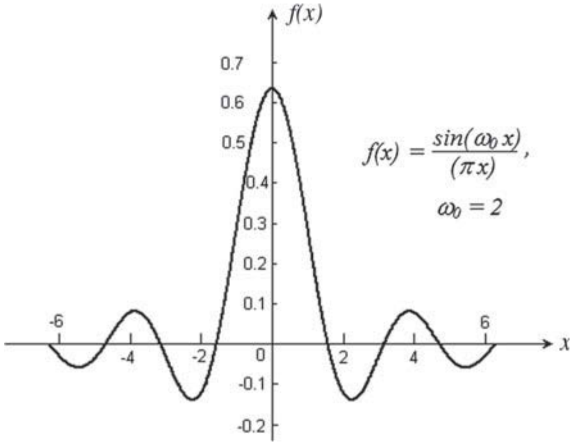


Fig. 1.22 Shannon sampling function

where ω_0 is a positive number, is called the Shannon sampling function. (See Fig. 1.22.)

Exponential function. A function of the form

$$y = f(x) = a^x ,$$

where $a > 0$ is a real constant, is called an **exponential function**. This name is chosen because the independent variable x appears in the exponent. Previously, we have defined a^x when x is a rational number, using roots like, for example, $x^{\frac{2}{3}} = \sqrt[3]{x^2}$. But how can we define, for example, the number $3^{\sqrt{2}}$? One way to do this is to observe that the function $f(x) = 3^x$ is increasing if we consider only rational numbers x as

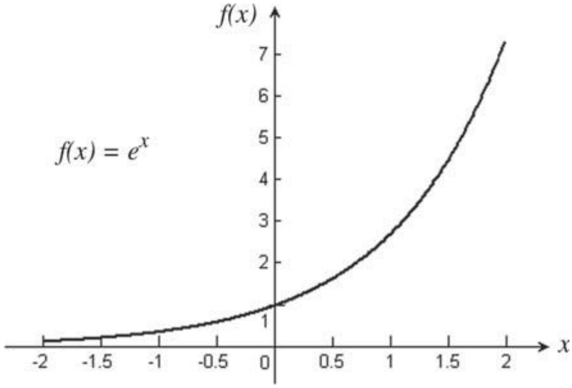


Fig. 1.23 Exponential function

arguments. Therefore, we require that $3^{\sqrt{2}} > 3^x$ if $\sqrt{2} > x$ and x is rational, and $3^{\sqrt{2}} < 3^x$ if $\sqrt{2} < x$ and x is rational. Indeed one can prove that this procedure defines a unique real number a^x for any real number x in the case $a > 1$. A similar argument works for the case $0 < a < 1$, and finally one sets $1^x = 1$ for all real numbers x. This approach is perfectly feasible.

Nowadays, mathematicians worldwide usually adopt a different approach which is directly related to important formulas. It is based on the famous **Euler number** e, an irrational number given by $e = 2.71828\ldots$. We will define e as the limit of a sequence in Chap. 2. Later, we will define the function $f(x) = e^x$ through an infinite series in Chap. 5. Because of its fundamental importance, both in mathematics itself and in applications of mathematics, the function $f(x) = e^x$ is usually called **the** exponential function. Its domain is $\mathbb{R} = (-\infty, \infty)$ and its range is $(0, \infty)$. Its graph is given in Fig. 1.23. The basic formula for the exponential function is

$$e^{x+y} = e^x \cdot e^y\,. \tag{1.5}$$

Naturally, this should be true, since otherwise the notation e^x would not make much sense, and indeed one can prove it from the series representation of e^x in Chap. 5. Setting $x = 0$ in (1.5) we see that $e^0 = 1$, and setting $y = -x$ in (1.5) we obtain that

$$1 = e^0 = e^{x-x} = e^x \cdot e^{-x}\,, \quad e^{-x} = \frac{1}{e^x}\,. \tag{1.6}$$

The number e itself is obtained from the exponential function as $e = e^1$. The exponential function will appear throughout this book in many examples where mathematics is applied.

The function $f(x) = e^{-x^2}$ is called a **Gaussian function**.

The other exponential functions $f(x) = a^x$, $a \neq e$ are related to the exponential function $f(x) = e^x$ through the logarithmic functions, to be explained in the next paragraph.

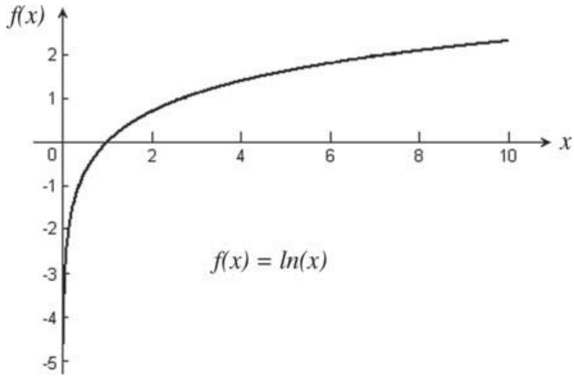


Fig. 1.24 Natural logarithm

Logarithmic function. If $x = e^y$, which is possible only if $x > 0$, then y is called the **natural logarithm** of x, and it is written as $y = \ln x$. In other words, $y = \ln x$ means the same as $x = e^y$. We have $\ln e = 1$ since $e = e^1$, and $\ln 1 = 0$ since $1 = e^0$. The exponential function is increasing and hence has an inverse function defined on its range $(0, \infty)$ due to Theorem 1.2. This inverse function is furnished by the natural logarithm, since $\ln e^y = y$ for all $y \in \mathbb{R}$, and $x = e^{\ln x}$ for all $x > 0$. Therefore, the graph of the natural logarithm (Fig. 1.24) is obtained from the graph of the exponential function (Fig. 1.23) by reflection across the line $y = x$. The graph of the natural logarithm crosses the x-axis at $(1, 0)$, and the graph of the exponential function crosses the y-axis at $(0, 1)$; this corresponds to the formulas $\ln 1 = 0$ and $e^0 = 1$ from above.

With the aid of the natural logarithm, a convenient definition of the exponential functions $f(x) = a^x$ for arbitrary real numbers $a > 0$ is given by

$$a^x = e^{x \ln a} \,. \tag{1.7}$$

The rules for powers with natural numbers extend to real numbers a, x, y where $a > 0$,

$$a^{x+y} = a^x \cdot a^y \,, \quad (a^x)^y = a^{xy} \,.$$

This can be seen by combining (1.7) with the basic formula (1.5) for the exponential function.

Due to (1.7), the functions $f(x) = a^x$ are increasing if $a > 1$ and decreasing if $a < 1$; hence, they can be inverted if $a > 0$ and $a \neq 1$. If $x = a^y$, which is possible only for $x > 0$, then y is called **logarithm** of x **to the base** a and it is written as $y = \log_a x$. In other words $y = \log_a x$ means the same as $x = a^y$. We give some examples:

$$\begin{aligned}
y = \log_2 1 \text{ means } 2^y &= 1, \text{that is }, y = 0.\\
y = \log_2 4 \text{ means } 2^y &= 4, \;\text{that is }, y = 2.\\
y = \log_a a \text{ means } a^y &= a, \;\text{that is }, y = 1.\\
y = \log_3 \frac{1}{3} \text{ means } 3^y &= \frac{1}{3}, \text{that is }, y = -1.
\end{aligned}$$

Besides e, the most common bases are $a = 10$ (due to the everyday use of the decimal system) and $a = 2$ (due to the binary system used in computers). The logarithm to the base 10 of x, that is $\log_{10} x$, is called the **common logarithm** and often written as $\log x$.

Let x, y, r be arbitrary real numbers, where $x, y > 0$. Then the following formulas hold.

$$\begin{aligned}
\log_a(xy) &= \log_a(x) + \log_a(y)\,,\\
\log_a\left(\frac{x}{y}\right) &= \log_a(x) - \log_a(y)\,,\\
\log_a(x^r) &= r \cdot \log_a x)\,,\\
\log_a x &= \frac{\log_b x}{\log_b a}\,.
\end{aligned}$$

In the last formula, b denotes any real number with $b > 0, b \neq 1$.

Hyperbolic functions. The hyperbolic functions are defined as follows.

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} && \text{(hyperbolic sine)}\\
\cosh x &= \frac{e^x + e^{-x}}{2} && \text{(hyperbolic cosine)}\\
\tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} && \text{(hyperbolic tan)}\\
\coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}} && \text{(hyperbolic cot)}\\
\operatorname{sech} x &= \frac{2}{e^x + e^{-x}} && \text{(hyperbolic sec)}\\
\operatorname{cosech} x &= \frac{2}{e^x - e^{-x}} && \text{(hyperbolic cosec).}
\end{aligned}$$

The terms “tanh”, “sech”, and “cosech” are pronounced as “tanch”, “seech”, and “coseech”, respectively.

1.4 Functions as Models

For a systematic study of the world around us, mathematics is required. As early as 1623, the Italian scientist and mathematician Galileo Galilei (1564–1642) wrote about “the all-encompassing book which is constantly open before our eyes” (he means the universe) that “it is written in mathematical language.” He adds that “its characters are triangles, circles, and other geometrical figures” and that without mathematical language, “one wanders around pointlessly in a dark labyrinth.” This is even more relevant today, when we have come a long way from the origins of mathematics in numbers and geometrical figures. Indeed, Galilei’s statement addresses what in the modern scientific view is termed as **mathematical modeling**. A situation in physics or another discipline is formulated in terms of mathematical concepts such as equations, functions, derivatives, and integrals. Such a formulation is then called a **mathematical model**.

In this section, we use functions as mathematical models for different situations. We address different situations from the real world, present mathematical models for them in terms of functions (algebraic, numerical, or graphical), and give some examples of results.

One of the most important steps in creating a mathematical model of a real-world situation is to decide which factors to consider and which to ignore. The more the factors one takes into account, the more complicated the expressions and equations of the model tend to become, so an appropriate balance is needed between keeping a model mathematically simple and considering enough factors to make the model realistic and useful. Moreover, the developers of the model often have a purpose, namely, some questions which the model should help to answer. A good mathematical model, at first, has to produce results that are consistent with the real world. Depending on the questions it should help to answer, it may be good for one purpose but not for another. If a mathematical model does not meet these requirements, it must be modified or even changed completely.

In this section, we consider models that involve only two variables, say x and y. We assume that the data for the phenomenon being modeled consists of a collection of ordered pairs of measurements $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ that relate corresponding values of the variables x and y. We will discuss the examples from the deterministic viewpoint only, that is, when a given value of x uniquely determines the value of y. Let us remark that in many other situations, a probabilistic (or stochastic) viewpoint has to be taken, that is, the value of y does not only depend on the value of x, but moreover probabilities are involved in some way or other.

Example 1.4 Express the area of a circular region as a function of its radius.

Solution: We know that the area of a circular region of radius r is given by the number πr^2. Thus, its area A is represented by the function $f(r) = \pi r^2$. In order not to introduce additional symbols, one often denotes the value of the area and the function by the same symbol A, so $A(r) = \pi r^2$. In this case, r is the independent variable, and the area A is the dependent variable. The domain of this function is $(0, \infty)$ and its range, too, is $(0, \infty)$.

Example 1.5 The area of a square region in the plane is a function of the length of its side. If α is the length of a side of the given square, then its area A is a function of α, namely, $A(\alpha) = \alpha^2$.

Example 1.6 The area of an equilateral triangle is a function of its side length.

Example 1.7 Let us consider a rectangular box such that the length plus the girth (the perimeter of a cross section) equals 108 cm. Express the volume V of the box as a function of the edge length of a cross section and find the domain of this function.

Solution: Let h denote the length of the box. All cross sections (including those at the end) are squares of identical edge length, let us denote this edge length by α. The girth equals the perimeter of such a square and is therefore equal to 4α. By assumption, we must have $4\alpha + h = 108$ or $h = 108 - 4\alpha$. The volume of the box is given by $V = \alpha^2 h$. Inserting the formula for h we obtain

$$V(\alpha) = \alpha^2(108 - 4\alpha)\,.$$

Since neither the edge length of the square nor the length of the box can be negative, we have $\alpha \geq 0$ and $h = 108 - 4\alpha \geq 0$. The latter inequality holds if and only if $\alpha \leq 27$. Therefore, the total restrictions on α are $0 \leq \alpha \leq 27$. Thus, the domain of the function V is the interval $[0, 27]$.

Example 1.8 A soft drink manufacturer wants to fabricate cylindrical cans. Each can should have a volume of 200 cubic cm. Express the total surface area of each can as a function of its radius and find the domain of this function.

Solution: The total surface area S of the can equals $2\pi r^2 + 2\pi r h$, and thus is a function of its radius r and height h. We can eliminate h through the volume constraint $V = \pi r^2 h = 200$, which gives $h = \frac{200}{\pi r^2}$ as a function of r. Inserting this into the formula for the surface area yields

$$S(r) = 2\pi r^2 + 2\pi r \frac{200}{\pi r^2} = 2\pi r^2 + \frac{400}{r}\,.$$

Since r can take any positive value, $D(S) = (0, \infty)$.

Example 1.9 The volume V of a ball is a function of its radius r,

$$V(r) = \frac{4}{3}\pi r^3.$$

Example 1.10 Suppose a computer shop sells x laptops each day at 500 Euro a piece, and suppose that the cost of manufacturing and selling the laptops is 400 Euro per laptop plus fixed daily operating costs (heat, rent, insurance, etc.) of 200 Euro. Express the daily profit as a function of x. How big is the profit if 20 laptops are sold daily?

Solution: The daily revenue equals $R(x) = 500x$, the daily cost equals $C(x) = 400x + 200$, the daily profit equals $P(x) = R(x) - C(x)$, all measured in Euro. Therefore, we obtain

$$P(x) = 500x - (400x + 200) = 100x - 200\,.$$

If 20 laptops are manufactured then

$$P(20) = 100 \cdot 20 - 200 = 1800\,.$$

In this case, the daily profit equals 1800 Euro.

Proportionality. We say that the quantity y is directly proportional to the quantity x if there is a constant k such that $y = kx$. This k is called the constant of proportionality. We also say that the quantity y is inversely proportional to the quantity x if y is proportional to the reciprocal of x, that is, there is a constant μ such that $y = \mu/x$. The constant μ is called the constant of inverse proportionality.

Example 1.11 (a) A motorist has to drive from town A to town B, taking a duration t of time at a constant speed v. Show that the speed he has to use is inversely proportional to the time t. Compute the constant μ if the towns are 100 km apart.

(b) Assume that the heart mass of an animal is proportional to its body mass (which is true approximately). Then

  (i) Write a formula for the heart mass, h, as a function of the body mass, b.
  (ii) An animal with a body mass of 80 kilograms has a heart mass of 0.48 kilograms. Find the constant of proportionality.
  (iii) Estimate the heart mass of a cow with a body mass of 600 kg.

Solution:

(a) Let s denote the distance between A and B, then $s = vt$ or $v = s/t$. If s is measured in km, t in hours and v in km/h, then

$$v = \frac{s}{t} = \frac{100}{t}\,, \quad \mu = s = 100\,.$$

(b) (i) By the definition of direct proportionality, $h = kb$. In this case, the proportionality constant is dimensionless, since both h and b have the dimension of a mass (to be taken as kilogram).

  (ii) Here $h = 0.48$ kg and $b = 80$ kg, so

$$0.48 = k \cdot 80\,, \quad k = \frac{0.48}{80} = 0.006\,.$$

  (iii) Since $h = kb$, we have $h = 0.006 \cdot 600 = 3.6$ kg. Thus, the heart mass of the cow is 3.6 kg.

Models for exponential growth and decay. We consider a function which is proportional to an exponential function,

$$P(t) = P_0 a^t \,.$$

It is a model for the growth or decay of a quantity P over time t, the initial amount being $P(0) = P_0$. The number $a > 0$ is called the growth factor. Indeed, since $P(t+1) = aP(t)$, when the time t increases by one unit, the new value $P(t+1)$ is obtained from the old value $P(t)$ by multiplying with a. If one writes $a = 1 + k$, then the number $kP(t)$ gives the absolute change of P from t to $t+1$, thus k is also termed the relative growth rate. If it is positive, then the quantity P indeed is growing, while if $k < 0$, it is decaying. We may also express the growth rate in percent, then, for example, a growth rate of 5% corresponds to $k = 0.05$.

This model is closely related to a basic situation in finance. Suppose P_0 is the initial amount of money deposited in an account which pays an annual interest rate of r percent, and $P(t)$ is the balance in the account after t years, then

$$P(t) = P_0(1+k)^t \,, \quad k = 0.01r \,, \tag{1.8}$$

if the interest is compounded annually. If the interest is compounded continuously, the model

$$P(t) = P_0 e^{kt} \tag{1.9}$$

applies. Here, $e = 2.71828\ldots$ is the Euler number as mentioned above. The meaning of this model will be explained in detail in Chap. 3.

Example 1.12 (a) If Rs. 10000 is deposited in a bank account paying 10% annual interest compounded continuously, how much will be the amount 10 years later?

(b) Suppose that a bank advertises an annual rate of 8% interest. If you deposit Rs. 5000, how much will be in the account 3 years later if the interest is compounded (i) annually, (ii) continuously?

Solution:

(a) We use the formula $P(t) = P_0 e^{kt}$. The annual interest rate is 10%, so $k = 0.1$, $t = 10$, and $P_0 = 10000$, the initial deposit. Therefore, $P(10) = 10000e^{0.1\cdot 10} = 10000e = 27182.81828$.

(b) (i) For annual compounding after $t = 3$ years,

$$P(3) = P_0(1+k)^3 = 5000(1+0.08)^3 = 6298.56\,.$$

(ii) For continuous compounding after $t = 3$ years,

$$P(3) = P_0 e^{3k} = 5000e^{0.08\cdot 3} = 6356.25\,.$$

We see that the amount in the account 3 years later is larger if the interest is compounded continuously (6356.25) than if the interest is compounded annually

(6298.56). This is to be expected since with annual compounding, the capital increases only at the end of the year, while with continuous compounding, the continuous increase of the capital during the year yields additional interest.

Example 1.13 A woman wants to invest for the education of her children in a certificate of deposit (CD). She wants it to be worth 12,000 in 10 years. How much should she invest if the CD pays 9% interest rate (a) compounded annually, (b) compounded continuously?

Solution:

(a) If the compound interest pays 9% interest over a period of 10 years, then $r = 0.09$ and $t = 10$. We find initial amount P_0 if the balance after 10 years compounded annually is $P = 12000$, we have

$$\begin{aligned} P &= P_0(1+r)^t \quad \text{or} \\ 12000 &= P_0(1+0.09)^{10} \quad \text{which gives} \\ P_0 &= \frac{12000}{(1.09)^{10}} = \frac{12000}{2.36736} = 5068.93. \end{aligned}$$

This amount should be invested to have the given balance after 10 years.

(b) In this case, we use the formula

$$\begin{aligned} 12000 &= P_0 e^{(0.09)(10)} \\ \text{or } P_0 &= \frac{12000}{e^{(0.09)(10)}} \\ &= \frac{12000}{e^{0.9}} \simeq \frac{12000}{2.45960} = 4878.84. \end{aligned}$$

Thus initial deposit should be Euro 4878.84.

Example 1.14 The population of a city equals 60,000 at the beginning of the year 2007 and is growing continuously at a yearly rate of 5%.

1. Determine the population of the city at the beginning of the year 2017.
2. Calculate the time after which the size of the population will have doubled since 2007.

Solution: We use the formula $P(t) = P_0 e^{kt}$.

(a) Here $k = 0.05$ and $t = 10$, hence $kt = 0.5$. As $P_0 = 60,000$, we obtain $P(10) = 60,000e^{0.5} = 98,923.28$.

(b) The time t to double satisfies $P(t) = 2P_0$. Since on the other hand $P(t) = P_0 e^{0.5t}$, we must have $2 = e^{0.05t}$. Taking the logarithm gives $\ln 2 = 0.05t$, hence $t = 20 \ln 2 = 13.8629$.

Example 1.15 Using the model $P(t) = P_0 a^t$, predict the size of the world population in 2010 with the help of the following table:

Year	Population(million)	Ratio
1986	4936	5023/4936 $\approx$ 1.0176
1987	5023	5111/5023 $\approx$ 1.0175
1988	5111	5201/5111 $\approx$ 1.0176
1989	5201	5329/5201 $\approx$ 1.0246
1990	5329	5422/5329 $\approx$ 1.0175
1991	5422	

Solution: The third column shows that the population in any year is about 1.018 times the population in the previous year, so let us take $a = 1.018$ for the value of the growth factor. If t years have passed after 1986, the world population would then be given by approximately $P(t) = 4936 \cdot 1.018^t$ million people. For 2010 we have $t = 24$, so our estimate yields $P(24) \approx 7573.9$, that is, approximately 7.6 billion people as the size of the world population in 2010. An example of **exponential decay** is furnished by the model $y(t) = Ae^{-0.00012t}$ which describes how the radioactive element carbon-14 decays over time. Here, A is the original amount of carbon-14, t is the time in years, and $y(t)$ is the amount of carbon-14 present after t years. Carbon-14 decay is used to date the remains of dead organisms such as shells, seeds, and wooden artifacts.

Models involving the logarithmic function. As the first example, consider the intensity of earthquakes. It is often characterized as a number on the logarithmic Richter scale. The formula for its magnitude R is given by

$$R = \log_{10}\left(\frac{a}{T}\right) + B\,,$$

where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic in seconds, and B is an empirical factor that allows for the weakening of the seismic wave with increasing distance from the epicenter of the quake. For an earthquake 10,000 km from the receiving station, $B = 6.8$. If the recorded vertical ground motion is $a = 10$ microns and the period is $T = 1$ seconds, the earthquake's magnitude is

$$R = \log_{10}(10/1) + 6.8 = 1 + 6.8 = 7.8\,.$$

An earthquake of this magnitude does great damage near its epicenter.

The logarithmic function is also used in measuring the intensity of sound. The so-called *sound intensity level* is defined by

$$L(I) = 10\log_{10}\left(\frac{I}{I_0}\right)\,.$$

Here, I is the sound intensity and I_0 is a reference value, usually taken as 10^{-12} Watts per square meter. The sound intensity arriving at a fixed point is directly proportional to the sound power emitted by the source. The sound intensity level $L(I)$ is a dimensionless number with decibel (abbreviated "db") as its unit.

Example 1.16 How many decibels does the sound intensity level increase if we double the power output of an amplifier?

Solution: We have

$$\begin{aligned} L(2I) &= 10\log_{10}\frac{2I}{I_0} = 10\left(\log_{10}\frac{I}{I_0} + \log_{10} 2\right) = L(p) + 10\log_{10} 2 \\ &= L(p) + 3.01\,. \end{aligned}$$

Thus, doubling the power output increases the sound intensity level by approximately 3 db.

Let us summarize the examples given in this section by saying that the exponential function and its inverse, the logarithm function, play an important role in the mathematical modeling of the real world. Their mathematical properties will be studied in detail in the following chapters.

1.5 Algebra of Functions

Let f and g be the two functions with domains $D(f)$ and $D(g)$, respectively. They can be added, subtracted, and multiplied on the intersection $D(f) \cap D(g)$ of their domains, that is,

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) && \text{(addition)}\,, \\ (f-g)(x) &= f(x) - g(x) && \text{(subtraction)}, \\ (fg)(x) &= f(x)g(x) && \text{(multiplication)}, \end{aligned}$$

in particular, if α is any real number,

$$(\alpha f)(x) = \alpha f(x) \qquad \text{(scalar multiplication)}.$$

At points x where $g(x) \neq 0$, f can be divided by g,

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}\,.$$

These statements may seem rather innocuous. Their mathematical meaning is as follows: From two given functions f and g, we build a new function, which (in the case of addition) we call $f + g$. Its value $(f + g)(x)$ at a point x is defined to be the number $f(x) + g(x)$, which we obtain by adding the values $f(x)$ and $g(x)$ of the original functions.

The **composition** of two functions f and g is again a function. It is denoted by $f \circ g$ and defined as

$$(f \circ g)(x) = f(g(x))\,.$$

In other words, we obtain its value $(f \circ g)(x)$ by inserting the value $g(x)$ as an argument into f. The domain of $f \circ g$ is given by

$$D(f \circ g) = \{x : x \in D(g) \text{ and } g(x) \in D(f)\} .$$

As an example, consider $f(x) = \ln x$ and $g(x) = 4 - x^2$. We have $D(g) = \mathbb{R}$ and $D(f) = (0, \infty)$, hence $D(f \circ g) = \{x : 4 - x^2 > 0\} = \{x : |x| < 4\}$. The function $f \circ g$ is given by $(f \circ g)(x) = f(g(x)) = \ln(4 - x^2)$.

1.6 Proofs, Mathematical Induction

Mathematical results are based on proofs. The notion of a proof and its rules have a long history, and they were formulated already by Greek philosophers and mathematicians, for example, by Aristotle. In mathematics, we work in a deductive system, where truth is argued on the basis of assumptions, definitions, and previously proved results. No one can righteously claim that a mathematical result is true without clearly stating the basis, either implicitly or explicitly, on which the claim is made, and presenting a proof.

In the previous sections, we already have stated definitions and theorems, and drawn conclusions in order to prove a theorem or to get other results out of a theorem. The ingredients of a theorem are its assumptions, the statements it claims to be true, and the proof which shows that the statements indeed follow from the assumptions in a strictly logical manner.

A simple way of stating a theorem is to say "if A holds then B holds," or equivalently "A implies B." This means that A is the assumption, and B is the statement which follows from the assumption. An equivalent formulation would be "A holds only if B holds."

Sometimes, a theorem tells us "if A holds then B holds" and conversely "if B holds then A holds." In this case, one usually merges those two statements and simply says "A holds if and only if B holds" or, more briefly, "A holds iff B holds." An example would be the theorem "the number n is odd if and only if the number $3n$ is odd", which is true.

It may be noted that there may be more than one set of assumptions under which a conclusion of a theorem holds. For example, if a and b are both positive, then ab is positive, and if a and b are both negative, then ab is positive.

Let us present three common schemes of a proof. For the first two, we consider the example theorem "if f is an even function, then $f + 1$ is an even function."

Direct proof. Let f be an even function, let x be an arbitrary element of $D(f)$. Then $(f + 1)(-x) = f(-x) + 1 = f(x) + 1 = (f + 1)(x)$; therefore, $f + 1$ is an even function.

Indirect proof (contraposition). Let the function $f + 1$ be not even. Then there is an $x \in D(f + 1) = D(f)$ such that $(f + 1)(-x) \neq (f + 1)(x)$. This implies

that $f(-x) + 1 \neq f(x) + 1$, hence, $f(-x) \neq f(x)$. Therefore, f is not an even function.

The third scheme is a bit more involved, and it is the proof by **contradiction**. Here, in order to prove "if A holds then B holds," we assume that A holds, but B does not hold. We proceed step by step and arrive at a contradiction. Since this is not allowed in mathematics, the statement "A holds, but B does not hold" must be wrong, so we conclude that if A holds, then B has to hold, too. Let us illustrate this scheme by a famous example, the proof that $\sqrt{2}$ is irrational, which is due to the Greek mathematician Euclid. Let us assume that $\sqrt{2}$ is rational. Then, there are natural numbers p, q with $\sqrt{2} = p/q$. We cancel common factors and arrive at $\sqrt{2} = r/s$, where r and s do not possess common factors. Squaring and rearranging, we get $2s^2 = r^2$. Therefore, 2 is a factor of r^2 and hence of r, let $r = 2t$. Furthermore, $2s^2 = 4t^2$, so $s^2 = 2t^2$, and 2 is a factor of s, too. But this means that r and s have the common factor 2, a contradiction. (Note that in this example, B corresponds to the statement "$\sqrt{2}$ is irrational," while the role of A is played by the rules of computation for rational numbers.)

Mathematical theorems often involve a statement about variables which may take infinitely many different values. For example, the statement "f is an even function" means that $f(-x) = f(x)$ holds for all (usually infinitely many) $x \in D(f)$. Another statement of this type is the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \tag{1.10}$$

which is true for all natural numbers n. A way to prove such a statement is offered by the principle of **mathematical induction**. In abstract form, it says

> Let S be a set of positive integers. If (i) $1 \in S$ and (ii) $n \in S$ implies that $n + 1 \in S$, then all positive integers are in S.

In other words, if a statement involving n is true for $n = 1$, and furthermore it is true for $n + 1$ whenever it is true for n, then it is true for all values of n.

Let us illustrate the principle of mathematical induction with the formula (1.10). Since

$$1 = \frac{1(1+1)}{2},$$

it is true for $n = 1$. Now assume that it is true for n. We then have

$$\sum_{k=1}^{n+1} k = (n+1) + \sum_{k=1}^{n} k = n + 1 + \frac{n(n+1)}{2} = \frac{(n+2)(n+1)}{2},$$

so it is true for $n + 1$. The principle of mathematical induction now implies that (1.10) is true for all natural numbers n.

1.7 Geometric Transformation of Functions

Translations of Functions
Suppose that $y = f(x)$ is a function and $c > 0$.

Translate Graph Horizontally
When you subtract a positive number c from x, you are translating horizontally the graph of the function c units to the right, that is, the graph of $y = f(x - c)$ is obtained by translating the graph of $y = f(x)$, c units to the right. The graph of $y = f(x + c)$ is obtained by translating the graph of $y = f(x)$, c units to the left.

Translate Graph Vertically
The graph of $y = f(x) + c$ is obtained by translating the graph of $y = f(x)$, c units upward, that is, when you add a positive number c to a function you are translating vertically the graph of the function c units upward.

The graph of $y = f(x) - c$ is obtained by translating the graph of $y = f(x)$, c units downward (Fig. 1.25).

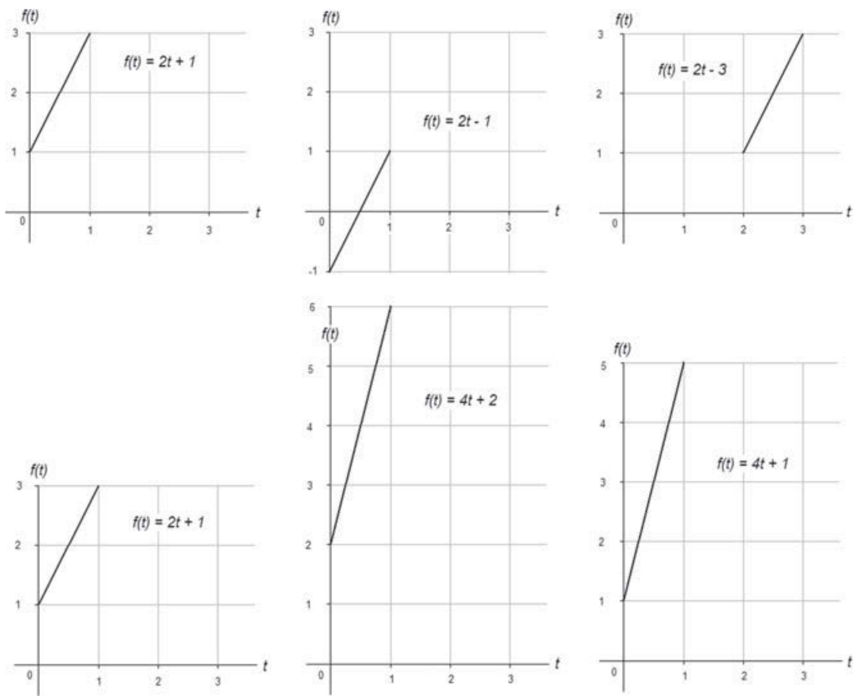


Fig. 1.25 Vertical translation and stretching of functions

Compression and Stretching of Functions
Suppose that $y = f(x)$ is a function and $c > 1$ (Fig. 1.25, 5th graph).

Horizontal Stretching or Compression
The graph of $y = f(cx)$ is obtained by compressing horizontally the graph of $y = f(x)$ by a factor of c units. The graph of $y = f(\frac{x}{c})$ is obtained by stretching horizontally the graph of $y = f(x)$ by a factor of c units.

Vertical Stretching or Compression
The graph of $y = cf(x)$ is obtained by stretching vertically the graph of $y = f(x)$ by a factor of c units. The graph of $y = (\frac{1}{c})f(x)$ is obtained by compressing vertically the graph of $y = f(x)$ by a factor of c units.

Reflection
Suppose that $y = f(x)$ is a function, then the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across the y-axis. The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ across the x-axis.

1.8 Exercises

1.8.1 Let $f(x) = \frac{1}{x}$. Find $f(\frac{3}{5})$, $f(-\frac{2}{7})$.
1.8.2 Let $f(x) = |x| - x$. Find $f(2)$, $f(-2)$, $f(50)$, $f(-40)$.
1.8.3 Find out for which numbers x the function $f(x) = \frac{1}{x^2-3}$ is defined. What is the value of this function for $x = 5$?
1.8.4 Find the domain and range of the function

(a) $f(x) = \sqrt{x} + 5$
(b) $f(x) = \sqrt{x - 3}$
(c) $f(x) = \frac{1}{\sqrt{9-x^2}}$
(d) $f(x) = |x - 2|$
(e) $f(x) = x^2 - x - 6$.

1.8.5 Indian postal service regulation requires that the length plus the girth (the perimeter of a cross section) of a package for mailing cannot exceed 208 mm. A rectangular box with square end is designed to meet the regulation exactly (see Fig. 1.26). Write down the volume V of the box as a function of the edge length of the square end and give the domain of this function.
1.8.6 A soft drink manufacturer wants to fabricate cylindrical cans for its product (see Fig. 1.27). The can is to have a volume of 12 fluid ounces, which is approximately 22 cubic mm. Express the total surface area S of the can as a function of its radius and give domain of this function.
1.8.7 Find the values of a^x for $a = 2$ and $x = 4$; $a = 5$ and $x = -1$; $a = 2$ and $x = \sqrt{2}$; $a = e$ and $x = \sqrt{2}$; $a = e$ and $x = \sqrt{\pi}$.

1.8.8 Examine whether the following functions are odd, even, or neither.
(a) $f(x) = x^{-5}$, (b) $f(x) = x^4 + 3x^2 - 1$, (c) $f(x) = \frac{x}{x^2-1}$, (d) $f(t) = |t^3|$, (e) $h(t) = \sqrt{t^4 + 3}$.

1.8.9 Let $f(x) = x^2 + x + 1$. Find $f(x - 4)$, $f(x + 4)$, $f(\frac{1}{2}x)$, $f(2x - 4)$.

1.8.10 Let $f(x) = x + 6$ and $g(x) = x^2 - 4$, find
(a) $f(g(x))$, (b) $f(f(2))$, (c) $g(g(3))$, (d) $f(f(x))$, (e) $g(f(x))$.

1.8.11 Let $f(x) = x + 1$ and $g(x) = \frac{1}{x+1}$, find
(a) $g(f(\frac{1}{2}))$, (b) $f(f(x))$, (c) $g(g(x))$, (d) $f(g(\frac{1}{3}))$.

1.8.12 (a) Let $(f \circ g)(x) = x$, where $g(x) = \frac{1}{x}$, find $f(x)$. ($\circ$ denotes composition.)
(b) Let $f(x) = \frac{x}{x-1}$, $g(x) = \frac{x}{x-1}$, find $(f \circ g)(x)$.
(c) Let $(f \circ g)(x) = x$, where $f(x) = 1 + \frac{1}{x}$, find $g(x)$.

1.8.13 Draw the graph of the following function:

$$f(x) = \frac{|x|}{x}, \quad x \neq 0, \quad f(0) = 0.$$

1.8.14 Draw the graph of the function $f(x) = ae^{b(x-c)} + d$ by putting different values of a, b, c, and d.

1.8.15 (a) Introduce the concept of random variable with the help of physical examples.
(b) Give examples of a function of random variable (stochastic function).

1.8.16 The average rate of change of a function $y = f(t)$ between time $t = t_0$ and $t = t_1$ is defined as $\frac{\Delta y}{\Delta t} = \frac{f(t_1)-f(t_0)}{t_1-t_0}$. As we know a linear function has the form

$$y = f(t) = mt + b,$$

where $m = \frac{f(t_1)-f(t_0)}{t_1-t_0}$ is the slope or rate of change of y with respect to t, and b is the vertical intercept or value of the function when $t = 0$. Suppose the solid waste generated each year in the cities of a country is a linear function of time, namely, $y = mt + b$. The amount of the solid waste was 82.3 million tons in 1960 and 139.1 in 1980. Find the linear function modeling this situation and use this model to predict the amount of the solid waste in 2020.

1.8.17 (a) The data in the following table lie on a line. Find a function y in terms of x (Table 1.1).
(b) Which of the following tables of values could represent linear equations (Tables 1.2, 1.3, 1.4 and 1.5).

1.8.18 A company wants to understand the relationship between the amount a spent on advertising and total sale S. The data collected is given in the following table: If an amount of 3500 Euro is spent on advertising, predict the sale.

1.8.19 The relationship between Fahrenheit temperature and Celsius temperature is linear, that is, $F = mC + b$. Find the Celsius equivalent of 90 °F and the Fahrenheit equivalent of −5 °C.

1.8.20 An open box is to be made from a rectangular piece of cardboard of dimensions 0 cm × 30 cm by cutting out identical squares of area x^2 from each corner turning up the sides. Express the volume V of the box as a function of x.

1.8.21 An aquarium, open on top, of height 1.5 m is to have a volume of 6 m^3. Let x denote the length of the base and let y denote the width.

(a) Express y as a function of x.
(b) Express the total number S of square meters of glass needed as a function of x.

1.8.22 The shape of a spacecraft be a frustum of a right circular cone, a solid formed by truncating a cone by a plane parallel to its base, see Fig. 1.28.
The radii a and b of the lower and upper part, respectively, are given.

(a) Express y as a function of the height h.
(b) Express the volume of the frustum as a function of h.
(c) Given $a = 6$ m, $b = 3$ m and $V = 600$ m^3, find h.

1.8.23 Explain the concept of periodic functions with the help of physical phenomena.

1.8.24 For any periodic function, the amplitude is defined as the half of the difference between its maximum and minimum values. The period is the time for the function to execute one complete cycle. Sketch the graph of $y = 3 \sin 2t$ and use the graph to determine the amplitude and period.

1.8.25 The constants A and B in the equation $y = A \sin Bt$ are called parameters. The amplitude is determined by the parameter A, while the period is determined by the parameter B. Functions $y = A \sin Bt + C$ and $y = A \cos Bt + C$ are periodic with period $\frac{2\pi}{|B|}$ and amplitude $|A|$, where C is the vertical shift.
On February 15, 1995(assume), high tide in Indian ocean was at midnight. The height of the water in the Bombay harbor is a periodic function, since it oscillates between high and low tide. The height is approximated by the function

$$y = 4 - 9 \cos\left(\frac{\pi}{6} t\right) + 5$$

where t is time in hours since midnight on February 15, 1995.

(a) Sketch a graph of this function on February 15, 1995 (from $t = 0$ to $t = 24$).
(b) What was the water level at high tide?
(c) When was low tide, and what was the water level at that time?
(d) What is the period of this function, and what does it represent in terms of tides?
(e) What is the amplitude of this function and what does it represent in terms of tides?

Fig. 1.26 A rectangular box with square end

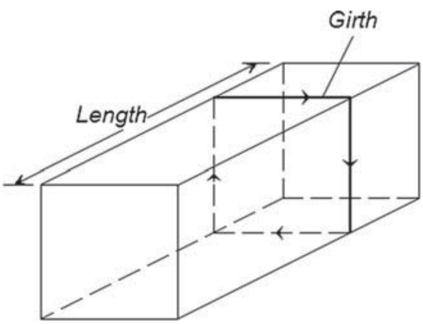


Fig. 1.27 A cylindrical can

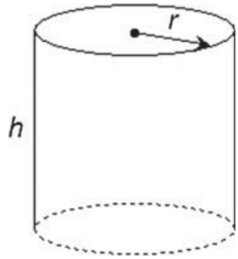


Table 1.1 Tabular representation of a function

x	10	20	30	40
y	200	180	160	140

Table 1.2 Does this table represent a linear function?

t	40	60	80	100
$f(t)$	2.4	2.2	2	1.8

Table 1.3 Does this table represent a linear function?

x	0	2	4	6
$g(x)$	10	16	26	40

Table 1.4 Does this table represent a linear function?

t	5	10	15	20
$h(t)$	100	90	80	70

Table 1.5 Does this table represent a linear function?

a (advertisement)	6	8	10	12
S (sales)	200	240	280	320

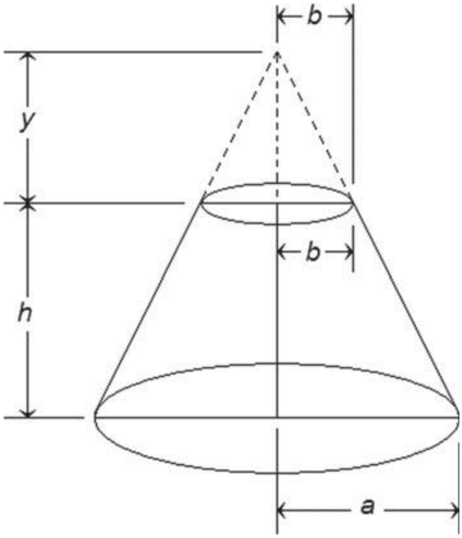


Fig. 1.28 A frustum of a circular cone

Chapter 2
Limit and Continuity

In this chapter, we introduce the concept of the limit of a function. This concept bridges the gap between the areas of algebra, geometry, and calculus. The limit is the most important concept of calculus, without limits calculus simply does not exist. Practically, every notion of calculus is a limit in one sense or another. Physical concepts and real-world situations can be expressed as limits of certain functions. For example, a circle is the limit of a polygon, the length of a curve is the limit of the lengths of polygonal paths, the area of a region bounded by a curve is the limit of the sum of areas of approximating rectangles, and instantaneous velocity (velocity at a particular time) is the limit of average velocities. We also discuss the concept of continuous and discontinuous functions in terms of limit.

2.1 Idea and Definition of the Limit

Originally, calculus was developed in order to solve problems of the following type.

Problem 2.1 A car moves on a straight line in such a way that its position s at time t is given by the equation $s = f(t)$, where f is a given function. Find the velocity of the car at any time t.

Problem 2.2 Let f be a given function, find the area bounded by the graph $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

Problem 2.1 is solved by the process of differentiation (Chap. 3), and Problem 2.2 by the process of integration (Chap. 6).

Both processes are based on the concept of a limit. Its mathematical definition gives a refined and formally precise meaning to an intuitive notion that occurs frequently in our everyday lives. Before giving this definition, let us first illustrate the concept with two examples. The first example involves the notion of instantaneous velocity.

M. Brokate et al., *Calculus for Scientists and Engineers*, Industrial and Applied Mathematics, https://doi.org/10.1007/978-981-13-8464-6_2

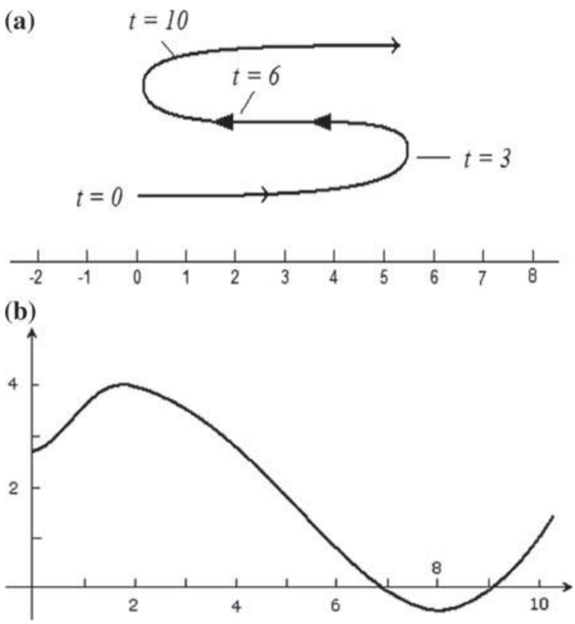


Fig. 2.1 **a** A particle moving along a straight line. **b** The position of the particle as a function of time

Example 2.1 (*Instantaneous velocity as a limit*) Consider an object (car, rocket, proton) moving along a straight line. Suppose we know the position s (relative to a fixed origin) of the object at each moment of time t, that is, $s = f(t)$ is given. Our experience with moving objects provides us with an intuitive idea of velocity, but what precisely does it mean to speak of the velocity of an object at time t (instantaneous velocity at t), and how can we calculate this quantity? Let the motion of the particle be given in Fig. 2.1a with the corresponding graph of f shown in Fig. 2.1b. Let us fix a time t_0, the object being at position $s_0 = f(t_0)$. For any other time t, the **average velocity** during the time interval between t_0 and t is given by

$$v_{av}(t_0, t) = \frac{s - s_0}{t - t_0} = \frac{f(t) - f(t_0)}{t - t_0}, \tag{2.1}$$

which is the ratio of the total distance traveled and the total time elapsed. For the situation shown in Fig. 2.1a, b, the average velocity may be either positive, negative, or zero, depending on the choice of t_0 and t.

Usually, more important than the average velocity is the **instantaneous velocity**, displayed, for example, on the speedometer in a moving car. Intuitively, the instantaneous velocity $v(t_0)$ should be obtained from the average velocity when t is close to t_0. We expect that the closer we choose t near t_0, the closer the average velocity $v_{av}(t_0, t)$ will be to the instantaneous velocity $v(t_0)$, or in other words, "as t approaches t_0, $v_{av}(t_0, t)$ will approach $v(t_0)$". With the definition of and the calculus rules for the limit, to be described below, mathematics provides a precise notion and ways to compute the instantaneous velocity.

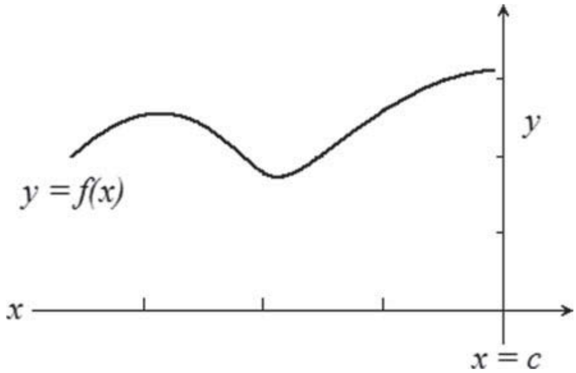


Fig. 2.2 Path of the object (projectile)

While obviously important, the notion of instantaneous velocity is already a somewhat complicated example of a limit, namely the derivative of a function. This will be discussed in the next chapter. Instead, we now present a second (simpler) example which will lead to the general definition of the limit in a straightforward manner.

Example 2.2 Suppose an object is moving through the air toward a solid wall, and we want to find out at which height L it hits the wall. See Fig. 2.2. Let the wall be represented by the vertical line $x = c$ (we look at it from the side), and suppose the path of the object in the air is described by a function $y = f(x)$, where $x < c$ (the object comes from the left). As x comes closer and closer to c, we expect the height $f(x)$ of the object to approach a number L, the height at which the object hits the wall. This expectation is natural if we imagine the object to move "continuously" through the air, without being "teleported" suddenly from one point to another or undergoing erratic fluctuations. We use the notation

$$\lim_{x \to c^-} f(x) = L \, . \tag{2.2}$$

This is read as "the limit of $f(x)$, as x approaches c from the left, equals L" or "the left-sided limit of $f(x)$ at $x = c$ equals L", and in this example, it expresses the fact that the object hits the wall at the height L. See Fig. 2.3. The minus sign within the expression "$x \to c^-$" indicates that we consider values of x which are close to c, but always strictly less than c.

In the same manner, we may think of an object approaching the wall $x = c$ from the right along a path $y = f(x)$, where now x is taken to be larger than c, see Fig. 2.4. If the number M equals the height at which the object hits the wall, we write

$$\lim_{x \to c^+} f(x) = M \tag{2.3}$$

to denote the right-sided limit of $f(x)$ at $x = c$. The only difference to (2.2), besides using a different letter for the height, is the expression "$x \to c^+$", where the plus

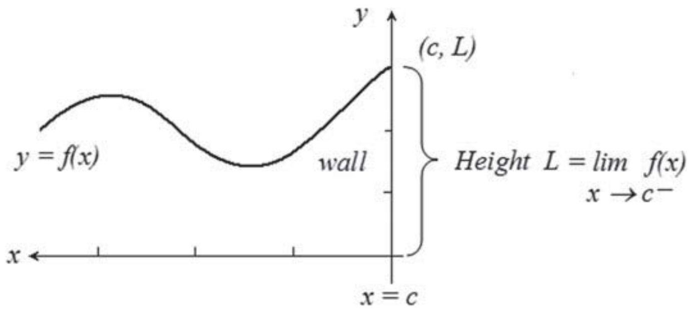


Fig. 2.3 Left-sided limit

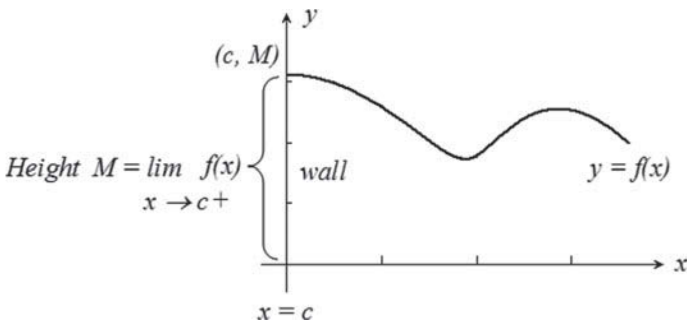


Fig. 2.4 Right-sided limit

sign indicates that we consider values of x close to c, but always strictly larger than c.

We now take a close look at the above description of the limit process: "As x approaches c from the left, $f(x)$ approaches L." We try to rephrase this as follows:

> We can enforce $f(x)$ to deviate from L by an amount less than a given $\varepsilon > 0$, if we restrict x to be taken from the open interval $(c - \delta, c)$ with a sufficiently small $\delta > 0$.

Since we want $f(x)$ to come "arbitrarily close" to L, we have to require that the quoted property holds no matter how small ε is prescribed.

Definition 2.1 (*Left-sided Limit*) We say that

$$\lim_{x \to c^-} f(x) = L\,, \tag{2.4}$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that: For all x with $c - \delta < x < c$, we have $|f(x) - L| < \varepsilon$. On the other hand, if there is no number L for which this is true, we say that $\lim_{x \to c^-} f(x)$ does not exist.

This definition looks very clumsy and complicated. It usually takes some time to comprehend, and no harm is done if one reads on, coming back to it at some later

time. On the other hand, it provides a firm foundation for the treatment of limits no matter how they occur. Indeed, it emerged more than 100 years ago as the final result of a long scientific struggle, when it was realized how easily one can arrive at wrong conclusions if one does not have a precise definition of the limit. This is even more important today, when mathematics lies at the heart of the description of complicated processes, and more and more trust is placed in machines and computers to work correctly.

The definition of the **right-sided limit**

$$\lim_{x \to c^+} f(x) = M \tag{2.5}$$

is exactly the same as Definition 2.1, except that the phrase "$c - \delta < x < c$" is replaced by "$c < x < c + \delta$", and L is replaced by M. Instead of "left-sided limit" and "right-sided limit", one may also use the notions "left hand limit" and "right hand limit". The notion **one-sided limit** refers to either the left-sided or the right-sided limit.

If both the left-sided and the right-sided limits exist and are moreover equal, that is, if

$$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)\,, \tag{2.6}$$

then we say that f has a **two-sided limit** or simply a **limit** at $x = c$ and write

$$\lim_{x \to c} f(x) = L\,. \tag{2.7}$$

Equivalently, we may write the definition of the limit in a way similar to Definition 2.1.

Definition 2.2 (*Limit*) We say that

$$\lim_{x \to c} f(x) = L\,, \tag{2.8}$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that: For all x with $c - \delta < x < c + \delta$ and $x \neq c$, we have $|f(x) - L| < \varepsilon$. On the other hand, if there is no number L for which this is true, we say that $\lim_{x \to c} f(x)$ does not exist.

Example 2.3 1. Let us consider the Heaviside function given by

$$f(x) = \begin{cases} 1\,, & x \geq 0\,, \\ 0\,, & x < 0\,, \end{cases} \tag{2.9}$$

Here, $f(x)$ approaches 0 (in fact, is equal to 0) as x approaches 0 from the left, that is

$$\lim_{x \to 0^-} f(x) = 0\,,$$

and $f(x)$ is equal to 1 as x approaches 0 from the right, that is

$$\lim_{x\to 0^+} f(x) = 1 .$$

Note that $\lim_{x\to 0} f(x)$ does not exist, since the left-sided limit and the right-sided limit are different from each other.

2. The function $f(x) = [x]$, denoting the integer part of x, possesses both one-sided derivatives for arbitrary real numbers x. They are equal if x is not an integer number, but they are not equal whenever x is an integer number, for example,

$$\lim_{x\to 1^+} [x] = 1 , \quad \lim_{x\to 1^-} [x] = 0 .$$

Remark 2.1 1. A one-sided or two-sided limit may or may not exist, but if it exists, it is uniquely determined. In other words, there can be at most one number L (or M) which has the property required in the definitions above.

2. For the question whether f has a one-sided or two-sided limit at a point c and what its value is, it does not matter whether f is defined at the point c itself or what the value $f(c)$ is. For the Heaviside function in (2.9), we have $f(0) = 1$ which happens to be equal to the right-sided, but unequal to the left-sided limit.
3. One also considers limits where either $x \to \infty$ or $f(x) \to \infty$ or both. These are called **improper limits** and will be treated in Sect. 2.4.

Remark 2.2 The following four statements are equivalent:

1. $\lim_{x\to c} f(x) = L$
2. $\lim_{h\to 0} f(c+h) = L$
3. $\lim_{x\to c}(f(x) - L) = 0$
4. $\lim_{x\to c} |f(x) - L| = 0$

2.2 Evaluating Limits

We summarize important properties of limits in the form of the following rules. With their aid, the knowledge of some elementary limits can be used to compute the limits of more and more complicated functions. Their proofs are based upon the exact definition of the limit as presented in the previous section. Interested readers may find them through solving selected exercises, given at the end of this chapter.

The simplest limit is that of a constant function, say $f(x) = A$ for all x. We have

$$\lim_{x\to c} f(x) = \lim_{x\to c} A = A ,$$

no matter at which point c we take the limit. Thus, the limit of a constant function with value A is equal to A itself. As an example, let $f(x) = 100$, then $\lim_{x\to 3} f(x) = \lim_{x\to 3} 100 = 100$.

Next, we consider the function $f(x) = x$. For all numbers c, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c \, .$$

Now, we look at elementary algebraic operations.

Theorem 2.1 *Let f and g be functions such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ at some given point c. Then*

$$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M \, , \tag{2.10}$$

$$\lim_{x \to c} \alpha f(x) = \alpha \lim_{x \to c} f(x) = \alpha L \, , \quad \textit{for any constant } \alpha, \tag{2.11}$$

$$\lim_{x \to c} [f(x) g(x)] = \left[\lim_{x \to c} f(x)\right] \cdot \left[\lim_{x \to c} g(x)\right] = LM \, , \tag{2.12}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \, , \quad \textit{provided } M \neq 0. \tag{2.13}$$

Rules (2.10)–(2.13) mean that we can interchange the limit with the elementary algebraic operations. This can be done repeatedly, as in the following examples.

Example 2.4 1. Let $f(x) = 3x + 5$. Then

$$\begin{aligned} \lim_{x \to 2} f(x) &= \lim_{x \to 2} (3x + 5) = \left(\lim_{x \to 2} 3x\right) + \lim_{x \to 2} 5 = 3 \cdot \left(\lim_{x \to 2} x\right) + 5 \\ &= 3 \cdot 2 + 5 = 11 \, . \end{aligned}$$

2. Let $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial. Then $\lim_{x\to c} P(x) = P(c)$, since

$$\begin{aligned} \lim_{x \to c} P(x) &= \lim_{x \to c} \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) \\ &= a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n \\ &= P(c) \, . \end{aligned}$$

3. Formula (2.13) for the quotient does not work when the denominator M is zero. But sometimes this can be avoided through cancelation. For example, let

$$f(x) = \frac{x^2 - 16}{x - 4} \, ,$$

and let $c = 3$. Then

$$\lim_{x \to 3} \frac{x^2 - 16}{x - 4} = \lim_{x \to 3} \frac{(x - 4)(x + 4)}{x - 4} = \lim_{x \to 3} (x + 4) = 7 \, .$$

4. Consider a rational function

$$R(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P \text{ and } Q \text{ are polynomials.}$$

Then

$$\lim_{x\to c} R(x) = \frac{P(c)}{Q(c)}, \quad \text{provided } Q(c) \neq 0.$$

Roots can also be interchanged with limits.

Theorem 2.2 *For every natural number n,*

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c} \tag{2.14}$$

holds for all real numbers c if n is odd, and for all real numbers $c \geq 0$ if n is even. (In the latter case, if $c = 0$, we have to replace the limit by the right-sided limit.)

Theorem 2.3 *If* $\lim_{x\to c} f(x) = L$, *then* $\lim_{x\to c} |f(x)| = |L|$. *For $L = 0$, the converse holds: If* $\lim_{x\to c} |f(x)| = 0$, *then* $\lim_{x\to c} f(x) = 0$.

Example 2.5 1. For the absolute value, we have $\lim_{x\to 0} |x| = 0$.

2. For the sign function, we have

$$\operatorname{sgn}(x) = \frac{x}{|x|}, \quad x \neq 0,$$

so

$$\lim_{x\to 0^-} \operatorname{sgn}(x) = -1, \quad \lim_{x\to 0^+} \operatorname{sgn}(x) = 1,$$

and therefore $\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist. Since on the other hand $|\operatorname{sgn}(x)| = 1$ for $x \neq 0$, this shows that in Theorem 2.3, the converse statement may not hold if $L \neq 0$.

Theorem 2.4 (Composition) *Let functions f and g be given with suitable domains such that the composition $g \circ f$ is defined. If*

$$\lim_{x\to c} f(x) = L, \quad \lim_{y\to L} g(y) = M,$$

then

$$\lim_{x\to c} g(f(x)) = M.$$

Theorem 2.5 (Sandwich Theorem) *Let the functions f, g, h have the property that $g(x) \leq f(x) \leq h(x)$ in some interval I which contains the point c. If*

$$\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x),$$

then also

$$\lim_{x\to c} f(x) = L\,.$$

The rules above (Theorems 2.1–2.5) are formulated for the limit (that is, the two-sided limit), but they are equally valid for the one-sided limits, one just has to replace the phrase "$x \to c$" by "$x \to c^-$" or "$x \to c^+$", respectively.

Example 2.6 1. We want to determine $\lim_{x\to 0} \sin x$. We know that $0 \le |\sin x| \le |x|$. Since $\lim_{x\to 0} |x| = 0$, see Example 2.5, we conclude from the Sandwich Theorem that $\lim_{x\to 0} |\sin x| = 0$ and hence, $\lim_{x\to 0} \sin x = 0$ by Theorem 2.3.

2. We want to determine $\lim_{x\to 0} \cos x$. We know that $\sin^2 x + \cos^2 x = 1$. For x near to 0, the cosine is positive, and therefore we have $\cos x = \sqrt{1 - \sin^2 x}$. As x tends to 0, $\sin x$ tends to 0, thus $\sin^2 x$ tends to 0. From Theorems 2.2 and 2.4, we see that $\cos x$ tends to 1. So $\lim_{x\to 0} \cos x = 1$.
3. We want to determine $\lim_{x\to 0} \frac{\sin x}{x}$. We cannot use the rule (2.13) for the quotient, because both denominator and numerator tend to 0 as x tends to 0. On the other hand, from Theorem C.1 in Appendix C.2 we have, for x close to 0,

$$\cos x < \frac{\sin x}{x} < 1\,.$$

Since $\lim_{x\to 0} \cos x = 1$ as shown just above, we obtain from the Sandwich Theorem that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,.$$

Example 2.7 This example is more complicated. We investigate whether

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) \tag{2.15}$$

exists. The function $f(x) = \sin\left(\frac{\pi}{x}\right)$ is defined for all $x \ne 0$; that it is not defined for $x = 0$ does not matter for the investigation of the limit. The values of f at successive points $x = 1$, $x = \frac{1}{2}$, $x = \frac{1}{3}$, …, coming closer and closer to 0, are

$$\sin\frac{\pi}{1} = \sin\pi = 0\,, \quad \sin\frac{\pi}{1/2} = \sin 2\pi = 0\,, \quad \sin\frac{\pi}{1/3} = \sin 3\pi = 0\,,$$

and so on. One might be misled to conclude that the limit (2.15) exists and is equal to 0. But if we compute the values of f at successive point $x = 2$, $x = \frac{2}{5}$, $x = \frac{2}{9}$, $x = \frac{2}{13}$, …, we see that those points, too, come closer and closer to 0. However,

$$\sin\left(\frac{\pi}{2}\right) = 1\,, \quad \sin\left(\frac{\pi}{2/5}\right) = \sin\left(\frac{5\pi}{2}\right) = 1\,, \quad \sin\left(\frac{\pi}{2/9}\right) = \sin\left(\frac{9\pi}{2}\right) = 1\,,$$

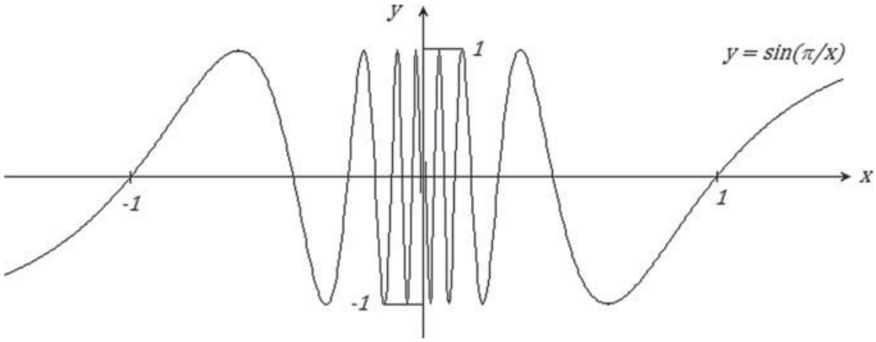


Fig. 2.5 Graph of $y = \sin(\pi/x)$

and so on. The graph of f is given in Fig. 2.5. It shows that the values of f oscillate between 1 and -1 faster and faster as x approaches 0, indeed an infinite number of times on the open interval (0, 1). The values of f do not approach a fixed number L, and the limit does not exist.

2.3 Continuous Functions

In daily life, when we speak of a continuous process, we mean that it goes somewhat smoothly, without any disruption or abrupt changes. In mathematics, the word "continuous" has much the same meaning. Intuitively, we imagine a function f to be continuous, if we can draw its graph in the x-y-plane in a single continuous movement. Thus, when x is close to a point c in the domain of f, the function values $f(x)$ should be close to $f(c)$. In terms of limits this means that f has a limit at c, which is equal to the value $f(c)$ of the function at this point.

A formal statement of the concept of continuity is the following.

Definition 2.3 (*Continuous Function, Open Interval*) Let f be a function defined on an open interval $I = (a, b)$. We say that f is **continuous** at a point $c \in I$, if

$$\lim_{x \to c} f(x) = f(c)\,. \tag{2.16}$$

We say that f is continuous on $I = (a, b)$, if it is continuous at every point $c \in I$.

If moreover f is defined at the left endpoint a, we say that f is **right-continuous** at a, if

$$\lim_{x \to a^+} f(x) = f(a)\,. \tag{2.17}$$

Correspondingly, if f is defined at the right endpoint b, we say that f is **left-continuous** at b, if

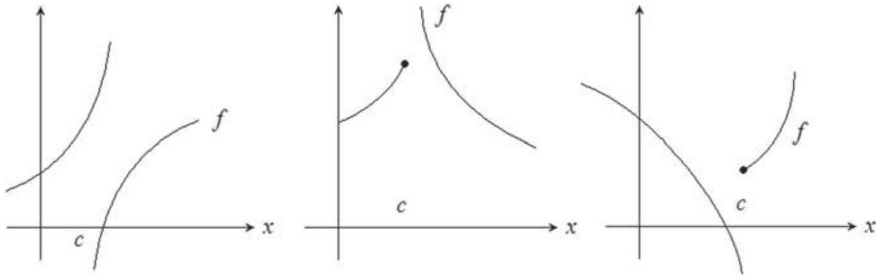


Fig. 2.6 Finite discontinuities

$$\lim_{x \to b^-} f(x) = f(b)\,. \tag{2.18}$$

Definition 2.4 (*Continuous Function, Closed Interval*) Let f be a function defined on a closed interval $[a, b]$. We say that f is **continuous** on $[a, b]$, if it is continuous on (a, b), and moreover right-continuous at a and left-continuous at b.

If f is not continuous at a point c, we say that f is **discontinuous** at c, or that c is a **point of discontinuity** of f. We distinguish the following situations.

If f has a limit as x approaches c (that is, $\lim_{x\to c} f(x)$ exists), but this limit is not equal to $f(c)$, then f is said to have a **removable discontinuity** at c. In fact, in this case, the discontinuity at c can be removed by redefining $f(c)$ as $f(x) = \lim_{x\to c} f(x)$. For example, the function

$$f(x) = \begin{cases} 1\,, & x \neq 0\,, \\ 2\,, & x = 0\,, \end{cases}$$

is discontinuous at $c = 0$, but redefining it as $f(0) = 1$ makes it continuous.

If $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist but are not equal, then f is said to have **jump discontinuity** at c. An example is given by the Heaviside function

$$f(x) = \begin{cases} 1\,, & x \geq 0\,, \\ 0\,, & x < 0\,, \end{cases}$$

which has a jump discontinuity at $c = 0$ (For other examples, see Fig. 2.6).

However, the situation can be even worse, since it may well happen that one or both of the one-sided (left or right) limits do not exist. For example, consider the Dirichlet function,

$$f(x) = \begin{cases} 1\,, & x \text{ is a rational number} \\ 0\,, & x \text{ is an irrational number} \end{cases}\,.$$

This function is discontinuous everywhere. Neither the left-sided nor the right-sided limit of f exists at any point, since for any number c there are both rational and irrational numbers arbitrarily close to c, both from the right as from the left.

Properties of continuous functions. By the definitions above, continuity is formulated in terms of limits. Therefore, the properties of limits as discussed in the preceding section give rise to corresponding properties of continuous functions.

Theorem 2.6 *Let f and g be continuous functions defined on an interval I, where $I = (a, b)$ or $I = [a, b]$. Then*

1. *$f + g$ is continuous on I,*
2. *$f - g$ is continuous on I,*
3. *αf is continuous on I for each number α,*
4. *$f \cdot g$ is continuous on I,*
5. *f/g is continuous at all points $c \in I$ where $g(c) \neq 0$.*

Let f and g as above, except that the domain of g is now contained in $f(I)$. Then

6. *The composition $g \circ f$ is continuous on I.*

Many of the functions which have been discussed in Chap. 1 are continuous on their domain of definition. Among them are polynomials, rational functions, trigonometric functions, exponential functions, and logarithmic functions. We will not give explicit proofs here. Their continuity will arise as a byproduct of other results in later chapters.

Theorem 2.7 (Intermediate Value Theorem) *Let f be a function which is continuous on a $[a, b]$, and let m be any number between $f(a)$ and $f(b)$ (either $f(a) < m < f(b)$ or $f(b) > m > f(a)$). Then there is at least one number c in the interval (a, b) such that $f(c) = m$.*

In Chap. 4, we will discuss maxima and minima extensively, mainly based on the concept and the properties of derivatives from Chap. 3. But let us mention here an important property of continuous functions relevant for optimization.

Theorem 2.8 (Extreme value theorem) *Let f be a function which is continuous on a closed and bounded interval $[a, b]$. Then, f attains a maximal value M and a minimal value m in this interval (Fig. 2.7).*

Note that if the domain of f is unbounded or not closed, then f may be unbounded, as for example $f(x) = \tan x$ on the domain $(-\pi/2, \pi/2)$.

Theorem 2.9 (Continuity of inverse functions) *Let f be a function which is increasing on (a, b). If moreover f is continuous on (a, b), then its inverse f^{-1} is also continuous on the respective domain.*

The assertion of Theorem 2.9 also holds if we replace "increasing" by "decreasing", or the open interval (a, b) by the closed interval $[a, b]$.

Let us finish this section by mentioning the so-called ε-δ-definition of continuity. It is equivalent to the one given above.

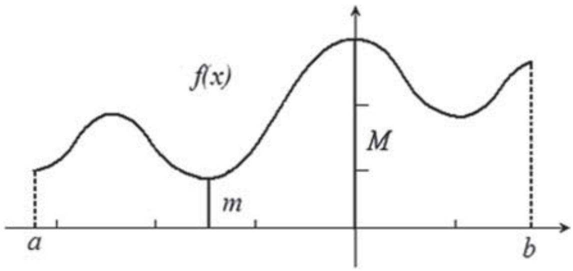


Fig. 2.7 Geometric illustration of Theorem 2.8

Definition 2.5 A function f is said to be continuous at a point c of its domain, if for each $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.

This is just another way of stating precisely that when x is close to c, then $f(x)$ is close to $f(c)$.

2.4 Improper Limits

There are two types of improper limits. For the first type, let us consider the function

$$f(x) = \frac{1}{x}.$$

When x tends to zero from the right, the values $f(x)$ increase without bound, and we say that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

In general, we say that

$$\lim_{x \to c^+} f(x) = \infty, \tag{2.19}$$

if for every $M > 0$, there is a $\delta > 0$ such that, for all x with $c < x < c + \delta$, we have $f(x) > M$. (In order that this makes sense, we have to assume that the interval $(c, c + \delta)$ is contained in the domain of f.) In an analogous manner one can consider the cases when $f(x)$ tends to $-\infty$, or when x comes from the left, to obtain the improper limits

$$\lim_{x \to c^+} f(x) = -\infty, \quad \lim_{x \to c^-} f(x) = \infty, \quad \lim_{x \to c^-} f(x) = -\infty. \tag{2.20}$$

For example,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

This type of improper limit typically arises at poles of rational functions, but also for other elementary functions. For example,

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \tan x = \infty\,, \quad \lim_{x\to 0^+} \ln x = -\infty\,.$$

The second type of improper limit occurs when we consider x to tend to ∞ or $-\infty$. For example, the values of $f(x) = 1/x$ tend to zero when x increases without bound. In general, we say that

$$\lim_{x\to\infty} f(x) = L\,, \tag{2.21}$$

if for every $\varepsilon > 0$ there is an $M > 0$ such that, for all x with $x > M$, we have $|f(x) - L| < \varepsilon$. (We have to assume that the interval (M, ∞) is contained in the domain of f.) Again, we may also consider the case when x tends to $-\infty$. For example,

$$\lim x \to \infty \arctan x = \frac{\pi}{2}\,, \quad \lim x \to -\infty \arctan x = -\frac{\pi}{2}\,.$$

As with standard limits, one must be careful to check whether such an improper limit really exists or not. For example, the function $f(x) = \sin x$ is bounded on the whole real line, but it has no limit when x tends to infinity, in fact, it "forever" oscillates between -1 and 1.

The two types of improper limits considered above may also be combined. For example,

$$\lim_{x\to\infty} 3x = \infty\,.$$

2.5 Exercises

2.5.1 Let

$$f(x) = \begin{cases} 1 - x^2\,, & x < 0\,, \\ \frac{1}{3}\,, & x = 0\,, \\ 1 - x\,, & x > 0\,. \end{cases}$$

Examine whether $\lim_{x\to 0} f(x)$ exists or not.

2.5.2 Find the following limits:

a. $\displaystyle\lim_{x\to 4} \frac{x^2 - x - 12}{x^2 - 4x}$

b. $\displaystyle\lim_{x\to -4} \frac{x^3 + 64}{x + 4}$

c. $\displaystyle\lim_{x\to\infty} \frac{3x^2 + 7x - 6}{4x^2 - 3x + 6}$

d. $\lim_{x \to 0} \frac{\sin x}{x}$
e. $\lim_{x \to 0} (x + \cos x)$
f. $\lim_{x \to 0} \left(e^x + \frac{\sin x}{x} \right)$

2.5.3 Let

$$f(x) = \begin{cases} 8\,, & x \text{ is rational,} \\ 3, & x \text{ is irrational.} \end{cases}$$

Show that $\lim_{x \to c} f(x)$ does not exist, no matter how we choose $c \in \mathbb{R}$.

2.5.4 Find $\lim_{x \to 0} \frac{1 - \cos x}{x}$.

2.5.5 Let $f(x) = \cos\left(\frac{\pi}{2}x\right) - x^2$ on $[0, 1]$. Show that there exists $c \in (0, 1)$ such that $f(c) = 0$.

2.5.6 Show that the absolute value function is continuous.

Chapter 3
Derivatives

The concept of the derivative was hidden in the problem of finding the line tangent to a given curve at a given point, a problem tackled by the Greek mathematicians more than two thousand years ago. This concept took definite shape during the years 1665–1666, when the famous English scientist Isaac Newton, the founder of modern physics, developed the process now known as differentiation. At that time, Newton did not publish his work, and the same concept was rediscovered independently by Gottfried Wilhelm Leibniz (1646–1716), a German scholar having expertise in philosophy, law, mathematics, and science. The discovery of the derivative as a fundamental ingredient of calculus completely changed the nature of scientific studies, particularly mathematics. Coupled with Newton's formulation of the laws of motion and gravitation, the calculus of Newton and Leibniz and their subsequent refinements and extensions revolutionized the modern world.

In this chapter, we introduce the concept of the derivative, discuss the method of computing derivatives of elementary functions, outline the results devoted to basic properties of differentiation, present a basic differential equation, and conclude with certain applications.

3.1 Definition of the Derivative

Average and instantaneous velocity. In Example 2.1, we have considered an object moving along a straight line from a position $s_0 = f(t_0)$ at time t_0 to a position $s = f(t)$ at time $t \neq t_0$. We have seen that

$$v_{av}(t_0, t) = \frac{s - s_0}{t - t_0} = \frac{f(t) - f(t_0)}{t - t_0} \tag{3.1}$$

gives the average velocity during a time interval from t_0 to t, and we have indicated that the instantaneous velocity $v(t_0)$ should arise as a limit when t tends to t_0 in (3.1). In Fig. 3.1, we examine this situation from a geometrical point of view. The points (t_0, s_0)

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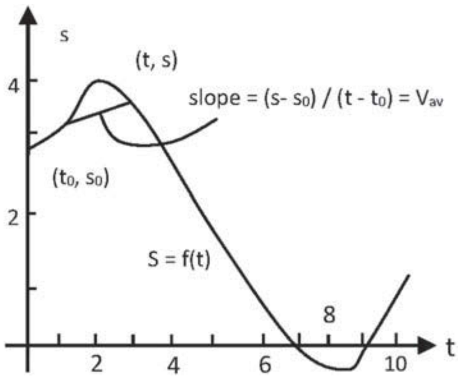


Fig. 3.1 Slope of a secant line

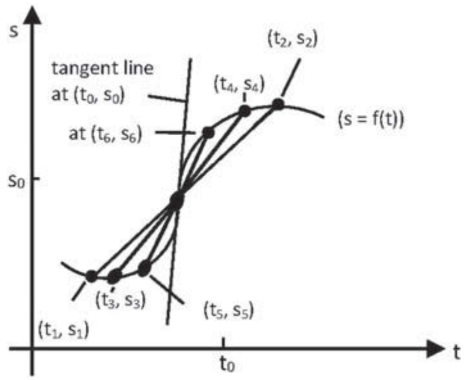


Fig. 3.2 Secant lines approach the tangent line as $t \to t_0$

and (t, s) on the graph of $s = f(t)$ are indicated. The quotient $(s - s_0)/(t - t_0)$ from Eq. (3.1) is precisely the slope of the straight line (the secant line) passing through these two points. In other words, given the position s of the particle as a function of time t, the average velocity during any time interval is the slope of the secant line joining the corresponding pair of points on the graph of $s = f(t)$. Now we interpret geometrically what happens if t tends to t_0 in (3.1). In Fig. 3.2, we have drawn the secant lines with slopes

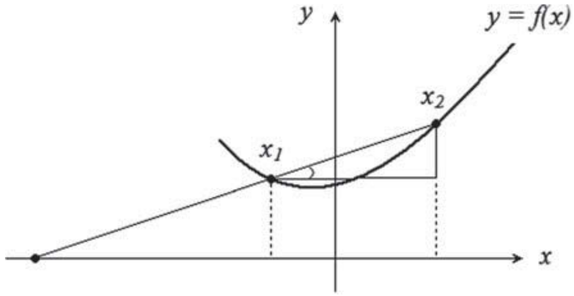
$$\frac{s_2 - s_0}{t_2 - t_0}, \quad \frac{s_4 - s_0}{t_4 - t_0}, \quad \frac{s_6 - s_0}{t_6 - t_0}, \quad \ldots$$

for $t_2, t_4, t_6, \ldots$ greater than but successively closer to t_0, and the secant lines with slopes

$$\frac{s_0 - s_1}{t_0 - t_1}, \quad \frac{s_0 - s_3}{t_0 - t_1}, \quad \frac{s_0 - s_5}{t_0 - t_5}, \quad \ldots$$

for $t_1, t_3, t_5, \ldots$ less than but successively closer to t_0.

Fig. 3.3 Secant line



It seems intuitively reasonable that these lines approach the slope of the line that is tangent to the graph of $s = f(t)$ at (t_0, s_0). In fact, we define the slope of the tangent line to the curve $s = f(t)$ as the limit

$$m = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

General definition of the derivative. Let us consider a function $y = f(x)$ defined on an interval I. Any change in x, say from x_1 to x_2, induces a corresponding change in y, namely, from $f(x_1)$ to $f(x_2)$. We define the **average rate of change** of f from x_1 to x_2 as the ratio of the change in y to the change in x,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{3.2}$$

This expression is also called a **difference quotient**. Sometimes, one writes symbolically

$$\frac{\Delta y}{\Delta x}, \quad \text{where } \Delta x = x_2 - x_1,\ \Delta y = f(x_2) - f(x_1).$$

From (3.2), we see that the average rate of change of a function from x_1 to x_2 equals the slope of the line which joins the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ of its graph. Such a line is called a **secant line**, see Fig. 3.3. The angle θ between the secant and the x-axis is given in terms of the slope as $\tan\theta = (f(x_2) - f(x_1))/(x_2 - x_1)$.

Definition 3.1 (*Derivative of a function*) Let $y = f(x)$ be a function defined on an interval I, and let c be an interior point of I. The derivative of f at c, denoted by $f'(c)$ and read "f prime of c" is defined as the number

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}, \tag{3.3}$$

provided this limit exists.

Remark 3.1 1. It is clear that the derivative is a unique number: For each c, for which the limit exists, the right-hand side of (3.3) specifies a unique number $f'(c)$. As a consequence, formula (3.3) defines a function f', called the derivative of f, whose value at c is $f'(c)$. The domain of f' consists of those points of $D(f)$ for which the limit in (3.3) exists. If $f'(c)$ is defined at a point c, then f is said to be **differentiable** at that point. If f is differentiable at each point of an open interval (a, b), then f is said to be differentiable on that interval.

2. According to (3.3), the derivative arises as a limit from difference quotients. Thus, we may interpret the derivative of f as the limit of its average rate of change when the corresponding interval shrinks to a point.
3. We define the **tangent line** to the graph of a function f at $c \in D(f)$ to be the line which passes through the point $(c, f(c))$ and has the slope

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}\,,$$

provided the limit exists.
4. Let $y = f(x)$ be differentiable. Sometimes y' is used to denote the value of its derivative at x, that is, $y' = f'(x)$. No matter which notation one uses, one has to distinguish between the derivative viewed as a function, denoted here as f', and specific values of the derivative, denoted here as $f'(x)$ or y'. It is most important to understand this distinction.
5. Sometimes, in discussing differentiation, it is helpful to emphasize the independent variable. Thus, if x is the independent variable, we may say "derivative with respect to x" instead of merely "derivative".
6. Above, we have interpreted the derivative as a **velocity** or as the slope of a tangent line. More generally, if $y = f(x)$, then its derivative f' is the **rate of change** of f with respect to the independent variable x, and $f'(x_0)$ is the rate of change of f at the point $x = x_0$.
7. Very often, Eq. (3.3) is written as

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}\,. \tag{3.4}$$

Substitute $h = x - c$ in (3.3), then $h \to 0$ as $x \to c$ and $x = h + c$. We thus obtain (3.4) from (3.3).
8. If f is defined on a closed interval $[a, b]$, we may consider the **right-sided** resp. **left-sided derivatives** at the end points, defined by

$$f'(a+) = \lim_{x\to a^+} \frac{f(x) - f(a)}{x - a}\,, \quad f'(b-) = \lim_{x\to b^-} \frac{f(x) - f(b)}{x - b}\,. \tag{3.5}$$

Example 3.1 A family travels by a car on a Saturday morning, starting at 5 a.m. and arriving at their destination at 9 a.m. When they began the trip, the car's odometer read 28,700 km, and when they arrived it read 29,000 km. Compute the average velocity of the car during the journey.

Solution: As we know, the average velocity equals the distance traveled, divided by time elapsed. In the present case, the distance traveled equals $29{,}000 - 28{,}700 = 300\,\text{km}$, and the time elapsed equals $9 - 5 = 4\,\text{h}$.

The average velocity therefore equals $300/4 = 75\,\text{km/h}$.

Example 3.2 Show that the rate of change of the area of a circle with respect to its radius is equal to its circumference.

Solution: The area A of a circle is related to its radius by $A(r) = \pi r^2$. We obtain the rate of change of A with respect to r as the derivative $A'(r)$ and compute

$$\begin{aligned} A'(r) &= \lim_{h\to 0} \frac{A(r+h) - A(r)}{h} = \lim_{h\to 0} \frac{\pi(r+h)^2 - \pi r^2}{h} \\ &= \lim_{h\to 0} \frac{\pi r^2 + 2\pi r h + \pi h^2 - \pi r^2}{h} = \lim_{h\to 0} \frac{\pi h(2r+h)}{h} = \lim_{h\to 0} \pi(2r+h) \\ &= 2\pi r\,, \end{aligned}$$

which equals the circumference of the circle.

Example 3.3 In a metabolic experiment, an amount of mass m of glucose decreases according to the formula $m(t) = 9 - 0.06t^2$, where m is measured in grams and the elapsed time t in hours. Find the reaction rate at the time when 1 h has elapsed.

Solution: The reaction rate at $t = 1$ is $m'(1)$. Thus

$$\begin{aligned} m'(1) &= \lim_{t\to 1} \frac{m(t) - m(1)}{t - 1} = \lim_{t\to 1} \frac{(9 - 0.06t^2) - (9 - 0.06)}{t - 1} \\ &= \lim_{t\to 1} \frac{-0.06 \cdot (t^2 - 1)}{t - 1} = \lim_{t\to 1} \frac{-0.06 \cdot (t-1)(t+1)}{t - 1} \\ &= -0.12\,. \end{aligned}$$

Thus, the reaction rate at $t = 1$ is -0.12, that is, at the time when 1 hour has elapsed, the rate of decrease equals 0.12 gram per hour.

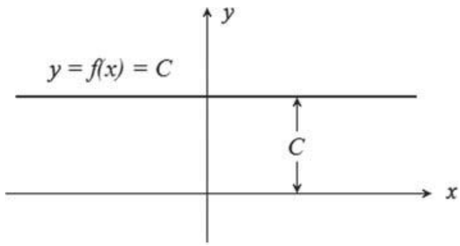
3.2 Derivative of Elementary Functions

Constant Function

Let $f(x) = c$ in some open interval I, c a given number. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} \\ &= \lim_{h\to 0} 0 = 0 \end{aligned}$$

Fig. 3.4 Constant function



Thus, the derivative of a constant function is everywhere zero. The graph of f equals the horizontal line $y = c$, the slope of the tangent at each point of the graph is zero, and hence all tangents coincide with the graph of f itself (Fig. 3.4).

Identity Function

Let $f(x) = x$ in some open interval I. Its graph on the interval $I = (-5, 5)$ is given in Fig. 3.5. For every $x \in I$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} 1 \\ &= 1 . \end{aligned}$$

Hence, the derivative of the identity function is everywhere equal to 1, and the tangent at a point (x, x) of the graph of f equals the straight line $y = x$. Thus, the tangent line again is simply the original line itself.

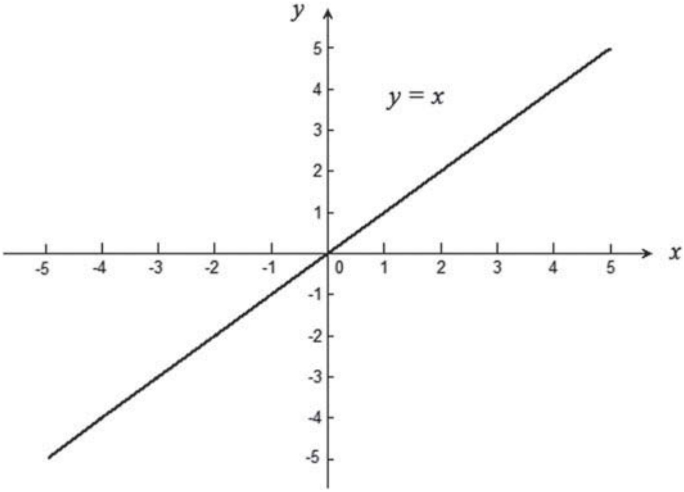


Fig. 3.5 Identity function

Power Function

Let $f(x) = x^n$ in some open interval I, for a given positive integer n. We will show that

$$f'(x) = nx^{n-1}\,. \tag{3.6}$$

For $n = 1$, we have seen that $f'(x) = 1$ for every x. For $n = 2$, we have at any point $c \in I$

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{x^2 - c^2}{x - c} = \lim_{x\to c}(x + c) = c + c = 2c\,.$$

For $n = 3$,

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{x^3 - c^3}{x - c} = \lim_{x\to c} \frac{(x - c)(x^2 + cx + c^2)}{x - c} = 3c^2\,.$$

Replacing the letter c by x, we see that (3.6) is satisfied for $n = 1, 2, 3$. We now check that (3.6) holds for arbitrary positive integers n. Indeed,

$$\begin{aligned}
\lim_{x\to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x\to c} \frac{x^n - c^n}{x - c} \\
&= \lim_{x\to c} \left[x^{n-1} + cx^{n-2} + \cdots + c^{n-1}\right] \\
&= \underbrace{c^{n-1} + c^{n-1} + \cdots + c^{n-1}}_{n \text{ times}} = nc^{n-1}\,.
\end{aligned}$$

In words, the derivative of x raised to the power n equals n times x raised to the power $n - 1$.

Another method to prove the same result starts from the alternative definition of the derivative as

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{(x + h)^n - x^n}{h}\,.$$

Applying the Binomial Theorem to $(x + h)^n$, we get

$$(x + h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\,.$$

Thus

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right) \\
&= nx^{n-1}.
\end{aligned}$$

Formula (3.6) can also be proved for negative integers n, in fact, it holds for any real number n.

Trigonometric Functions

We find here derivatives of $\sin x$ and $\cos x$. The derivatives of $\sec x$, $\csc x$, $\tan x$, and $\cot x$ will be given in Sect. 3.3.

Let $f(x) = \sin x$, then

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}\,.$$

Using the identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$, we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin x \cdot (\cos h - 1) + \cos x \sin h}{h} \\ &= -\lim_{h\to 0} \frac{\sin x \cdot (1 - \cos h)}{h} + \lim_{h\to 0} \cos x \frac{\sin h}{h}\,. \end{aligned}$$

We know that (see Example 2.6 (3))

$$\lim_{h\to 0} \frac{\sin h}{h} = 1\,,$$

so

$$\lim_{h\to 0} \cos x \frac{\sin h}{h} = \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} = \cos x\,.$$

Moreover,

$$\lim_{h\to 0} \sin x \frac{1 - \cos h}{h} = \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} = 0\,.$$

Putting these formulas together, we see that $f'(x) = 0 + \cos x = \cos x$.

We now consider $f(x) = \cos x$. Then

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}\,.$$

Using the identity $\cos(x+h) = \cos x \cos h - \sin x \sin h$, we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\cos x \cdot (\cos h - 1) - \sin x \sin h}{h} \\ &= \cos x \cdot \lim_{h\to 0} \frac{(\cos h - 1)}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\ &= -\sin x\,, \end{aligned}$$

where we have used the formulas

$$\lim_{h\to 0} \frac{\sin h}{h} = 1\,, \quad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0\,.$$

Absolute Value Function

We know that the absolute value function

$$f(x) = |x| = \begin{cases} x\,, & x > 0\,, \\ 0\,, & x = 0\,, \\ -x\,, & x < 0\,, \end{cases}$$

is continuous at all points x, in particular at $x = 0$. (See Exercise 2.5.6). On the other hand, f is not differentiable at $x = 0$. Indeed,

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1\,, & h > 0\,, \\ -1\,, & h < 0\,. \end{cases}$$

Therefore,

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1\,, \quad \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1\,,$$

so the right- and left-sided limits exist, but are different. Thus,

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist, so f is not differentiable at $x = 0$.

This example shows that it may happen that a function is **continuous, but not differentiable**. On the other hand, we have the following result.

Theorem 3.1 *Let f be a function which is differentiable on an interval I. Then f is continuous on I.*

Proof Let x be an arbitrary point of I. For $h \neq 0$ and $x + h \in I$

$$f(x+h) - f(x) = \frac{f(x+h) - f(x)}{h} \cdot h\,.$$

We have

$$\lim_{h \to 0} h = 0\,, \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)\,,$$

since f is differentiable at x. From the product rule for limits, we conclude that

$$\lim_{h \to 0} [f(x+h) - f(x)] = \left[\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right] \cdot \left[\lim_{h \to 0} h\right] = f'(x) \cdot 0 = 0\,.$$

It follows that $\lim_{h \to 0} f(x+h) = f(x)$, thus f is continuous at x.

3.3 Some Differentiation Formulas

A further remark on the notation of derivatives. In the previous sections, we have taken care to specify derivatives in terms of a general function f, as in the statement

Let $f(x) = \sin x$, then $f'(x) = \cos x$.

Having understood the concept of the derivative, one often writes more briefly

$$\frac{d}{dx}\sin x = \cos x\,, \qquad \text{or} \qquad \sin'(x) = \cos x\,,$$

or even $(\sin x)' = \cos x$.

Constant multiples. The derivative of a constant times a function equals the constant times the derivative of the function. That is, if c is a constant and f is a differentiable function, then

$$(cf)'(x) = cf'(x)\,. \tag{3.7}$$

Examples:

1. If $f(x) = x^5$, then $(6f)'(x) = 6f'(x) = 6 \cdot 5x^4 = 30x^4$. Or, more briefly, $(6x^5)' = 6(x^5)' = 6 \cdot 5x^4 = 30x^4$.
2. $(-2\sin)'(x) = -2\sin'(x) = -2\cos x$, or, with the same meaning, $(-2\sin x)' = -2(\sin x)' = -2\cos x$.

Derivative of a sum. The derivative of the sum of two differentiable functions equals the sum of their derivatives. That is,

$$(f+g)'(x) = f'(x) + g'(x)\,. \tag{3.8}$$

Verification:

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x)\,.
\end{aligned}$$

Example: $(\cos + 4\sin)'(x) = \cos'(x) + 4\sin'(x) = -\sin x + 4\cos x$.

Derivative of a difference. The derivative of the difference of two differentiable functions equals the difference of their derivatives. That is,

$$(f-g)'(x) = f'(x) - g'(x)\,. \tag{3.9}$$

Verification: We use the rules for sums and constant multiples,

$$(f-g)'(x) = (f+(-g))'(x) = f'(x) + (-g)'(x) = f'(x) - g'(x)\,.$$

Derivative of a product. The derivative of the product of two differentiable functions equals the first function times the derivative of the second plus the second function times the derivative of the first. That is,

$$(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x)\,. \tag{3.10}$$

We note that, in contrast to the case of sum and difference, the derivative of a product does **not** equal the product of the derivatives.

Verification: We analyze the difference quotient

$$\begin{aligned}
\frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] + g(x)\left[\frac{f(x+h)-f(x)}{h}\right].
\end{aligned}$$

Here we have added and subtracted $f(x+h)g(x)$ in the numerator and then regrouped the terms so as to display the difference quotients for f and g separately. Now, since f is differentiable at x, we know that f is continuous at x by Theorem 3.1 and thus $\lim_{h\to 0} f(x+h) = f(x)$. From the rules for limits and the definition of $f'(x)$ and $g'(x)$, we then obtain

$$\begin{aligned}
(f\cdot g)'(x) &= \lim_{h\to 0} \frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h} \\
&= \left[\lim_{h\to 0} f(x+h)\right] \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + g(x) \cdot \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] \\
&= f(x)g'(x) + g(x)f'(x)\,.
\end{aligned}$$

The reciprocal rule. If g is differentiable at x and $g(x) \neq 0$, then $1/g$ is differentiable and

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{(g(x))^2}\,. \tag{3.11}$$

Verification: Since g is differentiable at x, it is also continuous at x by Theorem 3.1. Since $g(x) \neq 0$, we know that the function $(1/g)(x)$ is continuous at x and moreover

$$\lim_{h\to 0} \frac{1}{g(x+h)} = \frac{1}{g(x)}\,.$$

We now obtain

$$\left(\frac{1}{g}\right)'(x) = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{g(x+h)} - \frac{1}{g(x)}\right] = \lim_{h\to 0}\frac{g(x)-g(x+h)}{g(x+h)g(x)}$$
$$= \left[-\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]\cdot\left[\lim_{h\to 0}\frac{1}{g(x+h)g(x)}\right]$$
$$= -\frac{g'(x)}{(g(x))^2}.$$

The quotient rule. The derivative of a quotient equals the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Let f and g be differentiable at x and $g(x) \neq 0$, then the quotient f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}. \tag{3.12}$$

To verify (3.12), one applies the rule for products and the reciprocal rule.

The chain rule. Let y be a differentiable function of u, and u be a differentiable function of x. Then the rate of change of y with respect to x equals the rate of change of y with respect to u times the rate of change of u with respect to x.

Let $y = f(u)$ be a differentiable function of u, and u a differentiable function of x, namely, $u = g(x)$. Then $y = f(u) = f(g(x)) = (f \circ g)(x)$, that is, in order to express y as a function of x, we have to use the composite function $f \circ g$. The derivative of the composite function is given by the so-called chain rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x). \tag{3.13}$$

For example, we want to find the derivative of $y = (\sin x)^2$. We introduce the intermediate variable u and write $y = f(u) = u^2$, $u = g(x) = \sin x$, then $(f \circ g)(x) = f(g(x)) = (\sin x)^2$. In order to obtain its derivative by (3.13), we have to compute $f'(u) = 2u$ and insert $u = g(x) = \sin x$, so $f'(g(x)) = 2\sin x$. Since moreover $g'(x) = \cos x$, we arrive at

$$(f \circ g)'(x) = 2\sin x \cos x$$

as the derivative of $(\sin x)^2$.

The chain rule is sometimes written symbolically as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \quad \text{or} \quad y'(x) = y'(u)u'(x).$$

Derivative of an inverse function. Let the function f have an interval I as its domain $D(f)$. Suppose that f is differentiable and that either $f'(x) > 0$ for all interior points of I, or that $f'(x) < 0$ for all such points. Then f is increasing and

continuous, its range $R(f)$ is again an interval, and it has an inverse function f^{-1} with domain $R(f)$ and range $D(f) = I$. Additionally, f^{-1} is differentiable in the interior of the interval $R(f)$. Its derivative at a point $y = f(x)$ is given by

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}\,. \tag{3.14}$$

As an example, let us consider $f(x) = x^2$ on the interval $I = [1, 3]$. We have $f'(x) = 2x > 0$ on I. The inverse of f is $f^{-1}(y) = \sqrt{y}$. To find its derivative at a point $y = x^2$, we have to compute $1/f'(x) = 1/2x$ and insert $x = f^{-1}(y) = \sqrt{y}$, so

$$(f^{-1})'(y) = (\sqrt{y})' = \frac{1}{2\sqrt{y}}\,.$$

To verify (3.14), we start from the equation

$$y = f(f^{-1}(y))$$

and differentiate both sides with respect to y. Assuming that f^{-1} is differentiable (which is true under the assumptions stated above, but we do not prove it here), we get

$$1 = f'(f^{-1}(y)) \cdot (f^{-1})'(y) \tag{3.15}$$

and divide both sides by $f'(f^{-1}(y))$ to obtain (3.14).

We note that the derivatives of f and f^{-1} are reciprocal to each other, if we take proper care to evaluate them at corresponding points $y = f(x)$ of the independent variables (x for f, y for f^{-1}). Symbolically, Eqs. (3.14) and (3.15) may also be written as

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = 1\,, \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}\,, \tag{3.16}$$

but let us remark again that nowadays standard mathematical notation is given by the formulas (3.14) and (3.15).

Derivatives of other trigonometric functions. We determine the derivative of the trigonometric functions $\sec x$, $\tan x$, $\cot x$, and $\csc x$.

(i) Let $f(x) = \sec x = \frac{1}{\cos x}$. By the reciprocal rule (3.11),

$$\frac{d}{dx}\left(\frac{1}{\cos x}\right) = -\frac{(-\sin x)}{(\cos x)^2} = \frac{\tan x}{\cos x} = \tan x \sec x\,.$$

(ii) Let $f(x) = \tan x = \frac{\sin x}{\cos x}$. By the quotient rule (3.12),

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\,.$$

(iii) Let $f(x) = \cot x = \frac{\cos x}{\sin x}$. Again, by the quotient rule (3.12),

$$\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x\,.$$

(iv)

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x} = -\frac{\cos t}{\sin x} = -\cot x \csc x\,.$$

Exponential and logarithmic functions. We determine the derivative of the exponential function and the logarithm.

(i) We consider first $f(x) = e^x$. One can prove, as a consequence of the definition of the exponential function to be given in Chap. 5, that

$$\frac{e^h - e^0}{h} = \frac{\left(1 + h + \frac{h^2}{2!} + \cdots\right) - 1}{h} = 1 + \frac{h}{2!} + \cdots\,.$$

This implies that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$, therefore

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - e}{h} = e^x\,,$$

so

$$\frac{d}{dx} e^x = e^x\,. \tag{3.17}$$

(ii) We consider the general exponential function $f(x) = a^x$ with $a > 0$. From the formula $a^x = e^{x \ln a}$ and the chain rule (3.13) we compute in view of (3.17)

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = \ln a \cdot e^{x \ln a} = \ln a \cdot a^x\,. \tag{3.18}$$

(iii) The logarithm function $f(x) = \log_a(x)$ and the general exponential function are inverse to each other,

$$x = a^{\log_a x}\,.$$

We compute the derivative of both sides, using the chain rule and (3.18),

$$1 = \frac{d}{dx}\left(a^{\log_a x}\right) = \ln a \cdot a^{\log_a x} \cdot \frac{d}{dx} \log_a x = \ln a \cdot x \cdot \frac{d}{dx} \log_a x\,,$$

so

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}\,. \tag{3.19}$$

In particular, taking $a = e$ we obtain, since $\ln e = 1$,

$$\frac{d}{dx}\ln x = \frac{1}{x}\,. \tag{3.20}$$

Implicit differentiation. Until now, we have discussed derivatives of functions which are given explicitly in terms of the independent variable, for example, functions such as $y = x^2 + 1$, $y = \sin x$, and $y = x^2 \sin x$. Now we discuss derivatives of functional relationships between two variables in which neither of the variables is given in terms of the other. For example, consider the equation

$$y^2 + y - x^2 = 1\,. \tag{3.21}$$

While it is possible to find y explicitly in the form $y = f(x)$ by the solution formula for quadratic equations, one may want to avoid this because it is somewhat cumbersome; in other situations, it may be impossible to write an explicit formula for y in terms of x. Nevertheless, we can obtain an equation for the derivative of y with respect to x by the process called implicit differentiation. This process is based on differentiating both sides of the equation satisfied by x and y. Let us assume that (3.21) defines a function $y = f(x)$, so we must have

$$f(x)^2 + f(x) - x^2 = 1\,.$$

Differentiating both sides with respect to x, we obtain in view of the chain rule

$$2f(x)f'(x) + f'(x) - 2x = 0\,.$$

Solving for $f'(x)$, we get

$$f'(x) = \frac{2x}{1 + 2f(x)}\,.$$

Usually one writes this in shorter form, with y and y' instead of $f(x)$ and $f'(x)$. Starting from (3.21), the computation then looks like

$$2yy' + y' - 2x = 0\,, \quad y' = \frac{2x}{1 + 2y}\,.$$

Note that we have to pay a price—the right-hand side involves not only x but also y. So, in order to determine the derivative y' at, for example, $x = 1$, we have to find the corresponding value of y first. In this example, there are two possible values, namely, $y = 1$ and $y = -2$. (This is related to the fact that (3.21), being a quadratic equation in y, is satisfied by two different functions of the form $y = f(x)$.) The corresponding values of the derivative are

$$y'(1) = \frac{2}{1+2} = \frac{2}{3}\,, \quad \text{respectively} \quad y'(1) = \frac{2}{1-4} = -\frac{2}{3}\,.$$

Example 3.4 Find $y' = \frac{dy}{dx}$, if x and y are related by the following equations:
(a) $x^2 + y^2 = 9$,
(b) $2x^2y - y^3 + 5 = x + 6y$,
(c) $\sin(x + y) = xy$,
(d) $y^3 + 2y - \cos \pi x - 6 = 0$.

Solution: (a) Differentiating both sides of the given equation, we get

$$2x + 2yy' = 0\,, \quad \text{or} \quad y' = -\frac{x}{y}\,.$$

(b) Differentiating both sides of the given equation, we get

$$4xy + 2x^2y' - 3y^2y' = 1 + 6y'\,, \quad (2x^2 - 3y^2 - 6)y' = 1 - 4xy\,,$$
$$y' = \frac{1 - 4xy}{2x^2 - 3y^2 - 6}\,.$$

(c) By differentiating both sides of $\sin(x + y) = xy$, we get

$$\cos(x + y)(1 + y') = y + xy'\,,$$
$$(\cos(x + y) - x)y' = y - \cos(x + y)\,,$$
$$y' = \frac{y - \cos(x + y)}{\cos(x + y) - x}\,.$$

(d) By differentiating both sides of the given equation, we get

$$3y^2y' + 2y' + \pi \sin x = 0\,,$$
$$y' = -\frac{\pi \sin x}{(3y^2 + 2)}\,.$$

Example 3.5 A spherical balloon is expanding. If the radius is increasing at the rate of 4 cm/min, at what rate does the volume increase when the radius is 10 cm?

Solution: The volume V and the radius r are related by $V = \frac{4}{3}\pi r^3$. At an arbitrary time t, we have $V(t) = \frac{4}{3}\pi r(t)^3$, and consequently

$$V' = \frac{4}{3}\pi \cdot 3r^2r' = 4\pi r^2 r'\,.$$

Here $r = 10$, and the rate of increase $r' = \frac{dr}{dt}$ of the radius with time equals 4. Therefore, $V' = \frac{dV}{dt} = 4\pi 10^2 \cdot 4 = 1600\pi$ cm/min. Note that it was not necessary to compute the time t at which $r(t) = 10$ and $r'(t) = 4$ holds.

Example 3.6 Consider a cylinder with variable radius r and height h. Suppose that the radius is changing, but the volume is kept constant. How is h changing with respect to r?

Solution: The volume V of the cylinder satisfies $V = \pi r^2 h$, thus $h = \frac{V}{\pi r^2}$. We obtain the rate of change of h with respect to r as

$$\frac{dh}{dr} = -\frac{2V}{\pi} r^{-3} = -\frac{2(\pi r^2 h)}{\pi} r^{-3} = -\frac{2h}{V}\,.$$

Example 3.7 How does the atmospheric pressure p vary (change) with respect to the height h?

Solution: We are required to find the derivative of p as a function of h. If h is measured in meters and p in kPa (kilo Pascal), then approximately

$$p(h) = p_0 \exp\left(-\frac{h}{7990}\right),$$

where p_0 is the atmospheric pressure at ground (=sea) level, $p_0 = 100$ approximately. The rate of change of pressure satisfies $p'(h) = -(7990)^{-1} p(h)$. Near the ground, we therefore have

$$p'(h) \cong p'(0) \cong -0.01 \text{ kPa/m}\,.$$

We see that for heights between 0 and 100 m, for example, the decrease in pressure is rather small.

Example 3.8 A projectile is dropped from a height of 98 m. After how many seconds does it hit the ground? What is the speed at the moment of impact?

Solution: According to Galileo's formula $y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$, where g is the gravitational acceleration, v_0 is the initial (upward) velocity of the projectile at time $t_0 = 0$, and y_0 is the height from which the projectile is dropped. In the present case, $g = 9.8$ m/sec^2 approximately, $y_0 = 98$ m and $v_0 = 0$, so

$$y(t) = -4.9t^2 + 98\,. \tag{3.22}$$

At the time t of impact there holds $y(t) = 0$, which gives

$$t = \left(\frac{98}{4.9}\right)^{1/2} = \pm\sqrt{20} = \pm 2\sqrt{5}\,.$$

Since the impact occurs at a time $t > 0$, $t = 2\sqrt{5}$ is the correct solution. The velocity at this time equals $y'(2\sqrt{5})$. From (3.22), we obtain $y'(t) = v(t) = -(4.9)2t = -9.8t$. Inserting $t = 2\sqrt{5}$ gives us the speed at impact as

$$|v(2\sqrt{5})| = |y'(2\sqrt{5})| \cong 43.83 \text{ m/s}\,.$$

Derivatives in economics. In business and economics, we want to know how changes in variables such as production, supply or price, effect changes in variables such as cost, revenue, or profit. If f is a function that describes the relationship between pairs of these variables, the term “marginal” is used when one wants to refer to the derivative of f.

For example, let $C = C(x)$ denote the cost of production x units of a certain commodity. Then C' is called the marginal cost. The marginal cost $C'(x)$ at the production level x is approximately equal to the cost of producing the $(x+1)st$ unit. If $R = R(x)$ is the revenue received for selling x units of the commodity, then R' is called the marginal revenue. The marginal revenue $R'(x)$ at a sales level x is approximately equal to the revenue obtained by selling one additional unit. For the cost and revenue functions $C = C(x)$ and $R = R(x)$, respectively, associated with producing and selling x units of a commodity,

$$P(x) = R(x) - C(x)$$

is called the profit function.

Values of x (if any) at which $C(x) = R(x)$, that is, values at which cost equals revenue, are called **break-even points**. The derivative P' of P is called the marginal profit.

Example 3.9 (a) A manufacturer of computer components determines that the total cost C of producing x components per week is

$$C(x) = 2000 + 100x - \frac{x^2}{10} \quad \text{dollars}.$$

Find the marginal cost at the production level of 40 units. What is the exact cost of producing the 41st component?

(b) A manufacturer of watches determines that the cost and revenue functions involved in producing and selling x watches are

$$C(x) = 1200 + 13x\,, \quad R(x) = 75x - \frac{x^2}{2}\,,$$

respectively. Find the profit function and determine the break-even points. Find the marginal profit and determine the production/sales level at which the marginal profit is zero.

Solution: (a) The marginal cost is $C'(x) = 100 - \frac{x}{5}$. Thus, the marginal cost at the production level of 40 components equals $C'(40) = 100 - \frac{40}{5} = 92$ dollars. The exact cost of producing the 41st component is given by

$$\begin{aligned} C(41) - C(40) &= \left[2000 + 100 \cdot 41 - \frac{41^2}{10}\right] - \left[2000 + 100 \cdot 40 - \frac{40^2}{10}\right] \\ &= 2000 + 4100 - 168.1 - 2000 - 4000 + 160 \\ &= 91.9 \text{ dollars.} \end{aligned}$$

(b) The profit function P is given by

$$\begin{aligned} P(x) = R(x) - C(x) &= 75x - \frac{x^2}{2} - (1200 + 13x) \\ &= 62x - \frac{x^2}{2} - 1200\,. \end{aligned}$$

In order to find the break-even point, set $P(x) = 0$. Multiplying by -2, we have

$$0 = x^2 - 124x + 2400 = (x - 24)(x - 100) = 0\,.$$

The break-even points are $x = 24$ and $x = 100$. The marginal profit is given by derivative $P'(x) = 62 - x$, and $P'(x) = 0$ when $x = 62$.

Example 3.10 Gravel is being poured by a conveyor onto a conical pile at a constant rate of 40π cubic meter per minute. The frictional forces within the pile are such that the height of the pile is always two-thirds of its radius. How fast is the radius of the pile changing at the moment when it equals 5 m?

Solution: The formula for the volume V of a right circular cone of radius r and height h is

$$V = \frac{1}{3}\pi r^2 h\,.$$

In the present case, it is prescribed that $h = \frac{2}{3}r$, and hence

$$V(r) = \frac{2}{9}\pi r^3\,, \quad V'(r) = \frac{2}{3}\pi r^2\,.$$

Since gravel is being poured onto the pile, both volume and radius are functions of time t. The chain rule implies that

$$\frac{d}{dt}V(r(t)) = V'(r(t))r'(t) = \left(\frac{2}{3}\pi r(t)^2\right) r'(t)\,,$$

or in abbreviated form

$$\frac{dV}{dt} = \frac{2}{3}\pi r^2 \cdot \frac{dr}{dt}\,. \tag{3.23}$$

It is prescribed that $\frac{dV}{dt} = 40\pi$. Our goal is to find $\frac{dr}{dt}$ at the time t when $r(t) = 5$. Inserting this information into (3.23), one obtains

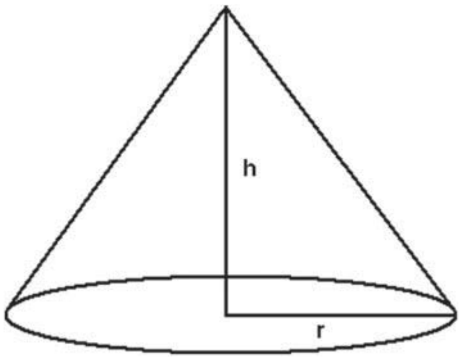


Fig. 3.6 A conical pile of sand

$$40\pi = \frac{50}{3}\pi \cdot \frac{dr}{dt}, \quad \text{therefore} \quad \frac{dr}{dt} = \frac{12}{5}.$$

We conclude that the radius is increasing at the rate of 2.4 m/min at the moment when the radius equals 5 m (Fig. 3.6).

Example 3.11 Sand is poured on a conical pile at a rate of 10 cubic meter per minute. The diameter of the base of the pile is always 50% greater than its height. Determine how fast the height of the pile is rising when the pile is 5 m high.

Solution: As in the previous example, $V = \frac{1}{3}\pi r^2 h$. This time, it is given that $2r = \frac{3}{2}h$, so $r = \frac{3}{4}h$. Thus

$$V(h) = \frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 = \frac{3}{16}\pi h^3, \quad V'(h) = \frac{9}{16}\pi h^2.$$

Since h is a function of t, the chain rule $\frac{d}{dt}V(h(t)) = V'(h(t))h'(t)$ thus yields

$$\frac{dV}{dt} = \frac{9}{16}\pi h^2 \cdot \frac{dh}{dt}.$$

We substitute the values of $\frac{dV}{dt}$ and h given in the problem and obtain

$$10 = \frac{9}{16}\pi 5^2 \frac{dh}{dt}, \quad \text{therefore} \quad \frac{dh}{dt} = \frac{32}{45\pi}.$$

At the moment when $h = 5$ m, the height rises at a rate of $32/45\pi$ m/min.

Example 3.12 A searchlight is mounted 1000 m offshore and rotates at a constant angular speed of four revolutions per minute. Determine how fast the spot of light is moving along the shoreline when it reaches a point 1500 m from the light.

Solution: We choose the origin at the searchlight, and the vertical line $x = 1000$ for the shoreline, thus the x-axis is perpendicular to the shoreline. Let y represent the

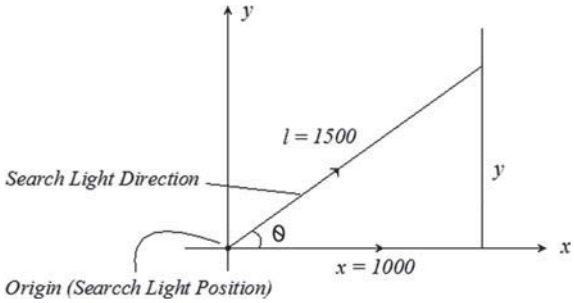


Fig. 3.7 Moving spot of light

current position of the spot of light on the shoreline, and let θ be the angle between the positive x-axis and the direction of the searchlight. We know that $\frac{d\theta}{dt} = 8\pi$ rad/min. We need to find $\frac{dy}{dt}$.

From Fig. 3.7, we see that $y = 1000 \tan\theta$, so $\frac{dy}{d\theta} = 1000 \sec^2\theta$. By the chain rule

$$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = 1000 \sec^2\theta \cdot 8\pi = 8000\pi \sec^2\theta\,. \tag{3.24}$$

At the moment of interest, we have $l = 1500$, so

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1500}{1000} = \frac{3}{2}\,.$$

Substituting this value into (3.24), we obtain

$$\frac{dy}{dt} = 8000\pi \left(\frac{3}{2}\right)^2 = 18000\pi\,.$$

The spot of light is moving at a speed of 18000π m/min at the moment in question.

Example 3.13 Find the derivatives of the following functions:

(a) $f(x) = x^{2/3}$
(b) $f(x) = x^{-3}$
(c) $f(x) = (2x^3 - x)(x^4 + 3x)$
(d) $f(x) = \frac{8}{x^2} - \frac{6}{x}$
(e) $f(x) = \frac{6x^2-1}{x^4+5x+1}$
(f) $f(x) = \frac{1}{9x^2+8x+10}$
(g) $f(x) = (x^2 - 1)^{100}$
(h) $f(x) = 2x^3(x^2 - 3)^4$
(i) $f(x) = x(x^2 + 1)^3$
(j) $f(x) = \sin x \cdot \cos x$
(k) $f(x) = e^{-x^2}$

Solution:

(a) For $f(x) = x^{2/3}$, we have

$$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{1}{2\sqrt[3]{x}}, \quad x > 0.$$

(The graph of $x^{2/3}$ suggests that the tangent line at $x = 0$ is the y-axis, which has an infinite slope.)

(b) For $f(x) = x^{-3}$, we have by the power rule

$$f'(x) = -3x^{-3-1} = -3x^{-4}.$$

(c) For $f(x) = (2x^3 - x)(x^4 + 3x)$, if we use the product rule, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}[2x^3 - x] \cdot (x^4 + 3x) + (2x^3 - x) \cdot \frac{d}{dx}[x^4 + 3x] \\ &= (6x^2 - 1)(x^4 + 3x) + (2x^3 - x)(4x^3 + 3) \\ &= (6x^6 - x^4 + 18x^3 - 3x) + (8x^6 - 4x^4 + 6x^3 - 3x) \\ &= 14x^6 - 5x^4 + 24x^3 - 6x. \end{aligned}$$

In this case, it is easier first to multiply out and then to differentiate with the power rule,

$$f(x) = 2x^7 - x^5 + 6x^4 - 3x^2, \quad f'(x) = 14x^6 - 5x^4 + 24x^3 - 6x.$$

(d) For $f(x) = \frac{8}{x^2} - \frac{6}{x} = 8x^{-2} - 6x^{-1}$, we obtain from the power rule

$$f'(x) = -16x^{-3} + 6x^{-2} = -\frac{16}{x^3} + \frac{6}{x^2}.$$

(e) For $f(x) = \frac{6x^2-1}{x^4+5x+1}$, we use the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^4 + 5x + 1)(12x) - (6x^2 - 1)(4x^3 + 5)}{(x^4 + 5x + 1)^2} \\ &= \frac{-12x^5 + 4x^3 + 30x^2 + 12x + 5}{(x^4 + 5x + 1)^2}. \end{aligned}$$

(f) For $f(x) = \frac{1}{9x^2+8x+10}$, we use the reciprocal rule,

$$f'(x) = \frac{-(18x + 8)}{(9x^2 + 8x + 10)^2}.$$

(g) For $f(x) = (x^2 - 1)^{100}$, we use the chain rule,

$$\begin{aligned} f'(x) &= 100(x^2-1)^{99}\frac{d}{dx}(x^2-1) = 100(x^2-1)^{99}\cdot 2x \\ &= 200x(x^2-1)^{99}\,. \end{aligned}$$

(h) For $f(x) = 2x^3(x^2-3)^4$, we use first the product rule and then the chain rule,

$$\begin{aligned} \frac{d}{dx}[2x^3(x^2-3)^4] &= 2x^3\cdot\frac{d}{dx}[(x^2-3)^4] + (x^2-3)^4\cdot\frac{d}{dx}(2x^3) \quad \text{(product rule)} \\ &= 2x^3\cdot 4(x^2-3)^3\cdot 2x + (x^2-3)^4\cdot 6x^2 \qquad \text{(chain rule)} \\ &= 16x^4(x^2-3)^3 + 6x^2(x^2-3)^4 \\ &= 2x^2(x^2-3)^3(11x^2-9)\,. \end{aligned}$$

(i) $f(x) = x(x^2+1)^3$

$$\begin{aligned} \frac{d}{dx}[x(x^2+1)^3] &= x\cdot\frac{d}{dx}[(x^2+1)^3] + (x^2+1)^3\cdot\frac{d}{dx}(x) \\ &= x\cdot 3(x^2+1)^2\cdot 2x + (x^2+1)^3\cdot 1 \\ &= 6x^2(x^2+1)^2 + (x^2+1)^3 \\ &= (x^2+1)^2[6x^2+x^2+1] \\ &= (x^2+1)^2(7x^2+1)\,. \end{aligned}$$

(j) For $f(x) = \sin x\cdot\cos x$, we use the product rule,

$$f'(x) = \sin'(x)\cdot\cos(x) + \sin(x)\cdot\cos'(x) = \cos^2 x - \sin^2 x\,.$$

(k) For $f(x) = e^{-x^2}$, we use the chain rule,

$$f'(x) = e^{-x^2}\cdot\frac{d}{dx}[-x^2] = -2xe^{-x^2}\,.$$

3.4 Derivatives of Higher Order

When we differentiate a function $y = f(x)$, its derivative f' is again a function. The derivative of f' is called the **second derivative** of f and is denoted by f''. The derivative of the function f'' is called the third derivative of f and denoted by f''', or alternatively $f^{(3)}$. The fourth derivative is denoted by $f^{(4)}$. In general, if n is a positive integer, the nth derivative of f is denoted by $f^{(n)}$.

Example 3.14 Find the fourth derivative of the following functions:

(a) $f(x) = x^4 - 3x^{-1} + 10$,
(b) $f(x) = \sin(2x)$.

Solution:

(a) To find the fourth derivative of $f(x) = x^4 - 3x^{-1} + 10$, we have to compute sequentially the first, second, and third derivatives,

$$f'(x) = 4x^3 + \frac{3}{x^2}\,, \quad f''(x) = 12x^2 - 6x^{-3}\,,$$
$$f'''(x) = 24x + 18x^{-4}\,, \quad f^{(4)}(x) = 24 - 72x^{-5}\,.$$

(b) The same procedure applies to $f(x) = \sin(2x)$,

$$f'(x) = 2\cos(2x)\,, \quad f''(x) = -4\sin(2x)\,,$$
$$f'''(x) = -8\cos(2x)\,, \quad f^{(4)}(x) = 16\sin(2x)\,.$$

As in the case of the first derivative, various different notations are in use to denote higher derivatives. Instead of f, f', f'', f''' ...we may write

$$f\,, \quad \frac{df}{dx}\,, \quad \frac{d^2 f}{dx^2}\,, \quad \frac{d^3 f}{dx^3}\,, \ldots$$

or, if we use y to denote a function $y = y(x)$, we may write

$$y\,,\ y'\,,\ y''\,,\ y'''\,,\ y^{(4)}\,, \ldots \qquad \frac{dy}{dx}\,,\ \frac{d^2 y}{dx^2}\,,\ \frac{d^3 y}{dx^3}\,, \ldots$$

So, for example, $y = x^{-1}$ has the derivatives

$$y' = -x^{-2}\,, \quad y'' = 2x^{-3}\,, \quad y''' = -6x^{-4}\,, \quad \frac{d^4 y}{dx^4} = y^{(4)}(x) = 24x^{-5}\,.$$

3.5 A Basic Differential Equation

A differential equation is an equation for one (or several) unknown functions, in which derivatives of those functions appear. Differential equations as mathematical models for phenomena in the real world have been developed for several hundred years, and are nowadays practically ubiquitous. In this subsection, we present a basic differential equation which leads to exponential growth or decay, and we discuss various situations where it arises.

The basic model. In different areas of science and technology, there arise many situations in which the following assertion is valid, in an exact or approximate manner:

$$\text{A quantity } Q \text{ varies at a rate proportional to itself.} \tag{3.25}$$

Since the mathematical expression for the rate of change of Q is the derivative of Q, the corresponding mathematical model takes the form of the differential equation

$$\frac{dQ}{dt} = kQ\,. \tag{3.26}$$

Here k is a real number, the constant of proportionality, and Q is a function of an independent variable which we denote by t, because in many applications it will stand for time. If $k > 0$, then (3.26) signifies that the time rate of change of Q is positive, thus the amount Q of the quantity is increasing with time. If $k < 0$, then (3.26) implies that the time rate of change of Q is negative, thus the amount Q of the quantity is decreasing with time.

We will study differential equations later in more detail (see Chap. 11), but we can check immediately that the function

$$Q(t) = Q_0 e^{kt}\,, \tag{3.27}$$

Q_0 being an arbitrary constant, solves (3.26), as we may simply differentiate both sides of (3.27) and obtain

$$Q'(t) = Q_0 \cdot ke^{kt} = kQ(t)\,.$$

Population Growth Model

In his 1798 "Essay on the Principle of Population", the English economist Thomas Malthus introduced (3.25) into the study of the growth of human populations. He assumed that the rate, at which a population grows at a certain time, is proportional to the total population at that time. Let $N(t)$ denote the population size (=number of individuals) of a country at time t. The mathematical model above (with Q replaced by N) then becomes

$$\frac{dN}{dt} = kN\,. \tag{3.28}$$

If the population size equals N_0 at the time t_0, we must have $N(t_0) = N_0$. The latter condition fixes the arbitrary constant in the solution, and we arrive at

$$N(t) = N_0 e^{k(t-t_0)}\,. \tag{3.29}$$

The model thus predicts the population size at any time t, if we know its size at some time t_0. It is a rather simple model which does not take into account other important factors like crowding, immigration, or emigration. Many more refined models have been studied in the life and social sciences, as well as by mathematicians. The basic model (3.28) is often used to model the growth of small populations over a short interval of time.

Example 3.15 Suppose the US population grows continuously at the rate of 4% per year. If the population is currently 275 million, what will it be in 1 year? In 10 years?

Solution: Let us fix the current time as $t_0 = 0$. With N measured in millions and t in years, the constants in (3.28), (3.29) become $k = 0.04$ and $N_0 = 275$, so the solution is

$$N(t) = 275e^{0.04t}\,.$$

After 1 year, the population size will be equal to $N(1) = 275e^{0.04} \simeq 286.223$ million, after 10 years, its size will be $N(10) = 275e^{0.4} \simeq 410.25$ million.

Example 3.16 Assume that the population of the United States grows at a rate proportional to the current population. According to census, the population size was 75.99 million in the year 1900 and 134 million in the year 1940.

(a) Find the annual growth rate during this period.
(b) Using the growth rate found in part (a), determine the population size predicted by the model for the year 2000.

Solution:

(a) With the constants $t_0 = 1900$ and $N_0 = 75.99$, the solution (3.29) becomes

$$N(t) = 75.99e^{k(t-1900)}\,. \tag{3.30}$$

Inserting the data for $t = 1940$, we get

$$134 = N(1940) = 75.99e^{40k}\,, \quad k = \frac{1}{40}\ln\left(\frac{134}{75.99}\right) = 0.01418\,.$$

Hence, the average growth rate was about 1.42% per year during the period from 1900 to 1940.
(b) For this part, $t = 2000$ and $k = 0.01418$. We insert these values into (3.30) and obtain $N(2000) = 75.99e^{100\cdot 0.01418} \simeq 313.76$ million.

Bacterial Growth Model

Example 3.17 Assume that a colony of bacteria increases at a rate proportional to the current number. If the number of the bacteria doubles in 5 h, how long will it take for their number to triple?

Solution: We fix $t_0 = 0$ and obtain the formula $N(t) = N_0e^{kt}$ for the number of bacteria as before. Since we know that this number doubles in 5 h, we get

$$2 = \frac{N(5)}{N_0} = e^{5k}\,, \quad \text{therefore} \quad k = \frac{1}{5}\ln 2\,.$$

The time t to triple has to satisfy

$$3 = \frac{N(t)}{N_0} = e^{kt}\,,$$

and we obtain t from

$$\ln 3 = kt\,, \quad t = \frac{\ln 3}{k} = 5\frac{\ln 3}{\ln 2} = 5\frac{1.0986}{0.6931} \simeq 7.925\,.$$

Thus, after 7.925 h the number of bacteria has tripled.

Financial Growth Model

Consider an amount y_0 of capital, deposited in a bank account. If the bank pays an annual interest rate of r percent at the end of the year (annual compounding), the capital will then amount to $y_0(1+k)$, where $k = 0.01r$. After t years, the total amount will be

$$y = y_0(1+k)^t\,.$$

Compounding monthly for t years yields

$$y(t) = y_0\left(1 + \frac{k}{12}\right)^{12t}\,.$$

Compounding daily for t years yields

$$y = y_0\left(1 + \frac{k}{365}\right)^{365t}\,.$$

If we compound kn times per year at equal time intervals, after t years, we will have an amount of

$$y(t) = y_0\left(1 + \frac{k}{kn}\right)^{knt} = y_0\left[\left(1 + \frac{1}{n}\right)^n\right]^{kt}\,.$$

We say that interest is compounded continuously, if we let n tend to infinity in the latter formula. Then

$$y(t) = y_0 e^{kt}\,, \quad \text{since} \quad e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n\,.$$

We see that continuous compounding results in capital growth according to our basic model

$$\frac{dy}{dt} = ky\,.$$

Example 3.18 How long will it take for money in the bank to double at 10% annual interest, if compounded continuously? Compounded annually?

Solution: We determine the time t from the equation

$$2 = \frac{y(t)}{y_0} = e^{kt}\,, \quad t = \frac{1}{k}\ln 2\,.$$

Since $k = 0.01$, the result is that the capital doubles after $t = 10 \ln 2 \simeq 6.92$ years. If we use the interest rate r (in percent) instead of k, the formula above becomes

$$t = \frac{100}{r} \ln 2 \simeq \frac{69.2}{r}.$$

This corresponds to the banker's "Rule of 70" which says that the doubling time can be estimated dividing 70 by the interest rate.

For interest compounded annually, $y(t) = y_0(1+k)^t = y_0(1.1)^t$. At doubling, $(1.1)^t = 2, t \ln 1.1 = \ln 2, t = \ln 2/\ln 1.1 \simeq 7.3$. Since annual compounding occurs only when t is an integer, the balance will not double before the eighth interest payment.

Radioactive Decay

For a radioactive substance, the rate of decay at a given time t is proportional to the amount present at that time. That is, if A represents the amount of radioactive substance at time t, again our basic model

$$\frac{dA}{dt} = kA$$

applies. Here, the constant k is negative and depends on the radioactive substance. The half-life of the radioactive substance is defined as the time when half of the substance has decayed.

In carbon dating, we use the fact that all living organisms contain two kinds of carbon, carbon-12 (a stable carbon) and carbon-14 (a radioactive carbon). As a result, when an organism dies, the amount of carbon-12 which is present within the organism remains unchanged, while the amount of carbon-14 begins to decrease. This change in the amount of carbon-14 relative to the amount of carbon-12 makes it possible to calculate the time at which the organism lived.

Example 3.19 In the skull of an animal found in an archaeological dig, it was determined that about 20% of the original amount of carbon-14 was still present. The half-life of carbon-14 is 5600 years. Find the approximate age of the animal.

Solution: Let $A(t)$ be the amount of carbon-14 present in the skull at time t. Then A satisfies the differential equation $dA/dt = kA$, whose solution is $A(t) = A_0 e^{kt}$, where A_0 is the amount of carbon-14 present at time $t = 0$, the time of death of the animal. To determine the constant k, we use the fact that when $t = 5600$, half of the original amount A_0 will remain. Thus,

$$\frac{1}{2} = \frac{A(t)}{A_0} = e^{5600k}, \quad k = \frac{1}{5600} \ln \frac{1}{2} = -0.000124.$$

Thus, at time t an amount of $A(t) = A_0 e^{-(0.000124)t}$ carbon-14 is still present. If this comprises 20% of the original amount A_0, we have

$$\ln 0.2 = -0.000124t\,, \quad \text{therefore} \quad t = \frac{1.6094}{0.000124} \simeq 12{,}979\,.$$

Thus, the animal lived approximately 13,000 years ago.

3.6 Differentials, Newton–Raphson Approximation

Differentials. Let f be a differentiable function whose graph and the tangent line at the point $(x, f(x))$ are shown in Fig. 3.8. We see that $f(x+h) - f(x)$, the change in f from x to $x+h$ can be approximated by the product $f'(x)h$ for small h, $h \neq 0$:

$$f(x+h) - f(x) \cong f'(x)h\,. \tag{3.31}$$

Definition 3.2 Let $h \neq 0$. The difference $f(x+h) - f(x)$ is called the **increment** of f from x to $x+h$ and is denoted by Δf,

$$\Delta f = f(x+h) - f(x)\,.$$

The product $f'(x)h$ is called the **differential** of f at x with increment h and is denoted by df,

$$df = f'(x)h\,.$$

As defined above, both Δf and df are functions which depend on x and h. In actual computation, one usually just writes Δf and df as above, instead of $\Delta f(x, h)$ or $df(x, h)$.

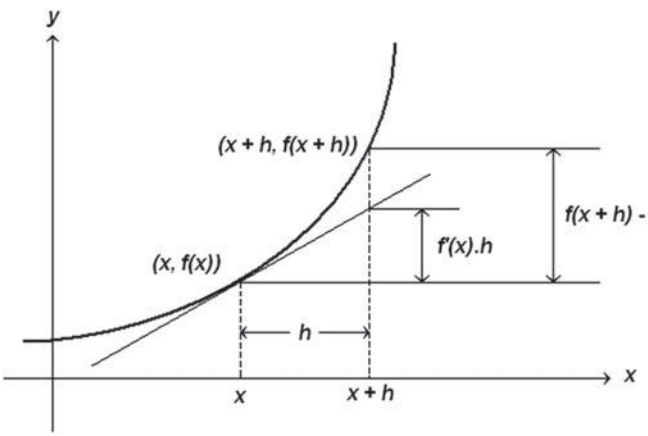


Fig. 3.8 Increment and differential

Figure 3.8 says that, for small h, the values of Δf and df are approximately equal,

$$\Delta f \cong df\,. \tag{3.32}$$

But the main point here is that, when h tends to 0, not only the difference $\Delta f - df$ tends to 0 but also the ratio

$$\frac{\Delta f - df}{h} \tag{3.33}$$

tends to 0, so $\Delta f - df$ tends to 0 **faster than h**. Indeed,

$$\frac{\Delta f}{h} = \frac{f(x+h) - f(x)}{h}, \quad \frac{df}{h} = \frac{f'(x)h}{h} = f'(x)$$

are the difference quotient and the derivative of f, respectively, and their difference tends to 0, since f is differentiable.

We now consider a simple case. The area of a square of sidelength $x > 0$ is given by the function

$$f(x) = x^2\,.$$

If the length of each side is increased from x to $x + h$, the area increases from $f(x)$ to $f(x + h)$. The change in area equals the increment Δf,

$$\begin{aligned}\Delta f &= f(x+h) - f(x) = (x+h)^2 - x^2 = (x^2 + 2xh + h^2) - x^2 \\ &= 2xh + h^2\,.\end{aligned}$$

See Fig. 3.9. As an estimate for this change, we can use the differential

$$df = f'(x)h = 2xh.$$

The error of this estimate, the difference between the actual change Δf and the estimated change df, is the difference

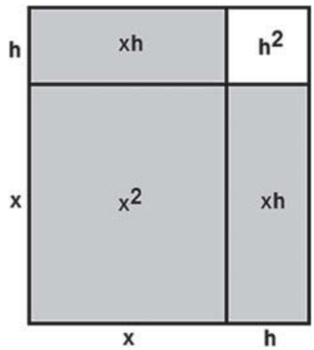


Fig. 3.9 Error of the estimated change df

$$\Delta f - df = h^2 .$$

The error of the estimate is small compared to h in the sense that

$$\frac{\Delta f - df}{h} = \frac{h^2}{h} = h$$

tends to 0 as h tends to 0.

Example 3.20 Use a differential to estimate the change in $f(x) = x^{2/5}$ if
(a) x is increased from 34 to 36,
(b) x is decreased from 1 to $\frac{7}{10}$.

Solution: Since $f'(x) = \frac{2}{5}x^{-3/5} = 2/(5x^{3/5})$, we have

$$df = f'(x)h = \frac{2h}{5x^{3/5}}.$$

(a) We set $x = 34$ and $h = 2$. The differential then becomes

$$df = \frac{2}{5 \cdot (34)^{3/5}} \cdot 2 = \frac{4}{41.48} = 0.096.$$

A change in x from 34 to 36 increases the value of f by approximately 0.096. We have

$$\Delta f = f(36) - f(34) = (36)^{2/5} - (34)^{2/5} \cong 4.193 - 4.098 = 0.095.$$

(b) Put $x = 1$ and $h = -\frac{3}{10}$. In this case, the differential is

$$df = \frac{2}{5 \cdot (1)^{3/5}} \cdot \left(-\frac{3}{10}\right) = -\frac{6}{50} = -0.12.$$

A change in x from 1 to $\frac{9}{10}$ decreases the value of f by approximately 0.12. We also have

$$\Delta f = f(0.7) - f(1) = (0.7)^{2/5} - (1)^{2/5} \cong 0.867 - 1 = -0.133 .$$

Example 3.21 Use a differential to estimate
(a) $\sqrt{105}$,
(b) $\cos 40°$,
starting from known values for nearby arguments.

Solution:

(a) We know that $\sqrt{100} = 10$. We want to obtain an estimate for the increase of

$$f(x) = \sqrt{x}\,,$$

when x increases from 100 to 105. Here,

$$f'(x) = \frac{1}{2\sqrt{x}}\,, \quad df = f'(x)h = \frac{h}{2\sqrt{x}}.$$

With $x = 100$ and $h = 5$, df becomes

$$\frac{5}{2\sqrt{100}} = \frac{1}{4} = 0.25\,.$$

A change in x from 100 to 105 increases the value of the square root by approximately 0.25. Hence,

$$\sqrt{105} \cong \sqrt{100} + 0.25 = 10 + 0.25 = 10.25\,.$$

Since $(10.25)^2 = 104.04$, so the estimate is not far off.

(b) Let $f(x) = \cos x$. We know that $\cos 45° = \cos\frac{\pi}{4} = \sqrt{2}/2$. The angle $40° = 45° - 5°$ can be written in radians as

$$\frac{\pi}{4} - 5 \cdot \frac{\pi}{180} = \frac{\pi}{4} - \frac{\pi}{36} \quad \text{rad.}$$

We use a differential to estimate the change in $\cos x$, when x decreases from $\pi/4$ to $(\pi/4) - (\pi/36)$. We have

$$f'(x) = -\sin x\,, \quad df = f'(x)h = -h \sin x\,.$$

With $x = \pi/4$ and $h = -\pi/36$, df becomes

$$df = -\left(-\frac{\pi}{36}\right)\sin\left(\frac{\pi}{4}\right) = \frac{\pi}{36}\frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{72} \cong 0.0617\,.$$

Thus, a decrease in x from $\pi/4$ to $(\pi/4) - (\pi/36)$ increases the value of the cosine function by approximately 0.0617. Therefore,

$$\cos 40° \cong \cos 45^0 = 0.0617 \cong 0.7071 + 0.0617 = 0.7688\,.$$

We see that $\cos 40° \cong 0.7688$.

Example 3.22 A metal sphere with a radius of 5 cm is to be covered with a 0.02 cm coating of silver. Approximately how much silver will be required?

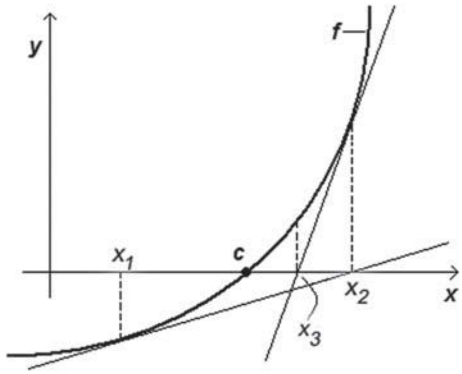


Fig. 3.10 Newton–Raphson method

Solution: We use a differential to estimate the increase in the volume of the sphere when the radius is increased from 5 to 5.02 cm. The formula for the volume of a sphere of radius r is

$$V(r) = \frac{4}{3}\pi r^3 .$$

The differential $dV = V'(r)h$ then becomes

$$dV = 4\pi r^2 h .$$

For $r = 5$ and $h = 0.02$, we get

$$dV = 4\pi(5)^2 \cdot 0.02 = 2\pi \cong 6.283 .$$

Thus, it will take approximately 6.283 cm^3 of silver to coat the sphere.

Newton–Raphson Method

Let f be a function whose graph is shown in Fig. 3.10. Since the graph of f crosses the x-axis at $x = c$, the number c is a solution (root) of the equation $f(x) = 0$. In the setup of Fig. 3.10, we can approximate c as follows: start at a point x_1 (see the figure). The tangent line at $(x_1, f(x_1))$ intersects the x-axis at a point x_2, which is closer to c than x_1. The tangent line at $(x_2, f(x_2))$ intersects the x-axis at a point x_3, which is closer to c than x_2. Continuing in this manner, we will obtain better and better approximations $x_4, x_5, \ldots, x_n$ to the root c.

We now develop an algebraic connection between x_n and x_{n+1}. The tangent line at $(x_n, f(x_n))$ has the equation

$$y - f(x_n) = f'(x_n)(x - x_n) .$$

The value x_{n+1}, where it intersects the x-axis, can be found by setting $y = 0$,

$$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n) .$$

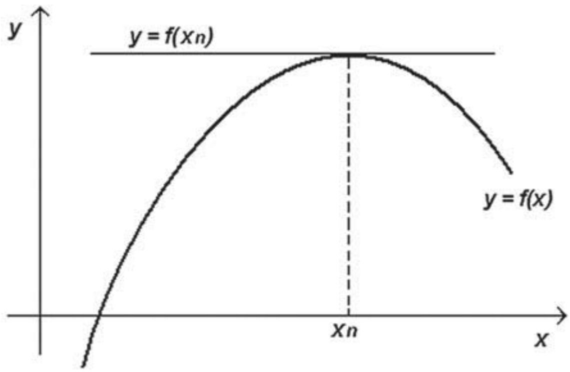


Fig. 3.11 Failure when the tangent becomes horizontal

Solving this equation for x_{n+1}, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The method described above of locating a root of an equation $f(x) = 0$ is called the **Newton–Raphson method**, or simply the **Newton method**. It works if the following conditions are satisfied:

(i) f is differentiable in some interval that includes the root c.
(ii) $f'(x) \neq 0$ in some interval including c. (See Fig. 3.11 for what happens if $f'(x_n) = 0$.)
(iii) The initial approximation x_1 is close enough to c.

Indeed, the method may fail if x_1 is not chosen properly. For instance, it can happen that the first approximation x_1 produces a second approximation x_2 which, in turn, gives the same x_1 as the third approximation, and so on—the approximations simply alternate between x_1 and x_2. See Fig. 3.12. Another type of difficulty can arise if $f'(x_1)$ is smaller than $f'(c)$. In this case, the second approximation x_2 could be worse than x_1, the third approximation x_3 could be worse than x_2, and so forth (see Fig. 3.13).

Let us consider a situation where the Newton–Raphson method is guaranteed to work no matter how far x_1 is away from the root c. Suppose that $x_1 > c$, f is twice differentiable and that $f(x)f''(x) > 0$ on the open interval I joining c and x_1. If $f''(x) > 0$ on I, then the graph of f is curved upward[1] on I, and we have the situation pictured in Fig. 3.14. On the other hand, if $f''(x) < 0$ on I, then the graph of f is curved downward on I, and we have the situation pictured in Fig. 3.15. In either case, the sequence of approximations $x_1, x_2, x_3, \ldots$ will converge to the root c.

Example 3.23 The number $\sqrt{5}$ is a root of the equation $x^2 - 5 = 0$. We will estimate $\sqrt{5}$ by applying the Newton–Raphson method to the function $f(x) = x^2 - 5$ starting

[1]The curvature of a graph and the role of the second derivative are treated in Chap. 4.

Fig. 3.12 Approximations alternate

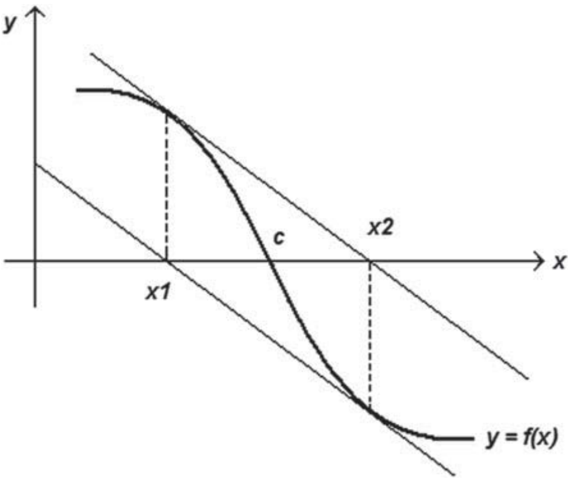


Fig. 3.13 Approximations get worse

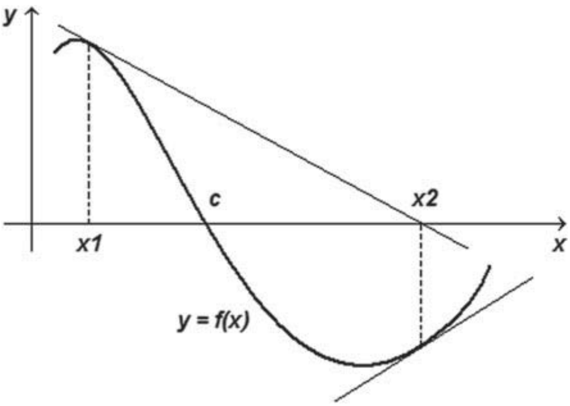
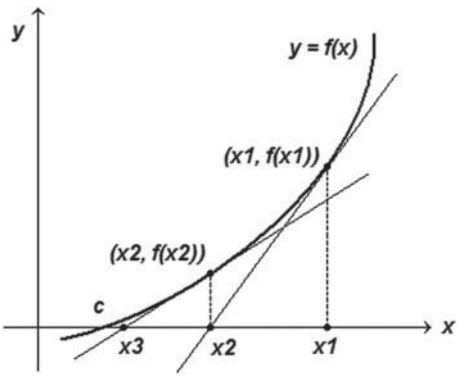


Fig. 3.14 Approximations converge to a root



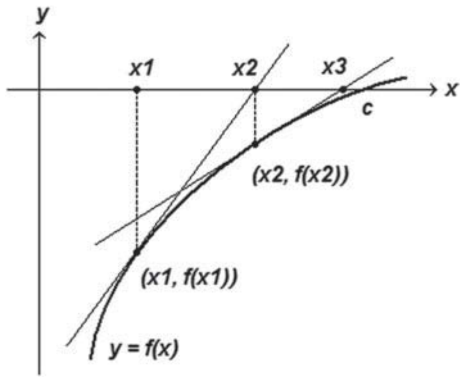


Fig. 3.15 Approximations converge to a root

at $x_1 = 4$. (As you can check, $f(x)f''(x) > 0$ on $(\sqrt{5}, 4)$, and therefore we can be sure that the method applies.) Since $f'(x) = 2x$, the Newton–Raphson formula gives

$$x_{n+1} = x_n - \left(\frac{x_n^2 - 5}{2x_n}\right) = \frac{x_n^2 + 5}{2x_n}\,.$$

Successive calculations with this formula (using a calculator) are given in the following table:

n	x_n	$x_{n+1} = \frac{x_n^2+4}{2x_n}$
1	4	2.625
2	2.625	2.2649
3	2.2649	2.2363

The approximation satisfies $(2.2363)^2 \cong 5.0010$, the exact solution is $\sqrt{5} = 2.23606...$ Thus, the method has generated a very accurate estimate in only three steps.

3.7 Indeterminate Forms and l'Hôpital's Rule

In Chap. 2, we already have discussed methods of finding limits. In this section, we present an additional technique which is known as l'Hôpital's rule, in honor of G. l'Hôpital, a French mathematician who lived during 1661–1704. This rule is used widely, both in theoretical work and practical calculations.

Let us illustrate the problem addressed by l'Hôpital's rule with a very simple example. We want to find

$$\lim_{x\to 0} \frac{2x}{x}\,. \tag{3.34}$$

Since we know that $2x/x = 2$ and obviously $\lim_{x\to 0} 2 = 2$, we see that the limit in (3.34) equals 2. On the other hand, we might compute the limits of the numerator and the denominator separately,

$$\lim_{x\to 0} x^2 = 0\,, \quad \lim_{x\to 0} x = 0\,,$$

but then we are stuck since there is no way from the quotient $0/0$ to arrive at the correct value 2 of the limit.

More generally, we want to compute the limit

$$\lim_{x\to c} \frac{f(x)}{g(x)}\,, \tag{3.35}$$

in situations where the formula

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

does not work, because the limits of numerator and denominator both have values 0 or $\pm\infty$. We say that f/g has **indeterminate form** $0/0$, respectively, ∞/∞ at $x = c$, if

$$\lim_{x\to c} f(x) = 0\,, \quad \lim_{x\to c} g(x) = 0\,, \tag{3.36}$$

respectively

$$\lim_{x\to c} f(x) = \pm\infty\,, \quad \lim_{x\to c} g(x) = \pm\infty\,. \tag{3.37}$$

The following theorem tells us how to evaluate the limit in such cases.

Theorem 3.2 (l'Hôpital's Rule) *Let f and g be differentiable on an open interval (a, b) containing c, except possibly at c itself. If f/g has the indeterminate form $0/0$ or ∞/∞ at $x = c$ and $f'(x)/g'(x) \neq 0$ for $x \neq c$, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}\,, \tag{3.38}$$

provided that $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists or $\lim_{x\to c} \frac{f'(x)}{g'(x)} = \infty$.

Typically, the condition that $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ has to exist is not checked separately, but is verified as a by-product of the computation when one uses l'Hôpital's Rule.

Example 3.24 Evaluate the following limits using l'Hôpital's rule:

1. $\lim_{x\to 3} \frac{x^2-9}{x-3}$,
2. $\lim_{x\to 0} \frac{\sin 3x}{x}$,
3. $\lim_{x\to 0} \frac{e^x-1}{x^2}$,
4. $\lim_{x\to 0} \frac{1-\cos x}{x}$,
5. $\lim_{x\to 0} \frac{\sin x}{x}$, (this limit we have already computed in Chap. 2 by more elementary arguments.)
6. $\lim_{x\to 0} \frac{\ln x}{\cot x}$.

Solution:

1. Here $f(x) = x^2 - 9$, $g(x) = x - 3$, and $\lim_{x\to 3} f(x) = 0$, $\lim_{x\to 3} g(x) = 0$. Therefore, f/g has indeterminate form $0/0$ at $x = 0$. By l'Hôpital's rule, we have
$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{2x}{1} = 6\,.$$
Note that in this example it is also possible to compute
$$\frac{x^2-9}{x-3} = x+3\,, \quad \lim_{x\to 3} x+3 = 6\,.$$

2. We have the form $0/0$. By L'Hôpital's rule,
$$\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{(\sin)'(3x)}{1} = \lim_{x\to 0} 3\cos 3x = 3\,.$$

3. We have the form $0/0$. By l'Hôpital's rule,
$$\lim_{x\to 0} \frac{e^x-1}{x^2} = \lim_{x\to 0} \frac{e^x}{2x} = \infty\,.$$

4. We have the form $0/0$. By l'Hôpital's rule,
$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1} = 0\,.$$

5. We have the form $0/0$. By l'Hôpital's rule,
$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1\,.$$

6. We have the form ∞/∞. By l'Hôpital's rule,
$$\lim_{x\to 0} \frac{\ln x}{\cot x} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\csc^2 x} = -\lim_{x\to 0} \frac{\sin^2 x}{x}\,,$$

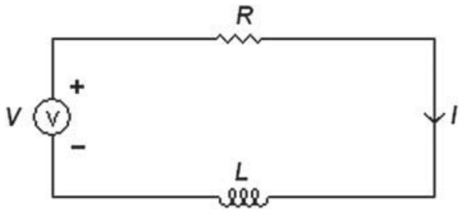


Fig. 3.16 An electrical circuit

if the latter limit exists. Now we may either use the product formula to obtain

$$-\lim_{x\to 0}\frac{\sin^2 x}{x} = -\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\sin x = 1\cdot 0 = 0\,,$$

or we may apply the rule a second time to get

$$-\lim_{x\to 0}\frac{\sin^2 x}{x} = -\lim_{x\to 0}\frac{2\sin x\cos x}{1} = 0\,.$$

The rule can also be applied at $c = \pm\infty$. Consider, for example

$$\lim_{x\to\infty}\frac{x^2}{\ln x}\,.$$

Applying l'Hôpital's rule, we get

$$\lim_{x\to\infty}\frac{x^2}{\ln x} = \lim_{x\to\infty}\frac{2x}{\frac{1}{x}} = \lim_{x\to\infty} 2x^2 = \infty\,.$$

Example 3.25 Figure 3.16 represents an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. The current I at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L}).$$

When the voltage is first applied at time $t = 0$, the inductor opposes the rate of increase of the current and I is small initially. As t increases to ∞, I approaches the value V/R, the value given by Ohm's law in the case $L = 0$.

1. If L is the only independent variable, find $\lim_{L\to 0^+} I$.
2. If R is the only independent variable, find $\lim_{R\to 0^+} I$.

Solution:

1. Let V, R, and t be constants and let L be variable. In this case, I is not indeterminate at $L = 0$. From the rules discussed in Chap. 2, we get

$$\begin{aligned}\lim_{L\to 0^+} I &= \lim_{L\to 0^+} \frac{V}{R}(1 - e^{-Rt/L}) = \frac{V}{R}\left(1 - \lim_{L\to 0^+} e^{-Rt/L}\right) \\ &= \frac{V}{R}(1-0) = \frac{V}{R}\,.\end{aligned}$$

   Thus, if $L \approx 0$, then the current can be approximated by the value $I = V/R$ given by Ohm's law, except for small times t.
2. If V, L, and t are constants and R is a variable, then I has the indeterminate form $0/0$ at $R = 0$. By l'Hôpital's rule, we get

$$\begin{aligned}\lim_{R\to 0^+} I &= V \lim_{R\to 0^+} \frac{1 - e^{-Rt/L}}{R} = V \cdot \lim_{R\to 0^+} \frac{0 - e^{-Rt/L}(-t/L)}{1} \\ &= V[0 - 1\cdot(-t/L)] = \frac{V}{L}t\,.\end{aligned}$$

   This shows that as $R \to 0^+$, the current is (approximately) directly proportional to time t. For $t = 1$, $I = \frac{V}{L}$. For $t = 4$, $I = \frac{4V}{L}$.

Example 3.26 The logistic model for population growth predicts the population size at time t by the formula $P(t) = \frac{K}{1+ce^{-rt}}$, where r and c are positive constants and $c = \frac{K-y(0)}{y(0)}$. K is called the carrying capacity and interpreted as the maximum number of individuals that the environment can sustain. Find $\lim_{t\to\infty} P(t)$ for K fixed, and $\lim_{K\to\infty} P(t)$ for t fixed, and interpret these limits.

Solution: We compute

$$\begin{aligned}\lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{K}{(1+ce^{-rt})} = \frac{K}{1 + c\lim_{t\to\infty} e^{-rt}} \\ &= \frac{K}{1 + c\cdot 0} = K\,.\end{aligned}$$

This shows that the population will essentially attain its carrying capacity after a sufficiently long period of time. The expression

$$\lim_{K\to\infty} P(t) = \lim_{K\to\infty} \frac{K}{1 + \frac{K-y(0)}{y(0)}e^{-rt}} = \lim_{K\to\infty} \frac{Ky(0)e^{rt}}{y(0)e^{rt} + K - y(0)}$$

has the indeterminate form ∞/∞. Applying l'Hôpital's rule, we get

$$\lim_{K\to\infty} P(t) = \lim_{K\to\infty} \frac{y(0)e^{rt}}{1} = y(0)e^{rt}\,.$$

This means that population will grow exponentially if the carrying capacity is infinite, and the logistic model reverts to the model of Malthus.

Other cases of indeterminate forms. Those can be converted to the forms $0/0$ or ∞/∞, and then l'Hôpital's rule can be applied.

The case $0 \cdot \infty$ means that we want to compute

$$\lim_{x\to c} f(x)g(x)\,,$$

where

$$\lim_{x\to c} f(x) = 0\,, \quad \lim_{x\to c} g(x) = \pm\infty\,.$$

Setting

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to\infty} \frac{f(x)}{\frac{1}{g(x)}}\,, \quad \text{or} \quad \lim_{x\to c} f(x)g(x) = \lim \frac{g(x)}{\frac{1}{f(x)}}\,,$$

we are back in the case $0/0$ or ∞/∞.

Next, we want to compute

$$\lim_{x\to c} f(x)^{g(x)}\,, \tag{3.39}$$

for the indeterminate forms 0^0, 1^∞, and ∞^0. In all those cases, we begin by setting

$$y(x) = f(x)^{g(x)}$$

and taking the natural logarithm of both sides to obtain

$$\ln y(x) = g(x) \ln f(x)\,. \tag{3.40}$$

We see that

the case 0^0 means $f(x) \to 0$, $g(x) \to 0$, $\ln f(x) \to -\infty$,
the case 1^∞ means $f(x) \to 1$, $g(x) \to \infty$, $\ln f(x) \to 0$,
the case ∞^0 means $f(x) \to 0$, $g(x) \to \infty$, $\ln f(x) \to -\infty$.

Therefore, we can apply our previous versions of l'Hôpital's rule to compute the limit in (3.40),

$$L := \lim_{x\to c} \ln y(x) = \lim_{x\to c} g(x) \ln f(x)\,.$$

After that, we come back to the original limit (3.39), since

$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c} y(x) = \lim_{x\to c} e^{\ln y(x)} = e^L\,.$$

Example 3.27 Evaluate the following limit, provided it exists:

1. $\lim_{x\to 0^+} x \ln x$,
2. $\lim_{x\to 0}(e^x - 1)^x$,
3. $\lim_{x\to\infty} x^{1/x}$,
4. $\lim_{x\to\infty}\left(\frac{x^2}{x-1} - \frac{x^2}{x+1}\right)$, *and*
5. $\lim_{x\to 1}(1-x)^{\ln x}$.

Solution:

1. Here we have the form $0 \cdot \infty$. We transform it to the form ∞/∞ and compute the limit,
$$\lim_{x\to 0^+} = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0\,.$$

2. The expression $(e^x - 1)^x$ has the form 0^0 for $x \to 0^+$. We set $y = (e^x - 1)^x$. Then
$$\ln y = x \ln (e^x - 1)$$
has the form $0 \cdot \infty$. We transform to the form ∞/∞ and compute the limit,
$$\lim_{x\to 0^+} x \ln(e^x - 1) = \lim_{x\to 0^+} \frac{\ln(e^x - 1)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{e^x}{e^x-1}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to 0^+} -\frac{x^2 e^x}{e^x - 1}\,.$$
The latter expression has the form $0/0$. We apply l'Hôpital's rule a second time and arrive at
$$\lim_{x\to 0^+} x \ln(e^x - 1) = \lim_{x\to 0^+} -\frac{x^2 e^x}{e^x - 1} = \lim_{x\to 0^+} -\frac{2xe^x + x^2 e^x}{e^x} = -\frac{0}{1} = 0\,.$$
Finally, we arrive at
$$\lim_{x\to 0}(e^x - 1)^x = e^0 = 1\,.$$

3. $\lim^{1/x}_{x\to\infty}$. Here we have the form ∞^0.
We set $y = x^{1/x}$. Then
$$\ln y = \frac{1}{x}\ln x$$
has the form $\frac{\infty}{\infty}$. We apply l'Hôpital's rule.
$$\lim x \to \infty \frac{1}{x} \ln x = \lim x \to \infty \frac{\frac{1}{x}}{1} = 0.$$

Therefore, $\lim x \to \infty x^{1/x} = e^0 = 1$.

4. $\lim_{x\to\infty}\left(\frac{x^2}{x-1} - \frac{x^2}{x+1}\right)$
   We write $\lim_{x\to\infty}\left(\frac{x^2}{x-1} - \frac{x^2}{x+1}\right) = \lim_{x\to\infty}\frac{x^2}{x-1} - \lim_{x\to\infty}\frac{x^2}{x+1}$
   Each term is of the form $\dfrac{\infty}{\infty}$.
   We use l'Hopital's rule to get

$$\lim_{x\to\infty}\left(\frac{x^2}{x-1} - \frac{x^2}{x+1}\right) = \lim_{x\to\infty} 2x - \lim_{x\to\infty} 2x.$$

5. $\lim_{x\to 1}(1-x)^{\ln x}$
   Here we have the form 0^0.
   We write $y = (1-x)^{\ln x}$. Then

$$\ln y = \ln x \ln(1-x)$$

which has the form $0.\infty$.
We transform it to the form $\dfrac{\infty}{\infty}$.

$$\begin{aligned}\lim_{x\to 1} \ln\ln(1-x) &= \lim_{x\to 1}\frac{\lim(1-x)}{\frac{1}{\ln x}} \\ &= \lim_{x\to 1}\frac{\frac{-1}{(1-x)}}{\frac{-1(\ln x)^2}{x}} = \lim_{x\to 1}\frac{x(\ln x)^2}{(1-x)} \\ &= \lim_{x\to 1}\frac{\frac{x}{x}2\ln x + (\ln x)^2}{-1} = 0.\end{aligned}$$

This implies $\lim_{x\to 1}\ln y = 0$. Therefore, $\lim_{x\to 1}(1-x)^{\ln x} = e^0 = 1$.

3.8 Sensitivity Analysis

We discuss in this section sensitivity to change. When a small change in x causes a large change in the value of a function $f(x)$, we say that the function is **highly sensitive** to changes in x. The derivative $f'(x)$ is the measure of this change.

As an example, we discuss a situation in genetics. Let us consider a population of peas, consisting of peas with smooth skin and with wrinkled skin. Let p (a number between 0 and 1) be the proportion of the gene for the smooth skin and $1-p$ the proportion of the gene for wrinkled skin. According to Mendel's theory of hybridization (the Austrian monk Gregor Johann Mendel (1822–1884) provided the first scientific explanation of hybridization), the proportion of smooth-skinned peas in the next generation will be

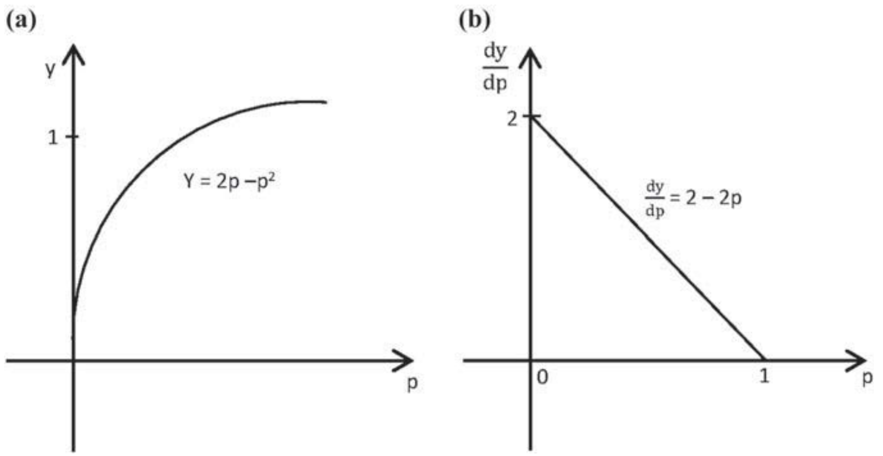


Fig. 3.17 **a** The graph of $y = 2p - p^2$ **b** The graph of $\frac{dy}{dp}$

$$y = 2p(1-p) + p^2 = 2p - p^2\,. \tag{3.41}$$

The graph of y versus p in Fig. 3.17a suggests that the value of y is more sensitive to a change in p where p is small than when p is large. Indeed, this fact is also visible through the graph of the derivative $y'(p) = 2 - 2p$, see Fig. 3.17b. It is clear that if $y'(p)$ is close to 2 when p is near to 0, and close to 0 when p is near 1.

The implication for genetics is that introducing a few more dominant genes into a highly recessive population (the proportion of wrinkled skin peas is large, in the example above) will have a more intense effect on later generations than a similar increase in a highly dominant population (the proportion of smooth-skinned peas is already large).

The number $f'(x)$ tells us how sensitive the output of f is with respect to a change in the input at any value of x. The larger the value of f' at x, the greater the effect of a given increment Δx of x.

In Sect. 3.6, we have already studied the effect of increments $h = \Delta x$ and seen that the differential $df = f'(x)h$ approximates the true change $\Delta f = f(x+h) - f(x)$ rather well when the increment is small. Indeed, the approximation error $\Delta f - df$ can be written as

$$\Delta f - df = f(x+h) - f(x) - f'(x)h = \left[\frac{f(x+h) - f(x)}{h} - f'(x)\right]h$$
$$= \varepsilon \cdot h\,,$$

where $\varepsilon = \frac{f(x+h)-f(x)}{h} - f'(x)$ is small if h is small. Consequently, the true change in f can be written as

$$\Delta f = f'(x)\Delta x + \varepsilon \Delta x\,,$$

and we see that the derivative $f'(x)$ is a good measure of sensitivity as described above.

Example 3.28 Suppose we want to calculate the depth s of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone dropped from above to splash into the water below. How sensitive will calculations be to an error of 0.1 s in measuring the time? (We measure s in ft. and t in seconds.)

Solution: Since $s = f(t) = 16t^2$, our estimate for the sensitivity becomes

$$f'(t) = 32t\,.$$

The effect of a measurement error $\Delta t = 0.1$ is approximately described by the differential $df = 32t\Delta t$ and thus depends on t. If $t = 2$, the error in depth caused by the measurement error equals $df = 64 \cdot 0.1 = 6.4$ ft. Four seconds later, the error in depth caused by the same Δt is $df = 192 \cdot 0.1 = 19.2$ ft.

3.9 Exercises

3.9.1 The distance s (in meters) covered by a particle in time t (in seconds) is given by $s = f(t) = 4t^2 + 3t$. Find the velocity at $t = 0$ and $t = 3$.

3.9.2 A circle of radius r has area $A = \pi r^2$ and circumference $c = 2\pi r$. If the radius changes from r to $r + dr$, find the

(i) change in area,
(ii) change in circumference.

3.9.3 Using the product rule for derivatives show that $\frac{d}{dx}(x^n) = nx^{n-1}$ for any integer n. Is this result true for any real number n?

3.9.4 Find the derivatives of the functions

(i) $f(x) = (2x^5 - x)(x^3 + 1)$,
(ii) $f(x) = 10x^{-4} + 3x^{-2}$,
(iii) $f(x) = \frac{-x^3-1}{2x^2+1}$,
(iv) $f(x) = (x^2 + 1)(x - 1)(x + 5)$,
(v) $f(x) = (1 + x^2)x^3e^x \ln x$,
(vi) $f(x) = \ln(1 + 3x^2)$,
(vii) $y = e^{x^2}$.

3.9.5 Find y''' if

(a) $y = x \ln x - x$,
(b) $y = \frac{1}{\sqrt{x^2+4}}$,
(c) $y = e^{2x}(e^{2x} - e^{-2x})$.

3.9.6 Find the first three derivatives of the following functions:

(a) $f(t) = t^{100} + t^{40} + t^2$,
(b) $f(t) = (3t + 5)^2$,
(c) $f(t) = t^5$,
(d) $f(t) = \frac{(t^2+1)(t^4+1)}{(t^3+1)(t^5+1)}$.

3.9.7 If $f = g^2$, prove that $f'(1) = 2g(1)g'(1)$.
3.9.8 If $y^5 + xy + x^2 = 3$, find y'.
3.9.9 Using the chain rule, find y' for $y = (3x + 5)^2$ and $y = (-5x^2 + x - 1)^2$.
3.9.10 Find $f'(x)$ for $f(x)$ given below:

(a) $f(x) = \frac{1}{(x^5-x+1)^3}$,
(b) $f(x) = \sin^3 x$,
(c) $f(x) = \frac{x}{\sqrt{1-x^2}}$.

3.9.11 Find $\frac{dy}{dx}$ for y given below:
(a) $y = x^3 \sin^2 5x$ (b)$y = \frac{\sin x}{\sec(3x+1)}$ (c) $y = \cos^3(\sin 2x)$.
3.9.12 If $y = 4x^3$ and the maximum percentage error in x is $\pm 15\%$, approximate the maximum percentage error in y.
3.9.13 A spherical balloon is being inflated with gas. Use differentials to approximate the increase in surface area of the balloon if the diameter changes from 2 to 2.02 m.
3.9.14 Examine whether the function $y^3 = 2x^2 + c$, c constant, satisfies the equation $y' = \dfrac{4x}{3y^2}$.
3.9.15 Show that any solution $y = y(x)$ of the equation $xy = c$ satisfies the equation $y + xy' = 0$.
3.9.16 Find derivatives of the following functions:
(a)$f(x) = \ln\left(\frac{x}{1+x^2}\right)$ (b)$f(x) = \ln(\ln x)$ (c)$f(x) = \sqrt{1 + \ln^2 x}$
(d)$f(x) = \cos(\ln x)$.

3.9.17 Find derivatives of the following functions:
(a)$f(x) = e^{1/x}$, (b)$f(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$, (c)$f(x) = \sin(e^x)$,
(d)$f(x) = \ln(\cos e^x)$.
3.9.18 Find $\frac{dy}{dx}$ by implicit differentiation
(a) $y + \ln xy - 2 = 0$ (b) $y = \ln(x \tan y)$.

3.9.19 Find $f'(x)$ for $f(x)$ given below:
(a)$f(x) = \sin^{-1}(\frac{1}{5}x)$, (b)$f(x) = (\tan x)^{-1}$,
(c)$f(x) = \sin^{-1} x + \cos^{-1} x$.

3.9.20 Show using l'Hôpital's rule that $\lim_{x\to 0}(1 + x)^{1/x} = e$.
3.9.21 Find

(a) $\lim_{x\to 0^+}\left[x^{(\ln a)/(1+\ln x)}\right]$,
(b) $\lim_{x\to +\infty}\left[x^{(\ln a)/(1+\ln x)}\right]$,
(c) $\lim_{x\to 0}\left[(x + 1)^{(\ln a)/x}\right]$.

3.9.22 Examine whether $\lim_{x\to 0^+} \frac{x \sin(1/x)}{\sin x}$ exists or not.

3.9.23 (a) Show that the function $y = e^{ax} \sin bx$ satisfies

$$y'' - 2ay' + (a^2 + y^2)y = 0.$$

(b) Show that $y = \tan^{-1} x$ satisfies $y'' = -2 \sin y \cos^3 y$.

3.9.24 Show that the rate of change of $y = 3^{2x} 5^{7x}$ is proportional to y.

Chapter 4
Optimization

Very often a real-world problem is equivalent to finding an element in the domain of a function at which the value of the function is larger or smaller than at all other elements of the domain. The techniques for finding such an element, that is, finding an element at which the maximal or minimal values of the function are attained, make up the field called *optimization*.

The goal of this chapter is to understand how maxima and minima of a function are related to its derivative. On practical aspects, we see how to analyze the relationship between average and marginal costs.

4.1 Extremum Values of Functions

Definition 4.1 (*Global maximum and minimum*) Let f be a real-valued function with domain D. If a point $c \in D$ satisfies

$$f(c) \leq f(x), \quad \text{for all } x \in D, \tag{4.1}$$

then c is called a **global minimizer** of f on D, and $f(c)$ is called the **global minimal value** of f on D. On the other hand, if a point $c \in D$ satisfies

$$f(c) \geq f(x), \quad \text{for all } x \in D, \tag{4.2}$$

then c is called a **global maximizer** of f on D, and $f(c)$ is called the **global maximal value** of f on D.

The notion "extremum" is generic for maximum and minimum, that is, a global extremum value is either a global maximal or a global minimal value.

M. Brokate et al., *Calculus for Scientists and Engineers*, Industrial and Applied Mathematics, https://doi.org/10.1007/978-981-13-8464-6_4

Alternatively, a minimizer is often termed a "minimum point" or simply "minimum".[1] Also, "absolute minimum" is synonymous to "global minimum".

Example 4.1 1. On $D = \mathcal{R}$, the functions $f(x) = x^2$ and $f(x) = |x|$ both have $c = 0$ as a global minimizer with minimal value $f(c) = 0$, but no global maximizer, since $f(x) \to \infty$ for $x \to \pm\infty$. Moreover, there is no other global minimizer.
2. For the function $f(x) = \cos x$, all points $c = 2k\pi$ with $k \in \mathbb{Z}$ are global maximizers with maximal value 1, and all points $c = (2k+1)\pi$ with $k \in \mathbb{Z}$ are global minimizers with minimal value -1.

Definition 4.2 (*Local Extrema*) Let f be a function with domain D. If there is an open interval I around some point $c \in D$ such that

$$f(c) \le f(x), \quad \text{for all } x \in D \cap I, \tag{4.3}$$

then c is called a **local minimizer** of f on D, and $f(c)$ is called the **local minimal value** of f on D. Reversing the inequality in (4.3) yields the definition of **local maximizer** and **local maximal value**.

The remarks above concerning the terminology for global extrema also apply to local extrema. Local extrema are also called relative extrema.

By definition, every global minimizer (maximizer) is also a local minimizer (maximizer). Hence, the collection of all local extrema will also include every global extremum (if existing). On the other hand, a local minimizer does not have to be a global minimizer.

The following observation provides a crucial link between extrema and derivatives. Let c be a local minimum point of a function f. Assume that $I = [c, d]$ is an interval contained in the domain D of f, such that $f(c) \le f(x)$ holds for all $x \in I$. Then

$$\frac{f(x) - f(c)}{x - c} \ge 0$$

holds for $c < x \le d$. By the properties of limits,

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \ge 0, \tag{4.4}$$

if this limit exists. But this limit is nothing else than the right-sided derivative $f'(c+)$ of f at c, see Remark 3.1, so

$$f'(c+) \ge 0 \tag{4.5}$$

[1]Note that the plural of "minimum", being a Latin word originally, is "minima", the same for maximum, extremum etc.

holds in this case. Analogously, if $I = [b, c]$ extends to the left of c, we obtain that

$$f'(c-) \le 0 \tag{4.6}$$

holds for the left-sided derivative. If, on the other hand, c is a local maximum of f, then c is a local minimum of $-f$, and we obtain (4.5) and (4.6) with inequalities reversed. Now, if f is differentiable at c, we must have $f'(c) = f'(c+) = f'(c-)$. Putting together (4.5) and (4.6), we have arrived at the single most important fact of optimization theory.

Theorem 4.1 *If f has a local extremum at an interior point c of its domain, and if $f'(c)$ exists, then*

$$f'(c) = 0\,, \tag{4.7}$$

that is, the tangent to the graph of f at the point $(c, f(c))$ is horizontal.

The theorem tells us that the only points where a function f with domain D can possibly have extrema are

- interior points of D at which $f' = 0$ (example: $f(x) = x^2$),
- interior points of D at which f' is undefined (example: $f(x) = |x|$),
- end points of D.

Note that the domain D plays a crucial role. The function $f(x) = x$ has no extremum on its natural domain $D = \mathbb{R}$. But if we restrict D to the nonnegative numbers $[0, \infty)$, for example, in an application where negative values make no sense, then $f(x) = x$ has the global minimizer $c = 0$, an endpoint of $[0, \infty)$.

Definition 4.3 (*Critical Point*) An interior point x of the domain D of a function f is called **critical** or **stationary**, if $f'(x) = 0$.

As we have seen above, if f is differentiable on an open interval I, then every local extremum in I must be a critical point. However, a point x can be critical without being an extremum. For example, the function $f(x) = x^3$ has no extremum, but $x = 0$ is a critical point of f.

At an extremum x of f, the slope of f typically changes its sign. The function $f(x) = x^2$, for example, has negative slope $f'(x) = 2x < 0$ for $x < 0$ and positive slope for $x > 0$. On the other hand, the slope $f'(x) = 3x^2$ of $f(x) = x^3$ is positive for $x \ne 0$, so it does not change sign at the critical point $x = 0$.

There arises the question whether we can guarantee that a given function has a maximizer or a minimizer.

Theorem 4.2 (Existence of Extrema) *Let f be a continuous function defined on a closed interval $I = [a, b]$, where $a < b$ are given numbers. Then f has a maximizer and a minimizer on I.*

To appreciate this result, let us show that it does not hold if we remove any of its assumptions.

Example 4.2 On the domain $D = \mathbb{R}$,

- the function $f(x) = x^2$ has the minimizer $c = 0$, but no maximizer,
- the function $f(x) = x^3$ has neither a minimizer nor a maximizer,
- the function $f(x) = \arctan x$ has neither a minimizer nor a maximizer, although all of its values remain in the bounded interval $[-\pi/2, \pi/2]$.

On the domain $D = [-2, 2]$, the discontinuous function

$$f(x) = \begin{cases} x\,, & |x| < 1\,, \\ 0\,, & |x| \geq 1\,, \end{cases}$$

has neither a minimizer nor a maximizer.

Again we note that the choice of the domain D is crucial. While the functions $f(x) = x^2$, $f(x) = x^3$, $f(x) = \arctan x$ of the preceding example fail to have maximizers on $D = \mathbb{R}$, they have maximizers on every closed interval $D = [a, b]$, by Theorem 4.2.

4.2 Monotonicity

In this section, we examine properties of a function and its graph in terms of its derivatives. First, let us recall from Chap. 1 that a function f with domain D is said to be

- increasing (respectively, nondecreasing), if $f(x_1) < f(x_2)$ (respectively, $f(x_1) \leq f(x_2)$) whenever $x_1 < x_2$,
- decreasing (respectively, nonincreasing), if $f(x_1) > f(x_2)$ (respectively, $f(x_1) \leq f(x_2)$) whenever $x_1 < x_2$

holds for arbitrary points $x_1, x_2 \in D$. Looking at the graphs of functions like the ones in Fig. 4.1, one realizes that increasing functions have tangents with positive slope, while decreasing functions have tangents with negative slope. (We already know that constant functions have zero slope.) The converse, too, is true.

Theorem 4.3 (Monotonicity Criterion) *Let f be function which is continuous on some interval $[a, b]$ and differentiable on its interior (a, b).*

1. *If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.*
3. *If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.*

This result is very important, because now we can check without doubt whether a function is increasing or decreasing, by computing its derivative (in fact, it suffices to compute the sign of the derivative.)

The proof of Theorem 4.3 is based on the following result, which is interesting by itself.

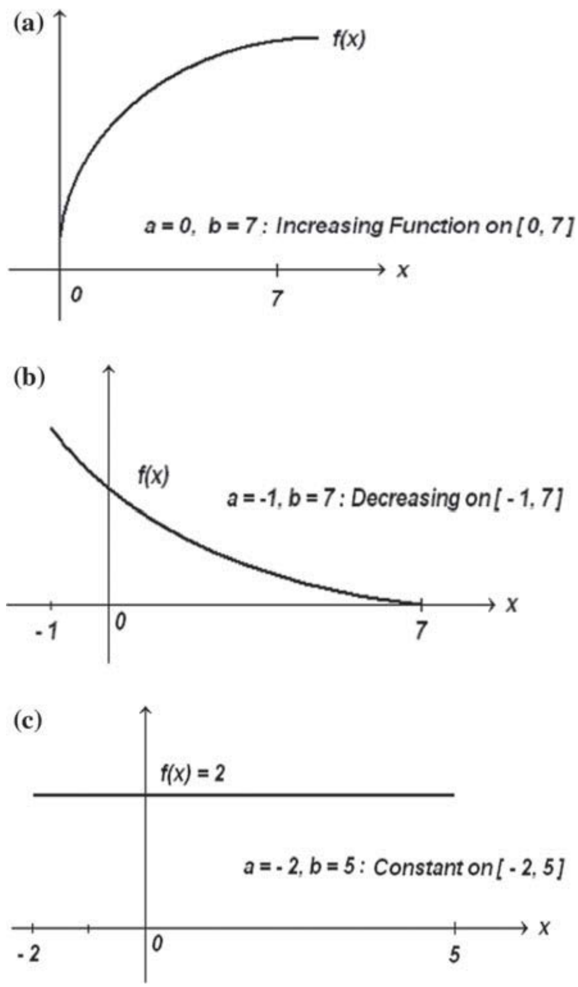


Fig. 4.1 (a) an increasing function, (b) a decreasing function, (c) a constant function

Theorem 4.4 (Mean Value Theorem) *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one point c in (a, b) where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{4.8}$$

In geometrical terms, the mean value theorem says that for every secant through two points of the graph of f, there is a tangent at an intermediate point, having the same slope as the secant.

The proof of Theorem 4.4 is yet based upon another theorem.

Theorem 4.5 (Rolle's Theorem) *let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is at least one point c in (a, b) where $f'(c) = 0$.*

We present here the proof of Theorem 4.3, while the proofs of Theorems 4.4 and 4.5 can be found in Appendix D.

Proof of Theorem 4.3: We only prove Part 1, the other parts are proved in the same manner. Let x_1, x_2 be in $[a, b]$ such that $x_1 < x_2$. By the mean value theorem applied on $[x_1, x_2]$, there is a point c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

or equivalently,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)\,. \tag{4.9}$$

By assumption, $f'(c) > 0$, as $c \in (x_1, x_2 \subset (a, b)$. Since $x_2 - x_1 > 0$, Eq. (4.9) implies that $f(x_2) - f(x_1) > 0$, so $f(x_1) < f(x_2)$. Since x_1 and x_2 have been chosen arbitrarily, f is increasing.

Remark 4.1 Let us point out explicitly that Eq. (4.8) of the Mean Value Theorem 4.4 implies that a function, whose derivative is identically zero, must be constant. Now, let f and g be two functions. If $f'(x) = g'(x)$ for all x in some interval I, then $(f - g)' = 0$ on I, hence there is a constant C such that $f(x) = g(x) + C$ for all x in I. That is, two functions with identical derivatives differ only by some constant value. This result will be important in the study of integration in Chap. 6.

Example 4.3 Find the intervals on which the following functions are increasing and the intervals on which they are decreasing:

1. $f(x) = x^2 - 4x + 9$,
2. $f(x) = x^3$.

Solution:

1. We have $f'(x) = 2x - 4 = 2(x - 2)$. It is clear that $f'(x) > 0$ if $x > 2$ and $f'(x) < 0$ if $x < 2$, therefore $f' > 0$ on the interval $(2, \infty)$ and $f' < 0$ on the interval $(-\infty, 2)$. Since f is differentiable (hence continuous) everywhere, it follows from Theorem 4.3 that f is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$.
2. In the same manner, one checks that f is increasing on $(-\infty, 0]$ and $[0, \infty)$. Therefore, it is increasing on all of $\mathbb{R}$.

4.3 Further Properties of Extremum Values

If we find a critical point, at first we do not know whether it is an extremum or not. However, if we know for example that f is increasing to the left of c and decreasing to the right of c, then c must be a local maximum. From the monotonicity criterion (Theorem 4.3) we, therefore, obtain the following test.

Theorem 4.6 (First Derivative Test) *Let f be a differentiable on an open interval which includes a critical point c.*

1. *If $f'(x) > 0$ on some open interval extending to the left from c,*[2] *and $f'(x) < 0$ on some open interval extending to the right from c, then f has a local maximum at c.*
2. *If $f'(x) < 0$ on some open interval extending to the left from c and $f'(x) > 0$ on some open interval extending to the right from c, then f has a local minimum at c.*
3. *If $f'(x)$ has the same sign (either $f'(x) > 0$ or $f'(x) < 0$) on some open intervals extending to the left and the right from c, then f does not have a local extremum at x_0.*

The previous test requires us to check the sign of f' in some interval around the critical point c, where $f'(c) = 0$. Since f'' is the derivative of f', we can interpret the value $f''(c)$ as the slope of the tangent to the function f' at the point c. Now, if we know that for example $f''(c) > 0$, we can conclude from Theorem 3.1 that $f'(x) > f'(c) = 0$ if $x > c$ is sufficiently close to c. This yields the following test.

Theorem 4.7 (Second Derivative Test) *Suppose that f is differentiable on an open interval which includes the point c, and that $f''(c)$ exists.*

1. *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
2. *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive, that is, f may have a local maximum, a local minimum, or neither at c.

Example 4.4 Find the extremum values of

$$f(x) = \frac{1}{\sqrt{9 - x^2}}.$$

Solution: $f'(x) = -\frac{1}{2}\frac{-2x}{(9 - x^2)^{3/2}} = \frac{x}{(9 - x^2)^{3/2}}$,
$f'(x) = 0$ when $x = 0$,
f has a stationary point at $x = 0$,
$f''(x) = \frac{(9 - x^2)^{3/2} - x.\frac{3}{2}(9 - x^2)^{1/2}}{(9 - x^2)^3}$,
$f''(0) > 0$,
So f has a minima at $x = 0$.

Example 4.5 Find the local extrema of the following functions:

1. $f(x) = 2x^2 - x^4$,
2. $f(x) = x^2 e^x$,
3. $f(x) = \cos^2 x, 0 < x < 2\pi$.

[2] That is, an interval of the form $(c - \delta, c)$ for some $\delta > 0$.

Solution:

1. $f(x) = 2x^2 - x^4$,
   $f'(x) = 4x - 4x^3 = 4x(1 - x^2)$,
   $f'(x) = 0$ gives $x = 0, 1, -1$,
   $f''(x) = 4 - 12x^2$,
   $f''(0) = 4 > 0$, $f''(1) = 4 - 12 = -8 < 0$, $f''(-1) = -8 < 0$,
   f has a minimum at $x = 0$ and maxima at $x = 1, -1$.
2. $f(x) = x^2e^x$,
   $f'(x) = 2xe^x + x^2e^x = xe^x(2 + x)$,
   $f'(x) = 0$ gives $x = 0, -2, x = -\infty$,
   We consider the first two values
   $f''(x) = 2xe^x + 2e^x + 2xe^x + x^2e^x = e^x(x^2 + 2 + 4x)$,
   $f''(0) = 2 > 0$; $f''(-2) = e^{-2}(4 + 2 - 8) = -2e^{-2} < 0$.
   Therefore, f has a minimum at $x = 0$ and a minimum at $x = -2$.
3. $f(x) = \cos^2 x$,
   $f'(x) = -2\sin x \cos x = -\sin 2x$,
   $f'(x) = 0$ gives $\sin 2x = 0$ or $2x = \pi$ or $x = \dfrac{\pi}{2}$.
   $f''(x) = -2\cos 2x$.
   $f''\left(\dfrac{\pi}{2}\right) = -2\cos(\pi) = 2 > 0$.
   Therefore, f has a minimum at $x = \dfrac{\pi}{2}$.

Example 4.6 Find values of a,b,c, and d, so that the function

$$f(x) = ax^3 + bx^2 + cx + d$$

satisfies $f(0) = 0$, $f(1) = 1$, has a local minimum at $c = 0$ and a local maximum at $c = 1$.

Solution: $f(x) = ax^3 + bx^2 + cx + d$,
$f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$.
Since f has a local minimum at $(0, 0)$ and a local maximum at $(1, 1)$,
$f'(0) = 0$ and $f'(1) = 0$. This gives $c = 0$ and $3a + 2b = 0$.
Since $f(0) = 0$ and $f(1) = 0$, we have $d = 0$ and $1 = a + b$.
Solve $3a + 2b = 0$ and $a + b = 1$ to get $a = -2$ and $b = 3$.
Therefore $a = -2$, $b = 3$, $c = d = 0$.

4.4 Convexity and Concavity

We have seen that the sign of derivative of f reveals where the graph of f is increasing or decreasing. However, it does not reveal the direction of its curvature.

In Fig. 4.2 the graph is increasing but on the left side it has an upwards curvature (holds water) and on the right side it has a downward curvature (spills water). On

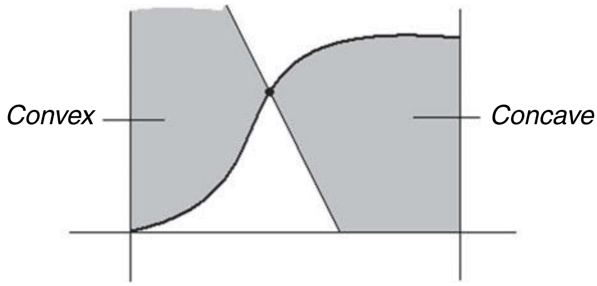


Fig. 4.2 A function which is convex on the left part and concave on the right part

intervals where the graph of a given function f has an upward curvature, we say that f is convex, and on intervals where the graph has a downward curvature, we say that f is concave.[3] See also Fig. 4.3 below. A formal definition can be made as follows.

Definition 4.4 A function f which is differentiable on an open interval $I = (a, b)$ is said to be **convex** on I if f' is increasing on I, and it is said to be **concave** on I if f' is decreasing on I.

The convexity or concavity of a function f can be characterized in terms of the second derivative f'' of f as follows.

Theorem 4.8 *Let f be twice differentiable on an open interval $I = (a, b)$.*

1. *If $f''(x) > 0$ on I, then f is convex on this interval.*
2. *If $f''(x) < 0$ on I, then f is concave on this interval.*

The proof of Theorem 4.8 follows from the observation that f' is increasing where its derivative f'' is positive and decreasing where f'' is negative.

Example 4.7 Find open intervals on which the following functions are convex or concave:

1. $f(x) = x^2$,
2. $f(x) = 3 + \sin x$,
3. $f(x) = x^2 - 4x + 6$.

Solution:

1. By Theorem 4.8, f is convex on $(-\infty, \infty)$ because its second derivative satisfies $f''(x) = 2$, which is positive everywhere.
2. f is concave on $(0, \pi)$, since there $f''(x) = -\sin x$ is negative. It is convex on $(\pi, 2\pi)$ as $f''(x) = -\sin x$ is positive on this interval.
3. $f''(x) = 2$, that is $f''(x) > 0$ on $(-\infty, \infty)$, hence f is convex on $(-\infty, \infty)$.

[3] Sometimes, the terms "concave up" and "concave down" are used instead of "convex" and "concave". We prefer to use the latter because it conforms to the standard use in mathematics.

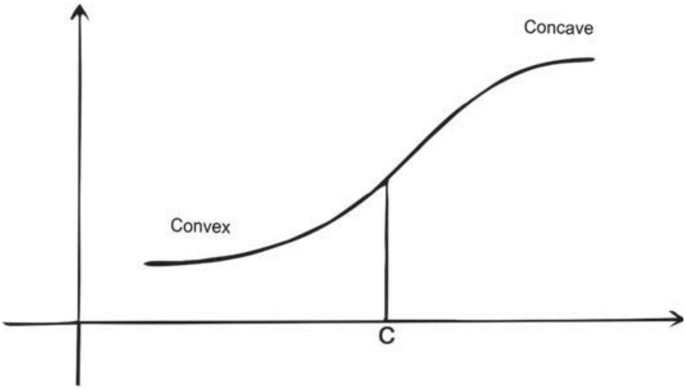


Fig. 4.3 Inflection point

The points where a function f changes its direction of curvature are called inflection points of f.

Definition 4.5 (*Inflection Points*) Let the function f be differentiable on an open interval (a, b). A point $c \in (a, b)$ is called an inflection point if f is either convex on some interval extending to the left of c and concave on some interval extending to the right of c (see Fig. 4.3), or concave on some interval extending to the left of c and convex on some interval extending to the right of c.

In Example 4.7, we have seen that the function $f(x) = 3 + \sin x$ has an inflection point at $c = \pi$.

From Theorem 4.8, we see that a point c is an inflection point of f if $f''(c) = 0$ and f'' has different signs to the left and to the right of c. However, not every point c with $f''(c) = 0$ is an inflection point. For example, the function $f(x) = x^4$ satisfies $f''(0) = 0$, but it is convex on the whole line $(-\infty, \infty)$.

Summary: properties of graphs of functions. Let us summarize at this point various concepts of calculus which are related to pictorial representations of functions (Fig. 4.4).

1. The *domain* and the *range* of f (see Chap. 1).
2. *Continuity* of f (see Chap. 2).
3. The points where the graph of f *meets the horizontal and the vertical axis*. For the x-axis, they are the solutions of the equation $f(x) = 0$. The y-axis is met at the point $(0, f(0))$, if 0 belongs to the domain of f.
4. *Symmetry*: If f is an even function, its graph is symmetric with respect to the y-axis. If f is an odd function, its graph is symmetric with respect to the origin.
5. *Local extrema* as discussed in Sect. 4.2.
6. *Inflection points* as presented in Definition 4.5.
7. *Convexity and Concavity* as discussed in Sect. 4.4.

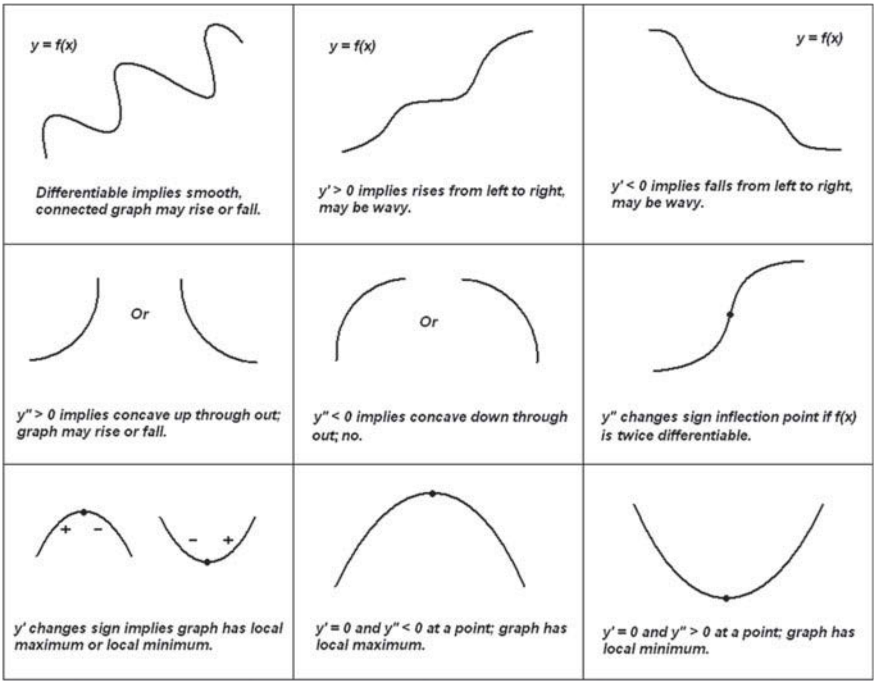


Fig. 4.4 Behavior of functions in terms of derivatives

8. *Horizontal and vertical asymptotes* are connected to improper limits. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote. If $\lim_{x\to c^+} f(x)$ or $\lim_{x\to c^-} f(x)$ equals either ∞ or $-\infty$, then the line $x = c$ is a vertical asymptote.
9. *Intervals of increase and decrease* as discussed in Sect. 4.2.

4.5 Applications of Optimization

If one wants to formulate and solve an optimization problem, one may go through the following steps.

- *Understand the problem.* Read the problem carefully and identify unknown quantities, known quantities, and the quantity to be sought.
- *Formulate a mathematical model.* Draw pictures and label the important parts. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, find a function f whose extreme values give the information sought.

- *Find the domain of the function f.* Determine what values of the variable make sense in the problem. Draw a graph of the function if feasible.
- *Identify the critical points and end points.* Find points where the derivative is zero or fails to exist. Use the first and second derivatives to identify and classify critical points.
- *Compute the solution.* If you are unsure of the result, support or confirm your solution with a different method.
- *Interpret the solution.* Translate the mathematical result into the problem setting and determine whether the result makes sense.

Applications to Business and Economics

We have already introduced in the examples of Sect. 3.3 the cost $C(x)$ of producing x units of a commodity, the revenue $R(x)$ received by selling them, and the profit $P(x) = R(x) - C(x)$. Moreover, we have defined the marginal cost, marginal revenue, and marginal profit as their rates $C'(x)$, $R'(x)$, and $P'(x)$, respectively. The term "marginal cost" is used because it is approximately equal to the cost of producing one more unit. Indeed, by definition the marginal cost equals

$$C'(x) = \lim_{h\to 0} \frac{C(x+h) - C(x)}{h}\,. \tag{4.10}$$

On the other hand, the cost $C(x+1) - C(x)$ of producing one more unit is obtained setting $h = 1$ in the difference quotient above. Example 3.9 furnishes a situation where those two quantities are in fact close to each other. Usually, this occurs when x is large. The same considerations apply to marginal revenue and marginal profit.

A common goal in business is to maximize profit. This is a question of optimization.

Theorem 4.9 *Let $x > 0$ be a production level which maximizes the profit $P(x)$ on $[0, \infty)$. Then at this level, the marginal cost equals marginal revenue, that is*

$$C'(x) = R'(x)\,. \tag{4.11}$$

(We have assumed that C and R, and hence also P, are differentiable functions of x.)

Proof Since x is a maximizer of P, we must have $P'(x) = 0$ by Theorem 4.1. Therefore

$$0 = P'(x) = (R - C)'(x) = R'(x) - C'(x)\,,$$

which proves the claim.

Note that in applying the differential calculus, we assume the argument x to take arbitrary real numbers as values, where the amount of a commodity like computers or tables is represented by an integer number. This is a typical situation in mathematical modeling, where often a continuum is used instead of a discrete set, in order to use the tools and concepts of the differential and integral calculus.

Let us consider a cost function $C(x)$ given by

$$C(x) = a + bx + dx^2 + kx^3 \,. \tag{4.12}$$

Here, the constant a represents a fixed overhead charge for items like rent, heat, and light that is independent of the number of units produced. If, except for this fixed overhead, the production costs are strictly proportional to the number x of units produced, then b is just the cost per additional unit, in this case ($d = k = 0$) equal to the marginal cost. The quadratic and cubic terms model situations where for large number of units the marginal cost is actually increasing, thus affecting production costs significantly.

Another relevant concept is the **average cost**

$$A(x) = \frac{C(x)}{x} \,.$$

The average cost describes the actual cost per unit, assuming that exactly x units are produced. It may be visualized as the slope of the line from the origin to point $(x, C(x))$ of the graph of the cost curve,

Example 4.8 Show that the critical points of the average cost occur when marginal cost equals average cost.

Solution: We have

$$A(x) = \frac{C(x)}{x} \,, \quad A'(x) = \frac{C'(x)x - C(x)}{x^2} \,.$$

At critical points of the average cost we have $A'(x) = 0$ which implies that $C'(x)x - C(x) = 0$. We conclude that

$$C'(x) = \frac{C(x)}{x} = A'(x) \,,$$

which is what we wanted to show.

Example 4.9 Let $R(x) = 9x$ and $C(x) = x^3 - 6x^2 + 15x$, where x represents one thousand units of some commodity. Find the production level at which the profit is maximal. Find also the maximal value of the profit.

Solution: For arbitrary x, the value of the profit is given by

$$P(x) = R(x) - C(x) = 9x - (x^3 - 6x^2 + 15x) = -x^3 + 6x^2 - 6x \,. \tag{4.13}$$

By Theorem 4.9 the profit is maximal for the level x at which $C'(x) = R'(x)$. Here we have $R(x) = 9x$ so $R'(x) = 9$, and $C(x) = x^3 - 6x^2 + 15x$ so $C'(x) = 3x^2 - 12x + 15$. Equating those expressions we get $9 = 3x^2 - 12x + 15$ or equivalently

$x^2 - 4x + 2 = 0$. Solving this quadratic equation with the standard formula yields the two solutions $x = 2 \pm \sqrt{2}$. One checks that $p(2 - \sqrt{2}) < p(2 + \sqrt{2})$. Thus the maximal profit is attained at $x = 2 + \sqrt{2} = 3.414$, and its value is

$$p(2 + \sqrt{2}) = -(3.414)^3 + 6 \cdot (3.414)^2 - 6 \cdot 3.414 = 9.657\,.$$

Remark 4.2 Normally, in the example above, we would have to check that $P''(2 + \sqrt{2}) < 0$ in order to ensure that $x = 2 + \sqrt{2}$ is indeed a maximizer, according to Theorem 4.7. But in this example (and also the following ones) the profit function has the properties

$$P(0) < 0\,, \quad \lim_{x \to \infty} P(x) = -\infty\,, \quad P(x) > 0 \quad \text{for some } x > 0. \tag{4.14}$$

In this case, Theorem 4.2 implies the existence of a maximizer with positive profit, and therefore any critical point which maximizes P *among the critical points* has to be a maximum of P on all of $[0, \infty)$.

Example 4.10 A company producing car components estimates that the cost (in Indian rupees) of producing x units of a certain component is given by

$$C(x) = 0.0001x^2 + 0.05x + 200\,.$$

1. Find the total as well as the average and marginal cost of producing 500 units and 1000 units.
2. Compare the marginal cost of producing 1000 units with the cost of producing the 1001st unit.

Solution:

1. The average and marginal costs are

$$A(x) = \frac{C(x)}{x} = 0.0001x + 0.05 + \frac{200}{x}\,, \quad C'(x) = .0002x + 0.05\,.$$

   The cost for producing 500 units is $C(500) = 250$ Indian rupees. The cost for producing 1000 units is $C(1000) = 350$ rupees. The average cost equals 0.50 for 500 units and 0.35 for 1000 units rupees per unit. The marginal costs for 500 units and 1000 units are 0.15 and 0.25 rupees per unit, respectively.
2. For 1001 units, we have

$$C(1001) = 200 + 0.05 \cdot 1001 + 0.0001x^2 \cdot 1001 \simeq 350.25\,.$$

   The cost of producing the 1001st unit becomes

$$C(1001) - C(1000) \simeq 350.25 - 350 = 0.025 = C'(1000)\,,$$

   thus it is very close to the marginal cost at the level $x = 1000$.

Example 4.11 A liquid form of medicine manufactured by a pharmaceutical company is sold in bulk at a price of 200 Indian rupees per unit (one bottle). The total production costs (in Indian rupees) for x units are

$$C(x) = 500{,}000 + 80x + 0.003x^2 .$$

Moreover, the production capacity of the firm is limited to 30,000 units during some specified time interval. How many units of medicine must be manufactured and sold to maximize the profit during that time interval?

Solution: Since the total revenue for selling x units is $R(x) = 200x$, the profit $P(x)$ on x units will be

$$P(x) = R(x) - C(x) = 200x - (1500{,}000 + 80x + 0.003x^2) . \tag{4.15}$$

Since the production capacity is limited 30,000 units, x must lie in the interval $I = [0, 30, 000]$. By (4.15)

$$P'(x) = 200 - (80 + 0.006x) = 120 - 0.006x .$$

Setting $P'(x) = 0$ gives $120 - 0.006x = 0$ or $x = 20{,}000$ as the only critical point of P in I. The maximizer of P on I, which exists by Theorem 4.2, must be either this critical point or one of the end points of I. Substituting these values into (4.15) we obtain $P(0) < 0$ (which we may ignore) and

$$P(20{,}000) = 700{,}000 , \quad P(30{,}000) = 400{,}000 .$$

This tells us that the maximal profit $P = 700{,}000$ occurs when $x = 20{,}000$ units are manufactured and sold during the specified time.

Example 4.12 Find the quantity which maximizes the profit if the total revenue and total cost in Indian rupees are given by $R(x) = 5x - 0.003x^2$ and $C(x) = 300 + 1.1x$, respectively. Moreover, production is restricted to at most 1000 units. Find the production levels at which profit is maximal and at which the profit is minimal.

Solution: We again determine the critical points of

$$P(x) = (5x - 0.003x^2) - (300 + 1.1x) = -300 + 3.9x - 0.003x^2 \tag{4.16}$$

from

$$0 = P'(x) = 3.9 - 0.006x .$$

This equation has the unique solution $x = \frac{3.9}{0.006} = 650$. The possible production levels lie in the interval $I = [0, 1000]$. By Theorem 4.2, the profit function P has a maximizer and a minimizer on I. They are to be found among the critical points and the end points. Substituting their values into (4.16) yields

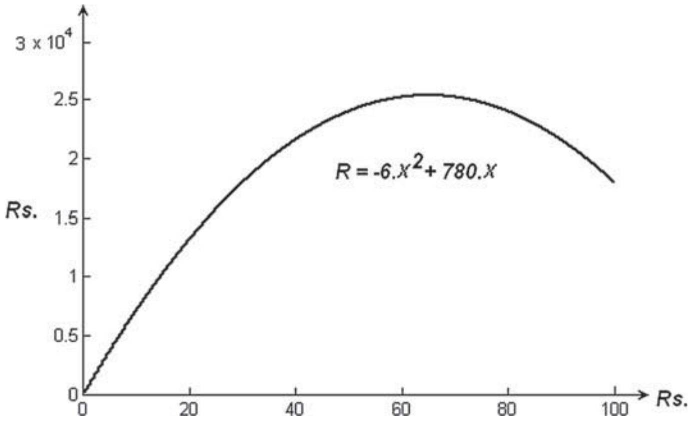


Fig. 4.5 Revenue of a travel agency as a function of price

$$P(0) = -300\,, \quad P(650) = 967.50\,, \quad P(1000) = 600\,.$$

Therefore, maximal profit occurs at a production level of 650 units, and minimal profit (in this case a loss) at $x = 0$ when there is no production at all.

Example 4.13 A travel agency knows that at a price of 80 Indian rupees for a half-day trip, they attract 300 customers. For every decrease of 5 rupees in price, they attract approximately 30 additional customers. What price should the agency charge in order to maximize its revenue?

Solution: This situation is different from the one in the examples above. We first set up a so-called **demand function** $D(x)$ which tells us the total number of customers at price level x. In order to understand its behavior, we make the following table:

Price x	No. of customers $D(x)$
80	300
75	330
70	360
65	390

These points all lie on the straight line with slope

$$\frac{300 - 330}{80 - 75} = -\frac{30}{5} = -6\,.$$

This line satisfies the equation

$$D(x) = -6x + 780\,,$$

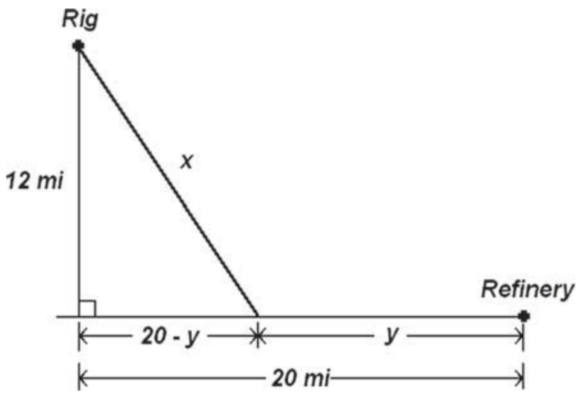


Fig. 4.6 Optimizing the pipe location

where we have determined the constant 780 from the condition $D(80) = 300$. The revenue of the agency is the product of the price (per customer) and the number of customers (Fig. 4.5),

$$R(x) = x \cdot D(x) = x(-6x + 780) = -6x^2 + 780x\,.$$

To find the maximal revenue, we differentiate the revenue function and find its critical points,

$$0 = R'(x) = -12x + 780\,, \quad \text{so} \quad x = \frac{780}{12} = 65\,.$$

The agency achieves the maximal revenue when it sets the price at 65 rupees.

Example 4.14 (*Piping oil from a drilling-rig to a refinery*) A drilling-rig positioned 12 km offshore is to be connected by a pipe to a refinery, located on the shoreline at 20 km distance from the coastal point closest to the rig. An underwater pipe costs 50,000, a land-based pipe 30,000 Indian rupees per km. What combination of the two will give the least expensive connection?

Solution: Figure 4.6 describes the geometrical situation, the horizontal line being the shoreline. Let x and y denote the length of underwater pipe and the length of land-based pipe, respectively. The Pythagorean theorem gives $x^2 = 12^2 + (20 - y)^2$,

$$x = \sqrt{144 + (20 - y)^2}\,. \tag{4.17}$$

The cost c of the whole pipeline is

$$c = 50{,}000x + 30{,}000y\,.$$

Substituting x from (4.17) yields

$$c(y) = 50{,}000\sqrt{144 + (20 - y)^2} + 30{,}000y\,.$$

We now find the minimal value of $c(y)$ on the interval $0 \le y \le 20$. The first derivative of c with respect to y is

$$\begin{aligned} c' &= 50{,}000 \cdot \frac{1}{2} \cdot \frac{2(20-y)(-1)}{\sqrt{144+(20-y)^2}} + 30{,}000 \\ &= -50{,}000 \frac{20-y}{\sqrt{144+(20-y)^2}} + 30{,}000\,. \end{aligned}$$

Setting c' equal to zero gives

$$50{,}000(20-y) = 30{,}000\sqrt{144+(20-y)^2}\,.$$

In order to solve this equation for y, we successively transform it as follows:

$$\begin{aligned} \frac{5}{3}(20-y) &= \sqrt{144+(20-y)^2}\,, \\ \frac{25}{9}(20-y)^2 &= 144+(20-y)^2\,, \\ \frac{16}{9}(20-y)^2 &= 144\,, \\ 20-y &= \pm\frac{3}{4}\cdot 12 = \pm 9\,, \\ y = 11 \quad &\text{or} \quad y = 29\,. \end{aligned}$$

Only $y = 11$ lies in the interval [0, 20]. The value of c at this critical point and at the end points are

$$\begin{aligned} c(11) &= 1{,}080{,}000 \\ c(0) &= 1{,}166{,}190 \\ c(20) &= 1.200{,}000\,. \end{aligned}$$

The least expensive connection costs 1,080,000 rupees and is achieved by running the line underwater to the point on the shore at 11 km distance from the refinery.

Example 4.15 (*The Stock Market*) The graph in Fig. 4.7 shows a hypothetical version of the Sensex Average. The Sensex Average is a stock market index, capturing the overall increase of the stock market along with local dips and rises.

One way to invest in the stock market is to buy shares of an index fund, which in turn buys a number of different stocks with the goal of tracking the index. The goal of an index fund director would certainly be to buy low (for example, at local minima) and sell high (for example, at local maxima). We may also attach a meaning to the curvature of this graph. On a convex part of the curve, the growth rate increases with time, while on the concave part it decreases. The inflection points mark the times where the growth rate changes this behavior, so one may view it as a signal of a trend reversal.

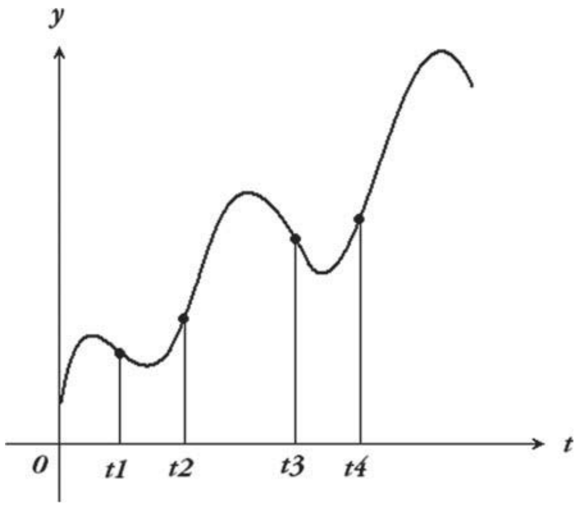


Fig. 4.7 Hypothetical version of Sensex

Other Applications

Example 4.16 (*Fabricating a Box*) A box open on top is made by cutting small congruent squares from the corners of a 12 by 12 cm sheet of tin and bending up the sides. How large should the squares be which we cut from the corners, in order to maximize the volume of the box?

Solution: Let the corner squares have x cm side length. By construction, the volume of the box becomes a function of x, namely (Fig. 4.8)

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3 \,. \tag{4.18}$$

Since the sides of the sheet of tin are 12 cm long, x must lie in the interval $[0, 6]$. Obviously, $V(x) > 0$ for $0 < x < 6$, and $V(0) = 0 = V(6)$, so the end points $x = 0$ and $x = 6$ are minima. By Theorem 4.2 there exists a maximum, which in this case has to be in the interior. Indeed, the graph of V (see Fig. 4.9) suggests a maximum near $x = 2$. We compute first derivative of V and obtain

$$V'(x) = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x) \,.$$

It has only one zero in the interior of $[0, 6]$, which, therefore, must be the maximum. Since $V(2) = 128$, the maximal volume of the box is $128\,\text{cm}^3$, and the squares cut out should have a side length of 2 cm.

Example 4.17 (*Designing an Efficient Oil Can*) You have to design a 1 L oil can shaped like a right circular cylinder (see Fig. 4.10). How should we choose the radius r and height h in order to use the least amount of material?

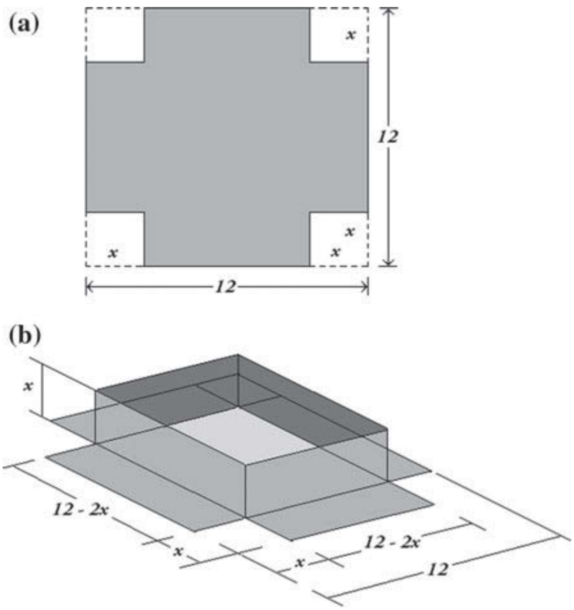


Fig. 4.8 An open box made by cutting the corners from a square sheet

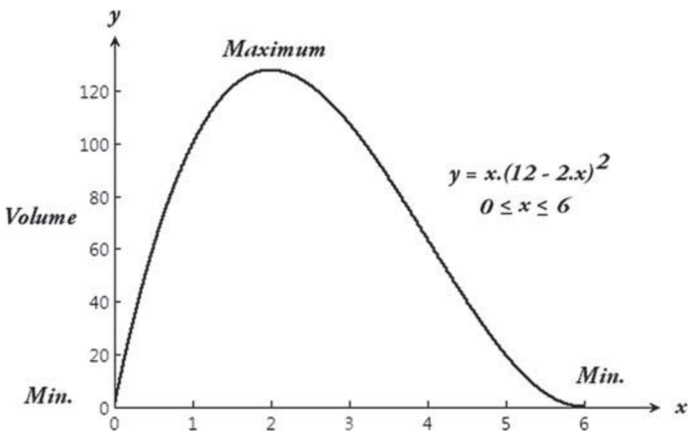


Fig. 4.9 Maximizing box volume

Solution: Because the volume should be equal to 1 L, we must have

$$\pi r^2 h = 1000\,,$$

if r and h are measured in centimeters. The surface area of the can is given by

$$A = 2\pi r^2 + 2\pi r h\,. \tag{4.19}$$

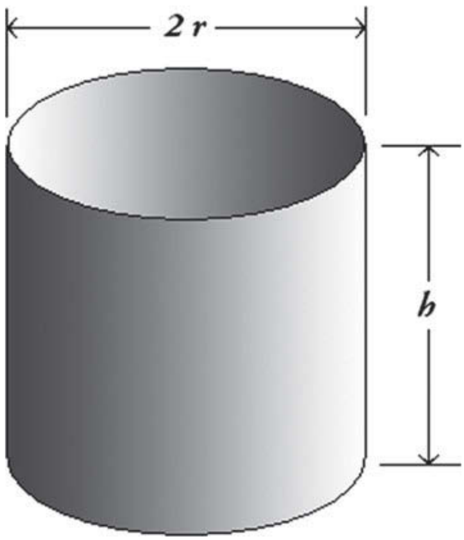


Fig. 4.10 Oil can to be designed

We assume that the amount of material we have to use is proportional to the surface area. We thus have to minimize A subject to the volume constraint $\pi r^2 h = 1000$. In order to remove the constraint from the computation, we express h in terms of r,

$$h = \frac{1000}{\pi r^2},$$

and substitute that expression into the surface area formula. Using (4.19) we obtain

$$A(r) = 2\pi r^2 + 2\pi r \cdot \left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}.$$

Our aim is to find the value of $r > 0$ that minimizes the value of A. Figure 4.11 suggests that such a value exists.

For small r (a tall thin container, like a piece of pipe), the term $2000/r$ dominates and A is large. For large r (a short wide container, like a pizza pan), the term $2\pi r^2$ dominates and A again is large. To compute the minimum, we set the derivative of A equal to zero,

$$0 = A'(r) = 4\pi r - \frac{2000}{r^2}.$$

Rearranging the terms, we obtain

$$4\pi r^3 = 2000, \quad r = \sqrt[3]{\frac{500}{\pi}} = 5.42.$$

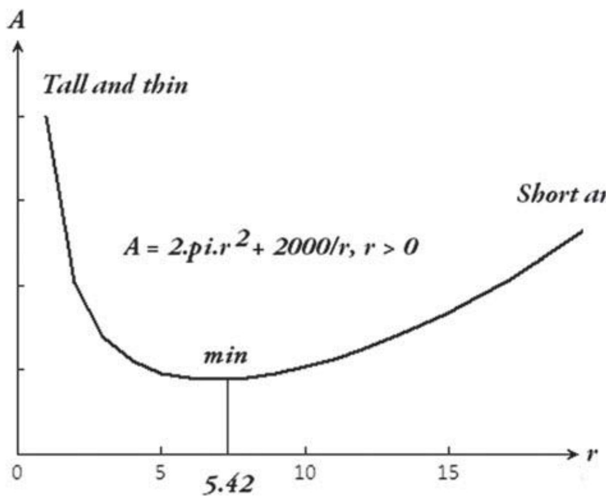


Fig. 4.11 Minimizing surface area at constant volume

In order to check the type of this critical point, we compute the second derivative,

$$A''(r) = 4\pi + \frac{4000}{r^3} \, .$$

It is positive throughout the domain $(0, \infty)$ of A. Therefore, A is convex and the critical point must be a global minimum. The corresponding value of h is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r \, .$$

The 1 L can that uses the least amount of material has height equal to its diameter, with $r = 5.42$ cm and $h = 10.84$ cm.

Example 4.18 (*Inscribing Rectangles*) A rectangle is to be inscribed into a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution: We place the circle and the rectangle in the plane as seen in Fig. 4.12. The lower right corner of the rectangle has coordinates $(x, 0)$, the upper right corner has coordinates $(x, \sqrt{4 - x^2})$. In terms of x, the rectangle has length $2x$, height $\sqrt{4 - x^2}$ and area

$$A(x) = 2x\sqrt{4 - x^2} \, . \tag{4.20}$$

Notice that the values of x are restricted to the interval $0 \le x \le 2$, since the rectangle has to lie inside the semicircle. Our goal is to find the global maximum of the function A from (4.20) on the domain $[0, 2]$, whose existence is guaranteed by Theorem 4.2.

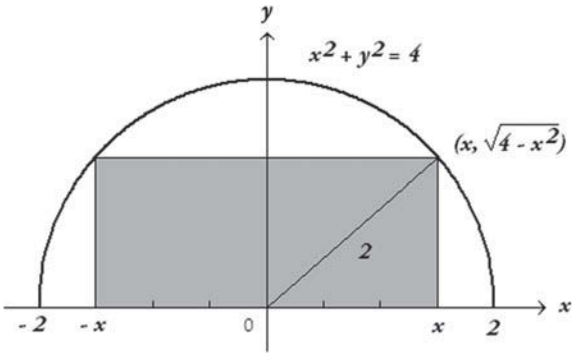


Fig. 4.12 Rectangle and semicircle

We have $A(x) > 0$ for $0 < x < 2$ and $A(0) = A(2) = 0$. The maximum, therefore, must be a critical point of the derivative

$$A'(x) = -\frac{2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

with $0 < x < 2$. Rearranging the equation

$$\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} = 0$$

we obtain

$$\begin{aligned} -2x^2 + 2(4 - x^2) &= 0 \\ 8 - 4x^2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2}. \end{aligned}$$

Of the two zeros $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ satisfies the restriction $0 < x < 2$, and

$$A(\sqrt{2}) = 2\sqrt{2}\sqrt{4 - 2} = 4\,.$$

The area has a maximal value of 4, the corresponding height is $\sqrt{4 - x^2} = \sqrt{2}$ and the length is $2x = 2\sqrt{2}$.

4.6 Exercises

4.6.1 Find the local extrema of f on the given interval

(i) $f(x) = \sec \frac{1}{2}x$; $[-\frac{\pi}{2}, \frac{\pi}{2}]$,
(ii) $f(x) = \tan x - 2\sec x$; $[-\frac{\pi}{4}, \frac{\pi}{4}]$,
(iii) $f(x) = \sin x - \cos x$; $[0, \pi]$,
(iv) $f(x) = |6 - 4x|$; $[-3, 3]$.

4.6.2 Find the global (absolute) extrema of f on $(-\infty, \infty)$.

(i) $f(x) = 3 - 4x - 2x^2$,
(ii) $f(x) = x^3 - 3x - 2$.

4.6.3 Show that among all rectangles with perimeter p, the square has the maximum area.

4.6.4 Show that among all rectangles with area A, the square has the minimum perimeter.

4.6.5 Find the point on the graph of $y = x^2 + 1$ that is closest to the point (3, 1).

4.6.6 Find the point on the graph of $y = x^3$ that is closest to the point (4, 0).

4.6.7 A pipeline for transporting oil will connect two points A and B that are 3 km apart and on opposite banks of a straight river 1 km wide (see Fig. 4.13). Part of the pipeline will run under water from A to a point C on the opposite bank, and then above ground from C to B. If the cost per km of running the pipeline underwater is four times the cost per km of running it above ground, find the location of C that will minimize the cost. (The slope of the of the river bed should be disregarded.)

4.6.8 Let $f(x) = x^2 + px + q$. Find the value of p and q such that $f(1) = 3$ is an extreme value of f on [0, 2]. Is this value a maximum or minimum?

4.6.9 Show that

$$f(x) = \frac{64}{\sin x} + \frac{27}{\cos x}$$

has a minimum value, but no maximum value on the interval $(0, \pi/2)$.

4.6.10 Find the extrema of f on the given interval

(a) $f(x) = -2x^3 - 6x^2 + 5$; $[-3, 1]$,
(b) $f(x) = x^4 - 5x^2 + 4$; $[0, 2]$.

4.6.11 A window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 15 m, find the dimensions that will allow the maximum amount of light to enter.

4.6.12 A farmer has 500 m of fencing to enclose a rectangular field. A barn will use part of one side of the field. Prove that the area of the field is maximal when the rectangle is a square.

4.6.13 Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius a, if two vertices lie on the diameter.

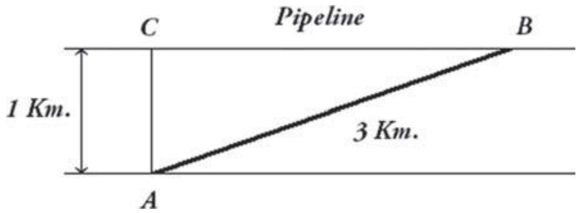


Fig. 4.13 Optimal location of a pipeline

4.6.14 A rectangle has its two lower corners on the x-axis and its two upper corners on the curve $y = 16 - x^2$. For all such rectangles, what are the dimensions of the one with largest area?

4.6.15 Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10.

4.6.16 Let the number of bacteria in a culture at time t be given by $N = 5000(25 + te^{-t/20})$.

(a) Find the largest and smallest number of bacteria in the culture during the time interval $0 \le t \le 100$.

(b) At what time during the time interval in part (a) is the number of bacteria decreasing most rapidly?

4.6.17 Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

Chapter 5
Sequences and Series

The concepts of sequence and series provide another basic tool of calculus. One may use them, among other things, to approximate functions by comparatively simple formulas. In 1668, Mercator published the formula for the logarithmic series. Since that time, series have been used for countless purposes in science and engineering. We give here a brief introduction into the definition and convergence properties of sequences and series, in particular, power series and Taylor series. Later in Chap. 10, we will treat Fourier series, a special type of series involving trigonometric functions.

5.1 Sequences and Their Limits

Everyone is familiar with the sequence 1, 2, 3, 4,... of positive integers, or with the sequence 2, 4, 6, 8,... of positive even integers. In the latter case, 2 is the first element of the sequence, 4 the second, and so on. If we denote the n-th element of this sequence by a_n, there is a simple formula for it, namely, $a_n = 2n$. In this manner, we may obtain other sequences of numbers as well, for example

$$a_n = \sqrt{n}, \qquad a_n = \frac{1}{n}, \qquad a_n = \frac{1}{n^2}, \tag{5.1}$$

$$a_n = (-1)^n, \qquad a_n = \frac{n-1}{n}, \qquad a_n = (-1)^{n+1}\frac{1}{n}. \tag{5.2}$$

In fact, a sequence of numbers is a special type of a function, namely, a function whose domain is the set of integers (usually the positive integers, with or without 0), and whose range is a subset of the real numbers. In other words, we can define a sequence as a function $f : \mathbb{N} \to \mathbb{R}$. Usually, however, sequences are denoted by $a_1, a_2, \ldots$ or briefly $\{a_n\}$, or with a different letter like $\{b_n\}$ or $\{s_n\}$. The elements a_n of the sequence $\{a_n\}$ are also called the terms of the sequence. Conceptually, one

M. Brokate et al., *Calculus for Scientists and Engineers*, Industrial and Applied Mathematics, https://doi.org/10.1007/978-981-13-8464-6_5

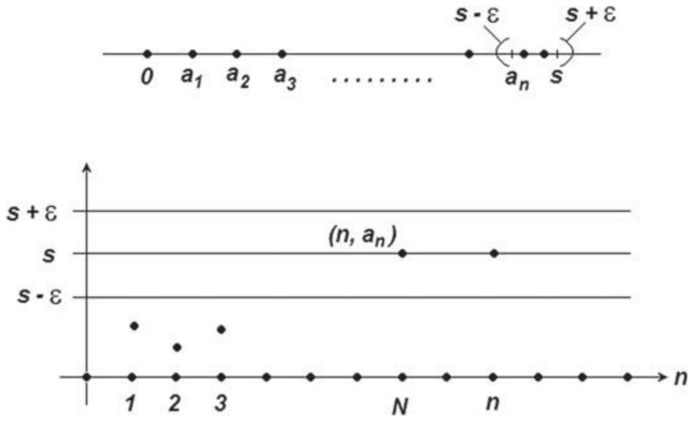


Fig. 5.1 Sequence and limit

has to distinguish between the sequence $a_1, a_2, a_3, \ldots$ and the set of its elements $\{a_n : n \in \mathbb{N}\} = \{a_1, a_2, \ldots\}$.[1] For example, for the sequence defined by $a_n = (-1)^n$, we have $\{a_n : n \in \mathbb{N}\} = \{-1, 1\}$.

Since sequences are special cases of functions, many of the notions developed in Chap. 1 for functions also apply to sequences. For example, $\{a_n\}$ is an increasing sequence whenever $a_n < a_{n+1}$ holds for all positive integers n (which is the same as saying that $a_n < a_m$ whenever $n < m$). Analogously, the notion of a decreasing sequence, a constant sequence, and a bounded sequence is introduced. The algebraic operations on sequences are defined elementwise. For example, the product of two sequences $\{a_n\}$ and $\{b_n\}$ is defined as the sequence whose n-th element is $a_n b_n$, compare Sect. 1.5 on the algebra of functions.

As for functions, the notion of a limit of a sequence is most important.

Definition 5.1 (*Convergence, Divergence, and Limit*) The sequence $\{a_n\}$ **converges** to the number s if to every positive number ε there corresponds an integer N such that

$$|a_n - s| < \varepsilon, \qquad \text{whenever } n > N.$$

In this case, we call $\{a_n\}$ a **convergent sequence**. If no such s exists, we call $\{a_n\}$ a **divergent sequence**. If $\{a_n\}$ converges to s, we write $\lim_{n\to\infty} a_n = s$, or simply $a_n \to s$, and call s the **limit** of the sequence.

Figure 5.1 illustrates two ways of describing limits geometrically. One may plot the points a_n on the real line in order to see how they approach the limit point s. Or one may plot the points (n, a_n). We have $a_n \to s$ if the line $y = s$ is a horizontal asymptote of those points. In this figure, all a_n's after a_N lie within the distance ε of s.

[1]This corresponds to the distinction between a function f and its range $R(f)$.

Example 5.1 1. We have $\lim_{n\to\infty} \frac{1}{n} = 0$, that is, the limit of the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$ is zero.

2. Consider the constant sequence with $a_n = k$ for all n, where k is a given number. Then $\lim_{n\to\infty} a_n = k$.
3. Let $a_n = 1 + \frac{1}{n}$, then $\lim_{n\to\infty} a_n = 1$.

Several properties of limits of functions, as developed in Chap. 2, hold for limits of sequences in an analogous manner. We present two such results.

Theorem 5.1 *Let $\{a_n\}$ and $\{b_n\}$ be sequences with $a_n \to s$ and $b_n \to t$. Then the sequences $\{a_n + b_n\}$, $\{a_n - b_n\}$, $\{a_n b_n\}$, and $\{a_n/b_n\}$ converge to $s + t$, $s - t$, st, and s/t, respectively, the latter if $t \neq 0$.*

Theorem 5.2 (Sandwich Theorem for sequences) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all n. If $a_n \to s$ and $c_n \to s$, then also $b_n \to s$.*

Continuity of functions is related to limits of sequences.

Theorem 5.3 *Let $\{s_n\}$ be a sequence of real numbers with $s_n \to s$. If f is a function that is continuous at s and defined at all s_n, then $f(s_n) \to f(s)$.*

In fact, a converse of Theorem 5.3 also holds: If $f(s_n) \to f(s)$ for all sequences $\{s_n\}$ which converge to a given point s, then f is continuous at s.

We may also express the limit of a sequence as an improper limit of a function,[2] see also Fig. 5.1.

Theorem 5.4 *Let f be a function with $f(x)$ defined for all $x \geq N$ with N given, let $\{a_n\}$ be a sequence of real numbers such that $a_n = f(n)$ for $n \geq N$. Then $\lim_{x\to\infty} f(x) = s$ implies $\lim_{n\to\infty} a_n = s$.*

We may exploit Theorem 5.4 in order to compute limits of sequences.

Example 5.2 Find the limit of the sequences

1. $\left\{\dfrac{\ln n}{n}\right\}$,
2. $\left\{(n)^{1/n}\right\}$,
3. $\left\{\left(\dfrac{n+1}{n-1}\right)^n\right\}$.

Solution:

1. The function $f(x) = \dfrac{\ln x}{x}$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. By Theorem 5.4, $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x}$ holds if the limit on the right-hand side exists. By l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

[2] See Sect. 2.4.

Therefore, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

2. Since

$$(n)^{\frac{1}{n}} = e^{\frac{\ln n}{n}},$$

we obtain from the previous example and Theorem 5.3 that $(n)^{1/n} \to e^0 = 1$, because the exponential function is continuous.

3. Let $a_n = \left(\frac{n+1}{n-1}\right)^n$. The limit is of indeterminate form ∞/∞. In order to apply l'Hôpital's rule, we change it to the form $\infty \cdot 0$ by taking the natural logarithm of a_n,

$$\ln a_n = \ln \left(\frac{n+1}{n-1}\right)^n = n \ln \left(\frac{n+1}{n-1}\right),$$

and set

$$f(x) = x \ln \left(\frac{x+1}{x-1}\right).$$

Then

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} x \ln \left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{1/x} \\
&= \lim_{x\to\infty} \frac{-2/(x^2-1)}{-1/x^2} \qquad \text{by l'Hôpital's rule} \\
&= \lim_{x\to\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\infty} \frac{2}{1-(1/x^2)} = 2.
\end{aligned}$$

By Theorem 5.4, $\lim_{n\to\infty} \ln a_n = \lim_{x\to\infty} f(x) = 2$. Since $f(x) = e^x$ is continuous, by Theorem 5.3 we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^2$. Therefore,

$$\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = e^2.$$

In the example $a_n = 1/n$, the distance of a_n to the limit 0 decreases monotonically with increasing n. The example $a_n = \ln n/n$ illustrates another possible behavior of a convergent sequence. Namely, we have $a_1 = 0 < a_2 = 0.347\cdots < a_3 = 0.366\ldots$, and $a_3 > a_4 > \ldots$, so the sequence at first moves away from its limit 0 before it comes closer and closer to it. Moreover, a convergent sequence may oscillate around its limit, like, for example

$$a_n = \begin{cases} (-1)^n \frac{1}{n}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Its limit is 0, but it moves away from zero in both directions an infinite number of times (but by smaller and smaller amounts).

Note also that if we add, delete, or alter a finite number of terms in a given sequence, its convergence behavior does not change—it will still converge to the same limit as before, or it will still be divergent.

In the examples considered so far, sequences $\{a_n\}$ were defined by specifying the n-th element (or term) a_n with a formula where we only have to insert n, like $a_n = 2n$. Alternatively, we may specify a_n with a formula which involves elements a_m with $m < n$, for example, $a_n = a_{n-1} + 1$. In that case, if we know a_1 (say, $a_1 = 1$), we can compute the elements a_n successively, since $a_2 = a_1 + 1 = 2$, $a_3 = a_2 + 1 = 3$, and so on.

Definition 5.2 (*Recursively Defined Sequences*) A **recursive definition** of a sequence consists of

1. the value(s) of the initial term or terms,
2. a rule, called a recursive formula, for calculating any later term from terms that precede it.

Note that for a given sequence we may have a direct definition as well as a recursive definition. For example, the sequence 1, 2, 3,... can be obtained by prescribing $a_n = n$, or from the recursive definition $a_1 = 1$, $a_n = a_{n-1} + 1$. Note also that the recursive definition could also be written as $a_1 = 1$, $a_{n+1} = a_n + 1$.

Example 5.3 1. Let $a_1 = 1$ and $a_n = na_{n-1}$. This is a recursive definition of the sequence whose terms are $a_1 = 1$, $a_2 = 2a_1 = 2$, $a_3 = 3a_2 = 3 \cdot 2 = 6$, $a_4 = 4 \cdot a_3 = 4 \cdot 6 = 24$, $a_5 = 5 \cdot a_4 = 5 \cdot 24 = 120$, and so on. A direct definition of this sequence is $a_n = n!$, the factorial of n.

2. Let $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$. The sequence generated from this recursive definition is known as the *Fibonacci sequence*, and the terms a_n of this sequence are called the *Fibonacci numbers*. Note that two initial terms are needed, since the recursive formula uses the two preceding terms. The next terms after a_2 are $a_3 = a_2 + a_1 = 2$, $a_4 = a_3 + a_2 = 2 + 1 = 3$, $a_5 = a_4 + a_3 = 3 + 2 = 5$, and $a_6 = a_5 + a_4 = 5 + 3 = 8$. There is also a direct definition of this sequence, by the formula of Moivre–Binet

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

3. Let $a_1 = -2$, $a_{n+1} = \dfrac{na_n}{(n+1)}$. This recursive formula defines a sequence whose terms are

$$a_1 = -2, \quad a_2 = \frac{a_1}{2} = -1, \quad a_3 = \frac{2a_2}{3} = -\frac{2}{3}, \quad a_4 = \frac{3a_3}{4} = -\frac{1}{2}, \quad \ldots$$

5.2 Infinite Series

General definition. Given a sequence $\{s_n\}$ of numbers, an expression of the form

$$s_1 + s_2 + s_3 + \cdots + s_n + \cdots \tag{5.3}$$

is called an **infinite series** or simply **series**. The number s_n is called the n-th term of the series. The finite sums S_n

$$\begin{aligned} S_1 &= s_1 \\ S_2 &= s_1 + s_2 \\ S_3 &= s_1 + s_2 + s_3 \\ \cdots \\ S_n &= \sum_{k=1}^{n} s_k \end{aligned}$$

are called **partial sums** of the series. If the sequence $\{S_n\}$ formed by the partial sums has a limit S, we say that the series (5.3) **converges**, we call S the **sum** of the series and write

$$s_1 + s_2 + s_3 + \cdots + s_n + \cdots = \sum_{k=1}^{\infty} s_k = S. \tag{5.4}$$

If the sequence $\{S_n\}$ diverges, we say that the series diverges.

Since it is convenient, it is the standard custom to write the series, too, in sigma notation. Therefore, expressions like

$$\sum_{n=1}^{\infty} s_n, \quad \sum_{k=1}^{\infty} a_k, \text{ or } \sum a_n$$

may refer either to the series itself (a collection of terms) or to its sum (a number). However, usually the meaning is clear from the context.

Geometric Series. A series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \tag{5.5}$$

where a and r are fixed real numbers with $a \neq 0$ is called **geometric series**. The ratio r can be positive or negative, for example, r is positive in the geometric series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{r-1}} + \cdots$$

while it is negative in

$$1 - \frac{1}{3} + \frac{1}{3^2} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots$$

If $|r| \neq 1$, we can determine the convergence or divergence of the geometric series in the following way. We start with the partial sums

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1},$$
$$r S_n = ar + ar^2 + \cdots + ar^n.$$

Subtracting $r S_n$ from S_n, we obtain

$$S_n(1-r) = a(1-r^n),$$

and dividing both sides by $1 - r$, we arrive at

$$S_n = \frac{a(1-r^n)}{1-r}. \quad \text{(recall } r \neq 1.)$$

Three cases arise

- If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ and so $\lim_{n\to\infty} S_n = \dfrac{a}{1-r}$.
- If $|r| > 1$, then $|r^n| \to \infty$ as $n \to \infty$ and the series diverges.
- If $r = 1$, we obtain

$$S_n = a + a \cdot 1 + a \cdot 1^2 + \cdots + a \cdot 1^{n-1} = na,$$

so the series diverges, as $\lim_{n\to\infty} S_n = \pm\infty$ depending on sign of a. If $r = -1$, the series diverges because the S_n alternate between the values a and 0.

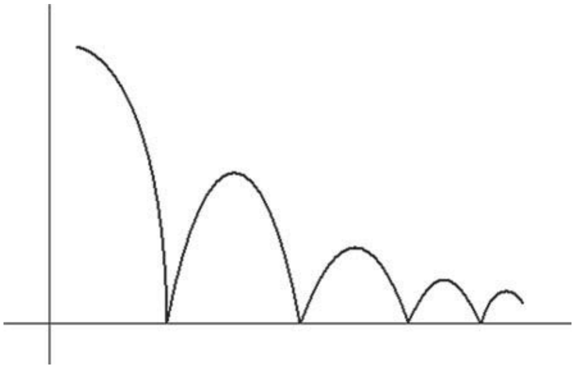
In short, the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to the sum $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Example 5.4 Let a ball be dropped from a meters above a flat surface. Each time the ball hits the surface after falling from a height h, it rebounds to the height rh, where r is positive but less than 1. Find the total distance the ball travels up and down (Fig. 5.2).

Solution: The total distance is

$$\begin{aligned} S &= a + 2ar + 2ar^2 + 2ar^3 + \cdots = a + \frac{2ar}{1-r} \\ &= a\frac{1+r}{1-r}. \end{aligned}$$

Fig. 5.2 A bouncing ball



If a equals 6 m and $r = 2/3$, for example, the distance is

$$S = 6 \cdot \frac{1 + 2/3}{1 - 2/3} = 6 \cdot \frac{5/3}{1/3} = 30\,\text{m}.$$

Example 5.5 Examine whether the following series converge or diverge:

1. $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$
2. $-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} \cdots - \left(-\frac{1}{2}\right)^{n-1} \cdots$
3. $\sum_{k=0}^{\infty} \left(\frac{3}{7}\right)^{k} = \sum_{k=1}^{\infty} \left(\frac{3}{7}\right)^{k-1}$
4. $\frac{\pi}{2} + \frac{\pi^2}{4} + \frac{\pi^3}{8} + \cdots$

Solution:

1. The series converges as $0 < r = \frac{1}{2} < 1$, its sum equals

$$S = \frac{5}{1 - (1/2)} = 10.$$

2. We have $a = -1$ and $r = -\frac{1}{2}$, so $|r| < 1$. The series converges to

$$S = \frac{-1}{1 - (-1/2)} = -\frac{2}{3}.$$

3. We have $a = 1$ and $r = \frac{3}{7}$, the series converges to

$$S = \frac{1}{1 - 3/7} = \frac{7}{4}.$$

4. The series diverges as $r = \frac{\pi}{2} \approx \frac{22}{14} > 1$.

General properties of series.

1. If $\sum_{n=1}^{\infty} s_n$ converges, then $s_n \to 0$ as $n \to \infty$.
2. $\sum_{n=1}^{\infty} s_n$ diverges if $\lim_{n\to\infty} s_n$ fails to exist or is different from zero.
3. Addition, deletion, or alteration of a finite number of terms of an infinite series does not change its nature (convergence or divergence). However, it may change its sum.
4. The series $s_1 + s_2 + s_3 + s_4 + \cdots + s_n + \cdots$, which we usually write as $\sum_{n=1}^{\infty} s_n$, can also be written as $\sum_{n=1+m}^{\infty} s_{n-m}$ or $\sum_{n=1-m}^{\infty} s_{n+m}$, with an arbitrary positive integer m. For example, the expressions

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \quad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}} \text{ and } \sum_{n=-3}^{\infty} \frac{1}{2^{n+3}}$$

denote the same series, namely, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$..

We already know that limits of sequences can be interchanged with addition, subtraction, and scalar multiplication. The same is true for series.

Theorem 5.5 *Consider two convergent series, $\sum_n s_n = S$ and $\sum_n t_n = T$, let k be a given real number. Then the series $\sum_n (s_n + t_n)$, $\sum_n (s_n - t_n)$, and $\sum_n ks_n$ also converge, and*

$$\sum_n (s_n + t_n) = \sum_n s_n + \sum_n t_n = S + T, \tag{5.6}$$

$$\sum_n (s_n - t_n) = \sum_n s_n - \sum_n t_n = S - T, \tag{5.7}$$

$$\sum_n ks_n = k \sum_n s_n = kS. \tag{5.8}$$

Convergence tests for series with nonnegative terms. It is often useful to know whether a given series converges or diverges, even if one does not (or not yet) know its sum. For this purpose, a variety of criteria have been developed.

Theorem 5.6 (Comparison principle) *Let $\sum_n s_n$ and $\sum_n t_n$ be series with $0 \le t_n \le s_n$ for all $n \ge N$, where N is a given positive integer. Then the following assertions hold:*

1. *If $\sum_n s_n$ converges, then $\sum_n t_n$ converges.*
2. *If $\sum_n t_n$ diverges, then $\sum_n s_n$ diverges.*

Theorem 5.7 (Limit comparison test) *Let $\sum_n s_n$ and $\sum_n t_n$ be series with $s_n > 0$ and $t_n > 0$ for all $n \ge N$, where N is a given positive integer. Then the following assertions hold:*

1. *If $\lim_{n\to\infty} \dfrac{s_n}{t_n} = \lambda$ for some $0 < \lambda < \infty$, then the series $\sum_n s_n$ and $\sum_n t_n$ either both converge or both diverge.*

2. *If* $\lim_{n\to\infty} \dfrac{s_n}{t_n} = 0$ *and* $\sum t_n$ *converges, then* $\sum_n s_n$ *converges.*
3. *If* $\lim_{n\to\infty} \dfrac{s_n}{t_n} = \infty$ *and* $\sum t_n$ *diverges, then* $\sum s_n$ *diverges.*

Theorem 5.8 (Ratio test) *Let* $\sum s_n$ *be a series with* $s_n > 0$ *for all* n *and suppose that*

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \rho. \tag{5.9}$$

Then the following assertions hold:

1. *The series converges if* $\rho < 1$.
2. *The series diverges if* $\rho > 1$ *or* ρ *is infinite.*
3. *The test is inconclusive if* $\rho = 1$.

Let us apply the ratio test to the series $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$, where x is a given real number. We have

$$s_n = \frac{x^n}{n!}, \quad \frac{s_{n+1}}{s_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}, \quad \text{so} \quad \lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 0.$$

Therefore, the series converges for all real numbers x. In this manner, we have shown that the definition

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{5.10}$$

makes sense. Indeed, (5.10) constitutes the standard definition of the exponential function on which the definitions of general exponentials and logarithms are based, see Chap. 1.

Let us add the remark that the ratio test can also be applied if the limit in (5.9) does not exist. In fact, if we can find an integer N and a real number ρ such that

$$\frac{s_{n+1}}{s_n} \le \rho < 1$$

holds for all $n \ge N$, then the series converges.

Theorem 5.9 (Root test) *Let* $\sum s_n$ *be a series with* $s_n \ge 0$ *for all* $n \ge N$*, where* N *is a given positive integer, and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{s_n} = \rho.$$

Then the following assertions hold:

1. *The series converges if* $\rho < 1$.
2. *The series diverges if* $\rho > 1$ *or* ρ *is infinite.*
3. *The test is inconclusive if* $\rho = 1$.

An important series is the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}. \tag{5.11}$$

It is divergent, as can be seen from the comparison principle (Theorem 5.6(2)), if we choose for $\sum_n t_n$ the series

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \underbrace{\left(\frac{1}{8} + \cdots + \frac{1}{8}\right)}_{\text{4 terms}} + \underbrace{\left(\frac{1}{16} + \cdots + \frac{1}{16}\right)}_{\text{8 terms}} + \cdots . \tag{5.12}$$

Indeed, all bracketed sums have the value $1/2$, so the series (5.12) diverges.

Example 5.6 Determine whether the following series converge or diverge:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$,
2. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$,
3. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$,
4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$.

Solution:

1. We see that this series converges when we compare it with the series $\sum \frac{1}{2^n}$ and apply Theorem 5.6(1) or alternatively Theorem 5.7(2). For the former note that $0 \le 1/(n2^n) \le 1/(2^n)$ and that $\sum \frac{1}{2^n}$ is a convergent geometric series, since $r = 1/2 < 1$.
2. The series diverges by Theorem 5.7(1) when we involve the divergent series $\sum \frac{1}{n}$. Let $t_n = \frac{1}{n}$ and $s_n = \frac{2n+1}{n^2+2n+1}$. We have

$$\frac{s_n}{t_n} = \frac{\frac{2n+1}{n^2+2n+1}}{\frac{1}{n}} = \frac{n^2(2+\frac{1}{n})}{n^2(1+\frac{2}{n}+\frac{1}{n^2})} = \frac{2+\frac{1}{n}}{1+\frac{2}{n}+\frac{1}{n^2}},$$

and therefore

$$\lim_{n\to\infty} \frac{s_n}{t_n} = 2,$$

since the three sequences $\frac{1}{n}, \frac{2}{n}, \frac{1}{n^2}$ converge to zero.

3. We apply the ratio test. We have

$$\frac{s_{n+1}}{s_n} = \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}} = \frac{2^{n+1}+5}{3(2^n+5)} = \frac{2+\frac{5}{2^n}}{3(1+\frac{5}{2^n})}.$$

Since $5/2^n \to 0$, we obtain

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \frac{2}{3} < 1.$$

Hence, the series is convergent.

4. It follows from the root test that the series is divergent. Indeed,

$$(s_n)^{1/n} = \left(\frac{2^n}{n^2}\right)^{1/n} = \frac{2}{(n^2)^{1/n}} = \frac{2}{(n^{1/n})^2}.$$

Since we know already that $n^{1/n} \to 1$ (see Example 5.2(2)), we obtain

$$\lim_{n\to\infty} (s_n)^{1/n} = 2 > 1.$$

Example 5.7 Let us consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots ,$$

where p is a nonnegative real number. This series is convergent if $p > 1$ and divergent if $0 \le p \le 1$, as we will see later in Chap. 6. For $p = 1$, the harmonic series arises, whose divergence we have already investigated above. When we consider the sum as a function of p, we obtain the famous **Riemann zeta function**

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}. \tag{5.13}$$

Here, ζ is the Greek letter "zeta". In 1734, the mathematician Euler has found a way to compute $\zeta(p)$ for even integers ζ, for example,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \textit{and} \quad \zeta(6) = \frac{\pi^6}{945}.$$

No simple formulas are known for $\zeta(p)$ when p is an odd integer.

5.3 Alternating Series, Absolute and Conditional Convergence

A series in which the terms are alternately positive and negative is called an **alternating series**. For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \tag{5.14}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots \tag{5.15}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} n = 1 - 2 + 3 - 4 + 5 - 6 + \cdots \tag{5.16}$$

are alternating series. The latter two are obviously divergent (just look at the partial sums). The first one converges, as a consequence of the following criterion.

Theorem 5.10 (Leibniz test for alternating series) *The series* $\sum_{n=1}^{\infty} (-1)^{n+1} s_n = s_1 - s_2 + s_3 - s_4 + \cdots$ *converges if all three of the following conditions are satisfied:*

1. *The terms s_n are nonnegative for all $n \geq N$, where N is a fixed integer.*
2. $s_n \geq s_{n+1}$ *for all* $n \geq N$.
3. $s_n \to 0$.

Moreover, in this case the sum S satisfies $|S - S_n| < s_{n+1}$ for all $n \geq N$, where S_n is the n-th partial sum.

Indeed, setting $s_n = 1/n$ in the theorem we see that the series (5.14) converges.

Definition 5.3 A series $\sum_n s_n$ **converges absolutely** (is absolutely convergent) if the corresponding series of absolute values, $\sum_n |s_n|$, converges. A series that converges but does not converge absolutely is called **conditionally convergent**.

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (as we have seen in the previous section), the series (5.14) furnishes an example of a conditionally convergent series.

Theorem 5.11 *Every absolutely convergent series is convergent.*

Example 5.8 1. We show that the series $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{n^2}$ is absolutely convergent, and hence convergent by Theorem 5.11. Indeed, we have

$$s_n = (-1)^{n+1} \frac{1}{n^2}, \quad |s_n| = \frac{1}{n^2},$$

and the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent according to Example 5.7.

2. We show that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is absolutely convergent. Indeed, we have

$$s_n = \frac{\sin nx}{n^2}, \quad |s_n| = \frac{|\sin nx|}{n^2} \leq \frac{1}{n^2}.$$

Again, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent according to Example 5.7. Now, the comparison principle (Theorem 5.6) implies that $\sum_{n=1}^{\infty} |s_n|$ is convergent.

Theorem 5.12 (Rearrangement of absolutely convergent series) *If $\sum_{n=1}^{\infty} s_n$ converges absolutely and $\{t_n\}$ is any rearrangement of the terms s_n, then $\sum_{n=1}^{\infty} t_n$ converges absolutely, and*

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n.$$

5.4 Power Series

Power series are a special type of infinite series. We will see that in an interval where a power series is convergent, the sum of the power series is a continuous function with derivatives of all orders. We will also examine the opposite question, that is, whether a function $f = f(x)$, which has derivatives of all orders on an interval I, can be expressed in the form of a power series.

Definition 5.4 (*Power series*) An infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

is called a **power series** centered at $x = 0$.
An infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is called a power series centered at $x = a$. Its n-th term is $c_n(x-a)^n$, the number a is the **center**.

Examples of power series. Our first example is the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \tag{5.17}$$

We have seen in Sect. 5.2 that it converges for $|x| < 1$ and has the sum

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{5.18}$$

We now look at this formula the other way round and see that the function $f(x) = 1/(1-x)$ is expressed in form of the power series (5.17), as long as $|x| < 1$.

Other examples of functions expressed by power series are the exponential, sine and cosine functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{5.19}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{5.20}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{5.21}$$

That these formulas are indeed correct will result from the following exposition up to Example 5.9. Moreover, the binomial series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \tag{5.22}$$

provides another example of a power series centered at 0.

Radius of convergence. We already know that the exponential series (5.19) converges for all real numbers x, while the geometric series (5.18) converges for $|x| < 1$ and diverges for $|x| > 1$. Such a behavior is not coincidental. It turns out that for every power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there is a nonnegative real number R (possibly $R = \infty$) such that the power series converges if $|x-a| < R$ and diverges if $|x-a| > R$. The case $R = \infty$ means that the power series converges for all x, the case $R = 0$ means that the series converges only for $x = a$ (which is trivial, since then its sum equals c_0 in any case). If $R > 0$ is finite, the series may or may not converge at either of the points $x = a - R$ or $x = a + R$.

Definition 5.5 The number R as defined above is called the **radius of convergence**, and the interval $(a-R, a+R)$ is called the **interval of convergence** of the power series.

Often, but not always, the radius of convergence can be determined by a variant (due to Euler) of the ratio test. Namely,

$$\text{if } \rho = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} \text{ exists, then } R = \frac{1}{\rho}. \tag{5.23}$$

Using this criterion, one can check that the power series for the sine and cosine functions given above have convergence radius $R = \infty$, while the binomial series (5.22) has $R = 1$.

Let us remark that the domain of definition of a function should not be confused with the radius of convergence of an associated power series. For example, the function $f(x) = 1/(1-x)$ is defined everywhere except at its poles $x = \pm 1$, but its power series (5.17) centered at 0 has $R = 1$, so it does not represent f for $|x| > 1$, that is, across the poles as seen from the center $x = 0$.

Theorem 5.13 (Derivative of power series) *Any function expressed as a power series,*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{5.24}$$

is differentiable (and hence, continuous) within its interval of convergence $(a - R, a + R)$. *Its derivative is obtained term by term,*

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \tag{5.25}$$

and the interval of convergence of (5.25) is again $(a - R, a + R)$.

As an example, for the exponential function we get from (5.19)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \frac{d}{dx} e^x = \sum_{n=1}^{\infty} n \frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = e^x,$$

an alternative way to compute its derivative. In an analogous manner, one may check that $\sin' = \cos$ and $\cos' = -\sin$.

Since by Theorem 5.13, the derivative of a power series has the same interval of convergence as the original series, we may repeat this process to obtain the power series for the second derivative,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}, \tag{5.26}$$

and so on for derivatives of arbitrary order. Inserting $x = a$ into the formulas (5.24)–(5.26), we get

$$f(a) = c_0, \quad f'(a) = c_1, \quad f''(a) = 2c_2.$$

Theorem 5.14 *Any function expressed as a power series,*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{5.27}$$

possesses derivatives $f^{(k)}$ of all orders k valid within its interval of convergence. Moreover, we have

$$f^{(k)}(a) = k!\, c_k. \tag{5.28}$$

Taylor and Maclaurin Series. We have just seen that, within its interval of convergence, a power series defines a function which has derivatives of all orders. Now we reverse this procedure. Let f be a function which has derivatives of all orders on an interval containing a as an interior point. Keeping in mind formula (5.28), we associate with it the power series

$$\sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \tag{5.29}$$

Definition 5.6 The power series (5.29) is called the **Taylor series** for f at a. In the special case $a = 0$, it takes the form

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \tag{5.30}$$

and is also called the **Maclaurin series** for f.

For an arbitrary point x in the domain of f with given $a \in D(f)$ there are three mutually exclusive possibilities:

1. The Taylor series of f in a converges at x, and we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k. \tag{5.31}$$

2. The Taylor series of f in a converges at x, but (5.31) does not hold.
3. The Taylor series of f in a diverges at x.

If the first case applies to all x in some interval I containing a, we say that the **Taylor series for f at a converges to f on I**. This situation arises when f can be expressed by a power series with interval of convergence I. (Then formula (5.31) says that the Taylor series for f is given by this power series.) In general, we may characterize the first case with the aid of the n-th partial sum of the Taylor series,

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k \qquad (5.32)\\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

Viewed as a function of x, P_n is a polynomial of order n, it is called the **Taylor polynomial** of order n for f at a. We interpret $P_n(x)$ as an approximation for $f(x)$ and define the remainder

$$R_n(x) = f(x) - P_n(x). \tag{5.33}$$

Obviously, (5.31) holds at a given x if and only if $\lim_{n\to\infty} R_n(x) = 0$.

Theorem 5.15 *Let f be differentiable up to order $n+1$ in an open interval I containing a point a. Then for each x in I, there exists a number c between x and a, that is, $x < c < a$ or $a < c < x$, such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^k(a)}{k}(x-a)^k + R_n(x) = P_n(x) + R_n(x) \tag{5.34}$$

holds with

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{5.35}$$

We present a particular situation where we can ensure that the remainder term converges to 0.

Theorem 5.16 *Let f be differentiable to all orders n in an open interval I containing a point a. If there are positive constants M and r such that*

$$|f^{(n+1)}(t)| \le Mr^{n+1}, \quad \textit{for all } t \in I, \tag{5.36}$$

then $\lim_{n\to\infty} R_n(x) = 0$ *for all $x \in I$, and consequently the Taylor series of f in a converges to f on I.*

Proof By Theorem 5.15 and (5.36), we have

$$|R_n(x)| \le Mr^{n+1}\frac{|x-a|^{n+1}}{(n+1)!}, \quad \text{for all } x \in I.$$

The assertion follows since $\lim_{n\to\infty} t^n/n! = 0$ for all numbers t.

Example 5.9 Find the Maclaurin series for $\sin x$, $\cos x$, and e^x and show that they converge to those functions.

Solution: For the sine function, we have $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and so on. We see that $0 = f(0) = f''(0) = f^{(4)}(0) = \cdots$, and that $f'(0) = 1$, $f'''(0) = -1$, $f^{(5)}(0) = 1$, and so on. Therefore, the Maclaurin series of $f(x) = \sin x$ becomes

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

In an analogous way, one obtains the Maclaurin series for $\cos x$ as

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots .$$

For the exponential function $f(x) = e^x$, we have $f^{(n)}(x) = f(x) = e^x$ for all n, and hence $f^{(n)}(0) = e^0 = 1$ for all n, and the Maclaurin series becomes

$$\begin{aligned}
&f(0) + f'(0)x + \frac{f''(0)}{2!} + \cdots + \frac{f^n(0)}{n!} + \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!}.
\end{aligned}$$

In all three cases, one may check that condition (5.36) holds on every bounded interval I, with $r = 1$ and a suitable M. Therefore, the Maclaurin series converges and yields a power series expansion of the sine, cosine, and the exponential function, respectively. We thus have proved that formulas (5.19)–(5.21) are correct.

Taylor series can be used to compute approximations to unknown quantities. We present a classical example from mechanics. It involves the notion and calculus of integrals treated in Chap. 6, so the student may want to come back to this example after having studied that chapter.

Example 5.10 A simple pendulum consists of a mass attached to the end of a weightless rod of length L, the other end of which is fixed. Its position is characterized by the angle it makes with the vertical axis. Suppose it is held initially at an angle α and then released from rest. It can be shown as a consequence of the laws of mechanics that, in the absence of friction, the time T it takes for the pendulum to perform one complete swing back and forth (called the **period** of the pendulum) is given by

$$T = \sqrt{\frac{8L}{g}} \int_0^{\alpha} \frac{1}{\sqrt{\cos\theta - \cos\alpha}}\, d\theta.$$

Putting $k = \sin\alpha/2$, $\cos\theta = 1 - 2\sin^2\dfrac{\theta}{2}$, and $\cos\alpha = 1 - 2\sin^2\alpha/2$ and making the substitution $\sin\phi = \dfrac{\sin\theta/2}{\sin\alpha/2} = \dfrac{1}{k}\sin\theta/2$, we obtain the integral[3]

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2\phi}}\, d\phi. \tag{5.37}$$

[3]Such an integral is called a complete elliptic integral of the first kind.

We are interested in the function $T = T(k)$, that is, the dependence of the period on the initial angular displacement. We compute the Maclaurin series for the integrand by expanding $(1 - k^2 \sin^2 \phi)^{-1/2}$ as a binomial series, according to (5.22) with $x = k^2 \sin^2 \phi$ and $p = -1/2$. Then

$$T(k) = \tag{5.38}$$
$$4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left[1 + \frac{1}{2} k^2 \sin^2 \phi + \frac{1}{2^2 2!} 3k^4 \sin^4 \phi + \frac{1 \cdot 3 \cdot 5}{2^3 3!} k^6 \sin^6 \phi + \cdots \right] d\phi.$$

If we integrate term by term, we get a power series w.r.t. the variable k. One can show that this power series is indeed the Maclaurin series for $T = T(k)$ which moreover converges to the period T. For small initial displacements, k is small, and it turns out that few terms of the series suffice in order to yield a good approximation of T. If we consider only the first (constant) term, we obtain the so-called first-order approximation of T which gives

$$T = 2\pi \sqrt{\frac{L}{g}}.$$

If we moreover take into account the next term in the Maclaurin series, we obtain the second-order approximation

$$T = 2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{k^2}{4} \right).$$

5.5 Exercises

5.5.1 Determine the following limits of sequences:

(a) $\lim_{n \to \infty} (3n)^{1/n}$,

(b) $\lim_{n \to \infty} (-1/2)^n$,

(c) $\lim_{n \to \infty} \left(\frac{n-2}{n} \right)^2$,

(d) $\lim_{n \to \infty} \frac{100^n}{n!}$,

(e) $\lim_{n \to \infty} \frac{\ln n^2}{n}$.

5.5.2 Show that $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2} \right) = \frac{3}{2}$.

5.5.3 In Exercises (a)–(d), check that the series is a geometric series and find its sum if it converges. In Exercises (e) and (f), find a formula for the nth partial sum of each series and use it to find the sum of the series if the latter converges.

(a) $2 + \dfrac{2}{3} + \dfrac{2}{9} + \dfrac{2}{27} + \cdots + \dfrac{2}{3^{n-1}} + \cdots$

(b) $\dfrac{9}{100} + \dfrac{9}{100^2} + \dfrac{9}{100^3} + \cdots + \dfrac{9}{100^n} + \cdots$

(c) $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \cdots + (-1)^{n-1}\dfrac{1}{2^{n-1}} + \cdots$

(d) $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}2^{n-1} + \cdots$

(e) $\dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \cdots + \dfrac{1}{(n+1)(n+2)} + \cdots$

(f) $\dfrac{5}{1\cdot 2} + \dfrac{5}{2\cdot 3} + \dfrac{5}{3\cdot 4} + \cdots + \dfrac{5}{n(n+1)} + \cdots$

5.5.4 Write out the first few terms of the following series to show how the series starts. Then find the sum of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{7}{4^n}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$,

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$,

(e) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$,

(f) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5^{n+1}}{2^n}\right)$.

5.5.5 Which of the following series converge, and which diverge? Give reasons for your answer. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$,

(b) $\displaystyle\sum_{n=0}^{\infty} (\sqrt{2})^n$,

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{3}{2^n}$,

(d) $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$,

(e) $\sum_{n=0}^{\infty} e^{-2n}$,

(f) $\sum_{n=0}^{\infty} \ln \frac{1}{n}$.

5.5.6 Discuss the convergence of the series

(a) $\sum \frac{1}{n^p}, 0 < p \leq 1$,

(b) $\sum \frac{1}{n^p}, p > 1$.

5.5.7 Test for convergence of the series

(a) $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$,

(b) $\sum_{n=0}^{\infty} \frac{1}{x^n + x^{-n}}$.

5.5.8 Test for convergence of the series

(a) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots$

(b) $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \cdots + \frac{(2^n - 2)x^{n-1}}{2^n + 1} + \cdots, x > 0$

5.5.9 Test for convergence of the series $\sum \frac{4 \cdot 7 ... (3n+1)}{1 \cdot 2 \cdots n} x^n$.

5.5.10 Test for convergence of the series

(i) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$,

(ii) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$.

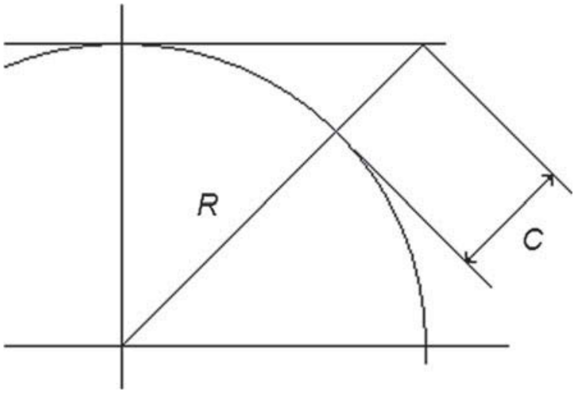
5.5.11 Test for convergence of the series $\sum s_n$, where

$$s_n = \begin{cases} \frac{n}{2^n}, & n \text{ s odd,} \\ \frac{1}{2^n}, & n \text{ is even.} \end{cases}$$

5.5.12 Find the Taylor series for $\frac{1}{1+x}$ at $a = 2$.

5.5.13 Compute the value of e with an error of less than 10^{-6}, using the series expansion of the exponential function.

Fig. 5.3 Planning a highway



5.5.14 In planning a highway across a desert, a surveyor must make compensations for the curvature of the earth when measuring differences in elevation, see the following figure.

a. If s is the length of highway and R is the radius of the earth, show that the correction C is given by $C = R[\sec(s/R) - 1]$.
b. Use the Maclaurin series for $\sec x$ to show that C is approximately $\frac{s^2}{2R} + \frac{5s^4}{24R^3}$ (Fig. 5.3).

Chapter 6
Integration

6.1 Introduction

In Chap. 1, we have seen how mathematical functions model real processes. The differential calculus deals with finding the rate of change of a function (thus, of a process). We have studied this subject in Chaps. 3 and 4. But this part reveals only half of the story. It is good to know, at any given instant, the rate of change of a function, but it is of additional interest and use if we could describe how such instantaneous changes accumulate over an interval to produce the original (given) function. In other words, we are interested in the process of studying how a change in behavior will yield behavior itself. The process we are looking for was discovered by Newton and Leibniz, it is called **integration**. Its precursors in human history date back to the early civilizations in China, India, Egypt, Mesopotamia, and Greece, related to the determination of length, area, and volume. In the meantime, however, the notion of an integral has by far surpassed these origins and nowadays constitutes a fundamental and general concept which permeates almost all areas where mathematics is applied.

The main goal of this chapter is to present those basic results of the integral calculus which are essential for its use in engineering, science, and economics to solve real-world problems. We introduce the definite integral as the limit of a sum of certain quantities, motivated by the classical problem of area computation. This is the theme of Sect. 6.2. In Sect. 6.3, we present the indefinite integral as the process inverse to differentiation, that is, given a function f, we want to find a function F (called the antiderivative) whose derivative F' is equal to f. Sections 6.4 and 6.5 are devoted to the method of substitution and of partial integration, powerful tools to transform and compute integrals. The Fundamental Theorem of Calculus, which exhibits that differentiation and integration are inverse to each other, is presented in Sect. 6.6, together with the calculus for definite integrals. In Sects. 6.7–6.10, derivatives and integrals are discussed for various classes of elementary functions.

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6.2 Integral and Area

Let $f(x)$ be a nonnegative function defined on a closed interval $[a, b]$. We want to find area of the region "below the graph of f", that is, of the region enclosed by the graph of f, the vertical lines $x = a$ and $x = b$, and the x-axis, shown as the shaded region in Fig. 6.1. If f is a constant function, say $f(x) = c$, the region in question becomes a rectangle whose area equals $c(b - a)$. Next, consider a subdivision (also called a **partition**) $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of the interval $[a, b]$. This partition divides $[a, b]$ into n subintervals $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$. Let s be a step function for this partition, that is, a function which has a constant value $s(x) = c_k$ on each open subinterval (x_{k-1}, x_k), see Sect. 1.3. If $c_k \geq 0$ holds for all those values, the area of the region below the graph of s equals the sum $\sum_{k=1}^{n} c_k(x_k - x_{k-1})$ of the area of the rectangles below s bounded by the verticals $x = x_{k-1}$ and $x = x_k$ from left and right.

Let now f be an arbitrary nonnegative function. We may approximate the area of the region below its graph by the sum

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \tag{6.1}$$

of the areas of the rectangles $\{(x, y) : x_{k-1} \leq x \leq x_k\ ,\ 0 \leq y \leq f(\xi_k)\}$. Here, the points ξ_k on the x-axis are arbitrarily chosen between x_{k-1} and x_k. We expect that the smaller the subintervals become (taking n larger and larger), the more accurate will be the estimate of the shaded area in Fig. 6.1. The exact value A of the area arises "in the limit of smaller and smaller partitions",

$$A = \lim \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})\,. \tag{6.2}$$

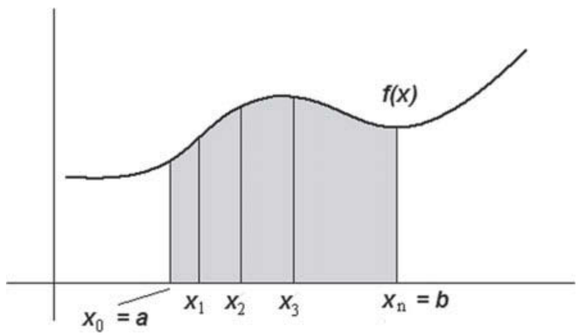

Fig. 6.1 Area and partition

When this limit A exists, it is called the **Riemann integral** or **definite integral** of f over $[a, b]$ and is denoted by

$$\int_a^b f(x)\, dx\,, \tag{6.3}$$

and the function f is called **Riemann integrable** or simply **integrable**. The approximate sums in (6.2) are called **Riemannian sums**. Thus, the Riemann integral is defined as the limit of Riemannian sums. For nonnegative functions f, it gives the area below $y = f(x)$ for $a \le x \le b$. The expression (6.3) is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x". The number a is called the lower limit of integration, the number b the upper limit, and x is called the integration variable.

Several remarks are in order.

1. The limit procedure in (6.2) is more complicated than the ones previously encountered (limits of functions and sequences), because it involves partitions. A precise definition is given in Appendix D.
2. The letter used for the integration variable is completely arbitrary. In place of $\int_a^b f(x)\, dx$, we may write $\int_a^b f(t)\, dt$ or $\int_a^b f(u)\, du$ or $\int_a^b f(y)\, dy$.
3. The value of the integral only depends on the function f, the lower limit a and the upper limit b.

So far, we only have considered nonnegative functions f, because we wanted to emphasize the close connection between the notion of an integral and the computation of areas. In fact, the Riemannian sums (6.1) and the Riemann integral (6.3)—if the limit (6.2) exists—are defined for arbitrary functions $f : [a, b] \to \mathbb{R}$, no matter whether the values of f are positive, negative, or both. As a consequence, the value of the integral may become negative. In which way, it is still related to areas will be discussed in Chap. 7.

Remark 6.1 The calculus of integrals was developed independently by Newton and Leibniz, as part of the infinitesimal calculus. They understood the relations and analogies between sums and integrals and between differences and derivatives. The symbol $\displaystyle\int$ was invented by Leibniz to represent the integral. People believe it is a stretched out "S", from the Latin word "summa" for sum. This summarizes the whole construction: Sum approaches integral, Σ approaches $\int$, and rectangular area approaches curved area. The notation

$$\int_a^b f(x)\, dx$$

has to be understood from this historical context—originally, the symbol "dx" had been interpreted as the length of an "infinitesimally small" rectangle with height $f(x)$, and the integral was seen as an infinite sum of such rectangles. Nowadays (that

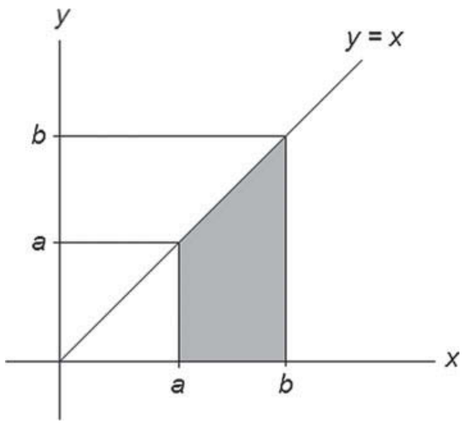

Fig. 6.2 Area of a trapezoid

is, since the end of the nineteenth century), this interpretation has been replaced in mathematics by the notion of the limit as it is used throughout this book.

Example 6.1 Express the area under the graph of the function $y = f(x) = x$ on $[a, b]$, $0 < a < b$, in the form of a definite integral and evaluate its value.

Solution: The shaded region in Fig. 6.2 is a trapezoid with height $b - a$ and bases a and b. Its area A equals the value of the corresponding definite integral, so we have

$$\int_a^b x\,dx = A = (b-a)\frac{a+b}{2} = \frac{b^2}{2} - \frac{a^2}{2},$$

as we know from elementary geometry. (Soon, we will be able to evaluate the integral without recourse to geometrical considerations.)

6.3 Antiderivatives and Rules of Integration

Definition 6.1 A function F is called an **antiderivative** or a **primitive** of the function f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example 6.2 1. Show that $\frac{2}{3}x^{3/2}$ is an antiderivative of $\sqrt{x}$.

2. Let $F(x) = \frac{1}{3}x^3 - 5x^2 + x - 1$. Show that F is an antiderivative of $f(x) = x^2 - 10x + 1$.
3. Let $F(x) = \frac{1}{n+1}x^{n+1}$ and $f(x) = x^n$. Show that $F(x)$ is an antiderivative of f.

Solution:

1. Let $F(x) = \frac{2}{3}x^{3/2}$, then $F'(x) = \frac{2}{3}\frac{3}{2}x^{1/2} = x^{1/2}$. Hence by Definition 6.1, $F(x) = \frac{2}{3}x^{3/2}$ is an antiderivative of $f(x) = x^{1/2}$.

2. In view of Definition 6.1, we must show that $F'(x) = f(x)$. We have $F'(x) = x^2 - 10x + 1$, which is equal to $f(x)$.
3. We have to show that $F'(x) = f(x)$. We have

$$F'(x) = \frac{1}{n+1} \cdot (n+1)x^n = x^n = f(x).$$

Theorem 6.1 *Let F be an antiderivative of f on an interval I. Then any antiderivative G of f has the form $G = F + C$ for some constant C, that is,*

$$G(x) = F(x) + C\,, \quad \textit{for all } x \in I.$$

Proof Since F and G both are antiderivatives of f, we have $F'(x) = f(x)$ and $G'(x) = f(x)$ for all $x \in I$. This implies that the derivatives of F and G are identically equal. By Remark 4.1 we find that F and G differ by a constant C. Hence $G = F + C$.

Definition 6.2 (*Indefinite Integral*) For any given function f defined on an interval I, the indefinite integral

$$\int f(x)\,dx$$

is defined as the family of all antiderivatives of f, that is, of all functions of the form

$$F(x) + C\,,$$

where $F' = f$ on I, and C is an arbitrary constant.

To express the statement above, one traditionally writes the single formula

$$\int f(x)\,dx = F(x) + C\,. \tag{6.4}$$

For example, the solution of Example 6.2(2) can now be written as

$$\int x^2 - 10x + 1\,dx = \frac{1}{3}x^3 - 5x^2 + x + C\,.$$

Note, however, that the letter "x" is used in two different meanings in (6.4). On the right-hand side, it denotes the argument of the function $F + C$, while on the left-hand side it denotes the integration variable.

In addition, let us point out a definite integral is a number, while an indefinite integral is a collection of functions (differing only by a constant).

Constant multiples, sums, and differences. Let F and G be antiderivatives of f and g respectively, let c be a constant. Since the formulas

$$(cF')(x) = cF'(x) = cf(x)\,, \quad (F+G)'(x) = F'(x) + G'(x) = f(x) + g(x)$$

hold by the rules of differentiation, cF is the antiderivative of cf, and $F + G$ is the antiderivative of $f + g$. Hence, the indefinite integral satisfies

$$\int cf(x)\,dx = c\int f(x)\,dx\,, \tag{6.5}$$

(in particular, for $c = -1$ we obtain $\int -f(x)\,dx = -\int f(x)\,dx$), and

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx\,. \tag{6.6}$$

These rules can be combined into the property of **linearity**, namely,

$$\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx \tag{6.7}$$

holds for functions f, g and constants α, β.

Example 6.3 Evaluate the following integrals:

1. $\int \pi^2\,dx$
2. $\int x^2 - 10x + 1\,dx$

Solution:

1. Since $\int dx = \int 1\,dx = x + C$, by (6.5) we have
$$\int \pi^2\,dx = \pi^2 \int dx = \pi^2 x + C\,.$$
2. This is just Example 6.2(2), seen in a different light. By the property of linearity (6.7) we have
$$\int x^2 - 10x + 1\,dx = \int x^2\,dx - 10\int x\,dx + \int 1\,dx = \frac{1}{3}x^3 - 5x^2 + x + C\,.$$

Tables of Integrals. Indefinite integrals are compiled in tables. Table 6.1 contains a small selection.

Example 6.4 Evaluate the following integrals using Table 6.1

1. $\int x^6\,dx$
2. $\int \frac{1}{\sqrt{x}}\,dx$

Table 6.1 Integral formulas

S.No.	Derivative $\frac{d}{dx}(F(x)) = F'(x)$	Indefinite integral $\int \frac{d}{dx}(F(x))\,dx = F(x) + C$
1	$\frac{d}{dx}(x) = 1$	$\int 1\,dx = \int dx = x + C$
2	$\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = x^n$, $n \neq -1$, n rational	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$, n rational
3	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
4	$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\,dx = -\cos x + C$
5	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
6	$\frac{d}{dx}(-\cot x) = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
7	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
8	$\frac{d}{dx}(-\csc x) = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
9	$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
10	$\frac{d}{dx}(\frac{b^x}{\ln b}) = b^x$	$\int b^x\,dx = \frac{b^x}{\ln b} + C$
11	$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
12	$\frac{d}{dx}(\ln\lvert\sec x + \tan x\rvert) = \sec x$	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$

3. $\int \frac{1}{x^3}\,dx$
4. $\int \frac{\tan x}{\sec x}\,dx$

Solution:

1. Put $n = 6$ in formula 2 of Table 6.1. This gives

$$\int x^6\,dx = \frac{x^7}{7} + C\,.$$

2. By Table 6.1(2) we have with $n = -1/2$

$$\int \frac{1}{\sqrt{x}}\,dx = 2x^{1/2} + C\,.$$

3.
$$\int \frac{1}{x^3}\,dx = -\frac{1}{2x^2} + C$$

by putting $n = -3$ in formula 2 of Table 6.1.

4.
$$\int \frac{\tan x}{\sec x}\,dx = \int \cos x \frac{\sin x}{\cos x}\,dx = \int \sin x\,dx = -\cos x + C\,.$$

6.4 Integration by Substitution

The rules of differentiation as discussed in Sect. 3.3 give rise to corresponding rules of integration. We have already seen this in the previous section for the rules related to linearity. In this section, we introduce the so-called method of substitution which is related to the chain rule for differentiation. In conjunction with the other rules of integration, the method of substitution is a powerful tool.

In order to understand this method, assume that we want to compute the indefinite integral $\int h(x)\,dx$, where the function h has the form

$$h(x) = f(g(x))g'(x) \tag{6.8}$$

for some other functions f and g. Let us assume that we can find an antiderivative F of f, that is, $F' = f$. Then it follows from the chain rule that

$$\frac{d}{dx}F(g(x)) = (F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x) = h(x)\,.$$

This means that the function $H = F \circ g$, $H(x) = F(g(x))$, is an antiderivative of h. In this manner, we can find the indefinite integral of h, provided we can decompose h as in (6.8) and we can compute the indefinite integral of f.

Let us consider for example

$$\int 8 \sin 8x\,dx\,.$$

Here $h(x) = 8 \sin 8x$. We try $g(x) = 8x$. Then $g'(x) = 8$, and (6.8) holds with $f(u) = \sin u$. We know that $F(u) = -\cos u + C$ is the indefinite integral of f. Hence $F(g(x)) = -\cos 8x + C$ is the indefinite integral of $h(x) = 8 \sin 8x$.

One may carry out this method according to the following scheme.

Step 1. Replace a suitable expression as part of $\int h(x)\,dx$ by a new variable u (so $u = g(x)$ for some function g).

Step 2. Compute $g'(x)$ and replace $g'(x)dx$ by du.

Step 3. Check whether the resulting integral now has the form $\int f(u)\,du$ for some function f.

Step 4. Evaluate the integral $\int f(u)\,du$.
Step 5. Replace u by $g(x)$ to obtain the indefinite integral $\int h(x)\,dx$ as a function of x.

The procedure just described is commonly written in the abbreviated form of the **substitution rule**

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du\,. \tag{6.9}$$

However, one must be careful in the interpretation of this formula, since it does not explicitly mention the necessary backsubstitution (Step 5 above).

Example 6.5 Evaluate the following integrals:

1. $\displaystyle\int \sin(x+10)\,dx$,
2. $\displaystyle\int \cos 15x\,dx$,
3. $\displaystyle\int \sin^2 x \cos x\,dx$,
4. $\displaystyle\int (3\sqrt{3x+1})\,dx$,
5. $\displaystyle\int \frac{x}{6x^2+1}\,dx$,
6. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$,
7. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$,
8. $\displaystyle\int \frac{(\ln x)^2}{3x}\,dx$.

Solution:

1. Let $u = x + 10 = g(x)$, then $g'(x) = 1$ and $du = dx$. Replace the expression $x + 10$ by u in the given integral, then

$$\int \sin(x+10)\,dx = \int \sin u\,du = -\cos u + C = -\cos(x+10) + C\,.$$

2. This example illustrates the handling of constants. Let $u = 15x = g(x)$, then $g'(x) = 15$ and $du = 15\,dx$. We have

$$\int \cos 15x\,dx = \frac{1}{15}\int 15\cos 15x\,dx = \frac{1}{15}\int \cos u\,du = \frac{1}{15}(\sin u + C)\,.$$

Since C is an arbitrary constant, we may replace $C/15$ by C in the last expression and obtain

$$\int \cos 15x\,dx = \frac{1}{15}\sin 15x + C.$$

We may abbreviate this computation somewhat. We transform "$du = 15\,dx$" into "$dx = \frac{1}{15}du$" and simply write

$$\int \cos 15x\,dx = \frac{1}{15}\int \cos u\,du = \frac{1}{15}\sin 15x + C\,.$$

3. Let $u = \sin x$ then $du = \cos x\,dx$ and

$$\int \sin^2 x \cos x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C\,.$$

4. Let $u = 3x + 1$, then $du = 3\,dx$ or $dx = \frac{1}{3}du$. With this substitution, we compute

$$\int 3\sqrt{(3x+1)}\,dx = \int u^{1/2}\,du = \frac{1}{\frac{3}{2}}.u^{3/2} + C = \frac{2}{3}(3x+1)^{3/2} + C\,.$$

5. Let $u = 6x^2 + 1$ then $du = 12x\,dx$ or $\frac{1}{12}du = x\,dx$. With this substitution, we compute

$$\int \frac{x}{6x^2+1}\,dx = \frac{1}{12}\int \frac{du}{u} = \frac{1}{12}\ln|u| + C = \frac{1}{12}\ln|6x^2+1| + C\,.$$

6. Let $u = \sqrt{x}$, then $du = \dfrac{1}{2}\dfrac{1}{\sqrt{x}}dx$ and $2\,du = \dfrac{1}{\sqrt{x}}dx$. Then

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int 2e^u\,du = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C\,.$$

7. Let $u = 1 + e^x$, then $du = e^x dx$ and the given integral takes the form

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+e^x| + C\,.$$

8. Let $u = \ln x$, then $du = \dfrac{1}{x}dx$. By this substitution, the integral takes the form

$$\int \frac{(\ln x)^2}{3x}\,dx = \frac{1}{3}\int u^2\,du = \frac{1}{9}u^3 + C = \frac{1}{9}(\ln x)^3 + C\,.$$

Note 6.1 The substitution technique yields

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \tag{6.10}$$

as well as

$$\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + C \tag{6.11}$$

for arbitrary functions f.

So far, we have used the substitution rule

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

in order to compute the integral on the left-hand side, knowing the integral on the right-hand side. We may also use the rule the other way round, that is, we want to find an antiderivative of f through evaluating the integral on the left-hand side. This approach is rather flexible since we may in principle choose any function $u = g(x)$ to work with, as long as we can evaluate the left-hand side.

Example 6.6 Evaluate $\displaystyle\int \frac{1}{1+e^u}\,du$.

Solution: In order to eliminate the exponential, we set $u = g(x) = \ln x$. Then $du = g'(x)\,dx = \frac{1}{x}\,dx$, and

$$\begin{aligned}\int \frac{1}{1+e^u}\,du &= \int \frac{1}{1+x}\cdot\frac{1}{x}\,dx = \int \left(\frac{1}{x} - \frac{1}{1+x}\right)dx \\ &= \ln x - \ln(1+x) + C = \ln e^u - \ln(1+e^u) + C \\ &= u - \ln(1+e^u) + C\,.\end{aligned}$$

Note that we have used the backsubstitution $x = g^{-1}(u) = e^u$. Indeed, this variant of the substitution method requires that the function g is invertible on the appropriate domain (here, $x > 0$) and range (here, all of $\mathbb{R}$).

6.5 Integration by Parts

The method of integration by parts is essentially the antiderivative version of the formula for differentiating a product of two functions which says that

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = f(x)g'(x) + g(x)f'(x) \tag{6.12}$$

holds whenever f and g are differentiable functions. Forming the indefinite integral on both sides, we obtain

$$\int \frac{d}{dx}\Big[f(x)g(x)\Big]\,dx = \int \Big[f(x)g'(x) + g(x)f'(x)\Big]\,dx\,.$$

Due to the linearity of the indefinite integral, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}\Big[f(x)g(x)\Big]\,dx - \int g(x)f'(x)\,dx\,.$$

Since the function $x \mapsto f(x)g(x)$ is an antiderivative of the function $x \mapsto d/dx\,[f(x)g(x)]$, we obtain the **formula for integration by parts**

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx\,. \tag{6.13}$$

It is not necessary to write a constant here, since both indefinite integrals in (6.13) are only determined up to a constant.[1]

The integration by parts formula is very useful as it converts a given indefinite integral into another one that may be easier to evaluate. For the purposes of actual computation, we may abbreviate it in the form

$$\int u\,dv = uv - \int v\,du\,, \tag{6.14}$$

which may also be easier to memorize. Here, u and v stand for $f(x)$ and $g(x)$, respectively, while du stands for $f'(x)dx$ and dv for $g'(x)dx$.

Example 6.7 Evaluate the following integrals applying the method of integration by parts:

1. $\displaystyle\int xe^x\,dx$,
2. $\displaystyle\int x^2e^{-x}\,dx$,
3. $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx$,
4. $\displaystyle\int \ln x\,dx$,
5. $\displaystyle\int x\ln x\,dx$,
6. $\displaystyle\int x^2\cos x\,dx$,
7. $\displaystyle\int e^x\cos x\,dx$,
8. $\displaystyle\int \sec^3 x\,dx$,
9. $\displaystyle\int \sin^n x\,dx$, where $n \geq 2$ is a natural number.

[1]Note that in (6.13), the letter 'x' is used in two different meanings; as an actual argument in the expression '$f(x)g(x)$' and as an integration variable in the two integrals.

Solution:

1. We want to write the given integral in the form $\int u\,dv$ and apply formula (6.14). To this purpose, we set $u = f(x) = x$ and $dv = g'(x)dx = e^x dx$. Then $du = dx$. Next, we choose $v = g(x) = e^x$ as an antiderivative of $x \mapsto e^x$. The formula

$$\int u\,dv = uv - \int v\,du$$

now becomes

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C\,.$$

The rightmost expression no longer contains an indefinite integral, therefore we have to introduce the constant C. (The reader may check that if we had chosen $v = e^x + 1$, for example, instead of $v = e^x$, we would have arrived at the same final result.)

2. Let $u = x^2$, $dv = e^{-x}dx$, then $du = 2x\,dx$, and we may choose $v = -e^{-x}$ as an antiderivative of $x \mapsto e^{-x}$. (The latter result is obtained by the substitution method. Let $-x = t$, then $-dx = dt$ or $dx = -dt$, so

$$\int e^{-x}\,dx = -\int e^t\,dt = -e^t = -e^{-x}\,,$$

where we have chosen the integration constant to be zero.) Thus

$$\int x^2e^{-x}\,dx = \int u\,dv = uv - \int v\,du = -x^2e^{-x} + 2\int e^{-x}x\,dx\,.$$

We still have an indefinite integral on the right-hand side, but with the factor of x instead of x^2. To evaluate that integral, we apply integration by parts once again. Put $u = x$, $dv = e^{-x}\,dx$, then $du = dx$. Again, we choose $v = -e^{-x}$. Then

$$\begin{aligned}\int xe^{-x}\,dx &= uv - \int v\,du = -xe^{-x} + \int e^{-x}\,dx\\ &= -xe^{-x} - e^{-x} + C_1\,.\end{aligned}$$

As final result, we get

$$\begin{aligned}\int x^2e^{-x}\,dx &= -x^2e^{-x} + 2(-xe^{-x} - e^{-x} + C_1)\\ &= -(x^2 + 2x + 2)e^{-x} + C\,,\end{aligned}$$

where we have written C instead of $2C_1$ for the arbitrary constant.

3. Let $u = xe^x$ and $dv = \dfrac{1}{(x+1)^2}\,dx$. We choose $v = -\dfrac{1}{x+1}$ as antiderivative and compute $du = (xe^x + e^x)\,dx = e^x(x+1)\,dx$. Now

$$\begin{aligned}\int \frac{xe^x}{(x+1)^2}\,dx &= \int u\,dv = uv - \int v\,du\\ &= xe^x\left(\frac{-1}{x+1}\right) - \int \left(\frac{-1}{x+1}\right) e^x(x+1)\,dx = -\frac{xe^x}{x+1} + e^x + C\\ &= \frac{e^x}{x+1} + C\,.\end{aligned}$$

4. $\int \ln x\,dx = \int u\,dv$, where we choose $u = \ln x$, $dv = dx$. We have $du = \dfrac{1}{x}\,dx$, and we take $v = x$ as antiderivative. Then

$$\begin{aligned}\int \ln x\,dx = \int u\,dv = uv - \int v\,du &= x\ln x - \int x\frac{1}{x}\,dx\\ &= x\ln x - x + C\,.\end{aligned}$$

5. Let $u = \ln x$ and $dv = x\,dx$, then $du = \dfrac{dx}{x}$ and $v = \dfrac{1}{2}x^2$. Thus,

$$\begin{aligned}\int x\ln x\,dx = \int u\,dv = uv - \int v\,du &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2.\frac{1}{x}\,dx\\ &= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C\\ &= \frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right) + C\,.\end{aligned}$$

6. $\int x^2\cos x\,dx = \int u\,dv$, where we set $u = x^2$ and $dv = \cos x\,dx$. We get $du = 2x\,dx$ and take $v = \sin x$ as antiderivative of $\cos x$. Then

$$\begin{aligned}\int x^2\cos x\,dx = \int u\,dv = uv - \int v\,du &= x^2\sin x - \int 2x\sin x\,dx\\ &= x^2\sin x - 2\int x\sin x\,dx\,.\end{aligned} \tag{6.15}$$

To find the indefinite integral $\int x\sin x\,dx$, we again apply integration by parts. Let $u = x$, $dv = \sin x\,dx$, then $du = dx$, and with $v = -\cos x$ we get

$$\begin{aligned}\int x\sin x\,dx = uv - \int v\,du &= -x\cos x - \int -\cos x\,dx\\ &= -x\cos x + \sin x + C_1\,.\end{aligned}$$

Inserting this formula into (6.15), we finally obtain

$$\int x^2 \cos x \, dx = x^2 \sin x - 2(-x \cos x + \sin x + C_1)$$
$$= x^2 \sin x + 2x \cos x - 2 \sin x + C \,,$$

where we have replaced $-2C_1$ by C.

7. Let $u = e^x$, $dv = \cos x \, dx$, then $du = e^x \, dx$ and we take $v = \sin x$. Integration by parts yields

$$\int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du \qquad = e^x \sin x - \int e^x \sin x \, dx$$
$$= e^x \sin x - \int e^x \sin x \, dx \,. \tag{6.16}$$

To evaluate $\int e^x \sin x \, dx$ we again use integration by parts. Let $u = e^x$, $dv = \sin x \, dx$, then $du = e^x dx$ and $v = -\cos x$. Then

$$\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = -e^x \cos x - \int -e^x \cos x \, dx$$
$$= -e^x \cos x + \int e^x \cos x \, dx \,.$$

Using this formula in (6.16), we get

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \,.$$

Consequently,

$$2 \int e^x \cos x \, dx = e^x (\sin x + \cos x) + C_1$$

and finally

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C \,.$$

8. $\int \sec^3 x \, dx = \int u \, dv$, where we set $u = \sec x$, $dv = \sec^2 x \, dx$. We get $du = \sec x \tan x \, dx$, and use $v = \tan x$ as an antiderivative. Thus

$$\int \sec^3 x \, dx = \int u \, dv = uv - \int v \, du = \sec x \tan x - \int \sec x \tan^2 x \, dx \,.$$

We use the trigonometric identity $1 + \tan^2 x = \sec^2 x$, or $\tan^2 x = \sec^2 x - 1$, to obtain

$$\int \sec^3 x\,dx = \sec x \tan x - \int (\sec^3 x - \sec x)\,dx\,.$$

Rearranging the integrals yields

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx\,.$$

We divide by 2 and look up the indefinite integral of $\sec x$ to obtain

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C\,.$$

9. $\int \sin^n x\,dx = \int u\,dv$, where we set $u = \sin^{n-1} x$ and $dv = \sin x\,dx$. We obtain $du = (n-1)\sin^{n-2} x \cos x\,dx$ and take $v = -\cos x$. Thus

$$\begin{aligned}\int \sin^n x\,dx &= \int \sin^{n-1} x \sin x\,dx = \int u\,dv = uv - \int v\,du\\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx\,.\end{aligned}$$

Since $\cos^2 x = 1 - \sin^2 x$, we may write

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx\,.$$

Moving the rightmost integral to the left-hand side, we get

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx\,,$$

and division by n finally yields

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx\,. \tag{6.17}$$

Remark 6.2 Formula (6.17) is called a **reduction formula** or a **recursion formula** for $\int \sin^n x\,dx$. For example, using it with $n = 4$ we get

$$\int \sin^4 x\,dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x\,dx\,.$$

Applying (6.17) for $n = 2$ we get

$$\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C\,.$$

Consequently,

$$\int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x - \frac{3}{8} \cos x \sin x + \frac{3}{8} x + D \,,$$

where $D = \frac{3}{4} C$. In this manner, we successively reduce even powers n to the constant function. Analogously, we may reduce odd powers n to the power 1, that is, to $\displaystyle\int \sin x \, dx = -\cos x + C$.

6.6 The Fundamental Theorem of Calculus

As its name indicates, this theorem is the cornerstone of calculus. It establishes the fact that the processes of integration and of differentiation are inverse to each other.

Since the theorem deals with general functions, not just functions given by explicit formulas as in the preceding sections, we need to know that the integrals to be written down make sense. For this purpose, we present a preliminary result, whose proof can be found in Appendix D.

Theorem 6.2 *If f is a continuous function defined on a closed interval $[a, b]$, then f is integrable on $[a, b]$, that is,*

$$\int_a^b f(x) \, dx$$

exists.

There are two parts of the Fundamental Theorem of Calculus. Here, we will state them and interpret them. Their proofs are given in Appendix D.5.

Theorem 6.3 (The Fundamental Theorem of Calculus, Part 1) *If f is a continuous function defined on $[a, b]$, then the function*

$$F(x) = \int_a^x f(t) \, dt \tag{6.18}$$

has a derivative at every point x in $[a, b]$ and

$$F'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \,. \tag{6.19}$$

Remark 6.3 1. Theorem 6.3 says that if we integrate a function f as in (6.18), and differentiate the resulting function F, we get back the function f from which we started. Conversely, if we differentiate some function G to obtain $f = G'$, and integrate f according to (6.18), the resulting F is an antiderivative of f, hence equal to G except for a constant difference.

2. Moreover, it says that every continuous function f has an antiderivative, namely F.
3. In addition, it implies that the differential equation $\dfrac{dF}{dx} = f$ has a solution for every continuous f.

Theorem 6.4 (The Fundamental Theorem of Calculus, Part 2) *If f is a continuous function defined on $[a, b]$, and if F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)\,. \tag{6.20}$$

Remark 6.4 1. Theorem 6.4 says that the definite integral of any continuous function f can be calculated without taking limits, without computing Riemann sums, as long as an antiderivative of f can be found (which often does not present any difficulty).
2. It relates the computation of area below the graph of a (nonnegative) function f to the concept of the tangent (namely, $f(x) = F'(x)$ is the slope of the tangent to the graph of F at $(x, F(x))$).
3. It can be viewed as the truly fundamental part of the theorem.
4. Often, $F(b) - F(a)$ is written as $F(x)\Big|_a^b$, or $[F(x)]_a^b$ or $F(x)\Big|_{x=a}^{x=b}$.

From Theorem 6.4 we see that we can **evaluate the definite integral of a function** f on the interval $[a, b]$, and thus (if f is nonnegative) **determine the area below its graph** as follows.

Step 1: Find an antiderivative F of f. (Any antiderivative F of f works.)
Step 2: Evaluate $F(b)$ and $F(a)$ and calculate

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Example 6.8 Using the Fundamental Theorem of Calculus, compute the area

1. below $y = x^2$ for the interval $0 \le x \le 1$,
2. below $y = \cos x$ for the interval $[0, \pi/2]$.

Solution:

1. Let $f(x) = x^2$. The general form of its antiderivative F (that is, its indefinite integral) is $F(x) = \frac{1}{3}x^3 + C$. Since we may choose any antiderivative, we set $C = 0$. (The constant C would cancel out in $F(b) - F(a)$ anyway.) Therefore, the sought-for area is given by

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}.$$

2. Let $f(x) = \cos x$. The function $F(x) = \sin x$ yields an antiderivative of f. Therefore the area below $y = \cos x$ on the interval $[0, \pi/2]$ is

$$\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1 \, .$$

Properties of the definite integral. Throughout this subsection, let f and g be integrable functions on the interval $[a, b]$. We have the rules for constant multiples, sums and differences,

$$\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx \, , \quad \text{for any constant } c, \tag{6.21}$$

$$\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \, . \tag{6.22}$$

They can be combined into the property of **linearity**,

$$\int_a^b (\alpha f(x) + \beta g(x)) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx \, , \tag{6.23}$$

where α, β are arbitrary constants. In addition, the property of **monotonicity** holds: If f is nonnegative on $[a, b]$, that is, $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f(x) \, dx \geq 0 \, , \tag{6.24}$$

and if $f \geq g$ on $[a, b]$, that is, $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \, . \tag{6.25}$$

Moreover, we may split a definite integral on $[a, b]$ into two integrals on $[a, c]$ and $[c, b]$ with $a < c < b$,

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \, . \tag{6.26}$$

The formulas (6.21)–(6.26) can be proved as a consequence of the formal definition of the definite integral.

So far, in accordance with its definition, we have assumed that $a < b$. However, for computations it can be convenient if we drop this restriction and define

$$\int_c^c f(x) \, dx = 0 \, , \tag{6.27}$$

whenever c belongs to the domain of f, as well as

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx\,. \tag{6.28}$$

It turns out that with these extended definitions, (6.26) holds no matter how a, b, and c are related, as long as f is integrable on the respective intervals.

We now present the **substitution rule for definite integrals**.

Theorem 6.5 *Let g be a continuously differentiable function on the interval $[a, b]$, let f be a continuous function on the range $R(g)$ of g. Then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\,. \tag{6.29}$$

Proof Let F be an antiderivative of f, then

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du\,.$$

Both equalities follow from the Fundamental Theorem, the first because $F \circ g$ is an antiderivative of the left integrand, the second because F is an antiderivative of f.

Setting $g(x) = x + c$ in Theorem 6.5, c being a fixed number, we obtain the formula

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx\,. \tag{6.30}$$

(Note that we have replaced the letter 'u' for the integration variable in the second integral by 'x'.) Setting $g(x) = cx$, $c \neq 0$ being a fixed number, we get

$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ca}^{cb} f(x)\,dx\,. \tag{6.31}$$

In the particular case $c = -1$, (6.31) becomes

$$\int_a^b f(-x)\,dx = -\int_{-a}^{-b} f(x)\,dx\,. \tag{6.32}$$

From this formula, we obtain

$$\int_{-a}^0 f(x)\,dx = \int_0^a f(x)\,dx\,, \qquad \text{if } f \text{ is an even function,} \tag{6.33}$$

$$\int_{-a}^0 f(x)\,dx = -\int_0^a f(x)\,dx\,, \qquad \text{if } f \text{ is an odd function.} \tag{6.34}$$

These formulas imply that

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\,, \qquad \text{if } f \text{ is an even function,} \tag{6.35}$$

$$\int_{-a}^{a} f(x)\,dx = 0\,, \qquad \text{if } f \text{ is an odd function.} \tag{6.36}$$

The next theorem states the **integration by parts** formula for **definite integrals**.

Theorem 6.6 *Let f and g be continuously differentiable functions on $[a, b]$. Then*

$$\int_{a}^{b} f(x)g'(x)\,dx = f(x)g(x)\Big|_{a}^{b} - \int_{a}^{b} f'(x)g(x)\,dx\,. \tag{6.37}$$

The Fundamental Theorem yields this result, too, if we apply it to the function $f \cdot g$ which is an antiderivative of the function $fg' + f'g$.

We close this section with the following theorem.

Theorem 6.7 (Mean Value Theorem for Integrals) *Let f be continuous on the interval $[a, b]$, then there is at least one number c in (a, b) such that*

$$\int_{a}^{b} f(x)\,dx = f(c)(b-a)\,, \quad \text{or} \quad f(c) = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx\,. \tag{6.38}$$

The proof of this theorem will be given in Appendix D.

6.7 Trigonometric Integrals

In this section we discuss the integration of functions which arise as products of sines, cosines, secants, and tangents, as well as integrals of hyperbolic trigonometric functions and inverse trigonometric functions. The method used is mainly integration by parts in combination with trigonometric identities. Moreover, integrals of other functions can often be converted to trigonometric integrals by a suitable substitution. Let us consider an example. We may evaluate

$$\int x^2\sqrt{1-x^2}\,dx$$

using the substitution $x = \sin\theta$ in the way described at the end of Sect. 6.4. This gives us

$$\int \sin^2\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta\,.$$

Let us evaluate the latter integral. We have

$$\int \sin^2\theta\cos^2\theta\,d\theta = \int \sin^2\theta(1-\sin^2\theta)\,d\theta = \int \sin^2\theta\,d\theta - \int \sin^4\theta\,d\theta\,. \tag{6.39}$$

By Example 6.7(9),

$$\int \sin^2\theta\,d\theta = -\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta + C_1$$

and

$$\int \sin^4\theta\,d\theta = -\frac{1}{4}\cos\theta\sin^3\theta - \frac{3}{8}\cos\theta\sin\theta + \frac{3}{8}\theta + C_2\,.$$

It follows that

$$\int \sin^2\theta\cos^2\theta\,d\theta = -\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\cos\theta\sin^3\theta + \frac{3}{8}\cos\theta\sin\theta - \frac{3}{8}\theta + C\,.$$

We substitute back $\sin\theta = x$, $\cos\theta = \sqrt{1-x^2}$ and $\theta = \sin^{-1}(x) = \arcsin x$ to obtain the final result

$$\begin{aligned}\int x^2\sqrt{1-x^2}\,dx = &-\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + \frac{1}{4}x^3\sqrt{1-x^2}\\ &+\frac{3}{8}x\sqrt{1-x^2} - \frac{3}{8}\sin^{-1}x + C\,.\end{aligned}$$

Formulas for Integration of Products of Trigonometric Functions

1. $\displaystyle\int \sin mx\sin nx\,dx = -\frac{\sin{(m+n)x}}{2(m+n)} + \frac{\sin{(m-n)x}}{2(m-n)} + C.$
2. $\displaystyle\int \cos mx\cos nx\,dx = \frac{\sin{(m+n)x}}{2(m+n)} + \frac{\sin{(m-n)x}}{2(m-n)} + C.$
3. $\displaystyle\int \sin mx\cos nx\,dx = -\frac{\cos{(m+n)x}}{2(m+n)} - \frac{\cos{(m-n)x}}{2(m-n)} + C.$
4. $$\begin{aligned}\int \sin^m x\cos^n x\,dx &= \frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx\\ &= \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx.\end{aligned}$$
5. $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

6. $\displaystyle\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx.$
7. $\displaystyle\int \sec^n x\,dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$
8. $\displaystyle\int \cot^n x\,dx = \frac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\,dx.$
9. $\displaystyle\int \csc^n x\,dx = \frac{\cot x \csc^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx\ .$

Integration and Differentiation of Hyperbolic Trigonometric Functions. We have introduced hyperbolic trigonometric functions in Sect. 1.3. Here, we discuss the differentiation and integration of such functions. Before doing this, we want to mention some practical situations where those functions arise. Hyperbolic cosine functions can be used to describe the shape of a uniform cable or chain, whose ends are fixed at the same height. Telephone and power lines may be strung between poles in this manner. As another example, let us consider a falling object. If we assume air resistance to be proportional to the square of its velocity, then the vertical distance y covered by the object in t seconds is given by $y = a\ln(\cosh bt)$, where a and b are constants.

Let us recall the two identities

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

From those and the other defining formulas for hyperbolic trigonometric functions as presented in Sect. 1.3, one may obtain the following identities:

(i) $\cosh x + \sinh x = e^x$,
(ii) $\cosh x - \sinh x = e^{-x}$,
(iii) $\cosh^2 x - \sinh^2 x = 1$,
(iv) $1 - \tanh^2 x = \operatorname{sech}^2 x$,
(v) $\coth^2 x - 1 = \operatorname{csch}^2 x$,
(vi) $\cosh(-x) = \cosh x$,
(vii) $\sinh(-x) = -\sinh x$,
(viii) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$,
(ix) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$,
(x) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$,
(xi) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$,
(xii) $\sinh 2x = 2\sinh x \cosh x$,
(xiii) $\cosh 2x = \cosh^2 x + \sinh^2 x$,
(xiv) $\cosh 2x = 2\sinh^2 x + 1$,
(xv) $\cosh 2x = 2\cosh^2 x - 1\ .$

Table 6.2 Integrals of hyperbolic functions

$\dfrac{d}{dx}\sinh x = \cosh x$	$\displaystyle\int \sinh x\,dx = \cosh x + C$
$\dfrac{d}{dx}\cosh x = \sinh x$	$\displaystyle\int \cosh x\,dx = \sinh x + C$
$\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$	$\displaystyle\int \operatorname{sech}^2 x\,dx = \tanh x + C$
$\dfrac{d}{dx}\coth x = \operatorname{csch}^2 x$	$\displaystyle\int \operatorname{csch}^2 x\,dx = -\coth x + C$
$\dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$	$\displaystyle\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$
$\dfrac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$	$\displaystyle\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C$

Table 6.3 Derivatives of inverse hyperbolic functions

$\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$	$\dfrac{d}{dx}\coth^{-1} x = \dfrac{1}{1-x^2},\ \lvert x\rvert > 1$
$\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}, x > 1$	$\dfrac{d}{dx}(\operatorname{sech}^{-1} x) = -\dfrac{1}{x\sqrt{1-x^2}},\ 0 < x < 1$
$\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2},\ \lvert x\rvert < 1$	$\dfrac{d}{dx}(\operatorname{csch}^{-1} x) = \dfrac{1}{\lvert x\rvert\sqrt{1+x^2}},\ x \neq 0$

Table 6.2 given below provides a complete list of the derivative formulas and corresponding integration formulas for the hyperbolic functions.
We check here the first line in Table 6.2. We have

$$\begin{aligned}\frac{d}{dx}\sinh x &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right] = \frac{e^x + e^{-x}}{2}\\ &= \cosh x\,.\end{aligned}$$

Therefore, we also have $\displaystyle\int \cosh x = \sinh x + C$ by definition of the indefinite integral. Tables 6.3 and 6.4 provide a list of derivative and integration formulas for inverse hyperbolic functions.

Table 6.4 Functions whose antiderivatives are inverse hyperbolic functions

If $a > 0$, then
$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad \text{or} \quad \ln\left(x + \sqrt{x^2+a^2}\right) + C$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad \text{or} \quad \ln\left(x + \sqrt{x^2-a^2}\right) + C$
$\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C\,, & \lvert x\rvert < a \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C, & \lvert x\rvert > a \end{cases} \quad \text{or} \quad \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + C,\ \lvert x\rvert \neq a$
$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left\lvert\frac{x}{a}\right\rvert + C \quad \text{or} \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{\lvert x\rvert}\right) + C\,, 0 < \lvert x\rvert < a$
$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left\lvert\frac{x}{a}\right\rvert + C \quad \text{or} \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+x^2}}{\lvert x\rvert}\right) + C\,, 0 < \lvert x\rvert < a$

Differentiation and Integration of Inverse Trigonometric functions. The inverse trigonometric functions (cyclometric functions) are introduced in Appendix C.3. We discuss here the differentiation and integration of these functions.

The following identities hold for the inverse trigonometric functions:

(i) $\cos(\arcsin x) = \sqrt{1-x^2}$.
(ii) $\sin(\arccos x) = \sqrt{1-x^2}$.
(iii) $\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$.
(iv) $\sec(\arctan x) = \sqrt{1+x^2}$.
(v) $\sin(\operatorname{arcsec} x) = \dfrac{\sqrt{1-x^2}}{x}$, if $x \geq 1$.

Let us check identity (i). We have

$$(\cos(\arcsin x))^2 + (\sin(\arcsin x))^2 = 1\,.$$

Since $\sin(\arcsin x) = x$, we get $(\cos(\arcsin x))^2 = 1 - x^2$, and therefore (i) holds. Similarly, the other identities are obtained.

Example 6.9 Find the derivatives of $\arcsin x$, $\arccos x$, and $\arctan x$.

Solution: We differentiate the identity $x = \sin(\arcsin x)$ and obtain from the chain rule that

$$1 = \cos(\arcsin x) \cdot (\arcsin)'(x)\,,$$

hence, using (i) above,

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}\,. \tag{6.40}$$

Similarly, from the identity $x = \cos(\arccos x)$ and (ii) above we obtain

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}} \,. \tag{6.41}$$

To find the derivative of $\arctan x$, we differentiate the identity $x = \tan(\arctan x)$ which yields, together with (iv) above,

$$1 = \sec^2(\arctan x) \cdot (\arctan)'(x) = (1 + x^2) \cdot (\arctan)'(x) \,,$$

so

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \,.$$

We can now derive the identity

$$\arcsin x + \arccos x = \frac{\pi}{2} \,. \tag{6.42}$$

Indeed, we have

$$\frac{d}{dx}(\arcsin x + \arccos x) = 0$$

by (6.40) and (6.41), and therefore the left-hand side of (6.42) must be constant. Since $\arcsin 0 = 0$ and $\arccos 0 = \pi/2$, (6.42) is proved.

Example 6.10 Evaluate the indefinite integral $\int \arcsin x \, dx$.

Solution: We use integration by parts. In the formula $\int u \, dv = uv - \int v \, du$ we set $u = \arcsin x$, $dv = dx$. This yields $du = \dfrac{dx}{\sqrt{1-x^2}}$, and we take $v = x$ as antiderivative of the constant 1. It follows that

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx \,.$$

We now substitute $w = 1 - x^2$. Then $dw = -2x \, dx$ and

$$\int \frac{x}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int \frac{dw}{\sqrt{w}} = -\sqrt{w} + C \,,$$

so finally

$$\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + C \,.$$

Example 6.11 Evaluate the indefinite integral $\displaystyle\int \frac{dx}{25x^2 - 1}$, where $x > \dfrac{1}{5}$.

Solution: Let $u = 5x$. Thus $du = 5\,dx$ and

$$\begin{aligned}\int \frac{dx}{\sqrt{25x^2-1}} &= \frac{1}{5}\int \frac{5}{\sqrt{25x^2-1}}\,dx = \frac{1}{5}\int \frac{du}{\sqrt{u^2-1^2}} = \frac{1}{5}\cosh^{-1} u + C \\ &= \frac{1}{5}\cosh^{-1} 5x + C\,.\end{aligned}$$

6.8 Partial Fractions and Integration

Here, we discuss the integration of rational functions

$$f(x) = \frac{P(x)}{Q(x)}\,,$$

which we have introduced in Definition 1.7 as the quotient of two polynomials P and Q. We present the method of **partial fraction expansion** by means of several examples.

Example 6.12 Find $\displaystyle\int \frac{-4x+16}{(x-2)(x+2)x}\,dx$.

Solution: The idea is to decompose the rational function as a sum of simpler fractions,

$$\frac{P(x)}{Q(x)} = \frac{-4x+16}{(x-2)(x+2)x} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}\,, \tag{6.43}$$

where the unknown coefficients A, B, and C have to be determined. To this purpose, we compute

$$\begin{aligned}\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x} &= \frac{Ax^2 + 2xA + Bx^2 - 2xB + Cx^2 - 4C}{(x-2)(x+2)x} \\ &= \frac{(A+B+C)x^2 + (2A-2B)x - 4C}{(x-2)(x+2)x}\,.\end{aligned}$$

In order that (6.43) holds, we must have

$$P(x) = -4x + 16 = (A+B+C)x^2 + (2A-2B)x - 4C\,.$$

Comparing coefficients we get the system

$$A + B + C = 0\,, \quad 2A - 2B = -4\,, \quad -4C = 16\,,$$

of 3 linear equations for the 3 unknowns A, B, C. The third equation gives $C = -4$, and we are left with

$$A + B = 4\,, \quad A - B = -2\,,$$

which we add to obtain $2A = 2$, $A = 1$, and therefore $B = 3$. We may use these numbers in (6.43) to obtain finally

$$\int \frac{-4x+16}{(x-2)(x+2)x}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{3}{x+2}\,dx + \int \frac{-4}{x}\,dx$$
$$= \ln|x-2| + 3\ln|x+2| - 4\ln|x| + C\,.$$

Example 6.13 Evaluate $\displaystyle\int \frac{3x^2+8x-4}{(x-2)(x+2)x}\,dx$.

Solution: As in the previous example, we determine A, B, C such that

$$\frac{3x^2+8x-4}{(x-2)(x+2)x} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x}$$
$$= \frac{A(x+2)x + B(x-2)x + C(x-2)(x+2)}{(x-2)(x+2)x}$$

holds. Comparing the coefficients of x^2, x and the constant term we find that $A = 3$, $B = -1$, $C = 1$. Thus,

$$\int \frac{3x^2+8x-4}{(x-2)(x+2)x}\,dx = \int \frac{3}{x-2}\,dx + \int \frac{-1}{x+2}\,dx + \int \frac{dx}{x}$$
$$= 3\ln|x-2| - \ln|x+2| + \ln|x| + C\,.$$

Example 6.14 Evaluate $\displaystyle\int \frac{2x+5}{(x+2)^2(x+3)^2}\,dx$.

Solution: Because of the double factors in the denominator, we use a slightly different partial fraction expansion, namely

$$\frac{2x+5}{(x+2)^2(x+3)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+3)} + \frac{D}{(x+3)^2} \tag{6.44}$$

with the 4 unknown coefficients A, B, C, D. They must satisfy

$$2x+5 = A(x+2)(x+3)^2 + B(x+3)^2 + C(x+3)(x+2)^2 + D(x+2)^2\,.$$

We have to compare the coefficients of the 4 terms x^3, x^2, x, and the constant. When we do this, we obtain $A = C = 0$, $B = 1$, $D = -1$. Therefore,

$$\int \frac{2x+5}{(x+2)^2(x+3)^2}\,dx = \int \frac{1}{(x+2)^2}\,dx - \int \frac{1}{(x+3)^2}\,dx$$
$$= -\frac{1}{x+2} + \frac{1}{x+3} + C\,.$$

Let us remark that often we have to carry out the factorization of the denominator Q as a preliminary step. If in Example 6.12, we had been asked for the indefinite integral

$$\int \frac{-4x+16}{x^3-4x}\,dx\,,$$

we first should have factorized $Q(x)$ as $x^3-4x=(x-2)(x+2)x$ and then continued as before. In general, it may also happen that $Q(x)$ not only contains linear factors, but also quadratic factors which cannot be decomposed further, like, for example, x^2+1. This case arises when the associated quadratic equation (here $x^2+1=0$) has no real solutions. We do not discuss this case here.

6.9 Improper Integrals

In many applications, we are required to evaluate integrals which have unbounded intervals of integration (that is, the lower limit of integration is $-\infty$, or the upper limit is $+\infty$, or both), or integrals whose integrand tends to infinity at an interior or boundary point of the integration interval. Such integrals are called **improper integrals**. To motivate their definition, assume we want to find the area of the region R below the curve $y=f(x)=\frac{1}{x^2}$, extending from the vertical line $x=1$ to the right without bound. For any number $b>1$, we may partition R into two subregions by the vertical line $x=b$. As b gets larger and larger, the area of the part where $x\geq b$ becomes smaller and smaller. If it tends to zero (which we will see below for this example), it makes sense to define the area $\int_1^\infty \frac{1}{x^2}\,dx$ of the region R as the limit (for $b\to\infty$) of the area $\int_1^b \frac{1}{x^2}\,dx$ to the left of the vertical $x=b$.

Definition 6.3 (*Infinite limit of integration*) Let f be a function which is integrable on every bounded interval $[a,b]$ with $b>a$ (this holds, for example, when f is continuous on $[a,\infty)$). Then the improper integral of f over $[a,\infty)$ is defined by

$$\int_a^\infty f(x)\,dx=\lim_{b\to\infty}\int_a^b f(x)\,dx \tag{6.45}$$

whenever the limit exists and is a finite number. In this case, the improper integral (6.45) is said to be convergent; otherwise, it is said to be divergent.

Note that the improper integral is a special case of the improper limit of a function as discussed in Sect. 2.4. Namely, if we set

$$F(b)=\int_a^b f(x)\,dx\,,$$

then the improper integral (6.45) equals the improper limit of $F(b)$ as $b \to \infty$.

Improper integrals of f over $(-\infty, b]$ are defined analogously.

The improper integral of f over the whole real line $(-\infty, \infty)$ is defined as

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx\,, \quad c \in \mathbb{R}\,, \tag{6.46}$$

if both integrals on the right-hand side are convergent. In this case, $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent. (It does not matter which value of c we choose in (6.46), since the sum of the two integrals on the right- hand side will always be the same.)

Example 6.15 Evaluate the following improper integrals:

1. $\displaystyle\int_2^{\infty} \frac{1}{x^2}\,dx$,
2. $\displaystyle\int_0^{\infty} e^{-3x}\,dx$,
3. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$,
4. $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$,
5. $\displaystyle\int_{-\infty}^{\infty} \frac{(\arctan x)^2}{1+x^2}\,dx$.

Solution: In all instances, the computations will show that the improper integrals exist, so the passage to the limit is justified.

1.
$$\int_2^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_2^b = -\lim_{b\to\infty} \frac{1}{b} + \lim_{b\to\infty} \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}\,.$$

2.
$$\int_0^{\infty} e^{-3x}\,dx = \lim_{b\to\infty} \int_0^{b} e^{-3x}\,dx = \lim_{b\to\infty} \left[-\frac{e^{-3x}}{3}\right]_0^b = -\frac{1}{3}[\lim_{b\to\infty} e^{-3b} - 1] = \frac{1}{3}\,.$$

3. Choosing $c = 0$ in (6.46), we partition the integral as

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx.$$

We compute the first integral on the right-hand side,

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_{a}^{0} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \tan^{-1} x\Big|_{a}^{0} = 0 - \lim_{a\to-\infty} \tan^{-1} x$$
$$= \frac{\pi}{2}\,.$$

Since the function $f(x) = 1/(1+x^2)$ is even, we have

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}\,,$$

and therefore $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi\,.$

4. Choosing $c = 0$ in (6.46), we have

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx\,. \tag{6.47}$$

We evaluate the first integral on the right-hand side,

$$\begin{aligned}
\int_{-\infty}^{0} xe^{-x^2}\,dx &= \lim_{a\to-\infty}\int_{-a}^{0} xe^{-x^2}\,dx \\
&= \lim_{a\to-\infty} \frac{1}{2}\int_{a^2}^{0} e^{-t}\,dt \qquad \text{(using the substitution } x^2 = t) \\
&= \lim_{a\to-\infty} -\frac{1}{2}e^{-t}\Big|_{a^2}^{0} = \lim_{a\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-a^2}\right) \\
&= -\frac{1}{2}\,.
\end{aligned}$$

Since the function $f(x) = xe^{-x^2}$ is odd, we have

$$\int_{0}^{\infty} xe^{-x^2}\,dx = -\int_{-\infty}^{0} xe^{-x^2}\,dx = \frac{1}{2}\,,$$

so

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0\,.$$

5. As above, we partition with $c = 0$,

$$\int_{-\infty}^{\infty} \frac{(\arctan x)^2}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{(\arctan x)^2}{1+x^2}\,dx + \int_{0}^{\infty} \frac{(\arctan x)^2}{1+x^2}\,dx\,.$$

The antiderivative we need is given by

$$\int \frac{(\arctan x)^2}{1+x^2}\,dx = \frac{1}{3}(\arctan x)^3 + C\,.$$

Now $\arctan b \to \frac{1}{2}\pi$ as $b \to \infty$, and $\arctan a \to -\frac{1}{2}\pi$ as $a \to -\infty$. Therefore,

$$\int_0^\infty \frac{(\arctan x)^2}{1+x^2}\,dx = \lim_{b\to\infty} \frac{1}{3}(\arctan x)^3\Big|_0^b = \frac{1}{3}\left(\frac{1}{2}\pi\right)^3 = \frac{1}{24}\pi^3\,.$$

Since the integrand is even,

$$\int_{-\infty}^0 \frac{(\arctan x)^2}{1+x^2}\,dx = \int_0^\infty \frac{(\arctan x)^2}{1+x^2}\,dx = \frac{1}{24}\pi^3\,,$$

hence

$$\int_{-\infty}^\infty \frac{(\arctan x)^2}{1+x^2}\,dx = \frac{1}{24}\pi^3 + \frac{1}{24}\pi^3 = \frac{1}{12}\pi^3\,.$$

Remark 6.5 The improper integral

$$\int_1^\infty \frac{1}{x}\,dx \tag{6.48}$$

is a divergent integral. Indeed, we have

$$\int_1^b \frac{1}{x}\,dx = \ln x\Big|_1^b = \ln b - \ln 1 = \ln b\,.$$

We know that $\ln b$ tends to ∞ as $b \to \infty$. Hence, the integral (6.48) is divergent.

We now discuss the case where the integrand tends to infinity at an end point of the integration interval. Let f be a continuous function on $(a, b]$ with $f(x) \to \infty$ (or $-\infty$) for $x \to a$. The improper integral of f on $[a, b]$ is defined as the right-hand limit

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b f(x)\,dx\,, \tag{6.49}$$

whenever this limit exists as a finite number, and the improper integral is then said to be convergent. As an example, consider

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx\,. \tag{6.50}$$

We have

$$\int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_{\varepsilon}^{1} = 2 - 2\sqrt{\varepsilon}\,,$$

so we get

$$\int_{0}^{1} \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0+}\int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\,dx = 2\,.$$

Analogously, when $f(x) \to \infty$ (or $-\infty$) for $x \to b$ on $[a, b]$, we define

$$\int_{a}^{b} f(x)\,dx = \lim_{\varepsilon\to 0+}\int_{a}^{b-\varepsilon} f(x)\,dx\,.$$

The integral test. We have discussed the notion of convergence for infinite series in Chap. 5. The so-called integral test relates the convergence behavior of a series to that of an improper integral.

Theorem 6.8 (Integral Test) *Let f be a continuous nonincreasing function defined for all $x \geq 1$. For $n = 1, 2, 3, \ldots$, we set $s_n = f(n)$. Then*

$$\textit{the series } \sum_{n=1}^{\infty} s_n \textit{ and the improper integral } \int_{1}^{\infty} f(x)\,dx$$

either both converge or both diverge.

The proof of this theorem will be given in Appendix D.

While this test works both ways, it is mainly used to deduce the convergence of a series from the convergence of the corresponding improper integral.

Example 6.16 Show that the improper integral $\displaystyle\int_{a}^{\infty} \frac{1}{x^p}\,dx$, where $a > 0$, diverges if $p \leq 1$ and converges if $p > 1$. As a consequence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$. In particular, the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution: Suppose $p \neq 1$. Then

$$\int_{a}^{b} \frac{1}{x^p}\,dx = -\frac{1}{p-1}.\frac{1}{x^{p-1}}\Big|_{a}^{b} = \frac{1}{p-1}\left(\frac{1}{a^{p-1}} - \frac{1}{b^{p-1}}\right).$$

We have

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0\,, & \text{if } p-1 > 0\,,\\ +\infty\,, & \text{if } p-1 < 0\,. \end{cases}$$

Hence, if $p > 1$ then

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_a^b \frac{1}{x^p}\,dx = \frac{1}{p-1}\cdot\frac{1}{a^{p-1}}\,.$$

If $p < 1$, then

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_a^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\frac{1}{1-p}(b^{1-p}-a^{1-p}) = +\infty\,.$$

In the case $p = 1$ we have (see Remark 6.5 above)

$$\int_a^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}(\ln b - \ln a) = +\infty\,.$$

Therefore, the improper integral converges for $p > 1$ and diverges for $p \le 1$.

Example 6.17 Use the integral test to determine whether the infinite series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$ is convergent.

Solution: The nth term of the series is $s_n = \dfrac{n}{e^{n^2}}$, so we take $f(x) = \dfrac{x}{e^{x^2}} = xe^{-x^2}$. This function is continuous and positive for all positive values of x. Moreover,

$$f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2}(1-2x^2)\,.$$

Since $f'(x)$ is negative for $x > 1$, f is decreasing on $[1,\infty)$. The assumptions of the integral test (Theorem 6.8) are satisfied. To apply the test, we compute

$$\int_1^b xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_1^b = -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1}\,.$$

Therefore, the improper integral (the limit as $b \to \infty$) converges,

$$\int_1^\infty xe^{-x^2}\,dx = \frac{1}{2}e^{-1}\,.$$

We thus conclude from the integral test that the given infinite series is convergent.

6.10 Additional Tables of Integrals

Forms involving $a + bx$

1. $\displaystyle\int \frac{x}{a+bx}\,dx = \frac{1}{b^2}[a + bx - a\ln|a+bx|] + C.$

2. $\displaystyle\int \frac{x^2}{a+bx}\,dx = \frac{1}{2b^3}[(a+bx)^2 - 4a(a+bx) + 2a^2\ln|a+bx|] + C.$
3. $\displaystyle\int \frac{x}{(a+bx)^2}\,dx = \frac{1}{b^2}\left[\frac{a}{a+bx} + \ln|a+bx|\right] + C.$
4. $\displaystyle\int \frac{x^2}{(a+bx)^2}\,dx = \frac{1}{b^3}\left[a+bx - \frac{a^2}{a+bx} - 2a\ln|a+bx|\right] + C.$
5. $\displaystyle\int \frac{x}{\sqrt{a+bx}}\,dx = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx} + C.$
6. $\displaystyle\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| + C,\ a > 0.$

Forms involving $\sqrt{a^2+x^2}$

7. $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \ln|x+\sqrt{a^2+x^2}| + C.$
8. $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+x^2}+a}{x}\right| + C.$
9. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C.$
10. $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C.$
11. $\displaystyle\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln|x+\sqrt{a^2+x^2}| + C$
12. $\displaystyle\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2} - \frac{a^4}{8}\ln|x+\sqrt{a^2+x^2}| + C$

Forms involving $\sqrt{x^2-a^2}$

13. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + C.$
14. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C.$
15. $\displaystyle\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}} + C.$
16. $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C.$
17. $\displaystyle\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln|x+\sqrt{x^2-a^2}| + C.$
18. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln|x+\sqrt{x^2-a^2}| + C.$

Forms involving $\sqrt{a^2-x^2}$

19. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C.$

20. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C.$
21. $\displaystyle\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C.$
22. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C..$

Forms involving e^{ax} and $\ln x$

23. $\displaystyle\int xe^{ax}\,dx = \frac{1}{a^2}(ax-1)e^{ax} + C$
24. $\displaystyle\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx.$
25. $\displaystyle\int \frac{dx}{1+be^{ax}} = x - \frac{1}{a}\ln(1+be^{ax}) + C.$
26. $\displaystyle\int \ln x\,dx = x\ln x - x + C.$
27. $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx.$
28. $\displaystyle\int x^n\ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C\,, \quad n \neq -1.$
29. $\displaystyle\int \frac{dx}{x\ln x} = \ln|\ln x| + C.$
30. $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$
31. $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}.$

We present some examples.

Example 6.18 Use the table of integrals to evaluate $\displaystyle\int \frac{x}{(1+2x)^2}\,dx.$

Solution: Since $(1+2x)^2$ is of the form $(a+bx)^2$ with $a=1$ and $b=2$, we use Formula 3 to obtain

$$\int \frac{x}{(1+2x)^2}\,dx = \frac{1}{4}\left[\frac{1}{1+2x} + \ln|1+2x|\right] + C\,.$$

Example 6.19 Use the table of integrals to evaluate $\displaystyle\int_3^4 \frac{dx}{x\sqrt{50-2x^2}}.$

Solution: We first evaluate the indefinite integral

$$I = \int \frac{dx}{x\sqrt{50-2x^2}}\,.$$

We have $\sqrt{50-2x^2} = \sqrt{2}\sqrt{25-x^2}$, so that

$$I = \frac{1}{\sqrt{2}} \int \frac{dx}{x\sqrt{25 - x^2}} \,.$$

Using Formula 19 with $a = 5$, we get

$$I = \frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{5}\right) \ln \left| \frac{5 + \sqrt{25 - x^2}}{x} \right| + C \,.$$

For $3 \le x \le 4$, the argument of the logarithm is positive, so we can compute

$$\begin{aligned}
\int_3^4 \frac{dx}{x\sqrt{50 - 2x^2}} &= -\frac{\sqrt{2}}{10} \ln \left(\frac{5 + \sqrt{25 - x^2}}{x} \right) \Bigg|_3^4 \\
&= -\frac{\sqrt{2}}{10} \ln \frac{5 + \sqrt{25 - 16}}{4} + \frac{\sqrt{2}}{10} \ln \frac{5 + \sqrt{25 - 9}}{3} \\
&= -\frac{\sqrt{2}}{10} \ln 2 + \frac{\sqrt{2}}{10} \ln 3 = \frac{\sqrt{2}}{10} (-\ln 2 + \ln 3) \\
&= \frac{\sqrt{2}}{10} \ln \frac{3}{2} \,.
\end{aligned}$$

Example 6.20 Use the table of integrals to evaluate $\displaystyle\int \frac{dx}{1 + e^{-x}}$.

Solution: Using Formula 25 with $a = -1$, $b = 1$, we get

$$\int \frac{dx}{1 + e^{-x}} = x + \ln(1 + e^{-x}) + C \,.$$

Example 6.21 Use the table of integrals to evaluate $\displaystyle\int_0^2 \frac{dx}{(5 - x^2)^{3/2}}$.

Solution: We first evaluate the indefinite integral

$$I = \int \frac{dx}{(5 - x^2)^{3/2}} \,.$$

Using Formula 21 with $a = \sqrt{5}$, we get

$$I = \frac{x}{5\sqrt{5 - x^2}} + C \,.$$

Now

$$\int_0^2 \frac{dx}{(5 - x^2)^{3/2}} = \frac{1}{5} \cdot \frac{x}{\sqrt{5 - x^2}} \Bigg|_0^2 = \frac{1}{5} \cdot \left[\frac{2}{\sqrt{5 - 4}} - 0 \right] = \frac{2}{5} \,.$$

6.11 Exercises

Evaluate the following integrals:

6.11.1 $\int (4x+5)\,dx$,

6.11.2 $\int (9t^2-5t+9)\,dt$,

6.11.3 $\int \left(3\sqrt{u}+\frac{2}{\sqrt{u}}\right)du$,

6.11.4 $\int (5z^{-7}+7z^{-3}-z)\,dz$,

6.11.5 $\int (x^{2/3}-4x^{-1/5}+4)\,dx$,

6.11.6 $\int u(1+u^2)\,du$,

6.11.7 $\int (2x^{-1}-\sqrt{2}e^x)\,dx$,

6.11.8 $\int (x^{2/3}-\sin x)\,dx$,

6.11.9 $\int \frac{\sec x}{\cos x}\,dx$,

6.11.10 $\int \frac{\sec u \sin u}{\cos u}\,du$,

6.11.11 $\int (1+\sin^2\theta\csc\theta)\,d\theta$,

6.11.12 $\int \frac{\sin 2\theta}{\cos\theta}\,d\theta$,

6.11.13 $\int \frac{(1+\cot^2 x)\cot x}{\csc x}\,dx$,

6.11.14 Show that $\sin^2 x$, $-\cos^2 x$ and $-\frac{1}{2}\cos 2x$ are antiderivatives of $2\sin x\cos x$.

Evaluate the following integrals using the given substitution and express the answer in terms of x:

6.11.15 $\int 2x(x^2+1)^9\,dx$; $u=x^2+1$,

6.11.16 $\int \frac{x}{(x^2+6)}\,dx$; $u=x^2+6$,

6.11.17 $\int \frac{\sin 3x}{(1+\cos 3x)}\,dx$; $u=1+\cos 3x$,

6.11.18 $\int e^{2x}\sqrt{1+e^{2x}}\,dx$, $u=1+e^{2x}$.

Evaluate the following integrals:

6.11.19 $\int \frac{\sin 2x}{\sqrt{1-\cos 2x}}\,dx$,

6.11.20 $\displaystyle\int \sin x(1+\cos x)^2\,dx$,

6.11.21 $\displaystyle\int \sin^5 3\theta \cos 3\theta\, d\theta$.

6.11.22 Evaluate the following integrals

(a) $\displaystyle\int_{-6}^{12} 12\,dx$ (b) $\displaystyle\int_{-1}^{4}(4-6x)\,dx$. (c) $\displaystyle\int_{3}^{3} 900\,dx$, (d) $\displaystyle\int_{-4}^{4} 2\,dx$.

(e) $\displaystyle\int_{2}^{8}(2x^2+5x+2)\,dx$, (f) $\displaystyle\int_{0}^{2}\sqrt{4-x^2}\,dx$, (g) $\displaystyle\int_{-\pi/3}^{\pi/3}\sin x\,dx$.

6.11.23 Express as a single integral

(a) $\displaystyle\int_{6}^{1} f(x)\,dx + \int_{-3}^{6} f(x)\,dx$

(b) $\displaystyle\int_{-2}^{6} f(x)\,dx - \int_{-2}^{2} f(x)\,dx$

(c) $\displaystyle\int_{d}^{h} f(x)\,dx - \int_{g}^{h} f(x)\,dx$.

6.11.24 Evaluate the following integrals:

(a) $\displaystyle\int_{-3}^{7} f(x)\,dx$, where $f(x)=|x-4|$.

(b) $\displaystyle\int_{-2}^{8} g(x)\,dx$, where $g(x)=\begin{cases}1 \text{ if } -2\le x\le 3\\ x \text{ if } 3\le x\le 8\end{cases}$

6.11.25 Use the integration by parts formula to evaluate the following integrals:

(a) $\displaystyle\int \ln x\,dx$, (b) $\displaystyle\int(\cos x)^2\,dx$, (c) $\displaystyle\int \tan^{-1}x\,dx$,

(d) $\displaystyle\int x^3e^x\,dx$, (e) $\displaystyle\int_0^1 xe^{-3x}\,dx$. (f) $\displaystyle\int_0^4 \ln(x^2+1)\,dx$

6.11.26 Evaluate the following improper integrals provided they are convergent:

(a) $\displaystyle\int_0^\infty e^{-x}\,dx$, (b) $\displaystyle\int_{-1}^\infty \frac{x}{1+x^2}\,dx$, (c) $\displaystyle\int_5^\infty \frac{2}{x^2-1}\,dx$,

(d) $\displaystyle\int_{-\infty}^0 \frac{e^x}{3-2e^x}\,dx$, (e) $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$.

6.11.27 Evaluate $\displaystyle\int \frac{6x^2+13x+6}{(x+2)(x+1)^2}\,dx$, using partial fractions.

6.11.28 Evaluate the following integrals:

(a) $\displaystyle\int \cos^5\theta\,d\theta$, (b) $\displaystyle\int \sin^3 x\cos^3 x\,dx$,

(c) $\displaystyle\int_0^{\pi/6}\sec^3\theta\tan\theta\,d\theta$, (d) $\displaystyle\int_0^{\pi/3}\sin^4 3x\cos^3 3x\,dx$,

(e) $\displaystyle\int \frac{\cos\theta}{\sqrt{2-\sin^2\theta}}\,d\theta$, (f) $\displaystyle\int_0^3 \frac{x^3}{(3+x^2)^{5/2}}\,dx$.

Chapter 7
Applications of Integration

In the previous chapter, we have introduced the notion of the integral as the limit of certain sums known as Riemannian sums, and we have seen that it yields the area below the graph of a given nonnegative function $f(x)$ defined over a closed interval $[a, b]$. In Sect. 7.1, we apply integration to find the area for several variants of this basic situation, involving curves below the x-axis, partly above and partly below the x-axis, two curves, or a curve and the y-axis. Sect. 7.2 is devoted to the determination of the length of plane curves, the area of surfaces of revolution as well as the volume of solids of revolution. The interpretation of the integral as an average is discussed in Sect. 7.3. Applications of integration to problems from finance and business are presented in Sect. 7.4. The modeling of basic concepts of mechanics like mechanical work and forces in terms of integrals is discussed in Sect. 7.5. In Sect. 7.6, we show that integrals arise in elementary probability theory as well. A fairly large number of exercises are given in Sect. 7.7.

7.1 Areas Under Curves

Curves Above the x-Axis. Let a curve $y = f(x)$ lie above the x-axis. We have already explained in the beginning of the previous chapter that the area below the curve between $x = a$ and $x = b$ is given by the definite integral

$$\int_a^b f(x)\,dx$$

as the result of a limiting process involving the areas $f(x)\delta x$ of small approximating rectangles (see Fig. 7.1).

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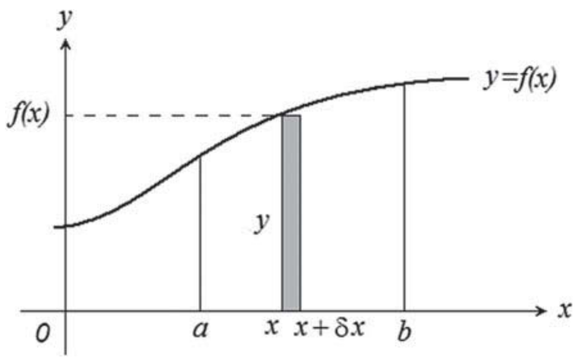


Fig. 7.1 A small approximating rectangle

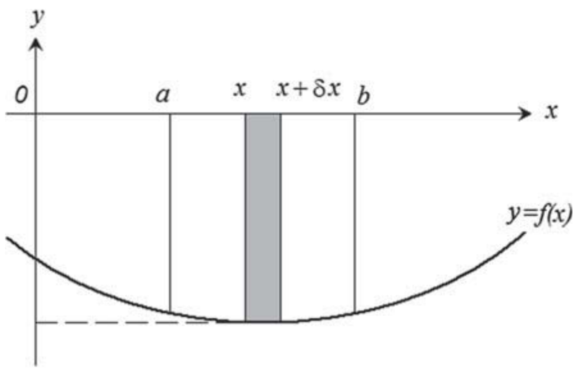


Fig. 7.2 Approximating rectangle for a curve below the x-axis

Example 7.1 Find the area under the curve $y = f(x) = x^2 - 4x + 5$ between $x = -1$ and $x = 2$.

Solution: The area is obtained from the Fundamental Theorem of Calculus as

$$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} (x^2 - 4x + 5)\,dx = \left[\frac{1}{3}x^3 - 2x^2 + 5x\right]_{x=-1}^{x=2}$$
$$= \left(\frac{8}{3} - 8 + 10\right) - \left(-\frac{1}{3} - 2 - 5\right) = 12\,.$$

Curves Below the x-Axis. If a curve $y = f(x)$ lies below the x-axis, the function values $y = f(x)$ are negative (Fig. 7.2). Thus, the numbers $f(x)\delta x$ are negative, and the area of a small approximating rectangle is given by $-f(x)\delta x$. In the limit, the number

$$-\int_a^b f(x)\,dx$$

gives the area between the curve and the x-axis from $x = a$ to $x = b$.

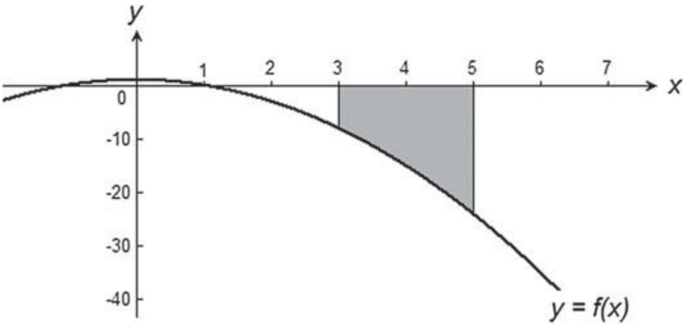


Fig. 7.3 Area between curve and x-axis

Example 7.2 Find the area between the curve $y = f(x) = 1 - x^2$ and the x-axis from $x = 3$ to $x = 5$.

Solution: Between $x = 3$ and $x = 5$, the curve lies below the x-axis (Fig. 7.3). We compute

$$\int_3^5 f(x)\,dx = \int_3^5 (1 - x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{x=3}^{x=5} = \left(5 - \frac{125}{3}\right) - \left(3 - \frac{27}{3}\right)$$
$$= -\frac{92}{3}.$$

Therefore, the required area is $\frac{92}{3}$.

Curves Partly Above and Partly Below the x-Axis. If a curve lies partly below and partly above the x-axis, we first determine the points where the curve intersects the x-axis. Then the areas above and below the x-axis are calculated separately.

Example 7.3 Find the total area between the curve $y = f(x) = x^2 - 4x + 3$ and the x-axis between $x = 0$ and $x = 4$.

Solution: The curve, shown in Fig. 7.4, intersects the x-axis when

$$0 = f(x) = x^2 - 4x + 3 = (x - 1)(x - 3)\,.$$

We therefore have two intersection points, namely, $x = 1$ and $x = 3$. We want to determine the area of the shaded region, which lies partly above and partly below the x-axis. To this purpose, we find the areas A_1, A_2, and A_3 separately. We compute

$$\int_0^1 f(x)\,dx = \int_0^1 (x^2 - 4x + 3)\,dx = \left[\frac{x^3}{3} - 4\frac{x^2}{2} + 3x\right]_{x=0}^{x=1} = \frac{1}{3} - 2 + 3 = \frac{4}{3},$$

therefore $A_1 = \frac{4}{3}$. Next,

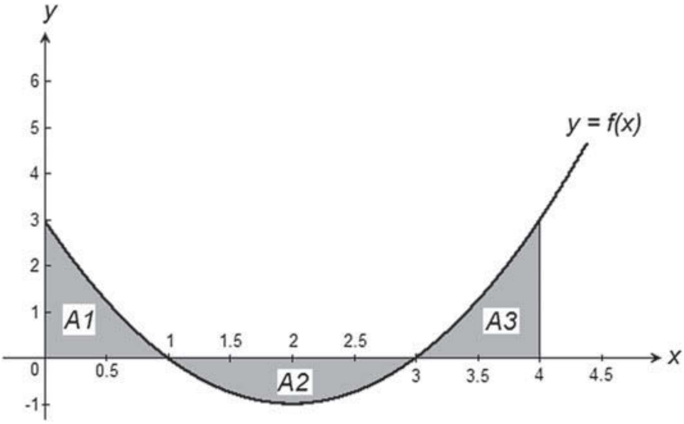


Fig. 7.4 Curve partly above and partly below the x-axis

$$\int_1^3 f(x)\,dx = \int_1^3 (x^2 - 4x + 3)\,dx = \left[\frac{x^3}{3} - 4\frac{x^2}{2} + 3x\right]_{x=1}^{x=3}$$
$$= (9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right) = -\frac{4}{3}\,.$$

Between $x = 1$ and $x = 3$, the curve lies below the x-axis, therefore $A_2 = -\int_1^3 f(x)\,dx = \frac{4}{3}$. Finally,

$$\int_3^4 f(x)\,dx = \int_3^4 (x^2 - 4x + 3)\,dx = \left[\frac{x^3}{3} - 4\frac{x^2}{2} + 3x\right]_{x=3}^{x=4}$$
$$= \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9) = \frac{4}{3}\,,$$

and $A_3 = \frac{4}{3}$.
The total area between the curve and the x-axis becomes

$$A_1 + A_2 + A_3 = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4\,.$$

Note that the definite integral from $x = 0$ to $x = 4$ equals

$$\int_0^4 f(x)\,dx = \int_0^4 (x^2 - 4x + 3)\,dx = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{x=0}^{x=4} = \frac{64}{3} - 32 + 12 = \frac{4}{3}$$
$$= 1\frac{1}{3}\,.$$

We may interpret this as the sum of the separate areas, where areas below the x-axis are counted as negative, that is,

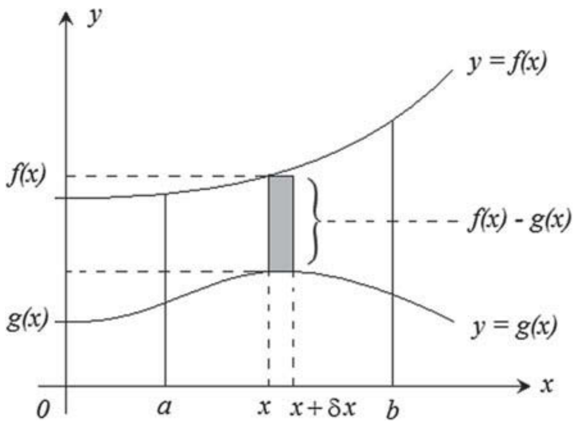


Fig. 7.5 Approximating rectangle between two curves

$$A_1 - A_2 + A_3 = \frac{4}{3} - \frac{4}{3} + \frac{4}{3} = 1\frac{1}{3} .$$

If we wish to find the actual area, we must calculate the areas separately as shown above.

Area Between Two Curves. Consider two curves $y = f(x)$ and $y = g(x)$ as shown in Fig. 7.5. If we divide the area between the two curves, from $x = a$ to $x = b$, into vertical strips, a typical approximating rectangle has height $f(x) - g(x)$ and area $(f(x) - g(x))\delta x$. The limiting process performed in the construction of the definite integral then yields the area between the two curves as

$$\int_a^b (f(x) - g(x))\, dx .$$

Note that as long as the curve $y = f(x)$ lies above the curve $y = g(x)$, the position of the x-axis in relation to the curves does not matter.

Example 7.4 Find the total area between the curves $y = f(x) = x^3 + x^2 - 5x$ and $y = g(x) = x^2 - x$, from $x = -2$ to $x = 2$.

Solution: The two curves intersect when $x^3 + x^2 - 5x = f(x) = g(x) = x^2 - x$, (see Fig. 7.6). Solving this equation for x yields

$$0 = x^3 - 4x = x(x^2 - 4) = x(x + 2)(x - 2) .$$

The intersection points are $x = -2$, $x = 0$ and $x = 2$. The required area equals the area of the shaded region, which consists of two parts whose areas we find separately. From $x = -2$ to $x = 0$, $y = f(x)$ lies above $y = g(x)$, so the area of this part is given by

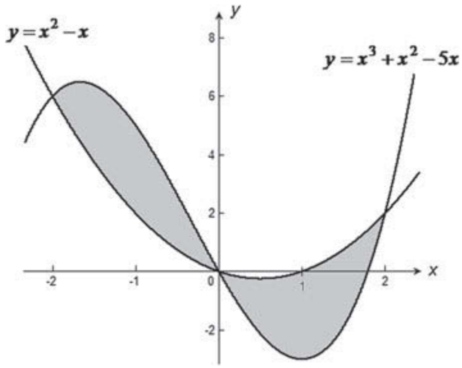


Fig. 7.6 Area between two curves

$$\int_{-2}^{0} f(x) - g(x)\,dx = \int_{-2}^{0} [(x^3 + x^2 - 5x) - (x^2 - x)]\,dx = \int_{-2}^{0} (x^3 - 4x)\,dx$$
$$= \left[\frac{x^4}{4} - 2x^2\right]_{x=-2}^{x=0} = 0 - (4 - 8) = 4\,.$$

From $x = 0$ to $x = 2$, $y = g(x)$ lies above $y = f(x)$ so the area of this part equals

$$\int_{0}^{2} g(x) - f(x)\,dx = \int_{0}^{2} [(x^2 - x) - (x^3 + x^2 - 5x)]\,dx = \int_{0}^{2} (4x - x^3)\,dx$$
$$= \left[2x^2 - \frac{x^4}{4}\right]_{x=0}^{x=2} = 4\,.$$

Therefore, the total area between the two curves equals $4 + 4 = 8$.

Area Between a Curve and the y-Axis. Consider the region bounded by a curve $x = g(y)$, the y-axis and the horizontal lines $y = c$ and $y = d$ (see Fig. 7.7). We divide the region into strips parallel to the x-axis. The area of a typical approximating rectangle equals $g(y)\delta y$. The limit process yields the total area between the curve and the y-axis as

$$\int_{c}^{d} g(y)\,dy\,.$$

Example 7.5 Find the area bounded by the curve given by $y^2 = 1 + 2x$ and the y-axis between $y = 1$ and $y = 3$.

Solution: We rewrite the equation of the curve as

$$x = g(y) = \frac{1}{2}(y^2 - 1)\,.$$

The required area is the area of the shaded region in Fig. 7.8, and computed as

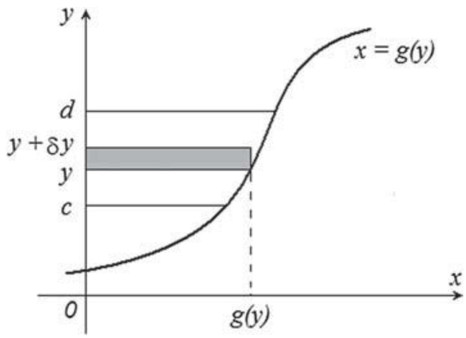


Fig. 7.7 Approximating rectangle for $x = g(y)$

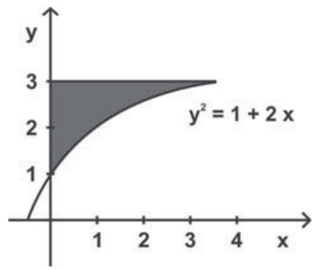


Fig. 7.8 Area between curve and y-axis

$$\int_1^3 g(y)\,dy = \int_1^3 \frac{1}{2}(y^2-1)\,dy = \left[\frac{y^3}{6} - \frac{y}{2}\right]_{y=1}^{y=3} = \left(\frac{9}{2} - \frac{3}{2}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)$$
$$= \frac{10}{3} = 3\frac{1}{3}\,.$$

We illustrate a different method by the following example.

Example 7.6 Find the area between the curve $y = f(x) = (x+1)^2$ and the y-axis, from $y = 4$ to $y = 16$.

Solution: As shown in Fig. 7.9, we may obtain the required area (called C) if we subtract the area of the region B and of the small rectangle A from the area of the large rectangle OMNP. We calculate

$$C = 48 - 4 - \int_1^3 (x+1)^2\,dx = 44 - \int_1^3 (x^2 + 2x + 1)\,dx$$
$$= 44 - \left[\frac{x^3}{3} + x^2 + x\right]_{x=1}^{x=3} = 44 - \left[(9+9+3) - \left(\frac{1}{3} + 1 + 1\right)\right]$$
$$= 44 - \frac{56}{3} = 25\frac{1}{3}\,.$$

Alternatively, we rewrite the equation as $x = g(y) = y^{1/2} - 1$ and apply the method described previously. This yields

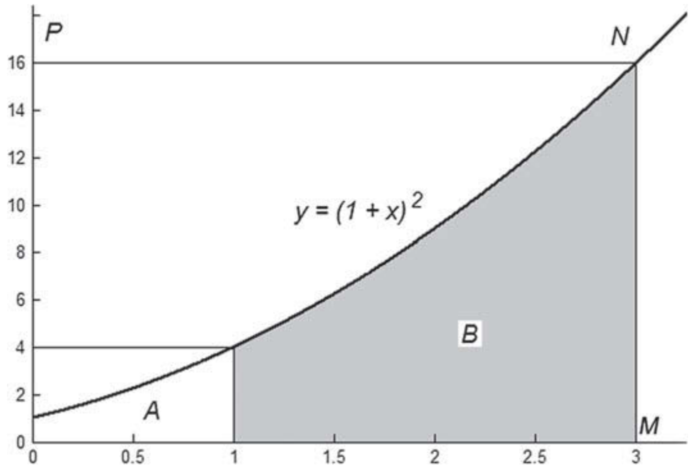


Fig. 7.9 Two ways to compute the area between a curve and the y-axis

$$C = \int_4^{16} g(y)\,dy = \int_4^{16} (y^{1/2} - 1)\,dy = \left[\frac{2}{3}y^{3/2} - y\right]_{y=4}^{y=16}$$
$$= \left(\frac{128}{3} - 16\right) - \left(\frac{16}{3} - 4\right) = 25\frac{1}{3}\,.$$

Area in Polar Coordinates. We want to compute the area of the region

$$G = \{(r, \theta) : \alpha \le \theta \le \beta\,,\ 0 \le r \le f(\theta)\}\,,$$

whose points are given in polar coordinates (r, θ). The region G is bounded by the curve $r = f(\theta)$ and the two lines which pass through the origin with angles α and β, respectively.

Theorem 7.1 *Let $r = f(\theta)$ be a continuous function defined in some interval $\alpha \le \theta \le \beta$, where $f(\theta) \ge 0$ and $\beta \le \alpha + 2\pi$. The area A of the region G is equal to*

$$A = \int_\alpha^\beta \frac{1}{2} f^2(\theta)\,d\theta\,.$$

Proof Consider a partition

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$$

of the interval $[\alpha, \beta]$. Let $f(s_i)$ be a maximum and $f(t_i)$ be a minimum of f in the interval $[\theta_i, \theta_{i+1}]$ (see Fig. 7.10). Let A_i be the area of the region between the curve and the lines $\theta = \theta_i$ and $\theta = \theta_{i+1}$. This region is enclosed between the two sectors

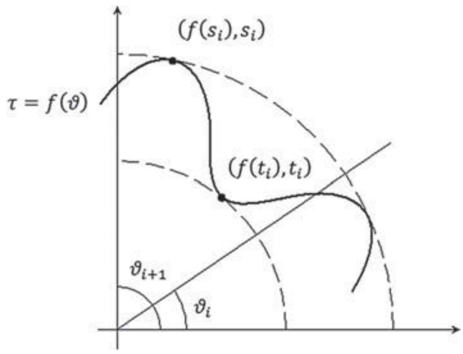


Fig. 7.10 Area in polar coordinates

bounded by those lines and the curves $r = f(t_i)$ and $r = f(s_i)$, respectively. The area of a sector having angle $\theta_{i+1} - \theta_i$ and radius R is equal to $\frac{\theta_{i+1}-\theta_i}{2\pi}$ times the total area of the circle of radius R, namely, πR^2. Hence, we have

$$\frac{\theta_{i+1} - \theta_i}{2\pi} \pi f^2(t_i) \le A_i \le \frac{\theta_{i+1} - \theta_i}{2\pi} \pi f^2(s_i) .$$

Let $g(\theta) = \frac{1}{2} f^2(\theta)$. Then the sum of the areas A_i of the small pieces satisfies the inequalities

$$\sum_{i=0}^{n-1} g(t_i)(\theta_{i+1} - \theta_i) \le \sum_{i=0}^{n-1} A_i \le \sum_{i=0}^{n-1} g(s_i)(\theta_{i+1} - \theta_i) .$$

The required area $A = \sum_{i=0}^{n-1} A_i$ is thus enclosed by the sums on the left and the right, which are Riemannian sums for the function g. Since g is continuous, it is integrable, and therefore the Riemannian sums converge to the integral of g according to the limit process which defines the integral. Therefore, we obtain

$$A = \int_\alpha^\beta g(\theta)\, d\theta = \int_\alpha^\beta \frac{1}{2} f^2(\theta)\, d\theta .$$

Example 7.7 Find the area of the region bounded by the curve $r = f(\theta) = 2 + \cos\theta$.

Solution: The required area is

$$\begin{aligned}\int_0^{\pi/2}\frac{1}{2}f^2(\theta)\,d\theta &= \frac{1}{2}\int_0^{\pi/2}(2+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(4+4\cos\theta+\cos^2\theta)\,d\theta\\ &= \frac{1}{2}\int_0^{\pi/2}\left(4+4\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta\\ &= \frac{1}{2}\left[4\theta+4\sin\theta+\frac{1}{2}\theta+\frac{\sin 2\theta}{4}\right]_{\theta=0}^{\theta=\pi/2} = \frac{1}{2}\left[4\frac{\pi}{2}+4+\frac{\pi}{4}\right]\\ &= \frac{9}{8}\pi+2\,.\end{aligned}$$

Example 7.8 Find the area bounded by one loop of the curve $r^2 = 2a^2\cos 2\theta, a > 0$.
Solution: The given curve is $r = f(\theta) = \sqrt{2}a\sqrt{\cos 2\theta}$. For $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, we have $\cos 2\theta \ge 0$, and the curve describes a loop which begins and ends at the origin. We compute the area of this loop as

$$A = \int_{-\pi/4}^{\pi/4}\frac{1}{2}f^2(\theta)\,d\theta = \int_{-\pi/4}^{\pi/4}\frac{1}{2}2a^2\cos 2\theta\,d\theta = a^2\,.$$

7.2 Determination of Length, Area, and Volume

In this section, we determine the length of curves, the area of revolving surfaces and the volume of revolving solids.

Length of Curves. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in the plane. The length of the line segment joining P_1 and P_2 equals the distance between P_1 and P_2, given by the theorem of Pythagoras as $\sqrt{(x_2-x_1)^2-(y_2-y_1)^2}$. Let us now consider a curve described by the graph of a function f over an interval $[a, b]$ (see Fig. 7.11). When f is differentiable and f' is continuous, its length, denoted by $L_a^b(f)$, is given by

$$L_a^b(f) = \int_a^b \sqrt{1+f'(x)^2}\,dx\,. \tag{7.1}$$

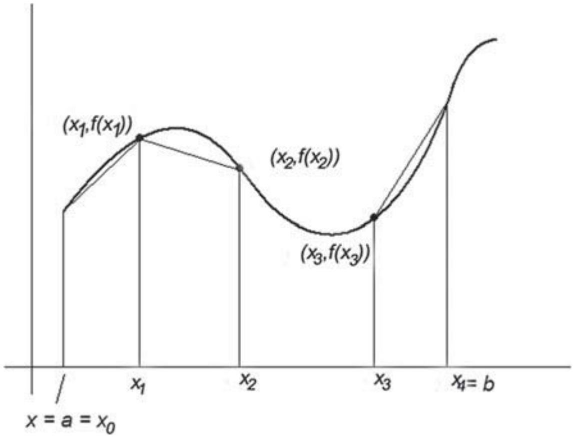
We will now argue that the mathematical definition (7.1) coincides with the intuitive notion of length. To this purpose, consider a partition of the interval $[a, b]$

$$a = x_0 < x_1 < \cdots < x_n = b\,.$$

For each x_i the point $(x_i, f(x_i))$ lies on the curve $y = f(x)$. Draw line segments between two successive points. The length of the line segment between $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ is

$$\sqrt{(x_{i+1}-x_i)^2+(f(x_{i+1})-f(x_i))^2}\,.$$

Fig. 7.11 Curve approximated by line segments



By the mean value theorem, we have

$$f(x_{i+1}) - f(x_i) = (x_{i+1} - x_i) f'(c_i)$$

for some number c_i between x_i and x_{i+1}. The length of our line segment therefore becomes

$$\sqrt{(x_{i+1} - x_i)^2 + (x_{i+1} - x_i)^2 f'(c_i)^2} = (x_{i+1} - x_i)\sqrt{1 + f'(c_i)^2}\,.$$

The sum of the lengths of these line segments is

$$\sum_{i=0}^{n-1} \sqrt{1 + f'(c_i)^2}(x_{i+1} - x_i)\,. \tag{7.2}$$

Let $h(x) = \sqrt{1 + f'(x)^2}$. The sum (7.2) can now be written as

$$\sum_{i=0}^{n-1} h(c_i)(x_{i+1} - x_i)\,.$$

This is a Riemannian sum for the function h. Since h is continuous, the Riemannian sums converge to the integral

$$\int_a^b h(x)\,dx = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

according to its construction, when the partition is made finer and finer. On the other hand, our intuitive notion of length says that the sums (7.2) of the lengths of the

approximating line segments should become closer and closer to the length of the curve as the partition becomes finer and finer. Therefore, formula (7.1) indeed is the correct way to define the length of a curve.

Let us remark that, according to the different possible notations of the derivative, formula (7.1) can also be written as

$$L_a^b(f) = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\,.$$

Example 7.9 Find the length of the curve $y = f(x) = e^x$ between $x = 1$ and $x = 2$.

Solution: The required length is

$$L_a^b(f) = \int_1^2 \sqrt{1 + f'(x)^2}\, dx = \int_1^2 \sqrt{1 + e^{2x}}\, dx\,.$$

To compute the integral, we make the substitution $1 + e^{2x} = u^2$, so $2e^{2x}\, dx = 2u\, du$. For $x = 1$ we get $u = \sqrt{1 + e^2}$, and for $x = 2$ we get $u = \sqrt{1 + e^4}$. Since $e^{2x} = u^2 - 1$, we obtain

$$\begin{aligned}
L_a^b(f) &= \int_{\sqrt{1+e^2}}^{\sqrt{1+e^4}} \frac{u^2}{u^2 - 1}\, du = \int_{\sqrt{1+e^2}}^{\sqrt{1+e^4}} \frac{u^2 - 1 + 1}{u^2 - 1}\, du \\
&= \int_{\sqrt{1+e^2}}^{\sqrt{1+e^4}} 1\, du + \int_{\sqrt{1+e^2}}^{\sqrt{1+e^4}} \frac{1}{u^2 - 1}\, du = \left[u + \frac{1}{2}\ln\frac{u-1}{u+1}\right]_{u=\sqrt{1+e^2}}^{u=\sqrt{1+e^4}} \\
&= \sqrt{1 + e^4} + \frac{1}{2}\ln\frac{\sqrt{1+e^4} - 1}{\sqrt{1+e^4} - 1} - \sqrt{1 + e^2} - \frac{1}{2}\ln\frac{\sqrt{1+e^2} - 1}{\sqrt{1+e^2} - 1}\,.
\end{aligned}$$

Curves in Parametric Form. Let a curve be given in parametric form $x = f(t)$, $y = g(t)$ with $a \le t \le b$, and assume that f and g are differentiable with continuous derivatives f' and g'. Its length is defined as

$$L_a^b(f, g) = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt\,. \tag{7.3}$$

In order to show that this formula is a reasonable definition of length, we argue in a similar manner as in the previous subsection. We consider a partition of the interval $[a, b]$

$$a = t_0 < t_1 < \cdots < t_n = b\,.$$

The distance between two successive points $(f(t_i), g(t_i))$ and $(f(t_{i+1}), g(t_{i+1}))$ is

$$\sqrt{(f(t_{i+1}) - f(t_i))^2 + (g(t_{i+1}) - g(t_i))^2}\,.$$

By the mean value theorem for f and g there exist numbers c_i and d_i between t_i and t_{i+1} such that

$$\begin{aligned} f(t_{i+1}) - f(t_i) &= f'(c_i)(t_{i+1} - t_i) \\ g(t_{i+1}) - g(t_i) &= g'(d_i)(t_{i+1} - t_i)\,. \end{aligned}$$

Substituting these values, the sum of the lengths of the line segments becomes

$$\sum_{i=0}^{n-1} \sqrt{f'(c_i)^2 + g'(d_i)^2}(t_{i+1} - t_i)\,. \tag{7.4}$$

Let $h(t) = \sqrt{f'(t)^2 + g'(t)^2}$. The sum (7.4) is close to

$$\sum_{i=0}^{n-1} h(c_i)(t_{i+1} - t_i)\,, \tag{7.5}$$

which is a Riemannian sum for h. The sums (7.4) and (7.5) are not necessarily equal (since c_i will, in general, be different from d_i), but their difference can be shown to converge to zero as the partition becomes finer and finer. Since the sums (7.5) converge to the integral (7.3) in the limit, it is indeed reasonable to define the length of the curve in parametric form by (7.3).
An alternative way would be to write

$$L_a^b = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,. \tag{7.6}$$

Note that the usual (nonparametric) form arises as a special case of the parametric form, if we set $f(t) = t$ in (7.3). (The letters t and g in (7.3) then correspond to x and f in (7.1).)

When we replace the fixed upper limit b in (7.3) by a variable t (and replace t by τ in the integral to avoid confusion), we obtain

$$s(t) = \int_a^t \sqrt{f'(\tau)^2 + g'(\tau)^2}\, d\tau\,. \tag{7.7}$$

The function s is called the **arc length** of the curve. From the Fundamental Theorem of Calculus, we immediately obtain

$$s'(t) = \sqrt{f'(t)^2 + g'(t)^2}\,.$$

Example 7.10 Find the length of the curve $x = f(\theta) = \cos^3\theta$, $y = g(\theta) = \sin^3\theta$ for $0 \le \theta \le \frac{\pi}{2}$.

Solution: We have $f'(\theta) = 3\cos^2\theta \cdot (-\sin\theta)$ and $g'(\theta) = 3\sin^2\theta\cos\theta$. From (7.3) we get

$$\begin{aligned} L_0^{\pi/2} &= \int_0^{\pi/2} \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\, d\theta \\ &= 3\int_0^{\pi/2} \sqrt{\sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= 3\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{3}{2}\int_0^{\pi/2} \sin 2\theta\, d\theta \\ &= \frac{3}{2}\left[-\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} = -\frac{3}{4}(\cos\pi - \cos 0) = \frac{3}{2}\,. \end{aligned}$$

Length of Curves in Polar Coordinates. Let $r = f(\theta)$ be the equation of a curve in polar coordinates, defined in the interval $a \le \theta \le b$. Its length is given by

$$L_a^b = \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta\,. \tag{7.8}$$

Proof From the formulas $x = r\cos\theta$, $y = r\sin\theta$ we obtain the parametric form $(x(\theta), y(\theta))$ of the curve in the usual (Cartesian) coordinates as

$$x(\theta) = f(\theta)\cos\theta\,, \quad y(\theta) = f(\theta)\sin\theta\,.$$

Using formula (7.6), its length is equal to

$$L_a^b = \int_a^b \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta\,. \tag{7.9}$$

Since

$$\begin{aligned} x'(\theta) &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ y'(\theta) &= f'(\theta)\sin\theta + f(\theta)\cos\theta\,, \end{aligned}$$

we obtain

$$\begin{aligned} x'(\theta)^2 + y'(\theta)^2 = &\ f(\theta)^2\sin^2\theta + f'(\theta)^2\cos^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta \\ &\ f(\theta)^2\cos^2\theta + f'(\theta)^2\sin^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta \\ = &\ f(\theta)^2 + f'(\theta)^2\,. \end{aligned}$$

Inserting this result into (7.9), we obtain (7.8). An alternative way to write this formula would be

$$L_a^b = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta\,.$$

From the computation above, we obtain the formula for the arc length in polar coordinates as

$$s(\theta) = L_b^\theta = \int_a^\theta \sqrt{x'(\eta)^2 + y'(\eta)^2}\, d\eta = \int_a^\theta \sqrt{f(\eta)^2 + f'(\eta)^2}\, d\eta\,,$$

so

$$s'(\theta) = \sqrt{f(\theta)^2 + f'(\theta)^2}\,. \tag{7.10}$$

Example 7.11 Find the length of the curve $r = f(\theta) = 1 - \cos\theta$ between $\theta = 0$ and $\theta = \frac{\pi}{4}$.

Solution: We have $f(\theta) = 1 - \cos\theta$, $f'(\theta) = \sin\theta$. We compute

$$\begin{aligned} L_0^{\pi/4} &= \int_0^{\pi/4} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \\ &= \int_0^{\pi/4} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\ &= \int_0^{\pi/4} \sqrt{2(1-\cos\theta)}\, d\theta = \int_0^{\pi/4} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \\ &= 2\int_0^{\pi/4} \sin\frac{\theta}{2}\, d\theta = -4\cos\frac{\theta}{2}\bigg|_{\theta=0}^{\theta=\pi/4} = 4\left(1 - \cos\frac{\pi}{8}\right)\,. \end{aligned}$$

Area of Surfaces of Revolution. Let $y = f(x)$ be a function which satisfies $f(x) \geq 0$ and has a continuous derivative on an interval $[a, b]$. Let S denote the area of the surface of revolution of the graph of f around the x-axis as shown in Fig. 7.12. Then

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx\,. \tag{7.11}$$

We explain how formula (7.11) arises.

We approximate the curve by line segments. Consider a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$. The length L_i of the line segment joining two successive points $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ is given by

$$L_i = \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}\,.$$

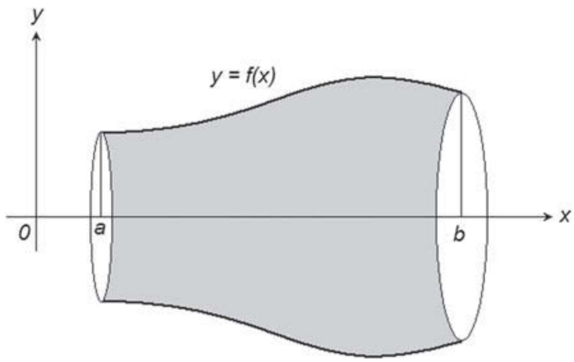


Fig. 7.12 A surface of revolution

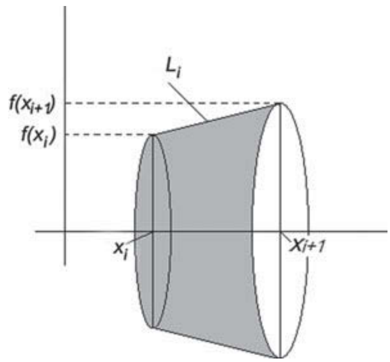


Fig. 7.13 Small approximating surface of revolution

We know that the length of a circle (its circumference) of radius y equals $2\pi y$. Therefore, if we revolve the line segment about the x-axis, we expect that the area of this surface of revolution will lie between $2\pi f(t_i)L_i$ and $2\pi f(s_i)L_i$, where $f(t_i)$ and $f(s_i)$ are the minimum and maximum of f, respectively, on the interval $[x_i, x_{i+1}]$ (see Fig. 7.13, here we have $t_i = x_i$ and $s_i = x_{i+1}$).

By the mean value theorem (Theorem 4.4), we have

$$f(x_{i+1}) - f(x_i) = f'(c_i)(x_{i+1} - x_i)$$

for some number c_i between x_i and x_{i+1}. From this we obtain

$$\begin{aligned} L_i &= \sqrt{(x_{i+1} - x_i)^2 + f'(c_i)^2(x_{i+1} - x_i)^2} \\ &= \sqrt{1 + f'(c_i)^2}(x_{i+1} - x_i). \end{aligned}$$

Therefore,

$$2\pi f(c_i)\sqrt{1 + f'(c_i)^2}(x_{i+1} - x_i)$$

is an approximation of the area of the surface of revolution of the curve over the small interval $[x_i, x_{i+1}]$. An approximation over the whole interval $[a, b]$ is given by the sum

$$\sum_{i=0}^{n-1} 2\pi f(c_i)\sqrt{1 + f'(c_i)^2}(x_{i+1} - x_i)\,.$$

This is a Riemannian sum for the function $h(x) = f(x)\sqrt{1 + f'(x)^2}$. It is therefore reasonable to define the area of the surface of revolution of the curve $y = f(x)$ between $x = a$ and $x = b$ by the integral

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx\,.$$

For curves in **parametric form** $x = f(t)$, $y = g(t)$, $a \le t \le b$, the length L_i between $(f(t_i), g(t_i))$ and $(f(t_{i+1}), g(t_{i+1}))$ is given by (again we use the mean value theorem)

$$\begin{aligned} L_i &= \sqrt{(f(t_{i+1}) - f(t_i))^2 + (g(t_{i+1}) - g(t_i))^2} \\ &= \sqrt{f'(c_i)^2 + g'(d_i)^2}(t_{i+1} - t_i)\,, \end{aligned}$$

where c_i, d_i are some numbers between t_i and t_{i+1}. Therefore, the approximation for the area of the surface of revolution of the curve in the small interval $[t_i, t_{i+1}]$ is given by

$$2\pi g(t_i)\sqrt{f'(c_i)^2 + g'(d_i)^2}(t_{i+1} - t_i)\,.$$

The arguments to obtain the length of a parametric curve can be used in an analogous manner to show that the area of the surface of revolution over the whole interval $[a, b]$ is given by

$$S = 2\pi \int_a^b g(t)\sqrt{f'(t)^2 + g'(t)^2}\,dt\,. \tag{7.12}$$

If we set $x = t$ and $y = g(t) = g(x)$, formula (7.11) becomes a special case of (7.12). Another form of (7.12) arises when we use the arc length $s(t)$ of the parametrized curve as defined in the previous subsection. Since $s'(t) = \sqrt{f'(t)^2 + g'(t)^2}$, (7.12) simply becomes

$$S = 2\pi \int_a^b g(t)s'(t)\,dt\,. \tag{7.13}$$

Note. Let the curve be given in polar coordinate form $r = f(\theta)$ with $a \le \theta \le b$. We write it in parametric form $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ and obtain from (7.13), taking into account (7.10),

$$S = 2\pi \int_a^b f(\theta)\sin\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = 2\pi \int_a^b f(\theta)\sin\theta s'(\theta)\,d\theta\,,$$

or, in an alternative form,

$$S = 2\pi \int_a^b r\sin\theta \frac{ds}{d\theta}\,d\theta\,, \quad \text{where} \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,.$$

Example 7.12 Find the area S of the surface of revolution which arises from rotating the curve $y = f(x) = x^3$ between $x = 0$ and $x = 1$ around the x-axis.

Solution: The given curve is $y = f(x) = x^3$ with $f'(x) = 3x^2$. We obtain

$$\begin{aligned} S &= 2\pi \int_0^1 x^3\sqrt{1+9x^4}\,dx = \frac{2\pi}{36}\int_0^1 36x^3\sqrt{1+9x^4}\,dx \\ &= \frac{\pi}{18}\left[\frac{2}{3}(1+9x^4)^{3/2}\right]_{x=0}^{x=1} = \frac{\pi}{27}[10\sqrt{10}-1]\,. \end{aligned}$$

Example 7.13 Find the area of a sphere of radius $a > 0$.

Solution: The sphere can be viewed as the surface of revolution of a half-circle of radius a. The equation of a half-circle in parametric form is

$$x = f(\theta) = a\cos\theta\,, \quad y = g(\theta) = a\sin\theta\,, \quad 0 \le \theta \le \pi\,.$$

Since $f'(\theta) = -a\sin\theta$, $g'(\theta) = a\cos\theta$, the required area is computed as

$$\begin{aligned} S &= 2\pi \int_0^\pi g(\theta)\sqrt{f'(\theta)^2 + g'(\theta)^2}\,d\theta \\ &= 2\pi \int_0^\pi a\sin\theta\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta \\ &= 2\pi a^2 \int_0^\pi \sin\theta\,d\theta = 2\pi a^2[-\cos\theta]_0^\pi \\ &= 4\pi a^2. \end{aligned}$$

Volume of Solids of Revolution. Let $y = f(x)$ be a continuous function of x on $[a, b]$ with $f(x) \ge 0$ for all x in $[a, b]$. If we revolve the curve $y = f(x)$ around the x-axis, we obtain a solid (or body), see Fig. 7.14, whose volume V is given by

$$V = \pi \int_a^b f(x)^2\,dx\,. \tag{7.14}$$

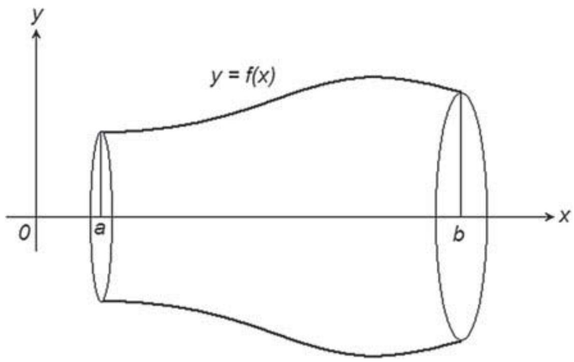


Fig. 7.14 A solid of revolution

To show that this is the correct formula, we again consider a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$. Let c_i and d_i be the minimum and maximum of f, respectively, in the interval $[x_i, x_{i+1}]$. Recall that the volume of a cylinder of radius r and height h equals $\pi r^2 h$. The volume of the solid of revolution formed by revolving the line segment, joining $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$, around the x-axis will lie between the volume of the small cylinder of radius $f(c_i)$ and height $x_{i+1} - x_i$ and the volume of the big cylinder of radius $f(d_i)$ and height $x_{i+1} - x_i$.
Taking the sum over all the line segments, we get

$$\sum_{i=0}^{n-1} \pi f(c_i)^2 (x_{i+1} - x_i) \le V \le \sum_{i=0}^{n-1} \pi f(d_i)^2 (x_{i+1} - x_i) \, .$$

The sums on the left and right are Riemannian sums for the function $h(x) = \pi f(x)^2$. It is therefore reasonable to define the volume to be

$$V = \pi \int_a^b f(x)^2 \, dx \, .$$

Example 7.14 Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = f_1(x) = x^2$ and $y = f_2(x) = 5x$ around x-axis.

Solution: The two curves intersect at the points (0, 0) and (5, 25), and f_2 lies above f_1, see Fig. 7.15. The required volume V is therefore equal to the difference of the volumes obtained by rotating $y = 5x$ and $y = x^2$ between $x = 0$ and $x = 5$. Consequently,

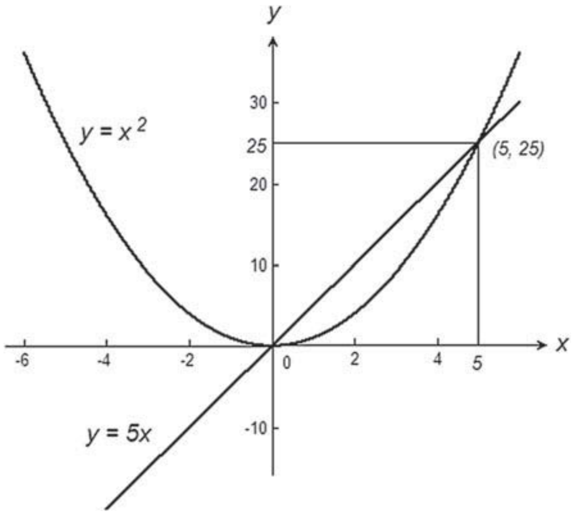


Fig. 7.15 A region which generates a solid of revolution

$$
\begin{aligned}
V &= \pi \int_0^5 (5x)^2\,dx - \pi \int_0^5 (x^2)^2\,dx = \pi \int_0^5 (25x^2 - x^4)\,dx \\
&= \pi \left[25\frac{x^3}{3} - \frac{x^5}{5}\right]_{x=0}^{x=5} = \pi \left[25 \cdot \frac{5^3}{3} - \frac{5^5}{5}\right] = \pi 5^4 \left(\frac{5}{3} - 1\right) \\
&= \frac{2}{3}\pi 5^4 .
\end{aligned}
$$

7.3 Definite Integral as Average

We are familiar with the average (or arithmetic mean) of n numbers, which we obtain by dividing the sum of those numbers by n. Here, we discuss the average value of a continuously varying function.

Let $f(t)$ denote the temperature at time t, measured in hours since midnight. We want to calculate the average temperature over a 24-h period. One way to start would be to average the temperatures at n equally spaced times $t_1, t_2, t_3, \ldots, t_n$ during the day. This gives the estimate

$$
\text{Average temperature} \;\simeq\; \frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{n} . \tag{7.15}
$$

Since the difference between two successive times equals $\Delta t = 24/n$ or $n = 24/\Delta t$, we may rewrite (7.15) as

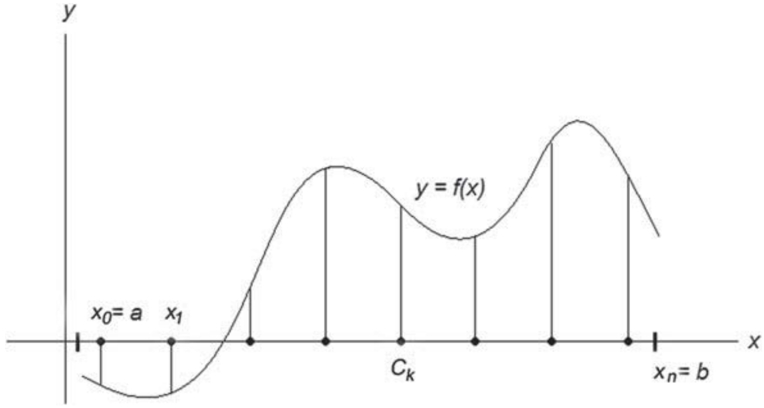


Fig. 7.16 A sample of values of a function on an interval $[a, b]$

$$\begin{aligned}\text{Average temperature} &\simeq \frac{f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t}{24} \\ &\simeq \frac{1}{24}\sum_{i=1}^{n} f(t_i)\Delta t\,.\end{aligned} \tag{7.16}$$

The larger we make n, the better we expect the estimate to be. Since the right-hand side of (7.16) is a Riemann sum for the function f and therefore converges to the integral of f for $n \to \infty$, it is natural to consider

$$\frac{1}{24}\int_0^{24} f(t)\,dt = \lim_{n\to\infty}\frac{1}{24}\sum_{i=1}^{n} f(t_i^n)\Delta t\,, \quad t_i^n = i\,\frac{24}{n}\,,$$

as the average of the function f over the interval $[0, 24]$.

In view of the discussion above we define the **average value (or mean value) of a function** f on the interval $[a, b]$ as

$$\mathrm{Av}(f) = \frac{1}{b-a}\int_a^b f(t)\,dt\,. \tag{7.17}$$

It is clear that $(b-a)\cdot \mathrm{Av}(f) = \int_a^b f(t)\,dt$.

The example above is an instance of a rather general situation. Let $f = f(x)$ be a continuous function on $[a, b]$. We partition $[a, b]$ into n subintervals of equal length $\Delta x = (b-a)/n$ and evaluate (or sample) f at a point c_k in each subinterval (see Fig. 7.16). The average of the n sampled values is

$$\frac{f(c_1)+f(c_2)+\cdots+f(c_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} f(c_i) = \frac{\Delta x}{b-a}\sum_{i=1}^{n} f(c_i)$$
$$= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x\,.$$

As $n \to \infty$, this expression converges to

$$\mathrm{Av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx\,.$$

Example 7.15 Find the average value (mean value) of $f(x) = \sqrt{9-x^2}$ on $[-3, 3]$.

Solution: The average value on the interval $[-3, 3]$ is defined as

$$\mathrm{Av}(f) = \frac{1}{3-(-3)}\int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{1}{6}\int_{-3}^{3}\sqrt{9-x^2}\,.$$

Let $x = 3\cos\theta$, then $dx = -3\sin\theta\,d\theta$, and

$$\begin{aligned}\mathrm{Av}(f) &= -\frac{1}{6}\int_{\pi}^{0} 3\sqrt{1-\cos^2\theta}\,3\sin\theta\,d\theta = \frac{9}{6}\int_0^{\pi}\sin^2\theta\,d\theta \\ &= \frac{3}{2}\cdot\frac{1}{2}\Big[\theta\Big]_{\theta=0}^{\theta=\pi} - \frac{3}{4}\Big[\sin 2\theta\Big]_{\theta=0}^{\theta=\pi} \qquad \left(\sin^2\theta = \frac{1-\cos 2\theta}{2}\right) \\ &= \frac{3}{4}\pi\,.\end{aligned}$$

Example 7.16 Find the average or mean value of

1. $f(x) = c$
2. $f(x) = x$

on the interval $[-4, 4]$.

Solution:

1. We have
$$\mathrm{Av}(f) = \frac{1}{8}\int_{-4}^{4} c\,dx = c\cdot\frac{8}{8} = c\,.$$

2. We have
$$\mathrm{Av}(f) = \frac{1}{8}\int_{-4}^{4} x\,dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_{-4}^{4} = \frac{1}{16}[16-16] = 0\,.$$

Note 7.1 The average value is used in economics to study the daily inventory $I(t)$ and the average daily inventory

$$\text{Av}(I) = \frac{1}{T}\int_0^T I(t)\,dt$$

over the time period $[0, T]$.

Example 7.17 Let the population of a country be modeled by the function

$$P(t) = 67.38 \cdot 1.026^t\,,$$

where P is measured in millions of people and t in years since 2000. Use this function to predict the average population of the country between the years 2020 and 2040.

Solution: We want to find average value of the function P between $t = 20$ and $t = 40$. We obtain

$$\begin{aligned}
\text{Av}(P) &= \frac{1}{40-20}\int_{20}^{40} P(t)\,dt = \frac{1}{20}\int_{20}^{40} 67.38(1.026)^t\,dt \\
&= 3.369\int_{20}^{40}(1.026)^t\,dt = 3.369\cdot\left[\frac{(1.026)^t}{\ln 1.026}\right]_{20}^{40} \\
&\simeq 147
\end{aligned}$$

by Table 6.1(10). Thus, the average population of the country between 2020 and 2040 is predicted to be 147 million people.

7.4 Applications to Business and Industry

7.4.1 Present and Future Values

A lot of business deals with payments in the future. For example, when buying a car on credit, payments are made over a period of time. If we are going to make or accept payments in the future under such an agreement, we should know how to compare the values of such payments made at different times. Being paid Rs 10,000 in the future is, under usual circumstances, clearly worse than being paid Rs 10,000 today, due to several reasons. For example, if we get money today, we can invest it in profitable shares, bank, and business. Therefore, even without considering inflation, in order to get the same value we should expect to be paid more when the payment is made in the future instead of now, in order to compensate for this loss of potential earnings. The important question is: How much more? To simplify matters we do not take inflation into consideration, but we consider only what we would lose by

not earning interest. Suppose we deposit Rs 10,000 in an account which earns 7% interest compounded annually, so in a year's time we will have Rs 10,700. We say that 10,700 is the future value of Rs 10,000, and that Rs 10,000 is the present value of Rs 10,700. In general we say the following.

The **future value** Rs B, of a present payment of Rs P, is the amount to which the Rs P would have grown if deposited today in an interest-bearing bank account.

The **present value** Rs P, of a future payment of Rs B, is the amount which would have to be deposited in a bank account today to produce Rs B in the account at the relevant time in the future.

It is clear that due to the interest earned, the future value is larger than the present value. The relation between the present value, denoted by PV, and the future value, denoted by FV, is as follows:

$$FV = PV \cdot (1+r)^t\,, \quad PV = \frac{FV}{(1+r)^t}\,. \tag{7.18}$$

For continuous compounding

$$FV = PV \cdot e^{rt}\,, \quad PV = FV \cdot e^{-rt}\,. \tag{7.19}$$

In both cases, it is assumed that the interest is compounded over a period of t years at an annual rate r, for example, $r = 0.07$ for an annual interest rate of 7%.

When we consider payments made to or by an individual, we normally think of discrete payments, that is, payments made at specific moments in time. However, when we analyze the overall money flow of a company or a bank, it makes sense to model it as a continuous stream of payments and earnings. (A similar modeling step—although the difference in scale is much larger—is performed when one analyzes the flow of a river, without looking at the behavior of the individual water molecules.)

Above we have considered the relation between the present and the future value of a single payment. We now want to calculate those quantities for a continuous stream of money, say an income stream, described by a rate of $S(t)$ Indian rupees per year, which varies continuously with time t during the time interval $[0, T]$, that is, from now until T years in the future. In order to use what we know about single deposits, we approximate the continuous income stream by a succession of many small deposits D_i made at times t_i of an equidistant partition

$$0 = t_0 < t_1 < \cdots < t_n = T\,, \quad \Delta t = t_{i+1} - t_i = \frac{T}{n}\,,$$

of the time interval $[0, T]$. If Δt is small, then the rate S (assumed to be a continuous function of t) does not vary much within one subinterval $[t_i, t_{i+1}]$, so the amount deposited by the continuous stream during that subinterval is approximately equal to $S(t_i)$ times its length Δt. Consequently, we set $D_i = S(t_i)\Delta t$. Assuming continuous compounding with a constant interest rate r, the present value of the deposit D_i at time t_i becomes

$$S(t_i)e^{-rt_i}\Delta t\,.$$

Summing over all subintervals gives

$$\sum_{i=0}^{n-1} S(t_i)e^{-rt_i}(t_{i+1}-t_i)$$

as the total present value of all successive small deposits. This is a Riemannian sum for the function $f(t) = S(t)e^{-rt}$. Since S is assumed to be continuous, so is f, and for $n \to \infty$ the sum converges to the integral of f. It is therefore natural to define the **present value of the continuous stream** of rate $S = S(t)$ during the time interval $[0, T]$ as

$$PV = \int_0^T S(t)e^{-rt}\,dt\,. \tag{7.20}$$

Consequently, the **future value** of the same continuous stream, evaluated at the final time after T years, becomes

$$FV = PV \cdot e^{rT} = e^{rT}\int_0^T S(t)e^{-rt}\,dt = \int_0^T S(t)e^{r(T-t)}\,dt\,. \tag{7.21}$$

Example 7.18 Find the present and future values of a constant income stream of 10,000 Indian rupees per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

Solution: We have $S(t) = 10{,}000$ and $r = 0.06$. By (7.20) and (7.21),

$$\begin{aligned} PV &= \int_0^{20} 10{,}000e^{-0.06t}\,dt \approx 116467.6 \text{ IR}\,, \\ FV &= 116467.6 \cdot e^{0.06(20)} = 386686.2 \text{ IR}\,. \end{aligned}$$

Example 7.19 Suppose we want to have an amount of 50,000 IR at the date 8 years in the future in a bank account earning 2% interest compounded continuously.

1. If we make one lump sum deposit now, how much should we deposit ?
2. If we deposit money continuously throughout the period of 8 years, at what rate should we deposit it?

Solution:

1. If we deposit a lump sum of P Indian rupees now, then P should be equal to the present value PV of 50,000 Indian rupees. Using the second equation from (7.19), we have

$$PV = 50{,}000 \cdot e^{-0.02 \cdot 8} = 50{,}000 \cdot e^{-0.16} \approx 42607.20\,.$$

We therefore should now deposit $P = 42607.20$ IR into the account so that we obtain 50,000 IR after 8 years.

2. Suppose we deposit money at the constant rate of S Indian rupees per annum. By (7.20) we have, since S is constant,

$$PV = \int_0^8 S(t)e^{-0.02t}\,dt = S\int_0^8 e^{-0.02t}\,dt \approx 7.39S\,.$$

But the present value of the continuous deposit must be the same as the present value of the lump sum deposits, that is 42607.20. So

$$42607.20 \approx 7.39S\,, \quad S \approx 5763.33\,.$$

7.4.2 *Annuity*

An **annuity** is a sequence of payments made at regular time intervals. The time period during which these payments are made is called the term of annuity. Although the payments need not be equal in size, in many situations they are indeed equal. We assume in our discussion here that they are equal. Let

P = size of each payment in the annuity,
r = interest rate compounded continuously,
T = term of annuity (in years), and
m = number of payments per year.

The payments into the annuity amount to mP rupees per year, which we model as an income stream with constant rate $S(t) = mP$. According to (7.20), its present value becomes

$$\int_0^T S(t)e^{-rt}\,dt = \int_0^T mPe^{-rt}\,dt = \left[\frac{-mPe^{-rt}}{t}\right]_{t=0}^{t=T}$$
$$= \frac{mP}{r}(1 - e^{-rT})\,.$$

The present value of an annuity is therefore defined as

$$PV = \frac{mP}{r}(1 - e^{-rT})\,. \tag{7.22}$$

The amount A of an annuity is defined as the corresponding future value at the end of the term and thus represents the sum of payments plus the interest earned. According to (7.21), it becomes

$$A = mP \int_0^T e^{r(T-t)}\, dt = \frac{mP}{r}(e^{rT} - 1)\,. \tag{7.23}$$

Example 7.20 A proprietor of a hardware store wants to establish a fund now, in order to withdraw 1000 IR per month for the next 10 years. The fund earns interest at the rate of 9% per year compounded continuously. Calculate how much money he needs to establish the fund.

Solution: The money he needs equals the present value of the annuity for the given values $P = 1000$, $r = .09$, $T = 10$ and $m = 12$. According to (7.22), we obtain

$$PV = \frac{12 \cdot 1000}{0.09}(1 - e^{-0.09 \cdot 10}) \simeq 79,124.04\ IR\,.$$

Example 7.21 On April 1, 1995, a person deposited 4000 IR into an individual retirement account paying interest at the rate of 10 percent per year compounded continuously. Assuming that he deposits 4000 IR annually into the account, how much will he have in his retirement account at the beginning of the year 2003?

Solution: We apply (7.23) where $P = 4000$, $r = 0.1$, $T = 8$ and $m = 1$ to get

$$A = 1 \cdot 4000 \cdot \int_0^8 e^{0.1(8-t)}\, dt = \frac{4000}{0.1}(e^{0.8} - 1) \simeq 158121.30\ IR\,.$$

The person has approximately 158121.30 IR in his account in the beginning of 2003.

7.4.3 Applications in Business

The determination of the variable cost of producing a consecutive number of units is important to manufacturers. Let us recall from Sect. 3.3 that $C = C(x)$ denotes the cost of producing x units of a certain commodity, and that its derivative C' is called the marginal cost. Assume that we want to produce $a - 1$ units anyway and ask how much more would it cost to additionally produce units a through b. By the Fundamental Theorem of Calculus, this cost (called variable cost) is given by

$$VC = C(b) - C(a-1) = \int_{a-1}^b C'(x)\, dx\,. \tag{7.24}$$

If we know the marginal cost $C'(x)$ which we also denote by $MC(x)$, we can compute the variable cost by evaluating the integral.

Example 7.22 Let $MC(x) = 2x^2 - 3x + 2$ be the marginal cost for a certain commodity. Find the variable cost of producing 12 through 16 units.

Solution: We insert the marginal cost function $MC = C'$ and the given values $a = 12$ and $b = 16$ into (7.24) and obtain

$$VC = \int_{11}^{16} (2x^2 - 3x + 2)\,dx = \left[2\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_{11}^{16} = 1650.84\,.$$

Depletion. Natural resources such as oil, gas, and coal are limited in quantity, and their total depletion depends on the rate at which each resource is being consumed. Let $A = A(t)$ denote the annual rate of depletion, let $A(0) = A_0$ at time $t = 0$ for some given value A_0 and suppose that $A(t)$ increases at a rate of k percent each year. If compounded continuously, we have

$$A(t) = A_0 e^{kt}\,,$$

and the total amount S of depletion after a time of T years becomes

$$S = \int_0^T A(t)\,dt = \int_0^T A_0 e^{kt}\,dt\,. \tag{7.25}$$

Example 7.23 Suppose the world use of oil in 1976 was 21 billion barrels and the annual percentage of increase of consumption equalled 8% in this and the following years.

1. How many barrels of oil did the world use from 1976 to 1996 ?
2. In 1976, there were 550 billion barrels of proven reserves. How long did it take to use all of them ?

Solution:

1. We apply Eq. (7.25) with the given values $T = 20$, $k = 0.08$ and $A_0 = 21$ billion barrels in 1976, and obtain the amount S of oil used between 1976 and 1996 as

$$S = \int_0^{20} 21e^{0.08t}\,dt = \frac{21}{0.08}e^{0.08t}\Big|_0^{20} = \frac{21}{0.08}(e^{1.6} - 1)$$
$$\simeq 1037.66 \text{ billion barrels.}$$

2. Here we need to know how long it did take to use 550 billion barrels of oil. We put $S = 550$, $A_0 = 21$ and $k = 0.08$ in Eq. (7.25) and get

$$550 = \int_0^T 21e^{0.08t}\,dt = \frac{21}{0.08}e^{0.08t}\Big|_0^T = \frac{21}{0.08}(e^{0.08T} - 1)\,.$$

We find the value of T from this equation,

$$e^{0.08T} = \frac{550 \cdot 0.08}{21} + 1 = 3.095\,, \quad \ln 3.095 = 0.08T\,, \quad T = 14.12\,.$$

Thus, 14.12 years from 1976 or by 1990, the world oil reserves would have been depleted according to this model if no new reserves were discovered.

Rate of Sales. When the rate of sales of a product is a known function $f = f(t)$ of time t, the number of sales $S(t)$ of this product up to time t satisfies $S'(t) = f(t)$, and hence the total sales over the period $[0, T]$ is given by

$$S(T) = \int_0^T f(t)\,dt\,. \tag{7.26}$$

Example 7.24 Suppose the rate of sales of a new model of Honda car is given by

$$f(t) = 100 - 90e^{-t}\,,$$

where t is the number of days the product is on the market. Find the total sales during the first 4 days.

Solution: From (7.26) we get

$$\begin{aligned} S(4) &= \int_0^4 f(t)\,dt = \int_0^4 (100 - 90e^{-t})\,dt = \Big[100t + 90e^{-t}\Big]_0^4 \\ &= 310 + 90 \cdot 0.018 = 311.62 \text{ units.} \end{aligned}$$

That is, 311 cars will be sold during first 4 days.

Example 7.25 A furniture manufacturing company has a current sales rate of 1,000,000 Indian rupees per month, and the profit to the company averages 10% of the sales. The company's past experience with a certain advertising strategy is that sales will increase 2% per month over the duration of the advertising campaign (12 months). The monthly rate of sales $f(t)$ during this advertising campaign obeys a growth curve of the type $f(t) = A_0 e^{rt}$, where A_0 is the current sales rate and r is constant determining its increase. The company now needs to decide whether to embark on a similar campaign that will cost 130,000 Indian rupees. The decision will be affirmative, provided the increase in sales due to the campaign yields more than 13,000 Indian rupees as a profit (which would be the standard 10% return on investments of the company). Should the company take an affirmative decision or not?

Solution: We calculate what happens when the company decides to start the advertising campaign. In this case, the monthly rate of sales during the campaign will be $f(t) = 10^6 e^{0.02t}$, where t is measured in months. The total sales after 12 months (the length of campaign) will be according to (7.26)

$$\begin{aligned} S(12) &= \int_0^{12} 10^6 \cdot e^{0.02t}\, dt = \frac{10^6}{0.02} e^{0.02t}\bigg|_0^{12} = 5 \cdot 10^7 \cdot (e^{0.24} - 1) \\ &= 13.55 \cdot 10^6 \text{ IR}. \end{aligned}$$

Without the campaign, the total sales will be $12 \cdot 10^6$ IR. The profit to the company amounts to 10% of the sales so that the profit due to an increase in sales by the campaign is

$$0.1 \cdot (13.55 \cdot 10^6 - 12 \cdot 10^6) = 155,000 \text{ IR}.$$

This 155,000 IR profit is achieved through the expenditure of 130,000 IR. Thus, the advertisement would yield an additional profit of

$$155,000 - 130,000 = 25,000 \text{ IR}.$$

Since this is more than 13,000 IR, the standard profit obtainable for the company from the money spent on advertisement, the decision should be affirmative.

Consumer Surplus. Here we introduce the notion of "consumer surplus" and show that it is represented by an integral. Assume that a company sells a certain commodity, and that the price $p(x)$ it gets is a function of the number x of units it sells. This function (or its inverse, the number of units the company can sell in dependence upon the price it offers) is called the demand function. Usually, p is a decreasing function of x as one needs lower prices for larger sales.

Let X be the amount of the commodity currently available and $P = p(X)$ its current selling price. Assume for the moment that we replace the function p by a step function s, according to a partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = X$ with subintervals of equal length $h = x_i - x_{i-1}$, having values $s(x) = p(x_i)$ for $x_{i-1} < x \le x_i$. The portion $(x_{i-1}, x_i]$ of the commodity is sold at price P, whereas it could have been sold at price $p(x_i)$ if only x_i instead of X units were on sale, because the function s tells us that there are customers prepared to pay the amount $s(x) = p(x_i)$. As a consequence, the amount saved by the customers "belonging" to that subinterval equals $(p(x_i) - P)(x_{i+1} - x_i)$. The total savings of all customers are

$$\sum_{i=1}^{n} (p(x_i) - P)(x_i - x_{i-1}).$$

This is a Riemannian sum for the function $f(x) = p(x) - P$. Taking the limit as $n \to \infty$, this sum approaches the integral

$$\int_0^X (p(x) - P)\, dx. \tag{7.27}$$

The integral in (7.27) is called the **consumer surplus** for the commodity. It represents the total amount of money saved by the customers in purchasing the commodity at

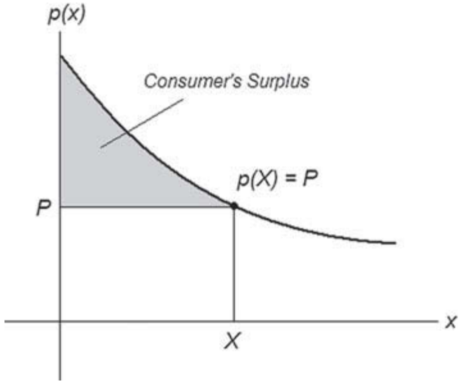


Fig. 7.17 The demand curve

price P corresponding to a demand level of X. Thus, the consumer surplus is equal to the area between the demand curve $p = p(x)$ and the line $p = P$, the shaded area in Fig. 7.17.

7.5 Applications to Mechanics and Engineering

In this section, we discuss how integrals arise in the modeling of mechanical work. The concept of mechanical work is of vital importance in engineering problems. Concrete examples are the work done when pumping water for a dam to function properly, or the work done when lifting an object (say, a bucket of water or a bag of sand). In lower classes, it is taught that the work W done by a constant force F when moving an object over a distance d along a line equals $W = Fd$, that is, the work done by a force equals force times distance. If the force acts along the x-axis and depends continuously upon the position x, and an object moves from a to b under its influence, then the work done by F in moving the object from a to b is

$$W = \int_a^b F(x)\,dx\,. \tag{7.28}$$

Note 7.2 In the English system, work is measured in foot-pounds (ft-lbs). In the metric system, the unit of work is a Newton meter (Nm), or a Joule (J).

Example 7.26 At each point of the x-axis, marked off in feet, a force of $5x^2 - x + 2$ pounds pulls an object. Determine the work done in moving the object from $x = 1$ to $x = 4$.

Solution: By (7.28) we have

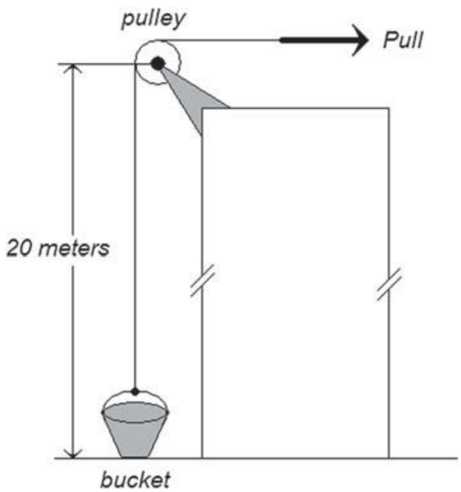


Fig. 7.18 Lifting a leaky bucket

$$\begin{aligned} W &= \int_1^4 (5x^2 - x + 2)\, dx = \left[\frac{5}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_1^4 \\ &= \frac{5}{3}(64-1) - \frac{1}{2}(16-1) + 2(4-1) \\ &= 103.5 \text{ ft-lbs.} \end{aligned}$$

Let us now consider what happens when we lift an object vertically upward. Assume that the object has mass $m = m(x)$ which depends on its height x above the ground, as for example a leaky bucket which gradually loses its contents. In lifting the object, we have to overcome the gravitational force of the earth which, according to Newton's law, is given by

$$F(x) = m(x)g\,.$$

Here $g = 9.81$ m/s^2 denotes the acceleration due to earth's gravity near its surface. By (7.28), the work needed to lift the object from height a to height b equals

$$W = \int_a^b F(x)\, dx = g\int_a^b m(x)\, dx\,. \tag{7.29}$$

Example 7.27 A leaky bucket of weight 5 kg is lifted vertically from the ground into the air by pulling in 20 m of rope at a constant speed (Fig. 7.18). The rope weighs 0.08 kg/m. The bucket starts with 8 L (=8 kg) of water and leaks at a constant rate. It finishes draining just as it reaches the top. Compute the work done in lifting for the bucket, the water and the rope separately as well as the total work done.

Solution: We first compute the masses as functions of height x. The mass of the bucket is constant, while the masses of water and rope decrease linearly with x. For

the water, the fraction of water still present when the bucket is x meters off the ground equals $(20 - x)/20$, so the mass of the water equals 8 times this fraction. In the same manner, we obtain the mass of the rope. Thus, we get the mass

$$\begin{aligned} &\text{for the bucket} && m_B(x) = 5 \text{ kg}, \\ &\text{for the water} && m_W(x) = 8\frac{20-x}{20} \text{ kg}, \\ &\text{for the rope} && m_R(x) = 0.08 \cdot (20 - x) \text{ kg}. \end{aligned}$$

The work done in lifting becomes, see (7.29),

$$\begin{aligned} W_B &= g\int_0^{20} 5\,dx = 100g = 981 \text{ Nm}, \\ W_W &= g\int_0^{20} 8\frac{20-x}{20}\,dx = g\int_0^{20}\left(8 - \frac{2}{5}x\right)dx = g\left[8x - \frac{x^2}{5}\right]_0^{20} \\ &= (160 - 80)g = 80g = 784.8 \text{ Nm}, \\ W_R &= g\int_0^{20} 0.08 \cdot (20 - x)\,dx = g\int_0^{20}(1.6 - 0.08x)\,dx \\ &= g\left[1.6x - 0.04x^2\right]_0^{20} = (32 - 26)g = 16g \\ &= 156.96 \text{ Nm}. \end{aligned}$$

The total work done is

$$W = W_B + W_W + W_R = 196g = 1922.76 \text{ Nm}.$$

Example 7.28 A bag of sand of mass 100 kg is lifted by a cable from the ground to the top of a 50 m high building. Sand leaks out of the bag at the rate of 0.5 kg for each meter the bag is raised. How much work is required to lift the bag of sand to the top of the building if

1. the masses of the cable and the bag are negligible,
2. the cable has a mass of 1.5 kg per m and the mass of the bag is negligible.

Solution:

1. At the point x m above the ground, the mass of the sand bag is

$$m_1(x) = 100 - 0.5x\,.$$

The work done in lifting the bag to the top of the building is

$$W_1 = g\int_0^{50} (100 - 0.5x)\,dx \qquad = g\left[100x - \frac{5x^2}{2}\right]_0^{50} = (5000 - 625)g$$
$$= 4375g = 42918.75 \text{ Nm}.$$

2. The mass of the fraction of the rope to be pulled at height x is $1.5(50 - x)$ kg. Thus, the work required to lift the cable to the top of the building is

$$W_2 = g\int_0^{50} 1.5(50 - x)\,dx = 1.5g\left[50x - \frac{x^2}{2}\right]_0^{50}$$
$$= 1.5g(2500 - 1250) = 1.5g \cdot 1250$$
$$= 1875g = 18393.75 \text{ Nm}.$$

Therefore, total work done in lifting is

$$W = W_1 + W_2 = (4375 + 1875)g = 6250g = 61312.5 \text{ Nm}.$$

Let us now consider an elastic spring. **Hooke's law** states that the force F required to stretch or compress a spring by a length x from its natural length (at $F = 0$) is proportional to x, that is,

$$F = kx\,. \tag{7.30}$$

Here, k is a constant which depends on the specific spring, it is called the spring constant and measured in units of force per length, for example, N/m.

Example 7.29 Find the work required to compress a spring from its natural length of 1 m to a length of 0.8 m if the spring constant equals $k = 16$ N/m.

Solution: Consider the uncompressed spring along the x-axis with its movable end at the origin and its fixed end at $x = 1$ m (see Fig. 7.19). The force required to compress the spring from 0 to x is $F = 16x$ Newton by Hooke's law. The work done by F

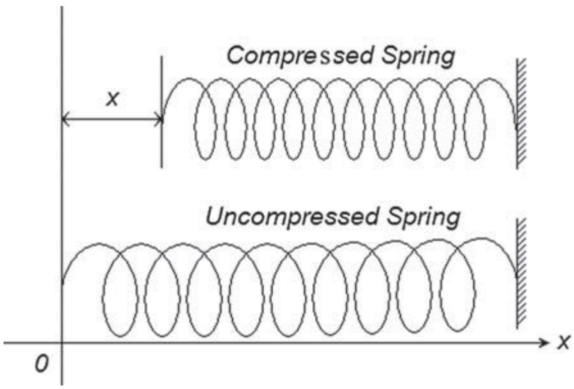


Fig. 7.19 Compressed and uncompressed spring

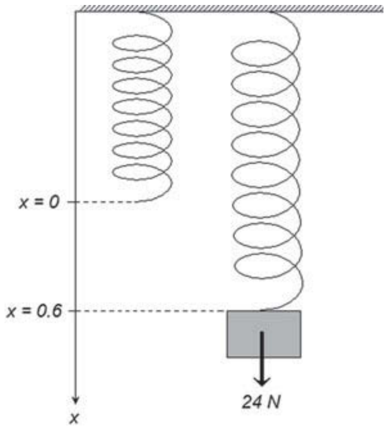


Fig. 7.20 Unstretched and stretched spring

over the interval from $x = 0$ to $x = 0.2$ m is

$$W = \int_0^{0.2} 16x\,dx = \Big[8x^2\Big]_0^{0.2} = 0.32 \text{ Nm}.$$

Example 7.30 A spring has a natural length of 1 m. A force of 24 N stretches the spring to a length of 1.6 m.

1. Find the spring constant k.
2. How much work is required to stretch the spring 2 m beyond its natural length?
3. How far will a 40 N force stretch the spring?

Solution:

1. Since a force of 24 N stretches the spring by 0.6 m, using (7.30) we get

$$24 = k \cdot 0.6\,, \quad k = \frac{24}{0.6} = 40\,\frac{\text{N}}{\text{m}}\,.$$

2. Consider the unstretched spring hanging along the x-axis with its free end at $x = 0$ (see Fig. 7.20). The force required to pull the spring x m beyond its natural length is just the force required to pull the free end of the spring x m downward from its original position.

   By Hooke's law this force is $F(x) = 40x$ N. The work done by pulling the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^{2} 40x\,dx = \Big[20x^2\Big]_0^{2} = 80 \text{ Nm}.$$

3. We have $F = 40x$. To find the elongation resulting from a force of 40 N, we substitute $F = 40$ into this equation and obtain

$$x = \frac{40}{40} = 1 \text{ m}.$$

A force of 40 N will stretch the spring by 1 m.

7.6 Integrals and Probability

Probability theory began to arise as a science in Europe during the sixteenth and seventeenth centuries, as marked by the book "Liber de ludo aleae" on games of dice by Gerolamo Cardano (published posthumously in 1663) and by a famous exchange of letters between the mathematicians Blaise Pascal and Pierre de Fermat in 1654). In the meantime, probability theory and its descendants have evolved into an important branch of mathematics with wide applications in practically every sphere of human endeavor in which an element of uncertainty is involved. Here, we present some examples of how integrals are involved in the computation of probabilities in some elementary situations.

A dependent variable whose values also depend on some random outcome is called a random variable. A random variable x that can assume any value in some given interval is called a continuous random variable. The life span of a light bulb, the length of a telephone call, the length of an infant at birth, the daily amount of rainfall in Delhi, and the life span of certain plant species are examples of continuous random variables.

Definition 7.1 (*Probability Density Function*) A function f defined on some interval $I = [A, B]$ (the values $A = -\infty$ and $B = \infty$ are also possible) is called a **probability density function** (or simply a **density function**) if the following conditions are satisfied:

1. f is nonnegative, that is, $f(x) \geq 0$ for all x.
2. f is integrable over I (in the cases $A = -\infty$ or $B = \infty$ we mean the improper integral).
3. The total area under the graph of f from A to B is equal to 1 (see Fig. 7.21), that is, $\int_A^B f(x)\,dx = 1$.

If the probability that an observed value of a given random variable x lies in some subinterval $[a, b]$ of I satisfies

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx \tag{7.31}$$

(see Fig. 7.22), we say that f is the density function belonging to this random variable.

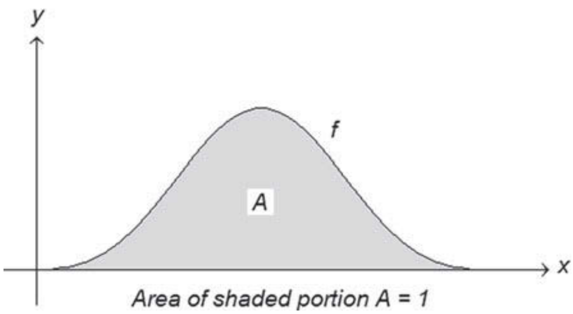


Fig. 7.21 Total area corresponds to probability equal to one

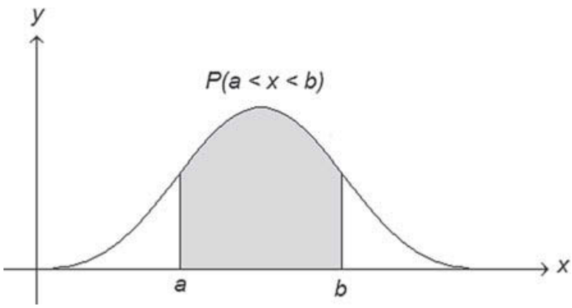


Fig. 7.22 Probability of the value of x lying between a and b

$P(a < x < b)$ is the probability that an outcome of an experiment will lie between a and b

Remark 7.1 1. According to property (3) of Definition 7.1, the probability that the continuous random variable takes on a value lying in its overall range $I = [A, B]$ equals 1. This corresponds to the fact that no other values are possible. (The probability of an event which always occurs is 1, by a basic convention of probability theory.)

2. According to formula (7.31), the probability that the random variable x assumes a value in an interval $a \le x \le b$ is given by the area of the region between the graph of f and the x-axis from $x = a$ to $x = b$.
3. Since $P(c - \varepsilon \le x \le c) = \int_{c-\varepsilon}^{c} f(x)\, dx$ tends to zero as ε tends to zero (the area below a single point of the graph of f is zero), the probability that x exactly attains an arbitrarily given value c is zero. The random variable x therefore must have the property that

$$P(a \le x < b) = P(a < x \le b) = P(a < x < b) = P(a \le x \le b)\,.$$

Consequently, a continuous random variable x which assigns a nonzero probability to certain discrete values cannot be modeled in the form (7.31).

Example 7.31 Show that the following functions are probability density functions on the intervals indicated:

1. $f(x) = \dfrac{3}{56}(5x - x^2), \quad I = [0, 4].$
2. $f(x) = \dfrac{2}{27}x(x-1), \quad I = [1, 4].$
3. $f(x) = \dfrac{1}{3}e^{-\frac{1}{3}x}, \quad I = [0, \infty).$

Solution: In all cases, it is clear that f is nonnegative on the respective interval I.

1. The function is continuous and hence integrable on I. We have

$$\int_0^4 \frac{3}{56}(5x - x^2)\,dx = \frac{3}{56}\left(5\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^4 = \frac{3}{56}\left[\frac{80}{2} - \frac{64}{3}\right] = 1\,,$$

   hence f satisfies requirements (1)–(3) of Definition 7.1 and therefore is a density function.
2. The function is continuous and hence integrable on I. We have

$$\begin{aligned}\int_1^4 f(x)\,dx &= \int_1^4 \frac{2}{27}(x^2 - x)\,dx = \frac{2}{27}\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_1^4 \\ &= \frac{2}{27}\left[\left(\frac{64}{3} - \frac{16}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right] = \frac{2}{27}\left(\frac{27}{2}\right). \\ &= 1\,.\end{aligned}$$

3. The function is continuous and hence integrable on any finite interval $[0, b]$. We obtain the improper integral as the limit

$$\begin{aligned}\int_0^\infty \frac{1}{3}e^{-\frac{1}{3}x}\,dx &= \lim_{b\to\infty}\int_0^b \frac{1}{3}e^{-\frac{1}{3}x}\,dx = \lim_{b\to\infty} -e^{-\frac{1}{3}x}\Big|_0^b \\ &= \lim_{b\to\infty}\left(-e^{-\frac{1}{3}b} + 1\right) = 1\,.\end{aligned}$$

Example 7.32 The Philips company manufactures a 200 watt light bulb. Laboratory tests showed that the life span of these light bulbs has a distribution described by the probability density function

$$f(x) = 0.001e^{-0.001x}\,,$$

where x is measured in hours. Determine the probability that a light bulb will have a life span of

1. 500 h or less,
2. more than 500 h, and
3. more than 1000 h, but less than 1500 h.

Solution: Let x denote the life span of the light bulbs. x is a continuous random variable whose value is different for each actual bulb.

1. The probability that a certain specific light bulb will have a life span of 500 h or less is given by

$$P(0 \le x \le 500) = \int_0^{500} 0.001e^{-0.001x}\,dx = -e^{-0.001x}\Big|_0^{500}$$
$$= -e^{-0.5} + 1 \simeq 0.3935\,.$$

2. The probability that a light bulb will have a life span of more than 500 h is given by

$$P(x > 500) = \int_{500}^{\infty} 0.001e^{-0.001x}\,dx = \lim_{b\to\infty}\int_{500}^{b} 0.001e^{-0.001x}\,dx$$
$$= \lim_{b\to\infty} -e^{-0.001x}\Big|_{500}^{b} = \lim_{b\to\infty}(-e^{-0.001b} + e^{-0.5})$$
$$= e^{-0.5} \simeq 0.6065\,.$$

Using (1), the result can also be obtained as

$$P(x > 500) = 1 - P(x \le 500) = 1 - (1 - e^{-0.5}) = e^{-0.5} \simeq 0.6065\,.$$

3. The probability that a light bulb will have a life span of more than 1000 h, but less than 1500 h is given by

$$P(1000 < x < 1500) = \int_{1000}^{1500} 0.001e^{-0.001x}\,dx$$
$$= -e^{-0.001x}\Big|_{1000}^{1500} = -e^{-1.5} + e^{-1}$$
$$\simeq -0.2231 + 0.3679 = 0.1448\,.$$

Example 7.33 1. Determine the value of the constant k such that the function $f(x) = kx^2$ becomes a probability density function on the interval [0, 5].

2. If x is a continuous random variable with the probability density function given by (1), compute the probability that x will assume a value between $x = 1$ and $x = 2$.

Solution:

1. We have

$$\int_0^5 kx^2\,dx = k\int_0^5 x^2\,dx = \frac{k}{3}x^3\Big|_0^5 = \frac{125}{3}k\,.$$

For $f(x) = kx^2$ to be a probability density function, we must have $\frac{125}{3}k = 1$, therefore $k = \frac{3}{125}$.

2. The required probability is given by

Fig. 7.23 Uniform density function

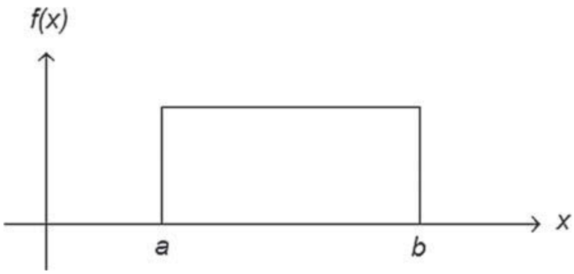
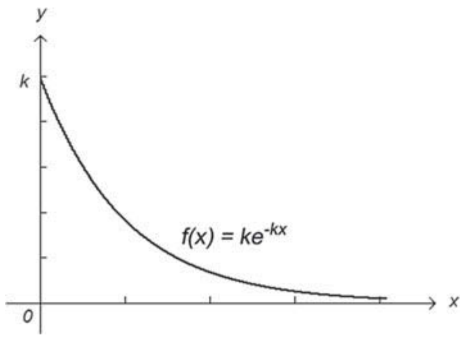


Fig. 7.24 Exponential density function



$$
\begin{aligned}
P(1 \leq x < 2) &= \int_1^2 f(x)\,dx = \frac{3}{125}\int_1^2 x^2\,dx = \frac{1}{125}x^3\Big|_1^2 \\
&= \frac{1}{125}(8-1) = \frac{7}{125}.
\end{aligned}
$$

Definition 7.2 (*Uniform and exponential density function*)

1. The probability density function f defined by

$$f(x) = \frac{1}{b-a}$$

   is called the uniform density function on $I = [a, b]$. In this case, we say that the random variable x is **uniformly distributed on** $[\mathbf{a}, \mathbf{b}]$ (Fig. 7.23).
2. A probability density function f defined by

$$f(x) = ke^{-kx},$$

   where k is a positive constant, is called an exponential density function on $I = [0, \infty)$. In this case, the random variable x is said to be **exponentially distributed on** $[\mathbf{0}, \infty)$. Note that the area below the graph of $f(x) = ke^{-kx}$ is equal to 1 (Fig. 7.24).

Example 7.34 Trains stop at a certain terminal regularly every 30 min. What is the probability that a passenger, who arrives at the terminal at a random time, will have to wait more than 10 min before he catches a train?

Solution: Let t denote the length of time the passenger has to wait for the next train. This is a continuous random variable with values in the interval $I = [0, 30]$. In the absence of other information, we assume that t is uniformly distributed on $[0, 30]$. The corresponding uniform density function is the constant function

$$f(x) = \frac{1}{30}\,.$$

The probability that a passenger will have to wait more than 10 min is

$$P(t \geq 10) = \int_{10}^{30} \frac{1}{30}\,dt = \frac{1}{30}(30 - 10) = \frac{2}{3}\,.$$

Example 7.35 Assume that airplanes departing from an airport follow a pattern described by an exponential density function, that is, when a plane has just left the next plane will depart after t minutes, where t is exponentially distributed. Find the probability that an airplane leaves within 6 min, if the constant of the exponential distribution has the value $k = 0.25$.

Solution: The random variable t is exponentially distributed according to

$$f(t) = 0.25e^{-0.25t}\,.$$

The required probability is

$$\begin{aligned} P(t \leq 6) &= \int_0^6 0.25e^{-0.25t}\,dt = e^{-0.25t}\Big|_0^6 = -e^{-1.5} + 1 \\ &\simeq 1 - 0.223 = 0.777\,. \end{aligned}$$

Therefore, the probability that the next plane departs within 6 min is equal to 0.777.

Example 7.36 A machine produces a successive stream of items of a certain commodity. An inspector tests the items and records when a defective item appears. Assume that it is known that, when a defective item has just appeared, the probability that the next x items are in order is given by an exponential density distribution with parameter $k = 1/200$. Find the probability that, after a defective item appears, the next 200 items are not defective.

Solution: The probability density function is given by

$$f(x) = ke^{-kx}\,, \quad k = \frac{1}{200}\,.$$

The required probability is

$$\int_0^{200} \frac{1}{200} e^{-\frac{x}{200}}\, dx = -e^{-\frac{x}{200}}\Big|_0^{200} = -e^{-1} + 1 \simeq 0.632\,.$$

The probability that the next defective item will not occur within the next 200 items is equal to $1 - 0.632 = 0.368$.

7.7 Exercises

7.7.1 Find the area of the region between the graphs of the following two functions on the interval from 0 to π.

(a) $y = 1 + \cos \frac{1}{3}x$ and $y = \sin 4x$.
(b) $y = 4 + \cos 2x$ and $y = 3 \sin \frac{x}{2}$.

7.7.2 Find the area of the region between the graphs of f and g if x belongs to the given interval.

(a) $f(x) = x^3 - 4x + 2$, $g(x) = 2$, on $[-1, 3]$.
(b) $f(x) = \sin x$, $g(x) = \cos x$, on $[0, 2\pi]$.

7.7.3 Sketch the region D bounded by the graphs of the functions of the following equations, and find the volume of the solid generated if D is revolved about the indicated axis.

(a) $y = \frac{1}{x}$, $x = 1$, $x = 3$, $y = 0$, x-axis.
(b) $y = 2x$, $y = 4x^2$, y-axis.

7.7.4 Show that the circumference C of the ellipse with the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given by

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta$$

where $e = \sqrt{1 - b^2/a^2}$ is the eccentricity (assume that $b \le a$).
If the planet Mercury travels in an elliptical orbit with $e = 0.206$ and $a = 1.387$, find the maximum and minimum distances between Mercury and the Sun.

7.7.5 A force of 25 N is required to compress a spring of natural length 0.80 m to a length of 0.75 m. Find the work done in compressing the spring from its natural length to a length of 0.70 m.

7.7.6 A construction worker pulls a 50 kg motor from ground level to the top of a 60 m high building using a rope that weighs $\frac{1}{4}$ kg per m. Find the work done.

7.7.7 The force (in Newtons) with which two electrons repel each other is inversely proportional to the square of the distance (in meters) between them.

(a) If one electron is held fixed at the point $(5, 0)$, find the work done in moving a second electron along the axis from the origin to the point $(3, 0)$.
(b) If two electrons are held fixed at the points $(5, 0)$ and $(-5, 0)$, respectively, find the work done in moving the third electron from the origin to $(3, 0)$.

7.7.8 A motorboat uses gasoline at the rate of $t\sqrt{9-t^2}$ gal/hr. If the motor is started at $t = 0$, how much gasoline has been used after 2 h?

7.7.9 Suppose the flow rate of blood at time t through a certain cross section of a blood vessel is given by

$$F(t) = \frac{F_1}{(1+\alpha t)^2} \quad \frac{\mathrm{cm}^3}{\mathrm{s}},$$

where F_1 and α are constants. Find the average flow rate $\overline{F}$ during the time interval $[0, T]$.

7.7.10 The cost (in Euro) of producing q units of a product is given by

$$c = 4000 + 10q + 0.1q^2 .$$

Find the average cost for the range from 100 to 500 units.

7.7.11 Suppose that colored dye is injected into the blood stream at a constant rate R. At time t, let $C(t) = \dfrac{R}{F(t)}$ the concentration of dye at a location different from the point of injection, where $F(t)$ is given as in Exercise 7.7.13. Show that the average concentration on a time interval $[0, T]$ is

$$\overline{C} = \frac{R(1 + \alpha T + \frac{1}{3}\alpha^2 T^2)}{F_1}.$$

7.7.12 Let the average lifetime of a DVD player be 4 years. A reasonable model for breakdown time is given by an exponential random variable. Let $p(x) = \frac{1}{4}e^{-x/4}$ be its density, for $0 \le x < \infty$ measured in years.

(a) Find the probability that the DVD player will eventually break.
(b) Find the probability that DVD player will break down within 12 years.

Hint: Use the fact that the probability of breakdown in the interval $a \le x \le b$ equals the integral of p from a to b.

7.7.13 Let the following figure (Fig. 7.25) gives the density function for the amount of waiting time at a doctor's clinic.

(a) What is the longest time one has to wait?
(b) Approximately what fraction of patients wait between 1 and 2 h?
(c) Approximately what fraction of patients wait less than an hour?

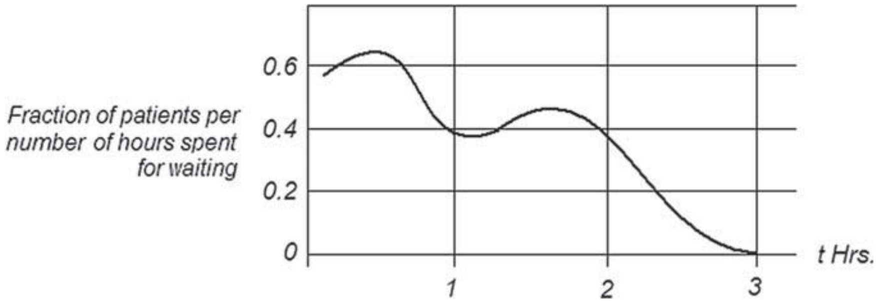


Fig. 7.25 Distribution of waiting time at a doctor's clinic

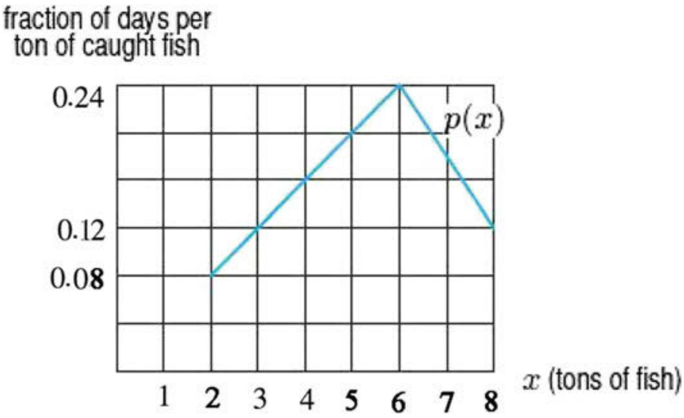


Fig. 7.26 Catching fish

7.7.14 Suppose we want to analyze the fishing industry in a town. Each day, boats bring back at least 2 tons (quintals) of fish, but never more than 8 tons (quintals).

(a) Applying the density function describing the catch in Fig. 7.26, find the cumulative distribution and graph the corresponding cumulative distribution function.
(b) What is the probability that the catch is between 5 and 7 tons?

7.7.15 After measuring the duration of many telephone cells, a telephone company found their data was well approximated by the density function $p(x) = 0.4e^{-0.4x}$, where x is the duration of a call in minutes.

(a) What percentage of calls last between 1 and 2 min?
(b) What percentage of calls last 1 min oe less?
(c) What percentage of calls last 2 min or less?
(d) What percentage of calls last 3 min or more?

7.7.16 A cumulative distribution function P of a density (probability) function p is defined by

$$P(t) = \int_{-\infty}^{t} p(x)\,dx\,.$$

The values $P(t)$ give the fraction of the population having values of x below t. Find the cumulative distribution function of the density function (probability density function) of Exercise 7.7.19.

7.7.17 At a bus stop, the time X (in minutes) that a randomly arriving person must wait for a bus is uniformly distributed with density function $f(x) = \frac{1}{10}$, where $0 \le x \le 10$ and $f(x) = 0$ otherwise. What is the probability that such a person must wait at most seven minutes? What is the average time that a person must wait?
Hint: The average or mean waiting time equals $\int_{-\infty}^{\infty} xf(x)\,dx$.

7.7.18 The length of life X (in years) of an electronic component has an exponential distribution with $k = \frac{1}{6}$. What is the probability that such a component will fail within 4 years of use? Find the probability that it will last more than 6 years.

7.7.19 Assume that in a specific hospital the length of time X (in hours) between successive arrivals at the emergency room is exponentially distributed with $k = 4$. What is the probability that it will take more than 2 h before the next arrival?

7.7.20 If an automobile starts from rest, what constant acceleration will enable it to travel 500 m in 10 s?

7.7.21 If a car is traveling at a speed of 60 km/hr, what constant (negative) acceleration will enable it to stop in 9 s?

7.7.22 Let a province of a country have a natural gas reserve of 100 billion m^3. Let $A(t)$ denote the total amount of natural gas consumed after t years, then $A'(t)$ is its rate of consumption. If the rate of consumption is predicted to be $5 + 0.01t$ b m^3/year, in approximately how many years will the province's natural gas reserve be depleted?

7.7.23 Let I be an alternating current of the form $I(t) = I_M \sin \omega t$ where t is the time, I_M is the current amplitude, and $\omega/2\pi$ is the frequency. Assume that the current flows through a resistor of R ohms. The rate P at which heat is being produced in the resistor is given by $P = I^2 R$. Compute the average rate of production of heat over one complete cycle from $t = 0$ to $t = 2\pi/\omega$.

Chapter 8
Functions of Several Variables

8.1 Introduction

In Sect. 1.1, we have introduced functions of one independent variable, expressed by $y = f(x)$. Functions of one variable represent various phenomena. A number of illustrative examples have been given in Sect. 1.4. Let us recall that a real-valued function f of one variable assigns to each point x of its domain $D(f)$ on the line $\mathbb{R}$ a unique point $f(x)$ on the line $\mathbb{R}$.

A real-valued function f of two variables differs from a real-valued function of one variable in the fact that its domain $D(f)$ lies in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$; as in the case of one variable, it assigns to each point in its domain a unique point on the line $\mathbb{R}$. Analogously, the domain $D(f)$ of a real-valued function f of three variables lies in the space $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$; again f assigns to each point in its domain a unique point in $\mathbb{R}$. In the same manner, we can introduce functions of n variables whose domain lies in n-dimensional space $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (with n factors) and whose values are real numbers.

We will confine ourselves mainly to functions of two variables for the sake of clarity and visualization. Many relevant concepts and results for 2 or 3 variables carry over to the general case of n variables, where n is an arbitrary (finite) natural number. In fact, the differential and integral calculus nowadays is well developed also in infinite-dimensional spaces. This, however, is beyond the scope of this book.

Very often one encounters situations which cannot be modeled by functions of one variable only, but require the use of functions of more than one variable. The cost of producing a certain item may depend on the simultaneous combination of variables such as labor and material. In economic theory, supply and demand of a commodity may depend not only on its own price but also on the prices of related commodities and some other factors such as income level or time of year. In physiology, we come across functions of several variables when we study the relationship between body surface area and the weight and height of the person. Other situations which demand the study of functions of more than one variable are as follows: The amount of food grown depends on the amount of water and fertilizer used; the rate of a chemical

M. Brokate et al., *Calculus for Scientists and Engineers*, Industrial and Applied Mathematics, https://doi.org/10.1007/978-981-13-8464-6_8

reaction depends on the temperature and pressure of the environment in which it takes place; the quantity of grocery purchased by a person depends on the price of different items and the income of the person concerned; the rate of fallout from a volcanic eruption depends on the distance from the volcano and the time since the eruption.

Examples of functions of several variables representing various phenomena will be discussed in Sect. 8.2. In the present section, we introduce the concepts of graph, level curves, and contours.

Definition 8.1 Suppose that D is a set of ordered pairs of real numbers (x, y). A real-valued **function f of two variables** on D is a rule that assigns a unique real number

$$z = f(x, y)$$

to each ordered pair (x, y) in D. The set D is the domain of f, and the set of values (or z-values) taken on by f is its range. As before, we also write $D(f)$ for the domain of f. The independent variables x and y are the components of the function's input variable (x, y), and the dependent variable z is the function's output variable.

Note.

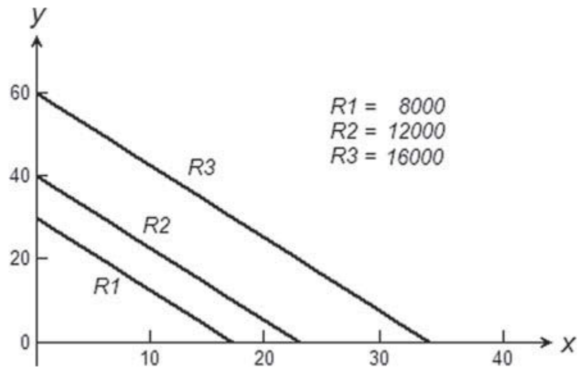
1. The notation "$z = f(x, y)$" is a convenient and short way to convey three pieces of information. These are the name of the function (here "f") and the letters used to denote the independent variables (here "(x, y)") and the dependent variable (here "z"), respectively.
2. In the same manner as in Definition 8.1, we can define functions of three independent variables $w = f(x, y, z)$, functions of four independent variables $u = f(x, y, z, w)$, and so on.
3. Definition 8.1 is a special case of the general definition of a function (Definition 1.2).

Definition 8.2 The set of points (x, y) in the plane where a function f of two variables has a fixed value $f(x, y) = c$, c being any constant, is called a **level set** or **contour** of f. Since in many cases the level sets are curves, they are also called **level curves** of f. A graph showing selected contours of a function is called a **contour diagram** or a **contour map**. The set of all points $(x, y, f(x, y))$ in the space $\mathbb{R}^3$, where (x, y) ranges over all points in the domain of f, is called the **graph** of f. The graph of f, described by the equation $z = f(x, y)$, is also called a **surface**.

For functions f of three variables, the level sets described by $f(x, y, z) = c$, where c is any constant, are called **level surfaces**, since in many cases they form a surface in space.

Example 8.1 1. Let $z = f(x, y) = \sqrt{y - x^2}$. The domain of this function is the set of points (x, y) such that $y \geq x^2$. For $y < x^2$ the z-values are not real. The range of f is $[0, \infty)$.

Fig. 8.1 A contour diagram



2. For $z = f(x, y) = \dfrac{1}{xy}$, the domain is the set of all (x, y) such that $xy \neq 0$, and the range is $(-\infty, 0) \cup (0, \infty)$.
3. Let $z = f(x, y) = \cos(xy)$. Its domain is the whole plane $\mathbb{R}^2$, and its range is $[-1, 1]$.
4. Let $f(x, y) = 2x + 4y$. The domain of this function is again the plane $\mathbb{R}^2$, and its range is $\mathbb{R}$.
5. Let $f(x, y, z) = x^2 + y^2 + z^2$. The domain of f is $\mathbb{R}^3$, and the range of f is $\mathbb{R}^+$, the set of nonnegative numbers.

When f is a potential function as used in physics, that is, it gives the value of the potential energy at each point of the space $\mathbb{R}^3$, the level surfaces $f(x, y, z) = c$ are called **equipotential surfaces**. When f represents a temperature distribution, the level surfaces $f(x, y, z) = c$ are called **isothermal surfaces**.

Example 8.2 Draw a contour diagram for the function $R(x, y) = 350x + 200y$. Include the contours for $R = 2000$, 4000, 8000, 12000, and 16000.

Solution: The contour for $R = 2000$ is given by

$$350x + 200y = 2000\,.$$

This is equation of a straight line with meets the x-axis at $x = \frac{2000}{350} = 5.71$ and the y-axis at $y = \frac{2000}{200} = 10$. The contour for $R = 4000$ is given by

$$350x + 200y = 4000.$$

This is the equation of a line parallel to the one above which meets the x- and y-axes at $x = \frac{4000}{350} = 11.43$ and $y = \frac{4000}{200} = 20$, respectively. The contours for $R = 8000$, $R = 12000$, and $R = 16000$ are again parallel lines drawn similarly (Fig. 8.1).

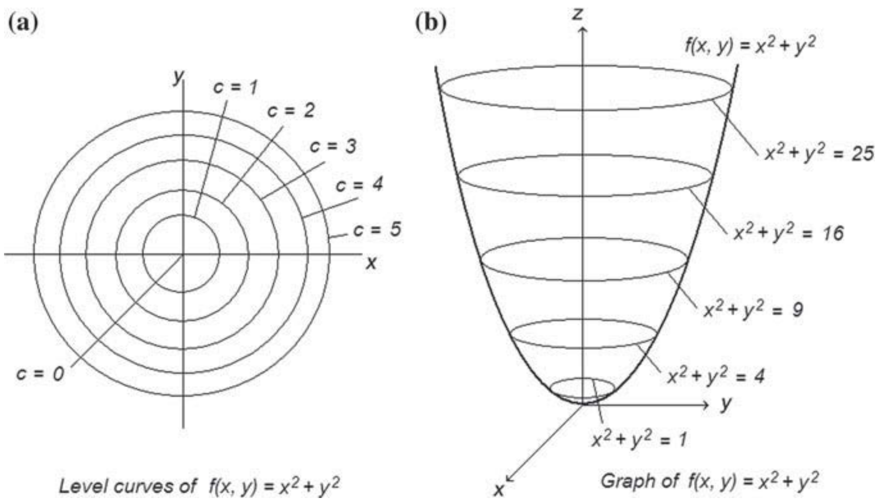


Fig. 8.2 **a** Level curves and **b** Graph of $f(x, y) = x^2 + y^2$

Example 8.3 Sketch a contour map for the function $f(x, y) = x^2 + y^2$.

Solution: The level curves are defined by the equation $x^2 + y^2 = c$ for nonnegative numbers c. Taking $c = 0, 1, 4, 9, 16$ and 25 for example, we get for

$c = 0$: $x^2 + y^2 = 0$
$c = 1$: $x^2 + y^2 = 1$
$c = 4$: $x^2 + y^2 = 4 = 2^2$
$c = 9$: $x^2 + y^2 = 9 = 3^2$
$c = 16$: $x^2 + y^2 = 16 = 4^2$
$c = 25$: $x^2 + y^2 = 25 = 5^2$

The level curves are concentric circles with center at the origin and radius given by $r = 0, 1, 2, 3, 4$, and 5, respectively (see Fig. 8.2a). A sketch of the graph of $f(x, y) = x^2 + y^2$ is shown in Fig. 8.2b

Computer Generated Graphs. As we know it is quite difficult to sketch an accurate graph of a function of two variables manually. However, powerful help is at hand; three-dimensional graphic programs for the computer (for example, MATLAB and MATHEMATICA) make it possible to visualize even quite complicated surfaces. These programs allow the user to view a surface from different perspectives. They show level curves and sections in various planes. Examples of computer generated graphs are shown below for the functions:

1. $f(x, y) = xy$, see Fig. 8.3.
2. $f(x, y) = \frac{-3y}{x^2+y^2+1}$, see Fig. 8.4.
3. $f(x, y) = e^{-x^2} + e^{-4y^2}$, see Fig. 8.5.

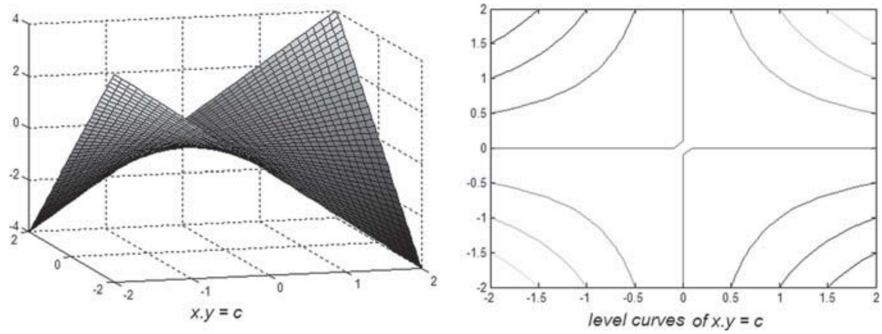


Fig. 8.3 Graph and level curves for $f(x, y) = xy$

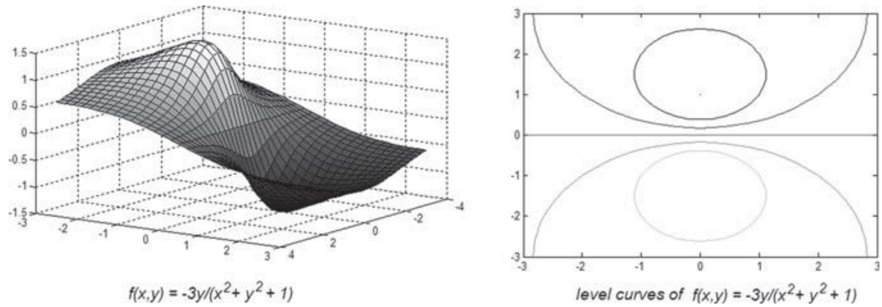


Fig. 8.4 Graph and level curves for $f(x, y) = (-3y)/(x^2 + y^2 + 1)$

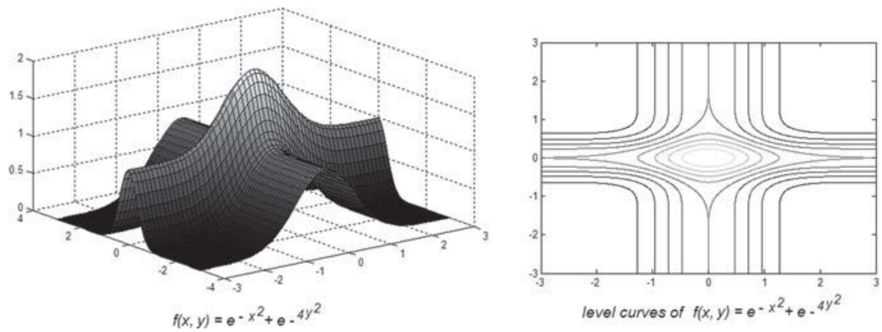


Fig. 8.5 Graph and level curves for $f(x, y) = e^{-x^2} + e^{-4y^2}$

4. $f(x, y) = \sin\sqrt{x^2 + y^2}$, see Fig. 8.6.
5. $f(x, y) = 2x^2 + 4y^2$, see Fig. 8.7.
6. $f(x, y) = xye^{-(x^2+y^2)/2}$, see Fig. 8.8.
7. $f(x, y) = \sin x \sin y$, see Fig. 8.9.

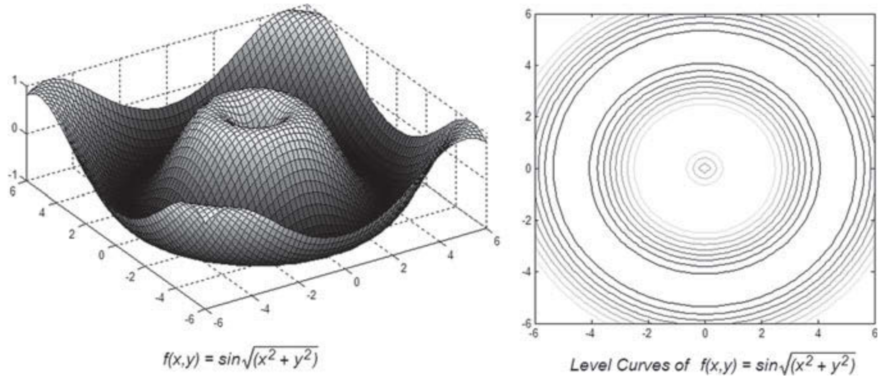


Fig. 8.6 Graph and level curves for $f(x, y) = \sin\sqrt{x^2 + y^2}$

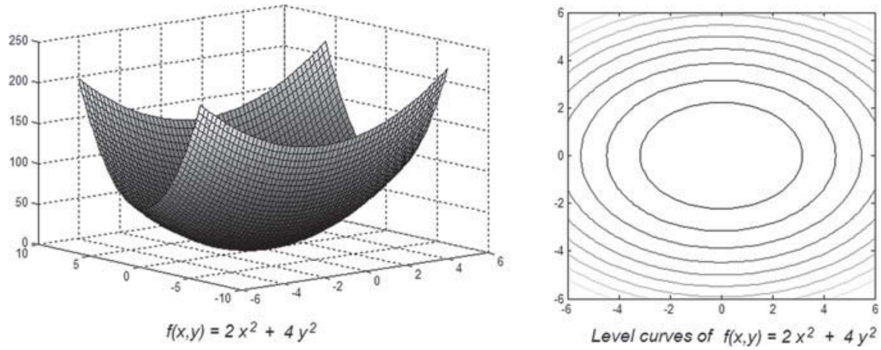


Fig. 8.7 Graph and level curves for $f(x, y) = 2x^2 + 4y^2$

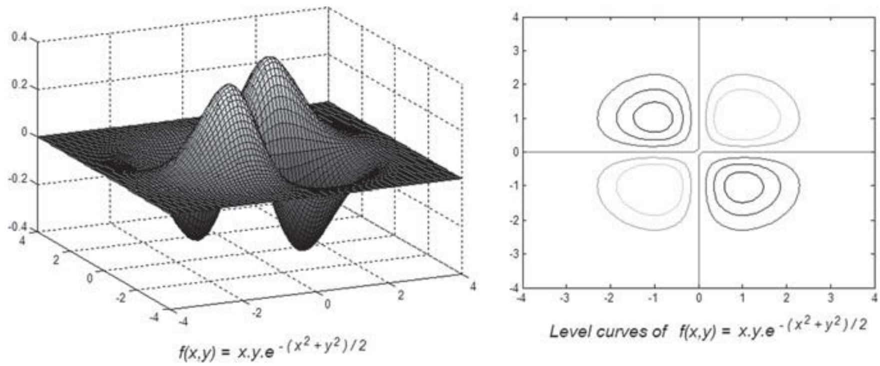


Fig. 8.8 Graph and level curves for $f(x, y) = xye^{-(x^2+y^2)/2}$

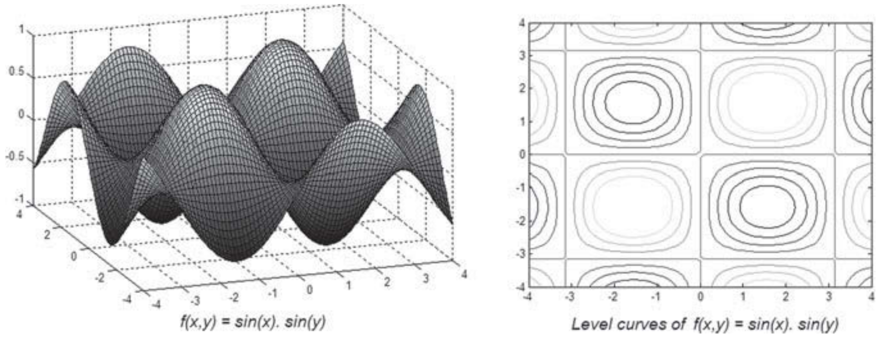


Fig. 8.9 Graph and level curves for $f(x, y) = \sin x \sin y$

8.2 Situations Modeled by Functions of More Than One Variable

Example 8.4 A computer manufacturing company determines that profits are 600 IR, 500 IR, and 400 IR, respectively, for a single unit of the type A, type B, and type C laptops it plans to produce. Let x, y, and z denote the number of type A, type B, and type C laptops to be made, then the profit of the company is modeled by the function of three variables

$$P(x, y, z) = 600x + 500y + 400z\,.$$

Example 8.5 An electronic company in India manufactures a TV set that may be bought fully assembled or in a kit. The demand equations that relate the unit prices, p and q to the weekly demanded quantities x and y of the assembled and the kit versions of TV set, are given by the functions of two variables:

$$\begin{aligned} p(x, y) &= 300 - \frac{1}{4}x - \frac{1}{8}y \\ q(x, y) &= 240 - \frac{1}{8}x - \frac{3}{8}y\,. \end{aligned}$$

The weekly total revenue function F is a function of two variables:

$$\begin{aligned} F(x, y) &= xp(x, y) + yq(x, y) \\ &= x(300 - \frac{1}{4}x - \frac{1}{8}y) + y(240 - \frac{1}{8}x - \frac{3}{8}y) \\ &= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y\,. \end{aligned}$$

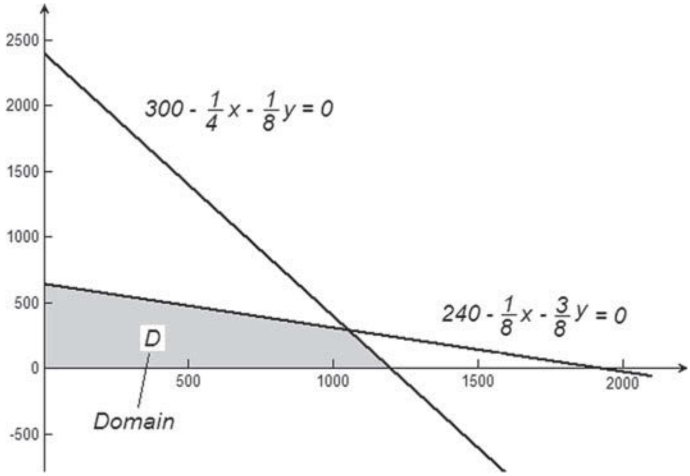


Fig. 8.10 Domain of the function F

To find the domain of the function $F(x, y)$, let us observe that the quantities x, y, p, and q must be nonnegative. This requirement leads to the following system of inequalities:

$$\begin{aligned} 300 - \frac{1}{4}x - \frac{1}{8}y &\geq 0 \\ 240 - \frac{1}{8}x - \frac{3}{8}y &\geq 0 \\ x &\geq 0 \\ y &\geq 0\,. \end{aligned}$$

The domain of this function is sketched in Fig. 8.10.

Example 8.6 A car rental company charges 400 IR a day and 15 IR per kilometer for its cars. Write a formula for the cost C of renting a car as a function of the number d of days and the number m of kilometers driven. Find $C(5, 300)$ and interpret it.

Solution: The total cost in Indian rupees of renting a car is 400 times the number d of days, plus 15 times the number m of kilometers, so that it equals the number $400d + 15m$. This gives the value of the function for given values of d and m, so

$$C(d, m) = 400d + 15m,$$

which is a function of two variables. We have

$$C(5, 300) = 400 \cdot 5 + 15 \cdot 300 = 2000 + 4500 = 6500.$$

This tells us that if we rent a car for 5 days and drive it over 300 km, it will cost 6500 IR.

Example 8.7 Let A denote the area of rectangle having sides of length x and y. Then A is a function of x and y, namely, $A(x, y) = xy$.

8.3 Continuity of Functions of Several Variables

In Chap. 2, we have discussed the notions of limit and continuity for functions of one variable. In this section, we extend those notions first for functions of two, and afterward for three and more variables.

We start with the definition of a limit of a function f of two variables. We want to give a precise meaning to the statement

> As (x, y) approaches a point (a, b) in the plane, $f(x, y)$ approaches the value L.

We rephrase this statement as follows:

> We can enforce $f(x, y)$ to deviate from L by an amount less than a given $\varepsilon > 0$, if we restrict (x, y) to be taken close enough to (a, b).

In the plane, the closeness of the point (x, y) to the point (a, b) is measured by the distance (see Appendix B)

$$d((x, y), (a, b)) = \sqrt{(x - a)^2 + (y - b)^2}. \tag{8.1}$$

For a given $\delta > 0$, the set of points whose distance from (a, b) is smaller than δ,

$$B_\delta(a, b) = \{(x, y) : d((x, y), (a, b)) < \delta\}, \tag{8.2}$$

is called the **$\boldsymbol{\delta}$-neighborhood** of the point (a, b). Thus we arrive at

> We can enforce $f(x, y)$ to deviate from L by an amount less than a given $\varepsilon > 0$, if we restrict (x, y) to be taken from a δ-neighborhood of (a, b) with a sufficiently small $\delta > 0$.

The formal definition of a limit is based upon these considerations.

Definition 8.3 (*Limit of a function of two variables*) Let f be a function defined in some neighborhood of (a, b) except possibly at the point (a, b) itself. We say that $f(x, y)$ tends to the limit L as (x, y) approaches (a, b) and write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L, \tag{8.3}$$

if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < d((x, y), (a, b)) < \delta, \tag{8.4}$$

where $d((x, y), (a, b)) = \sqrt{(x-a)^2 + (y-b)^2}$.

Remark 8.1 1. Definition 8.3 enforces that $f(x, y)$ tends to the same value L no matter from which direction (x, y) approaches (a, b), since only the distance to (a, b) plays a role.

2. As in the case of a single variable, statement (8.3) is equivalent to the statement

$$\lim_{(x,y)\to(a,b)} |f(x, y) - L| = 0. \tag{8.5}$$

3. The precise meaning of the requirement "f is defined in some neighborhood of (a, b)" is that there exists an $\eta > 0$ such that f is defined in the η-neighborhood of (a, b). In this case, the point (a, b) is called an **interior point** of the domain of f.

Sets D with the property that all elements of D are interior points of D are called open.

Definition 8.4 A subset D of the plane is called **open** if for every point (a, b) in D there exists an $\eta > 0$ such that $B_\eta(a, b) \subset D$, that is, every point whose distance to (a, b) is smaller than η belongs to D.

The definition of continuity is based on the definition of the limit in the same manner as in the case of a function of one variable. Namely, we require that the limit at a point has to be equal to the function value at that point.

Definition 8.5 In the situation of Definition 8.3, the function f is said to be **continuous at the point (a, b)**, if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b). \tag{8.6}$$

The function f is said to be **continuous in some subset D** of the plane, if it is continuous at every point (a, b) in D.

Example 8.8 1. The constant function defined by $f(x, y) = c$ for some number c is continuous at every point (a, b) of the plane. Indeed, condition (8.4) is satisfied for $L = f(a, b) = c$ no matter how we choose δ, because $|f(x, y) - c| = 0$ holds for all points (x, y).

2. The function $f(x, y) = x$ is continuous at every point (a, b) of the plane. To check this, we observe first that

$$|x - a| \le \sqrt{(x-a)^2 + (y-b)^2} = d((x, y), (a, b))$$

holds for every point (x, y). Condition (8.4) is therefore satisfied for $L = f(a, b)$ if, for any given $\varepsilon > 0$, we choose $\delta = \varepsilon$. Analogous considerations show that $f(x, y) = y$, too, yields a continuous function.

3. For the function

$$f(x, y) = \begin{cases} 1\,, & \text{for } (x, y) = (0, 0)\,, \\ 0\,, & \text{for all other } (x, y)\,, \end{cases}$$

we have

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0 \neq 1 = f(0, 0).$$

Hence, f is not continuous at $(a, b) = (0, 0)$. At all other points $(a, b) \neq (0, 0)$ it is continuous.

Fortunately, in order to check whether a function is continuous one usually does not have to verify condition (8.4) explicitly. As in the case of functions of a single variable, limits can be interchanged with the algebraic operations as well as the composition of functions, so the theorems regarding the sum, product, quotient, and composition of Sect. 2.2 hold for functions of two variables as well. Therefore, on the basis of parts 1 and 2 of Example 8.8, we see that functions like

$$f(x, y) = 3y^2 \sin(xy) - \frac{e^x}{3 + y^2} + |xy|$$

are continuous.

The Sandwich theorem can be stated as follows.

Theorem 8.1 (Sandwich Theorem, Functions of Two Variables) *Let f, g, h be functions of two variables such that $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (a, b)$ in some neighborhood of (a, b) and let*

$$\lim_{(x,y)\to(a,b)} g(x, y) = \lim_{(x,y)\to(a,b)} h(x, y) = L.$$

Then we have

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{8.7}$$

as well.

When f is defined by a quotient whose numerator and denominator both tend to zero, one has to take a closer look.

Example 8.9 1. Show that the function defined by $f(x, y) = \dfrac{x^2(x + y)}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ is continuous at $(0, 0)$ if we define $f(0, 0) = 0$.

2. Show that the function defined by $g(x, y) = \dfrac{x^2 - y^2 + 2x^3}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ is not continuous at $(0, 0)$ no matter how we define $g(0, 0)$.

Solution:

1. Since $x^2 \leq x^2 + y^2$,

$$0 \leq |f(x, y)| \leq \frac{x^2}{x^2 + y^2}|x + y| \leq |x| + |y|$$

holds whenever x and y are not both zero, the Sandwich theorem implies that

$$\lim_{(x,y)\to(a,b)} |f(x, y)| = 0.$$

Comparing (8.4) and (8.5) with $L = 0$, we see that $f(x, y)$ tends to 0 as (x, y) tends to $(0, 0)$, so f is continuous at this point provided $f(0, 0) = 0$.
2. Suppose $g(0, 0) = L$. If g is continuous at $(0, 0)$, then $g(x, y)$ must approach the value L as (x, y) approaches $(0, 0)$, and this must be true in particular when (x, y) approaches along an arbitrary line through $(0, 0)$. But along the x-axis we have $g(x, 0) = 1 + 2x$ for $x \neq 0$, which tends to 1 as $x \to 0$. On the other hand, along the y-axis we have $g(0, y) = -1$ for $y \neq 0$. The former result requires $L = 1$ and latter $L = -1$. Since these requirements are incompatible, it follows that g cannot be continuous at $(0, 0)$.

Remark 8.2 When we have a function f of two variables (x, y), we can consider it as a function of one of its variables, keeping the other variable fixed. For example, if we fix $y = b$, we obtain a function $g(x) = f(x, b)$ of x only, or if we fix $x = a$, we obtain a function $h(y) = f(a, y)$ of y only. Sometimes, these functions g and h are also called **partial functions** of f. We may check from the definitions that if f is continuous at (a, b), then g is continuous at a and h is continuous at b. We summarize this in saying that **a continuous function of two variables is continuous in each of its variables**. However, the converse is false as it is possible for a function of two variables to be continuous in each variable separately and yet fail to be continuous as a whole. Let us consider the function f defined by

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Since $f(x, 0) = 0$ for all x and $f(0, y) = 0$ for all y, we have $\lim\limits_{x\to 0} f(x, 0) = 0 = f(0, 0)$ and $\lim\limits_{y\to 0} f(0, y) = 0 = f(0, 0)$. Thus f is continuous in x as well as in y at the point $(0, 0)$. However, as a function of two variables f is not continuous at $(0, 0)$. To see this, we approach $(0, 0)$ along the line $x = y$ which consists of all points of the form (t, t). At such points we have, for $t \neq 0$,

$$f(t, t) = \frac{2t^2}{t^2 + t^2} = 1.$$

Hence, condition (8.4) cannot be satisfied for $L = f(0,0) = 0$, so f is not continuous at $(0,0)$.

For the remainder of this section, we discuss the case of **functions of 3 or more variables**. We can extend Definitions 8.3 and 8.5, if we extend the notion of distance accordingly. In the general case of n variables, consider points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n$ and define their distance as

$$d(\mathbf{x}, \mathbf{a}) = \left(\sum_{i=1}^{n}(x_i - a_i)^2\right)^{1/2}. \tag{8.8}$$

We say that $f(\mathbf{x})$ tends to a number L and write

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L, \tag{8.9}$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(\mathbf{x}) - L| < \varepsilon \qquad \text{whenever} \qquad d(\mathbf{x}, \mathbf{a}) < \delta,$$

where $d(\mathbf{x}, \mathbf{a})$ is given by (8.8). Presently, (8.9) is just a short (and very convenient) form of the statement

$$\lim_{(x_1,x_2,x_3,\ldots,x_n)\to(a_1,a_2,a_3,\ldots,a_n)} f(x_1, x_2, x_3, \ldots, x_n) = L.$$

When we will study the vector calculus in Chap. 9, we will consider $\mathbf{x}$, $\mathbf{a}$, and so on as vectors, and notations like (8.9) will be a natural way to write things down.

The theorems regarding the sum, product, quotient, and composition of continuous functions from Sect. 2.2 also extend to functions of three and more variables, so, for example, the function

$$f(x, y, z) = x^2 - \cos(xye^z)$$

is continuous.

8.4 Partial Derivatives with Applications

In Chap. 3, we have studied the derivative of a function of a single variable. Here, we study the concept of a partial derivative of a function of two or more variables, which means that we form the derivative with respect to one variable while keeping the other variable (resp. all variables) fixed. We present and explain its formal definition, give a geometrical interpretation as well as a lot of examples, including several situations in applications whose modeling and analysis involves partial derivatives. Moreover,

several forms of chain rule are discussed. Toward the end, we show how to compute derivatives in parameter-dependent integrals.

Definition 8.6 (*Partial Derivative*) Let f be a function of two variables (x, y). The **partial derivative** of f with respect to x at a point (x_0, y_0) is denoted by $\dfrac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$, and defined as

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, \tag{8.10}$$

provided that f is defined in some neighborhood of (x_0, y_0) and that the limit on the right-hand side of (8.10) exists.
Analogously, we define

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Remark 8.3 Let us consider f as a function of x only, that is, we fix $y = y_0$ and vary only x. For the resulting partial function $g(x) = f(x, y_0)$ we have, by the definitions,

$$g'(x_0) = \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = \frac{\partial f}{\partial x}(x_0, y_0).$$

In this manner, the partial derivative can be interpreted as an "ordinary" derivative of a function of a single variable. Thus, the partial derivative $\partial f / \partial x$ gives the rate of change of f with respect to x when y is held fixed at the value y_0, or in other words, the rate of change of f in the direction of the unit vector $\mathbf{i} = (1, 0)$ at (x_0, y_0).

Remark 8.4
1. Various notations are used for partial derivatives. As in $f_x(x_0, y_0)$, they must convey three pieces of information: The name of the function ("f"), the direction of the partial derivative ("x") and the point, at which the partial derivative is taken ("(x_0, y_0)"). Besides the expressions given in Definition 8.6, the following ones are also in use:

$$\frac{\partial}{\partial x} f(x_0, y_0), \quad \partial_x f(x_0, y_0), \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}.$$

   Their mathematical meaning is identical. Besides being read as "partial derivative of f with respect to x at (x_0, y_0)," they may also be read as "f sub x at (x_0, y_0)" or as "dee f by dee x at (x_0, y_0)." In order to familiarize the reader with this situation, in the remainder of this book we will not stick to one notation in particular, but will use several of them.
2. In Definition 8.6, we have denoted by (x_0, y_0) the point at which the partial derivative is taken, in order to distinguish it clearly from the names x, y of the variables. However, as in the case of functions of a single variable, one usually denotes the point as (x, y) and writes

$$\frac{\partial f}{\partial x}(x, y)\,, \quad \text{and so on.}$$

3. As in the case of ordinary derivatives, we may regard partial derivatives as functions of the point at which they are taken. Since a function is denoted just by its name (like "f" or "sin"), they are denoted accordingly as

$$\frac{\partial f}{\partial x}\,, \quad f_x\,, \quad \text{or} \quad \frac{\partial}{\partial x} f\,, \quad \partial_x f.$$

Any point (x, y) then becomes the argument, and $f_x(x, y)$ the value of this function.
4. Alternatively, we may emphasize just the variables. Consider $z = f(x, y)$. Instead of "f" we may just use the letter "z" and write

$$\frac{\partial z}{\partial x}\,, \quad z_x\,, \quad \text{and so on}$$

with or without the arguments (x, y). Again, the mathematical meaning of those expressions is the same as above.
5. The interpretation of partial derivatives as functions naturally leads us to the notion of **second partial derivatives** as well as partial derivatives of order higher than two. If f_x possesses a partial derivative with respect to x, it is denoted by

$$f_{xx}\,, \quad \frac{\partial^2 f}{\partial x^2}\,, \quad \text{or} \quad \partial_{xx} f.$$

If f_x possesses a partial derivative with respect to y, it is written as

$$f_{xy}\,, \quad \frac{\partial^2 f}{\partial y\, \partial x}\,, \quad \text{or} \quad \partial_y \partial_x f.$$

Analogously, f_{yx} and f_{yy} are defined. Third-order partial derivatives are

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3}\,, \quad f_{xxy} = \frac{\partial^3 f}{\partial y\, \partial x^2}\,, \quad \text{and so on.}$$

6. One can prove that $f_{yx} = f_{xy}$, provided the functions f, f_x, f_y, f_{xy}, and f_{yx} are continuous. Analogous results hold for higher partial derivatives. Thus, **partial derivatives can be interchanged** under those conditions.

Next, we discuss the **geometric interpretation** of partial derivatives. Since the partial derivative f_x for $y = y_0$ is equal to the derivative g' of the partial function $g(x) = f(x, y_0)$ (see Remark 8.3), the value $f_x(x_0, y_0)$ gives the slope of the curve $z = g(x) = f(x, y_0)$ at $x = x_0$, that is, the slope of the tangent to that curve at $x = x_0$. Let us now consider the full three-dimensional picture of the surface $z = f(x, y)$. Let C_1 be its intersection with the plane $y = y_0$ and C_2 be its intersection with the

plane $x = x_0$. Then $f_x(x_0, y_0)$ can be viewed as the **slope of the tangent** to the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ can be viewed as the slope of the tangent to the curve C_2 at the point (x_0, y_0). The number $f_x(x_0, y_0)$ is called the slope of the surface in the x-direction at (x_0, y_0), and $f_y(x_0, y_0)$ is called the slope of the surface in the y-direction at (x_0, y_0). Moreover, we can interpret $f_x(x, y_0)$, for varying x, as the rate of change of z with respect to x along the curve C_1, and $f_x(x_0, y_0)$ as its rate of change with respect to x at the point (x_0, y_0). Analogously, $f_y(x_0, y)$, for varying y, gives the rate of change of z with with respect to y along the curve C_2, and $f_y(x_0, y_0)$ the rate of change with respect to y at the point (x_0, y_0).

We now give examples of how to calculate partial derivatives. Note that, when forming the partial derivative with respect to one variable, the other variables are treated as constants (see Remark 8.3).

Example 8.10 Compute the partial derivatives f_x, f_y, f_{yx}, f_{xy}, f_{xx}, and f_{yy} of the following functions:

1. $f(x, y) = x^2 + 3xy + x + y - 2$.
2. $f(x, y) = y \cos xy$.
3. $f(x, y) = \dfrac{2y}{y + \cos x}$.
4. $f(x, y) = x^2 - xy^2 + y^3$.
5. $f(x, y) = \ln(x^2 + y^2)$.
6. $f(x, y) = (x^2 - xy + y^2)^5$.
7. $f(x, y) = e^{xy} + \ln(x + y)$.

Solution:

1. We have

$$\frac{\partial f}{\partial x}(x, y) = 2x + 3y + 1\,, \quad \frac{\partial f}{\partial y}(x, y) = 3x + 1,$$
$$\frac{\partial^2 f}{\partial y\,\partial x}(x, y) = 3 = \frac{\partial^2 f}{\partial x\,\partial y}(x, y), \quad \frac{\partial^2 f}{\partial x^2}(x, y) = 2, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = 0.$$

2. We have

$$\frac{\partial f}{\partial x}(x, y) = -y^2 \sin xy, \quad \frac{\partial f}{\partial y}(x, y) = \cos xy - xy \sin xy,$$
$$\frac{\partial^2 f}{\partial y\,\partial x}(x, y) = -xy^2 \cos xy - 2y \sin xy,$$
$$\frac{\partial^2 f}{\partial x\,\partial y}(x, y) = -y \sin xy - y \sin xy - xy^2 \cos xy.$$

We see that indeed $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$. Furthermore,

$$\frac{\partial^2 f}{\partial x^2}(x, y) = -y^3 \cos xy, \quad \frac{\partial^2 f}{\partial y^2} = -x \sin xy - x \sin xy - x^2 y \cos xy.$$

3. For $f(x, y) = 2y/(y + \cos x)$ we get

$$\begin{aligned}
\frac{\partial f}{\partial x}(x, y) &= \frac{-2y(-\sin x)}{(y+\cos x)^2} = \frac{2y \sin x}{(y+\cos x)^2}, \\
\frac{\partial f}{\partial y}(x, y) &= \frac{(y+\cos x)2 - 2y}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}, \\
\frac{\partial^2 f}{\partial y\, \partial x}(x, y) &= \frac{(y+\cos x)^2 2 \sin x - 4y \sin x (y + \cos x)}{(y+\cos x)^4} \\
&= \frac{(y+\cos x)\cdot(2\sin x)\cdot(y+\cos x - 2y)}{(y+\cos x)^4} \\
&= \frac{2\sin x(\cos x - y)}{(y+\cos x)^3}, \\
\frac{\partial^2 f}{\partial x\, \partial y}(x, y) &= \frac{(y+\cos x)^2(-2\sin x) - 4\cos x(y+\cos x)(-\sin x)}{(y+\cos x)^4} \\
&= \frac{-2\sin x(y + \cos x - 2\cos x)}{(y+\cos x)^3} \\
&= \frac{2\sin x(\cos x - y)}{(y+\cos x)^3},
\end{aligned}$$

$$\text{(again we see that } \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}\text{)}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(x, y) &= \frac{(y+\cos x)^2 2y \cos x - 4y \sin x(y+\cos x)(-\sin x)}{(y+\cos x)^4} \\
&= \frac{2(y+\cos x)[(y+\cos x)y\cos x + 2y\sin^2 x]}{(y+\cos x)^4} \\
&= \frac{2y(y\cos x + \cos^2 x + \sin^2 x)}{(y+\cos x)^3} = \frac{2y(y\cos x + 1)}{(y+\cos x)^3}.
\end{aligned}$$

4. For $f(x, y) = x^2 - xy^2 + y^3$ we get

$$\frac{\partial f}{\partial x}(x, y) = 2x - y^2, \quad \frac{\partial f}{\partial y}(x, y) = -2xy + 3y^2,$$

$$\frac{\partial^2 f}{\partial y\, \partial x}(x, y) = -2y = \frac{\partial^2 f}{\partial x\, \partial y}(x, y),$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 2, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = -2x + 6y.$$

5. For $f(x, y) = \ln(x^2 + y^2)$ we get

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x}{x^2+y^2}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{2y}{x^2+y^2},$$
$$\frac{\partial^2 f}{\partial y\,\partial x}(x,y) = \frac{-2x(2y)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2} = \frac{\partial^2 f}{\partial x\,\partial y}(x,y),$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)\cdot 2 - 2x\cdot 2x}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2},$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{(x^2+y^2)\cdot 2 - 2y\cdot 2y}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}.$$

6. For $f(x,y) = (x^2 - xy + y^2)^5$ we get

$$\frac{\partial f}{\partial x}(x,y) = 5(x^2-xy+y^2)^4(2x-y),$$
$$\frac{\partial f}{\partial y}(x,y) = 5(x^2-xy+y^2)^4(-x+2y),$$
$$\frac{\partial^2 f}{\partial y\,\partial x}(x,y) = 20(x^2-xy+y^2)^3(-x+2y)(2x-y) + 5(x^2-xy+y^2)^5(-1),$$
$$\frac{\partial^2 f}{\partial x\partial y}(x,y) = 20(x^2-xy+y^2)^3(2x-y)(-x+2y) + 5(x^2-xy+y^2)^4(-1),$$
$$\text{so } \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x},$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = 20(x^2-xy+y^2)^3(2x-y)^2 + 5(x^2-xy+y^2)^4\cdot 2,$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = 20(x^2-xy+y^2)^3(-x+2y)^2 + 5(x^2-xy+y^2)\cdot 2.$$

7. For $f(x,y) = e^{xy} + \ln(x+y)$ we get

$$\frac{\partial f}{\partial x}(x,y) = ye^{xy} + \frac{1}{x+y}, \quad \frac{\partial f}{\partial y}(x,y) = xe^{xy} + \frac{1}{x+y},$$
$$\frac{\partial^2 f}{\partial x\,\partial y}(x,y) = e^{xy} + xye^{xy} - \frac{1}{(x+y)^2} = \frac{\partial^2 f}{\partial y\,\partial x}(x,y),$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = y^2e^{xy} - \frac{1}{(x+y)^2}, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = x^2e^{xy} - \frac{1}{(x+y)^2}.$$

Example 8.11 1. Compute $f_x(1,1)$ and $f_y(-1,1)$ for $f(x,y) = e^{xy} + \ln(x^2 + y^2)$.
2. Compute $f_x(1,1,1)$ and $f_z(-1,1,-1)$ for $f(x,y,z) = x\sin(y+3z)$.
3. Compute $f_{xx}(-1,-1)$ and $f_{xy}(-1,-1)$ for $f(x,y) = 3x^2 - 5x\cos\pi y$.

Solution:

1. We get

$$f_x(x,y) = ye^{xy} + \frac{2x}{x^2+y^2}, \quad f_y(x,y) = xe^{xy} + \frac{2y}{x^2+y^2}.$$

Therefore,

$$f_x(1,1) = e + \frac{2}{2} = e + 1, \quad f_y(-1,1) = -e^{-1} + \frac{2}{2} = 1 - \frac{1}{e}.$$

2. We get $f_x(x, y, z) = \sin(y + 3z)$, $f_z(x, y, z) = 3x\cos(y + 3z)$, and therefore $f_x(1, 1, 1) = \sin 4$, $f_z(-1, 1, -1) = -3\cos(1 - 3) = -3\cos(-2) = -3\cos 2$.
3. For $f(x, y) = 3x^2 - 5x\cos\pi y$ we get

$$f_x(x, y) = 6x - 5\cos\pi y, \quad f_{xx}(x, y) = 6, \quad f_{xy}(x, y) = 5\pi\sin\pi y,$$

and therefore $f_{xx}(-1, -1) = 6$, $f_{xy}(-1, -1) = 5\pi\sin(-\pi) = 0$.

Remark 8.5 It may happen that the partial derivatives f_x and f_y exist at some point without f being continuous at that point. For example, let

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

One may check that f is not continuous at $(0, 0)$, but $f_x(0, 0)$ and $f_y(0, 0)$ both exist. The reason for this phenomenon to appear is that the partial derivatives of f are only concerned with the behavior of f along horizontal and vertical directions, whereas continuity of f at some point refers to the behavior of f in a whole neighborhood of that point.

On the other hand, the following result holds.

Definition 8.7 (*Continuous Differentiability*) Let $z = f(x, y)$ be defined in some neighborhood $B = B_\eta(x_0, y_0)$ of a point (x_0, y_0). We say that f is **continuously differentiable** in B if f_x and f_y exist for all points in B and are continuous in B. More general, let D be an open subset of the plane. We say that f is **continuously differentiable** in D if f_x and f_y exist for all points in D and are continuous in D.

Theorem 8.2 *Let $z = f(x, y)$ be continuously differentiable in some subset D of the plane. Then f is continuous in D.*

Note that all functions considered in Example 8.9 are continuously differentiable in their domain.

Applications of partial derivatives. We present several examples.

Example 8.12 The volume of the frustum of a cone (see Fig. 8.11) is given by the function

$$V(R, r, h) = \frac{1}{3}\pi h(R^2 + Rr + r^2).$$

Find the rate of change of the volume with respect to each of these variables if the other variables are held constant. Determine the values of these rates of changes when $R = 8$, $r = 4$, and $h = 6$.

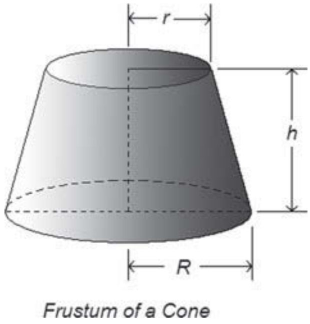


Fig. 8.11 Frustum of a cone

Solution: The partial derivatives of V are

$$V_R(R, r, h) = \frac{1}{3}\pi h(2R + r),$$
$$V_r(R, r, h) = \frac{1}{3}\pi h(R + 2r),$$
$$V_h(R, r, h) = \frac{1}{3}\pi (R^2 + Rr + r^2).$$

For $R = 8$, $r = 4$ and $h = 6$, the rate of change of V

with respect to R is $V_R(8, 4, 6) = \frac{1}{3}\pi 6(16 + 4) = 40\pi$,
with respect to r is $V_r(8, 4, 6) = 32\pi$,
with respect to h is $V_h(8, 4, 6) = \frac{1}{3}\pi(64 + 32 + 16) = \frac{112}{3}\pi$.

Example 8.13 The parallel connection of three resistors (see Fig. 8.12) acts like a single resistor with resistance R whose value is given in terms of the resistances R_1, R_2, and R_3 by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Find the rate of change of R with respect to R_1, when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90\,\Omega$.

Solution: The situation is modeled by the functions

$$R(R_1) = \frac{1}{g(R_1)}, \quad g(R_1) = f(R_1, R_2, R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

The rate of change of R with respect to R_1 is given by

$$R'(R_1) = -\frac{g'(R_1)}{(g(R_1))^2}, \quad g'(R_1) = \frac{\partial f}{\partial R_1}(R_1, R_2, R_3) = -\frac{1}{R_1^2},$$

therefore

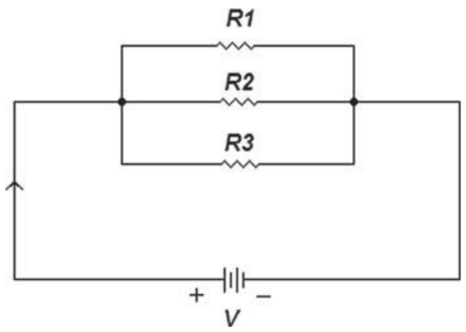


Fig. 8.12 Resistors connected in parallel

$$R'(R_1) = \frac{(R(R_1))^2}{R_1^2}.$$

For $R_1 = 30$, $R_2 = 45$, $R_3 = 90\,\Omega$ we have

$$\frac{1}{R} = f(30, 45, 90) = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{6}{90} = \frac{1}{15},$$

so $R = 15\,\Omega$ and

$$R'(R_1) = \frac{15^2}{30^2} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Another way to model this situation would be to set

$$R(R_1, R_2, R_3) = \frac{1}{f(R_1, R_2, R_3)}, \quad f(R_1, R_2, R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3},$$

and to apply the chain rule in the form

$$\frac{\partial R}{\partial R_1}(R_1, R_2, R_3) = -\frac{1}{f(R_1, R_2, R_3)^2}\frac{\partial f}{\partial R_1}(R_1, R_2, R_3),$$

which leads to the same computation as above.

Example 8.14 Suppose that the weight w in pounds is a function $w = f(c, t)$ of the number c of calories consumed daily and the number t of minutes you exercise daily. Using these units for w, c and t interpret the statements

$$\frac{\partial w}{\partial c}(2000, 15) = 0.02, \quad \frac{\partial w}{\partial t}(2000, 15) = -0.025.$$

Solution: The units of $\dfrac{\partial w}{\partial c}$ are pounds per calorie. Therefore, the statement $\frac{\partial w}{\partial c}(2000, 15) = 0.02$ means that if you are consuming 2000 cal daily and exercis-

ing 15 min daily, you will weigh approximately 0.02 pounds more if you consume daily one extra calorie. If this rate does not change much when c increases, that is, if w viewed as a function of c is approximately linear, then the increase in weight would be about 2 pounds for each extra 100 cal per day. The units of $\dfrac{\partial w}{\partial t}$ are pounds per minute. The statement $\frac{\partial w}{\partial t}(2000, 15) = -0.025$ means that, based on this calorie consumption and number of minutes of exercise, you will weigh 0.025 pounds less for each extra minute you exercise daily or 1 pound less for each extra 40 min per day. So if you eat 100 cal extra each day and exercise for 80 min more each day, your weight would remain roughly the same. (Again, this argument presumes that the relationships are approximately linear for the considered values of c and t.)

Example 8.15 Assume that the concentration C of bacteria in the blood (in millions of bacteria per milliliter) after the injection of an antibiotic is modeled by a function $C = f(x, t) = te^{-xt}$ of the dose x (in grams) injected and the time t (in hours) since the injection. Evaluate the following quantities and explain what each one means in practical terms.
(a) $f_x(1, 2)$, b) $f_t(1, 2)$.

Solution: (a) We have $f_x(x, t) = -t^2e^{-xt}$, so $f_x(1, 2) = -4e^{-2} \simeq -0.54$. The graph of $f(x, 2)$ as a function of x gives the concentration of bacteria two hours after the injection, as a function of the dose. The partial derivative $f_x(1, 2)$ gives the rate of change of bacteria concentration with respect to the dose, at the value of a dose of 1 g. It is the slope of the graph of $f(x, 2)$ (see Fig. 8.13) at the point $x = 1$; it is negative because a larger dose reduces the bacteria population. The value $f_x(1, 2) = -0.54$ means a rate of decrease in bacteria concentration of 0.54 million/ml per gram of additional antibiotic injected, near the nominal dose of 1 g.
(b) We have

$$f_t(x, t) = e^{-xt} - xte^{-xt}\,, \quad f_t(1, 2) = e^{-2} - 2e^{-2} \simeq -0.14.$$

The graph of $f(1, t)$ as a function of t (Fig. 8.14) gives the concentration of bacteria at time t for the dose 1 g of antibiotic. The derivative $f_t(1, 2)$ is the slope of the graph at the point $t = 2$. It is negative because 2 h after the injection, the concentration of bacteria is decreasing with time, due to the action of the antibiotic. The partial derivative $f_t(1, 2)$ gives the rate at which the bacteria concentration is changing with respect to time, namely, a rate of decrease in bacteria concentration of 0.14 million/ml per hour, near the nominal time of 2 h.

The Cobb Douglas Production Model. In the year 1928, Cobb and Douglas used a simple formula proposed by Wicksell to model the production of the entire US economy in the first quarter of twentieth century. Let P be the total yearly production between 1899 and 1922, K the total investment over the same period, and L the total labor force. They found that P was well approximated by the function

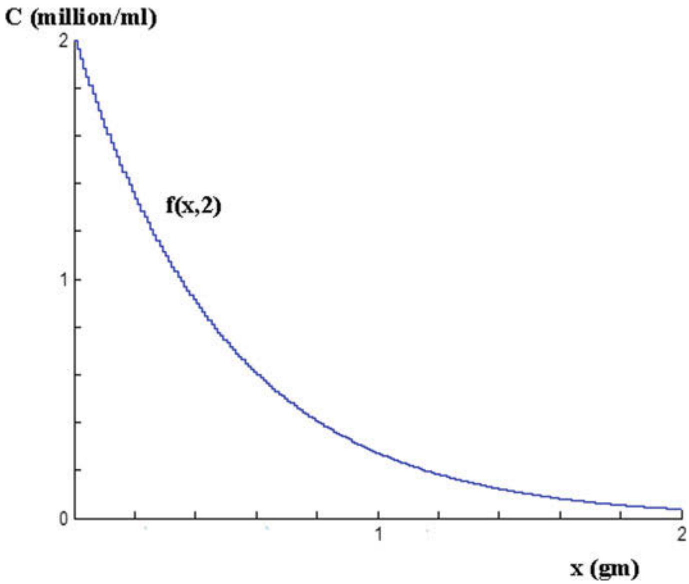


Fig. 8.13 Bacteria concentration after 2 h as a function of the quantity of antibiotic injected

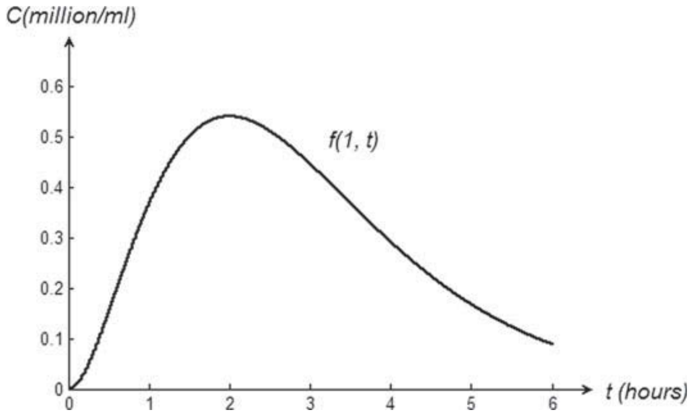


Fig. 8.14 Bacteria concentration as a function of time if 1 unit of antibiotic is injected

$$P = 1.01L^{0.75}K^{0.25}.$$

It modeled the US economy quite accurately for the period on which it was based as well as for some time afterward. This model has found a widespread use in the more general form of a so-called Cobb–Douglas production function

$$P = f(N, V) = cN^a V^b,$$

where P is the total quantity produced, c, a, and b are positive constants with $0 < a < 1$ and $0 < b < 1$ (often one assumes that $b = 1 - a$), N is the number of workers, and V is the cost of capital equipment or investment. The partial derivative f_N is called the **marginal productivity of labor**. It measures the rate of change of production with respect to a change in the expenditure for labor when capital expenditure is kept constant. The partial derivative f_V, called the **marginal productivity of capital**, measures the rate of change of production with respect to change in the amount expended on capital, keeping labor expenditure constant.

Example 8.16 Consider a factory manufacturing blades. Let N be the number of workers, V the value of the equipment (in units of Rs. 50, 000), and P the production measured in thousands of blades per day. Let the production function of this factory be given by

$$P = f(N, V) = 2N^{0.6}V^{0.4}.$$

(a) If the factory has a labor force of 100 workers and 200 units of equipment, what is the production output of the factory?
(b) Find $f_N(100, 200)$ and $f_V(100, 200)$. Interpret your answers in terms of production.

Solution:

(a) We have $N = 100$ and $V = 200$, so $P = 2 \cdot 100^{0.6} \cdot 200^{0.4} = 263.9$ thousand blades per day.
(b) We get

$$f_N(N, V) = 2 \cdot 0.6N^{-0.4}V^{0.4},$$
$$f_N(100, 200) = 1.2 \cdot 100^{-0.4} \cdot 200^{0.4} \simeq 1.583 \quad \text{thousand blades/worker.}$$

This means that if we have 200 units of equipment and increase the number of workers by 1 from 100 to 101, the production output will go up by approximately 1.58 units or 1580 blades per day. Furthermore, we get

$$f_V(N, V) = 2 \cdot 0.4N^{0.6}V^{-0.6},$$
$$f_V(100, 200) = 0.8 \cdot 100^{0.6} \cdot 200^{-0.6} \simeq 0.53$$

thousand blades per unit of equipment. This means that if we have 100 workers and increase the value of equipment by 1 unit (Rs 50, 000) from 200 units to 201 units, the production will go up by about 0.53 units, or 530 blades per day.

Example 8.17 The production in a certain country (in the early years following world war II) is described by the function

$$P = f(x, y) = 30x^{2/3}y^{1/3},$$

where x denotes the number of units of labor used and y the number of units of capital used.

1. Compute f_x and f_y.
2. What is the marginal productivity of labor and the marginal productivity of capital when the amount expended on labor and capital are 125 units and 27 units, respectively?
3. Assuming that one unit of labor and is somehow interchangeable with one unit of capital, should the government have encouraged capital investment rather than increasing expenditure on labor to increase the country's productivity?

Solution:

1. We have

$$f_x(x, y) = 30 \cdot \frac{2}{3} x^{-1/3} y^{1/3} = 20 \left(\frac{y}{x} \right)^{1/3},$$
$$f_y(x, y) = 30 \cdot \frac{1}{3} x^{2/3} y^{-2/3} = 10 \left(\frac{x}{y} \right)^{2/3}.$$

2. The required marginal productivity of labor is given by

$$f_x(125, 27) = 20 \left(\frac{27}{125} \right)^{1/3} = 20 \cdot \frac{3}{5} = 12,$$

that is, 12 units per unit increase in labor expenditure, keeping capital investment constant at 27. The required productivity of capital is given by

$$f_y(125, 27) = 10 \left(\frac{125}{27} \right)^{2/3} = 10 \cdot \frac{25}{9} = \frac{250}{9} = 27\frac{7}{9},$$

that is, $27\frac{7}{9}$ units per unit increase in capital expenditure, keeping labor constant at 125 units.
3. From (2) we see that a unit increase in capital expenditure would have resulted in a much faster increase in productivity than a unit increase in labor expenditure would have. Therefore, the government should have encouraged increased spending on capital rather than on labor during the early years of reconstruction.

The Chain Rule. Let us recall the chain rule for a function of a single variable. For $w = f(x)$ and $x = g(t)$, we obtain the derivative of the composite function $w(t) = (f \circ g)(t) = f(g(t))$ of the functions f and g as

$$w'(t) = (f \circ g)'(t) = f'(g(t))g'(t). \tag{8.11}$$

Symbolically, this may also be written as

$$\frac{dw}{dt} = \frac{df}{dx}\frac{dx}{dt}. \tag{8.12}$$

Let now w be a function of two variables $w = f(x, y)$, and let $x = g(t)$ and $y = h(t)$ be functions of yet another variable t. We consider the composite function w defined by

$$w(t) = f(g(t), h(t)).$$

Theorem 8.3 (Chain Rule for functions of two variables) *Let $w = f(x, y)$, $x = g(t)$ and $y = h(t)$ be continuously differentiable functions. Then w is a continuously differentiable function of t and*

$$w'(t) = \frac{\partial f}{\partial x}(g(t), h(t))g'(t) + \frac{\partial f}{\partial y}(g(t), h(t))h'(t). \tag{8.13}$$

Symbolically, this formula may be written as

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{8.14}$$

or

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}. \tag{8.15}$$

Remark 8.6 In the context of the theorem above, w is the dependent variable and t the independent variable. The variables x and y are called intermediate variables; they play the role of dependent variables (with respect to t) as well as of independent variables (with respect to w). The *tree diagram* (Fig. 8.15) provides a convenient way to remember the chain rule (8.15). Start at w and go down both routes to t, multiplying derivatives along the way. Then add the products.

Note.

1. Formulas (8.14) and (8.15) do not specify at which arguments the various derivatives are evaluated. A more detailed form of (8.15) is

$$\frac{dw}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0).$$

   Here, x_0 is the value of the function $x = g(t)$ at $t = t_0$, and y_0 is the value of the function $y = h(t)$ at $t = t_0$.
2. The chain rule for a function of three independent variables can be stated as follows: If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable functions of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}. \tag{8.16}$$

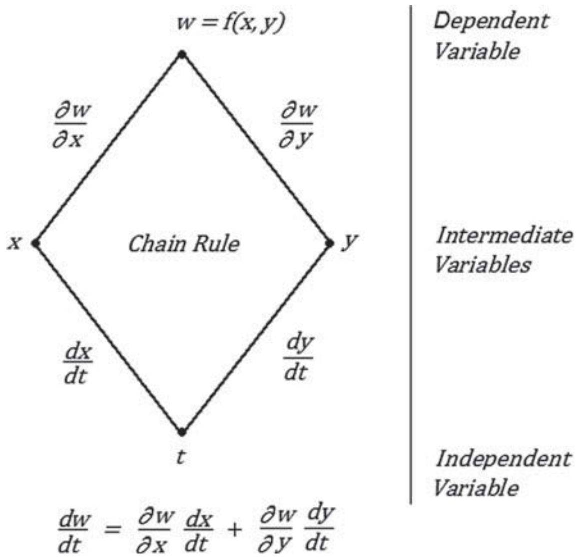


Fig. 8.15 Illustration of the chain rule

3. The chain rule for two independent variables and three intermediate variables reads as follows. Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable then w has partial derivatives with respect to r and s, given by formulas

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}, \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}. \end{aligned} \tag{8.17}$$

Example 8.18 Let $w = f(x, y) = e^{x(x-y)}$, $x = g(t) = 2t\cos t$, $y = h(t) = 2t\sin t$. Evaluate $w' = \dfrac{dw}{dt}$ at $t = \pi$.

Solution: We have

$$\frac{\partial f}{\partial x}(x, y) = e^{x(x-y)}(2x - y), \qquad \frac{dx}{dt}(t) = 2\cos t - 2t\sin t,$$
$$\frac{\partial f}{\partial y}(x, y) = -xe^{x(x-y)}, \qquad \frac{dy}{dt}(t) = 2\sin t + 2t\cos t.$$

For $t = \pi$ we have $x = g(\pi) = -2\pi$, $y = h(\pi) = 0$. The chain rule yields

$$\begin{aligned} w'(\pi) = \frac{dw}{dt}(\pi) &= \frac{\partial f}{\partial x}(-2\pi, 0)\frac{dx}{dt}(\pi) + \frac{\partial f}{\partial y}(-2\pi, 0)\frac{dy}{dt}(\pi) \\ &= (-4\pi)e^{4\pi^2}(-2) - (-2\pi)e^{4\pi^2}(-2\pi) \\ &= 4\pi(2-\pi)e^{4\pi^2}. \end{aligned}$$

Example 8.19 Let $w = f(x, y, z) = x + yz$, $x = g(t) = \cos t$, $y = h(t) = \sin t$, $z = k(t) = t$. Find $w' = \dfrac{dw}{dt}$ at $t = 0$.

Solution: We have

$$\begin{aligned} &\frac{\partial f}{\partial x}(x, y, z) = 1, && \frac{\partial f}{\partial y}(x, y, z) = z, && \frac{\partial f}{\partial z}(x, y, z) = y, \\ &\frac{dx}{dt}(t) = -\sin t, && \frac{dy}{dt}(t) = \cos t\,, && \frac{dz}{dt}(t) = 1. \end{aligned}$$

For $t = 0$ we have $x = \cos(0) = 1$, $y = \sin(0) = 0$, $z = 0$. Thus, the partial derivatives of f have to be evaluated at $(1, 0, 0)$. Using (8.16), that is,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}\,,$$

we obtain

$$w'(0) = \frac{dw}{dt}(0) = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0.$$

Example 8.20 Let $w = f(x, y, z) = x + 2y + z^2$, $x = g(r, s) = \dfrac{r}{s}$, $y = h(r, s) = r^2 + \ln s$, $z = k(r, s) = 2r$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ as functions of r and s.

Solution: We use (8.17) and get, in abbreviated notation,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2.$$

Since $z = 2r$ we obtain

$$\frac{\partial w}{\partial r}(r, s) = \frac{1}{s} + 4r + 4r \cdot 2 = \frac{1}{s} + 12r.$$

In the same manner,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = 1 \cdot \Big(-\frac{r}{s^2}\Big) + 2 \cdot \frac{1}{s} + 2z \cdot 0\,,$$

and therefore

$$\frac{\partial w}{\partial s}(r,s) = \frac{2}{s} - \frac{r}{s^2}.$$

Another application of the chain rule is concerned with homogeneous functions. A function f of n variables is called **homogeneous of degree *n***, if

$$f(tx_1, tx_2, \dots, tx_n) = t^n f(x_1, x_2, \dots, x_n) \tag{8.18}$$

holds for all values of t, x_1, $\dots x_n$.

Theorem 8.4 (Euler) *Let f be a function of n variables which is homogeneous of degree n. Then*

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \cdots + x_n \frac{\partial f}{\partial x_n} = nf \tag{8.19}$$

holds for all values of x_1,...,x_n. (All functions in (8.19) are evaluated at $(x_1, \dots, x_n)$.)

Solution: Let x_1,...,x_n be arbitrary real numbers which will be kept fixed in the following. We introduce intermediate variables $X_1 = g_1(t) = tx_1$,...,$X_n = g_n(t) = tx_n$, and define

$$w(t) = f(g_1(t), \dots, g_n(t)) = f(tx_1, \dots, tx_n).$$

Since f is homogeneous of degree n, we have

$$w(t) = t^n f(x_1, \dots, x_n).$$

Differentiating both sides as functions of t yields

$$w'(t) = nt^{n-1} f(x_1, \dots, x_n). \tag{8.20}$$

We evaluate $w'(t)$ with the chain rule and obtain

$$\begin{aligned} w'(t) &= \frac{\partial f}{\partial x_1}(g_1(t), \dots, g_n(t))g_1'(t) + \cdots + \frac{\partial f}{\partial x_n}(g_1(t), \dots, g_n(t))g_n'(t) \\ &= \frac{\partial f}{\partial x_1}(tx_1, \dots, tx_n)x_1 + \cdots + \frac{\partial f}{\partial x_n}(tx_1, \dots, tx_n)x_n \end{aligned} \tag{8.21}$$

Putting together (8.20) and (8.21) and setting $t = 1$ yields the assertion.

Example 8.21 Verify Euler's theorem for

$$f(x, y, z) = 2x^3 + yz^2 - xyz.$$

Solution: The given function f is homogeneous of degree 3. By direct calculation

$$
\begin{aligned}
xf_x + yf_y + zf_z &= x(6x^2 - yz) + y(z^2 - xz) + z(2yz - xy) \\
&= 6x^3 - 3xyz + 3yz^2 = 3f.
\end{aligned}
$$

Hence (8.19) is verified.

Example 8.22 For $w = f(x, y)$, the two-dimensional Laplace equation

$$
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{8.22}
$$

describes, for example, steady-state temperature distributions in the plane. Its three-dimensional version for $w = f(x, y, z)$ is given by

$$
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0.
$$

(See Fig. 8.16.) Show that the following functions are solutions of the two-dimensional Laplace equation:

1. $w(x, y) = e^{-2y} \cos 2x$,
2. $w(x, y) = \ln \sqrt{x^2 + y^2}$.

Solution:

1. For $w(x, y) = e^{-2y} \cos 2x$ we get

$$
\begin{aligned}
\frac{\partial w}{\partial x}(x, y) &= -2e^{-2y} \sin 2x\,, & \frac{\partial w}{\partial y}(x, y) &= -2e^{-2y} \cos 2x\,, \\
\frac{\partial^2 w}{\partial x^2}(x, y) &= -4e^{-2y} \cos 2x\,, & \frac{\partial^2 w}{\partial y^2}(x, y) &= 4e^{-2y} \cos 2x.
\end{aligned}
$$

   We see that (8.22) is satisfied.
2. We get

$$
\begin{aligned}
\frac{\partial w}{\partial x}(x, y) &= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{x^2 + y^2}\,, \\
\frac{\partial^2 w}{\partial x^2}(x, y) &= \frac{x^2 + y^2 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}\,, \\
\frac{\partial w}{\partial y}(x, y) &= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{x^2 + y^2}\,, \\
\frac{\partial^2 w}{\partial y^2}(x, y) &= \frac{x^2 + y^2 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.
\end{aligned}
$$

   We see that (8.22) is satisfied.

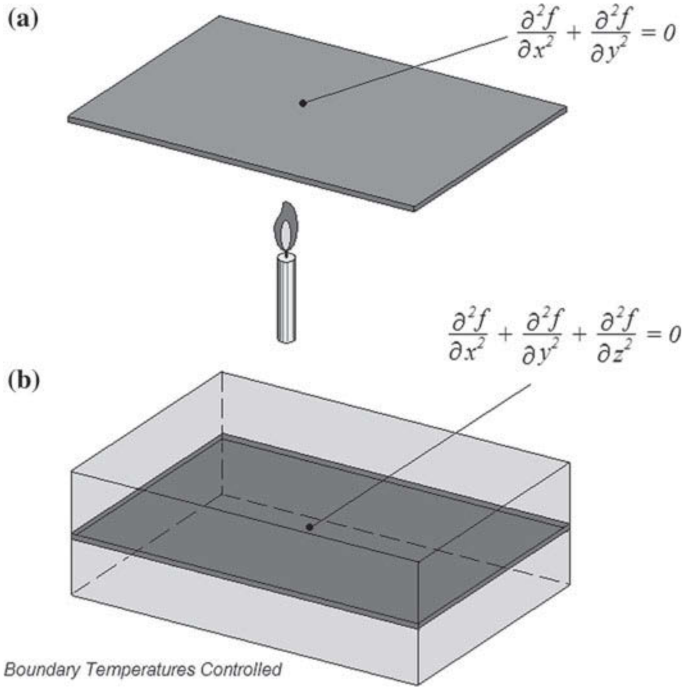


Fig. 8.16 Steady state temperature distributions in planes and solids satisfy Laplace equations. The plane (a) may be treated as a thin slice of the solid (b) Perpendicular to the z-axis

Example 8.23 For functions $w = f(t, x)$, the one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \tag{8.23}$$

is a basic model for wave propagation along a line (forward and backward). For example, w stands for air pressure (when describing sound waves) or for displacement (when describing elastic waves, as in a vibrating string). The independent variables are the time t and the distance x. The constant c corresponds to the speed of the wave, and it varies with the medium; for example, the speed of sound is greater in water than in air.

Show that the following functions are solutions of wave equation:

1. $w = \sin(x + ct)$,
2. $w = \cos(2x + 2ct)$,
3. $w = f(u)$,
   where f is a twice differentiable function of u and $u(t, x) = a(x + ct)$ with some constant a,
4. $w = 5\cos(3x + 3ct) + e^{x+ct}$.

Solution:

1. For $w = \sin(x + ct)$ we get

$$\frac{\partial w}{\partial x}(t,x) = \cos(x+ct), \qquad \frac{\partial^2 w}{\partial x^2}(t,x) = -\sin(x+ct),$$
$$\frac{\partial w}{\partial t}(t,x) = c\cos(x+ct), \qquad \frac{\partial^2 w}{\partial t^2}(t,x) = -c^2\sin(x+ct).$$

We see that (8.23) is satisfied.

2. We get

$$\frac{\partial w}{\partial x}(t,x) = -2\sin(2x+2ct), \qquad \frac{\partial^2 w}{\partial x^2}(t,x) = -4\cos(2x+2ct),$$
$$\frac{\partial w}{\partial t}(t,x) = -2c\sin(2x+2ct), \qquad \frac{\partial^2 w}{\partial t^2}(t,x) = -4c^2\cos(2x+2ct).$$

We see that (8.23) is satisfied.

3. We have $w(t,x) = f(u(t,x)) = f(ax+act)$. Therefore

$$\frac{\partial w}{\partial t}(t,x) = f'(ax+act)\cdot ac, \qquad \frac{\partial^2 w}{\partial t^2}(t,x) = f''(ax+act)\cdot a^2c^2,$$
$$\frac{\partial w}{\partial x}(t,x) = f'(ax+act)\cdot a, \qquad \frac{\partial^2 w}{\partial x^2}(t,x) = f''(ax+act)\cdot a^2.$$

We see that (8.23) is satisfied.

4. For $w(t,x) = 5\cos(3x+3ct) + e^{x+ct}$ we get

$$\frac{\partial w}{\partial t}(t,x) = -15c\sin(3x+3ct) + ce^{x+ct},$$
$$\frac{\partial^2 w}{\partial t^2}(t,x) = -45c^2\cos(3x+3ct) + c^2e^{x+ct},$$
$$\frac{\partial w}{\partial x}(t,x) = -15\sin(3x+3ct) + e^{x+ct},$$
$$\frac{\partial^2 w}{\partial x^2}(t,x) = -45\cos(3x+3ct) + e^{x+ct}.$$

We see that (8.23) is satisfied.

Example 8.24 The **heat equation** (or **diffusion equation**)

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}\,, \tag{8.24}$$

where $\kappa = c^2 > 0$ is a constant, describes instationary (that is, time-dependent) diffusion processes, in particular, heat conduction. In the latter case, $u = u(t, x)$ stands for the temperature, and the number κ represents the heat conduction coefficient. Show that the following functions are solutions of the heat equation.

1. $u = e^{-t} \sin(x/c)$
2. $u = e^{-t} \cos(x/c)$

Solution:

1. We have, in short notation,

$$\frac{\partial u}{\partial t} = -e^{-t} \sin \frac{x}{c}, \quad \frac{\partial u}{\partial x} = \frac{1}{c} e^{-t} \cos \frac{x}{c}\,, \quad \frac{\partial^2 u}{\partial x^2} = -\frac{1}{c^2} e^{-t} \sin \frac{x}{c}.$$

   Hence, $u(t, x) = e^{-t} \sin(x/c)$ is a solution of the heat equation.
2. We have

$$\frac{\partial u}{\partial t} = -e^{-t} \cos \frac{x}{c}\,, \quad \frac{\partial u}{\partial x} = -\frac{1}{c} e^{-t} \sin \frac{x}{c}\,, \quad \frac{\partial^2 u}{\partial x^2} = -\frac{1}{c^2} e^{-t} \cos \frac{x}{c}.$$

   Hence, $u(t, x) = e^{-t} \cos(x/c)$ is a solution of the heat equation.

Parameter-dependent integrals. Let $f = f(x, y)$ be a real-valued function defined on a rectangular domain

$$Q = \{(x, y) : a \le x \le b\,,\ c \le y \le d\}. \tag{8.25}$$

We consider, for a fixed $x \in [a, b]$, the integral

$$\int_c^d f(x, y)\, dy.$$

Such an integral is called a **parameter-dependent integral**, since integration is performed with respect to y, while x plays the role of a parameter. Next, we may consider this integral as a function of the parameter,

$$F(x) = \int_c^d f(x, y)\, dy. \tag{8.26}$$

We can compute the derivative of F as

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_c^d f(x, y)\, dy = \int_c^d \frac{\partial f}{\partial x}(x, y)\, dy\,, \tag{8.27}$$

provided that it is correct to interchange the integral with the derivative. This is asserted in the following theorem. Note that the present situation is different from the one encountered in the fundamental theorem of calculus, since differentiation and integration are performed here with respect to different variables.

Theorem 8.5 *Let f be a continuous real-valued function with domain Q as in (8.25). Then (8.26) defines a function $F : [a, b] \to \mathbb{R}$ which is continuous. If moreover f has a continuous partial derivative with respect to x on Q, then F is differentiable on (a, b) and*

$$F'(x) = \int_c^d \frac{\partial f}{\partial x}(x, y)\, dy. \tag{8.28}$$

We will not prove this theorem.

Example 8.25 Verify (8.28) for $f(x, y) = \sin(x + y)$ and $Q = [0, 1] \times [0, \pi]$.

Solution: We have

$$\begin{aligned} F(x) &= \int_0^\pi \sin(x + y)\, dy = \Big[-\cos(x + y) \Big]_{y=0}^{y=\pi} \\ &= -cos(x + \pi) + \cos x = 2 \cos x\,, \end{aligned}$$

so $F'(x) = -2 \sin x$. On the other hand,

$$\begin{aligned} \int_0^\pi \frac{\partial f}{\partial x}(x, y)\, dy &= \int_0^\pi \cos(x + y)\, dy = \Big[\sin(x + y) \Big]_{y=0}^{y=\pi} \\ &= \sin(x + \pi) - \sin x = -2 \sin x. \end{aligned}$$

Remark 8.7 In order for the assertions of Theorem 8.5 to hold, it is not necessary that f resp. $\partial_x f$ are continuous on Q. Actually, it suffices that they are bounded by integrable functions g resp. h,

$$|f(x, y)| \le g(y)\,, \quad |\partial_x f(x, y)| \le h(y)\,,$$

for all $x \in [a, b]$.

We finally consider the situation where not only the integrand but also the lower and upper limits depend on the parameter,

$$F(x) = \int_{c(x)}^{d(x)} f(x, y)\, dy. \tag{8.29}$$

The derivative of F is given by the **formula of Leibniz**

$$F'(x) = \int_c^d \frac{\partial f}{\partial x}(x, y)\, dy + f(x, d(x))d'(x) - f(x, c(x))c'(x)\,. \tag{8.30}$$

In order to derive this formula, one sets

$$G(x, p, q) = \int_p^q f(x, y)\, dy$$

and applies the chain rule as well as the fundamental theorem of calculus to

$$F(x) = G(x, c(x), d(x))\,.$$

Example 8.26 Compute the derivative of

$$F(x) = \int_{x^2}^{3-x} \sin(xy)\, dy\,.$$

Solution: We have $c(x) = x^2$ and $d(x) = 3 - x$. From (8.30) we obtain that

$$F'(x) = \int_{x^2}^{3-x} y\cos(xy)\, dy + \sin(x(3-x)) \cdot (-1) - \sin(x^3) \cdot 2x\,.$$

8.5 Optimization of Functions of Two Variables

In Chap. 4, we have discussed optimization of functions of one variable. In this section we extend those results to the case of functions of two or more variables. Again, optimization means that we want to choose the most favorable value – here, the maximum or the minimum of a function of several variables. As in the case of a single variable, the differential calculus helps us to find those maximum and minimum values as well as the location of their occurrence.

In optimization, one usually distinguishes between "unconstrained" and "constrained" optimization. In contrast to the former, the latter means that we explicitly take into account certain restrictions imposed when finding the maximum or minimum.

8.5.1 Unconstrained Optimization

Definition 8.8 Let f be a function of two variables with domain $D \subset \mathbb{R}^2$, that is, its domain is a subset of the plane $\mathbb{R}^2$.

1. A point (x_0, y_0) in D is called a **global minimizer** or **global minimum** of f if $f(x_0, y_0) \leq f(x, y)$ holds for all points (x, y) in D.
2. A point (x_0, y_0) in D is called a **global maximizer** or **global maximum** of f if $f(x_0, y_0) \geq f(x, y)$ holds for all points (x, y) in D.
3. A point (x_0, y_0) in D is called a **local minimizer** or **local minimum** of f if

$$f(x_0, y_0) \leq f(x, y) \tag{8.31}$$

holds for all (x, y) in D sufficiently close to (x_0, y_0). This means that there exists an $\varepsilon > 0$ such that (8.31) holds for all (x, y) in D with $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \varepsilon$.
4. A point (x_0, y_0) in D is called a **local maximizer** or **local maximum** of f if $f(x_0, y_0) \geq f(x, y)$ holds for all (x, y) in D sufficiently close to (x_0, y_0) (which means the same as in 3.)
5. A point (x_0, y_0) in D is called a (**local** resp. **global**) **extremum** of f if it is either a (local resp. global) maximum or a (local resp. global) minimum of f.

In all these cases, the corresponding function values $f(x_0, y_0)$ are called **global** (respectively **local**) **minimal** or **minimum** (respectively **maximal** or **maximum**) **values** of f.

Let us recall from Remark 8.1 that (x, y) is called an interior point of the domain of a function f, if f is defined in a whole neighborhood of this point.

Definition 8.9 Let (x, y) be an interior point of the domain of a function f of two variables at which the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ both exist. If

$$f_x(x, y) = 0 \quad \text{and} \quad f_y(x, y) = 0\,, \tag{8.32}$$

then (x, y) is called a **critical point** or a **stationary point** of f. If (x, y) is a critical point, but neither a local maximum nor a local minimum point, it is called a **saddle point**.

First and second derivative test for a local extremum. Here, we look for extrema which lie in the interior of the domain D of the function f to be extremized. This means that near the extremum we are not constrained by any restriction. Extrema on the boundary of D are treated in Sect. 8.5.2.

Theorem 8.6 *Let the function f of two variables have a local maximum or minimum at a point (x_0, y_0) which is an interior point of the domain D of f.*

1. ***(First Derivative Test for a Local Extremum)***
Assume that both f_x and f_y exist at (x_0, y_0). Then

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0, \tag{8.33}$$

that is, (x_0, y_0) is a critical point of f.

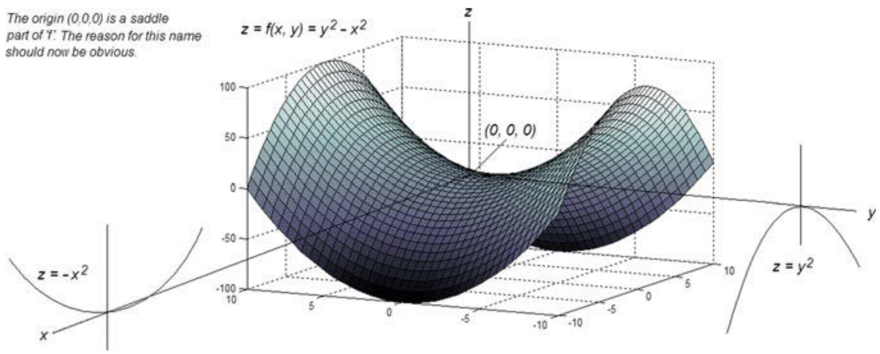


Fig. 8.17 A saddle point

2. ***(Second Derivative Test for a Local Extremum)***
Let (x_0, y_0) *be a critical point of* f *found from part* (1)*, and assume that moreover the second partial derivatives* f_{xx}*,* f_{xy}*,* f_{yx}*, and* f_{yy} *exist and are continuous. Let*

$$\Delta = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2. \tag{8.34}$$

Then the following holds.

a. *If* $\Delta > 0$ *and* $f_{xx}(x_0, y_0) > 0$*, then* f *has a local minimum at* (x_0, y_0)*.*
b. *If* $\Delta > 0$ *and* $f_{xx}(x_0, y_0) < 0$*, then* f *has a local maximum at* (x_0, y_0)*.*
c. *If* $\Delta < 0$*, then* f *has a saddle point at* (x_0, y_0)*.*
d. *If* $\Delta = 0$*, then the test is inconclusive.*

Remark 8.8 1. In Fig. 8.17 we see the function $z = f(x, y) = y^2 - x^2$ with its saddle point at the origin $(0, 0, 0)$
2. The only points where $f(x, y)$ can assume extremum values are critical points and boundary points.

Example 8.27 Find the local extrema of

1. $f(x, y) = x^2 + y^2$,
2. $f(x, y) = x^2 - xy + y^2 + 3x$,
3. $f(x, y) = x^2 + xy + y^2 - 6x + 6$.

Solution:

1. The domain D of $f(x, y) = x^2 + y^2$ is the whole plane, so all points (x, y) are interior points of D. We have

$$f_x(x, y) = 2x\,, \quad f_y(x, y) = 2y.$$

To find the critical points of f we set $f_x = 0$ and $f_y = 0$, which gives $x = 0$ and $y = 0$. Therefore, $(0, 0)$ is the only critical point of f. To check whether

(0, 0) gives a maximum, a minimum, or a saddle point, we compute the second derivatives. They turn out to be the constant functions

$$f_{xx} = 2\,, \quad f_{xy} = 0\,, \quad f_{yy} = 2.$$

So $f_{xx}(0,0) = 2$, $f_{xy}(0,0) = 0$ and $f_{yy}(0,0) = 2$, and we get

$$\Delta = f_{xx}(0,0) f_{yy}(0,0) - (f_{xy}(0,0))^2 = 2 \cdot 2 - 0 = 4 > 0\,.$$

Since $\Delta > 0$ and $f_{xx}(0,0) = 2 > 0$, we see that f has a local minimum at the point (0, 0) with the minimal value $f(0,0) = 0$. Since $f(x, y) > 0$ for all other points (x, y), in this case the local minimum is also a global minimum of f, see Fig. 8.18.

2. The domain D of $f(x, y) = x^2 - xy + y^2 + 3x$ is the whole plane, so all points (x, y) are interior points of D. We have

$$f_x(x, y) = 2x - y + 3\,, \quad f_y(x, y) = -x + 2y.$$

The conditions $f_x = 0$ and $f_y = 0$ yield the system

$$\begin{aligned} 2x - y + 3 &= 0 \\ -x + 2y &= 0 \end{aligned}$$

of two equations for the two unknowns x and y. Solving these gives $x = -2$ and $y = -1$, so f has exactly one critical point, namely, $(-2, -1)$. To check whether $(-2, -1)$ is a maximum, a minimum, or a saddle point, we calculate the second partial derivatives. They turn out to be the constant functions

$$f_{xx} = 2\,, \quad f_{yy} = 2\,, \quad f_{xy} = -1.$$

At the critical point $f_{xx}(-2, -1) = 2 > 0$ and

$$\Delta = f_{xx}(-2,-1) f_{yy}(-2,-1) - (f_{xy}(-2,-1))^2 = 2 \cdot 2 - 1 = 3 > 0.$$

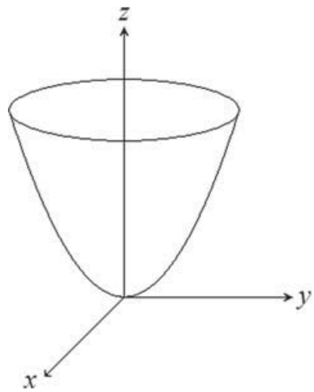
Therefore, f has a local minimum at $(-2, -1)$ with the minimal value $f(-2, -1) = -3$. Since f has partial derivatives everywhere and has no other critical point, there can be no other local minimum or maximum.

3. For $f(x, y) = x^2 + xy + y^2 - 6x + 6$ we compute the first partial derivatives

$$f_x(x, y) = 2x + y - 6\,, \quad f_y(x, y) = x + 2y.$$

Critical points must satisfy $2x + y - 6 = 0$ and $x + 2y = 0$. Solving these equations simultaneously, we get $x = 4$ and $y = -2$, so that $(4, -2)$ is the unique critical point. To determine whether this critical point gives a minimum, a max-

Fig. 8.18 A global minimum



imum or a saddle point we find f_{xx}, f_{yy}, f_{xy} at $(4, -2)$. Again, these partial derivatives are constant functions, namely, $f_{xx} = 2$, $f_{yy} = 2$ and $f_{xy} = 1$ so that in particular

$$f_{xx}(4, -2) = 2\,, \quad f_{yy}(4, -2) = 2\,, \quad f_{xy}(4, -2) = 1.$$

Since

$$\Delta = f_{xx}(4, -2) f_{yy}(4, -2) - (f_{xy}(4, -2))^2 = 2 \cdot 2 - 1 = 3 > 0$$

and $f_{xx}(4, -2) = 2 > 0$, the function f has a local minimum at $(4, -2)$ with minimum value $f(4, -2) = -6$.

Example 8.28 Find and analyze the critical points of

1. $f(x, y) = -x^2 + 2x - y^2 - 4y - 5$,
2. $f(x, y) = xy$.

Solution:

1. We have
$$f_x(x, y) = -2x + 2\,, \quad f_y(x, y) = -2y - 4.$$

To find the critical points we set $f_x = 0$ and $f_y = 0$, that is $-2x + 2 = 0$ and $-2y - 4 = 0$. Solving these equations gives $x = 1$, $y = -2$. Hence, f has exactly one critical point, namely, $(1, -2)$. To determine the type of the critical point we compute the second partial derivatives and obtain the constant functions $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$. Since

$$\Delta = f_{xx}(1, -2) f_{yy}(1, -2) - (f_{xy}(1, -2))^2 = 2 \cdot 2 - 0 = 4 > 0$$

and $f_{xx}(1, -2) < 0$, the function f has a local maximum at the point $(1, -2)$ with a maximum value $f(1, -2) = 0$.

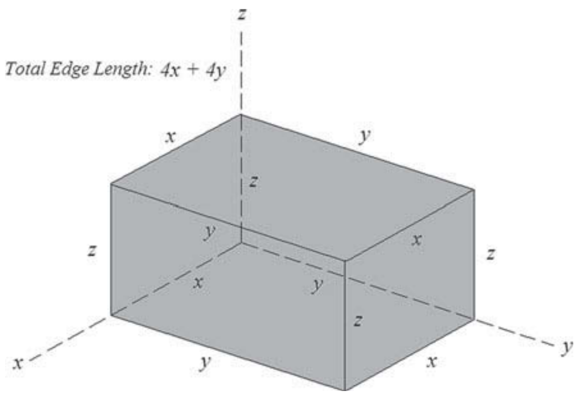


Fig. 8.19 A solid whose volume is maximized

2. We have $f_x(x, y) = y$ and $f_y(x, y) = x$. The conditions $f_x = 0$ and $f_y = 0$ give $x = 0$ and $y = 0$, so $(0, 0)$ is the only critical point. To determine the type of this critical point, we compute

$$f_{xx}(0, 0) = 0, \; f_{yy}(0, 0) = 0, \; f_{xy}(0, 0) = 1$$

so

$$\Delta = 0 \cdot 0 - 1^2 = -1 < 0.$$

Therefore, the function f has a saddle point at $(0, 0)$, and f has no local extrema.

Example 8.29 Find the rectangular three-dimensional solid of maximal volume (see Fig. 8.19), when the sum of the length of all edges is equal to a given constant.

Solution: Let x, y, and z be the length of the edges in the corresponding directions. The sum of the length of those 12 edges equals $4x + 4y + 4z$ (see Fig. 8.19). Let $4x + 4y + 4z = 4k$ with a fixed constant $k > 0$, so $x + y + z = k$. The volume of the solid is $V = xyz$. Since $z = k - x - y$, we can write V as a function of x and y only,

$$V(x, y) = xy(k - x - y) = kxy - x^2y - xy^2.$$

Since edges must have nonnegative lengths, the domain D of V is described by the inequalities $0 \le x, 0 \le y$ and $0 \le z = k - x - y$ which form a triangle in the plane, see Fig. 8.20. Along the sides of the triangle we have $V = 0$, so the maximum, if it exists, must be an interior point of D. We therefore compute the critical points of f in the interior of D. They satisfy

$$\begin{aligned} 0 &= V_x(x, y) = ky - 2xy - y^2, \\ 0 &= V_y(x, y) = kx - x^2 - 2xy. \end{aligned}$$

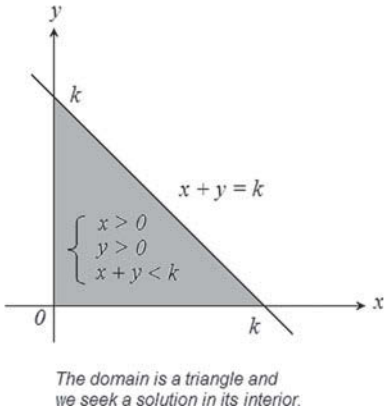


Fig. 8.20 Restrictions on edge lengths

Since $x > 0$ and $y > 0$ for interior points, we may divide by y and x, respectively, and obtain

$$k - 2x - y = 0\,, \quad k - x - 2y = 0.$$

Solving these equations simultaneously, we get

$$x = \frac{1}{3}k\,, \quad y = \frac{1}{3}k$$

as the unique critical point of f which moreover lies in the interior of D because $k - \frac{1}{3}k - \frac{1}{3}k = \frac{1}{3}k > 0$. Now we determine whether the point $(\frac{1}{3}k, \frac{1}{3}k)$ is a local maximum, a local minimum, or a saddle point. We compute the second partial derivatives as

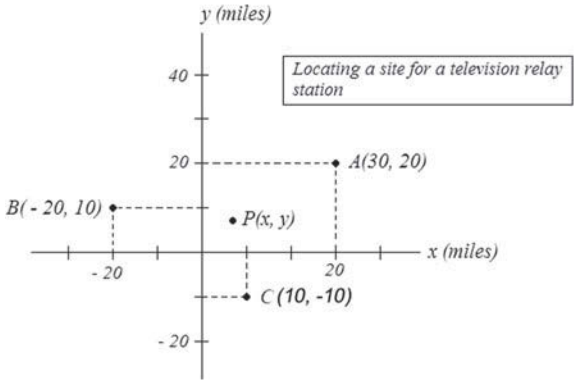
$$V_{xx}(x, y) = -2y\,, \quad V_{yy}(x, y) = -2x, \quad V_{xy}(x, y) = k - 2x - 2y\,.$$

At the point $(\frac{1}{3}k, \frac{1}{3}k)$, we get

$$\begin{aligned} V_{xx}V_{yy} - V_{xy}^2 &= -\frac{2}{3}k \cdot \left(-\frac{2}{3}k\right) - \left(-\frac{1}{3}k\right)^2 = \frac{4}{9}k^2 - \frac{1}{9}k^2 \\ &= \frac{1}{3}k^2 > 0. \end{aligned}$$

Therefore, V has in D a unique local maximum at $x = \frac{1}{3}k$ and $y = \frac{1}{3}k$. One can prove by other means that V must have a global maximum on D. Since the global maximum is also a local maximum, it must be the point we just computed. We also see that $z = k - x - y = \frac{1}{3}k$ for this point, so the solid with maximal volume is in fact a cube with side length $\frac{1}{3}k$.

Fig. 8.21 Locating a site for a television relay station



Example 8.30 A television relay station will serve towns A, B, and C whose relative locations are shown in Fig. 8.21. Determine a site for the location of the station such that the sum of the squares of the distances from each town to the site is minimized.

Solution: Let the required site be located at the point $P = (x, y)$. The square of the distance of P from town A is $(x-30)^2 + (y-20)^2$. For B and C, the squares of the distances are $(x+20)^2 + (y-10)^2$ and $(x-10)^2 + (y+10)^2$, respectively. The sum of the squares of the distances from P to the three towns is given by

$$f(x, y) = (x-30)^2 + (y-20)^2 + (x+20)^2 + (y-10)^2 + (x-10)^2 + (y+10)^2.$$

In order to find the minimum, we determine the critical points of f. We must have $f_x = 0$ and $f_y = 0$, so

$$0 = f_x(x, y) = 2(x-30) + 2(x+20) + 2(x-10) = 6x - 40,$$
$$0 = f_y(x, y) = 2(y-20) + 2(y-10) + 2(y+10) = 6y - 40.$$

Solving these equations we get $x = \frac{20}{3}$ and $y = \frac{20}{3}$, so that $(\frac{20}{3}, \frac{20}{3})$ is the only critical point of f. Now

$$f_{xx}\left(\frac{20}{3}, \frac{20}{3}\right) = 6, \quad f_{xy}\left(\frac{20}{3}, \frac{20}{3}\right) = 0, \quad f_{yy}\left(\frac{20}{3}, \frac{20}{3}\right) = 6.$$

We get

$$D = f_{xx}\left(\frac{20}{3}, \frac{20}{3}\right) \cdot f_{yy}\left(\frac{20}{3}, \frac{20}{3}\right) - f_{xy}^2\left(\frac{20}{3}, \frac{20}{3}\right) = 36 > 0,$$

and $f_{xx}(\frac{20}{3}, \frac{20}{3}) > 0$. Therefore, f has a local minimum at $(\frac{20}{3}, \frac{20}{3})$. One can prove by other means that f must have a global minimum. Thus, the point just computed yields the unique global minimum. Hence, the required site has coordinates

$$x = \frac{20}{3} \quad \text{and} \quad y = \frac{20}{3}.$$

Example 8.31 A manufacturing company produces two products which are sold in two separate markets. The quantities q_1 and q_2 demanded by the consumers and the prices p_1 and p_2 (in dollars) of each item are related by the equations

$$p_1 = 600 - 0.3q_1\,, \quad p_2 = 500 - 0.2q_2.$$

(Increasing price corresponds to decreasing demand.) The company's total production cost is given by

$$C(q_1, q_2) = 16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2.$$

If the company wants to maximize its total profits, how much of each product should it produce? What is the maximum profit?

Solution: The revenue from the each product sold in the corresponding market equals p_1q_1 and p_2q_2, respectively. The total revenue R equals its sum $p_1q_1 + p_2q_2$. Written as a function of q_1 and q_2 we get

$$\begin{aligned} R(q_1, q_2) &= (600 - 0.3q_1)q_1 + (500 - 0.2q_2)q_2 \\ &= 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2. \end{aligned}$$

The total profit becomes

$$\begin{aligned} P(q_1, q_2) &= R(q_1, q_2) - C(q_1, q_2) \\ &= 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2 - (16 + 1.2q_1 + 1.5q_2 + .2q_1q_2) \\ &= -16 + 598.8q_1 - 0.3q_1^2 + 498.5q_2 - 0.2q_2^2 - 0.2q_1q_2. \end{aligned}$$

The critical points of P are obtained by equating to zero both partial derivatives of P, that is,

$$\begin{aligned} 0 &= \frac{\partial P}{\partial q_1}(q_1, q_2) = 598.8 - 0.6q_1 - 0.2q_2, \\ 0 &= \frac{\partial P}{\partial q_2}(q_1, q_2) = 498.5 - 0.4q_2 - 0.2q_1. \end{aligned}$$

Solving these two equations simultaneously, we get

$$q_1 = 699.1 \simeq 699 \quad \text{and} \quad q_2 = 896.7 \simeq 897,$$

and therefore $p_1 \simeq 390.30$, $p_2 \simeq 320.60$. We use the second derivative test to check whether this critical point $(699.1, 896.7)$ gives a local maximum. The second-order

partial derivatives are constant functions,

$$\frac{\partial^2 P}{\partial q_1^2} = -0.6, \quad \frac{\partial^2 P}{\partial q_2^2} = -0.4, \quad \frac{\partial^2 P}{\partial q_1 \partial q_2} = -0.2.$$

Since

$$\Delta = \frac{\partial^2 P}{\partial q_1^2}\frac{\partial^2 P}{\partial q_2^2} - \left(\frac{\partial^2 P}{\partial q_1 \partial q_2}\right)^2 = (-0.6)(-0.4) - (-0.2)^2 = 0.2 > 0,$$

and

$$\frac{\partial^2 P}{\partial q_1^2} = -0.6 < 0,$$

the profit function has a unique local maximum at (699, 897). One can prove by other means that a global maximum must exist, which is therefore equal to this point. As the final result, the company should produce 699 units of the first product priced at \$ 390.30 per unit and 897 units of the second product priced at \$ 320.60 per unit. The maximal profit is

$$P(699, 897) = \$\,432,797.$$

Least Squares Fit, Regression. Let us assume that two scalar quantities x and y are related by the linear model $y = g(x) = ax + b$, and that we want to determine the parameters a and b from a set of known data $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ of corresponding values of x and y. (Once we now a and b, we may predict the values of y for other values of x from the equation $y = g(x)$.) In principle, two pairs are sufficient ($n = 2$), since we then may determine a and b from the linear system

$$ax_1 + b = y_1\,, \quad ax_2 + b = y_2,$$

whenever $x_1 \neq x_2$, and $y = ax + b$ for those values of a and b yields the unique straight line through the points (x_1, y_1) and (x_2, y_2) in the plane. Since the data or the model may not be accurate, one usually wants to use more data ($n > 2$) in order to get the most useful values for a and b. In that case, however, we have more equations $ax_i + b = y_i$ than the two unknowns a and b, and the problem arises to find the "best" line with respect to the data, see Fig. 8.22. The usual solution is to determine a and b such that the sum of the squares of the deviations $y_i - g(x_i)$ becomes as small as possible. In other words, one minimizes the function of two variables

$$f(a, b) = \sum_{i=1}^{n} [(ax_i + b) - y_i]^2.$$

The resulting values of a and b are called the **least squares fit** to the given data set of pairs (x_i, y_i), see Fig. 8.23. In order to determine those values we compute

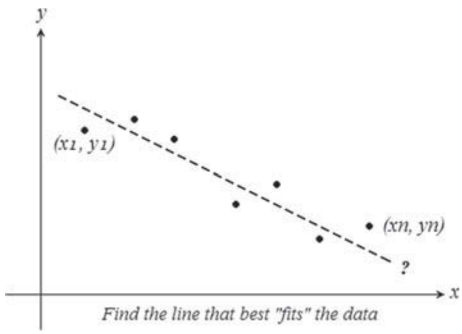


Fig. 8.22 Find the line that best "fits" the data

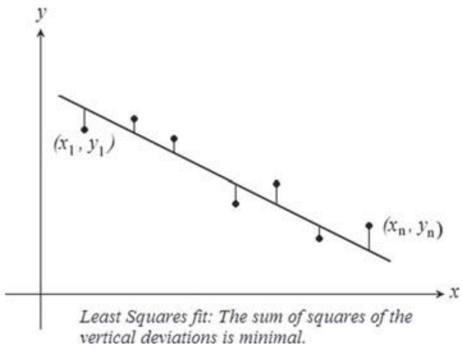


Fig. 8.23 Least squares fit: The sum of squares of the vertical deviations is minimal

$$\begin{aligned}\frac{\partial f}{\partial a}(a,b) &= 2\sum_{i=1}^{n}(ax_i + b - y_i)x_i = 2\Big[a\sum_{i=1}^{n}x_i^2 + b\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}x_i y_i\Big],\\ \frac{\partial f}{\partial b}(a,b) &= 2\sum_{i=1}^{n}(ax_i + b - y_i) = 2\Big[a\sum_{i=1}^{n}x_i + nb - \sum_{i=1}^{n}y_i\Big].\end{aligned} \tag{8.35}$$

At the minimum both partial derivatives must be zero, and we arrive at the system

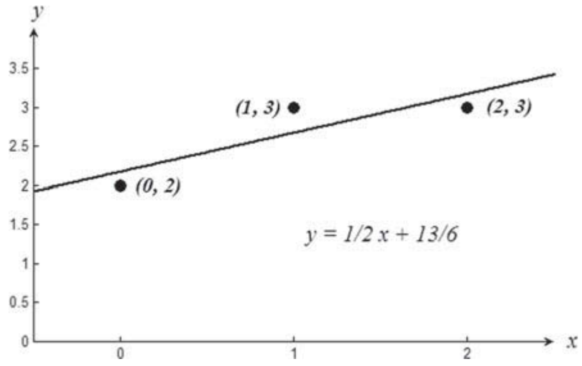
$$\begin{aligned}a\sum_{i=1}^{n}x_i^2 + b\sum_{i=1}^{n}x_i &= \sum_{i=1}^{n}x_i y_i,\\ a\sum_{i=1}^{n}x_i + nb &= \sum_{i=1}^{n}y_i,\end{aligned} \tag{8.36}$$

of two equations, linear with respect to a and b, from which we determine a and b.

Example 8.32 Find the line $y = ax + b$ that gives the least squares fit to the points $(0, 2)$, $(1, 3)$, $(2, 3)$, see Fig. 8.24.

Solution: We have $n = 3$, and the (x_i, y_i) are given successively by $(0, 2)$, $(1, 3)$ and $(2, 3)$. According to (8.36), the necessary conditions $\partial f/\partial a = 0 = \partial f/\partial b$ become

Fig. 8.24 Least squares fit, Example 8.32



$$\begin{aligned} a(0+1+4)+b(0+1+2) &= 2\cdot 0+3\cdot 1+3\cdot 2,\\ a(0+1+2)+3b &= 2+3+3, \end{aligned}$$

that is,

$$\begin{aligned} 5a+3b &= 9\\ 3a+3b &= 8. \end{aligned}$$

Its unique solution is $a=\frac{1}{2}$ and $b=\frac{13}{6}$, so the least squares solution yields the straight line

$$y=\frac{1}{2}x+\frac{13}{6}.$$

With the second derivative test, one may check that the computed values for a and b indeed yield a local minimum. Namely, we get $f_{aa}=10>0$, $f_{bb}=6$ and $f_{ab}=f_{ba}=6$, so $\Delta=f_{aa}f_{bb}-(f_{ab})^2=60-36=24>0$. One may ask whether the system (8.36) has a unique solution for a and b. The answer is "yes" unless we have $x_i=x_j$ for all $1\le i,j\le n$. We will not give the proof here.

Relation to statistics. Statistical analysis provides a reason why the least squares method appears in many cases as the natural approach when one wants to fit data. Let us briefly mention this connection; for further explanation and exposition we refer to courses in probability and statistics. One interprets the measured values $y_1,\dots,y_n$ as solutions of

$$y_i=ax_i+b+z_i,$$

where z_i are random perturbations, assumed to be independent and normally distributed with mean zero. One then can prove that the least squares solution computed above yields the so-called maximum likelihood estimate of a and b. In this context, the resulting line $y=ax+b$ is called the **regression line**, and its slope a is called the **regression coefficient**. Moreover, the **mean values**

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i,$$

are related by

$$a\overline{x} + b = \overline{y}.$$

This follows if we divide the second line in (8.36) by n.

8.5.2 Constrained Optimization

In most cases, practical optimization problems include constraints due to external circumstances and conditions. For example, a city administration desiring to build a modern public transport system has limited resources. A nation trying to maintain its balance of trade must not spend more on imports than it earns on exports.

We discuss here how to find an optimum value under such constraints. We restrict ourselves to the basic situation when want to find the extrema of a function $z = f(x, y)$ of two variables subject to a single constraint of the form $g(x, y) = 0$. (Such constrained extrema are also called relative extrema.)

The Method of Lagrange Multipliers. To find the constrained extrema of the function $z = f(x, y)$ subject to the constraint $g(x, y) = 0$, one carries out the following steps.

1. One forms an auxiliary function of three variables,

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y). \tag{8.37}$$

The function F is called the Lagrange function (or, in short, the Lagrangian), and the variable λ is called the Lagrange multiplier.

2. Determine the critical points of F, that is, solve the system of three equations

$$F_x = 0, \quad F_y = 0, \quad F_\lambda = 0, \tag{8.38}$$

with respect to the three unknowns x, y, and λ.

3. For each critical point (x, y, λ) of F found in step 2, the point (x, y) is a candidate for a constrained local maximum (or minimum) of f.
4. Apply the second derivative test in the following form: If

$$\Delta = F_{xx}F_{yy} - F_{xy}^2$$

satisfies $\Delta > 0$ at a critical point (x, y, λ), then (x, y) is a local minimum if $F_{xx}(x, y, \lambda) > 0$, and a local maximum if $F_{xx}(x, y, \lambda) < 0$. The test is inconclusive in the cases $\Delta = 0$ and (in contrast to the situation without constraints) $\Delta < 0$.

Example 8.33 Using the method of Lagrange multipliers, find the local extrema of

1. $f(x, y) = 2x^2 + y^2$ subject to the constraint $x + y = 1$.
2. $f(x, y) = xy$ subject to the constraint $x + y = 16$.

Solution:

1. We write the constraint equation $x + y = 1$ in the form $0 = g(x, y) = x + y - 1$ and form the Lagrange function

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = 2x^2 + y^2 + \lambda(x + y - 1)\,.$$

To find the critical points of the function F, we solve the system of equations

$$\begin{aligned} 0 &= F_x = 4x + \lambda, \\ 0 &= F_y = 2y + \lambda, \\ 0 &= F_\lambda = x + y - 1. \end{aligned}$$

Solving the first two equations for x and y in terms of λ, we obtain

$$x = -\frac{1}{4}\lambda\,, \quad y = -\frac{1}{2}\lambda.$$

Substituting x and y in the third equation, we get

$$-\frac{1}{4}\lambda - \frac{1}{2}\lambda - 1 = 0\,, \quad \text{so } \lambda = -\frac{4}{3}.$$

Therefore, $x = \frac{1}{3}$, $y = \frac{2}{3}$, and thus the point $(\frac{1}{3}, \frac{2}{3}, -\frac{4}{3})$ is the only critical point of F. Concerning the second derivative test, we see that the second derivatives of F are the constant functions

$$F_{xx} = 4\,, \quad F_{yy} = 2\,, \quad F_{xy} = 0,$$

so $\Delta = F_{xx}F_{yy} - (F_{xy})^2 = 4 \cdot 2 - 0 = 8 > 0$ and $F_{xx} = 4 > 0$. Therefore, the point $(\frac{1}{3}, \frac{2}{3})$ is a local minimum of f relative to the constraint $x + y = 1$
2. We set $g(x, y) = x + y - 16$ and obtain the Lagrange function F as

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = xy + \lambda(x + y - 16).$$

The critical points are obtained by solving the system of equations

$$\begin{aligned} 0 &= F_x = y + \lambda, \\ 0 &= F_y = x + \lambda, \\ 0 &= F_\lambda = x + y - 16 = 0. \end{aligned}$$

From the first and second equation, we get $y = -\lambda$ and $x = -\lambda$. Substituting these in the third equation, we get $-2\lambda - 16 = 0$, so $\lambda = -8$ and $x = y = 8$. Therefore, the unique critical point of F is $(8, 8, -16)$. Again, the second derivatives of F are constant functions, namely, $F_{xx} = F_{yy} = 0$ and $F_{xy} = F_{yx} = 1$, so $\Delta = F_{xx}F_{yy} - (F_{xy})^2 = -1 < 0$. Thus, the second derivative test is inconclusive. In fact, the point $(8, 8)$ is a local maximum of f relative to the constraint $x + y = 16$.

Note that in the two examples above we could have eliminated the constraint $g(x, y) = 0$ by solving for y, thus reducing the problem to an unconstrained optimization problem, as we have done previously. In the first problem, $g(x, y) = 0$ gives $y = 1 - x$ and we are left with finding the extrema of $h(x) = f(x, 1 - x) = 2x^2 + (1 - x)^2$, which indeed would be much simpler than the computation above. However, such a direct elimination may be cumbersome or infeasible in other examples, in particular, when more variables and more constraints are involved, a situation which we do not discuss here.

Example 8.34 Suppose that x units of labor and y units of capital are required to produce

$$f(x, y) = 100x^{3/4}y^{1/4}$$

units of a certain product. Assume that each unit of labor costs \$200, each unit of capital costs \$ 300, and a total amount of \$ 60, 000 is available for production. Determine how the funds should be allocated to labor and capital in order to maximize production.

Solution: The total cost of x units of labor at \$ 200 per unit and y units of capital at \$ 300 per unit equals $200x + 300y$ dollars. At our disposal, we have $200x + 300y = 60,000$ dollars. We thus write

$$g(x, y) = 200x + 300y - 60000$$

as the constraint function. To maximize $f(x, y) = 100x^{3/4}y^{1/4}$ subject to the constraint $g(x, y) = 0$ we form the Lagrange function

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) + \lambda g(x, y) \\ &= 100x^{3/4}y^{1/4} + \lambda(200x + 300y - 60000). \end{aligned}$$

To find the critical points of F, we solve the system of equations

$$\begin{aligned} 0 &= F_x = 75x^{-1/4}y^{1/4} + 200\lambda = 0\,, \\ 0 &= F_y = 25x^{3/4}y^{-3/4} + 300\lambda\,, \\ 0 &= F_\lambda = 200x + 300y - 60000. \end{aligned} \tag{8.39}$$

Solving the first equation for λ, we get

$$\lambda = -\frac{75x^{-1/4}y^{1/4}}{200} = -\frac{3}{8}\Big(\frac{y}{x}\Big)^{1/4}.$$

Substituting in the second equation gives

$$25\Big(\frac{x}{y}\Big)^{3/4} + 300 \cdot \Big(-\frac{3}{8}\Big)\Big(\frac{y}{x}\Big)^{1/4} = 0.$$

Multiplying the above equation by $(\frac{x}{y})^{1/4}$ gives

$$25 \cdot \frac{x}{y} - \frac{900}{8} = 0\,, \quad \text{so } x = \frac{900}{8} \cdot \frac{1}{25}y = \frac{9}{2}y.$$

Substituting this value of x in the third equation of the system (8.39), we have

$$200 \cdot \frac{9}{2}y + 300y - 60000 = 0,$$

so $y = 50$ and hence $x = 225$. We leave the verification of the second derivative test to the reader. Thus, maximum production is obtained when 225 units of labor and 50 units of capital are used.

8.6 Taylor Expansion in Two Variables

For a function f of a single variable, we have discussed in Sect. 5.4 its Taylor expansion of order n at $x = a$

$$f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k + R_n(h)\,, \quad R_n(h) = \frac{f^{(n+1)}(a+ch)}{(n+1)!}h^{n+1},$$

the remainder term R_n being evaluated at some point $a + th$ with $t \in [0, 1]$, which lies between a and $a + h$. For functions f of more than one variable, an analogous formula holds which involves the partial derivatives of f. We present here the Taylor expansion in the case of two independent variables.

Theorem 8.7 *Let a function f of two variables be defined in an open rectangle D centered at (a, b), suppose that f and its partial derivatives up to order $(n + 1)$ are continuous in D. Then for all points $(x, y) = (a + h, b + k)$ in D,*

$$
\begin{aligned}
f(a+h, b+k) = f(a, b) &+ \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)\bigg|_{(a,b)} \\
&+ \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)\bigg|_{(a,b)} \\
&+ \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial y^2 \partial x} + k^3\frac{\partial^3 f}{\partial y^3}\right)\bigg|_{(a,b)} \\
&+ \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\bigg|_{(a,b)} + R_n(a+th, b+tk),
\end{aligned}
\tag{8.40}
$$

where the terms up to order n are evaluated at the point (a, b) *as indicated, and the remainder*

$$
R_n(x, y) = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\bigg|_{(x,y)}
$$

is evaluated at some point $(x, y) = (a+th, b+tk), \in [0, 1]$*, on the line segment joining* (a, b) *and* $(a+h, b+k)$.

In the last line of formula (8.40), and analogously in the formula for R_n, we have used the abbreviation

$$
\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f
$$

with n to be understood as an exponent. The reader should compare with the term for $n = 3$ in formula (8.40) to interpret its meaning.

In the case $(a, b) = (0, 0)$ we have $(x, y) = (h, k)$ in the theorem above, and Taylor's formula for the expansion of f at the origin becomes

$$
\begin{aligned}
f(x, y) = f(0, 0) &+ \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)\bigg|_{(0,0)} \\
&+ \frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y2}\right)\bigg|_{(0,0)} \\
&+ \frac{1}{3!}\left(x^3\frac{\partial^3 f}{\partial x^3} + 3x^2y\frac{\partial^3 f}{\partial x^2 \partial y} + 3xy^2\frac{\partial^3 f}{\partial y^2 \partial x} + y^3\frac{\partial^3 f}{\partial x^3}\right)\bigg|_{(0,0)} + \dots \\
&+ \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f\bigg|_{(0,0)} + \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f\bigg|_{(tx,ty)}.
\end{aligned}
\tag{8.41}
$$

The terms containing the first n derivatives are evaluated at $(0, 0)$. The remainder term is evaluated at some point (tx, ty) on the line segment joining the origin and (x, y).

Example 8.35 Write down the expansion at the origin of the following functions of two variables with derivative terms up to order 3.

1. $f(x, y) = \sin x \sin y$.
2. $f(x, y) = e^x \sin y$.
3. $f(x, y) = e^{xy}$.

Solution:

1. The partial derivatives up to order 3 are given by (we list them without writing the argument (x, y) on the left side)

$$\frac{\partial f}{\partial x} = \cos x \sin y, \qquad \frac{\partial f}{\partial y} = \sin x \cos y, \qquad \frac{\partial^2 f}{\partial x^2} = -\sin x \sin y,$$
$$\frac{\partial^2 f}{\partial x \partial y} = \cos x \cos y, \qquad \frac{\partial^2 f}{\partial y^2} = -\sin x \sin y, \qquad \frac{\partial^3 f}{\partial x^3} = -\cos x \sin y,$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} = -\sin x \cos y, \qquad \frac{\partial^3 f}{\partial y^2 \partial x} = -\cos x \sin y, \qquad \frac{\partial^3 f}{\partial y^3} = -\sin x \cos y\,.$$

Since all those terms except $\partial^2 f/\partial x \partial y$ are equal to 0 when evaluated at $(0, 0)$, the Taylor expansion of f at $(0, 0)$ up to order 3 therefore becomes

$$f(x, y) = \frac{1}{2!} 2xy \frac{\partial^2 f}{\partial x \partial y}(0, 0) + R_2(x, y) == xy + R_3(x, y)\,.$$

2. For $f(x, y) = e^x \sin y$, we have $f(0, 0) = 0$ and

$$\frac{\partial f}{\partial x}(x, y) = e^x \sin y, \qquad \frac{\partial f}{\partial x}(0, 0) = 0,$$
$$\frac{\partial f}{\partial y}(x, y) = e^x \cos y, \qquad \frac{\partial f}{\partial y}(0, 0) = 1,$$
$$\frac{\partial^2 f}{\partial x^2}(x, y) = e^x \sin y, \qquad \frac{\partial^2 f}{\partial x^2}(0, 0) = 0,$$
$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = e^x \cos y, \qquad \frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1,$$
$$\frac{\partial^2 f}{\partial y^2}(x, y) = -e^x \sin y, \qquad \frac{\partial^2 f}{\partial y^2}(0, 0) = 0,$$
$$\frac{\partial^3 f}{\partial x^3}(x, y) = e^x \sin y, \qquad \frac{\partial^3 f}{\partial x^3}(0, 0) = 0,$$
$$\frac{\partial^3 f}{\partial x^2 \partial y}(x, y) = e^x \cos y, \qquad \frac{\partial^3 f}{\partial x^2 \partial y}(0, 0) = 1,$$
$$\frac{\partial^3 f}{\partial x \partial y^2}(x, y) = -e^x \sin y, \qquad \frac{\partial^3 f}{\partial x \partial y^2}(0, 0) = 0,$$
$$\frac{\partial^3 f}{\partial y^3}(x, y) = -e^x \cos y, \qquad \frac{\partial^3 f}{\partial y^3}(0, 0) = -1\,.$$

Inserting these values into (8.41) we get

$$\begin{aligned} f(x, y) &= 0 + y + xy + \frac{1}{3!}3x^2y - \frac{1}{6}y^3 + R_3(x, y) \\ &= y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + R_3(x, y)\,. \end{aligned}$$

3. For $f(x, y) = e^{xy}$, we have $f(0, 0) = 1$. The partial derivatives up to order 3 are given by (we again omit the argument (x, y) on the left side)

$$\begin{aligned} &\frac{\partial f}{\partial x} = ye^{xy}, && \frac{\partial f}{\partial y} = xe^{xy}, && \frac{\partial^2 f}{\partial x^2} = y^2e^{xy}, \\ &\frac{\partial^2 f}{\partial x \partial y} = e^{xy} + xye^{xy}, && \frac{\partial^2 f}{\partial y^2} = x^2e^{xy}, && \frac{\partial^3 f}{\partial x^3} = y^3e^{xy}, \\ &\frac{\partial^3 f}{\partial x^2 \partial y} = 2ye^{xy} + xy^2e^{xy}, && \frac{\partial^3 f}{\partial y^2 \partial x} = 2xe^{xy} + yx^2e^{xy}, && \frac{\partial^3 f}{\partial y^3} = x^3e^{xy}\,. \end{aligned}$$

Evaluated at $(0, 0)$, all those partial derivatives except $\partial^2 f/\partial x \partial y$ are equal to zero, so we obtain

$$\begin{aligned} f(x, y) &= f(0, 0) + \frac{1}{2!}xy\frac{\partial^2 f}{\partial x \partial y}(0, 0) + R_3(x, y) \\ &= 1 + \frac{1}{2}xy + R_3(x, y)\,. \end{aligned}$$

8.7 Integration of Functions of Several Variables

In Chap. 6, we have discussed integrals of functions of one variable, and some applications have been presented in Chap. 7. In this section, we extend the notion of integration to functions of several variables. We consider in detail the case of two variables. Subsequently, we present the case of more than two variables which is analogous.

Definition of integrals in two dimensions. Consider a function f which is defined in a rectangular region $Q = [a, b] \times [c, d]$ of the xy-plane. Let Q be partitioned into rectangular subregions as in Fig. 8.25, let us denote by A_P the area of such a small rectangle P, by f_P the value of f at an arbitrarily chosen point of P, and by Δ the partition of Q consisting of all such P. If the sums $\sum_{P \in \Delta} f_P A_P$ approach a finite limit when the partition Δ is made finer and finer, this limit is called the **integral** (or **double integral**) of f over Q, and it is written as

$$\iint_Q f\, dA\,, \quad \text{or} \quad \iint_Q f(x, y)\, dA\,. \tag{8.42}$$

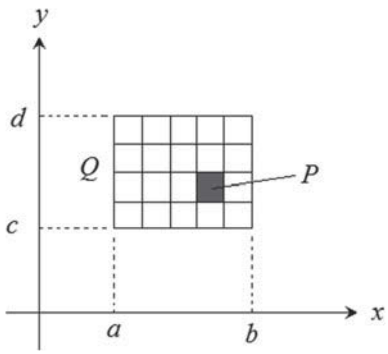


Fig. 8.25 Partition of a rectangle

The function f is then called **integrable over Q**. A formally precise definition of this limit will be given in Appendix D.10. It always exists when f is continuous, but it also exists for many discontinuous functions.

We now consider an arbitrary bounded region D of the plane. Let $Q = [a, b] \times [c, d]$ be an arbitrary rectangle which encloses D (that is, $D \subset Q$), and let us denote by 1_D the function whose value is 1 for points in D and 0 at all other points. The **integral of f over D** is defined as

$$\iint_D f(x, y)\,dA = \iint_Q f(x, y) 1_D(x, y)\,dA, \tag{8.43}$$

provided the integral on the right-hand side exists, that is, the product $f 1_D$ is integrable over Q. (Note that the function 1_D is discontinuous if $D \neq Q$, so that $f 1_D$ usually is discontinuous, too.) In particular, setting $f = 1$ we obtain the area of D,

$$\text{area of } D = \iint_Q 1_D(x, y)\,dA. \tag{8.44}$$

Computation of two-dimensional integrals. The main tool is provided by Fubini's theorem, which reduces the computation of such an integral to two successive computations of one-dimensional integrals.

Theorem 8.8 (Fubini) *Let f be integrable over the rectangle $Q = [a, b] \times [c, d]$. Then*

$$\iint_Q f(x, y)\,dA = \int_c^d \left(\int_a^b f(x, y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx. \tag{8.45}$$

*The integrals in the middle and on the right-hand side are called **repeated** (or **iterated**) integrals.*

To evaluate a repeated integral like $\int_c^d (\int_a^b f(x, y)\,dx)\,dy$, one first computes the inner integral $\int_a^b f(x, y)\,dx$. The result will be an expression which usually contains

y. In the second step, this expression is integrated with respect to y. After having understood this procedure, one may also omit the brackets on the right-hand side of (8.45) and simply write $\int_c^d \int_a^b f(x, y)\, dx\, dy$ for the repeated integral.

Example 8.36 1. Evaluate $\iint_Q f(x, y)\, dA$, where $f(x, y) = x + 2y$ and Q is the rectangle defined by $1 \leq x \leq 4$ and $1 \leq y \leq 2$.

2. Let Q be the rectangle given by $0 \leq x \leq 1$ and $-1 \leq y \leq 2$. Express the double integral of $x^2y^2(x^2 - y^3)$ over Q as a repeated integral in two different ways and evaluate each.

Solution:

1. By Fubini's theorem, the double integral becomes a repeated integral,

$$\iint_Q x + 2y\, dA = \int_1^2 \left(\int_1^4 (x + 2y)\, dx \right) dy.$$

We first evaluate the inner integral,

$$\int_1^4 (x + 2y)\, dx = \left[\frac{x^2}{2} + 2yx \right]_1^4 = \frac{15}{2} + 6y.$$

We insert this result into the outer integral and obtain

$$\begin{aligned} \iint_Q x + 2y\, dA &= \int_1^2 \frac{15}{2} + 6y\, dy = \left[\frac{15}{2} y + 3y^2 \right]_1^2 \\ &= (15 + 12) - (\frac{15}{2} + 3) = \frac{33}{2}. \end{aligned}$$

2. In the first variant, we put the y-integration inside and the x-integration outside. This gives

$$\begin{aligned} \iint_Q x^2y^2(x^2 - y^3)\, dA &= \int_0^1 \left(\int_{-1}^2 x^2y^2(x^2 - y^3)\, dy \right) dx \\ &= \int_0^1 \left(\int_{-1}^2 x^4y^2 - x^2y^5\, dy \right) dx = \int_0^1 \left[\frac{x^4y^3}{3} - \frac{1}{6}x^2y^6 \right]_{y=-1}^{y=2} dx \\ &= \int_0^1 3x^4 - \frac{21}{2}x^2\, dx = \left[\frac{3}{5}x^5 - \frac{7}{2}x^3 \right]_0^1 = \frac{3}{5} - \frac{7}{2} = -\frac{29}{10}. \end{aligned}$$

In the second variant, we do it the other way round,

$$\begin{aligned}\iint_Q x^2y^2(x^2-y^3)\,dA &= \int_{-1}^{2}\left(\int_0^1 x^2y^2(x^2-y^3)\,dx\right)dy\\ &= \int_{-1}^{2}\left(\int_0^1 x^4y^2-x^2y^5\,dx\right)dy = \int_{-1}^{2}\left[\frac{x^5y^2}{5}-\frac{1}{3}x^3y^5\right]_{x=0}^{x=1}dy\\ &= \int_{-1}^{2}\frac{y^2}{5}-\frac{1}{3}y^5\,dx = \left[\frac{y^3}{15}-\frac{y^6}{18}\right]_{-1}^{2} = \frac{8}{15}-\frac{64}{18}+\frac{1}{15}+\frac{1}{18} = -\frac{29}{10}.\end{aligned}$$

Integrals over non-rectangular domains D also often can be reduced to a repeated integral. Suppose, for example, that each line "$x =$ constant" which crosses the boundary of D does so in just two points $y_1(x)$ and $y_2(x)$, where $y_1(x) < y_2(x)$. Then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx\,, \tag{8.46}$$

where a and b are the smallest and largest values of x in D. This follows from Fubini's theorem, applied to the function $f1_D$ according to (8.43), since we have for the inner integral

$$\int_c^d f(x,y)1_D(x,y)\,dy = \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy.$$

Example 8.37 Evaluate $\iint_D f(x,y)\,dA$, where $f(x,y) = xe^y$ and D is the plane region bounded by the graphs of $y = x^2$ and $y = x$.

Solution: The region D is shown by the shaded portion in Fig. 8.26. The points of intersection of the two curves are obtained by solving the equation $x^2 = x$, giving $x = 0$ and $x = 1$. Thus we obtain $D = \{(x,y) : 0 \le x \le 1\,,\ x^2 \le y \le x\}$. Applying ... we get

$$\begin{aligned}\iint_D xe^y\,dA &= \int_0^1\left(\int_{x^2}^{x} xe^y\,dy\right)dx = \int_0^1 x\left(\int_{x^2}^{x} e^y\,dy\right)dx\\ &= \int_0^1 x\Big[e^y\Big]_{y=x^2}^{y=x}dx = \int_0^1 xe^x - xe^{x^2}\,dx\\ &= \int_0^1 xe^x\,dx - \int_0^1 xe^{x^2}\,dx\,.\end{aligned}$$

The first of these two integrals is computed with integration by parts,

$$\int_0^1 xe^x\,dx = \Big[xe^x\Big]_0^1 - \int_0^1 e^x\,dx = e^1 - \Big[e^x\Big]_0^1 = 1\,,$$

the second with the substitution $t = x^2$,

$$-\int_0^1 xe^{x^2}\,dx = -\frac{1}{2}\int_0^1 e^t\,dt = -\frac{1}{2}e + \frac{1}{2}\,.$$

Thus

$$\iint_D xe^y\,dA = 1 - \frac{1}{2}e + \frac{1}{2} = \frac{1}{2}(3-e)\,.$$

The properties of linearity and monotonicity, formulated in (6.21)–(6.25) for one-dimensional integrals, are also valid for two-dimensional integrals, so

$$\begin{gathered}\iint_D \alpha f + \beta g\,dA = \alpha\iint_D f\,dA + \beta\iint_D g\,dA\,,\\ \iint_D f\,dA \geq \iint_D g\,dA\,,\quad \text{if } f \geq g,\end{gathered} \tag{8.47}$$

hold for integrable functions f, g and scalars α, β.

From the very beginning, we have seen that the integral of a nonnegative function $y = f(x)$ defined on a one-dimensional interval gives the (two-dimensional) area below the graph of f. In an analogous manner, we can obtain the (three-dimensional) volume V of the solid bounded from above by the surface $z = f(x, y)$ (assuming f to be nonnegative) and from below by some two-dimensional region D by

$$V = \iint_D f(x, y)\,dA\,.$$

Integrals and polar coordinates. To evaluate the integral of a function f of a single variable, one may use the substitution rule, Theorem 6.5. For the substitution $x = g(r)$, it reads as

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(r))g'(r)\,dr\,. \tag{8.48}$$

A corresponding rule can be used for functions of two variables. We explain this procedure for the case where Cartesian coordinates (x, y) are substituted by polar coordinates (r, θ), that is, $x = r\cos\theta$ and $y = r\sin\theta$. It connects the integral over a plane region G expressed in polar coordinates to the integral over the corresponding region

$$D = \{(r\cos\theta, r\sin\theta) : (r, \theta) \in G\} \tag{8.49}$$

in Cartesian coordinates.

Theorem 8.9 *Let G and D be as above, let f be integrable over D. Then*

$$\iint_D f(x, y)\,dA = \iint_G f(r\cos\theta, r\sin\theta)r\,dA\,. \tag{8.50}$$

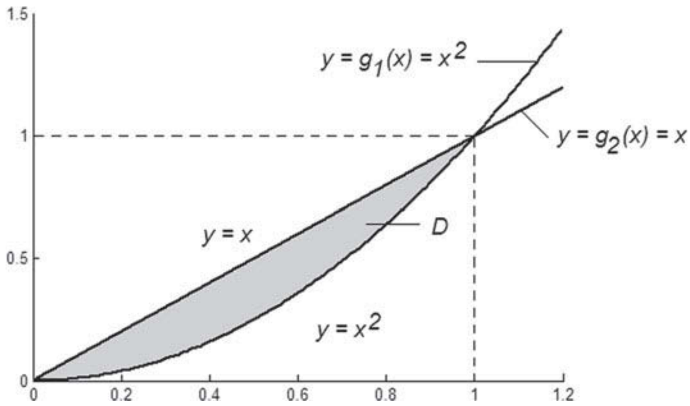


Fig. 8.26 Illustration of Theorem 8.9

The plane regions D and G in (8.50) correspond to the intervals $[g(a), g(b)]$ and $[a, b]$ in (8.48), and the factor r in (8.50) corresponds to the factor $g'(r)$ in (8.48). We will not prove this theorem.

Example 8.38 Evaluate the integral $\iint_D f(x, y)\,dA$, where D is the unit circle and $f(x, y) = x^2$.

Solution: Expressed in polar coordinates, the unit circle D in the plane takes on the form of the rectangular region $G = \{(r, \theta) : 0 \le r \le 1\,,\ 0 \le \theta \le 2\pi\}$. We compute, using Theorem 8.9 and Fubini's theorem,

$$\begin{aligned}\iint_D f(x, y)\,dA &= \iint_D x^2\,dA = \iint_G (r\cos\theta)^2 r\,dA \\ &= \int_0^1 \int_0^{2\pi} r^3 \cos^2\theta\,d\theta\,dr = \int_0^1 r^3\,dr \cdot \int_0^{2\pi} \cos^2\theta\,d\theta \\ &= \frac{1}{4} r^4 \Big|_{r=0}^{r=1} \cdot \pi = \frac{\pi}{4}\,.\end{aligned}$$

As an alternative solution, one might proceed along the lines of Example 8.37. This leads to

$$\iint_D f(x, y)\,dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2\,dy\,dx = \int_{-1}^{1} x^2 \cdot 2\sqrt{1-x^2}\,dx = \dots\,,$$

but this procedure is more involved.

Integrals in three dimensions. To define such integrals, one carries out a procedure analogous to that described above for two dimensions. One first defines the integral

$$\iiint_Q f(x, y, z)\, dV \tag{8.51}$$

for functions f defined on rectangular solids $Q = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ by a limit process involving small such solids. In a second step, one defines the integral over an arbitrary bounded domain D by

$$\iiint_D f(x, y, z)\, dV = \iiint_Q f(x, y, z) 1_D(x, y, z)\, dV\,, \tag{8.52}$$

where Q is any rectangular solid enclosing D, and 1_D is the function equal to 1 on D and to 0 elsewhere. We refer to Appendix D.10 for a more detailed exposition. The integrals in (8.51) and (8.52) are called **three-dimensional** or **triple** integrals. The triple integral can be expressed as a repeated integral just as a double integral. For $Q = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ we get

$$\iiint_Q f(x, y, z) dV = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z)\, dz\, dy\, dx\,.$$

The sequence of the one-dimensional integrations on the right-hand side can be interchanged arbitrarily.

Integrals and spherical coordinates. In three dimensions, spherical coordinates are often convenient (like polar coordinates in two dimensions). We denote them by (r, φ, θ); r has the meaning of a radius, while φ and θ stand for angles. Transformation from Cartesian to spherical coordinates is done via the formulas

$$\begin{aligned} x &= r \sin\varphi \cos\theta \\ y &= r \sin\varphi \sin\theta \\ z &= r \cos\varphi\,. \end{aligned} \tag{8.53}$$

If r is kept fixed, and if φ varies in $[0, \pi]$ and θ in $[0, 2\pi]$, we get all points of the sphere of radius r centered at the origin. Indeed, one may check that $x^2 + y^2 + z^2 = r^2$. The angle $\varphi = 0$ corresponds to the “north pole” $(0, 0, r)$, the angle $\varphi = \pi$ to the “south pole” $(0, 0, -r)$, and for $\varphi = \pi/2$ we get the “equator”, that is, the circle $(r\cos\theta, r\sin\theta, 0)$ in the xy-plane. The transformation formula analogous to (8.50) connects the integral over a spatial region G expressed in spherical coordinates to the integral over the corresponding region

$$D = \{(r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r \cos\varphi) : (r, \varphi, \theta) \in G\} \tag{8.54}$$

in Cartesian coordinates according to the following theorem.

Theorem 8.10 *Let G and D be as above, let f be integrable over D. Then*

$$\iiint_D f(x, y, z)\, dV = \iiint_G f(r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r \cos\varphi) r^2 \sin\varphi \, dV \,. \tag{8.55}$$

Again, we will not prove this theorem.

Example 8.39 Evaluate the integral $\iiint_D f(x, y, z)\, dV$, where D is the unit ball and $f(x, y, z) = x^2 + y^2 + z^2$.

Solution: Expressed in spherical coordinates, the unit ball D in three-dimensional space takes on the form of the rectangular region

$$G = \{(r, \varphi, \theta) : 0 \le r \le 1 \,,\ 0 \le \varphi \le \pi \,,\ 0 \le \theta \le 2\pi\} \,.$$

We compute, using Theorem 8.10 and Fubini's theorem,

$$\begin{aligned}
\iiint_D f(x, y, z)\, dA &= \iiint_D x^2 + y^2 + z^2 \, dA = \iiint_G r^2 \cdot r^2 \sin\varphi \, dA \\
&= \int_0^1 \int_0^\pi \int_0^{2\pi} r^4 \sin\varphi \, d\theta \, d\varphi \, dr \\
&= \int_0^1 r^4 \, dr \cdot \int_0^\pi \sin\varphi \, d\varphi \cdot \int_0^{2\pi} 1 \, d\theta \\
&= \frac{1}{5} r^5 \Big|_{r=0}^{r=1} \cdot 2 \cdot 2\pi = \frac{4}{5}\pi \,.
\end{aligned}$$

8.8 Applications of Double Integrals

8.8.1 Population of a City

Let the rectangular region D of Fig. 8.27 represent a certain district of a city, and let $f(x, y)$ be the population density function (the number of people per unit area) defined at all points $(x, y) \in D$. Then the double integral $\iint_D f(x, y)\, dA$ gives the actual number of people living in the district under consideration.

Example 8.40 The population density of a certain city equals

$$f(x, y) = 20000 e^{-0.2|x| - 0.1|y|}$$

people per square kilometer, where x and y are measured in kilometers and the origin $(0, 0)$ gives the location of the city hall. Determine the total population inside the rectangular area described by

$$D = \{(x, y) : -10 \le x \le 10 \,,\ -5 \le y \le 5\} \,.$$

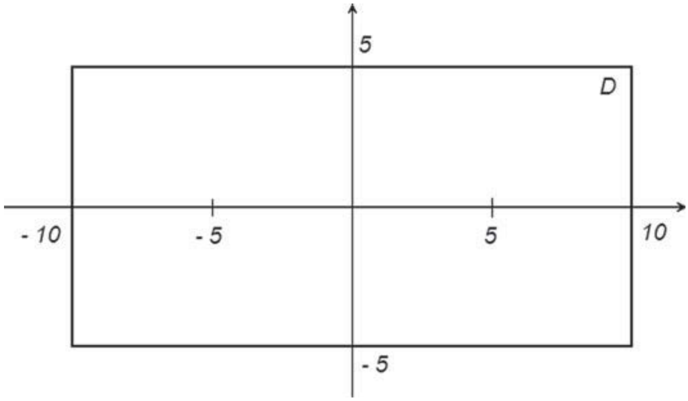


Fig. 8.27 A rectangular region D representing a district of a big city

Solution: By symmetry, it suffices to compute the population in the first quadrant of Fig. 8.27. Therefore the population in the district equals

$$\iint_D f(x, y)\, dA = 4 \int_0^{10} \left(\int_0^5 2 \cdot 10^4 e^{-0.2x} e^{-0.1y}\, dy \right) dx$$
$$= 4 \int_0^{10} \Big[-2 \cdot 10^5 e^{-0.2x} e^{-0.1y} \Big]_{y=0}^{y=5} dx = 8 \cdot 10^5 (1 - e^{-0.5}) \int_0^{10} e^{-0.2x}\, dx$$
$$= 4 \cdot 10^6 (1 - e^{-0.5})(1 - e^{-2})$$

or approximately 1, 360, 876 people.

8.8.2 Average Value of a Function of Two Variables

In Sect. 7.3, we have shown that the average value or mean value of a function of one variable over an interval can be represented by its integral divided by the length of the interval. An analogous result holds for functions of two variables, that is, if such a function f is integrated over a plane region D, then its average value over D is given by the quotient

$$\frac{\iint_D f(x, y)\, dA}{\text{area of } D} = \frac{\iint_D f(x, y)\, dA}{\iint_D 1\, dA}, \tag{8.56}$$

see (8.44). In particular, if $D = \{(x, y) : a \le x \le b, c \le y \le d\}$ is a rectangle, then the average value of f over D is given by

$$\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y)\, dx\, dy = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dy\, dx\,.$$

Example 8.41 Find the average value of the function $f(x, y) = xy$ over the plane region defined by $y = e^x$, $0 \le x \le 2$.

Solution: The integral of f over D is given by

$$\begin{aligned}\iint_D f(x, y)\, dA &= \int_0^2 \left(\int_0^{e^x} xy\, dy \right) dx = \int_0^2 \left[\frac{1}{2} y^2 x \right]_{y=0}^{y=e^x} dx \\ &= \int_0^2 \frac{1}{2} x e^{2x}\, dx = \left[\frac{1}{4} x e^{2x} \right]_0^2 - \int_0^2 \frac{1}{4} e^{2x}\, dx \\ &= \frac{1}{2} e^4 - \left[\frac{1}{8} e^{2x} \right]_0^2 = \frac{1}{2} e^4 - \frac{1}{8} e^4 + \frac{1}{8} = \frac{1}{8}(3e^4 + 1)\,.\end{aligned}$$

The area of the region D is

$$\iint_D 1\, dA = \int_0^2 \int_0^{e^x} 1\, dy\, dx = \int_0^2 \Big[y \Big]_{y=0}^{y=e^x} dx = \int_0^2 e^x\, dx = e^2 - 1\,.$$

By (8.56), the average value of f over D equals the quotient

$$\frac{\iint_D f(x, y)\, dA}{\iint_D 1\, dA} = \frac{\frac{1}{8}(3e^4 + 1)}{e^2 - 1} = \frac{1}{8} \cdot \frac{3e^4 + 1}{e^2 - 1}\,.$$

8.8.3 Joint Probability Density Functions

In Sect. 7.6, we have established a relationship between the integral of a function f of a single variable and the probability of an event to occur. Let us recall that f is the density function belonging to some random variable x if the probability that an observed value of x lies in an interval $[a, b]$ is given by

$$\int_a^b f(x)\, dx\,.$$

Here, we consider two real-valued random variables x and y. Let x take values in some interval I_x and y take values in some interval I_y. Let f be a nonnegative function of two variables with domain $D(f) = \{(x, y) : x \in I_x\,,\ y \in I_y\}$, that is, $f(x, y) \ge 0$ for all $(x, y) \in D(f)$. f is called the **joint probability density function** of x and y, if

$$\iint_{D(f)} f(x, y)\, dA = 1\,, \tag{8.57}$$

and if for every planar region $D \subset D(f)$, the probability that the pair (x, y) of the observed values of the random variables lies in D is given by

$$P((x, y) \in D) = \iint_D f(x, y)\, dA\,. \tag{8.58}$$

Example 8.42 A new car manufactured by some company carries a 50, 000 km warranty on its engine and its transmission. Preproduction tests indicate that the life spans of the engine and the transmission are described by random variables x and y with joint probability density function

$$f(x, y) = 0.00004e^{-0.005x-0.008y}\,, \quad x, y \geq 0\,,$$

and the unit of measurement for x and y is 1000 km.

1. What is the probability that a new car chosen at random will have an engine breakdown before the 50, 000 km warranty expires?
2. What is the probability that a new car chosen at random will have a breakdown of both its engine and its transmission before the 50, 000 km warranty expires?

Solution: 1. The required probability is given by

$$P((x, y) \in D)\,, \quad \text{where} \quad D = \{(x, y) : 0 \leq x \leq 50\,,\ 0 \leq y\}\,.$$

We compute

$$\begin{aligned} P((x, y) \in D) &= \iint_D f\, dA = \int_0^\infty \int_0^{50} f(x, y)\, dx\, dy \\ &= \int_0^\infty \int_0^{50} 0.00004e^{-0.005x-0.008y}\, dx\, dy \\ &= 4 \cdot 10^{-5} \int_0^\infty \int_0^{50} e^{-0.005x} e^{-0.008y}\, dx\, dy \\ &= 4 \cdot 10^{-5} \int_0^{50} e^{-0.005}\, dx \cdot \int_0^\infty e^{-0.008y}\, dy\,. \end{aligned}$$

These integrals give

$$\int_0^{50} e^{-0.005}\, dx = 200(1 - e^{-0.25})\,, \quad \int_0^\infty e^{-0.008y}\, dy = \frac{1}{0.008}\,,$$

so

$$P((x, y) \in D) = 4 \cdot 10^{-5} \cdot 200(1 - e^{-0.25})\frac{1}{0.008} = 1 - e^{-0.25}) \simeq 0.2212\,.$$

2. The required probability is given by

$$P((x, y) \in D)\,, \quad \text{where} \quad D = \{(x, y) : 0 \le x \le 50\,,\ 0 \le y \le 50\}\,.$$

We compute

$$\begin{aligned} P((x, y) \in D) &= \iint_D f\,dA = \int_0^{50}\int_0^{50} f(x, y)\,dx\,dy \\ &= \int_0^{50}\int_0^{50} 0.00004e^{-0.005x-0.008y}\,dx\,dy \\ &= 4 \cdot 10^{-5}\int_0^{50}\int_0^{50} e^{-0.005x}e^{-0.008y}\,dx\,dy \\ &= 4 \cdot 10^{-5}\int_0^{50} e^{-0.005}\,dx \cdot \int_0^{50} e^{-0.008y}\,dy\,. \end{aligned}$$

These integrals give

$$\int_0^{50} e^{-0.005}\,dx = 200(1 - e^{-0.25})\,, \quad \int_0^{50} e^{-0.008y}\,dy = \frac{1}{0.008}(1 - e^{-0.4})\,,$$

so as before

$$P((x, y) \in D) = 1 - e^{-0.25})(1 - e^{-0.4}) \simeq 0.0729\,.$$

8.9 Exercises

8.9.1 Give a formula for the function $m = f(b, t)$ where m is the amount of money in a bank account t years after an initial investment of b Indian rupees, if interest occurs at a rate of 5% per year compounded
(a) annually, (b) continuously.

8.9.2 Suppose the concentration C (in mg per liter) of a drug in the blood is a function of two variables x, the amount (in mg) of the drug given in the injection, and t, the time (in hours) since the injection was administered. Let C be given by

$$C = f(x, t) = te^{-t(9-x)}\,, \quad \text{for } 0 \le x \le 8 \text{ and } t \ge 0.$$

Explain the meaning of the cross sections
(a) $f(8, t)$, (b) $f(x, 1)$.

8.9.3 Let $f(x) = x \sin x$. Evaluate
(a) $f(x-y)$, (b) $f(\frac{x}{y})$, (c) $f(xy)$.

8.9.4 Let $h(x, y, z) = xy^2z^3 + 4$. Evaluate
(a) $h(a+b, a-b, b)$, (b) $h(0, 0, 0)$, (c) $h(t, t^2, -t)$, (iv) $h(-6, 4, 2)$.

8.9.5 Describe in words the domain of the following functions.
(a) $f(x, y) = xe^{-\sqrt{y+2}}$, (b) $f(x, y, z) = e^{xyz}$,
(c) $f(x, y, z) = \dfrac{xyz}{x+y+z}$.

8.9.6 Sketch the graph of the functions given below.
(a) $f(x, y) = \sqrt{x^2 + y^2}$ (b) $f(x, y) = 4 - x^2 - y^2$
(c) $f(x, y) = \sqrt{x^2 + y^2 - 1}$.

8.9.7 (a) Sketch the level curve $z = k$ for the specified values of k.
(i) $z = x^2 + y^2$, $k = 0, 1, 2, 3$
(ii) $z = x^2 - y^2$, $k = -2, -1, 0, 1, 2$.
(b) Sketch the level surface $f(x, y, z) = c$.
(i) $f(x, y, z) = 4x^2 + y^2 + 4z^2$, $c = 16$
(ii) $f(x, y, z) = 4x - 2y + z$, $c = 1$.

8.9.8 Let $T(x, y)$ be the temperature at a point (x, y) on a flat metal plate situated in the xy-plane. The level curves of T are called isothermal curves. Assume that $T(x, y)$ is inversely proportional to the distance of (x, y) from the origin.

(a) Sketch the isothermal curves on which $T = 1$, $T = 2$, and $T = 3$.
(b) If the temperature at the point $(4, 3)$ is $40\,°C$, find an equation for the isothermal curve belonging to a temperature of $20\,°C$.

8.9.9 If $V(x, y)$ is a potential (for example, the voltage) at a point (x, y) in the xy-plane, the level curves of V are called equipotential curves. The potential remains constant along such curves. For

$$V(x, y) = \frac{8}{\sqrt{16 + x^2 + y^2}}$$

sketch the equipotential curves at which $V = 2$, $V = 1$, and $V = 0.5$.

8.9.10 According to the ideal gas law, the pressure P, volume V, and temperature T of a confined gas are related by the formula $PV = nkT$, where n is the number of particles in the gas and k is the Boltzmann constant. Express P as a function of V and T, and describe the level curves associated with this function. What is the physical significance of these level curves?

8.9.11 The power P generated by a wind rotor is proportional to the product of the area A swept out by the blades and the third power of the wind velocity v.

(a) Express P as a function of A and v.
(b) Describe the level curves of P and explain their physical meaning.
(c) Consider a rotor whose blades sweep out a circular area with a diameter of 10 ft and which produces a power of 3000 watts at a wind velocity of 20 m/sec. Find an equation of the level curve $P = 4000$.

8.9.12 Assume that the atmospheric pressure near ground level in certain region is given by

$$p(x, y) = ax^2 + by^2 + c\,,$$

where a, b, and c are positive constants.

(a) Describe the isobars in this region for pressures greater than c.
(b) How are the low and the high pressures distributed in this region ?

8.9.13 Find

(a) $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y}$,
(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$.

8.9.14 Suppose

$$f(x, y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}\,, & (x, y) \neq (0, 0)\,, \\ 1\,, & (x, y) = (0, 0)\,. \end{cases}$$

Show that f is continuous at $(0, 0)$.

8.9.15 Describe the set of all points in the xy-plane at which f is continuous.
(a) $f(x, y) = \ln(x + y - 1)$, (b) $f(x, y) = \sqrt{x}e^{\sqrt{1-y^2}}$.

8.9.16 (a) $z = y^2e^{x^2} + \frac{1}{x^2y^3}$. Find $\dfrac{\partial^2 z}{\partial x\partial y}$ and $\dfrac{\partial^2 z}{\partial y\partial x}$.
(b) Let $w = \dfrac{x^2}{y^2+z^2}$, find $\dfrac{\partial^3 w}{\partial z\partial y^2}$.

8.9.17 (a) Show that $f(x, y) = \arctan\dfrac{y}{x}$ satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\,.$$

(b) Show that $f(x, t) = (x - at)^4 + \cos(x + at)$ satisfies

$$\frac{\partial^2 f}{\partial t^2} = a^2\frac{\partial^2 f}{\partial x^2}\,.$$

8.9.18 Show that the functions $u = u(x, y)$ and $v = v(x, y)$ given below satisfy the following relations, known as the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad\text{and}\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

(a) $u = x^2 - y^2$, $v = 2xy$,
(b) $u = e^x\cos y$, $v = e^x\sin y$,

(c) $u = \ln(x^2 + y^2)$, $v = 2\tan^{-1}(\frac{y}{x})$.

8.9.19 Let $z = f(u)$ and $u = g(x, y)$. Show that

(a) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{d^2 z}{du^2}(\dfrac{\partial u}{\partial x})^2$,

(b) $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y^2} + \dfrac{d^2 z}{du^2}(\dfrac{\partial u}{\partial y})^2$,

(c) $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y \partial x} + \dfrac{d^2 z}{du^2}\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}$.

8.9.20 Show that among all parallelograms with perimeter l, a square with sides of length $l/4$ has maximum area.

8.9.21 Determine the dimensions of a rectangular box, open at the top, having volume V, and requiring the least amount of material for its construction.

8.9.22 A company plans to manufacture closed rectangular boxes that have a volume of 8 cubic feet. Find the dimensions that will minimize the cost if the material for the top and bottom costs twice as much as the material for the sides.

8.9.23 A window has the shape of a rectangle surmounted by an isosceles triangle as shown in Fig. 8.28. If the perimeter of the window is 4 m, what values of x, y, and θ will maximize the total area?

8.9.24 The following Table 8.1 lists the relationship between semester averages and scores on the final examination for ten students in a mathematical class.
Fit the data to a line and use the line to estimate the final examination grade of a student with an average of 70.

Table 8.1 Relationship between semester averages and examination scores

Semester average	40	55	62	68	72	76	80	86	90	94
Final examination	30	45	65	72	60	82	76	92	88	98

8.9.25 In studying the stress–strain diagram of an elastic material, an engineer finds that part of the curve appears to be linear. Experimental values are listed in the following Table 8.2.
Fit these data to a straight line and estimate the strain when the stress is 2.5%.

Table 8.2 Data relating stress and strain of an elastic material

Stress (MPa)	4	4.4	4.8	5.2	5.6	6
Strain (%)	0.2	0.6	0.8	1.2	1.4	1.8

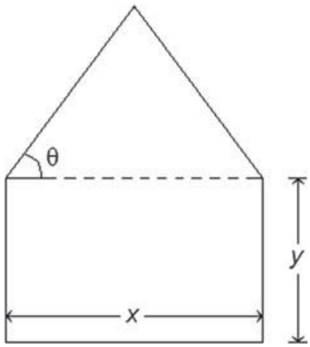


Fig. 8.28 Maximizing the area of a window

8.9.26 Apply the method of Lagrange multipliers to find maximum and minimum values of

(a) $f(x, y) = xy$, where $4x^2 + 8y^2 = 16$.
(b) $f(x, y) = x - 3y - 1$, where $x^2 + 3y^2 = 16$.

8.9.27 Find the point on the line $2x - 4y = 3$ that is closest to the origin.

8.9.28 (a) Find the Taylor series of the function $e^x \ln y$ around the point $(0, 1)$ up to the terms of order 4.
(b) Expand $e^x \cos y$ in a Taylor series at $(1, \frac{\pi}{2})$. (Compute the terms up to order 3.)

8.9.29 Evaluate the following integrals:

(a) $\int_0^3 \int_{-2}^{-1} (4xy^3 + y)\, dx\, dy$
(b) $\int_0^{\pi/6} \int_0^{\pi/2} (x \cos y - y \cos x)\, dy\, dx$.

8.9.30 Let an artificial lake be created bordering one side of a straight dam. The shape of the lake surface is that of a region in the xy-plane bounded by the graph of $2y = 16 - x^2$ and $x + 2y = 4$. Find the area A of the surface of the lake.

8.9.31 Find the area of the region D that lies outside the circle $r = 100$ and inside the circle $r = 200 \sin\theta$.

8.9.32 Find the area of the region R bounded by one loop of the lemniscate $r^2 = a^2 \sin 2\theta$.

8.9.33 Use the polar coordinates to evaluate the following integrals

(a) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dy\, dx.$$
(b) $$\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} (x^2 + y^2)^{3/2}\, dy\, dx.$$

Chapter 9
Vector Calculus

9.1 Introduction

In the previous chapters, we have studied properties of functions defined on $\mathbb{R}$ (the line), $\mathbb{R}^2$ (the plane), or $\mathbb{R}^3$ (the space) with values in $\mathbb{R}$, which are called real-valued or scalar functions, or scalar fields. Here, we would like to study the calculus of functions taking values in $\mathbb{R}^2$ or $\mathbb{R}^3$, instead of $\mathbb{R}$. Those functions are called vector-valued functions or vector fields. In Sect. 9.2 the concept of a vector will be introduced along with its basic algebraic properties. Vector fields and their continuity and differentiation properties are discussed in Sect. 9.3 along with the notions of gradient, divergence, and curl. Moreover, we explain how curves and surfaces are described by such functions. Integrals of vector fields are introduced in Sect. 9.4, first the line integral for scalar and vector fields, and then the surface integral for scalar fields. In Sect. 9.5 we present three fundamental theorems of vector calculus, namely, the Green–Ostrogradski theorem, the Gauss divergence theorem, and the theorem of Stokes. Section 9.6 is devoted to certain applications of the vector calculus to science and engineering, with an emphasis on problems from various parts of mechanics.

The concept of a vector can be traced to the development of affine and analytic geometry in the seventeenth century and, later, the invention of complex numbers and quaternions. Vector calculus itself was developed mainly during the second half of the nineteenth century; contributions stem from, among others, Sir William Rowan Hamilton (1805–1865), William Kingdon Clifford (1845–1879), James Clerk Maxwell (1831–1879), Hermann Günter Grassmann (1809–1877), and Josiah Willard Gibbs (1839–1903). However, the fundamental results mentioned above were already discovered by George Green (1793–1841), Carl Friedrich Gauss (1777–1855), Gabriel Stokes (1819–1903), and Mikhail Ostrogradski (1801–1862). Nowadays, vector calculus serves as a basic mathematical tool in all areas of science and engineering, where mechanical, electromagnetic and thermodynamic forces determine the behavior of solids, fluids, electric conductors or semiconductors, and magnetic materials.

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9.2 Vectors

We know that a number x is used to represent a point on a line, and a pair (x, y) of numbers x and y is used to represent a point in the plane, see Fig. 9.1a and b.

Moreover, in the previous chapter, we have already represented points in space, that is, three-dimensional space, by a triple (x, y, z) of numbers, see Fig. 9.2. The number of components in such tuples equals the dimension of the corresponding space (1 for the real line $\mathbb{R}$, 2 for the plane $\mathbb{R}^2$, and 3 for the space $\mathbb{R}^3$). Irrespective of the dimension, those tuples are called **vectors**. Instead of (x, y) or (x, y, z), we also denote them by (x_1, x_2) or (x_1, x_2, x_3), respectively. This notation is more convenient when we want to go to higher dimensions and consider, for example, a vector (x_1, x_2, x_3, x_4) as an element of the 4-space $\mathbb{R}^4$, or even—when we do not want to specify the dimension as a fixed number—a vector $(x_1, x_2, x_3, \ldots, x_n)$ as an element of n-space $\mathbb{R}^n$, where n stands for an arbitrary natural number.

Since the main goal of this chapter is vector calculus in 3-space, when we speak of vectors we will mean vectors in 3-space; otherwise, it will be made explicit.

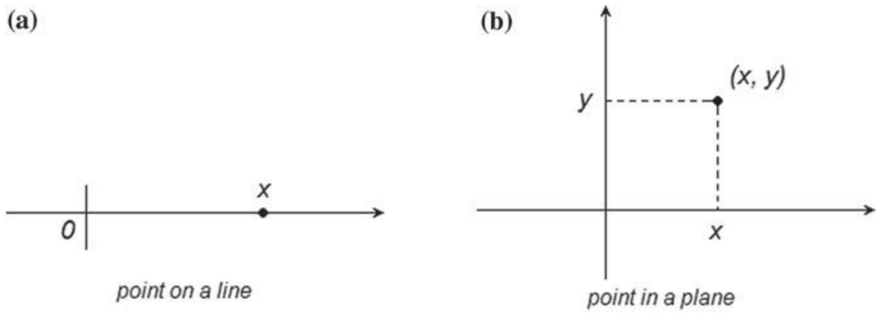
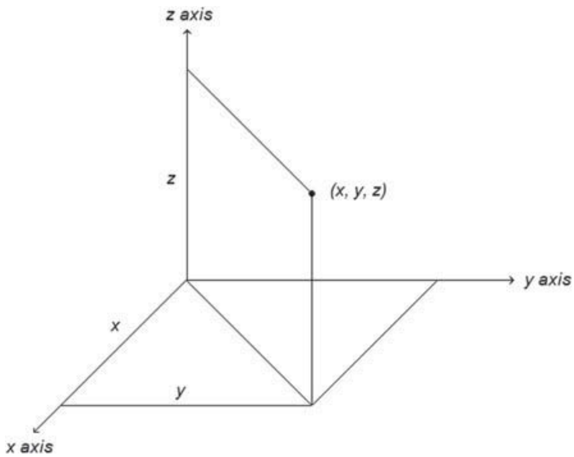


Fig. 9.1 **a** Point on a line, **b** point in a plane

Fig. 9.2 A vector (or point) in three-dimensional space



The basic operations with vectors are **addition**, defined componentwise by

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3),$$

and **multiplication by scalars** (real numbers) α,

$$\alpha(x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3).$$

It is also very convenient to denote a vector by a single letter. We will use $\mathbf{x}$ (in boldface type) to denote the vector (x_1, x_2, x_3).

In arbitrary dimension, addition and scalar multiplication are defined analogously by

$$\begin{aligned}(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\\ \alpha(x_1, x_2, \cdots, x_n) &= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n),\end{aligned}$$

where $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ and α are real numbers.

Let us note at this point that two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are **equal**, that is $\mathbf{x} = \mathbf{y}$, if all their components are equal, that is, $x_1 = y_1$, $x_2 = y_2$, and $x_3 = y_3$.

Definition 9.1 Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be **parallel** if $\mathbf{x} = \alpha\mathbf{y}$ (or $\mathbf{y} = \alpha\mathbf{x}$) for some real number α. If $\alpha > 0$ then $\mathbf{x}$ and $\mathbf{y}$ are said to have the same direction; if $\alpha < 0$ then $\mathbf{x}$ and $\mathbf{y}$ are said to have opposite direction.

As an immediate consequence (take $\alpha = 0$), we note that the vector $\mathbf{0} = (0, 0, 0)$ is parallel to every vector. Moreover, if $\mathbf{x}$ and $\mathbf{y}$ are vectors which are parallel to a vector $\mathbf{z}$, then any linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$ of $\mathbf{x}$ and $\mathbf{y}$, where α and β are real numbers, is also parallel to $\mathbf{z}$.

Addition and scalar multiplication have the property that

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= \mathbf{x} + \mathbf{y}, \quad (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}),\\ \alpha(\mathbf{x} + \mathbf{y}) &= \alpha\mathbf{x} + \alpha\mathbf{y}, \quad (\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}, \quad (\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x}),\end{aligned}$$

hold for all vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ and all scalars α and β. This follows from the corresponding properties of ordinary addition and multiplication.

We know from elementary geometry that the length (or magnitude) of the vector $\mathbf{x} = (x_1, x_2, x_3)$ is given by the expression

$$\sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Definition 9.2 The **Euclidean norm** (or simply **norm**) $\|\mathbf{x}\|$ of a vector $\mathbf{x}$ is defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}\,.$$

The norm has the following properties, which are analogous to the properties of the absolute value of a real number.

$$\|\mathbf{x}\| \geq 0\,, \text{ and } \|\mathbf{x}\| = 0 \text{ if and only if } \mathbf{x} = 0\,, \tag{9.1}$$

$$\|\alpha\mathbf{x}\| \;= |\alpha|\|\mathbf{x}\| \quad \text{for scalars } \alpha\,, \tag{9.2}$$

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|\,. \tag{9.3}$$

Properties (9.1) and (9.2) can be checked immediately from the definition. It will be seen below that property (9.3) is a consequence of the Schwarz inequality. Setting $\alpha = -1$ in property (9.2), we obtain that

$$\| - \mathbf{x}\| = \|\mathbf{x}\| \tag{9.4}$$

holds for every vector $\mathbf{x}$.

The inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ is called the **triangle inequality**. Consider the triangle formed by the points $A = 0$, $B = \mathbf{x}$, and $C = \mathbf{x} + \mathbf{y}$. The triangle inequality states that the length of the side connecting A and C cannot be larger than the sum of the length of the other two sides.

Example 9.1 Given $\mathbf{x} = (1, -2, 2)$ and $\mathbf{y} = (-1, 2, -2)$, compute
(i) $\|\mathbf{x}\|$, (ii) $\|\mathbf{y}\|$, (iii) $\|\mathbf{x} + \mathbf{y}\|$, (iv) $\|\mathbf{x} - \mathbf{y}\|$, and (v) $\|4\mathbf{x}\|$.

Solution:

(i) $\|\mathbf{x}\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$.
(ii) $\|\mathbf{y}\| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$.
(iii) $\mathbf{x} + \mathbf{y} = (1 - 1, -2 + 2 - 2) = (0, 0, 0)$, $\|\mathbf{x} + \mathbf{y}\| = 0$.
(iv) $\mathbf{x} - \mathbf{y} = (2, -4, 4)$, $\|\mathbf{x} - \mathbf{y}\| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{36} = 6$.
(v) $4\mathbf{x} = (4, -8, 8)$, $\|4\mathbf{x}\| = \sqrt{4^2 + (-8)^2 + 8^2} = \sqrt{16 + 64 + 64} = \sqrt{144}{=}12$.
Alternatively, $\|4\mathbf{x}\| = 4\|\mathbf{x}\| = 4 \cdot 3 = 12$, by (i).

A vector $\mathbf{x}$ with norm $\|\mathbf{x}\| = 1$ is called a **unit vector**. If $\mathbf{x}$ is a nonzero vector, then

$$\mathbf{e} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

is a unit vector. The unit vectors

$$\mathbf{i} = (1, 0, 0)\,, \quad \mathbf{j} = (0, 1, 0)\,, \quad \mathbf{k} = (0, 0, 1)$$

(they obviously satisfy $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$) are also called **unit coordinate vectors** or **standard unit vectors**, see Fig. 9.3. Every vector can be expressed as a linear combination of the unit coordinate vectors: For $\mathbf{x} = (x_1, x_2, x_3)$, we have

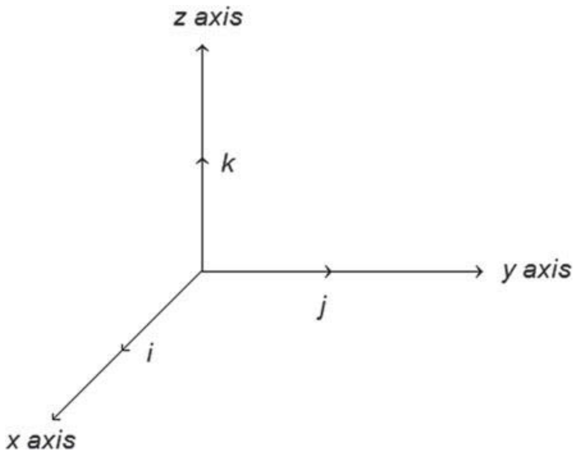


Fig. 9.3 Standard unit vectors

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\,.$$

Indeed, from the rules of addition and scalar multiplication we see that

$$\begin{aligned}\mathbf{x} &= (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)\\ &= x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\,.\end{aligned}$$

Example 9.2 Let $\mathbf{x} = (3, -1, 1)$ and $\mathbf{y} = (2, 3, -1)$.

(i) Express $2\mathbf{x} - \mathbf{y}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.
(ii) Calculate $\|3\mathbf{x} - \mathbf{y}\|$.

Solution:

(i) We have $2\mathbf{x} = (6, -2, 2)$ and

$$\begin{aligned}2\mathbf{x} - \mathbf{y} &= (6, -2, 2) - (2, 3, -1) = (6 - 2, -2 - 3, 2 + 1) = (4, -5, 3)\\ &= 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}\,.\end{aligned}$$

(ii) We have $3\mathbf{x} - \mathbf{y} = (9, -3, 3) - (2, 3, -1) = (7, -6, 4)$ and

$$\|3\mathbf{x} - \mathbf{y}\| = \sqrt{7^2 + (-6)^2 + 4^2} = \sqrt{49 + 36 + 16} = \sqrt{101}\,.$$

Let us consider a vector $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. If $x_3 = 0$, we have $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$, so $\mathbf{x}$ lies in the plane spanned by the first two unit coordinate vectors. We may identify this plane with the standard xy-plane spanned by the unit coordinate vectors $(1, 0)$ and $(0, 1)$, which we continue to denote by $\mathbf{i}$ and $\mathbf{j}$. Every vector $\mathbf{x} = (x_1, x_2)$ in the plane can be expressed as a linear combination

$$\mathbf{x} = x_1(1,0) + x_2(0,1) = x_1\mathbf{i} + x_2\mathbf{j}$$

of the unit coordinate vectors.

Example 9.3 (a) Simplify the linear combinations $(2\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$ and $2(\mathbf{i} - \mathbf{j}) + 6(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.
(b) Calculate the norm of the vector $(\mathbf{i} - \mathbf{j}) + 2(\mathbf{j} - \mathbf{i}) + (\mathbf{k} - \mathbf{j})$.
(c) Find the unit vector in the direction of $\mathbf{x} = (3, -4, 0)$ and $\mathbf{y} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.
(d) Prove that $|\|\mathbf{x}\| - \|\mathbf{y}\|| \le \|\mathbf{x} - \mathbf{y}\|$ holds for all vectors $\mathbf{x}$ and $\mathbf{y}$. (This inequality is called the **reverse triangle inequality**.)

Solution:

(a) We have

$$(2\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) = 3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}\,,$$
$$2(\mathbf{i} - \mathbf{j}) + 6(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} - 2\mathbf{j} + 12\mathbf{i} + 6\mathbf{j} - 12\mathbf{k} = 14\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}\,.$$

(b) We have

$$\begin{aligned}\|(\mathbf{i} - \mathbf{j}) + 2(\mathbf{j} - \mathbf{i}) + (\mathbf{k} - \mathbf{j})\| &= \|-\mathbf{i} + \mathbf{k}\| = \|(-1,0,0) + (0,0,1)\| \\ &= \|(-1,0,1)\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}\,.\end{aligned}$$

(c) Let us denote by $\mathbf{e}$ the corresponding unit vector. We get

$$\mathbf{e} = \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{(3,-4,0)}{\sqrt{9 + (-4)^2 + 0}} = \frac{(3,-4,0)}{\sqrt{25}} = \left(\frac{3}{5}, -\frac{4}{5}, 0\right) = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\,,$$
$$\mathbf{e} = \frac{\mathbf{y}}{\|\mathbf{y}\|} = \frac{(1,-2,2)}{\sqrt{1 + (-2)^2 + (2)^2}} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\,.$$

(d) From the triangle inequality, we get $\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{y}\| \le \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$, so $\|\mathbf{x}\| - \|\mathbf{y}\| \le \|\mathbf{x} - \mathbf{y}\|$. Interchanging the role of $\mathbf{x}$ and $\mathbf{y}$ we get $\|\mathbf{y}\| - \|\mathbf{x}\| \le \|\mathbf{y} - \mathbf{x}\| = \|-(\mathbf{y} - \mathbf{x})\| = \|\mathbf{x} - \mathbf{y}\|$ by (9.4). Putting those two inequalitites together we obtain $|\|\mathbf{x}\| - \|\mathbf{y}\|| \le \|\mathbf{x} - \mathbf{y}\|$, the desired result.

The Scalar Product

Definition 9.3 (*Scalar Product*) Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be two vectors. The **scalar product** (or **dot product**) of $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{x} \cdot \mathbf{y}$ and defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3\,.$$

Thus, the scalar product yields a scalar (a real number).

Example 9.4 Find the scalar product of the following vectors:

(a) $\mathbf{x} = (2, -1, 3)$ and $\mathbf{y} = (-3, 1, 4)$,
(b) $\mathbf{y} = (-3, 1, 4)$ and $\mathbf{z} = (1, 3, 0)$,
(c) $\mathbf{x} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ and $\mathbf{y} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

Solution:

(a) $\mathbf{x} \cdot \mathbf{y} = 2 \cdot (-3) + (-1) \cdot 1 + 3 \cdot 4 = -6 - 1 + 12 = 5$.
(b) $\mathbf{y} \cdot \mathbf{z} = -3 + 3 + 0 = 0$.
(c) $\mathbf{x} \cdot \mathbf{y} = 2 \cdot 1 + (-4) \cdot (-1) + 1 \cdot 3 = 9$.

The scalar product and the norm are related by

$$\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2, \quad \|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}, \tag{9.5}$$

which holds for every vector $\mathbf{x} = (x_1, x_2, x_3)$.

The scalar product is **commutative**, that is,

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$$

holds for all vectors $\mathbf{x}$ and $\mathbf{y}$. Moreover, it is **distributive** in the sense that

$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z} \tag{9.6}$$

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z} \tag{9.7}$$

$$(\alpha\mathbf{x}) \cdot \mathbf{y} = \alpha(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (\alpha\mathbf{y}) \tag{9.8}$$

holds for all vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ and all scalars α. In particular, setting $\alpha = 0$ we have for every vector $\mathbf{x}$

$$\mathbf{x} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{x}.$$

All these rules follow immediately from the definition of the scalar product.

Geometric interpretation of the scalar product. We look at the triangle in Fig. 9.4 with vertices A, B, and C which has a right angle at B. Setting $\mathbf{x} = B - A$, $\mathbf{y} = C - B$, the theorem of Pythagoras tells us that

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2. \tag{9.9}$$

Using (9.5) and the properties of the scalar product, we compute

$$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 + 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2.$$

Comparing with (9.9), we see that we must have $\mathbf{x} \cdot \mathbf{y} = 0$.

Definition 9.4 Two vectors $\mathbf{x}$ and $\mathbf{y}$ are called **orthogonal** or **perpendicular** if $\mathbf{x} \cdot \mathbf{y} = 0$, and we write $\mathbf{x} \perp \mathbf{y}$ in this case.

Example 9.5 Examine whether the vectors $\mathbf{x} = (2, 1, 1)$ and $\mathbf{y} = (1, 1, -3)$ are orthogonal.

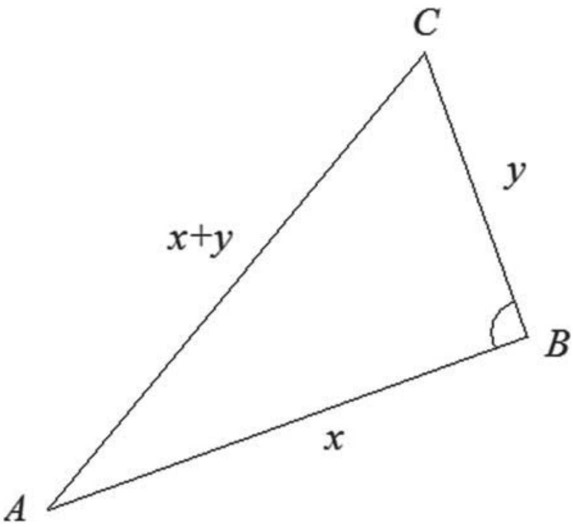


Fig. 9.4 Triangle with a right angle at B

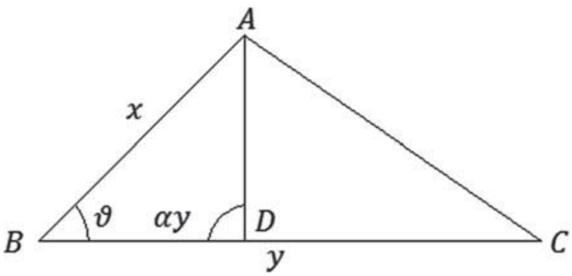


Fig. 9.5 The projection $p_y(x) = \alpha y$

Solution: We have $\mathbf{x} \cdot \mathbf{y} = 2 \cdot 1 + 1 \cdot 1 + 1 \cdot (-3) = 2 + 1 - 3 = 0$. This implies $\mathbf{x} \perp \mathbf{y}$. Let us now look at the triangle with vertices A, B, and C in Fig. 9.5, setting $\mathbf{x} = B - A$ and $\mathbf{y} = C - B$. We want to determine the point D on the side BC such that the line AD becomes perpendicular to BC. We have $D - B = \alpha\mathbf{y}$ for some scalar α, and $A - D = \mathbf{x} - \alpha\mathbf{y}$. In order to determine α, we exploit the orthogonality relation

$$0 = \alpha\mathbf{y} \cdot (\mathbf{x} - \alpha\mathbf{y}) = \alpha\mathbf{y} \cdot \mathbf{x} - \alpha^2\mathbf{y} \cdot \mathbf{y}\,.$$

Solving for α, we obtain

$$\alpha = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\,. \tag{9.10}$$

Definition 9.5 Let $\mathbf{x}$, $\mathbf{y}$ be vectors with $\mathbf{y} \neq \mathbf{0}$. The **projection of $\mathbf{x}$ on $\mathbf{y}$**, denoted by $\mathbf{p}_\mathbf{y}(\mathbf{x})$, is defined by

$$\mathbf{p}_\mathbf{y}(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\mathbf{y}\,. \tag{9.11}$$

The length of the projection is given by

$$\|\mathbf{p_y}(\mathbf{x})\| = \frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|}\,, \tag{9.12}$$

since

$$\|\mathbf{p_y}(\mathbf{x})\| = \left\|\frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}\right\| = \left|\frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\right|\|\mathbf{y}\| = \frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|^2}\|\mathbf{y}\| = \frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|}\,.$$

In the special case where $\mathbf{y}$ is a unit vector, $\|\mathbf{y}\| = 1$, (9.11) becomes

$$\mathbf{p_y}(\mathbf{x}) = (\mathbf{x}\cdot\mathbf{y})\mathbf{y}\,. \tag{9.13}$$

In this manner, setting

$$\mathbf{x}_I = \mathbf{p_y}(\mathbf{x})\,, \quad \mathbf{x}_{II} = \mathbf{x} - \mathbf{p_y}(\mathbf{x})\,, \tag{9.14}$$

we can decompose an arbitrary vector $\mathbf{x}$ into a sum $\mathbf{x} = \mathbf{x}_I + \mathbf{x}_{II}$ of a vector $\mathbf{x}_I$ parallel to $\mathbf{y}$ and a vector $\mathbf{x}_{II}$ perpendicular to $\mathbf{y}$, provided $\mathbf{y}$ is nonzero. Indeed, (9.14) is the only way to get a decomposition with those properties, so the decomposition is unique. This appears natural when we recall the geometric construction made above; we will not write down a formal proof here. One may check directly, however, that the vectors $\mathbf{x}_I$ and $\mathbf{x}_{II}$ are orthogonal. We have

$$\begin{aligned}\mathbf{x}_I\cdot\mathbf{x}_{II} &= \mathbf{p_y}(\mathbf{x})\cdot(\mathbf{x}-\mathbf{p_y}(\mathbf{x})) = \frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}\cdot\left(\mathbf{x}-\frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}\right)\\ &= \frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}(\mathbf{y}\cdot\mathbf{x}) - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{(\mathbf{y}\cdot\mathbf{y})^2}(\mathbf{y}\cdot\mathbf{y}) = 0\,,\end{aligned}$$

We now relate the projection vector to the angle θ in Fig. 9.5. There θ is an acute angle (less than 90°), and we have by (9.12)

$$\cos\theta = \frac{\|\mathbf{p_y}(\mathbf{x})\|}{\|\mathbf{x}\|} = \frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|\,\|\mathbf{x}\|}\,.$$

If the angle θ is larger than 90°, then both the cosine and the scalar product $\mathbf{x}\cdot\mathbf{y}$ become negative. (The reader is urged to draw a picture similar to Fig. 9.5 for this situation.) The general formula relating the **angle $\boldsymbol{\theta}$ between x and y** to the **scalar product** is

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\,. \tag{9.15}$$

Example 9.6 Find the angle between the vectors $\mathbf{x} = (2, 3, 2)$ and $\mathbf{y} = (1, 2, -1)$.

Solution: We determine the angle from formula (9.15) just above. We have $\mathbf{x}\cdot\mathbf{y} = 2\cdot 1 + 3\cdot 2 + 2\cdot(-1) = 6$ and

$$\|\mathbf{x}\| = \sqrt{4+9+4} = \sqrt{17}, \quad \|\mathbf{y}\| = \sqrt{1+2^2+(-1)^2} = \sqrt{6}.$$

Thus

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{6}{\sqrt{17}\sqrt{6}} = \frac{1}{17}\sqrt{102} \cong \frac{10.1}{17} \cong 0.594,$$

and $\theta \cong 0.935$ rad, which is about 54°. Since the cosine ranges between -1 and 1, we immediately get from (9.15) the **Schwarz inequality** (or **Cauchy–Schwarz** or **Cauchy–Bunyakovski–Schwarz** inequality)

$$|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|, \tag{9.16}$$

which is valid for any two vectors $\mathbf{x}$ and $\mathbf{y}$. (Let us remark that one can prove this inequality directly from the properties of the scalar product, without recourse to the geometric construction used above.) Using the Schwarz inequality, we may now prove the triangle inequality. Indeed, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ we have

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x}+\mathbf{y}\cdot\mathbf{x}+\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y}\\ &\le \|\mathbf{x}\|^2 + 2|\mathbf{x}\cdot\mathbf{y}| + \|\mathbf{y}\|^2\\ &\le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 \qquad \text{by Schwarz's inequality}\\ &= (\|\mathbf{x}\|+\|\mathbf{y}\|)^2.\end{aligned}$$

Taking the square root of both sides, we get

$$\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|.$$

The Cross Product

Definition 9.6 (*Cross Product*) Let $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $\mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$. The **cross product** (or **vector product**) of $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{x}\times\mathbf{y}$ and defined as

$$\mathbf{x}\times\mathbf{y} = (x_2y_3 - x_3y_2)\mathbf{i} + (x_3y_1 - x_1y_3)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}. \tag{9.17}$$

Thus, the cross product of two vectors yields a vector.

The formula (9.17) for $\mathbf{x}\times\mathbf{y}$ is related to determinants as follows:

$$\mathbf{x}\times\mathbf{y} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix} = \begin{vmatrix} x_2 & x_3\\ y_2 & y_3\end{vmatrix}\mathbf{i} - \begin{vmatrix} x_1 & x_3\\ y_1 & y_3\end{vmatrix}\mathbf{j} + \begin{vmatrix} x_1 & x_2\\ y_1 & y_2\end{vmatrix}\mathbf{k}.$$

Actually, the expression between the two equality signs is not a true determinant, since the first line consists of vectors, not of numbers, but it is a convenient way to memorize formula (9.17). The standard rule for computing the three 2×2 determinants on the

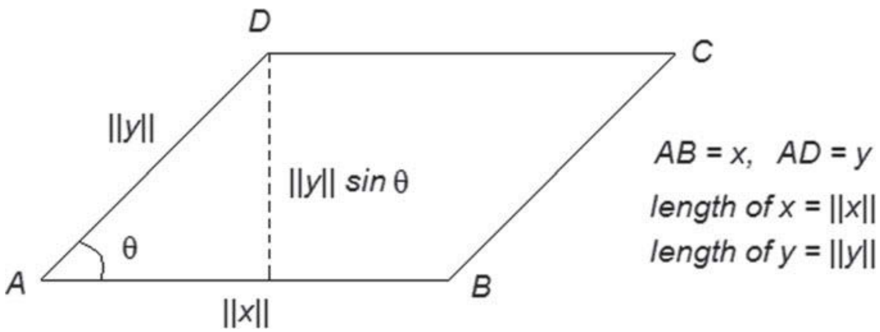


Fig. 9.6 Geometric interpretation of the cross product

right-hand side gives (9.17). Another way to memorize (9.17) is to observe the cyclic behavior of the indices.

Geometric interpretation of the cross product. The vector $\mathbf{x} \times \mathbf{y}$ is geometrically related to the vectors $\mathbf{x}$ and $\mathbf{y}$ as follows. Whenever the vectors $\mathbf{x}$ and $\mathbf{y}$ are not parallel, they form the sides of a parallelogram, see Fig. 9.6. The side AB is formed by the vector $\mathbf{x}$ and has length $\|\mathbf{x}\|$, AD is formed by $\mathbf{y}$ with length $\|\mathbf{y}\|$. As we know, the area of a parallelogram with base b and height h is given by bh; here $b = \|\mathbf{x}\|$ and $h = \|\mathbf{y}\| \sin\theta$. It turns out (see the discussion of Lagrange's identity below) that the magnitude of $\mathbf{x} \times \mathbf{y}$ equals the area of that parallelogram, so

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \|\mathbf{y}\| \sin\theta \,. \tag{9.18}$$

Moreover, $\mathbf{x} \times \mathbf{y}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$. Finally, from the two remaining possibilities the direction of $\mathbf{x} \times \mathbf{y}$ is chosen according to the so-called **right-hand rule**: Point the index finger in the direction of $\mathbf{x}$ and the middle finger in the direction of $\mathbf{y}$. The thumb then points in the direction of $\mathbf{x} \times \mathbf{y}$.

Below and in the exercises, we will show that these geometric properties follow from Definition 9.6, if the coordinate system has a right-hand orientation (index finger points to $\mathbf{i}$, middle finger points to $\mathbf{j}$, thumb points to $\mathbf{k}$.)

Algebraic properties of the cross product. In contrast to the scalar product, the cross product is **anticommutative**, that is,

$$\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y} \tag{9.19}$$

holds for all vectors $\mathbf{x}$ and $\mathbf{y}$. As a consequence, for all vectors $\mathbf{x}$ we have

$$\mathbf{x} \times \mathbf{x} = \mathbf{0} \,. \tag{9.20}$$

On the other hand, the cross product shares some properties of the scalar product. The **distributive** laws

$$\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z} \tag{9.21}$$

$$(\mathbf{x} + \mathbf{y}) \times \mathbf{z} = \mathbf{x} \times \mathbf{z} + \mathbf{y} \times \mathbf{z} \tag{9.22}$$

$$(\alpha\mathbf{x}) \times \mathbf{y} = \alpha(\mathbf{x} \times \mathbf{y}) = \mathbf{x} \times (\alpha\mathbf{y}) \tag{9.23}$$

hold for all vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ and all scalars α. In particular, setting $\alpha = 0$ we have for every vector $\mathbf{x}$

$$\mathbf{x} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{x}\,.$$

The properties (9.19)–(9.23) can be verified directly from the definition of the cross product.

Example 9.7 (a) Calculate $\mathbf{x} \times \mathbf{y}$ where $\mathbf{x} = (1, -2, 3)$ and $\mathbf{y} = (2, 1, -1)$.
(b) Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
(c) Compute $\mathbf{i} \times (\mathbf{i} \times \mathbf{j})$ and $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$.
(d) Compute $\mathbf{x} \times \mathbf{y}$ where $\mathbf{x} = \mathbf{i} - \mathbf{j}$ and $\mathbf{y} = \mathbf{i} + \mathbf{k}$.

Solution: (a) We have

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \mathbf{k} = -\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}\,.$$

(b) We have $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$. We compute

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 1\mathbf{k} = \mathbf{k}\,,$$

so we have shown that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Similarly, we can prove the other two identities.
(c) We obtain

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}\,, \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}\,.$$

(d) We calculate as before

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \mathbf{k} = -\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Alternatively we may use part (b) above together with properties (9.19)–(9.23) to compute

$$\mathbf{x} \times \mathbf{y} = (\mathbf{i} - \mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{k} - \mathbf{j} \times \mathbf{i} - \mathbf{j} \times \mathbf{k} = -\mathbf{j} + \mathbf{k} - \mathbf{i}\,.$$

We note that, as we see from part (c) above, it may occur that

$$(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} \neq \mathbf{x} \times (\mathbf{y} \times \mathbf{z}) .$$

Thus, the cross product is **not associative**.

We come back to formula (9.18), $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \|\mathbf{y}\| \sin\theta$. From formula (9.15), we see that

$$\|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 (1 - \cos^2\theta) = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \sin^2\theta ,$$

so (9.18) is equivalent to **Lagrange's identity**

$$\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2 . \tag{9.24}$$

Lagrange's identity, in turn, is a special case ($\mathbf{z} = \mathbf{x}$, $\mathbf{w} = \mathbf{y}$) of the more general formula

$$(\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{z} \times \mathbf{w}) = (\mathbf{x} \cdot \mathbf{z})(\mathbf{y} \cdot \mathbf{w}) - (\mathbf{y} \cdot \mathbf{z})(\mathbf{x} \cdot \mathbf{w}) . \tag{9.25}$$

Therefore, formula (9.18) for the magnitude of the cross product follows from (9.25). A derivation of (9.25) will be done in Exercise 9.7.2.

The scalar triple product. If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors, the expression $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$ is called the **scalar triple product** of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Its absolute value $|(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}|$ represents the volume of the parallelogram with edges $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Example 9.8 Show that the scalar triple product satisfies

$$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} .$$

Solution: We have

$$\begin{aligned} (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} &= \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \mathbf{k} \right) \cdot (z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}) \\ &= z_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - z_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + z_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} . \end{aligned}$$

The latter expression is the expansion of the 3×3 determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} .$$

Scalar and vector products often appear in physics and engineering. Work is the scalar product of force and displacement. Torque and angular momentum are the cross products of force and displacement, resp., force and linear momentum. Maxwell's

equations, which provide the foundation of electromagnetic theory, involve both scalar and cross products of electrical and magnetic variables.

9.3 Differential Calculus of Vector Fields

Let us begin with a specific situation. Let f_1, f_2, f_3 be real-valued functions defined on some interval I, then for each $t \in I$ we can form the vector

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}\,. \tag{9.26}$$

All those points (that is, the points in the range of $\mathbf{F}$) form a curve in 3-space; if we think of t as a time variable, we may imagine an object moving along this curve, passing the point $\mathbf{F}(t)$ at time t. The functions f_1, f_2, f_3 are called the **component functions**, or simply the **components** of $\mathbf{F}$.

In general, let us consider a function $\mathbf{F}$ defined on some set S. Whenever the values of $\mathbf{F}$ are vectors, the function $\mathbf{F}$ is called a **vector field** (or a **vector-valued function**, or a **vector function**). Its domain S may be an interval as above, or it may be a subset of the plane $\mathbb{R}^2$ or the space $\mathbb{R}^3$. In the latter case, the vector field has the form

$$\mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}\,. \tag{9.27}$$

As an example, $\mathbf{F}$ may be the gravity field, which to every point $P = (x, y, z)$ in space associates the vector $\mathbf{F}(x, y, z)$ representing the gravity force at P. The same situation occurs with the electric field and the magnetic field. Moreover, the flow of a fluid gives rise to a velocity field which associates with each point $P = (x, y, z)$ the velocity vector of the fluid at this point. The form (9.27) describes the situation when the velocity at a point P does not change with time (this is called stationary flow); for instationary flow the vector field $\mathbf{F}$ and its components f_i depend on time, too,

$$\mathbf{F}(x, y, z, t) = f_1(x, y, z, t)\mathbf{i} + f_2(x, y, z, t)\mathbf{j} + f_3(x, y, z, t)\mathbf{k}\,,$$

so in this case the domain S of $\mathbf{F}$ is a subset of 4-space. Many more examples of vector fields will appear in the following.

In the two-dimensional situation, (9.27) simplifies to

$$\mathbf{F}(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}\,.$$

In this case, the values of $\mathbf{F}$ are vectors in $\mathbb{R}^2$ instead of $\mathbb{R}^3$.

As in Sect. 9.2 above, for the argument vector (x, y, z) we may also write (x_1, x_2, x_3), or $\mathbf{x}$ in compact notation. The value of $\mathbf{F}$ at $\mathbf{x}$ is then simply denoted by $\mathbf{F}(\mathbf{x})$.

9.3.1 Curves

In this subsection, we consider vector fields which are defined on an interval I of the real line, as in (9.26). Such vector fields are commonly called **curves**, since one imagines the points $\mathbf{F}(t)$ to form a curve in space as t varies in I. Note that in Sect. 7.2 we have already encountered curves in 2-space of the form $\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, which we have called "curves in parametric form".

Example 9.9
(a) The function $\mathbf{F}$ defined by $\mathbf{F}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + e^t\,\mathbf{k}$ is a vector field defined on the whole line $\mathbb{R}$.
(b) Let $\mathbf{F}(t) = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, where x_0, y_0, z_0 are constants. Since the right-hand side does not depend on t, the range of $\mathbf{F}$ consists of a single point, and $\mathbf{F}$ is called a constant vector field.
(c) Let $f_1(t) = 2\cos t$, $f_2(t) = 2\sin t$, $f_3(t) = t$. Write down the associated vector field having f_1, f_2 and f_3 as components.

Solution: (c) The vector field is $\mathbf{F}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$, or $\mathbf{F}(t) = (2\cos t, 2\sin t, t)$ in vector notation.

Definition 9.7 Let $\mathbf{F}$ be a vector field defined on an interval I, let $t_0 \in I$ be given. We say that the vector $\mathbf{L}$ is the limit of $\mathbf{F}$ as t tends to t_0, and we write

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{L}\,,$$

if $\lim\limits_{t\to t_0} \|\mathbf{F}(t) - \mathbf{L}\| = 0$. We say that $\mathbf{F}$ is continuous at t_0 if

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{F}(t_0)\,.$$

We say that $\mathbf{F}$ is continuous on I if it is continuous at every point $t_0 \in I$.

The limit process can be carried out by components.

Theorem 9.1 *Let* $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ *and* $\mathbf{L} = l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k}$. *Then* $\lim\limits_{t\to t_0} \mathbf{F}(t) = \mathbf{L}$ *if and only if*

$$\lim_{t\to t_0} f_1(t) = l_1\,, \quad \lim_{t\to t_0} f_2(t) = l_2\,, \quad \lim_{t\to t_0} f_3(t) = l_3\,. \tag{9.28}$$

$\mathbf{F}$ *is continuous at* t_0 *if and only if all* f_i *are continuous at* t_0. $\mathbf{F}$ *is continuous on* I *if and only if all* f_i *are continuous on* I.

Verification: Since

$$\|\mathbf{F}(t) - \mathbf{L}\| = \sqrt{\sum_{i=1}^{3} (f_i(t) - l_i)^2}$$

and therefore $0 \leq |f_i(t) - l_i| \leq \|\mathbf{F}(\mathbf{t}) - \mathbf{L}\|$ for all i, the properties of limits and the Sandwich theorem imply the first assertion. The other assertions are then a consequence of Definition 9.7.

Remark 9.1 If $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{L}$ then $\lim_{t\to t_0} \|\mathbf{F}(t)\| = \|\mathbf{L}\|$. As for scalar functions, the converse of this result is wrong, namely $\lim_{t\to t_0} \|\mathbf{F}(t)\| = \|\mathbf{L}\|$ does not imply $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{L}$.

Verification: By the reverse triangle inequality (see Example 9.3(d))

$$0 \leq |\|\mathbf{F}(t)\| - \|\mathbf{L}\|| \leq \|\mathbf{F}(t) - \mathbf{L}\| .$$

It follows from the Sandwich theorem that if $\lim_{t\to t_0} \|\mathbf{F}(t) - \mathbf{L}\| = 0$, then $\lim_{t\to t_0} (\|\mathbf{F}(t)\| - \|\mathbf{L}\|) = 0$. For the converse choose $\mathbf{F}(t) = \mathbf{x}$ and $\mathbf{L} = -\mathbf{x}$, where $\mathbf{x}$ is any fixed nonzero vector. We have $\lim_{t\to t_0} \|\mathbf{F}(t)\| = \|\mathbf{x}\| = \|\mathbf{L}\|$, but $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{x} \neq -\mathbf{x} = \mathbf{L}$.

The usual properties of the limit (see Chap. 2) can be extended for limits of vector fields. Let $\mathbf{F}(t) \to \mathbf{L}$, $\mathbf{G}(t) \to \mathbf{M}$ and $\alpha(t) \to A$ as $t \to t_0$. Then

$$\lim_{t\to t_0} (\mathbf{F}(t) + \mathbf{G}(t)) = \lim_{t\to t_0} \mathbf{F}(t) + \lim_{t\to t_0} \mathbf{G}(t) = \mathbf{L} + \mathbf{M}\,, \tag{9.29}$$

$$\lim_{t\to t_0} (\alpha(t)\mathbf{F}(t)) = \left(\lim_{t\to t_0} \alpha(t)\right) \cdot \left(\lim_{t\to t_0} \mathbf{F}(t)\right) = A\mathbf{L}\,, \tag{9.30}$$

in particular, if β is a scalar and $\mathbf{x}$ is a vector,

$$\lim_{t\to t_0} (\beta\mathbf{F}(t)) = \beta \lim_{t\to t_0} \mathbf{F}(t) = \beta\mathbf{L}\,, \quad \lim_{t\to t_0} (\alpha(t)\mathbf{x}) = \left(\lim_{t\to t_0} \alpha(t)\right) \mathbf{x} = A\mathbf{x}\,, \tag{9.31}$$

and moreover

$$\lim_{t\to t_0} (\mathbf{F}(t) \cdot \mathbf{G}(t)) = \left(\lim_{t\to t_0} \mathbf{F}(t)\right) \cdot \left(\lim_{t\to t_0} \mathbf{G}(t)\right) = \mathbf{L} \cdot \mathbf{M}\,, \tag{9.32}$$

$$\lim_{t\to t_0} (\mathbf{F}(t) \times \mathbf{G}(t)) = \mathbf{L} \times \mathbf{M}\,. \tag{9.33}$$

Definition 9.8 (*Derivative of a vector field*) We define the derivative of $\mathbf{F}$ at t_0 by

$$\mathbf{F}'(t_0) = \lim_{h\to 0} \frac{\mathbf{F}(t_0 + h) - \mathbf{F}(t_0)}{h}\,,$$

if this limit exists. In that case, $\mathbf{F}$ is called **differentiable** at t_0. Alternatively, $\mathbf{F}'(t_0)$ is also denoted as $\dfrac{d\mathbf{F}}{dt}(t_0)$.

Geometrically, $\mathbf{F}'(t_0)$ points along the tangent of the curve in the point $\mathbf{F}(t_0)$, see Fig. 9.7.

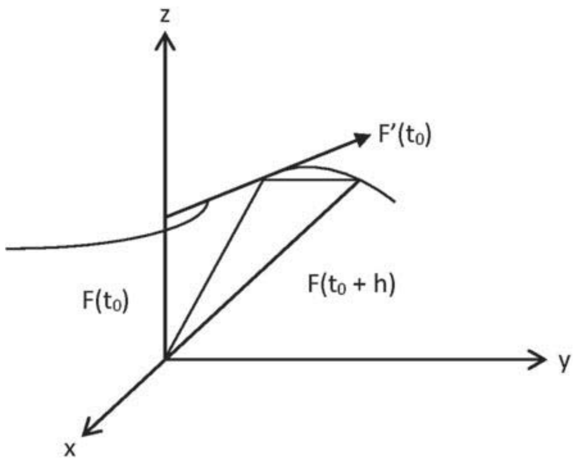


Fig. 9.7 Tangent vector

Differentiation can be carried out component by component, that is, if $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ is differentiable at t, then

$$\mathbf{F}'(t) = f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}\,. \tag{9.34}$$

Indeed, applying rules (9.29) and (9.31), we see that

$$\begin{aligned}
\mathbf{F}'(t) &= \lim_{h\to 0} \frac{\mathbf{F}(t+h) - \mathbf{F}(t)}{h} \\
&= \lim_{h\to 0} \left[\frac{f_1(t+h) - f_1(t)}{h}\mathbf{i} + \frac{f_2(t+h) - f_2(t)}{h}\mathbf{j} + \frac{f_3(t+h) - f_2(t)}{h}\mathbf{k} \right] \\
&= \left[\lim_{h\to 0} \frac{f_1(t+h) - f_1(t)}{h} \right]\mathbf{i} + \left[\lim_{h\to 0} \frac{f_2(t+h) - f_2(t)}{h} \right]\mathbf{j} \\
&\quad + \left[\lim_{h\to 0} \frac{f_3(t+h) - f_2(t)}{h} \right]\mathbf{k} \\
&= f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}\,.
\end{aligned}$$

Differentiation properties of vector fields. We have already seen in Sect. 1.5 that we may define, for example, the sum $f + g$ of two functions by $(f + g)(x) = f(x) + g(x)$. In this manner, operations performed on function values give rise to operations on the functions themselves. The same applies to vector fields. Let $\mathbf{F}$ and $\mathbf{G}$ be two vector fields defined on an interval I, and let α, β be scalars. We define

$$(\mathbf{F} + \mathbf{G})(t) = \mathbf{F}(t) + \mathbf{G}(t)\,, \quad (\alpha\mathbf{F})(t) = \alpha\mathbf{F}(t)\,, \tag{9.35}$$

$$\text{thus} \quad (\alpha\mathbf{F} + \beta\mathbf{G})(t) = \alpha\mathbf{F}(t) + \beta\mathbf{G}(t)\,, \tag{9.36}$$

$$(\mathbf{F}\cdot\mathbf{G})(t) = \mathbf{F}(t)\cdot\mathbf{G}(t)\,, \tag{9.37}$$

$$(\mathbf{F} \times \mathbf{G})(t) = \mathbf{F}(t) \times \mathbf{G}(t) . \tag{9.38}$$

Moreover, when v is a scalar function defined on I, we set

$$(v\mathbf{F})(t) = v(t)\mathbf{F}t . \tag{9.39}$$

We also consider the composition

$$(\mathbf{F} \circ u)(t) = \mathbf{F}(u(t))$$

for a scalar function u whose range is contained in I.

The following rules hold for differentiation.

$$(\mathbf{F} + \mathbf{G})'(t) = \mathbf{F}'(t) + \mathbf{G}'(t) \tag{9.40}$$

$$(\alpha\mathbf{F})'(t) = \alpha\mathbf{F}'(t) \quad \text{for constants } \alpha, \tag{9.41}$$

$$(v\mathbf{F})'(t) = v(t)\mathbf{F}'(t) + v'(t)\mathbf{F}(t) , \tag{9.42}$$

$$(\mathbf{F} \cdot \mathbf{G})'(t) = \mathbf{F}(t) \cdot \mathbf{G}'(t) + \mathbf{F}'(t) \cdot \mathbf{G}(t) , \tag{9.43}$$

$$(\mathbf{F} \times \mathbf{G})'(t) = \mathbf{F}(t) \times \mathbf{G}'(t) + \mathbf{F}'(t) \times \mathbf{G}(t) , \tag{9.44}$$

and the chain rule for vector functions

$$(\mathbf{F} \circ u)'(t) = \mathbf{F}'(u(t))u'(t) = u'(t)\mathbf{F}'(u(t)) . \tag{9.45}$$

The formulas (9.40)–(9.45) are a consequence of the definitions, the componentwise formula (9.34) and the corresponding differentiation rules for scalar functions (see Chap. 3). They can also be written in Leibniz' notation, for example,

$$\frac{d}{dt}(\mathbf{F} + \mathbf{G}) = \frac{d\mathbf{F}}{dt} + \frac{d\mathbf{G}}{dt} ,$$

$$\frac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \left(\mathbf{F} \cdot \frac{d\mathbf{G}}{dt}\right) + \left(\frac{d\mathbf{F}}{dt} \cdot \mathbf{G}\right) ,$$

$$\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \left(\mathbf{F} \times \frac{d\mathbf{G}}{dt}\right) + \left(\frac{d\mathbf{F}}{dt} \times \mathbf{G}\right) .$$

Example 9.10 Let $\mathbf{F}(t) = 2t^2\mathbf{i} - 3\mathbf{j}, \mathbf{G}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, u(t) = \frac{1}{3}t^3$. Verify (9.43)–(9.45) for these functions.

Solution: We first compute the derivatives

$$\mathbf{F}'(t) = 4t\mathbf{i} , \quad \mathbf{G}'(t) = \mathbf{j} + 2t\mathbf{k} , \quad u'(t) = t^2 .$$

We verify (9.43). We have

$$(\mathbf{F} \cdot \mathbf{G})(t) = (2t^2\mathbf{i} - 3\mathbf{j}) \cdot (\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}) = 2t^2 - 3t ,$$

therefore, the left-hand side becomes $(\mathbf{F}\cdot\mathbf{G})'(t) = 4t - 3$. For the right-hand side, we get

$$\mathbf{F}(t)\cdot\mathbf{G}'(t) + \mathbf{F}'(t)\cdot\mathbf{G}(t) = (2t^2\mathbf{i} - 3\mathbf{j})\cdot(\mathbf{j} + 2t\mathbf{k}) + 4t\mathbf{i}\cdot(\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}) = -3 + 4t\,,$$

so indeed both sides are equal.

We now consider (9.44). We have

$$(\mathbf{F}\times\mathbf{G})(t) = (2t^2\mathbf{i} - 3\mathbf{j})\times(\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}) = -3t^2\mathbf{i} - 2t^4\mathbf{j} + (2t^3 + 3)\mathbf{k}\,,$$

so the left-hand side equals

$$(\mathbf{F}\times\mathbf{G})'(t) = -6t\mathbf{i} - 8t^3\mathbf{j} + 6t^2\mathbf{k}\,.$$

We compute the right-hand side

$$\begin{aligned}\mathbf{F}(t)\times\mathbf{G}'(t) + \mathbf{F}'(t)\times\mathbf{G}(t) &= (2t^2\mathbf{i} - 3\mathbf{j})\times(\mathbf{j} + 2t\mathbf{k}) + 4t\mathbf{i}\times(\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})\\ &= (2t^2\mathbf{k} - 4t^3\mathbf{j} - 6t\mathbf{i}) + (4t^2\mathbf{k} - 4t^3\mathbf{j}) = -6t\mathbf{i} - 8t^3\mathbf{j} + 6t^2\mathbf{k}\,,\end{aligned}$$

therefore, both sides are equal.

Finally, we investigate (9.45). We have

$$(\mathbf{F}\circ u)(t) = 2\left(\frac{1}{3}t^3\right)^2\mathbf{i} - 3\mathbf{j}\,,\quad (\mathbf{F}\circ u)'(t) = 2\cdot\frac{1}{9}\cdot 6t^5\mathbf{i} = \frac{4}{3}t^5\mathbf{i}$$

on the left-hand side, and

$$\mathbf{F}'(u(t))u'(t) = [4u(t)\mathbf{i}]\cdot u'(t) = \left[4\left(\frac{1}{3}t^3\right)\mathbf{i}\right]t^2 = \frac{4}{3}t^5\mathbf{i}$$

on the right-hand side.

Example 9.11 (a) Let $\mathbf{F}(t) = t\mathbf{i} + \sqrt{t+1}\,\mathbf{j} - e^t\mathbf{k}$. Find $\mathbf{F}'(t)$, $\mathbf{F}'(0)$, $\mathbf{F}''(1)$, and $\mathbf{F}(t)\cdot\mathbf{F}'(t)$.
(b) Find $\mathbf{F}''(t)$ if $\mathbf{F}(t) = t\sin t\,\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}$.

Solution: (a) We have $f_1(t) = t$, $f_2(t) = \sqrt{t+1}$, and $f_3(t) = -e^t$. We get

$$\mathbf{F}'(t) = f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} - f_3'(t)\mathbf{k} = \mathbf{i} + \frac{1}{2\sqrt{t+1}}\mathbf{j} - e^t\mathbf{k}\,,$$

$$\mathbf{F}'(0) = \mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k},\quad \text{as}\ \ e^0 = 1,\ \ \sqrt{t+1} = 1\ \ \text{for}\ \ t = 0\,,$$

$$\mathbf{F}''(t) = 0\mathbf{i} - \frac{1}{4}\frac{1}{(t+1)^{3/2}}\mathbf{j} - e^t\mathbf{k}\,,$$

$$\mathbf{F}''(0) = -\frac{1}{4}\mathbf{j} - e^0\mathbf{k} = -\frac{1}{4}\mathbf{j} - \mathbf{k}\,,$$
$$\mathbf{F}(t)\cdot\mathbf{F}'(t) = \left(t\mathbf{i} + \sqrt{t+1}\mathbf{j} - e^t\mathbf{k}\right)\cdot\left(\mathbf{i} + \frac{1}{2\sqrt{t+1}}\mathbf{j} - e^t\mathbf{k}\right) = t + \frac{1}{2} + e^{2t}\,.$$

(b) We have $f_1(t) = t\sin t$, $f_2(t) = e^{-t}$, and $f_3(t) = t$. We get

$$\mathbf{F}'(t) = f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} - f_3'(t)\mathbf{k} = (t\cos t + \sin t)\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}\,,$$
$$\mathbf{F}''(t) = (-t\sin t + \cos t + \cos t)\mathbf{i} + e^{-t}\mathbf{j} + 0\mathbf{k} = (2\cos t - t\sin t)\mathbf{i} + e^{-t}\mathbf{j}\,.$$

9.3.2 Vector Fields in Several Dimensions

In this section, we introduce the concepts of gradient, divergence, and curl. These are based on the notion of partial derivatives, which we have introduced in Sect. 8.4. Since any kind of derivatives of a function f at a certain point $\mathbf{x}$ are defined as limits involving function values at nearby points, it is most convenient if the domain of f "always includes nearby points". This is made precise in the following definition.

Definition 9.9 (*Open Set*) A subset D of $\mathbf{R}^n$ is called **open** if for every $\mathbf{x} \in D$ there exists a ball B with center $\mathbf{x}$ which is contained in D.

Thus, if $\mathbf{x} \in D$ and if the ball B in question has radius r, then every point $\mathbf{z}$ whose distance from $\mathbf{x}$ is smaller than r must also belong to D. For example, the interior $D = \{(x, y, z) : 0 < x, y, z < 1\}$ of a cube is open, while if we include its boundary, the corresponding set $R = \{(x, y, z) : 0 \leq x, y, z \leq 1\}$ is not open.

Let us note that the definition of an open set in the plane (Definition 8.4) is a special case of Definition 9.9.

The gradient. Let f be a scalar function of three variables. The **gradient** of f at a point (x, y, z) is defined as the vector

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z)\mathbf{i} + \frac{\partial f}{\partial y}(x, y, z)\mathbf{j} + \frac{\partial f}{\partial z}(x, y, z)\mathbf{k}\,. \tag{9.46}$$

If f and its partial derivatives are defined on some open subset D of $\mathbb{R}^3$, the gradient thus becomes a vector field $\nabla f : D \to \mathbb{R}^3$ whose component functions are just the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$. According to the notation of vectors, we may also write

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z)\right)\,.$$

Example 9.12 For $f(x, y, z) = x^2 y \sin z$ we have

$$\frac{\partial f}{\partial x}(x, y, z) = 2xy \sin z\,, \quad \frac{\partial f}{\partial y}(x, y, z) = x^2 \sin z\,, \quad \frac{\partial f}{\partial z}(x, y, z) = x^2 y \cos z\,,$$

and thus

$$\nabla f(x, y, z) = 2xy \sin z\mathbf{i} + x^2 \sin z\mathbf{j} + x^2 y \cos z\mathbf{k}\,,$$

or

$$\nabla f(x, y, z) = (2xy \sin z, x^2 \sin z, x^2 y \cos z)\,.$$

For a function f of two variables (x, y),

$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}(x, y)\mathbf{j}\,.$$

In n dimensions, we define

$$\nabla f(\mathbf{x}) = \sum_{i=1}^{n} \partial_i f(\mathbf{x})\mathbf{e}_i\,, \tag{9.47}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\partial_i f = \partial f/\partial x_i$ and $\mathbf{e}_i$ is the ith unit vector.

The gradient obeys the rules

$$\nabla(f + g)(x) = \nabla f(x) + \nabla g(x)\,, \tag{9.48}$$

$$\nabla(\alpha f)(x) = \alpha \nabla f(x)\,, \quad \text{if } \alpha \text{ is a scalar,} \tag{9.49}$$

$$\nabla(fg)(x) = f(x)\nabla g(x) + g(x)\nabla f(x)\,. \tag{9.50}$$

Those rules can be verified componentwise by the corresponding rules for partial derivatives.

Linearization. In the case of a function f of a single variable we have, as a consequence of the definition of the derivative,

$$f(x + h) = f(x) + f'(x)h + r(h) \tag{9.51}$$

for some remainder term $r(h)$ with $r(h)/h \to 0$ as $h \to 0$. For functions of several variables, the gradient plays the corresponding role.

Definition 9.10 Let f be a real-valued function of n variables defined in an open subset D of $\mathbb{R}^n$. We say that f is **differentiable** at a point $\mathbf{x} \in D$, if all partial derivatives of f exist at $\mathbf{x}$ and

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot \mathbf{h}|}{\|\mathbf{h}\|} = 0\,. \tag{9.52}$$

As a consequence, for a differentiable function f we obtain the analogue of (9.51) in several dimensions as

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + r(\mathbf{h})\,, \tag{9.53}$$

where $r(\mathbf{h})$ is a remainder term with $r(\mathbf{h})/\|\mathbf{h}\| \to 0$ as $\mathbf{h} \to 0$. For this reason, the function g defined by

$$g(\mathbf{z}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (\mathbf{z}-\mathbf{x})$$

is called the **linearization** of f at $\mathbf{x}$. Note that the difference

$$g(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}+\mathbf{h}) = r(\mathbf{h})$$

converges to 0 faster than $\|\mathbf{h}\|$. The notation

$$r(\mathbf{h}) = o(\|\mathbf{h}\|)$$

(read as "$r(\mathbf{h})$ is small o of $\|\mathbf{h}\|$") is a common way of stating that "$r(\mathbf{h})/\|\mathbf{h}\| \to 0$ as $\mathbf{h} \to 0$".

It would be very inconvenient if, whenever we want to apply linearization, we had to check the validity of (9.52) explicitly. Usually, this is not necessary because of the following theorem.

Theorem 9.2 *Let f be a real-valued function of n variables, which is continuously differentiable in an open subset D of $\mathbb{R}^n$. Then f is differentiable at every point $\mathbf{x} \in D$.*

This means that we only have to check whether all partial derivatives of f are continuous in D, which is often obvious.

The directional derivative. We present its definition in the general case of n dimensions.

Definition 9.11 (*Directional Derivative*) Let f be a real-valued function of n variables. For each vector $\mathbf{u} \in \mathbb{R}^n$, the limit

$$f'_{\mathbf{u}}(\mathbf{x}) = \lim_{t\to 0} \frac{f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x})}{t}\,,$$

if it exists, is called the **directional derivative** of f at $\mathbf{x}$ in the direction $\mathbf{u}$. Instead of $f'_{\mathbf{u}}(\mathbf{x})$, we also write $\partial_{\mathbf{u}} f(\mathbf{x})$.

Remark 9.2 1. When $\mathbf{u} = \mathbf{e}_i$ is the ith canonical unit vector, the directional derivative $f'_{\mathbf{u}}$ becomes the partial derivative $\partial_i f$. In particular, for a function f of three variables (x, y, z), we obtain

$$\frac{\partial f}{\partial x}(x,y,z) = f'_{\mathbf{i}}(x,y,z) = \partial_x f(x,y,z)\,, \quad \mathbf{i} = (1,0,0)\,,$$

similarly $\partial f/\partial y = f'_{\mathbf{j}} = \partial_y f$ and $\partial f/\partial z = f'_{\mathbf{k}} = \partial_z f$.

2. As we know already the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ give the rates of change of f in the directions $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$, respectively. Analogously, if $\|\mathbf{u}\|$ is a unit vector, the directional derivative $f'_{\mathbf{u}}(\mathbf{x})$ gives the rate of change of f in the direction $\mathbf{u}$.
3. The geometrical interpretation of the directional derivative is essentially the same as that of the partial derivative, presented in Sect. 8.4, except that we now look at tangents to the graph of f in arbitrary directions $\mathbf{u}$, not only in the direction of the coordinate axes.

The following theorem gives the connection between the gradient of f at $\mathbf{x}$ and its directional derivative at $\mathbf{x}$.

Theorem 9.3 *If f is differentiable at $\mathbf{x}$, then f has a directional derivative at $\mathbf{x}$ in every direction $\mathbf{u}$, and*

$$f'_{\mathbf{u}}(\mathbf{x}) = \partial_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}\,. \tag{9.54}$$

In particular, all partial derivatives $\partial_i f$ exist.

Proof Let $\mathbf{u} \in \mathbb{R}^n$ be given, assume $\mathbf{u} \neq 0$ (otherwise, the assertion is trivially satisfied). In order to verify the definition of the directional derivative, we consider the identity, valid for all $t \neq 0$,

$$\begin{aligned}\left|\frac{f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x})}{t} - \nabla f(\mathbf{x}) \cdot \mathbf{u}\right| &= \frac{|f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x}) - t\nabla f(\mathbf{x}) \cdot \mathbf{u}|}{|t|} \\ &= \frac{|f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x}) - t\nabla f(\mathbf{x}) \cdot \mathbf{u}|}{\|t\mathbf{u}\|} \cdot \|\mathbf{u}\|\,.\end{aligned}$$

Since f is differentiable at $\mathbf{x}$, the fraction on the right-hand side converges to 0, see Definition 9.2. Therefore, the left-hand side, too, converges to 0, which yields the assertion.

Theorem 9.3 yields an important geometric property of the gradient. According to (9.15), the angle θ between $\nabla f(\mathbf{x})$ and any vector $\mathbf{u}$ satisfies

$$\nabla f(\mathbf{x}) \cdot \mathbf{u} = \|\nabla f(\mathbf{x})\| \|\mathbf{u}\| \cos\theta\,.$$

If moreover $\mathbf{u}$ is a unit vector and $\nabla f(\mathbf{x})$ is nonzero, we get from (9.54) that

$$f'_{\mathbf{u}}(\mathbf{x}) = \|\nabla f(\mathbf{x})\| \cos\theta\,.$$

Since $-1 \leq \cos\theta \leq 1$, we have

$$-\|\nabla f(\mathbf{x})\| \leq f'_{\mathbf{u}}(\mathbf{x}) \leq \|\nabla f(\mathbf{x})\|\,.$$

In particular, if $\mathbf{u}$ points in the direction of $\nabla f(\mathbf{x})$, then $\theta = 0$ and $\cos\theta = 1$, so

$$f'_{\mathbf{u}}(\mathbf{x}) = \|\nabla f(\mathbf{x})\|\,, \tag{9.55}$$

whereas, if $\mathbf{u}$ points in the direction of $-\nabla f(\mathbf{x})$, then $\theta = \pi$ and $\cos\theta = -1$, so

$$f'_{\mathbf{u}}(\mathbf{x}) = -\|\nabla f(\mathbf{x})\|\,. \tag{9.56}$$

Since the directional derivative gives the rate of change of the function in that direction, it follows that the function f increases most rapidly in the direction of the gradient and decreases most rapidly in the opposite direction.

Example 9.13 Let $f(x, y, z) = xyz$. Compute $\nabla f(1, 1, 1)$ and determine the maximum and minimum rate of change of f at $(1, 1, 1)$.

Solution: We have

$$\frac{\partial f}{\partial x}(x, y, z) = yz\,, \quad \frac{\partial f}{\partial y}(x, y, z) = xz\,, \quad \frac{\partial f}{\partial z}(x, y, z) = xy\,,$$

therefore

$$\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = (yz, xz, xy)\,,$$

and hence $\nabla f(1, 1, 1)) = \mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 1, 1)$. According to (9.55) and (9.56), the maximum and minimum rates of change are $\|\nabla f(1, 1, 1)\| = \sqrt{3}$ and $-\|\nabla f(1, 1, 1)\| = -\sqrt{3}$, respectively.

Example 9.14 Calculate the directional derivative of $\varphi(x, y, z) = 8xy^2 - xz$ at an arbitrary point (x, y, z) in the direction of the vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\,.$$

Solution: According to Theorem 9.3, we have

$$\partial_{\mathbf{u}}\varphi(x, y, z) = \varphi'_{\mathbf{u}}(x, y, z) = \nabla\varphi(x, y, z) \cdot \mathbf{u}\,.$$

We compute

$$\frac{\partial \varphi}{\partial x}(x, y, z) = 8y^2 - z\,, \quad \frac{\partial \varphi}{\partial y}(x, y, z) = 16xy\,, \quad \frac{\partial \varphi}{\partial z}(x, y, z) = -x\,,$$

hence $\nabla\varphi(x, y, z) = (8y^2 - z, 16xy, -x)$ and

$$\partial_{\mathbf{u}}\varphi(x, y, z) = (8y^2 - z, 16xy, -x) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{8y^2 - z + 16xy - x}{\sqrt{3}}\,.$$

Example 9.15 Let the temperature at a point (x, y) on a metallic plate in the xy-plane be given by $T(x, y) = \dfrac{xy}{1 + x^2 + y^2}$ degrees Celsius.
(a) Find the rate of change of temperature at $(1, 1)$ in the direction of $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$.
(b) An insect at $(1, 1)$ wants to walk in the direction in which the temperature drops most rapidly. Find a unit vector in that direction.

Solution: (a) We have

$$\frac{\partial T}{\partial x}(x, y) = \frac{y(1 - x^2 + y^2)}{(1 + x^2 + y^2)^2}, \quad \frac{\partial T}{\partial y}(x, y) = \frac{x(1 + x^2 - y^2)}{(1 + x^2 + y^2)^2},$$

so at $(x, y) = (1, 1)$ we have

$$\nabla T(1, 1) = \frac{\mathbf{i}}{9} + \frac{\mathbf{j}}{9} = \left(\frac{1}{9}, \frac{1}{9}\right).$$

The directional derivative gives the correct rate of change only after we normalize the vector $\mathbf{u}$ to unit length, so we set $\mathbf{e} = \mathbf{u}/\|\mathbf{u}\| = (2\mathbf{i} - \mathbf{j})/\sqrt{5}$ and compute the rate of change as

$$\partial_{\mathbf{e}} T(1, 1) = \nabla T(1, 1) \cdot \mathbf{e} = \left(\frac{\mathbf{i}}{9} + \frac{\mathbf{j}}{9}\right) \cdot \frac{2\mathbf{i} - \mathbf{j}}{\sqrt{5}} = \frac{1}{9\sqrt{5}}.$$

(b) The temperature drops most rapidly in the direction of of $-\nabla T(1, 1)$. Using part (a), we compute a unit vector $\mathbf{e}$ in that direction as

$$-\nabla T(1, 1) = -\frac{1}{9}\mathbf{i} - \frac{1}{9}\mathbf{j}, \quad \| - \nabla T(1, 1)\| = \frac{\sqrt{2}}{9},$$

so

$$\mathbf{e} = \frac{-\nabla T(1, 1)}{\| - \nabla T(1, 1)\|} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

Example 9.16 Let the temperature at each point of a metal plate be given by the function

$$T(x, y) = e^x \cos y + e^y \cos x.$$

In what direction does the temperature increase most rapidly at the point (0,0)? What is the rate of increase?
(b) In what direction does the temperature decrease most rapidly at (0,0)?

Solution: We first compute

$$\begin{aligned}\nabla T(x, y) &= \frac{\partial T}{\partial x}(x, y)\mathbf{i} + \frac{\partial T}{\partial y}(x, y)\mathbf{j} \\ &= (e^x \cos y - e^y \sin x)\mathbf{i} + (e^y \cos x - e^x \sin y)\mathbf{j}.\end{aligned}$$

(a) At (0, 0) the temperature increases most rapidly in the direction of the gradient $\nabla T(0,0) = \mathbf{i} + \mathbf{j}$. The rate of increase is obtained as $\|\nabla T(0,0)\| = \|\mathbf{i} + \mathbf{j}\| = \sqrt{2}$.
(b) The temperature decreases most rapidly in the direction of $-\nabla T(0,0) = -\mathbf{i} - \mathbf{j}$.
We repeat that the gradient vector $\nabla f(\mathbf{x})$ tells us the direction of the steepest climb of f at the point $\mathbf{x}$, and its length, $\|\nabla f(\mathbf{x}\|$, gives the steepness.

Polar coordinates and the gradient. Let a function "$\psi = \psi(r,\theta)$" be defined on the plane with arguments given in polar coordinates. We want to determine its gradient in Cartesian coordinates (x, y). One way to do this is to first transform ψ explicitly to Cartesian coordinates, and then compute the gradient as above. Usually, however, it is easier to compute the partial derivatives w.r.t. Cartesian coordinates directly from the partial derivatives w.r.t. polar coordinates, as we now explain. In order to clearly understand what is going on, we use the symbol $\tilde{\psi}$ to denote the function in Cartesian coordinates, "$\tilde{\psi} = \tilde{\psi}(x, y)$". We emphasize that ψ and $\tilde{\psi}$ are different functions from the standpoint of the general definition of a function as presented in Chap. 1, since the rule by which we associate the function value to (x, y) will be different from the rule applied to (r,θ). The fact that ψ and $\tilde{\psi}$ only differ through a change of coordinates is expressed by the equation

$$\psi(r,\theta) = \tilde{\psi}(r\cos\theta, r\sin\theta)\,. \tag{9.57}$$

Our task is to determine $\nabla\tilde{\psi}$. From (9.57) and from the chain rule (see Sect. 8.4), we obtain the relation between the partial derivatives of ψ and $\tilde{\psi}$ as

$$\frac{\partial\psi}{\partial r}(r,\theta) = \cos\theta\cdot\frac{\partial\tilde{\psi}}{\partial x}(r\cos\theta, r\sin\theta) + \sin\theta\cdot\frac{\partial\tilde{\psi}}{\partial y}(r\cos\theta, r\sin\theta). \tag{9.58}$$

$$\frac{\partial\psi}{\partial\theta}(r,\theta) = -r\sin\theta\cdot\frac{\partial\tilde{\psi}}{\partial x}(r\cos\theta, r\sin\theta) + r\cos\theta\cdot\frac{\partial\tilde{\psi}}{\partial y}(r\cos\theta, r\sin\theta)\,. \tag{9.59}$$

For any given point (r,θ), this is a system of two linear equations for the two unknowns $\partial\tilde{\psi}/\partial x$ and $\partial\tilde{\psi}/\partial y$ at the corresponding point $(x, y) = (r\cos\theta, r\sin\theta)$. Solving this, we obtain

$$\frac{\partial\tilde{\psi}}{\partial x}(x,y) = \cos\theta\frac{\partial\psi}{\partial r}(r,\theta) - \frac{1}{r}\sin\theta\frac{\partial\psi}{\partial\theta}(r,\theta) \tag{9.60}$$

$$\frac{\partial\tilde{\psi}}{\partial y}(x,y) = \sin\theta\frac{\partial\psi}{\partial r}(r,\theta) + \frac{1}{r}\cos\theta\frac{\partial\psi}{\partial\theta}(r,\theta)\,. \tag{9.61}$$

(We obtain (9.60) if we multiply (9.58) by $\cos\theta$, (9.59) by $-(1/r)\sin\theta$ and add the resulting equations, and similarly for (9.61).) This is the result we were looking for. In shortened symbolic form (and no longer distinguishing between ψ and $\tilde{\psi}$) it reads

$$\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) = \left(\cos\theta\frac{\partial\psi}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial\psi}{\partial\theta}, \sin\theta\frac{\partial\psi}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial\psi}{\partial\theta}\right). \tag{9.62}$$

Example 9.17 Let $\psi(r,\theta) = 1/r$. Compute $\nabla\psi$ at $(x, y) = (r\cos\theta, r\sin\theta)$.

Solution: Since ψ does not depend on θ, we have $\partial\psi/\partial\theta = 0$, and (9.62) becomes

$$\nabla\psi(x, y) = \left(-\frac{1}{r^2}\cos\theta, -\frac{1}{r^2}\sin\theta\right) = -\frac{1}{r^3}(x, y).$$

Sometimes, a notation yet shorter than (9.62) is in use. Let us define vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ by

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}. \tag{9.63}$$

Then (9.62) becomes

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta. \tag{9.64}$$

Here, again, the right-hand side is evaluated at (r, θ) and the left-hand side is evaluated at $(x, y) = (r\cos\theta, r\sin\theta)$.

Divergence and rotation. Here, we present the definition and some properties of the differential expressions termed "divergence" and "curl" (or "rotation").

Definition 9.12 Let $\mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ be a vector field with components f_1, f_2, f_3 defined on some open subset D of $\mathbb{R}^3$ which we assume to possess partial derivatives.

(a) The divergence of $\mathbf{F}$, denoted by $\operatorname{div}\mathbf{F}$ or $\nabla\cdot\mathbf{F}$ (read as nabla dot $\mathbf{F}$), is defined as

$$\operatorname{div}\mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}. \tag{9.65}$$

It is a scalar field with domain D.

(b) The curl of $\mathbf{F}$, denoted by $\mathbf{curl}\,\mathbf{F}$ or $\nabla\times\mathbf{F}$ (read as nabla cross $\mathbf{F}$), is defined as

$$\mathbf{curl}\,\mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}. \tag{9.66}$$

It is a vector field with domain D. In analogy to the definition of a 3×3-determinant, the curl can be expressed symbolically as

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}. \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix},$$

a traditional way to memorize the definition of the curl. Alternatively, the notion "rotation" is used instead of "curl", and **rot F** is used instead of **curl F**.

In contrast to the gradient, the meaning of divergence and curl cannot be explained from this definition in a direct and intuitive manner. The notion of divergence is best understood in context with the divergence theorem of Gauss (Theorem 9.9) and will be explained in Sect. 9.5.2. Likewise, the theorem of Stokes (Theorem 9.12) helps to clarify the meaning of the curl, see Sect. 9.5.3. Moreover, in Example 9.50 we will see that for a rotating body, angular velocity is proportional to the curl of the tangential velocity.

Example 9.18 Compute $\nabla \cdot \mathbf{F}$ (the divergence) and $\nabla \times \mathbf{F}$ (the curl) of the vector field

$$\mathbf{F}(x, y, z) = 2xy\mathbf{i} + xe^y\mathbf{j} + 2z\mathbf{k}\,.$$

Solution: The components of $\mathbf{F}$ are $f_1(x, y, z) = 2xy$, $f_2 = xe^y$, $f_3 = 2z$. Therefore

$$(\operatorname{div}\mathbf{F})(x, y, z) = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)(x, y, z) = 2y + xe^y + 2\,.$$

To evaluate

$$\mathbf{curl\,F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}\,,$$

we compute

$$\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)(x, y, z) = 0 - 0 = 0\,, \quad \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)(x, y, z) = 0 - 0 = 0\,,$$
$$\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)(x, y, z) = e^y - 2x\,,$$

and obtain

$$\mathbf{curl\,F}(x, y, z) = 0\mathbf{i} + 0\mathbf{j} + (e^y - 2x)\mathbf{k} = (e^y - 2x)\mathbf{k}\,.$$

Example 9.19 (i) Let φ be a scalar field which possesses continuous first and second partial derivatives. Show that

$$\nabla \times (\nabla\varphi) = 0 \quad \text{or} \quad \mathbf{curl}(\nabla\varphi) = 0\,,$$

that is, the curl of the gradient of φ is identically equal to the zero vector field.
(ii) Let $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ be a vector field which has continuous first and second partial derivatives. Show that

$$\operatorname{div}\mathbf{curl\,F} = 0 \quad \text{or} \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0\,.$$

This means that the divergence of the curl of $\mathbf{F}$ is identically equal to the zero scalar field.

Solution: (i) In order to compute the curl of $\nabla\varphi$, we apply (9.66) to the vector field $\mathbf{F} = \nabla\varphi$, that is, $f_1 = \partial\varphi/\partial x$, $f_2 = \partial\varphi/\partial y$, $f_3 = \partial\varphi/\partial z$. Inserting those expressions into (9.66), we obtain

$$\nabla \times (\nabla\varphi) = \left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right)\mathbf{i} + \left(\frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z}\right)\mathbf{j} + \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right)\mathbf{k}\,.$$

We know from Remark 8.4 that we can interchange the sequence of partial derivatives if the second partial derivatives are continuous. Therefore

$$\frac{\partial^2\varphi}{\partial y\partial z} = \frac{\partial^2\varphi}{\partial z\partial y}\,, \quad \frac{\partial^2\varphi}{\partial z\partial x} = \frac{\partial^2\varphi}{\partial x\partial z}\,, \quad \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x}\,,$$

so all components of the vector field $\nabla \times (\nabla\varphi)$ are zero, and thus $\nabla \times (\nabla\varphi)$ is identically equal to the zero vector field.
(ii) By the definition

$$\nabla \times \mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}\,.$$

Forming the divergence on both sides, we get

$$\begin{aligned}
\operatorname{div}(\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}\left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right] \\
&= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} + \frac{\partial^2 f_1}{\partial y\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y} \\
&= \left[\frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_3}{\partial y\partial x}\right] + \left[\frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_2}{\partial x\partial z}\right] + \left[\frac{\partial^2 f_1}{\partial y\partial z} - \frac{\partial^2 f_1}{\partial z\partial y}\right] \\
&= 0 + 0 + 0 = 0\,,
\end{aligned}$$

because, for the same reason as in (i),

$$\frac{\partial^2 f_3}{\partial x\partial y} = \frac{\partial^2 f_3}{\partial y\partial x}\,, \quad \frac{\partial^2 f_2}{\partial z\partial x} = \frac{\partial^2 f_2}{\partial x\partial z}\,, \quad \frac{\partial^2 f_1}{\partial y\partial z} = \frac{\partial^2 f_1}{\partial z\partial y}\,.$$

Example 9.20 Consider the vector field $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (the identity mapping on $\mathbb{R}^3$). We let $r = \|\mathbf{r}\|$, that is, we also consider the scalar field r given by $r(x, y, z) = \|\mathbf{r}(x, y, z)\|$.
(a) Let n be any integer (positive or negative). Prove that $\nabla(r^n) = nr^{n-2}\mathbf{r}$.
(b) Let φ be a real-valued function of one variable. Prove that $\mathbf{curl}(\varphi(r)\mathbf{r}) = 0$.

Solution: (a) We have $r(x, y, z) = \|\mathbf{r}(x, y, z)\| = (x^2 + y^2 + z^2)^{1/2}$, so $r^n(x, y, z) = (x^2 + y^2 + z^2)^{n/2}$. This gives

$$\frac{\partial(r^n)}{\partial x}(x, y, z) = \frac{n}{2}(x^2 + y^2 + z^2)^{(\frac{n}{2}-1)} \cdot 2x = nxr^{n-2}(x, y, z)\,,$$

and similarly

$$\frac{\partial(r^n)}{\partial y} = nyr^{n-2}\,, \quad \frac{\partial(r^n)}{\partial z} = nzr^{n-2}\,.$$

Thus, in abbreviated notation,

$$\begin{aligned}\nabla r^n &= \frac{\partial r^n}{\partial x}\mathbf{i} + \frac{\partial r^n}{\partial y}\mathbf{j} + \frac{\partial r^n}{\partial z}\mathbf{k} = nxr^{n-2}\mathbf{i} + nyr^{n-2}\mathbf{j} + nzr^{n-2}\mathbf{k} \\ &= nr^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nr^{n-2}\mathbf{r}\,.\end{aligned}$$

(b) Recall the formula for the curl of a vector field $\mathbf{F}$,

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}\,.$$

In the present case (again we use abbreviated notation) $\mathbf{F} = \varphi(r)\mathbf{r} = \varphi(r)x\mathbf{i} + \varphi(r)y\mathbf{j} + \varphi(r)z\mathbf{k}$, that is,

$$f_1 = \varphi(r)x\,, \quad f_2 = \varphi(r)y\,, \quad f_3 = \varphi(r)z\,,$$

and $r(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$. Hence, using the chain rule,

$$\begin{aligned}\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} &= z\varphi'(r)\frac{\partial r}{\partial y} - y\varphi'(r)\frac{\partial r}{\partial z} \\ &= \varphi'(r)[zy(x^2 + y^2 + z^2)^{-1/2} - yz(x^2 + y^2 + z^2)^{-1/2}] = 0\,.\end{aligned}$$

Similarly, we obtain

$$\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} = 0\,, \quad \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0\,.$$

This proves that $\mathbf{curl}(\varphi(r)\mathbf{r}) = 0$. Finally, we note that divergence and curl satisfy the property of linearity, that is,

$$\begin{aligned}\operatorname{div}(\mathbf{F} + \mathbf{G}) &= \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}\,, \quad \operatorname{div}(\alpha\mathbf{F}) = \alpha\operatorname{div}\mathbf{G}\,, \\ \mathbf{curl}(\mathbf{F} + \mathbf{G}) &= \mathbf{curl}\,\mathbf{F} + \mathbf{curl}\,\mathbf{G}\,, \quad \mathbf{curl}(\alpha\mathbf{F}) = \alpha\,\mathbf{curl}\,\mathbf{G}\,,\end{aligned}$$

hold for all vector fields $\mathbf{F}$ and $\mathbf{G}$ and all scalars α.

9.3.3 Surfaces

Many surfaces Σ in the space $\mathbb{R}^3$ can be described parametrically by vector fields defined on a domain D in the plane $\mathbb{R}^2$ as

$$\Sigma = \mathbf{r}(D) = \{r(u, v) : (u, v) \in D\}\,. \tag{9.67}$$

Example 9.21
(a) Let $\mathbf{x}$, $\mathbf{y}$ be vectors in $\mathbb{R}^3$. Describe the parallelogram with corners $\mathbf{0}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} + \mathbf{y}$ as a parametric surface Σ.
(b) Consider a vertical cylinder of height H whose bottom is formed by a circle of radius R and midpoint 0 in the xy-plane. Describe its side as a parametric surface Σ.

Solution: (a) We set $D = [0, 1] \times [0, 1]$ and $\mathbf{r}(u, v) = u\mathbf{x} + v\mathbf{y}$. Then $\Sigma = \mathbf{r}(D)$ yields the parallelogram.
(b) We set $D = [0, 2\pi] \times [0, H]$ and define

$$\mathbf{r}(u, v) = (R\cos u, R\sin u, v) = R\cos u\mathbf{i} + R\sin u\mathbf{j} + v\mathbf{k}\,.$$

Then $\Sigma = \mathbf{r}(D)$ yields the side of the cylinder. Often, a surface in xyz-space is given as a graph of a function S defined on a subset D of the xy-plane. In that case, we may identify $u = x$ and $v = y$ and write its parametric representation as

$$\mathbf{r}(x, y) = (x, y, S(x, y)) = x\mathbf{i} + y\mathbf{j} + S(x, y)\mathbf{k}\,, \quad (x, y) \in D\,. \tag{9.68}$$

We have already seen many examples of this type in Sect. 8.1.

Let us consider a surface Σ given in the parametric form (9.67), let $P_0 = \mathbf{r}(u_0, v_0)$ be a point on Σ. Varying u_0 with v_0 fixed, we obtain a curve $\mathbf{c}(u) = \mathbf{r}(u, v_0)$ through the point P_0 which lies completely within the surface Σ, since all of its points are also points of Σ. According to Sect. 9.3.1, the vector

$$\mathbf{c}'(u_0) = \lim_{h\to 0} \frac{\mathbf{c}(u_0 + h) - \mathbf{c}(u_0)}{h}$$

(if nonzero) is tangent to C in the point P_0. Instead of $\mathbf{c}'(u_0)$ we write $\partial_u\mathbf{r}(u_0, v_0)$; this is natural since we then have

$$\partial_u\mathbf{r}(u_0, v_0) = \lim_{h\to 0} \frac{\mathbf{r}(u_0 + h, v_0) - \mathbf{r}(u_0, v)}{h}\,. \tag{9.69}$$

Analogously, if we fix u_0 and vary v_0, we obtain another curve through P_0 within Σ with

$$\partial_v\mathbf{r}(u_0, v_0) = \lim_{h\to 0} \frac{\mathbf{r}(u_0, v_0 + h) - \mathbf{r}(u_0, v)}{h} \tag{9.70}$$

as a tangent vector. The plane through P_0 spanned by the two vectors $\partial_u \mathbf{r}(u_0, v_0)$ and $\partial_v \mathbf{r}(u_0, v_0)$ is called the **tangent plane** for Σ at P_0, or the plane tangent to Σ at P_0. In order that all of this works, we assume that the parametrization $\mathbf{r}$ is regular in the sense of the following definition.

Definition 9.13 A parametrization $\mathbf{r} : D \to \Sigma$ is called **regular**, if it is continuously differentiable on the interior of D, and if the vectors $\partial_u \mathbf{r}(u, v)$ and $\partial_v \mathbf{r}(u, v)$ are not parallel at all interior points (u, v) of D. A surface Σ is called **smooth** if it has a regular parametrization.

Example 9.22 Consider the vertical cylinder with height H and a circular base of radius R, see Example 9.21(b) above. Find the plane tangent to the side surface at the point $P_0 = ((1/\sqrt{2})R, (1/\sqrt{2})R, 1)$.

Solution: The parametrization $\mathbf{r}(u, v) = R\cos u\mathbf{i} + R\sin u\mathbf{j} + v\mathbf{k}$ yields $P_0 = \mathbf{r}(u_0, v_0)$ with $u_0 = v_0 = \pi/4$, as well as

$$\partial_u \mathbf{r}(u, v) = -R\sin u\mathbf{i} + R\cos u\mathbf{j}\,, \quad \partial_v \mathbf{r}(u, v) = \mathbf{k}\,.$$

The tangent plane for Σ at P_0 is, therefore, spanned by the vectors

$$\partial_u \mathbf{r}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\,, \quad \partial_v \mathbf{r}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \mathbf{k}\,.$$

Let P_0 be a point on a surface Σ. Any vector which is perpendicular to the tangent plane for Σ at P_0 is called a **normal vector** (or simply a **normal**) for Σ at P_0, or a vector normal to Σ at P_0. Note that any scalar multiple of a normal vector is again a normal vector. A normal vector of length 1 is called a **unit normal**. Unit normals are commonly denoted by $\mathbf{n}$.

Since the vector product $\mathbf{x} \times \mathbf{y}$ of two vectors is perpendicular to both vectors $\mathbf{x}$ and $\mathbf{y}$, we see that the vector

$$\mathbf{N}(P_0) = \partial_u \mathbf{r}(u_0, v_0) \times \partial_v \mathbf{r}(u_0, v_0)$$

is a normal vector for Σ at $P_0 = \mathbf{r}(u_0, v_0)$.

Let us return to the special situation where Σ is given as the graph of a (continuously differentiable) function S,

$$\mathbf{r}(x, y) = (x, y, S(x, y)) = x\mathbf{i} + y\mathbf{j} + S(x, y)\mathbf{k}\,, \quad (x, y) \in D\,. \tag{9.71}$$

In this case, the vectors

$$\begin{aligned} \partial_x \mathbf{r}(x_0, y_0) &= \mathbf{i} + \partial_x S(x_0, y_0)\mathbf{k}\,, \\ \partial_y \mathbf{r}(x_0, y_0) &= \mathbf{j} + \partial_y S(x_0, y_0)\mathbf{k}\,, \end{aligned} \tag{9.72}$$

span the tangent plane at $P_0 = (x_0, y_0, S(x_0, y_0))$. Note that the parametrization (9.71) is regular, since the vectors in (9.72) are not parallel. The vector

$$\mathbf{N}(P_0) = \partial_x \mathbf{r}(x_0, y_0) \times \partial_y \mathbf{r}(x_0, y_0) = -\partial_x S(x_0, y_0)\mathbf{i} - \partial_y S(x_0, y_0)\mathbf{j} + \mathbf{k} \quad (9.73)$$

is normal to Σ at P_0. A corresponding unit normal is given by

$$\mathbf{n} = \frac{1}{\nu}\left(\partial_x \mathbf{r}(x_0, y_0) \times \partial_y \mathbf{r}(x_0, y_0) = -\partial_x S(x_0, y_0)\mathbf{i} - \partial_y S(x_0, y_0)\mathbf{j} + \mathbf{k}\right), \quad (9.74)$$

where

$$\nu = \sqrt{1 + (\partial_x S(x_0, y_0))^2 + (\partial_y S(x_0, y_0))^2}\,.$$

As we know from geometry, any plane in the space $\mathbb{R}^3$ can be described by an equation involving a vector perpendicular to it. In our case, this means that the tangent plane to Σ at P_0 is equal to the set of all points (x, y, z) which satisfy the equation

$$-\partial_x S(x_0, y_0) \cdot (x - x_0) - \partial_y S(x_0, y_0) \cdot (y - y_0) + (z - z_0) = 0\,, \quad (9.75)$$

or alternatively

$$z - z_0 = \partial_x S(x_0, y_0) \cdot (x - x_0) + \partial_y S(x_0, y_0) \cdot (y - y_0)\,. \quad (9.76)$$

As an alternative to the parametric description (9.67), we can describe a surface as a level set in the form

$$\Sigma = \{(x, y, z) : f(x, y, z) = c\}\,, \quad (9.77)$$

where f is a scalar vector field and c a constant. For example,

$$\Sigma = \{(x, y, z) : x^2 + y^2 + z^2 = c\}\,, \quad c > 0\,,$$

yields the sphere centered at the origin of radius $\sqrt{c}$. Let now $\mathbf{q}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be any curve which lies in Σ, that is,

$$f(\mathbf{q}(t)) = f(x(t), y(t), z(t)) = c\,. \quad (9.78)$$

We differentiate both sides with respect to t and obtain from the chain rule that

$$\partial_x f(\mathbf{q}(t))x'(t) + \partial_y f(\mathbf{q}(t))y'(t) + \partial_z f(\mathbf{q}(t))z'(t) = \nabla f(\mathbf{q}(t)) \cdot \mathbf{q}'(t) = 0\,. \quad (9.79)$$

Since the vector $\mathbf{q}'(t)$ is tangent to the surface Σ at the point $\mathbf{q}(t)$, (9.79) means that the gradient of f at some point of the surface is perpendicular to any tangent vector in that point, that is, $\nabla f(x_0, y_0, z_0)$ is normal to Σ at $P_0 = (x_0, y_0, y_0)$ whenever P_0 lies in Σ. This is consistent with the fact that ∇f, if nonzero, points in the direction of the steepest increase (and $-\nabla f$ in the direction of steepest decrease) of f, since the tangent vectors point in directions where f, to first order, is constant.

9.4 Integration in Vector Fields

In this section, we discuss integrals of vector fields over curves and surfaces. Since they describe important aspects of processes taking place in space and time, they constitute another basic tool for solving problems of science and engineering, see also Sect. 9.6. There are two subsections of this section dealing, respectively, with line integrals and surface integrals.

9.4.1 Line Integrals

Suppose a curve C in $\mathbb{R}^3$ is given by the parametric equations

$$x = x(t)\,, \quad y = y(t)\,, \quad z = z(t)\,, \quad \text{for} a \le t \le b. \tag{9.80}$$

The resulting vector function $\mathbf{r} : [a, b] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\,, \quad t \in [a, b]\,. \tag{9.81}$$

is called a **parametrization** of the curve C.

We will think of C not only as a geometric locus of points $(x(t), y(t), z(t))$ but also as having a direction or an **orientation** induced by its parametrization. The point $\mathbf{r}(a) = (x(a), y(a), z(a))$ is called the **initial point**, while $\mathbf{r}(b) = (x(b), y(b), z(b))$ is called the **terminal** (or **end**) **point**. A curve is called **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$, that is, if initial and terminal points coincide. In Fig. 9.8a and b we display curves as the range of $\mathbf{r}$, that is, as the set of all points $\mathbf{r}(t) = (x(t), y(t), z(t))$ as t varies over the parameter interval.

A curve C is called **continuous** if its parametrization $\mathbf{r}$ is a continuous function (or equivalently, if its components in (9.80) are continuous functions) on the parameter interval $[a, b]$. C is called **differentiable** if $\mathbf{r}$ is differentiable, and C is called **smooth** if the derivative $\mathbf{r}'$ is a continuous function and the vectors $\mathbf{r}'(t)$ are not zero for all t. A continuous curve C is called **piecewise smooth**, if the parameter interval $[a, b]$ admits a partition $a = t_0 < \ldots t_n = b$ such that C is smooth on every interval $[t_{j-1}, t_j]$ of this partition.

Line Integrals of Scalar Fields. Let us first consider a straight rod of length L which we think of as a one-dimensional object with mass density ρ. If L is measured in centimeters and ρ in grams per centimeter, its total mass will be ρL grams. Now let us consider a curved rod as a curve C in space, whose points are given by the parametrization $\mathbf{r}(t) = (x(t), y(t), z(t))$ for $t \in [a, b]$, and assume that its mass density varies along the rod as a function of (x, y, z). Then the total mass of the rod is given by the integral

$$\int_a^b \rho(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, dt\,. \tag{9.82}$$

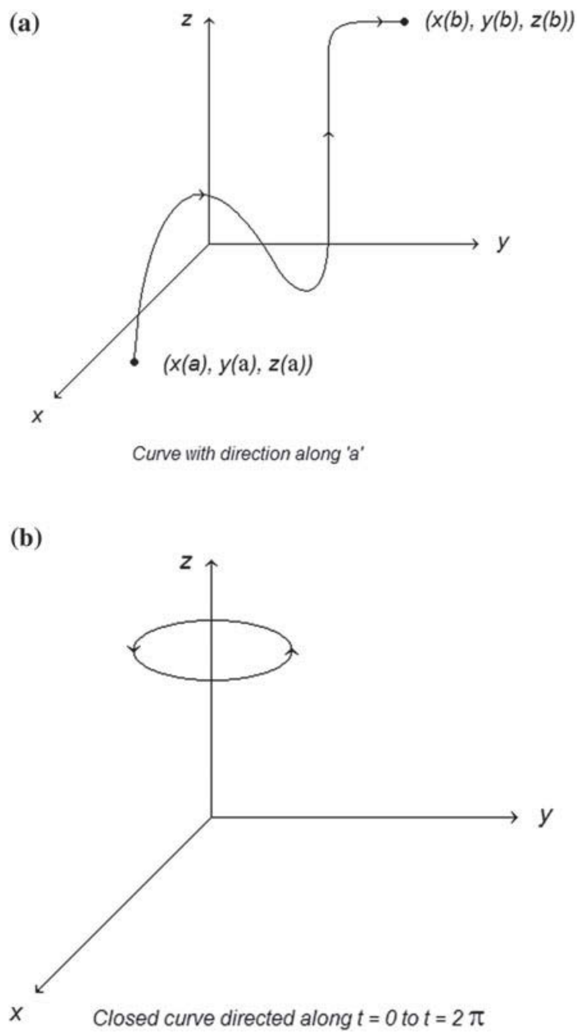


Fig. 9.8 **a** Curve between initial and terminal point, **b** closed curve

Indeed, a limit passage like the one explained in Sect. 7.2 (based on an approximation of the rod consisting of straight pieces) shows that (9.82) is the correct formula to compute the total mass of the rod. This motivates the following definition.

Definition 9.14 (*Line Integral of a Scalar Field*) Let C be a smooth curve given by (9.81), let f be a continuous scalar field whose domain contains the curve C. Then the line integral of f over C is denoted by $\int_C f\, ds$ and defined by

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, dt\,. \tag{9.83}$$

Example 9.23 Evaluate the line integral

$$\int_C (x+y)\,ds\,,$$

where C is given by $x(t) = y(t) = t$, $z(t) = t^2$ for $0 \le t \le 2$.

Solution: Since $\mathbf{r}(t) = (x(t), y(t), z(t)) = (t, t, t^2)$, we get

$$\begin{aligned} \mathbf{r}'(t) &= (x'(t), y'(t), z'(t)) = (1, 1, 2t)\,, \\ \|\mathbf{r}'(t)\| &= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{1+1+(2t)^2} = \sqrt{2+4t^2}\,. \end{aligned}$$

For $f(x, y, z) = x + y$ we get $f(x(t), y(t), z(t)) = f(t, t, t^2) = t + t = 2t$, so

$$\int_C (x+y)\,ds = \int_0^2 2t\sqrt{2+4t^2}\,dt = \frac{1}{6}(2+4t^2)^{3/2}\Big|_0^2 = \frac{26\sqrt{2}}{3}\,.$$

In the special case $f = 1$, the line integral (9.83) equals the length of the curve C. Therefore, the function

$$s(t) = \int_a^t \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\,d\tau = \int_0^t \|\mathbf{r}'(\tau)\|\,d\tau \tag{9.84}$$

yields the length of C from the initial point $\mathbf{r}(a)$ up to the point $\mathbf{r}(t)$, it is called the **arc length** of the curve C; compare (7.7) for the corresponding situation in two dimensions. From (9.84) we see that $s'(t) = \|\mathbf{r}'(t)\|$. This explains the notation $\int_C f\,ds$ in (9.83), where "ds" stands for "$s'(t)\,dt$" or "$\|\mathbf{r}'(t)\|\,dt$". Accordingly, "ds" is termed the **line element**.

Line Integrals of Vector Fields. We consider a vector field $\mathbf{F}$ with components f_1, f_2, and f_3,

$$\mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}\,, \tag{9.85}$$

whose domain contains the curve C, that is, the points $\mathbf{r}(t)$ belong to the domain of $\mathbf{F}$ for all $t \in [a, b]$.

Definition 9.15 (*Line Integral of a Vector Field*) Let C be a smooth curve given by (9.81), let $\mathbf{F}$ be a continuous vector field of the form (9.85) whose domain contains the curve C. Then the line integral of $\mathbf{F}$ over C is denoted by $\int_C \mathbf{F}\cdot d\mathbf{r}$ and defined by

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt\,. \tag{9.86}$$

When C is closed, one also writes

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

instead of $\int_C \mathbf{F} \cdot d\mathbf{r}$. In this case, the value of the line integral is also called the **circulation** of $\mathbf{F}$ around C.

We give some explanations concerning the integral on the right-hand side of (9.86). The integrand is a real-valued function of the single variable t, obtained as the scalar product of the vector $\mathbf{F}(\mathbf{r}(t))$ (the value of the vector field at the curve point $\mathbf{r}(t)$) and the vector $\mathbf{r}'(t)$ (a vector tangent to the curve at that point). If we expand the scalar product into components according to its definition, we obtain, since $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b f_1(\mathbf{r}(t))x'(t) + f_2(\mathbf{r}(t))y'(t) + f_3(\mathbf{r}(t))z'(t)\,dt \\ &= \int_a^b f_1(x(t), y(t), z(t))x'(t) + f_2(x(t), y(t), z(t))y'(t) \\ &\qquad + f_3(x(t), y(t), z(t))z'(t)\,dt\,.\end{aligned} \tag{9.87}$$

One may ask why the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is defined in this way. In fact, it expresses certain quantities which are important in applications, for example, the mechanical work done by a force field $\mathbf{F}$, moving a point mass along the curve C from the initial to the terminal point. This will elaborated in more detail in Sect. 9.6.2.

Example 9.24 Let the curve C be given by $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + t\mathbf{k}$ on the interval $0 \le t \le \pi$. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $F(x, y, z) = y\mathbf{i} - x\mathbf{j} + 2\mathbf{k}$.

Solution: We use formula (9.86), or (9.87). In order to form $\mathbf{F}(\mathbf{r}(t))$, we have to insert $\cos t$ for x, $-\sin t$ for y and t for z. This gives $\mathbf{F}(\mathbf{r}(t)) = (-\sin t, -\cos t, 2)$. Moreover, $\mathbf{r}'(t) = (-\sin t, -\cos t, 1)$. Computing the scalar product of those two vectors and integrating, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^\pi \sin^2 t + \cos^2 t + 2\,dt = \pi + 2\pi = 3\pi\,.$$

The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is also frequently denoted as

$$\int_C f_1(x, y, z)\,dx + f_2(x, y, z)\,dy + f_3(x, y, z)\,dz\,, \tag{9.88}$$

or shorter as

$$\int_C f_1\,dx + f_2\,dy + f_3\,dz\,.$$

The definition of those expressions is, of course, the same as above in (9.86) or (9.87). If we formally replace "x" by "$x(t)$" and "dx" by "$x'(t)\,dt$" (and analogously for y and z) in (9.88), we arrive at (9.87).

Example 9.25 Evaluate $\int_C xy\,dx - y\cos x\,dy$, where C is given by $x(t) = t^2$ and $y(t) = t$ on the interval $-2 \le t \le 5$.

Solution: We have $x'(t) = 2t$ and $y'(t) = 1$, therefore

$$\begin{aligned}
\int_C xydx - y\cos xdy &= \int_{-2}^{5} \left[xyx' - y(\cos x)y'\right](t)\,dt \\
&= \int_{-2}^{5} t^2 t 2t - t\cos t^2\,dt = \left[2\frac{t^5}{5}\right]_{-2}^{5} - \frac{1}{2}\left[\sin t^2\right]_{-2}^{5} \\
&= 2\left(625 + \frac{32}{5}\right) - \frac{1}{2}(\sin 25 - \sin 4) \\
&= \frac{6314}{5} - \frac{1}{2}(\sin 25 - \sin 4)\,.
\end{aligned}$$

Example 9.26 Find the circulation of the field $\mathbf{F}(x, y) = (-y + x)\mathbf{i} + x\mathbf{j}$ around the circle $x^2 + y^2 = 1$.

Solution: The parametric equation of the given circle is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}\,, \quad 0 \le t \le 2\pi\,.$$

We have

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) = (-\sin t + \cos t)\mathbf{i} + \cos t\mathbf{j}\,, \quad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}\,, \\
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (\sin^2 t - \sin t\cos t) + \cos^2 t = 1 - \sin t\cos t\,.
\end{aligned}$$

Therefore, the circulation of $\mathbf{F}$ around the circle (denoted as C) becomes

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} (1 - \sin t\cos t)\,dt = \left[t - \frac{\sin^2 t}{2}\right]_0^{2\pi} \\
= 2\pi\,.
\end{aligned}$$

The result of the following example will be used later in the proof of the theorem of Green and Ostrogradski.

Example 9.27 Let C be the graph of a function $y = h(x)$, where $h : [a, b] \to \mathbb{R}$ is differentiable, let $\mathbf{F}$ be a vector field whose second component is zero, that is, $\mathbf{F}(x, y) = (f(x, y), 0)$ for some continuous real-valued function f. Show that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b f(x, h(x))\,dx\,. \tag{9.89}$$

Solution: We parametrize the curve C as $\mathbf{r}(t) = (t, h(t))$. Then $\mathbf{r}'(t) = (1, h'(t))$ and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b f(\mathbf{r}(t)) \cdot 1 + 0 \cdot h'(t)\, dt = \int_a^b f(t, h(t))\, dt\,,$$

so (9.89) is proved. (Recall that it does not play any role whether the integration variable is denoted by "t" or by "x".) The following properties hold for the line integral. Let $\mathbf{F}$ and $\mathbf{G}$ be continuous vector fields defined on some region in space containing a curve C, which is given as the range of some vector function $\mathbf{r}$ as above. Then

$$\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}\,. \tag{9.90}$$

$$\int_C \alpha\mathbf{F} \cdot d\mathbf{r} = \alpha \int_C \mathbf{F} \cdot d\mathbf{r}\,, \quad \alpha\text{being any scalar.} \tag{9.91}$$

This means that the line integral is linear with respect to the vector fields, that is, the line integral of the sum of two vector fields (respectively, the scalar multiple of a vector field) is equal to the sum of the line integrals (respectively, the scalar multiple of the line integral). These formulas are a direct consequence of the definition of the line integral, using the corresponding properties of the ordinary integral.

For piecewise smooth curves, the line integral is evaluated on each piece separately, as stated in the following definition.

Definition 9.16 (*Line Integral, Piecewise Smooth Curves*) Let C be a piecewise smooth curve given by (9.81), let $\mathbf{F}$ be a continuous vector field of the form (9.85) whose domain contains the curve C. Then the line integral of $\mathbf{F}$ over C is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt\,, \tag{9.92}$$

where $a = t_0 < \cdots < t_n = b$ is a partition of $[a, b]$ such that C is smooth on every interval $[t_{i-1}, t_i]$.

Path Independence, Potential Functions, and Conservative Fields

Definition 9.17 Let $\mathbf{F}$ be a vector field defined on an open region D in $\mathbb{R}^3$. If $\mathbf{F} = \nabla\psi$ for some scalar function ψ defined on D, then ψ is called a **potential function** (or simply **potential**) for $\mathbf{F}$ in D, and the vector field $\mathbf{F}$ is called **conservative**.

It will be explained in Remark 9.10 why the term "conservative" is used here.

Theorem 9.4 *Let* $\mathbf{F}$ *be a continuous vector field defined on an open region D in* $\mathbb{R}^3$ *which possesses a potential function ψ in D. Let C be a smooth curve in D with initial point* $\mathbf{A}$ *and terminal point* $\mathbf{B}$. *Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \psi(\mathbf{B}) - \psi(\mathbf{A})\,. \tag{9.93}$$

The proof will be given in Appendix D.9.

Let us take a look at formula (9.93). Given the potential ψ, the value of the right-hand side depends only on the points **A** and **B**. This means in particular that the value of the line integral on the left-hand side does not change if we replace C by a different curve $\tilde{C}$, as long as $\tilde{C}$ has the same initial and terminal point as C. This property is called **path independence**. Theorem 9.4 thus states that for conservative vector fields, line integrals are path independent.

We present another related definition.

Definition 9.18 A continuous vector field **F** defined on an open region D in $\mathbb{R}^3$ is called **circulation free on *D*** if we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \tag{9.94}$$

for every closed curve C in D.

When the curve C is closed, its initial and terminal points coincide, and the right-hand side of (9.93) becomes zero. Therefore, a conservative field is circulation free by Theorem 9.4. Actually, the converse is true, too, so the following theorem holds.

Theorem 9.5 (Closed-Loop Property of Conservative Fields) *Let* **F** *be a continuous vector field defined on an open region D in $\mathbb{R}^3$. Assume that D is connected, that is, any two points in C can be connected by a smooth curve. The following statements are equivalent:*

1. **F** *is conservative on D.*
2. **F** *is circulation free in D.*

The proof will be given in Appendix D.9.

Another way of stating Theorem 9.5 would be to say that a vector field **F** possesses a potential ψ in D if and only if the line integral over any closed curve in D is zero.

Example 9.28 Let $\psi(x, y, z) = xyz$, let C be a smooth curve with initial point $(-1, 6, 18)$ and terminal point $(2, 12, -8)$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \nabla\psi$.

Solution: Setting $\mathbf{A} = (-1, 6, 18)$ and $\mathbf{B} = (2, 12, -8)$ in Theorem 9.4, we get

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \psi(\mathbf{B}) - \psi(\mathbf{A}) = 2 \cdot 12 \cdot (-8) - (-1) \cdot 6 \cdot 18 = -192 + 108 \\ &= 84\,.\end{aligned}$$

9.4.2 Surface Integrals

From Sect. 9.3.3, we know how a surface Σ in space is described by a parametrization $\mathbf{r} : D \to \Sigma$ defined on some domain D of the plane. Our first task is to compute the area of the surface. As an introduction to the general formula, we consider the parallelogram with corners $\mathbf{0}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} + \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are two nonparallel vectors in space. We know that its area equals $\|\mathbf{x} \times \mathbf{y}\|$. We interpret this expression in terms of the parametrization

$$\mathbf{r}(u, v) = u\mathbf{x} + v\mathbf{y}\,, \quad \Sigma = \mathbf{r}(D)\,, \quad D = [0, 1] \times [0, 1]\,.$$

Indeed, since

$$\partial_u \mathbf{r}(u, v) = \mathbf{x}\,, \quad \partial_v \mathbf{r}(u, v) = \mathbf{y}\,,$$

are constant as functions of u and v, we have

$$\|\mathbf{x} \times \mathbf{y}\| = \int_0^1 \int_0^1 \|\mathbf{x} \times \mathbf{y}\|\, du\, dv = \iint_D \|\partial_u \mathbf{r}(u, v) \times \partial_v \mathbf{r}(u, v)\|\, dA\,,$$

so we have expressed the area in terms of a two-dimensional integral. If we just want to determine the area of a parallelogram, there is no need for such a complicated formula, but for a general surface, this is just the way to go.

Definition 9.19 (*Area of a Surface*) Let Σ be a smooth surface given as $\Sigma = \mathbf{r}(D)$ with a regular parametrization $\mathbf{r}$. The area of Σ is defined as

$$A(\Sigma) = \iint_D \|\partial_u \mathbf{r}(u, v) \times \partial_v \mathbf{r}(u, v)\|\, dA\,. \tag{9.95}$$

Since $\mathbf{r}$ is regular, the integral on the right-hand side is well defined as the limit of the corresponding Riemannian sums, according to Sect. 8.7. Those sums can be interpreted as the area of small parallelograms which approximate corresponding portions of Σ. (We do not carry out the details, which in fact would be rather cumbersome to do). Thus, (9.95) is a natural definition of the area of an arbitrary curved surface.

The most important special case arises when Σ is the graph of a function S.

Theorem 9.6 *Let Σ be a smooth surface given as $z = S(x, y)$, where S is a continuously differentiable function defined on some domain D of the plane. Then we have*

$$A(\Sigma) = \iint_D \sqrt{1 + \partial_x S(x, y)^2 + \partial_y S(x, y)^2}\, dA\,. \tag{9.96}$$

Verification: From Sect. 9.3.3, we know that with $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + S(x, y)\mathbf{k}$ we obtain

$$\partial_x \mathbf{r}(x, y) \times \partial_y \mathbf{r}(x, y) = -\partial_x S(x, y)\mathbf{i} - \partial_y S(x, y)\mathbf{j} + \mathbf{k}\,.$$

Taking the length of this vector, we see that (9.96) is indeed a special case of (9.95).

Formulas (9.95) and (9.96) are analogous to the formula for the length of a curve as the integral over the length of tangent vectors. Similarly, the definition of the surface integral of a scalar function f defined on Σ is analogous to that of a line integral as given in Definition 9.14.

Definition 9.20 (*Surface Integral*) Let Σ be a smooth surface, given as $\Sigma = \mathbf{r}(D)$ with a regular parametrization $\mathbf{r}$. Let f be a scalar continuous function defined on Σ. Then the **surface integral** of f over the surface Σ is denoted by $\iint_\Sigma f(x, y, z)\, d\sigma$ or simply $\iint_\Sigma f\, d\sigma$, and defined by

$$\begin{aligned}\iint_\Sigma f\, d\sigma &= \iint_\Sigma f(x, y, z)\, d\sigma \\ &= \iint_D f(\mathbf{r}(u, v))\|\partial_u \mathbf{r}(u, v) \times \partial_v \mathbf{r}(u, v)\|\, dA\,.\end{aligned} \tag{9.97}$$

For the case where Σ is the graph of a function S, we obtain the following formula for the surface integral from (9.97) in the same manner as we obtained the area in Theorem 9.6 from Definition 9.19.

Theorem 9.7 *Let Σ be a smooth surface given as $z = S(x, y)$, where S is a continuously differentiable function defined on some domain D of the (x, y)-plane. Let f be a scalar continuous function defined on Σ. Then we have*

$$\iint_\Sigma f\, d\sigma = \iint_D f(x, y, S(x, y))\sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2 (x, y) + \left(\frac{\partial S}{\partial y}\right)^2 (x, y)}\, dA\,. \tag{9.98}$$

Remark 9.3 (i) In the special case $f = 1$ we obtain the area of the surface, $A(\Sigma) = \iint_\Sigma 1\, d\sigma$, as one sees from the definitions.

(ii) For a given surface Σ, it is possible to choose different parametrizations $\mathbf{r}$ of it. One can prove that this does not change the value of the area, respectively, of the surface integral.

(iii) Let a surface Σ consist of smooth components $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ which are mutually disjoint or intersect each other only in a set of zero area, for example, along a curve. Such a surface is called **piecewise smooth**. The surface integral over Σ is then defined as

$$\iint_\Sigma f\, d\sigma = \iint_{\Sigma_1} f\, d\sigma + \cdots + \iint_{\Sigma_n} f\, d\sigma\,. \tag{9.99}$$

For example, the boundary Σ of a cube is not smooth (the normal vector jumps across its edges), but it is piecewise smooth, since it consists of six square pieces which are smooth.

(iv) The formal expression "$d\sigma = \|\partial_u \mathbf{r}(u, v) \times \partial_v \mathbf{r}(u, v)\|\, dA$", respectively "$d\sigma = \sqrt{1 + \partial_x S(x, y)^2 + \partial_y S(x, y)^2}\, dA$" is often called the **surface element**.

Example 9.29 Evaluate the surface integral $\iint_\Sigma f\,d\sigma$, where $f(x, y, z) = y^2$ and Σ is the part of the plane $z = x$ given by $0 \le x \le 2$, $0 \le y \le 4$.

Solution: We have $S(x, y) = x$ and $S : D \to \Sigma$ with $D = [0, 2] \times [0, 4]$, so

$$\sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2 (x, y) + \left(\frac{\partial S}{\partial y}\right)^2 (x, y)} = \sqrt{1 + 1^2 + 0^2} = \sqrt{2}\,,$$

and therefore

$$\begin{aligned}\iint_\Sigma f\,d\sigma &= \iint_D y^2\sqrt{2}\,dA = \int_0^2 \int_0^4 y^2\sqrt{2}\,dy\,dx \\ &= \sqrt{2}\int_0^2 dx \cdot \int_0^4 y^2 dy = \frac{128}{3}\sqrt{2}\,.\end{aligned}$$

Example 9.30 Evaluate the surface integral $\iint_\Sigma f\,d\sigma$, where $f(x, y, z) = z^2$, and Σ is the portion of the boundary surface of the vertical cone $z = \sqrt{x^2 + y^2}$ which lies between the planes $z = 1$ and $z = 2$.

Solution: We present two different approaches. For the first one, we use a parametrization adapted to the cone, namely,

$$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\mathbf{k}\,,$$

so $\Sigma = \mathbf{r}(D)$ with $D = \{(u, v) : 1 \le u \le 2\,,\ 0 \le v \le 2\pi\}$. In order to use Definition 9.20 we compute

$$\partial_u\mathbf{r}(u, v) = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}\,, \quad \partial_v\mathbf{r}(u, v) = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}\,.$$

We get

$$\begin{aligned}(\partial_u\mathbf{r} \times \partial_v\mathbf{r})(u, v) &= (0 - u\cos v)\mathbf{i} - (-u\sin v + 0)\mathbf{j} + (u\cos^2 v + u\sin^2 v)\mathbf{k} \\ &= -u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\mathbf{k}\end{aligned}$$

and moreover

$$\|(\partial_u\mathbf{r} \times \partial_v\mathbf{r})(u, v)\| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2} = u\sqrt{2}\,.$$

Inserting this into (9.97), we get, since $f(\mathbf{r}(u, v)) = u^2$,

$$\iint_\Sigma f\,d\sigma = \iint_D u^2 \cdot u\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_1^2 u^3\,du\,dv = \frac{15}{2}\pi\sqrt{2}\,.$$

For the second approach, we use Theorem 9.7 with $S(x, y) = \sqrt{x^2 + y^2}$ and the annular domain $D = \{(x, y) : 1 \leq \sqrt{x^2 + y^2} \leq 2\}$. We have $f(x, y, S(x, y)) = S(x, y)^2 = x^2 + y^2$, so

$$\iint_\Sigma f\, d\sigma = \iint_D (x^2 + y^2)\sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2 (x, y) + \left(\frac{\partial S}{\partial y}\right)^2 (x, y)}\, dA\,.$$

The partial derivatives of S are

$$\frac{\partial S}{\partial x}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\,, \quad \frac{\partial S}{\partial y}(x, y) = \frac{y}{\sqrt{x^2 + y^2}}\,,$$

so

$$\sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2 (x, y) + \left(\frac{\partial S}{\partial y}\right)^2 (x, y)} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}\,.$$

This yields

$$\iint_\Sigma f\, d\sigma = \iint_D (x^2 + y^2)\sqrt{2}\, dA\,.$$

Since D is an annular region, it is best to evaluate this two-dimensional integral in polar coordinates. According to Theorem 8.9, we get with $G = \{(r, \theta) : 1 \leq r \leq 2\,,\ 0 \leq \theta \leq 2\pi\}$ that

$$\begin{aligned}\iint_\Sigma f\, d\sigma &= \sqrt{2}\iint_D (x^2 + y^2)\, dA = \sqrt{2}\iint_G (r^2\cos^2\theta + r^2\sin^2\theta) r\, dA \\ &= \sqrt{2}\int_0^{2\pi}\int_1^2 r^3\, dr\, d\theta = \frac{15}{2}\pi\sqrt{2}\,,\end{aligned}$$

which is the same as above.

Example 9.31 Evaluate the surface integral $\iint_\Sigma f\, d\sigma$, where $f(x, y, z) = x^2$ and Σ is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: First we apply Theorem 9.7. We have $S(x, y) = \sqrt{a^2 - x^2 - y^2}$ and

$$\frac{\partial S}{\partial x}(x, y) = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}\,, \quad \frac{\partial S}{\partial y}(x, y) = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}\,.$$

Thus

$$\sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2 (x, y) + \left(\frac{\partial S}{\partial y}\right)^2 (x, y)} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,.$$

The integration domain becomes the circular disk $D = \{(x, y) : 0 \leq x^2 + y^2 \leq a^2\}$, and therefore,

$$\iint_\Sigma f \, d\sigma = \iint_D x^2 \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA \, .$$

Again, this two-dimensional integral is best evaluated in polar coordinates, and Theorem 8.9 yields

$$\begin{aligned}\iint_\Sigma f \, d\sigma &= \int_0^{2\pi} \int_0^a r^2 \cos^2\theta \cdot \frac{a}{\sqrt{a^2 - r^2}} r \, dr \, d\theta \\ &= a \int_0^{2\pi} \cos^2\theta \, d\theta \cdot \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} \, dr \, .\end{aligned}$$

The rightmost integral can be evaluated with the substitution $r = a \sin t$, $dr = a \cos t \, dt$, and (we omit the details)

$$\int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} \, dr = \int_0^{\pi/2} a^3 \sin^3 t \, dt = \frac{2}{3} a^3 \, .$$

We finally obtain

$$\iint_\Sigma f \, d\sigma = a \int_0^{2\pi} \cos^2\theta \, d\theta \cdot \frac{2}{3} a^3 = \frac{2}{3} \pi a^4 \, .$$

An alternative solution procedure arises from the use of spherical coordinates,

$$\mathbf{r}(u, v) = a \sin u \cos v \mathbf{i} + a \sin u \sin v \mathbf{j} + a \cos u \mathbf{k} \, .$$

Here $\Sigma = \mathbf{r}(D)$ with $D = \{(u, v) : 0 \leq u \leq \pi/2 \, , \; 0 \leq v \leq 2\pi\}$. A straightforward but somewhat lengthy computation yields

$$\|(\partial_u \mathbf{r} \times \partial_v \mathbf{r})(u, v)\| = a^2 \sin u \, .$$

Since $f(x, y, z) = x^2$, we have $f(\mathbf{r}(u, v)) = a^2 \sin^2 u \cos^2 v$, and from (9.97) we get

$$\begin{aligned}\iint_\Sigma f \, d\sigma &= \iint_D a^2 \sin^2 u \cos^2 v \cdot a^2 \sin u \, dA \\ &= a^4 \int_0^{\pi/2} \sin^3 u \, du \cdot \int_0^{2\pi} \cos^2 v \, dv = \frac{2}{3} \pi a^4\end{aligned}$$

as before.

9.5 Fundamental Theorems of Vector Calculus

We discuss here three very important results of vector calculus. The first result bears the names Green and Ostrogradski, in honor of the British scientist George Green and of the Ukrainian mathematician Mikhail Ostrogradski. This theorem establishes a relationship between a double integral over a domain in the plane and a line integral over the boundary of that domain. The second result which we present here is known as the divergence theorem of Gauss in honor of the German mathematician and scientist Carl Friedrich Gauss. This theorem relates a volume (triple) integral over a region in space to a surface integral over the boundary surface of that region. The third theorem is known as Stokes' theorem, named for Sir George Gabriel Stokes, an Irish mathematician who worked at Cambridge University. This theorem connects a surface integral to a line integral over the boundary curve of the surface.

Thus, all those theorems relate integrals over objects of different (adjacent) dimensions. In some sense, they can be viewed as extensions of the fundamental theorem of calculus; recall that the latter relates an integral over an interval (a one-dimensional object) to function values on the boundary (a zero-dimensional object).

There are numerous applications of this theorem in science and engineering, some of them will be described in the next section.

9.5.1 The Theorem of Green and Ostrogradski

Let C be a piecewise smooth curve in the plane $\mathbb{R}^2$ parametrized as $\mathbf{r}(t) = (x(t), y(t))$ for $a \le t \le b$. Let C be closed, that is, $\mathbf{r}(b) = \mathbf{r}(a)$ holds for the initial and terminal points. The curve C is called **positively oriented** if $\mathbf{r}(t)$ traverses C counterclockwise (anticlockwise) as t varies from a to b. If $\mathbf{r}(t)$ traverses C clockwise, then C is said to be **negatively oriented**.

Example 9.32 The parametrization $\mathbf{r}(t) = (\cos t, \sin t)$ traverses the unit circle C anticlockwise as t varies from 0 to 2π, and hence C is positively oriented by $\mathbf{r}$. On the other hand, the parametrization $\mathbf{r}(t) = (-\cos t, \sin t)$ defines a negatively oriented curve C (whose graph is again the unit circle), since it traverses C clockwise.

A non-closed curve C in the plane is called **simple** if it does not intersect itself, that is, if it has a parametrization $\mathbf{r} : [a, b] \to \mathbb{R}^2$ which is one-to-one, so $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ can hold only if $t_1 = t_2$ (recall Definition 1.12). If we imagine the graph of a curve as a train track, this means that the track does not cross itself. A closed curve is called simple if it does not intersect itself except for the initial and final point, that is, $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ holds for $t_1 < t_2$ only if $t_1 = a$ and $t_2 = b$.

Recall that by $\oint_C \mathbf{F} \cdot d\mathbf{r}$ we denote the line integral of a vector field $\mathbf{F} = (f, g)$ over a closed curve C, and also write it as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f(x, y)\, dx + g(x, y)\, dy\,.$$

Theorem 9.8 (Green–Ostrogradski) *Let D be a bounded domain in the plane whose boundary C is a closed, simple, and positively oriented curve. Let $\mathbf{F} = (f, g)$ be a vector field whose components are continuously differentiable. Then*

$$\oint_C f(x, y)\, dx + g(x, y)\, dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA\,. \tag{9.100}$$

The proof of this theorem will be given in Appendix D.6.

Remark 9.4

(i) The theorem relates a line integral, which is one-dimensional, to an integral over a 2-dimensional region.
(ii) The theorem is used for theoretical purposes as well as for the computation of specific integrals. In particular, it may happen that one of the integrals is much easier to calculate directly than the other one.

Example 9.33 (a) A particle moves counterclockwise once around the triangle D with vertices $(0, 0)$, $(4, 0)$, and $(1, 6)$, under the influence of the force $\mathbf{F}(x, y) = xy\mathbf{i} + x\mathbf{j}$. Calculate the work done by this force, if the units of length and force are meters and Newton, respectively.
(b) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (\cos 2y - e^{3y} + 4x)\mathbf{j}$ and C is (the boundary of) any square with sides of length 5. Assume C is oriented counterclockwise.

Solution: (a) Let us denote by C the curve formed by the three sides of the triangle. The total work done by the force $\mathbf{F}$ equals the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$. We have $\mathbf{F} = (f, g)$ with $f(x, y) = xy$, $g(x, y) = x$, so $\partial f/\partial y = x$ and $\partial g/\partial x = 1$. From Theorem 9.8 we know that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)(x, y)\, dA = \iint_D (1 - x)\, dA\,.$$

With the methods of Sect. 8.7, we compute the two-dimensional integral over the triangular region D as a double integral,

$$\begin{aligned}\iint_D (1 - x)\, dA &= \int_0^1 \int_0^{6x} (1 - x)\, dy\, dx + \int_1^4 \int_0^{8-2x} (1 - x)\, dy\, dx \\ &= \int_0^1 6x(1 - x)\, dx + \int_1^4 (8 - 2x)(1 - x)\, dx = -8\,.\end{aligned}$$

Therefore, the work done equals $-8\,\mathrm{Nm}$.
(b) We have $\mathbf{F} = (f, g)$ with $f(x, y) = x^2 - y$, $g(x, y) = \cos 2y - e^{3y} + 4x$, so $\partial f/\partial y = -1$ and $\partial g/\partial x = 4$. The two-dimensional region D is a square of side length 5. We calculate

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)(x, y)\, dA = \iint_D 4 - (-1)\, dA = 5 \iint_D dA\,.$$

Since $\iint_D dA$ equals the area of the square, which is equal to 25, we obtain $\oint_C \mathbf{F} \cdot d\mathbf{r} = 125$.

Example 9.34 (a) Use the Green–Ostrogradski theorem to evaluate the line integral $\oint_C y^2\, dx + x^2\, dy$, where C is the boundary of the square with vertices (0, 0), (1, 0), (0, 1), and (1, 1), oriented counterclockwise. Check the answer by evaluating the line integral directly.

(b) Do the same for $\oint_C xydx + (y + x)dy$, where C is the unit circle $x^2 + y^2 = 1$, oriented counterclockwise.

(c) Verify the validity of the Green–Ostrogradski theorem for the vector field $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j}$ and the curve C taken as the circle of radius 4 with center (1, 3).

Solution: (a) Here $f(x, y) = y^2$, $g(x, y) = x^2$, so $\partial f/\partial y = 2y$ and $\partial g/\partial x = 2x$, moreover $D = [0, 1] \times [0, 1]$. Using (9.100) we get

$$\begin{aligned} \oint_C f(x, y)\, dx + g(x, y)\, dy &= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)(x, y)\, dA = \iint_D 2x - 2y\, dA \\ &= \int_0^1 \int_0^1 2x - 2y\, dy\, dx = 1 - 1 = 0\,. \end{aligned} \tag{9.101}$$

For the line integral, we have to evaluate $\int y(t)^2 x'(t) + x(t)^2 y'(t)\, dt$ separately along all four sides of the square, traversed by a suitable parametrization $\mathbf{r}(t) = (x(t), y(t))$. Along the side $x = 0$ we have $y(t) = 0 = x'(t)$, and along the side $y = 0$ we have $x(t) = 0 = y'(t)$, so the corresponding line integrals are zero. The side $x = 1$ is parametrized by $\mathbf{r}(t) = (x(t), y(t)) = (1, t)$ for $0 \leq t \leq 1$, so the line integral becomes

$$\int_0^1 t^2 \cdot 0 + 1^2 \cdot 1\, dt = 1\,.$$

Analogously, the line integral along $y = 1$ yields the value -1, so the overall integral gives $0 + 0 + 1 - 1 = 0$, which is the same as the result in (9.101).

(b) We have $f(x, y) = xy$, $g(x, y) = y + x$, so $\partial f/\partial y = x$ and $\partial g/\partial x = 1$, moreover $D = \{(x, y) : 0 \leq x^2 + y^2 \leq 1\}$ is the unit disk. Thus with $\mathbf{F} = (f, g)$ we obtain

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)(x, y)\, dA = \iint_D 1 - x\, dA\,.$$

Using polar coordinates we get, according to Theorem 8.9,

$$\iint_D 1 - x\, dA = \int_0^{2\pi} \int_0^1 (1 - r\cos\theta) r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{3}\cos\theta \right) d\theta = \pi\,.$$

In order to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly, we parametrize C by $\mathbf{r}(t) = (x(t), y(t))$ with $x(t) = \cos t$, $y(t) = \sin t$, where $0 \le t \le 2\pi$. We obtain

$$\begin{aligned}
\oint_C xy\,dx + (y+x)\,dy &= \int_0^{2\pi} [xyx' + (y+x)y'](t)\,dt \\
&= \int_0^{2\pi} [\cos t \sin t(-\sin t) + (\sin t + \cos t)\cos t]\,dt \\
&= -\int_0^{2\pi} \cos t \sin^2 t\,dt + \int_0^{2\pi} \sin t \cos t\,dt + \int_0^{2\pi} \cos^2 t\,dt \\
&= 0 + 0 + \int_0^{2\pi} \cos^2 t\,dt = \pi\ ,
\end{aligned}$$

recall that $\int_0^{2\pi} \cos^2 t\,dt = \pi$.

(c) We have $\mathbf{F} = (f, g)$ with $f(x, y) = 2y$, $g(x, y) = -x$, so $\partial f/\partial y = 2$, $\partial g/\partial x = -1$, and D is the disk of radius 4 with center $(1, 3)$. We get ($A(D)$ denotes the area of D)

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)(x, y)\,dA = \iint_D -3\,dA \\
= -3 \cdot A(D) = -3 \cdot 4^2\pi = -48\pi\ .
\end{aligned}$$

To evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly, we parametrize C by $\mathbf{r}(t) = (x(t), y(t))$ with $x(t) = 1 + 4\cos t$, $y(t) = 3 + 4\sin t$, where $0 \le t \le 2\pi$. We obtain

$$\begin{aligned}
\oint_C 2y\,dx - x\,dy &= \int_0^{2\pi} [2yx' - xy'](t)\,dt \\
&= \int_0^{2\pi} 2(3 + 4\sin t)\cdot(-4\sin t) - (1 + 4\cos t)4\cos t\,dt \\
&= \int_0^{2\pi} -24\sin t - 32\sin^2 t - 4\cos t - 16\cos^2 t\,dt \\
&= 0 - 32\pi - 0 - 16\pi = -48\pi\ ,
\end{aligned}$$

recall that $\int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \cos^2 t\,dt = \pi$.

9.5.2 The Divergence Theorem of Gauss

We consider the following situation in the space $\mathbb{R}^3$. Let D be an open set which is bounded by a surface Σ such that D lies "completely inside" Σ. For example, the open unit ball $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3, \|\mathbf{x}\| < 1\}$ is bounded by and lies completely inside the unit sphere $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^3, \|\mathbf{x}\| = 1\}$; the same is true for the (full) cube whose boundary

surface consists of its 6 sides. In the case of the ball the boundary surface (the sphere) is smooth, in the case of the cube the boundary surface is piecewise smooth.

At every point $\mathbf{x}$ of a smooth surface, normal vectors can be defined (see Sect. 9.3.3). In the present situation, we can distinguish between **outer normals** pointing away from D and inner normals pointing into D. Thus, while we have two unit normals (normals whose length equals 1) at each point $\mathbf{x} \in \Sigma$, there is exactly one outer unit normal which we denote by $\mathbf{n}(\mathbf{x})$. In this way, we obtain a vector field $\mathbf{n}$, which is defined on the boundary surface Σ. As a consequence of our definition of a smooth surface (Definition 9.13), the field $\mathbf{n}$ is continuous.

When the boundary surface consists of smooth pieces (as in the case of the cube), the outer unit normal field $\mathbf{n}$ is continuous within each piece, but has jumps across their connecting boundaries (the edges and corners, in the case of the cube) where it is not defined.

Theorem 9.9 (Divergence Theorem of Gauss) *Let D be a domain in $\mathbb{R}^3$ with boundary Σ and outer unit normal field $\mathbf{n}$ as described above. Suppose that $\mathbf{F}$ is a vector field with values in $\mathbb{R}^3$ whose components are continuous and have continuous first partial derivatives in D and up to the boundary Σ. Then*

$$\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_{D} div\, \mathbf{F}\, dV\,. \tag{9.102}$$

Remark 9.5 (i) A proof will be given in Appendix D.7.
(ii) The theorem relates an integral over a two-dimensional surface to an integral over a three-dimensional volume. The integral on the left side is a surface integral (see Definition 9.20), the integrand being the scalar function $\mathbf{F} \cdot \mathbf{n} = f_1 n_1 + f_2 n_2 + f_3 n_3$, where f_j and n_j are the component functions of the vector fields $\mathbf{F}$ and $\mathbf{n}$. The integral on the right side is a volume integral, which we have treated in Sect. 8.7, of the scalar function $\operatorname{div} \mathbf{F} = \partial f_1/\partial x + \partial f_2/\partial y + \partial f_3/\partial z$.
(iii) Its interpretation and some applications will be discussed in Sect. 9.6. Indeed, the divergence theorem is a fundamental tool in the analysis of partial differential equations and of the phenomena described by them. Some identities useful for that purpose will be presented below, following the examples.
(iv) Moreover, Eq. (9.102) is sometimes helpful when one wants to compute surface or volume integrals, because the evaluation of one side may be more convenient than the evaluation of the other side.

Example 9.35 For each of the following data calculate the right-hand side and left-hand side of Eq. (9.102), whichever is convenient.
(i) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$, Σ is the sphere of radius 4 centered at $(1, 1, 1)$.
(ii) $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, Σ is the sphere of radius 1 with center at the origin.
(iii) $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, Σ is the rectangular box bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and the planes $x = 6$, $y = 2$ and $z = 7$.

Solution: In all cases, D denotes the region enclosed by Σ. Recall that

$$\operatorname{div}\mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\,, \quad \text{where} \quad \mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}\,.$$

(i) We have $f_1(x, y, z) = x$, $f_2(x, y, z) = y$, $f_3(x, y, z) = -z$, so

$$\operatorname{div}\mathbf{F}(x, y, z) = 1 + 1 - 1 = 1\,.$$

We evaluate the right-hand side of (9.102) as (here, $\operatorname{vol}(D)$ denotes the volume of D, a ball of radius 4)

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D dV = \operatorname{vol}(D) = \frac{4}{3}\pi \cdot 4^3 = \frac{256}{3}\pi\,.$$

(ii) Here $f_1(x, y, z) = x^3$, $f_2(x, y, z) = y^3$, $f_3(x, y, z) = z^3$. So

$$\frac{\partial f_1}{\partial x}(x, y, z) = 3x^2\,, \quad \frac{\partial f_2}{\partial y}(x, y, z) = 3y^2\,, \quad \frac{\partial f_3}{\partial x}(x, y, z) = 3z^2\,,$$
$$\operatorname{div}\mathbf{F}(x, y, z) = 3x^2 + 3y^2 + 3z^2\,.$$

The right-hand side of (9.102) becomes

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = 3\iiint_D (x^2 + y^2 + z^2)\,dV\,.$$

Using spherical coordinates, we have already evaluated this integral in Example 8.39 so

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = 3 \cdot \frac{4}{5}\pi = \frac{12}{5}\pi\,.$$

(iii) Here $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, so

$$\operatorname{div}\mathbf{F}(x, y, z) = 2x + 2y + 2z\,.$$

Again we evaluate the right-hand side of (9.102), this time with Fubini's theorem applied to the rectangular box,

$$\begin{aligned}
\iiint_D \operatorname{div}\mathbf{F}\,dV &= 2\int_0^7 \int_0^2 \int_0^6 x + y + z\,dx\,dy\,dz \\
&= 2\int_0^7 \int_0^2 \left[\frac{1}{2}x^2 + xy + xz\right]_{x=0}^{x=6} dy\,dz = \int_0^7 \int_0^2 36 + 12y + 12z\,dy\,dz \\
&= \int_0^7 \left[36y + 6y^2 + 12yz\right]_{y=0}^{y=2} dz = \int_0^7 72 + 24 + 24z\,dz \\
&= \left[96z + 12z^2\right]_{z=0}^{z=7} = 672 + 12 \cdot 49 = 1260\,.
\end{aligned}$$

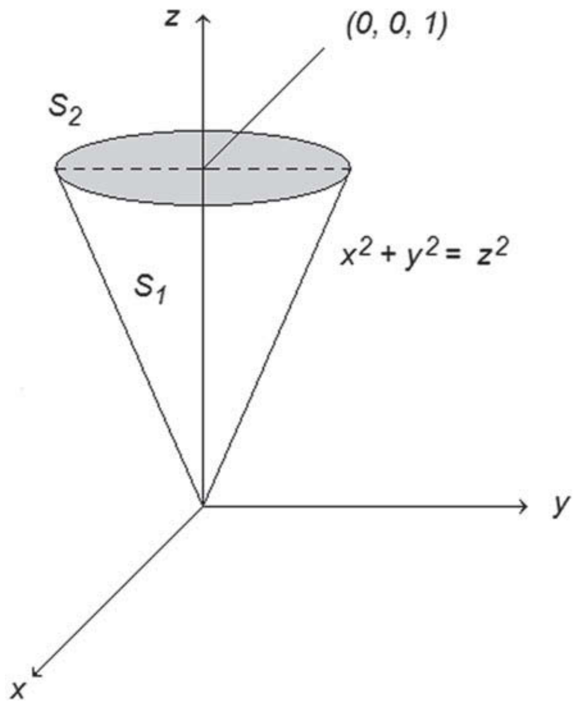


Fig. 9.9 The cone D

Example 9.36 Verify the statement of Gauss' divergence theorem for the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the cone whose interior is given by

$$D = \{(x, y, z) : 0 < z < 1\,,\ 0 < x^2 + y^2 < z^2\}\,.$$

Solution: The boundary surface Σ of D consists of two parts: see Fig. 9.9. Its lateral part Σ_1 is described by $z = S(x, y)$ with $S(x, y) = \sqrt{x^2 + y^2}$ for $x, y \in [-1, 1]$, its upper part Σ_2 is the flat circular disk described by $x^2 + y^2 \leq 1$ lying in the plane $z = 1$. The surface integral on the left-hand side of (9.102) thus decomposes into two parts,

$$\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{\Sigma_1} \mathbf{F} \cdot \mathbf{n_1}\, d\sigma + \iint_{\Sigma_2} \mathbf{F} \cdot \mathbf{n_2}\, d\sigma\,,$$

where $\mathbf{n_1}$ and $\mathbf{n_2}$ are the vector fields of outer unit normals to Σ_1 respectively Σ_2. According to (9.73), the vector

$$\mathbf{N_1}(x, y, z) = -\frac{\partial S}{\partial x}(x, y)\mathbf{i} - \frac{\partial S}{\partial y}(x, y)\mathbf{j} + \mathbf{k} = -\frac{x}{z}\mathbf{i} - \frac{y}{z}\mathbf{j} + \mathbf{k}$$

is a normal vector to Σ_1 in (x, y, z)), where $z = \sqrt{x^2 + y^2}$. The corresponding outer unit normal $\mathbf{n_1}$ is then given by

$$\mathbf{n_1}(x, y, z) = \frac{1}{\sqrt{2}}\left(\frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{j} - \mathbf{k}\right).$$

Moreover, $\mathbf{n_2} = \mathbf{k}$ as Σ_2 is a flat horizontal surface. For the given vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we now compute

$$\begin{aligned}
\iint_{\Sigma} \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iint_{\Sigma_1} \mathbf{F}\cdot\mathbf{n_1}\,d\sigma + \iint_{\Sigma_2} \mathbf{F}\cdot\mathbf{n_2}\,d\sigma \\
&= \iint_{\Sigma_1} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \frac{1}{\sqrt{2}}\left(\frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{j} - \mathbf{k}\right) d\sigma + \iint_{\Sigma_1} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{k}\,d\sigma \\
&= \frac{1}{\sqrt{2}} \iint_{\Sigma_1} \left(\frac{x^2}{z} + \frac{y^2}{z} - z\right) d\sigma + \iint_{\Sigma_2} z\,d\sigma = 0 + \iint_{\Sigma_2} 1\,d\sigma \\
&= \pi\,,
\end{aligned}$$

as $z^2 = x^2 + y^2$ on Σ_1, $z = 1$ on Σ_2 and the area of the disk Σ_2 equals π. The right-hand side of (9.102) becomes

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D 1 + 1 + 1\,dV = 3\iiint_D 1\,dV = 3\mathrm{vol}\,(D)\,.$$

Since the volume of the cone of height 1 and radius 1 is equal to $\pi/3$, the right-hand side, too, is equal to π. We now present Green's identities. Here Δ denotes the Laplace operator,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,.$$

Theorem 9.10 (Green's first identity) *Let D, Σ, and* $\mathbf{n}$ *as in Theorem 9.9. Let f and g be continuous scalar fields whose first and second partial derivatives are continuous in D and up to the boundary Σ. Then*

$$\iint_{\Sigma} f\nabla g\cdot\mathbf{n}\,d\sigma = \iiint_D (f\Delta g + \nabla f\cdot\nabla g)\,dV\,. \tag{9.103}$$

One can derive Green's first identity from the divergence theorem by choosing $\mathbf{F} = f\nabla g$ in (9.102).

Theorem 9.11 (Green's second identity) *Let D, Σ,* $\mathbf{n}$*, f, and g as in Theorem 9.10. Then*

$$\iint_{\Sigma} (f\nabla g - g\nabla f)\cdot\mathbf{n}\,d\sigma = \iiint_D (f\Delta g - g\Delta f)\,dV\,. \tag{9.104}$$

Green's second identity is obtained from Green's first identity by interchanging the roles of f and g, and then subtracting the resulting formula from the original one.

As a particular case of Green's identities, setting $f = 1$ we get

$$\iint_{\Sigma} \nabla g \cdot \mathbf{n}\, d\sigma = \iiint_{D} \Delta g \, dV\,. \tag{9.105}$$

9.5.3 The Theorem of Stokes

Let $\mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ be a vector field whose component functions are continuously differentiable, let Σ be a smooth surface bounded by a piecewise smooth closed curve C, let $\mathbf{n}$ be a vector field of unit normals to Σ whose orientation fits the orientation of C (this will be described below). Stokes' theorem relates the line integral of $\mathbf{F}$ over C to a surface integral involving $\mathbf{curl}\,\mathbf{F}$ over Σ by the formula

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{\Sigma} (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\, d\sigma\,. \tag{9.106}$$

Line integrals of vector fields have been studied in Sect. 9.4.1 surface integrals in Sect. 9.4.2. The integrand of the surface integral on the right-hand side is a scalar field, obtained as the scalar product of the vector fields $\mathbf{curl}\,\mathbf{F}$ and $\mathbf{n}$. From Definition 9.12, we recall that the notation as a vector product or as a determinant

$$\mathbf{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \tag{9.107}$$

is a convenient way to memorize the componentwise definition of the rotation $\mathbf{curl}\,\mathbf{F}$ of the vector field $\mathbf{F}$.

We assume that the surface Σ is given as $z = S(x, y)$ with a function S defined on the corresponding domain D in the xy-plane, and we assume that S and its first and second partial derivatives are continuous. We have seen in (9.74) that

$$\mathbf{n} = \frac{1}{\nu}\left(-\partial_x S\mathbf{i} - \partial_y S\mathbf{j} + \mathbf{k}\right)\,, \quad \nu = \sqrt{1 + (\partial_x S)^2 + (\partial_y S)^2}\,, \tag{9.108}$$

defines a unit normal in the point $(x, y, S(x, y))$ of Σ, if the right-hand side is evaluated at $(x, y) \in D$. We fix a suitable orientation of the boundary curve C of Σ as follows. Let Γ be the boundary of D, positively oriented by a parametrization $\mathbf{q} : [a, b] \to \mathbb{R}^2$. Then

$$\mathbf{r}(t) = q_1(t)\mathbf{i} + q_2(t)\mathbf{j} + S(q_1(t), q_2(t))\mathbf{k}$$

defines a parametrization $\mathbf{r} : [a, b] \to \mathbf{R}^3$ of C which orients C in the needed way.

Theorem 9.12 (Stokes) *Under the assumptions above, we have*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\Sigma (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\, d\sigma\,. \tag{9.109}$$

Remark 9.6 (i) A proof will be given in Appendix D.8.
(ii) The theorem also holds for more general surfaces, that is, surfaces which cannot be described as the graph of a function S.
(iii) The theorem relates a one-dimensional integral (the line integral) to a two-dimensional integral (the surface integral).

Example 9.37 Verify the statement of Stokes' theorem by evaluating both sides of (9.109) for the following data.
(a) $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$, the surface Σ is the portion of the plane $x + y + z = 1$ which lies in the first octant.
(b) $\mathbf{F}(x, y, z) = (y - z)\mathbf{i} + (z + x)\mathbf{j} + (y - x)\mathbf{k}$, the surface Σ is the portion of the paraboloid $z = 9 - x^2 - y^2$ which lies above the xy-plane.

Solution: (a) The surface Σ forms a planar triangular region in space whose vertices are the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, the boundary curve C is the triangle connecting those points, thus it is piecewise smooth. The corresponding domain D in the xy-plane is the triangular region enclosed by $x + y = 1$, $x = 0$, and $y = 0$. Thus, if we go through the corners of C in the sequence $\mathbf{i} \to \mathbf{j} \to \mathbf{k} \to \mathbf{i}$, the orientation corresponds to a positive orientation of Γ, the boundary of D. We parametrize the first piece $C_1 : \mathbf{i} \to \mathbf{j}$ with $\mathbf{r}(t) = (1 - t)\mathbf{i} + t\mathbf{j}$, $t \in [0, 1]$. For $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$, the line integral becomes

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^1 ((1 - 2t)\mathbf{i} + t\mathbf{j} + (t - 1)\mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j})\, dt \\ &= \int_0^1 (3t - 1)\, dt = \frac{1}{2}\,.\end{aligned}$$

The other two pieces of C are parametrized by $\mathbf{r}(t) = (1 - t)\mathbf{j} + t\mathbf{k}$ and $\mathbf{r}(t) = (1 - t)\mathbf{k} + t\mathbf{i}$, $t \in [0, 1]$, respectively. An analogous computation as above shows that

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}\,,$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}\,.$$

The parametrization S of Σ is given by $S(x, y) = 1 - x - y$, so $\partial_x S = \partial_y S = -1$ are constant. The unit normal according to (9.108) is

$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) ,$$

Moreover, $\mathbf{curl}\,\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Thus, both $\mathbf{n}$ and $\mathbf{curl}\,\mathbf{F}$ are constant on Σ. Since $\sqrt{1 + (\partial_x S)^2 + (\partial_y S)^2} = \sqrt{3}$, the surface integral becomes

$$\iint_\Sigma (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_\Sigma \sqrt{3}\, d\sigma = \iint_D \sqrt{3} \cdot \sqrt{3}\, dA = \frac{3}{2} ,$$

because the area of the triangular region D equals $1/2$.

(b) The boundary curve C is the circle lying in the xy-plane with radius 3 and center at the origin, it can be parametrized as $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$ for $0 \le t \le 2\pi$. We have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} 9(-\sin^2 t + \cos^2 t)\, dt = 0 .$$

Since the line integral equals 0, it does not matter which orientation we choose. Moreover, we compute that $\mathbf{curl}\,\mathbf{F} = 0$, so

$$\iint_\Sigma (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_\Sigma 0\, d\sigma = 0 .$$

Hence, we have verified Stokes' theorem for both sets of data.

Remark 9.7 A vector field $\mathbf{F}$ defined on some open set G of space $\mathbb{R}^3$ is called **irrotational** in G if $\mathbf{curl}\,\mathbf{F} = 0$ at all points of G. This terminology is motivated by Stokes' theorem, since then the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ equals zero for all closed curves which arise as a boundary of a surface Σ within the domain G. If all closed curves in G arise as boundaries, we then can conclude from Theorem 9.5 that

$$\mathbf{curl}\,\mathbf{F} = \mathbf{0} \text{ in } G \text{ implies that } \mathbf{F} \text{ is conservative on } G.$$

This is the case, for example, when G is the whole space $\mathbb{R}^3$ or when G is a ball. On the other hand, if $\mathbf{curl}\,\mathbf{F}(x, y, z)$ is not zero at some point (x, y, z), then for small disks around this point whose normal is parallel to $\mathbf{curl}\,\mathbf{F}(x, y, z)$, we have $\oint_C \mathbf{F} \cdot d\mathbf{r} \neq 0$ and hence $\mathbf{F}$ cannot be conservative.

However, note that for some types of regions G there may exist closed curves C which are not boundaries of such a surface Σ and have nonzero circulation (and hence, $\mathbf{F}$ is not circulation free and not conservative on G according to Definition 9.18 and Theorem 9.5, even though we might have $\mathbf{curl}\,\mathbf{F} = 0$ on G). Consider, for example, the vector field

$$\mathbf{F}(x, y, z) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} .$$

Its domain G is the whole space $\mathbb{R}^3$ except the z-axis, it satisfies $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ on G, but $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi \neq 0$ for the circle C defined by $x^2 + y^2 = 1$. Indeed, if Σ is a surface with boundary C, it must intersect the z-axis, so it cannot be contained in G.

Example 9.38 For the following data, evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ or $\iint_\Sigma (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{n}\, d\sigma$, whichever is easier.
(a) $\mathbf{F} = yx^2\mathbf{i} - xy^2\mathbf{j} + z^2\mathbf{k}$, the surface Σ is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.
(b) $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$, the surface Σ is the cone $z = \sqrt{x^2 + y^2}$ for $0 \leq z \leq 4$.

Solution: (a) The boundary curve C of Σ can be described by $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$ for $0 \leq t \leq 2\pi$, so along C we have

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos^2 t \sin^2 t - 16\cos^2 t \sin^2 t = -8\sin^2(2t) = -4(1 - \cos 4t)\,,$$

and therefore

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -4(1 - \cos 4t)dt = -8\pi\,.$$

(b) The parametric equation of C is $\mathbf{r}(t) = 4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 4\mathbf{k}$ for $0 \leq t \leq 2\pi$, so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 16\cos t - 16\sin^2 t\,,$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 16\cos t - 16\sin^2 t\, dt = -16\pi\,.$$

Note that the parametrization $z = S(x, y) = \sqrt{x^2 + y^2}$ is not differentiable at 0; we remark that Stokes' theorem remains valid in this particular case.

9.6 Applications of Vector Calculus to Engineering Problems

A major part of science and engineering deals with the analysis of forces such as the force of water on a dam or of air on a wing, stresses within buildings and bridges, electric and magnetic forces in the power industry or in computer hardware, and so on. These forces vary with position, time and other circumstances. Vector calculus provides basic tools for analyzing and understanding these situations.

In the first subsection, we exhibit relationships between elements of vector calculus and physical concepts like velocity, acceleration, momentum, angular momentum, and temperature. The second subsection presents applications of line integrals. The third subsection contains some basic examples of fluid flow, with an application to hurricane modeling.

9.6.1 Elements of Vector Calculus and the Physical World

We present here some physical phenomena modeled by vector fields.

Example 9.39 (*Gravitation*) According to Newton's law of gravitation, two objects with masses m and M attract each other with a force $\mathbf{F}$ of magnitude

$$\|\mathbf{F}\| = \frac{GmM}{r^2}, \tag{9.110}$$

where r is the distance between the two objects (treated as point masses), and G is the gravitational constant ($G = 6.673\,\mathrm{m}^3/(\mathrm{kg}\,\mathrm{sec}^2)$). Assume that the object with mass M is located at the origin in 3-space and $\mathbf{r}$ is the position vector of the object of mass m, then $r = \|\mathbf{r}\|$, and the force $\mathbf{F}(\mathbf{r})$ exerted by the object of mass M on the object of mass m points in the direction of the unit vector $-\mathbf{r}/\|\mathbf{r}\|$. Thus, from (9.110)

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r}, \tag{9.111}$$

or, in Cartesian coordinates,

$$\mathbf{F}(x, y, z) = -\frac{GmM}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

This defines a vector field whose domain D equals the whole space $\mathbb{R}^3$ except the origin. It describes the gravitational force of a point mass M located at the origin, as a function of the position of the point mass m.

Example 9.40 (*Electric Force Field*) Coulomb's law states that the electrostatic force exerted by one charged particle on another is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. Let two particles of charge Q and q be located at the origin of $\mathbb{R}^3$ and at the position given by a vector $\mathbf{r}$, respectively. Then the force $\mathbf{F}(\mathbf{r})$ that the particle of charge Q exerts on the particle of charge q equals

$$\mathbf{F}(\mathbf{r}) = \frac{qQ}{4\pi\varepsilon_0\|\mathbf{r}\|^3}\mathbf{r}, \tag{9.112}$$

where ε_0 is a positive constant (called the permittivity constant or the dielectric constant). Note that the force is repellent (directed outward) if Q and q have the same sign, and attractive otherwise. Formula (9.112) defines the vector field $\mathbf{F}$ of the electrostatic force generated by a the point charge Q at the origin, as a function of the position of the point charge q. If we divide by q we obtain the electrostatic force per unit charge, which is called the **electric field**

$$\mathbf{E} = \frac{\mathbf{F}}{q}. \tag{9.113}$$

Because of their form (9.110) and (9.112), both Newton's law and Coulomb's law are instances of what is termed an inverse square law.

Recall that, for a scalar vector field $\psi = \psi(x, y, z)$, the gradient is defined as

$$\nabla\psi = \frac{\partial\psi}{\partial x}\mathbf{i} + \frac{\partial\psi}{\partial y}\mathbf{j} + \frac{\partial\psi}{\partial z}\mathbf{k},$$

and that ψ is called a potential of the vector field $\mathbf{F}$ if $\mathbf{F} = \nabla\psi$. In this case, $\mathbf{F}$ is called a **gradient field**.

Example 9.41 (*Gravitational and Electric Potential*) The gravitational force field (9.111) possesses the potential (called gravitational potential)

$$\psi(\mathbf{r}) = \frac{GmM}{\|\mathbf{r}\|}, \tag{9.114}$$

Indeed, we have computed in Example 9.20(a), setting $n = -1$ there, that $\mathbf{F} = \nabla\psi$. Analogously, the electric field (9.113) possesses the potential $-\psi$ (called electric potential, the minus sign is conventional) with

$$\psi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0\|\mathbf{r}\|}. \tag{9.115}$$

We have seen in Sect. 9.3 that at each point in a gradient field $\mathbf{F}$ where the gradient is nonzero, the latter points in the direction in which the rate of increase of the corresponding potential ψ is maximal, and that moreover the gradient is perpendicular to the tangent plane of the level surface $\psi(x, y, z) = c$ through that point.

Vector fields $\mathbf{F}$ with domain and range in the plane $\mathbb{R}^2$ can be represented graphically by drawing the vectors $\mathbf{F}(\mathbf{r})$ at some points $\mathbf{r}$ of the plane. We give some examples.

Example 9.42 (i) Setting $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j}$ as above, we consider the vector field $\mathbf{F}(\mathbf{r}) = \mathbf{r}$ as well as

$$\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}, \quad (x, y) \neq (0, 0).$$

These vector fields are shown in Fig. 9.10a and b, respectively.

(ii) The vector field defined as $\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ is called the **spin field** or **rotation field** or **turning field**. $\mathbf{F}(\mathbf{r})$ is perpendicular to $\mathbf{r}$, since

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{r} = -yx + xy = 0$$

Moreover, we have $\|\mathbf{F}(\mathbf{r})\| = \sqrt{x^2+y^2} = \|\mathbf{r}\|$. The vector fields $\mathbf{F}$ and $\mathbf{F}\|/\mathbf{F}\|$ are, respectively, shown in Fig. 9.11a and b.

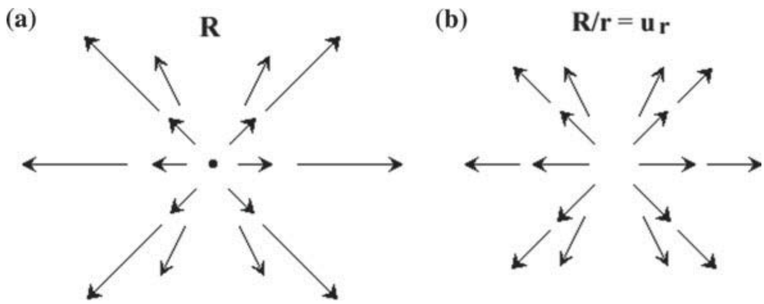


Fig. 9.10 The vector fields **(a)** $F(r) = R$ and **(b)** $F(r) = R/r$

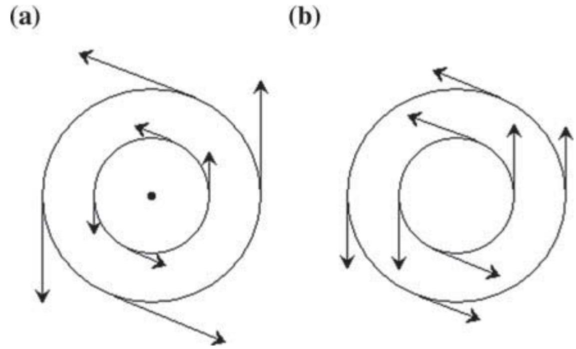


Fig. 9.11 The spin fields **(a)** F and **(b)** $F/\|F\|$

Example 9.43 (*Velocity and Acceleration*) Let a particle be moving along a path having position $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ as t varies from a to b. Its velocity field $\mathbf{v}$ also has the domain $[a, b]$ and is defined as

$$\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}\,. \tag{9.116}$$

The speed of the particle is defined as the magnitude of the velocity, it, therefore, is equal to

$$\|\mathbf{v}(t)\| = (x'(t)^2 + y'(t)^2 + z'(t)^2)^{1/2} = \|\mathbf{r}'(t)\|\,. \tag{9.117}$$

Let $s(t)$ denote the total distance traveled up to time t, that is, $s(t) = \int_a^t \|\mathbf{F}'(\tau)\|\, d\tau$, see (9.84). Since $s'(t) = \|\mathbf{r}'(t)\|$, we see from (9.117) that the speed of the particle is equal to $s'(t)$.

The acceleration $\mathbf{a}$ of the particle is defined as the rate of change of the velocity with respect to time,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}\,.$$

Example 9.44 (*Momentum*) The momentum $\mathbf{p}$ of an object is defined as the mass of the object times its velocity,

$$\mathbf{p}(t) = m\mathbf{v}(t) = m\mathbf{r}'(t)\,.$$

By Newton's law, its time derivative

$$\mathbf{p}'(t) = m\mathbf{r}''(t) = m\mathbf{a}(t)$$

equals the total (or net) force acting upon the object. We see, therefore, that $\mathbf{p}'(t) = 0$ if the net force is zero at time t.

Remark 9.8 As a consequence, the momentum $\mathbf{p}$ of an object stays constant during time intervals where no force is applied to it. This statement is called the law of **conservation of momentum**. Indeed conservation laws (which assert that certain quantities stay constant under certain conditions) are a basic ingredient of the sciences, since in particular when analyzing situations with several (or many) changing quantities it can be very helpful to identify quantities which are not changing. Instead of saying "quantity X stays constant" one also says "quantity X is invariant" or "quantity X remains invariant".

Example 9.45 (*Angular Momentum*) If an object of mass m has velocity $\mathbf{v}(t)$ at position $\mathbf{r}(t)$ at time t, its angular momentum $\mathbf{L}(t)$ w.r.t. the origin is defined by the equation

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t) = \mathbf{r}(t) \times m\mathbf{v}(t) = \mathbf{r}(t) \times m\mathbf{r}'(t)\,.$$

Note that the angular momentum $\mathbf{L}(t)$ is perpendicular to $\mathbf{r}(t)$ and $\mathbf{v}(t)$. According to (9.18), its magnitude is given by

$$\|\mathbf{L}(t)\| = \|\mathbf{r}(t)\|\,\|m\mathbf{v}(t)\|\,\sin\theta(t)\,,$$

where $\theta(t)$ is the angle between $\mathbf{r}(t)$ and $\mathbf{v}(t)$.

Example 9.46 A particle in the plane moves along the circle with radius 2 centered at the origin in such a way that its x- and y-coordinates are given by $x(t) = 2\cos t$, $y(t) = 2\sin t$.

(a) Find the velocity, the speed, and the acceleration of the particle at an arbitrary time t.

(b) Sketch the path of the particle and show the position and velocity vectors at $t = \pi/4$.

Solution: (a) The position is described by the vector function

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}\,.$$

Its velocity and speed at time t are, therefore,

$$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}\,,$$
$$\|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = \sqrt{4(\sin^2 t + \cos^2 t)} = 2\,.$$

Its acceleration at time t is $\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$. At time $t = \pi/4$, we have

$$\mathbf{r}\left(\frac{\pi}{4}\right) = 2\cos\frac{\pi}{4}\mathbf{i} + 2\sin\frac{\pi}{4}\mathbf{j} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}\,,$$
$$\mathbf{v}\left(\frac{\pi}{4}\right) = \mathbf{r}'\left(\frac{\pi}{4}\right) = -2\sin\frac{\pi}{4}\mathbf{i} + 2\cos\frac{\pi}{4}\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}\,.$$

Example 9.47 A particle of charge q moving in a magnetic field $\mathbf{B}$ is subject to the so-called **Lorentz force**

$$\mathbf{F} = \frac{q}{c}\mathbf{v} \times \mathbf{B}\,, \tag{9.118}$$

where c is the speed of light and $\mathbf{v}$ is the velocity of the particle. Assume that the magnetic field is constant and vertically oriented, $\mathbf{B}(t) = B_0\mathbf{k}$ with $B_0 \neq 0$. Find the path

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

of the particle, given its initial position $\mathbf{r}(0) = \mathbf{r}_0$ and velocity $\mathbf{v}(0) = \mathbf{v}_0$, as well as its mass m.

Solution: By Newton's law and (9.118), we have

$$m\mathbf{v}'(t) = m\mathbf{r}''(t) = \mathbf{F}(t) = \frac{q}{c}\mathbf{v}(t) \times \mathbf{B}(t)\,.$$

Since $\mathbf{B}(t) = B_0\mathbf{k}$, we get

$$\mathbf{v}'(t) = \lambda\mathbf{v}(t) \times \mathbf{k}\,, \quad \text{where } \lambda = \frac{qB_0}{mc}\,. \tag{9.119}$$

Written in components $\mathbf{v}(t) = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}$, (9.119) becomes

$$v_1'(t)\mathbf{i} + v_2'(t)\mathbf{j} + v_3'(t)\mathbf{k} = \lambda[v_2(t)\mathbf{i} - v_1(t)\mathbf{j}]\,.$$

This implies that

$$v_1'(t) = \lambda v_2(t)\,, \quad v_2'(t) = -\lambda v_1(t)\,, \quad v_3'(t) = 0\,. \tag{9.120}$$

Since $v_3'(t) = 0$ for all t, v_3 is a constant function, say $v_3(t) = C$. From the first two equations of (9.120), we get

$$v_1''(t) = \lambda v_2'(t) = -\lambda^2 v_1(t)\,,$$

or

$$v_1''(t) + \lambda^2 v_1(t) = 0\,. \tag{9.121}$$

A solution of (9.121) is given by

$$v_1(t) = A\sin(\lambda t + \varphi)\,,$$

because

$$v_1'(t) = \lambda A\cos(\lambda t + \varphi)\,, \quad v_1''(t) = -\lambda^2 A\sin(\lambda t + \varphi)\,.$$

Since $v_1'(t) = \lambda v_2(t)$, we get

$$v_2(t) = \frac{v_1'(t)}{\lambda} = A\cos(\lambda t + \varphi)\,.$$

Therefore

$$\mathbf{r}'(t) = \mathbf{v}(t) = A\sin(\lambda t + \varphi)\mathbf{i} + A\cos(\lambda t + \varphi)\mathbf{j} + C\mathbf{k}\,.$$

A final integration with respect to t gives us

$$\begin{aligned}\mathbf{r}(t) = &\left[-\frac{A}{\lambda}\cos(\lambda t + \varphi) + K_1\right]\mathbf{i} \\ &+ \left[\frac{A}{\lambda}\sin(\lambda t + \varphi) + K_2\right]\mathbf{j} + [Ct + K_3]\mathbf{k}\,,\end{aligned} \tag{9.122}$$

where K_1, K_2, K_3 are constants of integration. All six constants A, φ, C, K_1, K_2, and K_3 can be evaluated from the six initial conditions given by the vector equations $\mathbf{r}(0) = \mathbf{r}_0$ and $\mathbf{v}(0) = \mathbf{v}_0$, but we omit this computation here. Instead, we observe that the path of the particle is a circular helix with axis parallel to $\mathbf{B}$, that is, parallel to $\mathbf{k}$. One can see this from Eq. (9.122): The z-component of $\mathbf{r}$ varies linearly with t, while the x- and y-components represent uniform circular motion with angular velocity λ and radius $|A_1/\lambda|$ around the center (K_1, K_2). Thus, charged particles spiral around the magnetic field lines. Qualitatively, this behavior still occurs even if the magnetic field lines are curved, as in the case of the Earth's magnetic field. Charged particles trapped by the Earth's magnetic field spiral around the magnetic field lines that run from pole to pole.

Example 9.48 A heat-seeking particle is located at the point (2, 3) on a flat metal sheet whose temperature at a point (x, y) is

$$T(x, y) = 10 - 8x^2 - 2y^2\,.$$

Find an equation for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

Solution: Let the trajectory (a curve) be parametrized by $\mathbf{r}(t) = (x(t), y(t))$ with $\mathbf{r}(0) = (2, 3)$. Since the direction of maximum temperature increase at any point (x, y) is given by $\nabla T(x, y)$, the velocity vector $\mathbf{v}(t)$ of the particle at time t points in the direction of the gradient at its current position $\mathbf{r}(t)$. Thus, there is a scalar μ that may depend on t such that

$$\mathbf{v}(t) = \mu(t)\nabla T(x(t), y(t)) .$$

Since $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\nabla T(x, y) = (-16x, -4y)$, we get

$$x'(t)\mathbf{i} + y'(t)\mathbf{j} = \mu(t)(-16x(t)\mathbf{i} - 4y(t)\mathbf{j}) ,$$

or, equating components,

$$x'(t) = -16\mu(t)x(t) , \quad y'(t) = -4\mu(t)y(t) . \tag{9.123}$$

Let $M = M(t)$ be any antiderivative of μ. We may check that

$$x(t) = e^{-16M(t)}x_0 , \quad y(t) = e^{-4M(t)}y_0 ,$$

are solutions of (9.123) with initial values $x(0) = x_0$ and $y(0) = y_0$. Using the initial values $x_0 = 2$ and $y_0 = 3$, we see that the trajectory $(x(t), y(t))$ satisfies, for all values of t, the equation

$$y = \frac{3}{\sqrt[4]{2}}\sqrt[4]{x} . \tag{9.124}$$

The graph of the trajectory and contour plot of the temperature function are shown in Fig. 9.12.

Remark 9.9 The preceding example exhibits dynamics controlled by the gradient of some scalar field. This is called **gradient flow**. There are numerous applications where such a situation arises.

Example 9.49 Assume that a certain distribution of electric charges in the plane produces the electric potential $\psi(x, y) = e^{-2x}\cos(2y)$.
(i) Find the electric field vector $\mathbf{E} = -\nabla\psi$ at $(\pi/4, 0)$.
(ii) Find the direction in which the potential decreases most rapidly at this point.

Solution: (i) We have

$$\nabla\psi(x, y) = -2e^{-2x}\cos 2y\mathbf{i} - 2e^{-2x}\sin 2y\mathbf{j} ,$$
$$\mathbf{E}(x, y) = -\nabla\psi(\pi/4, 0) = -2e^{-\pi/2}\mathbf{j} .$$

(ii) At $(\pi/4, 0)$, ψ decreases most rapidly in the direction of $-\nabla\psi(\pi/4, 0)$ which was computed in (i).

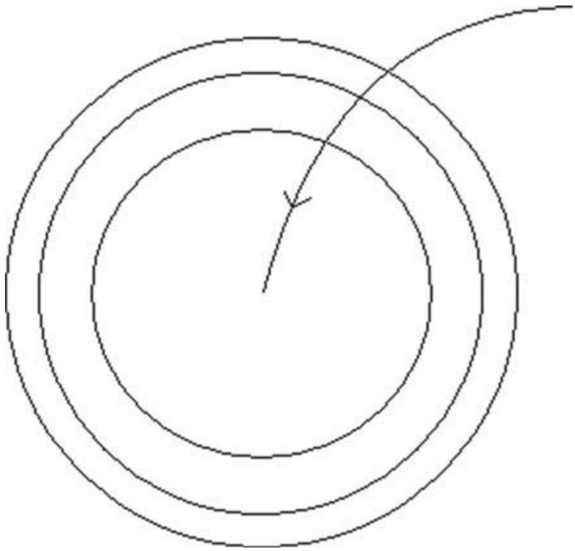


Fig. 9.12 Trajectory of the particle and level curves of the temperature

Example 9.50 Suppose a rigid object rotates with constant angular speed around an axis through the origin with direction $\mathbf{a} \in \mathbb{R}^3$. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be any point of space. We decompose $\mathbf{r} = \mathbf{r}_\perp + \alpha\mathbf{a}$ into a radial component $\mathbf{r}_\perp$ and a component $\alpha\mathbf{a}$ parallel to $\mathbf{a}$, where α is a scalar. Since any material point of the object which passes through a fixed space point $\mathbf{r}$ will have the same velocity vector, we can associate with this motion a velocity field $\mathbf{v} = \mathbf{v}(x, y, z)$. We see that $\mathbf{v}$ is perpendicular to $\mathbf{r}_\perp$ as well as to the rotation axis $\mathbf{a}$ and that the speed $\|\mathbf{v}\|$ is proportional to $\|\mathbf{r}_\perp\| = \|\mathbf{r}\| \sin\theta$. Therefore, there exists a unique vector $\boldsymbol{\omega}$ which is parallel to $\mathbf{a}$ such that

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_\perp = \boldsymbol{\omega} \times \mathbf{r}\,.$$

This vector $\boldsymbol{\omega}$ is called the **angular velocity** of the object; its length $\|\boldsymbol{\omega}\|$ equals the angular speed. Let $\boldsymbol{\omega} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then

$$\begin{aligned} \mathbf{v}(x, y, z) &= (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= (Bz - Cy)\mathbf{i} + (Cx - Az)\mathbf{j} + (Ay - Bx)\mathbf{k}\,. \end{aligned}$$

We compute

$$\begin{aligned} \mathbf{curl}(\mathbf{v}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ Bz - Cy & Cx - Az & Ay - Bx \end{vmatrix} = 2A\mathbf{i} + 2B\mathbf{j} + 2C\mathbf{k} \\ &= 2\boldsymbol{\omega}\,. \end{aligned} \tag{9.125}$$

Thus the angular velocity of a uniformly rotating body equals one-half the curl of the "linear" velocity, as the latter is called in this context to emphasize its different character.

9.6.2 *Applications of Line Integrals*

Line integrals have been introduced in Sect. 9.4.1. Here, we present some applications in the area of mechanics.

Line Integrals of Scalar Fields. Let us consider a bent wire as a (one-dimensional) smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in space, where $\mathbf{r} : [a, b] \to \mathbb{R}^3$. We assume that the distribution of its mass is described by a continuous density function ρ (in units of mass per unit length) defined on the set $C = r([a, b])$ of the curve points. We have already mentioned in Sect. 9.4.1 that its total mass M can be expressed as the line integral

$$M = \int_C \rho \, ds = \int_a^b \rho(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt \, .$$

Its center of mass is another quantity of interest in mechanics. We define the coordinates of the so-called first moment of ρ as the line integrals

$$M_x = \int_C x\rho \, ds \, , \quad M_y = \int_C y\rho \, ds \, , \quad M_z = \int_C z\rho \, ds \, ,$$

that is,

$$M_x = \int_a^b x(t)\rho(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt \, , \quad M_y = \dots \, , \quad M_z = \dots \, .$$

The coordinates of the center of mass are then given by

$$\bar{x} = \frac{M_x}{M} \, , \quad \bar{y} = \frac{M_y}{M} \, , \quad \bar{z} = \frac{M_z}{M} \, .$$

Let us now assume that the wire rotates around an axis. Its moment of inertia with respect to this axis is given by the line integral

$$I = \int_C d^2 \rho \, ds \, ,$$

here $d(x, y, z)$ denotes the distance of the point (x, y, z) from the axis. In particular, we obtain the moment of inertia of the wire with respect to the x-axis as

$$I_x = \int_C (y^2 + z^2)\rho \, ds \, .$$

Line Integrals of Vector Fields. Let C be the straight line connecting two points $\mathbf{r}_0$ and $\mathbf{r}_1$ in space. We know from elementary mechanics that if a mass is moved from $\mathbf{r}_0$ to $\mathbf{r}_1$ under the influence of a constant force $\mathbf{F}$, the work W done by the force is given by

$$W = \mathbf{F} \cdot (\mathbf{r}_1 - \mathbf{r}_0) = \|\mathbf{F}\| \, \|\mathbf{r}_1 - \mathbf{r}_0\| \cos\alpha \, ,$$

where α is the angle between the vectors $\mathbf{F}$ and $\mathbf{r}_1 - \mathbf{r}_0$. Let us now parametrize C as $\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0)$ with $\mathbf{r} : [0, 1] \to \mathbb{R}^3$. Since $\mathbf{r}'(t) = \mathbf{r}_1 - \mathbf{r}_0$, we have

$$\mathbf{F} \cdot (\mathbf{r}_1 - \mathbf{r}_0) = \int_0^1 \mathbf{F} \cdot \mathbf{r}'(t) \, dt \, ,$$

The latter integral is nothing else than the line integral of Definition 9.15, so we can express the work W as the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} \, . \tag{9.126}$$

This line integral also yields the correct value of the total work done by an arbitrary force field Let $\mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ on a mass which has moved along an arbitrary curve C from its initial point $\mathbf{r}(a)$ to its final point $\mathbf{r}(b)$, if C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ with $\mathbf{r} : [a, b] \to \mathbb{R}^3$. This can be seen if we approximate C by a curve consisting of pieces of straight lines and pass to the limit. The procedure is analogous to the one used to compute the length of a curve in Sect. 7.2, we will not carry out the details.

Example 9.51 Let $\mathbf{F}(x, y, z) = \mathbf{i} - y\mathbf{j} + xyz\mathbf{k}$ be a force field. Calculate the work done when moving a particle from $(0, 0, 0)$ to $(1, -1, 1)$ along the curve $x = t$, $y = -t^2$, $z = t$, $0 \le t \le 1$.

Solution: The work done is equal to the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$. We compute

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C dx - y\,dy + xyz\,dz = \int_0^1 [1x' - yy' + xyzz'](t)\,dt \\ &= \int_0^1 1 + t^2(-2t) - t^4\,dt = \left[t - \frac{1}{2}t^4 - \frac{1}{5}t^5\right]_0^1 \\ &= 1 - \frac{1}{2} - \frac{1}{5} = \frac{3}{10} \, . \end{aligned}$$

Example 9.52 Find the work done by $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2yz\mathbf{j} + z\mathbf{k}$ in moving an object along the line segment from $(1, 1, 1)$ to $(4, 4, 4)$.

Solution: First parametrize the line segment as $x = y = z = 1 + 3t$ with $0 \le t \le 1$, so $x'(t) = y'(t) = z'(t) = 3$. The work done by $\mathbf{F}$ moving the object along C is given by

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [x^2 x' - 2yzy' + zz'](t)\, dt \\ &= \int_0^1 [(1+3t)^2 - 2(1+3t)^2 + (1+3t)] \cdot 3\, dt \\ &= \left[\frac{(1+3t)^2}{2} - \frac{(1+3t)^3}{3} \right]_0^1 = -\frac{27}{2}\,. \end{aligned}$$

In Definition 9.17, we have called a vector field $\mathbf{F}$ conservative if it possesses a potential function, that is, if $\mathbf{F} = \nabla \psi$ for some scalar field ψ. The following remark explains this terminology.

Remark 9.10 (*Conservation of Mechanical Energy*) Let an object of mass m move in space according to Newton's law $\mathbf{F} = m\mathbf{a} = m\mathbf{r}''$, see Example 9.43. Assume that the force field $\mathbf{F}$ has a potential function ψ, so $\mathbf{F} = \nabla \psi$. In mechanics, one considers the potential energy $-\psi$ and the kinetic energy $-(1/2)m\|\mathbf{v}\|^2$, where $\mathbf{v} = \mathbf{r}'$. The total mechanical energy $E(t)$ of the object at time t is given as the sum of its kinetic and its potential energy at that time, so

$$E(t) = \frac{1}{2} m\mathbf{r}'(t) \cdot \mathbf{r}'(t) - \psi(\mathbf{r}(t))\,. \tag{9.127}$$

Let us compute its time derivative $E'(t)$. Using the product rule and the chain rule we get

$$\begin{aligned} E'(t) &= \frac{1}{2} m\mathbf{r}''(t) \cdot \mathbf{r}'(t) + \frac{1}{2} m\mathbf{r}'(t) \cdot \mathbf{r}''(t) - \nabla\psi(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \\ &= \left[m\mathbf{r}''(t) - \nabla\psi(\mathbf{r}(t)) \right] \cdot \mathbf{r}'(t)\,. \end{aligned}$$

Since $m\mathbf{r}''(t) = \mathbf{F}(r(t)) = \nabla\psi(\mathbf{r}(t))$, we conclude that $E'(t) = 0$. This means that the total energy remains constant (or, is conserved) along the trajectory of the object.

Remark 9.11 As we have seen in Example 9.41, the gravitational field possesses a potential and hence it is conservative.

9.6.3 An Example of Planar Fluid Flow-Hurricane

We begin with two basic situations, sink flow and vortex flow, in a situation made as simple as possible. Consider an incompressible fluid (that is, its density ρ is constant) whose flow takes place in the plane and is stationary, that is, its velocity

field $\mathbf{v} = \mathbf{v}(x, y)$ does not depend on time. It is also assumed that the fluid is inviscid, that is, its internal friction (the viscosity) is zero.

Sink flow. We imagine that there is a hole at the origin (the sink) where the fluid leaves the plane and that

(i) the fluid flows toward the origin, that is, the velocity vector at every point (x, y) is directed toward the origin,
(ii) the flow is radially symmetric, that is, the speed of the fluid is the same at all points of every given circle centered at the origin.

Conditions (i) and (ii) above are modeled by a velocity field of the form

$$\mathbf{v}(x, y) = \beta(r)(x\mathbf{i} + y\mathbf{j}), \quad r = \sqrt{x^2 + y^2}, \tag{9.128}$$

with a function $\beta = \beta(r) < 0$ yet to be fixed. One can show that one must have $\operatorname{div} \mathbf{v} = 0$ in order to satisfy the principle of mass conservation. Using the chain rule, we compute

$$\frac{\partial \mathbf{v}_1}{\partial x} = \beta(r) + x\beta'(r)\frac{\partial r}{\partial x} = \beta(r) + \beta'(r)\frac{x^2}{\sqrt{x^2 + y^2}},$$

In the same manner, we obtain

$$\frac{\partial \mathbf{v}_2}{\partial y} = \beta(r) + \beta'(r)\frac{y^2}{\sqrt{x^2 + y^2}}.$$

Since in two dimensions

$$\operatorname{div} \mathbf{v} = \frac{\partial \mathbf{v}_1}{\partial x} + \frac{\partial \mathbf{v}_2}{\partial y},$$

we get

$$\operatorname{div} \mathbf{v} = 2\beta(r) + \beta'(r)\frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = 2\beta(r) + r\beta'(r).$$

The condition $\operatorname{div} \mathbf{v} = 0$ leads to

$$\beta(r) = \frac{c}{r^2}$$

for some constant c which must be negative by the above. Setting $c = -q/2\pi$ (q is called the sink strength), (9.128) becomes

$$\mathbf{v}(x, y) = -\frac{q}{2\pi(x^2 + y^2)}(x\mathbf{i} + y\mathbf{j}). \tag{9.129}$$

Since it follows that $\|\mathbf{v}(x, y)\| = q/2\pi r$, the sink flow has the further property that

(iii) the speed of the fluid at any point $P = (x, y)$ is inversely proportional to the distance of P from the origin; in particular, it tends to $+\infty$ as that distance tends to zero.

Vortex flow. Here the fluid flows along concentric circles around the origin in the counterclockwise direction, that is,

(i) the velocity vector $\mathbf{v}(x, y)$ at a point (x, y) is tangent to the circle centered at the origin which passes through (x, y),
(ii) $\mathbf{v}(x, y)$ points in the counterclockwise direction.

Moreover, the speed $\|\mathbf{v}\|$

(iii) is constant along those circles, and
(iv) is inversely proportional along any such circle to the radius r of the latter,

and hence it tends to $+\infty$ as r tends to 0. The vector field

$$\mathbf{v}(x, y) = \frac{k}{2\pi(x^2 + y^2)}(-y\mathbf{i} + x\mathbf{j}) \tag{9.130}$$

possesses those four properties. (The constant $k > 0$ is called the vortex strength.) Indeed, by (9.130) we have $\|\mathbf{v}(x, y)\| = k/2\pi\sqrt{x^2 + y^2} = k/2\pi r$. Moreover, $\mathbf{v}(x, y) \cdot (x\mathbf{i} + y\mathbf{j}) = 0$, so $\mathbf{v}(x, y)$ is perpendicular to the radius vector of the circle, and by drawing a picture one sees that the direction is counterclockwise.

Streamlines and stream functions. The paths followed by the fluid particles in a fluid flow are called the **streamlines** of the flow. If the streamlines can be represented as the level curves of some function $\psi = \psi(x, y)$, then ψ is called the **stream function** of the flow. In this case, every particle path must satisfy $\psi(x(t), y(t)) = c$ for a suitable constant c. By the chain rule,

$$\frac{\partial\psi}{\partial x}(x(t), y(t))\dot{x}(t) + \frac{\partial\psi}{\partial y}(x(t), y(t))\dot{y}(t) = 0$$

must hold along particle paths. Therefore, ψ and the velocity field $\mathbf{v}$ of the flow are related by

$$\nabla\psi(x, y) \cdot \mathbf{v}(x, y) = 0 \tag{9.131}$$

at all points (x, y) in the domain of the flow; the velocity vectors are tangent, while the vectors $\nabla\psi(x, y)$ are perpendicular to the streamlines.

Combined sink and vortex flow. Here we have the velocity field

$$\mathbf{v}(x, y) = \frac{1}{2\pi(x^2 + y^2)}\left[(-qx - ky)\mathbf{i} + (-qy + kx)\mathbf{j}\right]. \tag{9.132}$$

The particles in this flow rotate while moving toward the sink, so we expect that they spiral inward. In order to find the streamlines, we will compute its stream function ψ, using (9.131). It is convenient to do this in polar coordinates, using the vectors introduced in (9.63) as

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\,, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\,.$$

Setting $x = r\cos\theta$, $y = r\sin\theta$, the velocity field in polar coordinates becomes

$$\begin{aligned}\mathbf{v}(r,\theta) &= \frac{1}{2\pi r^2}\big[(-qr\cos\theta - kr\sin\theta)\mathbf{i} + (-qr\sin\theta + kr\cos\theta)\mathbf{i}\big]\\ &= \frac{1}{2\pi r}(-q\mathbf{e}_r + k\mathbf{e}_\theta)\,,\end{aligned}$$

and $\nabla\psi$ transforms according to (9.64) into

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta\,.$$

Condition (9.131) for the stream function now becomes in polar coordinates

$$\begin{aligned}0 &= \left(\frac{\partial\psi}{\partial r}(r,\theta)\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}(r,\theta)\mathbf{e}_\theta\right)\cdot\mathbf{v}(r,\theta)\\ &= \frac{1}{2\pi r}\left(-q\frac{\partial\psi}{\partial r}(r,\theta) + \frac{k}{r}\frac{\partial\psi}{\partial\theta}(r,\theta)\right)\,.\end{aligned} \tag{9.133}$$

We see that (9.133) is satisfied if we choose ψ such that

$$\frac{\partial\psi}{\partial r}(r,\theta) = \frac{k}{r}\,, \quad \frac{\partial\psi}{\partial\theta}(r,\theta) = q\,. \tag{9.134}$$

This is indeed possible, we set

$$\psi(r,\theta) = k\ln r + q\theta\,. \tag{9.135}$$

We want to compute r as a function of θ for the streamlines $\psi = c$. From $k\ln r + q\theta = c$ we get

$$\ln r = \frac{1}{k}(c - q\theta)\,, \quad r = e^{(c-q\theta)/k} = e^{c/k}\cdot e^{-q\theta/k}\,.$$

Since c is an arbitrary constant, we may replace $e^{c/k}$ by c and finally obtain that

$$r = ce^{-q\theta/k}\,, \quad c > 0\,, \tag{9.136}$$

holds along the streamlines. Thus, the spirals are determined by the value of q/k.

Modeling of a Hurricane. Let us assume that the preceding flow model can be used for a hurricane and that only a single measurement of velocity is available.

Example 9.53 Find the strength of the parameters k and q of the flow model (9.132) for a hurricane from the report that at 20 km distance from the eye the wind velocity has a component of 15 km/h toward the eye and a counterclockwise tangential component of 45 km/h. Estimate the size of the hurricane by finding a radius beyond which the wind speed is less than 5 km/h.

Solution: The velocity component toward the eye is equal to the speed of the sink part of the flow. We have already seen that the latter is given by $q/(2\pi r)$, so at $r = 20$ we have

$$15 = \frac{q}{2\pi \cdot 20}, \quad \text{so } q = 600\pi$$

with unit $1/\mathrm{h}$. The tangential velocity component is equal to the speed of the vortex part of the flow, and we have already seen that the latter is given by $k/(2\pi r)$, so at $r = 20$ we have

$$45 = \frac{q}{2\pi \cdot 20}, \quad \text{so } q = 1800\pi\,,$$

again with unit $1/\mathrm{h}$. To estimate the size, we determine r from the condition that $\|v\| = 5\,\mathrm{km/h}$. Since tangential and inward velocity are perpendicular, we have

$$5 = \|v\| = \frac{1}{r}\sqrt{\left(\frac{q}{2\pi}\right)^2 + \left(\frac{k}{2\pi}\right)^2},$$

therefore

$$r = \frac{1}{5}\sqrt{300^2 + 900^2} = \frac{100}{5}\sqrt{9+81} = 60\sqrt{10} \approx 189.7\,\mathrm{km}.$$

9.7 Exercises

9.7.1 Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be vectors of three-dimensional space, let λ be a scalar. Show that

(a) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$,
(b) $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$,
(c) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$,
(d) $\lambda(\mathbf{x} \cdot \mathbf{y}) = (\lambda\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (\lambda\mathbf{y})$,
(e) $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = 0$, that is, $\mathbf{x} \times \mathbf{y}$ is perpendicular to $\mathbf{x}$.
(f) $\mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = 0$, that is, $\mathbf{x} \times \mathbf{y}$ is perpendicular to $\mathbf{y}$.
(g) $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$
(h) $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}), = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$

(i) $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}), = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})$
(j) $\mathbf{x} \times \mathbf{x} = 0$.

9.7.2 The goal of this exercise is to derive the identity

$$(\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{z} \times \mathbf{w}) = (\mathbf{x} \cdot \mathbf{z})(\mathbf{y} \cdot \mathbf{w}) - (\mathbf{y} \cdot \mathbf{z})(\mathbf{x} \cdot \mathbf{w}) \,, \qquad (*)$$

for arbitrary vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{w}$ of three-dimensional space.

(a) Show that (*) holds for $\mathbf{x} = \mathbf{y} = \mathbf{k}$ and arbitrary vectors $\mathbf{z}$ and $\mathbf{w}$.
(b) Show that (*) holds for $\mathbf{x} = \mathbf{k}$, $\mathbf{y} = \alpha\mathbf{i} + \beta\mathbf{j}$ and arbitrary vectors $\mathbf{z}$ and $\mathbf{w}$, where α and β are arbitrary scalars.
(c) Show that (*) holds for $\mathbf{x} = \mathbf{k}$ and arbitrary vectors $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{w}$.
(d) Show that (*) holds for for arbitrary vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{w}$.

9.7.3 The line in $\mathbb{R}^2$ that passes through the point $\mathbf{r}_0 = (x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = (a, b) = a\mathbf{i} + b\mathbf{j}$ has parametric equations

$$x(t) = x_0 + at \,, \quad y(t) = y_0 + bt \,,$$

or, in vector form $\mathbf{r}(t) = (x(t), y(t))$,

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} \,.$$

In $\mathbb{R}^3$, the vector form is the same, but now $\mathbf{r}(t) = (x(t), y(t), z(t))$, $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{v} = (a, b, c)$ and the component equations are

$$x(t) = x_0 + at \,, \quad y(t) = y_0 + bt \,, \quad z(t) = z_0 + ct \,.$$

(a) Find the parametric equations of the line
  (i) passing through $(4, 2)$ and parallel to $\mathbf{v} = (-1, 5)$,
  (ii) passing through $(1, 2, -3)$ and parallel to $\mathbf{v} = (4, 5, -7)$.
(b) Find parametric equations for the line whose vector equation is given as
  (i) $\mathbf{r}(t) = 2\mathbf{i} - 3\mathbf{j} + t\mathbf{i} - 4t\mathbf{j}$,
  (ii) $\mathbf{r}(t) = -\mathbf{i} + 0\mathbf{j} + 2\mathbf{k} - t\mathbf{i} + 3t\mathbf{j}$.

9.7.4 The equation of the plane passing through a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ in $\mathbb{R}^3$ and perpendicular to a vector $\mathbf{N} = (a, b, c)$, called a normal for the plane, is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \,,$$

that is, the plane consists of the points $\mathbf{r} = (x, y, z)$ satisfying this equation. Its vector form is

$$\mathbf{N} \cdot (\mathbf{r} - \mathbf{r_0}) = 0 \,.$$

Find an equation of the plane that passes through the point $\mathbf{r}_0 = (2, 6, 1)$ having the vector $\mathbf{N} = (1, 4, 2)$ as a normal.

9.7.5 Let $\mathbf{F}$ and $\mathbf{G}$ be two vector fields defined on an interval with values in $\mathbb{R}^3$. Prove that

(a) $\dfrac{d}{dt}[\mathbf{F} \cdot \mathbf{G}] = \mathbf{F} \cdot \dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt} \cdot \mathbf{G}$.

(b) $\dfrac{d}{dt}[\mathbf{F} \times \mathbf{G}] = \mathbf{F} \times \dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt} \times \mathbf{G}$.

(c) Let $\mathbf{r} = \mathbf{r}(t)$ be a vector-valued function with values in $\mathbb{R}^3$ such that $\|\mathbf{r}(t)\| = c$ for some constant c. Show that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = \mathbf{r}(t) \cdot \dfrac{d\mathbf{r}}{dt}(t) = 0$, that is, $\mathbf{r}(t)$ is orthogonal to $\dfrac{d\mathbf{r}}{dt}(t) = \mathbf{r}'(t)$, for all t.

9.7.6 (a) Verify explicitly that $\operatorname{div}(\mathbf{curl}\,\mathbf{F}) = 0$, where

(i) $\mathbf{F} = \sinh(x - z)\mathbf{i} + 2y\mathbf{j} + (z - y^2)\mathbf{k}$,

(ii) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.

(b) Verify explicitly that $\mathbf{curl}\,(\nabla\varphi) = 0$, where

(i) $\varphi(x, y, z) = -2x^3yz^2$,

(ii) $\varphi(x, y, z) = e^{x+y+z}$.

9.7.7 (a) Let $\mathbf{F}$ and $\mathbf{G}$ be two vector fields with domain in $\mathbb{R}^3$ and values in $\mathbb{R}^3$. Prove that

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

(b) Let $\varphi = \varphi(x, y, z)$ and $\psi = \psi(x, y, z)$ be scalar fields. Prove that $\operatorname{div}(\nabla\varphi \times \nabla\psi) = 0$.

9.7.8 Let $\mathbf{C}$ be a constant vector and $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Prove that

(a) $\nabla(\mathbf{C} \cdot \mathbf{r}) = \mathbf{C}$,

(b) $\operatorname{div}(\mathbf{r} - \mathbf{C}) = 3$,

(c) $\mathbf{curl}(\mathbf{r} - \mathbf{C}) = 0$.

9.7.9 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \mathbf{i} - x\mathbf{j} + \mathbf{k}$ and C is parametrized by $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + t\mathbf{k}$ for $0 \le t \le \pi$.

9.7.10 Evaluate the surface integral $\iint_\Sigma f\, d\sigma$, where

(a) $f(x, y, z) = y^2$, Σ is the part of the cone $z = \sqrt{x^2 + y^2}$ lying in the first octant and between the plane $z = 2$ and $z = 4$.

(b) $f(x, y, z) = xyz$, Σ is the part of the surface $z = 1 + y^2$ for $0 \le x \le 1$, $0 \le y \le 1$.

9.7.11 A particle moves once counterclockwise around the circle of radius 12 centered at the origin under the influence of the force $\mathbf{F}(x, y, z) = (e^x - y + x\cosh x)\mathbf{i} + (y^{3/2} + x)\mathbf{j}$. Calculate the work done.

9.7.12 Apply the Green–Ostrogradski theorem to evaluate the line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following data. The curves are oriented counterclockwise.

(a) $\mathbf{F} = x^2 y\mathbf{i} - xy^2\mathbf{j}$, C is the boundary of the region defined by $x^2 + y^2 \leq 4$, $x \geq 0$ and $y \geq 0$.
(b) $\mathbf{F} = (e^{\sin x} - y)\mathbf{i} + (\sinh(y^3) - 4x)\mathbf{j}$, C is the circle of radius 4 with center $(-8, 0)$.
(c) $\mathbf{F} = (x^2 + y^2)\mathbf{i} + (x^2 - y^2)\mathbf{j}$, C is the ellipse $4x^2 + y^2 = 10$.

9.7.13 Let D be a region bounded by the surface Σ. Evaluate $\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, d\sigma$ or $\iiint_D \operatorname{div}(\mathbf{F})\, dV$ for the following data, whichever is more convenient.

(a) $\mathbf{F} = 4x\mathbf{i} - 6y\mathbf{j} + \mathbf{k}$, Σ is the surface of the solid cylinder $x^2 + y^2 \leq 4$, $0 \leq z \leq 6$ (the surface includes both caps of the cylinder).
(b) $\mathbf{F} = 2yz\mathbf{i} - 4xz\mathbf{j} + xy\mathbf{k}$, Σ is the sphere of radius 6 with center $(-1, 3, 1)$.

9.7.14 Find the value of $\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, d\sigma$ for the data $\mathbf{F} = 3xy\mathbf{i} + z^2\mathbf{k}$, Σ the sphere of radius 1 centered at the origin.

9.7.15 Let Σ be a smooth surface enclosing some region D, let $\mathbf{C}$ be a constant vector. Show that $\iint_\Sigma \mathbf{C} \cdot \mathbf{n}\, d\sigma = 0$.

9.7.16 Evaluate $\iint_\Sigma (\mathbf{curl\, F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$ and Σ is the part of the plane $2x + 4y + z = 8$ in the first octant.

9.7.17 Calculate the circulation of $\mathbf{F} = (x - y)\mathbf{i} + x^2 y\mathbf{j} + xza\mathbf{k}$ counterclockwise around the circle $x^2 + y^2 = 1$, where a is a positive constant.

9.7.18 Examine whether the following vector fields are conservative or not.

(a) $\mathbf{F} = \cosh(x + y)(\mathbf{i} + \mathbf{j} - \mathbf{k})$,
(b) $\mathbf{F} = 2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k}$,
(c) $\mathbf{F} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

9.7.19 Let Σ be the portion of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$, and let C be the circle $x^2 + y^2 = 1$ that forms the boundary of Σ. Verify Stokes' theorem for the vector field

$$\mathbf{F} = (x^2 y - z^2)\mathbf{i} + (y^3 - x)\mathbf{j} + (2x + 3z - 1)\mathbf{k}$$

by evaluating the line integral as well as the surface integral.

Chapter 10
Fourier Methods with Applications

10.1 Introduction

Fourier methods or Fourier analysis is a branch of mathematics that was developed formally some 150 years after Newton's and Leibniz' calculus and heavily depended on integral and differential calculus. Jean Baptiste Joseph Fourier was born in 1768 in Auxerre, a town between Paris and Dijon. He became fascinated by mathematics at the age of 13 years. After the French revolution, Fourier taught in Paris, then accompanied Napoleon to Egypt and served as permanent secretary of the Institute of Egypt. He wrote a book on Egypt and in certain quarters he is famous as an Egyptologist rather than for his contributions to mathematics and physics. In the world of science, he is famous for, among other things, the ideas and thoughts he set forth in a memoir in 1807 and published in 1822 in his book in French entitled "The Analytic Theory of Heat".

Fourier analysis shows that we can represent periodic functions, even very jagged and irregular-looking ones, in form of a finite or infinite sum of sine and cosine functions called **Fourier series**. Nonperiodic functions can be treated with the **Fourier transform**. Fourier himself showed how these mathematical tools can be used to study natural phenomena such as heat diffusion, making it possible to solve equations that had until then remained intractable. Under the action of the Fourier transform, derivatives are transformed into multiplications, thus turning differential equations into equations containing algebraic expressions. In this way, many important differential equations are transformed into equations which are much easier to study and to solve.

If a phenomenon is described by a function of time or of space, Fourier analysis tells us, loosely speaking, how much of each frequency this phenomenon contains. In many cases, this frequency information is not simply a mathematical trick to make calculations easier, but corresponds to relevant properties of the phenomenon under study.

During the second half of the previous century, Fourier analysis has been refined in various directions in order to make it easier and more efficient to transmit, compress,

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and analyze information or to separate information from surrounding noise. One of those techniques, the wavelet transform which was invented during the 80s, will be presented at the end of this chapter.

Fourier analysis and its descendants have been applied to a broad range of areas of science and engineering, let us mention telecommunications and signal processing, physics, imaging in the biomedical sciences (EEG, ECG, CAT, MRI, NMR) and elsewhere, meteorology, oceanography, seismology, economics, and finance. A few of them will be touched in this chapter. Interested readers may find details of some of these applications in the references Prestini [28] and Cartwright [7]. There are also many interactions within mathematics itself, in areas as diverse as statistics and number theory.

For several reasons, we have chosen not to present proofs (detailed or sketched) for the majority of results of this chapter, and we refer the reader to the literature on the subject.

Finally, we want to acknowledge the contribution of Eng. A.K. Verma to this chapter, whose program we have used in order to give pictures of the partial sums of several of the Fourier series discussed below.

10.2 Orthonormal Systems and Fourier Series

10.2.1 Orthonormal Systems

In Chap. 9, we have introduced the concept of orthogonal (or perpendicular) vectors. Namely, two vectors $\mathbf{u}$ and $\mathbf{v}$ are called orthogonal if their scalar product (dot product) $\mathbf{u} \cdot \mathbf{v}$ equals zero. Recall also that $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 > 0$ if $\mathbf{u} \neq \mathbf{0}$. We now extend this concept in two respects. We define orthogonality for a system of more than two vectors, and we define orthogonality for functions instead of vectors.

During all of this chapter, the functions considered will be piecewise continuous. Such functions are formed by putting together continuous functions defined on separate intervals, for example, the sign function or the greatest integer function as considered in Sect. 1.3, or the "sawtooth" function defined by

$$f(x) = x - k, \quad \text{if } k - \frac{1}{2} \leq x < k + \frac{1}{2}. \tag{10.1}$$

The formal definition runs as follows. A function f defined on an interval $I = [a, b]$ is called **piecewise continuous**, if we can decompose I into finitely many subintervals $I_k = [x_{k-1}, x_k]$ such that f is continuous in the interior of I_k and the one-sided limits $\lim_{x \to x_{k-1}+} f(x)$ and $\lim_{x \to x_k-} f(x)$ exist for all such subintervals. If the domain of f is unbounded, we require this property to hold for every bounded interval in the domain. As a consequence, we can integrate piecewise continuous functions, and

$$\int_a^b f(x)\,dx = \sum_k \int_{x_{k-1}}^{x_k} f(x)\,dx\,.$$

Definition 10.1 (*Orthogonal functions*) Let f_1, f_2 be piecewise continuous real-valued functions defined on an interval $[a, b]$. Their **scalar product** (or **inner product**) is denoted by $\langle f_1, f_2\rangle$ and defined by

$$\langle f_1, f_2\rangle = \int_a^b f_1(x) f_2(x)\,dx\,. \tag{10.2}$$

The functions f_1 and f_2 are said to be **orthogonal on [*a*, *b*]** if

$$\langle f_1, f_2\rangle = \int_a^b f_1(x) f_2(x)\,dx = 0\,. \tag{10.3}$$

Example 10.1 (a) The functions $f_1(x) = e^x$ and $f_2(x) = \sin x$ are orthogonal on the interval $[\pi/4, 5\pi/4]$.
(b) The functions $f_1(x) = x$ and $f_2(x) = \cos 2x$ are orthogonal on the interval $[-\pi/2, \pi/2]$.

In analogy to the case of vectors, for functions $f : [a, b] \to \mathbb{R}$, we define

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b f(x)^2\,dx} \tag{10.4}$$

and call it the **norm** of f. (We might also call $\|f\|$ the "length" of f, although its geometric meaning is no longer obvious.) We then have

$$\|f\|^2 = \langle f, f\rangle = \int_a^b f(x)^2\,dx\,.$$

As in the case of vectors, we have $\|0\| = 0$ for the zero function. On the other hand, a piecewise continuous function f satisfies $\|f\| = 0$ only if f is the zero function; in other words, if f is nonzero, we must have $\|f\| > 0$.

If two nonzero functions f_1 and f_2 are orthogonal, we say that the set $\{f_1, f_2\}$ formed by those two functions is an orthogonal system. If we have more than two functions, we require that the functions are mutually (or pairwise) orthogonal in the sense of the following definition.

Definition 10.2 (*Orthogonal system, orthonormal system*) A system (or set) consisting of finitely or infinitely many nonzero functions $f_1, f_2, \ldots$ is said to be **orthogonal on the interval [*a*, *b*]** if

$$\langle f_m, f_n\rangle = \int_a^b f_m(x) f_n(x) = 0\,, \quad \text{whenever } m \neq n.$$

If moreover

$$\langle f_n, f_n \rangle = 1 = \|f_n\|, \quad \text{for all } n,$$

the system is called **orthonormal on $[a, b]$**.

Remark 10.1 An orthogonal system $\{f_1, f_2, \ldots\}$ can be made into an orthonormal system by replacing each function f_n with its scalar multiple

$$\frac{f_n}{\|f_n\|}.$$

Example 10.2 (a) The set $\{1, \cos x, \cos 2x, \ldots, \cos nx\}$ is an orthogonal system on $[-\pi, \pi]$.

(b) The set $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}\right\}$ is an orthonormal system on $[-\pi, \pi]$.

(c) Examine whether the following systems are orthonormal or not on the intervals indicated:

(i) $\{\sin x, \sin 3x, \sin 5x, \ldots\}$, $I = [0, \pi/2]$,

(ii) $\{\sin nx\} : n = 1, 2, 3, \ldots\}$, $I = [0, \pi]$,

(iii) $\{1, \frac{\cos n\pi x}{l} : n = 1, 2, 3, \ldots\}$, $I = [0, l]$,

(iv) $\{1, \frac{\cos n\pi x}{l}, \frac{\sin m\pi x}{l}\} : n, m = 1, 2, 3, \ldots\}$, $I = [-l, l]$.

Solution: We discuss here part (c).
(i) We set $f_n(x) = \sin(2n+1)x$. For $m \neq n$, we obtain

$$\begin{aligned}
\langle f_n, f_m \rangle &= \int_0^{\pi/2} \sin(2n+1)x \sin(2m+1)x\, dx \\
&= \frac{1}{2}\int_0^{\pi/2} [\cos 2(n-m)x - \cos 2(n+m+1)x]\, dx
\end{aligned}$$

(using the trigonometric identity $2\sin Ax \sin Bx = \cos(A-B)x - \cos(A+B)x$)

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{\sin 2(n-m)x}{2(n-m)}\right]_0^{\pi/2} - \frac{1}{2}\left[\frac{\sin 2(n+m+1)x}{2(n+m+1)}\right]_0^{\pi/2} \\
&= 0 - 0 = 0\,.
\end{aligned}$$

For $n = m$, we get

$$\int_0^{\pi/2} \sin^2(2n+1)x\, dx = \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2(2n+1)x\right) dx$$

(using the identity $2\sin^2 Ax = 1 - \cos 2Ax$)

$$
\begin{aligned}
&= \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2} \cos 2(2n+1)x\, dx \\
&= \Big[\frac{1}{2}x\Big]_0^{\pi/2} - \frac{1}{2}\Big[\frac{1}{2(2n+1)} \sin 2(2n+1)x\Big]_0^{\pi/2} \\
&= \frac{\pi}{4}\,.
\end{aligned}
$$

Hence, the given system is orthogonal but not orthonormal. However, if we multiply each function by $2/\sqrt{\pi}$, the new system obtained in this way is orthonormal, that is,

$$
\left\{\frac{2\sin x}{\sqrt{\pi}}, \frac{2\sin 3x}{\sqrt{\pi}}, 2\frac{\sin 5x}{\sqrt{\pi}}, \dots, \frac{2\sin(2n+1)x}{\sqrt{\pi}}, \dots, \right\}
$$

is an orthonormal system on $[0, \pi/2]$.
(ii) Set $f_n(x) = \sin nx$. For $m \neq n$, we have

$$
\langle f_n, f_m\rangle = \int_0^{\pi} \sin nx \sin mx\, dx = \frac{1}{2}\int_0^{\pi} \Big[\cos(n-m)x - \cos(n+m)x\Big]\, dx
$$

(by the trigonometric identity $2\sin Ax \sin Bx = \cos(A-B)x - \cos(A+B)x$)

$$
\begin{aligned}
&= \frac{1}{2}\int_0^{\pi} [\cos(n-m)x]\, dx - \frac{1}{2}\int_0^{\pi} [\cos(n+m)x]\, dx \\
&= \frac{1}{2}\Big[\frac{\sin(n-m)x}{(n-m)}\Big]_0^{\pi} - \frac{1}{2}\Big[\frac{\sin(n+m)x}{(n+m)}\Big]_0^{\pi} = 0\,.
\end{aligned}
$$

For $m = n$, we have

$$
\begin{aligned}
\langle f_n, f_n\rangle &= \int_0^{\pi} \sin^2 nx\, dx = \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2nx\right) dx \\
&= \frac{1}{2}\pi - \frac{1}{4n}\Big[\sin 2nx\Big]_0^{\pi} = \frac{\pi}{2}\,,
\end{aligned}
$$

where again we have used that $2\sin^2 Ax = 1 - \cos 2Ax$. Hence, $\{\sin nx\}$ is an orthogonal, but not an orthonormal system on $[0, \pi]$. But the system $\left\{\frac{\sqrt{2}\sin nx}{\sqrt{\pi}}\right\}$ is an orthonormal system on $[0, \pi]$.
(iii) For $m \neq n$, we have

$$
\int_0^{l} \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l}\, dx = \frac{1}{2}\int_0^{l} \left[\cos\frac{(n-m)\pi x}{l} + \cos\frac{(n+m)\pi x}{l}\right] dx
$$

(by the trigonometric identity $2\cos Ax \cos Bx = \cos(A-B)x + \cos(A+B)x$)

$$
\begin{aligned}
&= \frac{1}{2}\int_0^l \cos\frac{(n-m)\pi x}{l}\,dx + \frac{1}{2}\int_0^l \cos\frac{(n+m)\pi x}{l}\,dx \\
&= \frac{l}{2(n-m)\pi}\left[\sin\frac{(n-m)\pi x}{l}\right]_0^l + \frac{l}{2(n+m)\pi}\left[\sin\frac{(n+m)\pi x}{l}\right]_0^l \\
&= 0+0=0\,.
\end{aligned}
$$

For $m = n$, we have

$$
\begin{aligned}
\int_0^l \cos^2\frac{n\pi x}{l}\,dx &= \int_0^l\left(\frac{1}{2}+\frac{1}{2}\cos\frac{2n\pi x}{l}\right)dx = \left[\frac{1}{2}x\right]_0^l + \frac{l}{4n\pi}\left[\sin\frac{2n\pi x}{l}\right]_0^l \\
&= \frac{l}{2}\,.
\end{aligned}
$$

Moreover,

$$
\int_0^l 1\cdot\cos\frac{n\pi x}{l}\,dx = \left[\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right]_0^l = 0\,,\quad \int_0^l 1^2dx = l\,.
$$

The given system is orthogonal but not orthonormal. However, the system

$$
\left\{\frac{1}{\sqrt{l}},\frac{\sqrt{2}}{\sqrt{l}}\frac{\cos n\pi x}{l}\right\}
$$

is orthonormal.
(iv) We know already from part (iii) that

$$
\int_{-l}^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = 2\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = 0\,.
$$

Concerning the sine, for $m \neq n$, we get

$$
\int_{-l}^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = 2\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx
$$

(by the identity $2\sin Ax\sin Bx = \cos(A-B)x - \cos(A+B)x$)

$$
\begin{aligned}
&= \int_0^l\left[\cos\frac{(n-m)\pi x}{l} - \cos\frac{(n+m)\pi x}{l}\right]dx \\
&= \left[\sin\frac{(n-m)\pi x}{l}\cdot\frac{l}{n-m}\right]_0^l - \left[\sin\frac{(n+m)\pi x}{l}\cdot\frac{l}{n+m}\right]_0^l \\
&= 0-0=0\,.
\end{aligned}
$$

Using the same identities as in the previous computations one derives that

$$\int_{-l}^{l} \sin^2 \frac{n\pi x}{l}\,dx = l\,, \quad \int_{-l}^{l} \cos^2 \frac{n\pi x}{l}\,dx = l\,,$$
$$\int_{-l}^{l} \sin \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\,dx = 0\,, \quad \text{for } n \neq m,$$

moreover

$$\int_{-l}^{l} 1 \cdot \cos \frac{n\pi x}{l}\,dx = 0\,, \quad \int_{-l}^{l} 1 \cdot \sin \frac{n\pi x}{l}\,dx = 0\,, \quad \int_{-l}^{l} 1^2\,dx = 2l\,.$$

The given system is orthogonal but not orthonormal. However, the system

$$\left\{ \frac{1}{\sqrt{l}}, \frac{1}{\sqrt{l}} \frac{\cos n\pi x}{l}, \frac{1}{\sqrt{l}} \frac{\sin n\pi x}{l} \right\}$$

is orthonormal.

Remark 10.2 The systems named after Rademacher, Haar, and Walsh are other well-known orthonormal systems.

A function f is said to be a **linear combination** of the functions $f_1, f_2, \ldots, f_n$ if

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$$

holds for suitably chosen scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Definition 10.3 (*Linear Dependence and Independence*) A system $\{f_1, f_2, \ldots, f_n\}$ of functions is said to be **linearly independent** if $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \cdots + \alpha_n f_n = 0$ implies that $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$. The system is called **linearly dependent** if it is not linearly independent. In other words, at least one element of the system is a linear combination of the remaining $n - 1$ elements. An infinite system $\{f_1, f_2, \ldots\}$ of functions is said to be linearly independent, if every finite subset taken from it forms a linearly independent system in the sense above, and it is said to be linearly dependent if this is not the case.

Remark 10.3 It turns out that every orthogonal (and, hence, every orthonormal) system is linearly independent. However, the converse is not true, so a system may be linearly independent without being orthogonal.

10.2.2 Fourier Series

A series of the form

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

that is,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{10.5}$$

is called an (infinite) **trigonometric series**.

Let f be a piecewise continuous function defined on the interval $[-\pi, \pi]$. The numbers

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\,, \tag{10.6}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\,, \tag{10.7}$$

where $n = 1, 2, 3, \ldots$, are called the **Fourier coefficients** (more precisely, the Fourier cosine resp. sine coefficients) of f on the interval $[-\pi, \pi]$, and the series (10.5) with the coefficients from (10.6) and (10.7) is called the **Fourier series** of f on this interval. Note that $a_0/2$ is just the average of the function f over $[-\pi, \pi]$.

The Fourier coefficients arise when we want to represent a function $f : [-\pi, \pi] \to \mathbb{R}$ as a trigonometric series,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,, \tag{10.8}$$

for the following reason. For an arbitrary trigonometric series, we define the partial sums

$$s_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)\,. \tag{10.9}$$

(We remind the reader that we have studied series and their partial sums in Chap. 5.) Now let us compute, for $N \geq n \geq 1$, using the orthogonality of the system $\{\cos nx, \sin nx : n = 1, 2, \ldots\}$ on $[-\pi, \pi]$,

$$\begin{aligned}\int_{-\pi}^{\pi} s_N(x)\cos nx\,dx &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\cos nx\,dx + \sum_{k=1}^{N} a_k \int_{-\pi}^{\pi}\cos nx \cos kx\,dx \\ &\quad + \sum_{k=1}^{N} b_k \int_{-\pi}^{\pi}\cos nx \sin kx\,dx \\ &= 0 + a_n\pi + 0 = \pi a_n\,.\end{aligned} \tag{10.10}$$

If f can be represented as in (10.8), that is, if

$$f(x) = \lim_{N\to\infty} s_N(x)\,,$$

and if moreover

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \lim_{N\to\infty}\int_{-\pi}^{\pi} s_N(x)\cos nx\,dx\,,$$

then we see from (10.10) that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,.$$

Similarly, we get the corresponding formula for b_n by using $\sin nx$ instead of $\cos nx$ in (10.10).

Instead of the interval $[-\pi, \pi]$ we may consider an interval $[-l, l]$, where $l > 0$ is an arbitrary number.

Definition 10.4 (*Fourier series*) Let f be a piecewise continuous function defined on the interval $[-l, l]$, where $l > 0$. Then the Fourier series of f on $[-l, l]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)\,, \quad x \in [-l, l]\,,$$

where

$$a_0 = \tfrac{1}{l}\textstyle\int_{-l}^{l} f(x)\,dx\,, \tag{10.11}$$

$$a_n = \tfrac{1}{l}\textstyle\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx\,, \quad b_n = \tfrac{1}{l}\textstyle\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\,. \tag{10.12}$$

If the Fourier series at x converges to $f(x)$, we write as usual

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)\,. \tag{10.13}$$

The following theorem gives conditions under which (10.13) holds, that is, when the Fourier series of f on $[-l, l]$ converges to f.

Theorem 10.1 (Dirichlet Convergence Theorem) *Let f and f' be piecewise continuous on the interval $[-l, l]$, let $x \in (-l, l)$.*
(i) If f is continuous at x, the Fourier series of f at x converges to $f(x)$.

(ii) If f is discontinuous at x, the Fourier series of f at x converges to the mean value

$$\frac{f(x+)+f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the right- and left-hand limits, respectively, of f at x.

Note that in fact (i) can be viewed as a special case of (ii), since $f(x)=f(x+)=f(x-)$ whenever f is continuous at x.

Example 10.3 Expand the following functions in Fourier series:

(a) $$f(x)=\begin{cases}0, & -\pi<x<0,\\ \pi-x, & 0\le x<\pi,\end{cases} \qquad \text{on } [-\pi,\pi],$$

(b) $$f(x)=\begin{cases}0, & -\pi<x<0,\\ 1, & 0\le x<\pi,\end{cases} \qquad \text{on } [-\pi,\pi],$$

(c) $$f(x)=\begin{cases}0, & -1<x<0,\\ x, & 0\le x<1,\end{cases} \qquad \text{on } [-1,1],$$

(d) $$f(x)=\begin{cases}0, & -\pi<x<0,\\ \sin x, & 0\le x<\pi,\end{cases} \qquad \text{on } [-\pi,\pi],$$

(e) $$f(x)=\begin{cases}-\frac{\pi}{4}, & -\pi<x<0,\\ 0, & x=0,\\ \frac{\pi}{4}, & 0\le x<\pi,\end{cases} \qquad \text{on } [-\pi,\pi],$$

(f) $$f(x)=\begin{cases}-1, & -4<x<0,\\ 1, & 0\le x<4,\end{cases} \qquad \text{on } [-4,4],$$

(g) $$f(x)=e^x, \qquad \text{on } [-\pi,\pi].$$

Solution: (a) Here we have $l=\pi$. We get

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]\\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi-x)\,dx\right] = \frac{1}{\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}.\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi-x)\cos nx\,dx\right]\\
&= \frac{1}{\pi}\left[(\pi-x)\frac{\sin nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx dx\\
&= \left[-\frac{1}{n\pi}\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-\cos n\pi+1}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}.
\end{aligned}$$

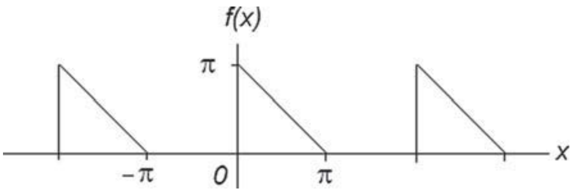


Fig. 10.1 The function from Example 10.3(a)

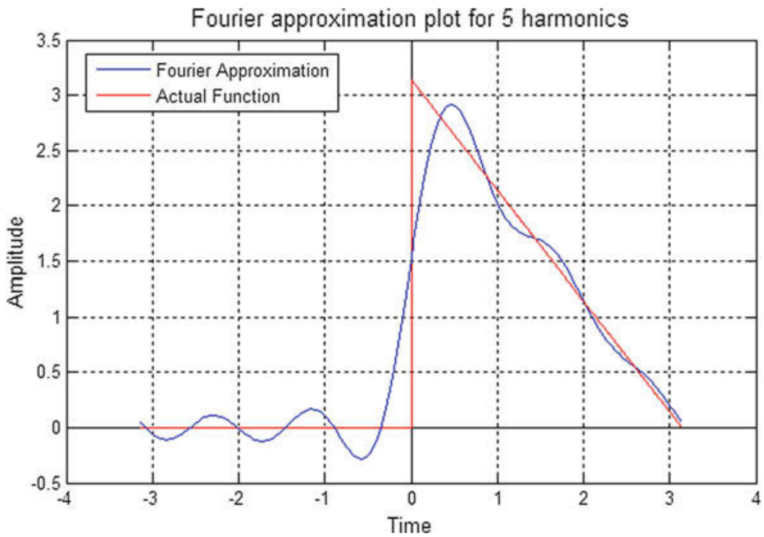


Fig. 10.2 Fourier approximation of the function in Fig. 10.1

Similarly, we can calculate that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$
$$= \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{n}\,.$$

Since the given function is continuous except at $x = 0$, the Fourier series converges. See the graphs of the function and of a partial sum of its Fourier series in Figs. 10.1 and 10.2, respectively.

(b) We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right] = 0 + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx$$

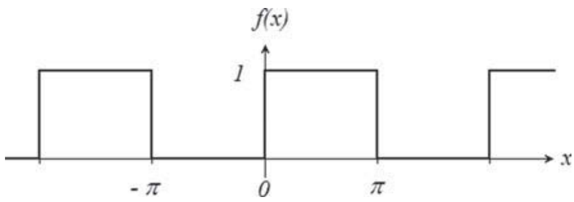


Fig. 10.3 The function from Example 10.3(b)

$$
\begin{aligned}
&= 1\,,\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]\\
&= 0 + \int_0^{\pi} 1\cos nx\,dx = \left[\frac{\sin nx}{n}\right]_0^{\pi} = 0\,,\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]\\
&= 0 + \frac{1}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\\
&= \frac{1}{\pi}\left[-\frac{\cos n\pi}{n} + \frac{1}{n}\right]\\
&= -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi}\,, \qquad \text{as } \cos n\pi = (-1)^n,\\
&= \frac{1}{n\pi}(1-(-1)^n)\,.
\end{aligned}
$$

Since the given function is continuous except at $x = 0$, the Fourier series converges to the given function for $x \neq 0$, and we have

$$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx\,.$$

For the graphs of f and of a partial sum of its Fourier series see Figs. 10.3 and 10.4, respectively.

(c) Here $l = 1$. The given function is continuous also at $x = 0$, and hence its Fourier series converges everywhere. We get

$$
\begin{aligned}
a_0 &= \int_{-1}^{1} f(x)\,dx = \frac{1}{\pi}\left[\int_{-1}^{0} f(x)\,dx + \int_0^1 f(x)\,dx\right] = \left[\int_{-1}^{0} 0\,dx + \int_0^1 x\,dx\right]\\
&= 0 + \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \left[\frac{1}{2}\cdot 1^2 - 0\right] = \frac{1}{2}\,,
\end{aligned}
$$

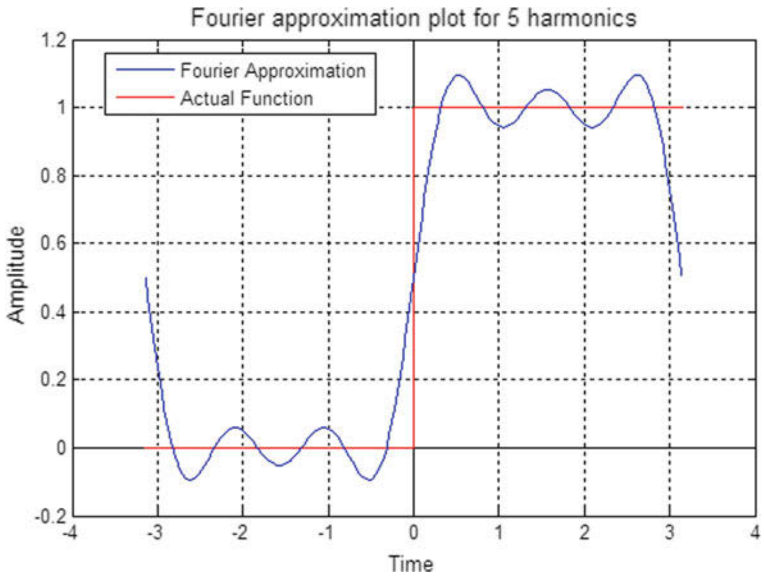


Fig. 10.4 Fourier approximation of the function in Fig. 10.3

$$
\begin{aligned}
a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos nx\,dx = \int_{-1}^{0}\cos n\pi x\,dx + \int_{0}^{1} f(x)\cos n\pi x\,dx \\
&= \int_{-1}^{0} 0\cos n\pi x\,dx + \int_{0}^{1} x\cos n\pi x\,dx \\
&= 0 + \left[\frac{x\sin n\pi x}{n}\pi\right]_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx \\
&= \left[\frac{\sin n\pi}{n\pi} - 0\right] + \left[\frac{\cos n\pi x}{\pi^2 n^2}\right]_0^1 = 0 + \left[\frac{\cos n\pi}{n^2\pi^2} - \frac{\cos n\pi 0}{\pi^2 n^2}\right] \\
&= \frac{(-1)^n}{\pi^2 n^2} - \frac{1}{n^2\pi^2} = \frac{1}{n^2\pi^2}[(-1)^n - 1]\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} f(x)\sin n\pi x\,dx + \int_{0}^{1} f(x)\sin n\pi x\,dx \\
&= 0 + \int_0^1 x\sin n\pi x\,dx = \frac{(-1)^{n+1}}{n\pi}\,,
\end{aligned}
$$

through integration by parts. See Figs. 10.5 and 10.6 for the graphs of the function and of a partial sum of its Fourier series.

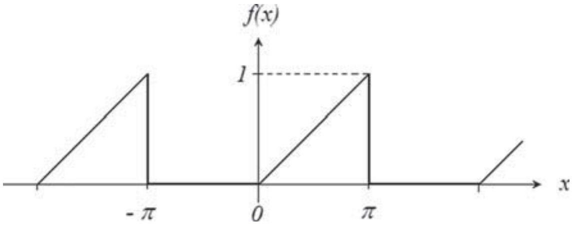


Fig. 10.5 The function from Example 10.3(c)

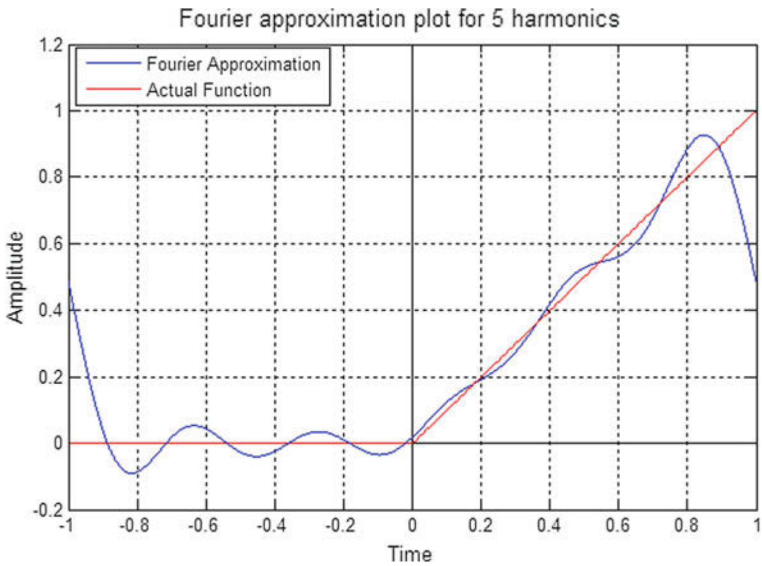


Fig. 10.6 Fourier approximation of the function in Fig. 10.5

(d) We have

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx \\
&= 0 + \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}\,. \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right] \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\,dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} [\sin(n+1)x + \sin(1-n)x]dx
\end{aligned}$$

(by the identity $2\sin Ax\cos Bx = \sin(A+B)x + \sin(A-B)x$)

$$
\begin{aligned}
&= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1}\right]_0^{\pi} + \frac{1}{2\pi}\left[\frac{-\cos(n-1)x}{1-n}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{1}{n+1}\right] + \frac{1}{2\pi}\left[\frac{-\cos(1-n)\pi}{1-n} + \frac{1}{1-n}\right] \\
&= \frac{1+(-1)^n}{\pi(1-n^2)}\,, \quad \text{for } n = 2, 3, 4, \ldots \\
a_1 &= \frac{1}{2\pi}\int_0^{\pi} \sin 2x\, dx = 0\,. \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\, dx \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos(1-n)x - \cos(1+n)x]\, dx \\
&= 0\,, \quad \text{for } n = 2, 3, 4, \ldots \\
b_1 &= \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x)\, dx = \frac{1}{2}\,.
\end{aligned}
$$

Thus,

$$
f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{1+(-1)^n}{\pi(1-n^2)}\cos nx
$$

is the Fourier series for f.

See the graph of the function in Fig. 10.7 and the graph of a partial sum of its Fourier series in Fig. 10.8.

(e) We have

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{4}\, dx + \frac{1}{\pi}\int_0^{\pi} \frac{\pi}{4}\, dx - \frac{\pi}{4} + \frac{\pi}{4} = 0\,.
\end{aligned}
$$

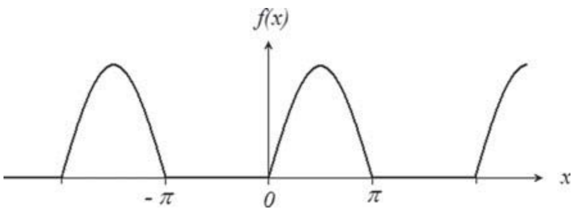


Fig. 10.7 The function from Example 10.3(d)

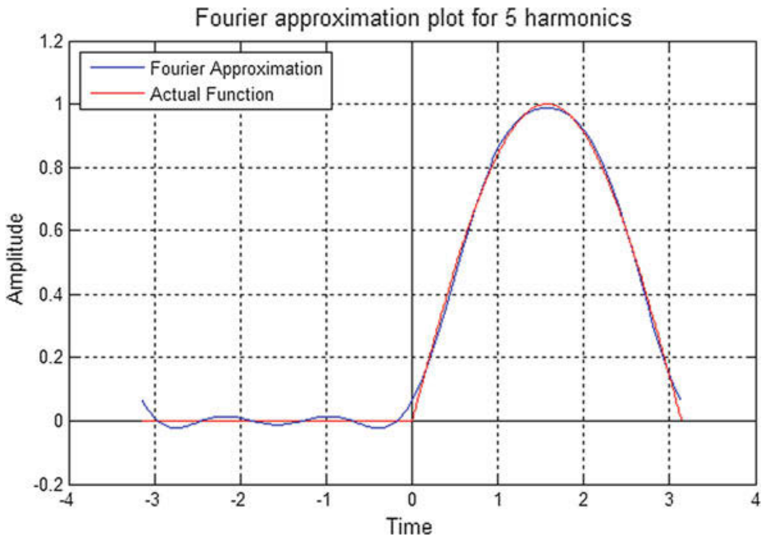


Fig. 10.8 Fourier approximation of the function in Fig. 10.7

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \\
&= -\frac{1}{\pi}\frac{\pi}{4}\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{\pi}\frac{\pi}{4}\int_{0}^{\pi}\cos nx\,dx \\
&= -\frac{1}{4}\Big[\frac{\sin nx}{n}\Big]_{-\pi}^{0} + \frac{1}{4}\Big[\frac{\sin nx}{n}\Big]_{0}^{\pi} \;=\; 0\,.
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{4}\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{\pi}{4}\sin nx\,dx \\
&= \frac{1}{4}\Big[\frac{\cos nx}{n}\Big]_{-\pi}^{0} - \frac{1}{4}\Big[\frac{\cos nx}{n}\Big]_{0}^{\pi} \\
&= \frac{1}{4n} - \frac{\cos n\pi}{4n} - \frac{1}{4n}\cos n\pi + \frac{1}{4n} = \frac{2}{4n} - \frac{2(-1)^n}{4n}\,.
\end{aligned}
$$

Thus, the Fourier series for f is

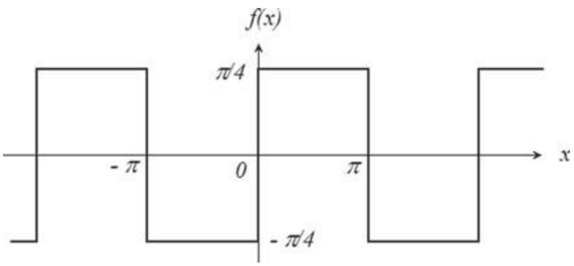


Fig. 10.9 The function from Example 10.3(e)

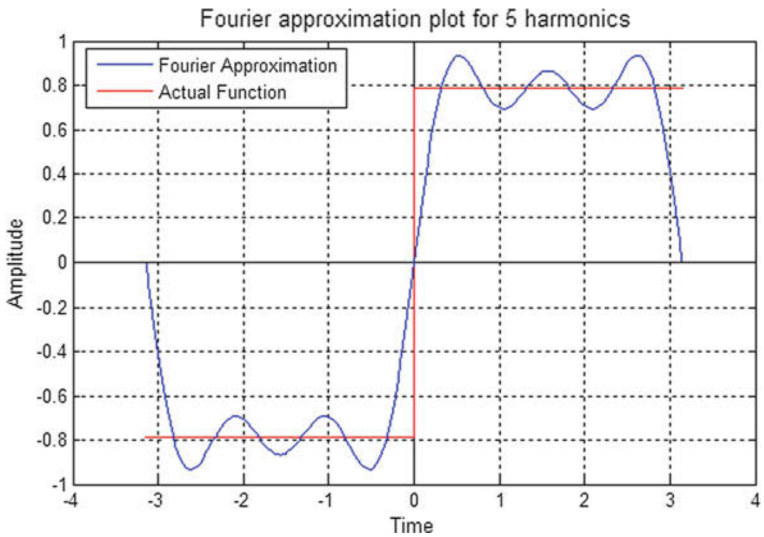


Fig. 10.10 Fourier approximation of the function in Fig. 10.9

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x\,.$$

The graph of this function and of a partial sum of its Fourier series are given in Figs. 10.9 and 10.10, respectively.

(f) We have

$$\begin{aligned} a_0 &= \frac{1}{4}\int_{-4}^{4} f(x)\,dx = \frac{1}{4}\int_{-4}^{0} f(x)\,dx + \frac{1}{4}\int_{0}^{4} f(x)\,dx = \frac{1}{4}[-x]_{-4}^{0} + \frac{1}{4}[x]_0^4 \\ &= -1 + 1 = 0\,. \\ a_n &= -\frac{1}{4}\int_{-4}^{0} \cos\frac{n\pi x}{4}\,dx + \frac{1}{4}\int_{0}^{4} \cos\frac{n\pi x}{4}\,dx \end{aligned}$$

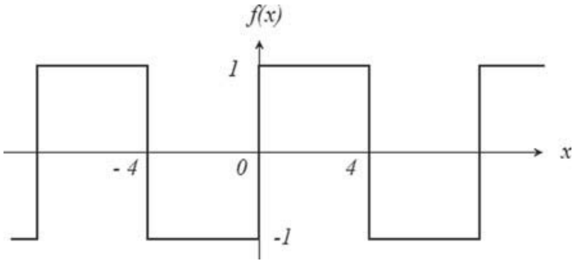


Fig. 10.11 The function from Example 10.3(f)

$$= -\frac{1}{4}\left[\frac{4}{n\pi}\sin\frac{n\pi x}{4}\right]_{-4}^{0} + \frac{1}{4}\left[\frac{4}{n\pi}\sin\frac{n\pi x}{4}\right]_{0}^{4} = 0\,.$$

$$\begin{aligned}
b_n &= \frac{1}{4}\int_{-4}^{0} f(x)\sin\frac{n\pi x}{4}\,dx + \frac{1}{4}\int_{0}^{4} f(x)\sin\frac{n\pi x}{4}\,dx \\
&= -\frac{1}{4}\int_{-4}^{0}\sin\frac{n\pi x}{4}\,dx + \frac{1}{4}\int_{0}^{4}\sin\frac{n\pi x}{4}\,dx = \frac{1}{4}\left[\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right]_{-4}^{0} \\
&\quad - \frac{1}{4}\left[\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right]_{0}^{4} = \frac{1}{n\pi}[1-(-1)^n] - \frac{1}{n\pi}[(-1)^n - 1] \\
&= \frac{2}{n\pi} - \frac{2(-1)^n}{n\pi} = \frac{4}{\pi}\frac{1}{2n-1}\,.
\end{aligned}$$

Therefore, the Fourier series of f is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[\frac{(2n-1)\pi x}{4}\right].$$

The graph of this function and of a partial sum of its Fourier series are given in Figs. 10.11 and 10.12.

(g)

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi} - e^{-\pi})\,, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx = \frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}\,, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx = \frac{(-1)^n(e^{-\pi} - e^{\pi})}{\pi(1+n^2)}\,.
\end{aligned}$$

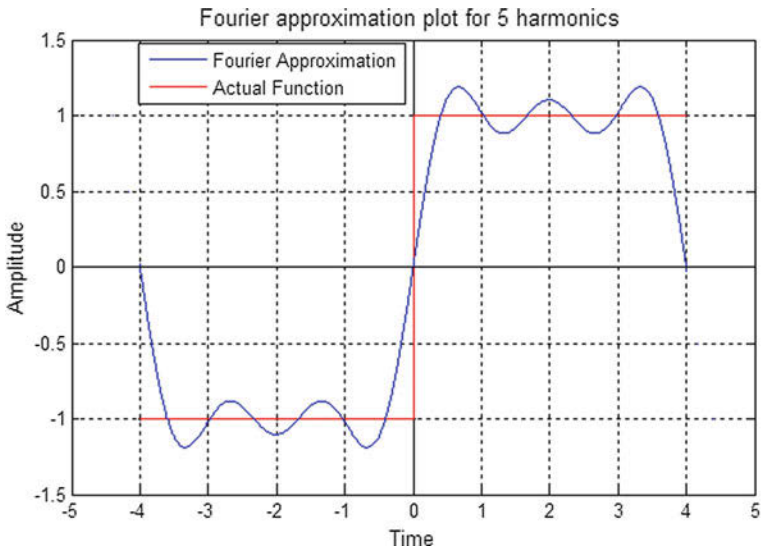
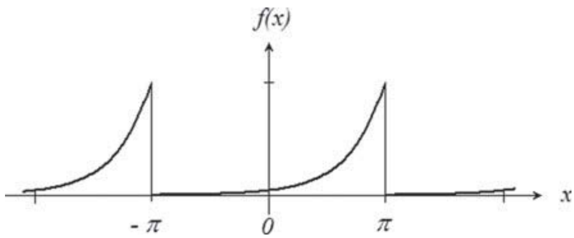


Fig. 10.12 Fourier approximation of the function in Fig. 10.11

Fig. 10.13 The function from Example 10.3(g)



The Fourier series of f is

$$f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(1+n^2)} \cos nx + \frac{(-1)^n (e^{-\pi} - e^{\pi})}{\pi(1+n^2)} \sin nx \right].$$

See Figs. 10.13 and 10.14 for the graph of the function and of a partial sum of its Fourier series.

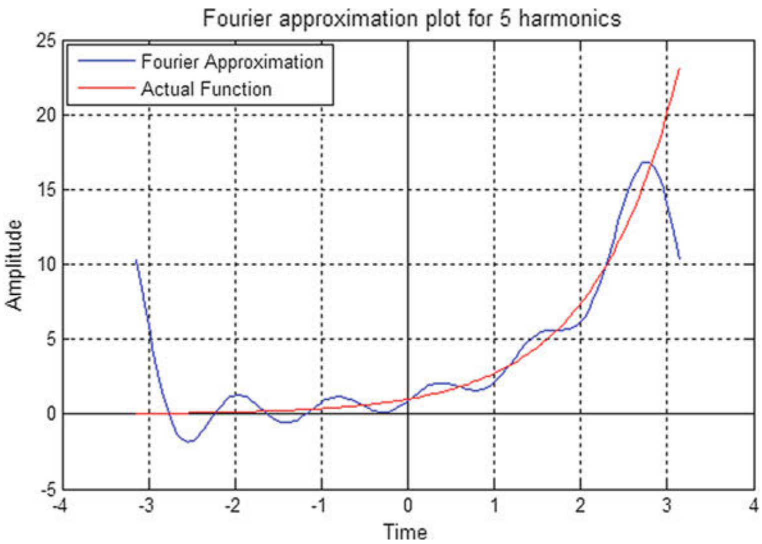


Fig. 10.14 Fourier approximation of the function in Fig. 10.13

10.2.3 Further Properties of Fourier Series

Periodic Extension. Let us recall the notion of a periodic function given in Chap. 1. A real function of a single variable is called periodic with period p if $f(x+p) = f(x)$ for all x. For example, 8π is a period of the sine function as $\sin(x+8\pi) = \sin x$ for all x. The smallest value of p for which $f(x+p) = f(x)$ holds for all x is called the fundamental period. For example, $p = 2\pi$ is the fundamental period of the sine function as 2π is the smallest value of p which satisfies $f(x+p) = f(x)$ for all x. Let us point out that often "period" is defined to be the fundamental period.

Let f be an arbitrary function defined on $(-l, l)$. Its Fourier series (if convergent)

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

is a periodic function of x of period $p = 2l$ and thus (if convergent to f) not only represents f on $(-l, l)$ but also gives the periodic extension of f on the real line.

Approximation by Partial Sums. Let

$$s_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}\left(a_n \cos nx + b_n \sin nx\right)$$

denote the Nth partial sum of the Fourier series of a function f defined on $[-\pi, \pi]$. One may ask how well s_N approximates f. We present without proof the following

two results. If f is continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$, then

$$|f(x) - s_N(x)| \leq \frac{1}{\sqrt{N}} \frac{1}{\sqrt{\pi}} \sqrt{\int_{-\pi}^{\pi} |f'(t)|^2 dt}\,, \quad \text{for all } x.$$

If f is continuous, piecewise differentiable, and satisfies $f(-\pi) = f(\pi)$, then

$$|f(x) - s_N(x)| \leq \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,, \quad \text{for all } x.$$

The Gibbs Phenomenon. If we examine the graphs of the partial sums of the Fourier series in Example 10.2.2, we observe that all of them are overshooting the true values of the function f near its point of discontinuity. In fact, this phenomenon always occurs when we approximate a discontinuous function with Fourier series. It is known as the **Gibbs phenomenon**, in honor of Josiah William Gibbs, a mathematical physicist working at Yale, who analyzed it just prior to 1900, after it had been discovered by Henry Wilbraham already in 1848. One can show that the overshooting amounts to approximately 9% of the size of the jump, that is, of the difference $|f(x+) - f(x-)|$ of the one-sided limits at the discontinuity point x. The main point is that the amount of overshooting of the partial sums s_N does **not** decrease when N tends to infinity.

As an example, let us consider the Fourier series of the step function

$$f(x) = \begin{cases} -1\,, & \text{if } x \in [-\pi, 0), \\ 0\,, & \text{if } x = 0, \\ 1\,, & \text{if } x \in (0, \pi], \end{cases}$$

extended to a 2π-periodic function on $\mathbb{R}$. Proceeding along the lines of the solution of Example 10.2.3(e), we obtain that

$$f(x) = \frac{4}{\pi} \sum_{x=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x\,.$$

The graph of s_{15} is given in Fig. 10.15, and the graph of s_{100} is given in Fig. 10.16. We observe the overshooting of the partial sum around the points where f is discontinuous.

Complex Form of Fourier Series. For this paragraph, we assume the reader to have some basic familiarity with complex numbers. The sine and cosine are related to the complex exponential function by

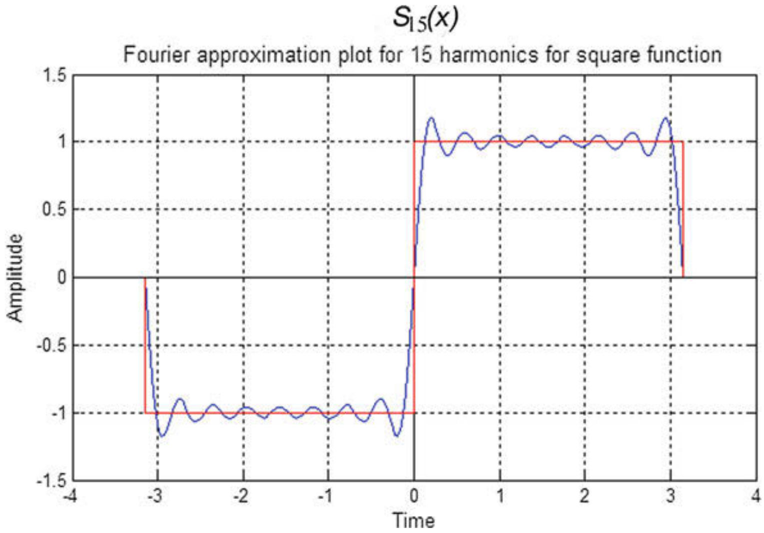


Fig. 10.15 Fourier approximation for 15 harmonics

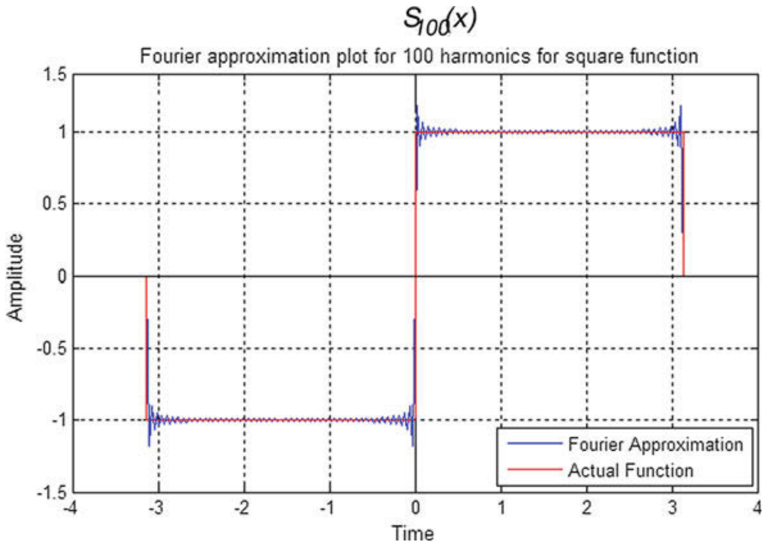


Fig. 10.16 Fourier approximation for 100 harmonics

$$e^{i\theta} = \cos\theta + i\sin\theta\,, \quad e^{-i\theta} = \cos\theta - i\sin\theta\,,$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\,, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\,,$$

where θ is any real number. For a real-valued function f with domain $(-\pi, \pi)$, we define the **complex Fourier coefficients** by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \,, \tag{10.14}$$

where n is any integer $0, \pm 1, \pm 2, \ldots$. Since

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx &= \int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx)\, dx \\ &= \int_{-\pi}^{\pi} f(x) \cos nx \, dx - i \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \pi a_n - i\pi b_n \,, \quad n > 0 \,, \end{aligned}$$

and, analogously,

$$\int_{-\pi}^{\pi} f(x) e^{inx}\, dx = \pi a_n + i\pi b_n \,, \quad n > 0 \,,$$

we see that the complex Fourier coefficients are related to the Fourier sine and cosine coefficients by

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) \,, \quad c_{-n} = \frac{1}{2}(a_n + ib_n) \,, \\ a_n &= c_n + c_{-n} \,, \quad b_n = i(c_n - c_{-n}) \,, \end{aligned} \tag{10.15}$$

for any $n > 0$. Moreover,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{a_0}{2} \,. \tag{10.16}$$

From (10.15), we obtain

$$\begin{aligned} c_n e^{inx} + c_{-n} e^{-inx} &= c_n(\cos nx + i \sin nx) + c_{-n}(\cos nx - i \sin nx) \\ &= a_n \cos nx + b_n \sin nx \,. \end{aligned}$$

Therefore, the partial sums s_N of the Fourier series can also be represented as

$$s_N(x) = \frac{1}{2} a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) = \sum_{n=-N}^{N} c_n e^{inx} \,,$$

and the Fourier series of f can be written in complex form as

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} .$$

Over an interval $(-l, l)$, $l > 0$, the complex form of the Fourier series is defined as

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} , \quad \text{where} \quad c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} \, dx , \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

It can be verified that the set of functions $\{\frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in Z}$ is an orthonormal set.

Sine and Cosine Fourier Series. We consider Fourier series of even and odd functions. Recall that if g is an even function on $(-\pi, \pi)$, then

$$g(-x) = g(x) \quad \text{for } x \in [0, \pi), \quad \int_{-\pi}^{0} g(x) \, dx = \int_{0}^{\pi} g(x) \, dx ,$$

and if h is an odd function on $(-\pi, \pi)$, then

$$h(-x) = h(x) \quad \text{for } x \in [0, \pi), \quad \int_{-\pi}^{0} h(x) \, dx = -\int_{0}^{\pi} h(x) \, dx .$$

Thus, if f is even on $(-\pi, \pi)$, then its Fourier coefficients satisfy

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \, dx , \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx , \end{aligned}$$

since the function $g(x) = f(x) \cos nx$ is even, and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx \\ &= 0 , \end{aligned}$$

since the function $h(x) = f(x) \sin nx$ is odd. Consequently, the Fourier series of an even function f is a cosine series,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \,,$$

where the coefficients a_n are given above.

Similarly, we find that the Fourier series of an odd function f contains only sine terms, and hence it is a sine series

$$\sum_{n=1}^{\infty} b_n \sin nx \,, \quad \text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \,.$$

Indeed, in this case $g(x) = f(x) \cos nx$ is an odd function, while $h(x) = f(x) \sin nx$ is an even function, so in particular

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = -\frac{1}{\pi} \int_0^{\pi} f(x)\, dx + \frac{1}{\pi} \int_0^{\pi} f(x)\, dx = 0 \,, \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\
&= -\frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = 0 \,.
\end{aligned}$$

As a consequence, if we know that f is odd or even on $(-\pi, \pi)$, we need to compute only the b_k's if f is odd function and the a_k's if f is an even function, respectively.

Example 10.4 Find the Fourier series of the function $f(x) = x^2$, $x \in (-\pi, \pi)$.

Solution: Since $f(-x) = (-x)^2 = x^2 = f(x)$, f is even and so $b_n = 0$ for $n = 1, 2, 3, \ldots$ We compute

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2}{\pi} \cdot \frac{1}{3} \big[x^3\big]_0^{\pi} = \frac{2}{3} \pi^2 \,,$$

and

$$\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx \\
&= \frac{2}{\pi} \left[\frac{x^2}{n} \sin nx \right]_0^{\pi} - \frac{4}{n\pi} \int_0^{\pi} x \sin nx \, dx \\
&= 0 + \frac{4}{n^2 \pi} \big[x \cos nx\big]_0^{\pi} - \frac{4}{n^2 \pi} \int_0^{\pi} \cos nx \, dx \\
&= \frac{4}{n^2} (-1)^n - \frac{4}{n^3 \pi} \big[\sin nx\big]_0^{\pi} = \frac{4}{n^2} (-1)^n \,.
\end{aligned}$$

Thus, the Fourier series of f is given as the cosine series

$$\frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \,.$$

Phase Angle Form and Frequency Spectrum. Let f be a periodic function defined on the real line which has the fundamental period l, that is, $f(x+l) = f(x)$ for all x, and l is the smallest number satisfying this condition. We define $\omega = 2\pi/l$ as the frequency corresponding to l. Let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + b_n \sin(n\omega x) \tag{10.17}$$

be the Fourier series of f on $[-l/2, l/2]$. The series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega x + \delta_n) \tag{10.18}$$

is called the **phase angle** of the Fourier series. Indeed, if two pairs (a, b) and (d, δ) of numbers are related by

$$\begin{aligned} a &= d\cos\delta\,, & b &= -d\sin\delta\,, \\ d &= \sqrt{a^2+b^2}\,, & \delta &= \arctan\left(-\frac{b}{a}\right), \end{aligned} \tag{10.19}$$

then

$$a\cos(n\omega x) + b\sin(n\omega x) = d\cos(n\omega x + \delta)$$

holds for all x, as a consequence of the trigonometric identities, so the series (10.17) and (10.18) correspond term by term. The phase angle form of the Fourier series is also called its **harmonic form**. It represents a periodic function as a superposition of cosine waves. The term $\cos(n\omega x + \delta_n)$ is the nth **harmonic**, d_n is the nth **harmonic amplitude**, and δ_n is the nth **phase angle** of f. Note that the harmonic amplitude satisfies

$$d_n = 2|c_n|\,, \tag{10.20}$$

where $c_n = (a_n - ib_n)/2$ is the nth complex Fourier coefficient of f as introduced earlier.

The **amplitude spectrum** or **frequency spectrum** of the periodic function f is a plot of $|c_n| = d_n/2$ on the vertical versus n along the horizontal axis. The **phase spectrum** of f is a plot of the points (n, δ_n) for $n = 0, 1, 2, \ldots$, where $\delta_n = \arctan(-b_n/a_n)$ is nth phase angle of f.

Example 10.5 Find the complex Fourier series of f on the given intervals. Furthermore, find the frequency spectrum of the function.
(a)

$$f(x) = \begin{cases} 0\,, & -\pi < x < 0\,, \\ x\,, & 0 \le x < \pi\,, \end{cases} \qquad \text{on } [-\pi, \pi].$$

(b)

$$f(x) = \begin{cases} -1\,, & -2 < x < 0\,, \\ 1\,, & 0 \le x < 2\,, \end{cases} \qquad \text{on } [-2, 2].$$

(c)

$$f(x) = \begin{cases} \cos x\,, & 0 < x < \pi/2\,, \\ 0\,, & \pi/2 \le x < \pi\,, \end{cases} \qquad \text{on } [0, \pi].$$

Solution: (a) We have

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)e^{-inx}\,dx + \int_0^{\pi} f(x)e^{-inx}\,dx\right] \\ &= \frac{1}{2\pi}\left[0 + \int_0^{\pi} xe^{-inx}\,dx\right] = \frac{1}{2\pi}\left[-xe^{-inx}\cdot\frac{1}{in}\right]_0^{\pi} - \frac{1}{2\pi}\int_0^{\pi} -e^{-inx}\frac{1}{in}\,dx \\ &= \frac{1}{2}\frac{i}{n}e^{-in\pi} - \frac{1}{2\pi}\left[e^{-inx}\frac{1}{in}\frac{1}{in}\right]_0^{\pi} \\ &= \frac{1}{2}\frac{i}{n}e^{-in\pi} + \frac{1}{2\pi n^2}e^{-in\pi} - \frac{1}{2\pi n^2} \\ &= \frac{1+in\pi}{2\pi n^2}e^{-in\pi} - \frac{1}{2n^2\pi}\,, \quad \text{for } n \neq 0, \\ c_0 &= \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}\,. \end{aligned}$$

The complex Fourier representation of f becomes

$$f(x) = \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n^2}[(1+in\pi)e^{-in\pi} - 1]e^{inx}\,.$$

Here we have used the formulas

$$e^{in\pi} = (-1)^n = e^{-in\pi}\,, \quad e^{-2\pi in} = 1\,, \quad e^{-in\pi/2} = (-i)^n)\,.$$

(b) We have

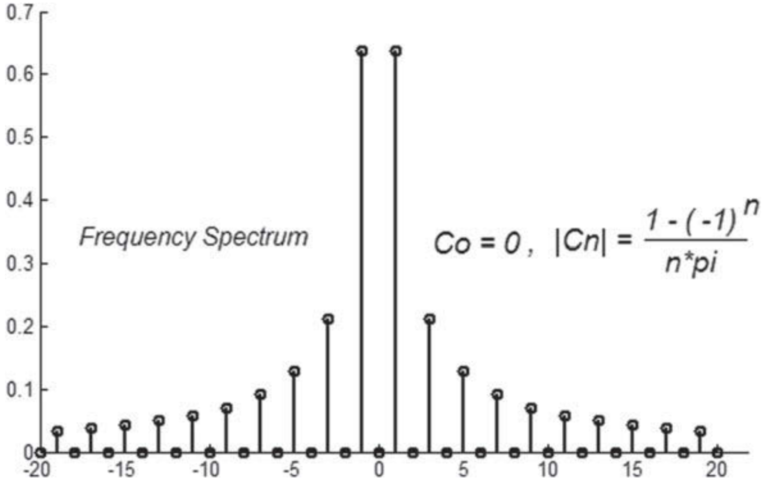


Fig. 10.17 Frequency spectrum

$$\begin{aligned}
c_n &= \frac{1}{4}\int_{-2}^{2} f(x)e^{-in\pi x/2}\,dx = \frac{1}{4}\left[\int_{-2}^{0}(-1)e^{-in\pi x/2}\,dx + \int_{0}^{2} e^{-in\pi x/2}\,dx\right] \\
&= \frac{i}{2n\pi}[-1 + e^{in\pi} + e^{-in\pi} - 1] = \frac{1}{2n\pi}[-2 + (-1)^n + (-1)^n] \\
&= \frac{1-(-1)^n}{n\pi i}, \quad \text{for } n \neq 0, \\
c_0 &= \frac{1}{4}\int_{-2}^{2} f(x)\,dx = 0\,.
\end{aligned}$$

Thus

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1-(-1)^n}{n\pi i} e^{in\pi x/2}\,.$$

The fundamental period is equal to 4 so $\omega = 2\pi/4 = \pi/2$, $c_0 = 0$, and $|c_n| = (1-(-1)^n)/n\pi$ (Fig. 10.17).

(c) Using $\cos x = (e^{ix} - e^{-ix})/2$ we get

$$\begin{aligned}
c_n &= \frac{1}{\pi}\int_{0}^{\pi} f(x)e^{-2inx}\,dx = \frac{1}{\pi}\int_{0}^{\pi/2} \cos x e^{-2inx}\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi/2} \frac{1}{2}(e^{ix} - e^{-ix})e^{-2inx}\,dx = \frac{1}{2\pi}\int_{0}^{\pi/2}(e^{(1-2n)ix} - e^{-(1+2n)ix})\,dx \\
&= \frac{1}{2\pi}\left[\frac{1}{i(1-2n)}e^{(1-2n)ix} + \frac{1}{i(1+2n)}e^{(1+2n)ix}\right]_0^{\pi/2} = \frac{2ne^{-in\pi} + i}{\pi(1-4n^2)}\,.
\end{aligned}$$

10.3 The Fourier Transform

The Fourier transform is a mathematical procedure that breaks up a function into the frequencies that compose it, as a prism breaks up light into colors. It transforms a function f into a new function, $\hat{f}$ or $\mathscr{F}[f]$ (read as "f hat" or "script f") which is called the Fourier transform of f. Depending on the context, the argument of f is a time variable or a spatial variable. The argument of $\hat{f}$ usually has the meaning of a frequency.

A function and its Fourier transform are two faces of the same information. The function exhibits the time (or spatial) information and hides the information about frequencies. The Fourier transform displays information about frequencies and hides the time (or spatial) information. Nevertheless, the function and its Fourier transform both contain all the information about the function. One can compute the transform from the original function as well as reconstruct the function from its transform, that is, one can invert the transform.

In the previous section, we have studied the decomposition of a function into its Fourier series, which is a periodic function. This works well for functions defined on a bounded interval, as we can always think of them as periodically extended to the whole real line. In contrast to that, the Fourier transform acts on arbitrary (nonperiodic) functions.

10.3.1 Basic Properties of the Fourier Transform

When dealing with the Fourier transform, one constantly encounters integrals over the whole real line $(-\infty, \infty)$, that is, improper integrals as introduced in Sect. 6.9. In order to simplify the exposition during this section, we call a function f defined on $\mathbb{R} = (-\infty, \infty)$ **integrable on** $\mathbb{R}$, respectively, **square integrable on** $\mathbb{R}$, if the improper integral

$$\int_{-\infty}^{\infty} |f(x)|\, dx\,, \quad \text{resp.} \quad \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

converges.

For the definition of the Fourier transform, several variants are in common use, see Remark 10.11 below. We choose the following one.

Definition 10.5 (*Fourier Transform*) Let the function f be defined on $\mathbb{R}$ and assume that it is integrable on $\mathbb{R}$. The function $\hat{f}$, defined on $\mathbb{R}$ by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i\xi t}\, dt\,, \tag{10.21}$$

is called the **Fourier transform** of f.

In signal analysis, t is understood as time variable and ξ is understood as the frequency variable, see Sect. 10.4 below.

Remark 10.4 1. According to (10.21), integrals of complex-valued functions are involved in the definition of the Fourier transform. They are defined as

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt &= \int_{-\infty}^{\infty} f(t)(\cos(\xi t) - i\sin(\xi t))\,dt\\ &= \int_{-\infty}^{\infty} f(t)\cos(\xi t)\,dt - i\int_{-\infty}^{\infty} f(t)\sin(\xi t)\,dt\,,\end{aligned}$$

that is, the real and imaginary parts of the integrand are evaluated separately and yield the real and imaginary parts of the integral, which is a complex number (in this case, the number $\hat{f}(\xi)$).

2. The integrand in (10.21) satisfies

$$|f(t)e^{-i\xi t}| = |f(t)| \cdot |e^{-i\xi t}| = |f(t)|\,,$$

since $|e^{ix}| = 1$ for all real numbers x. Therefore and since f is assumed to be integrable on $\mathbb{R}$, the improper integral in (10.21) is defined. One then infers from the properties of parameter-dependent integrals (Theorem 8.5 and the remark following it) that the Fourier transform $\hat{f}$ is a continuous function. In addition, one can prove that $\hat{f}(\xi)$ tends to zero as ξ tends to $\pm\infty$. (The latter result is called the Riemann–Lebesgue lemma.)

3. As it stands, the requirement that f has to be integrable on the whole line is rather restrictive. For example, Definition 10.5 does not cover the case when f is a constant function. Indeed, the Fourier transform of the constant 1 is defined, but it is no longer a function defined on $\mathbb{R}$, but a more general mathematical object (although it is called the Dirac function). This, however, is outside the scope of this book.

Example 10.6 Find the Fourier transform of the following functions:

(a) $f(t) = e^{-|t|}$

(b) $f(t) = \begin{cases} 0, & t < 0\\ e^{-t}, & t \geq 0\end{cases}$

(c) Let a and k be positive numbers, let

$$f(t) = \begin{cases} k, & -a \leq t < a\\ 0, & \text{otherwise}\end{cases}$$

Solution: (a) For $f(t) = e^{-|t|}$, we get

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-|t|-it\xi}\,dt = \int_{-\infty}^{0} e^{-|t|-it\xi}\,dt + \int_{0}^{\infty} e^{-|t|-it\xi}\,dt \\
&= \int_{-\infty}^{0} e^{t}e^{-i\xi t}dt + \int_{0}^{\infty} e^{-t}e^{-i\xi t}\,dt \\
&= \left[\frac{1}{1-i\xi}e^{(1-i\xi)t}\right]_{t=-\infty}^{t=0} + \left[\frac{-1}{1+i\xi}e^{-(1+i\xi)t}\right]_{t=0}^{t=\infty} \\
&= \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1^2+\xi^2} = \frac{2}{1+\xi^2}\,.
\end{aligned}$$

(b) Let

$$H(t) = \begin{cases} 1\,, & t \geq 0 \\ 0\,, & t < 0 \end{cases}$$

be the Heaviside function (see Chap. 1). Then $f(t) = H(t)e^{-t}$ and

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(t)e^{-i\xi t}\,dt = \int_{-\infty}^{\infty} H(t)e^{-t}e^{-i\xi t}\,dt = \int_{0}^{\infty} e^{-t}e^{-i\xi t}\,dt \\
&= \int_{0}^{\infty} e^{-(1+i\xi)t}dt = \left[-\frac{1}{1+i\xi}e^{-(1+i\xi)t}\right]_{t=0}^{t=\infty} = \frac{1}{1+i\xi}\,.
\end{aligned}$$

(c) We obtain

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(t)e^{-i\xi t}\,dt = \int_{-a}^{a} ke^{-i\xi t}\,dt = \left[\frac{-k}{i\xi}\,e^{-i\xi t}\right]_{t=-a}^{t=a} \\
&= -\frac{k}{i\xi}\left[e^{-i\xi a} - e^{i\xi a}\right] = \frac{2k}{\xi}\sin(a\xi)\,,
\end{aligned}$$

since

$$\sin(a\xi) = \frac{e^{i\xi a} - e^{-i\xi a}}{2i}\,.$$

Remark 10.5 **(i)** For a given function f, the Fourier transform (if defined) yields a new function $\hat{f}$. We may thus view the Fourier transform as a mapping whose domain and range are certain sets of functions. Such a mapping is commonly called an **operator**. Let us denote it by $\mathscr{F}$, so

$$\mathscr{F}(f) = \hat{f}\,. \tag{10.22}$$

From Definition 10.5, we see that $\mathscr{F}$ is linear, that is,

$$\mathscr{F}(\alpha f + \beta g) = \alpha\mathscr{F}(f) + \beta\mathscr{F}(g)$$

holds for functions f, g and scalars α, β.
(ii) Instead of $\mathscr{F}(f) = \hat{f}$, one often writes

$$\mathscr{F}[f(t)] = \hat{f}(\xi)\,.$$

Although this is, in a strict sense, mathematically not correct (it confuses the functions f and $\hat{f}$ with their function values $f(t)$ and $\hat{f}(\xi)$), it leads to a concise way of writing formulas. In this notation, the result of Example 10.3.1 (a) becomes

$$\mathscr{F}[e^{-|t|}] = \frac{2}{1+\xi^2}\,.$$

In the next two theorems, we state some important properties of the Fourier transform.

Theorem 10.2 *Let f be a function which is integrable on $\mathbb{R}$.*
***(a) Time shift**. Let t_0 be a real number, let $g(t) = f(t-t_0)$. Then*

$$\hat{g}(\xi) = e^{-i\xi t_0}\hat{f}(\xi)\,, \quad \xi \in \mathbb{R}\,. \tag{10.23}$$

This means that the Fourier transform of the translated function f equals the Fourier transform of f multiplied by a factor. In shorter notation, without introducing the function g explicitly, (10.23) becomes

$$\mathscr{F}[f(t-t_0)] = e^{-i\xi t_0}\hat{f}(\xi)\,.$$

***(b) Frequency shift**. Let ξ_0 be a real number, let $g(t) = e^{i\xi_0 t}f(t)$. Then*

$$\hat{g}(\xi) = \hat{f}(\xi-\xi_0)\,, \quad \xi \in \mathbb{R}\,, \tag{10.24}$$

or, in shorter notation,

$$\mathscr{F}[e^{i\xi_0 t}f(t)] = \hat{f}(\xi-\xi_0)\,.$$

***(c) Scaling or Dilation**. Let a be a real number with $a \neq 0$, let $g(t) = f(at)$. Then*

$$\hat{g}(\xi) = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right), \quad \xi \in \mathbb{R}\,. \tag{10.25}$$

This states that the Fourier transform of the scaled function is obtained by replacing ξ by ξ/a in the Fourier transform of the original function and dividing by the magnitude of the scaling factor.

The formulas in the theorem above are obtained from properties of the integral. For example, (10.23) results from the computation

$$\begin{aligned}\hat{g}(\xi) &= \int_{-\infty}^{\infty} f(t-t_0)e^{-i\xi t}\,dt = e^{-i\xi t_0}\int_{-\infty}^{\infty} f(t-t_0)e^{-i\xi(t-t_0)}\,dt \\ &= e^{-i\xi t_0}\int_{-\infty}^{\infty} f(s)e^{-i\xi s}\,ds = e^{-i\xi t_0}\hat{f}(\xi)\,,\end{aligned}$$

and (10.25) from the computation

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(at)e^{-i\xi t}\,dt = \int_{-\infty}^{\infty} f(u)e^{-iu\xi/a}\frac{1}{|a|}\,du = \frac{1}{|a|}\hat{f}\Big(\frac{\xi}{a}\Big)\,.$$

Remark 10.6 The properties given in Theorem 10.2 can also be written in operator form. For example, the time shift can be expressed by the **translation operator** T_{t_0} which maps a function f to its translate $T_{t_0}f$ defined by $(T_{t_0}f)(t) = f(t-t_0)$. Equation (10.23) then takes the form

$$\widehat{T_{t_0}f}(\xi) = e^{-\xi t_0}\hat{f}(\xi)\,.$$

Theorem 10.3 ***(a)*** *Suppose that f is continuous, f' is piecewise continuous and both f and f' are integrable on $\mathbb{R}$. Then*

$$\widehat{f'}(\xi) = \widehat{\mathscr{F}[f']}(\xi) = i\xi\hat{f}(\xi)\,. \tag{10.26}$$

(b) *Suppose that f satisfies $\int_{-\infty}^{\infty}|f(t)|\,dt < \infty$ and $\int_{-\infty}^{\infty}|tf(t)|\,dt < \infty$. Then (see Remark 10.5 for the notation)*

$$\mathscr{F}[tf(t)] = i\frac{d}{d\xi}\hat{f}(\xi)\,. \tag{10.27}$$

Remark 10.7 If we apply Theorem 10.3 to derivatives of f, we obtain the formulas

$$\widehat{f^{(n)}}(\xi) = \mathscr{F}[f^{(n)}](\xi) = (i\xi)^n\hat{f}(\xi)\,, \tag{10.28}$$

$$\mathscr{F}[t^nf(t)] = i^n\frac{d^n}{d\xi^n}\hat{f}(\xi)\,, \tag{10.29}$$

provided f and its derivatives up to order n satisfy the corresponding assumptions. In particular, for $n=2$ we have

$$\mathscr{F}[t^2f(t)] = -\frac{d^2}{d\xi^2}\hat{f}(\xi)\,. \tag{10.30}$$

Theorem 10.4 (Plancherel's Identity) *If f is integrable as well as square integrable on $\mathbb{R}$, and if the same holds for g, then*

$$\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi = 2\pi \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt\,. \tag{10.31}$$

(Here $\overline{c}$ denotes the complex conjugate of the complex number c.)

Setting $g = f$ in the preceding theorem, we obtain the following.

Theorem 10.5 (Parseval's Identity) *If f and f^2 are integrable on $\mathbb{R}$, then*

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt\,. \tag{10.32}$$

If we interpret f as a signal, the norm

$$\|f\| = \Big(\int_{-\infty}^{\infty} |f(t)|^2 dt\Big)^{1/2}$$

represents the energy of the signal.

The Inverse Fourier Transform. In the beginning of this section, we have defined the Fourier transform $\hat{f}$ of a function f. It turns out that we can reverse this procedure—if we know $\hat{f}$, we can obtain f according to the following result.

Theorem 10.6 *Suppose that f is continuous and that f and $\hat{f}$ are integrable on $\mathbb{R}$. Then*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi t}\, d\xi \tag{10.33}$$

holds for all $t \in \mathbb{R}$.

In abstract terms, the right-hand side of (10.33) defines the inverse $\mathscr{F}^{-1}$ of the Fourier transform $\mathscr{F}$. It is called the **inverse Fourier transform**

$$(\mathscr{F}^{-1}[g])(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi) e^{i\xi t}\, d\xi\,. \tag{10.34}$$

Indeed, we see that $\mathscr{F}^{-1}[\mathscr{F}[f]] = f$.

Remark 10.8 The following interpretation of Theorem 10.6 is fundamental for many applications of the Fourier transform. Consider t as a time variable. For fixed ξ, the values $e^{i\xi t}$ traverse along the unit circle at constant speed. Since $\xi = 2\pi$ corresponds to the completion of one cycle in one unit of time, the number $\xi/2\pi$ gives the number of cycles per unit time, which is called the frequency. Its unit is Hertz if t is measured in seconds. The number ξ is called the angular frequency, it gives the number of radians traversed per unit time. Seen in this light, formula (10.33) is a decomposition of the original function f into a weighted sum of oscillations in form of an integral.

The weight of the angular frequency ξ is given by the value $\hat{f}(\xi)$ of the Fourier transform of f.

Remark 10.9 If f is twice differentiable and if f, f', and f'' are integrable on $\mathbb{R}$, then $\hat{f}$ is integrable on $\mathbb{R}$, so in this case we can apply Theorem 10.6.

For piecewise continuous functions, one has the following result.

Theorem 10.7 *Let f and f' be piecewise continuous and assume that f is integrable on $\mathbb{R}$. Then*

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \hat{f}(\xi) e^{i\xi t}\, d\xi = \frac{1}{2}(f(t+) - f(t-)) \tag{10.35}$$

holds for all $t \in \mathbb{R}$.

Remark 10.10 For a function h, the limit

$$\lim_{R\to\infty} \int_{-R}^{R} h(x)\, dx\,,$$

if it exists, is called the **principal value** (or **Cauchy principal value**) of $\int_{-\infty}^{\infty} h(x)\, dx$. Thus, under the assumptions of Theorem 10.7 we also obtain the formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi t}\, d\xi$$

at points t where f is continuous, provided we interpret the integral as its principal value.

Remark 10.11 If one wants the frequency variable ξ to denote ordinary frequency instead of angular frequency, one defines the Fourier transform by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t\xi}\, dt\,.$$

The inverse formula then becomes

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i t\xi}\, d\xi\,.$$

In this case, the frequency ξ is measured in Hertz (cycles per second), if t is measured in seconds. If one keeps the angular frequency, but wants a more symmetric relation between the transform and its inverse, one uses

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-it\xi}\, dt\,, \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{it\xi}\, d\xi\,.$$

Less common is an interchange of the sign in the exponent,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{it\xi}\,dt\,, \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-it\xi}\,d\xi\,.$$

It is also possible to mix these variants. Therefore, when dealing with the Fourier transform, one has to make sure which convention is used.

Localization and the Uncertainty Principle. In this subsection, we explain the fact that a function f and its Fourier transform $\hat{f}$ cannot both be concentrated on a small interval.

First, consider the dilation $g(t) = f(at)$. For $a > 1$, g represents a compression of f around $t = 0$ by the factor a. On the other hand, Theorem 10.2(c) says that

$$\hat{g}(\xi) = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right),$$

that is, we have to stretch out $\hat{f}$ by a factor a in order to obtain $\hat{g}$. If $a < 1$, g is a stretched version of f while $\hat{g}$ is a compressed version of $\hat{f}$.

Second, assume that $\hat{f}(\xi) = 0$ outside some interval $[-l, l]$. We say in this case that f has **bandwidth** $\boldsymbol{l}$. From the Fourier inversion formula which becomes

$$f(t) = \frac{1}{2\pi}\int_{-l}^{l} \hat{f}(\xi)e^{i\xi t}\,d\xi$$

one concludes, with the aid of a result of complex function theory which we cannot present here, that f can be zero only at isolated points, so it spreads out to infinity and in particular cannot have the property that $f(t) = 0$ outside some interval $[-M, M]$.

Third, one can quantify this phenomenon. Consider the expression

$$\Delta f = \left(\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt\right)\cdot\left(\int_{-\infty}^{\infty} |f(t)|^2\,dt\right)^{-1}.$$

If the large values of f arise only at small values of t and f decays rapidly as t gets large, the numerator will be small in comparison with the denominator, so Δf somehow measures the concentration (or localization) of f around $t = 0$. It can be proved that

$$(\Delta f)\cdot(\Delta\hat{f}) \geq \frac{1}{4}\,,$$

that is, if Δf is small then $\Delta\hat{f}$ has to be large and vice versa. This is called the **uncertainty principle**.

10.3.2 Convolution

The convolution of two sequences $a = \{a_n\}$ and $b = \{b_n\}$, where n ranges over all integers, is defined as

$$(a * b)_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j} ,$$

provided the infinite series converges. If a_j and b_j are nonzero only for $j \geq 0$, the convolution becomes the finite sum

$$(a * b)_k = \sum_{j=0}^{k} a_j b_{k-j} .$$

For example, for $k = 2$ and $k = 3$, we have

$$(a * b)_2 = \sum_{j=0}^{2} a_j b_{2-j} = (a_0 b_2 + a_1 b_1 + a_2 b_0)$$
$$(a * b)_3 = \sum_{j=0}^{3} a_j b_{3-j} = (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) .$$

For functions, convolution involves the integral instead of the sum.

Definition 10.6 Let f and g be two functions defined on the real line $\mathbb{R}$, assume that f and g are integrable on $\mathbb{R}$. The **convolution** of f and g is denoted by $f * g$ and defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y)\, dy \tag{10.36}$$

(read as f star g, or f convolved with g).

One can prove that, under the stated assumptions, the improper integral in (10.36) indeed converges (we will not do it here), so that $f * g$ is integrable on $\mathbb{R}$, too.

Remark 10.12 (a) The convolution has the following properties:

(i) For all functions f, g, h as in Definition 10.6,

$$(f + g) * h = (f * h) + (g * h) ,$$

that is, $[(f + g) * h](x) = (f * h)(x) + (g * h)(x)$ for all $x \in \mathbb{R}$.

(ii) $(\lambda f) * g = \lambda(f * g)$ for functions f, g and scalars λ.

(iii) $f * (g + h) = (f * g) + (f * h)$.

(iv) $f * (\lambda g) = \lambda(f * g)$.

(v) $f * (g * h)] = (f * g) * h$.
(vi) $f * g = g * f$.

These properties tell us that the convolution $f * g$ is linear with respect to f and g separately, and that it is commutative (property (vi)).

(c) The convolution can also be interpreted as a moving weight average of the function f, where the weighting is determined by the function g. In view of (a) (vi), $f * g$ can also be interpreted as a moving weight average of g, where the weighting is determined by f.

(d) If the function $f(x)$ has large oscillations, sharp peaks, or discontinuities, then averaging about each point x will tend to decrease the oscillations, lower the peaks, and smooth out the discontinuities. In view of all this, convolution acts as a smoothing operator. Let us mention two particular results in this direction.

(i) If $\sup_{t\in\mathbb{R}} |f(t)|$ and $\int\limits_{-\infty}^{\infty} |g(t)|\,dt$ are finite, then the function $f * g$ is continuous on $\mathbb{R}$.
(ii) If $\int_{-\infty}^{\infty} |f(t)|^2\,dt$ and $\int_{-\infty}^{\infty} |g(t)|^2\,dt$ are finite, then the function $f * g$ is continuous on $\mathbb{R}$.

(e) Convolutions arise as a basic tool to describe input–output systems. Such a system transforms a time-dependent input function $u = u(t)$ into a time-dependent output function $w = w(t)$ according to

$$w(t) = (f * u)(t) = \int_{-\infty}^{\infty} f(t-s)u(s)\,ds\,. \tag{10.37}$$

In signal analysis, such an input–output system is called a **filter**. For example, an electrical circuit with an input and an output line can be described in this way, and indeed much of mathematical systems theory has been developed in this context. A filter may serve various purposes such as letting through certain frequencies while blocking other ones, or removing noise or blurring in pictures. We may write the system (10.37) in operator form as

$$w = \mathscr{S}[u]\,.$$

The system is linear,

$$\mathscr{S}[\alpha u + \beta v] = \alpha\mathscr{S}[u] + \beta\mathscr{S}[v]\,,$$

and it is time-invariant, that is, if $\tilde{u}(t) = u(t-h)$ is a translate of u, then $\tilde{w} = \mathscr{S}[\tilde{u}]$ satisfies $\tilde{w}(t) = w(t-h)$, that is, $\tilde{w}$ is the corresponding translate of w. This means that the behavior of the system does not change when time passes.

We state some important properties of the convolution.

Theorem 10.8 *Suppose that f and g are integrable on $\mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} |(f * g)(t)|\, dt \le \int_{-\infty}^{\infty} |f(t)|\, dt \cdot \int_{-\infty}^{\infty} |g(t)|\, dt\,. \tag{10.38}$$

For convolution in the time domain, we have

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)\,. \tag{10.39}$$

For convolution in the frequency domain, we have

$$\widehat{fg}(\xi) = \frac{1}{2\pi}(\hat{f} * \hat{g})(\xi)\,. \tag{10.40}$$

Thus, under the action of the Fourier transform or its inverse, multiplication becomes convolution and vice versa. This is a major reason why convolution plays a prominent role in the calculus of Fourier transforms.

10.3.3 The Discrete Fourier Transform

In digital signal processing, signals are represented by sequences $\{x_n\}$, also written as $\{x(n)\}$, where n ranges over all integers. In other words, we consider functions x whose domain are the integers, instead of functions defined on the real numbers $\mathbb{R}$.

More specifically, let x be a periodic sequence with period N, that is, $x(n + N) = x(n)$ for all n. Any such sequence is completely specified by the values $x(0)$, $x(1), \ldots, x(N-1)$. The (N-point) **discrete Fourier transform** (DFT) of x, denoted by $\hat{x}$, is the N-periodic sequence defined by

$$\hat{x}(n) = \sum_{j=0}^{N-1} x(j)e^{-2\pi ijn/N}\,, \quad 0 \le n \le N-1\,, \tag{10.41}$$

and extended by periodicity to all integer values of n.

The following theorem yields the inverse of the discrete Fourier transform.

Theorem 10.9 *Let x be an N-periodic sequence $x(n)$ with DFT $\hat{x}$. Then*

$$x(j) = \frac{1}{N}\sum_{n=0}^{N-1} \hat{x}(n)e^{2\pi inj/N}\,, \quad 0 \le j < N-1\,. \tag{10.42}$$

We define the convolution of N-periodic sequences.

Definition 10.7 Let x and y be N-periodic sequences. The circular convolution of x and y is defined by

$$(x * y)(n) = \sum_{k=0}^{N-1} x(k)y(n-k)\,. \tag{10.43}$$

One may check immediately from (10.43) that $x * y$ is also N-periodic.

Theorem 10.10 *Let x and y be N-periodic sequences with DFT's $\hat{x}$ and $\hat{y}$. Then*

$$\widehat{x * y}(n) = \hat{x}(n)\hat{y}(n)\,, \tag{10.44}$$

where $\widehat{x * y}$ denotes the DFT of $x * y$.

If one computes the discrete Fourier transform of an N-periodic sequence directly from its definition, one needs N multiplications (and N additions) for each element $\hat{x}(n)$, thus N^2 multiplications for all elements $x(0), \ldots, fx(N-1)$. However, due to specific properties of the factors $e^{-2\pi ijn/N}$, it is possible to compute the DFT with only $cN \log_2 N$ multiplications and additions, where c is a small constant. An algorithm for this purpose was discovered by James Cooley and John Tukey, published in 1965, and is known since as the **fast Fourier transform** (FFT). Its history goes back to Carl Friedrich Gauss. The algorithm is based on the recursive factorization of a particular matrix. When N is large, the speedup from N^2 to $cN \log_2 N$ is enormous. Indeed, the FFT is most widely used in computations involving the Fourier transform in all areas of science and technology.

10.4 Application of Fourier Methods to Signal Analysis

Signals are ubiquitous. The train's whistle, the blinking of a car's beam can be viewed as quantities varying in time which contain information; they are examples of time-dependent signals. Early in human history, signals of smoke by day and of fire by night have been used to transmit information. In recent times, telegraph, telephone, radio, television, and radar have been, respectively, are used as signal transmitters. A radio signal consists of a sine or cosine wave with radio frequency, called the carrier wave, which has been modulated with the information to be transmitted.

Signals can be divided into two categories, analog signals (functions defined on a continuum of numbers, for example, an interval in $\mathbb{R}$) and digital signals, which are defined on a discrete set like the integers. In 1949, Claude E. Shannon of Bell Telephone Laboratories published a mathematical result now known as the Shannon sampling theorem. This result provided the foundation for digital signal processing. It tells us that if the range of frequencies of a signal measured in cycles per second does not exceed n, then the time-continuous signal can be reconstructed with complete accuracy by measuring its amplitude $2n$ times a second.

The study of signals is relevant not only in telecommunication but also in telemetry, astronomy, oceanography, optics, crystallography, geophysics, bioengineering, bioinformatics, and medicine, to mention a few.

The Shannon Sampling Theorem. Let us assume that a signal f is **band-limited**, that is,

$$\hat{f}(\xi) = 0\,, \quad \text{for all } |\xi| > l, \tag{10.45}$$

holds for some $l > 0$, and the smallest such number l is called the bandwidth of the signal. This means that the total frequency content of the signal f lies in the band (or interval) $[-l, l]$. Moreover, let us assume that f is integrable and square integrable on $\mathbf{R}$, that is, the signal f has finite energy. Let it be recovered from its Fourier transform as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi t}\,d\xi\,.$$

Because of (10.45),

$$f(t) = \frac{1}{2\pi}\int_{-l}^{l} \hat{f}(\xi)e^{i\xi t}\,d\xi\,. \tag{10.46}$$

We expand $\hat{f}$ on $[-l, l]$ in a complex Fourier series, so

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} c_n e^{n\pi i\xi/l}\,, \tag{10.47}$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} \hat{f}(\xi)e^{-n\pi i\xi/l}\,d\xi\,. \tag{10.48}$$

We insert (10.47) in (10.46) and obtain

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-l}^{l} \hat{f}(\xi)e^{i\xi t}\,d\xi = \frac{1}{2\pi}\int_{-l}^{l} \sum_{n=-\infty}^{\infty} c_n e^{n\pi i\xi/l}e^{i\xi t}\,d\xi \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{n\pi i\xi/l}e^{i\xi t}\,d\xi\,. \end{aligned} \tag{10.49}$$

(Interchanging the integral with the sum is possible by a general property of Fourier series of functions of finite energy; note that $\hat{f}$ has finite energy by Parseval's formula (10.32).) Next, we compare (10.46) for $t = -n\pi/l$ with (10.48) and see that

$$c_n = \frac{\pi}{l} f\Big(-\frac{n\pi}{l}\Big)\,. \tag{10.50}$$

We insert this value into (10.49) and compute

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\pi}{l} f\Big(-\frac{n\pi}{l}\Big) \int_{-l}^{l} e^{n\pi i\xi/l} e^{i\xi t}\, d\xi \\ &= \frac{1}{2l} \sum_{n=-\infty}^{\infty} f\Big(\frac{n\pi}{l}\Big) \int_{-l}^{l} e^{-n\pi i\xi/l} e^{i\xi t}\, d\xi \end{aligned}$$

(we have replaced n by $-n$, as n ranges over all integers)

$$\begin{aligned} &= \frac{1}{2l} \sum_{n=-\infty}^{\infty} f\Big(\frac{n\pi}{l}\Big) \int_{-l}^{l} e^{i\xi(t-n\pi/l)}\, d\xi \\ &= \frac{1}{2l} \sum_{n=-\infty}^{\infty} f\Big(\frac{n\pi}{l}\Big) \frac{1}{i(t-n\pi/l)} \Big[e^{i\xi(t-n\pi/l}\Big]_{\xi=-l}^{\xi=l} \\ &= \sum_{n=-\infty}^{\infty} f\Big(\frac{n\pi}{l}\Big) \frac{1}{lt-n\pi} \frac{1}{2i} [e^{i(lt-n\pi)} - e^{-i(lt-n\pi)}]\,, \end{aligned}$$

so finally we arrive at the Whittaker–Shannon interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f\Big(\frac{n\pi}{l}\Big) \frac{\sin(lt-n\pi)}{lt-n\pi}\,. \tag{10.51}$$

This is the main content of Shannon's sampling theorem which states that a function of bandwidth l can be completely recovered by (10.51) from its values at the points $n\pi/l$, where $n = 0, \pm 1, \pm 2, \ldots$. This result forms the basis for the conversion between analog and digital signals. If we convert an analog signal f of bandwidth l into a digital signal by evaluating it at times $t = 0, \pm\pi/l, \pm 2\pi/l$, we can convert it back to an analog signal without loss of information, at least in principle—note that an exact evaluation of (10.51) involves an infinite sum of values of f at arbitrarily large positive and negative times. To develop suitable approximations, both from the theoretical and the practical standpoint, is one of the subjects of the area of signal analysis and signal processing.

10.5 Exercises

10.5.1 Discuss the relationship between linear independence and orthonormality. Can you convert an orthogonal system into an orthonormal system?

10.5.2 (a) Show that $f(x) = e^x$ and $g(x) = \sin x$ are orthogonal on the interval $\Big[\frac{\pi}{4}, \frac{5\pi}{4}\Big]$.

(b) Show that $\{\cos x, \cos 3x, \cos 5x, \ldots\}$ is an orthogonal set on $[0, \pi/2]$.

10.5.3 Expand the following functions into Fourier series on the given interval:

(a) $f(x) = x + \pi$, $-\pi < x < \pi$.
(b) $f(x) = e^{-8x}$ for $-4 \le x \le 4$.
(c) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$.

10.5.4 Show that

(a) $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \qquad \text{(using 10.5.3 (a))},$$
(b) $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \qquad \text{(using 10.5.3 (c))}.$$

10.5.5 Compare the graph of the function $f(x) = x^2$ with the 3rd, 6, 10, and 13th partial sums of its Fourier series on the interval $[-2, 2]$.

10.5.6 Write the complex Fourier series of the following functions:

(a) $f(x) = \cos x$, $0 \le x < 1$, and f has period 1.
(b) f has period 4, and $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 4 \end{cases}$.

10.5.7 Expand the following functions in an appropriate cosine and sine Fourier series:

(a) $f(x) = x^3$, $-\pi < x < \pi$.
(b) $f(x) = \begin{cases} x - 1, & -\pi < x < 0\,, \\ x + 1, & 0 \le x < \pi\,. \end{cases}$

10.5.8 Let $f(x) = 4 \sin x, 0 < x < \pi, f(x + \pi) = f(x)$. Sketch this function and its Fourier series. Find the frequency spectrum of f.

10.5.9 Let f be integrable on $[-l, l]$.

(a) Prove that
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \le \frac{1}{l}\int_{-l}^{l} |f(x)|^2\,dx\,,$$
where a_0, a_n, and b_n are the Fourier coefficients of f.

(b) Prove Parseval's identity
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{l}\int_{-l}^{l} |f(x)|^2\,dx$$
if $\int_{-l}^{l} |f(x)|^2\,dx$ is finite and a_0, a_n, and b_n are its Fourier coefficients.

(c) Show that $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

10.5.10 Find the Fourier transform of the following functions:

(a) $f(t) = te^{-|t|}$ for all real t.

(b) $f(t) = \begin{cases} \sin(\pi t), & -5 \le t \le 5 \\ 0, & |t| > 5 \end{cases}$

(c) $f(t) = \begin{cases} 1, & 0 \le t \le k \\ -1, & -k \le t < 0 \\ 0, & |t| > k \end{cases}$, where k is a positive constant.

10.5.11 Prove Theorem 10.2 (b): Let ξ_0 be a real number, let $g(t) = e^{i\xi_0 t} f(t)$, then $\hat{g}(\xi) = \hat{f}(\xi - \xi_0)$.

10.5.12 Show that

(a) For $g(t) = f(-t)$ show that $\hat{g}(\xi) = \hat{f}(-\xi)$.
(b) For $g(t) = \hat{f}(t)$ show that $\hat{g}(\xi) = 2\pi f(-\xi)$.

10.5.13 Prove that $(f * g)(x) = (g * f)(x)$ for all x.

10.5.14 Let $f(x) = e^{-ax}\chi_{(0,\infty)}(x)$ and $g(x) = e^{-bx}\chi_{(0,\infty)}(x)$, where

$$\chi_{(0,\infty)}(x) = \begin{cases} 1, & x \in (0,\infty) \\ 0, & x \notin (0,\infty) \end{cases}$$

Calculate $(f * g)(x)$.

10.5.15 Find $\hat{f}$ and $\hat{g}$ if

$$f(t) = \chi_{[-1/2,1/2]}(t) = \begin{cases} 1, & -1/2 \le t \le 1/2 \\ 0, & t < -1/2 \text{ or } t > 1/2 \end{cases},$$

10.5.16 Let f and g be square integrable. Show that the convolution $f * g$ is a continuous function on $\mathbb{R}$.

10.7.17 Let $x = x(n)$ and $y = y(n)$ be N-periodic signals. Prove that $(x * y)(n) = (y * x)(n)$.

Chapter 11
Differential Equations

A differential equation is an equation relating a function with its derivatives. In these equations, the functions often represent physical quantities, the derivatives represent their rates of change and the equation defines their relationship. Differential equations have been and still are a major and important branch of pure and applied mathematics since their invention in the mid-seventeenth century. Differential equations began with the German mathematician Leibniz and the Swiss brother mathematicians Jacob and Johann Bernoulli and some others from 1680 on, not long after Newton's fluxional equations in the 1670s. Applications were made to geometry, mechanics, and optimization. Most part of the eighteenth century was devoted to the consolidation of the Leibnizian tradition, extending it to several independent variables which led to partial differential equations. New scholars known as experts of mathematics, physics, astronomy, and philosophy, namely Euler, Daniel Bernoulli (the son of Jacob Bernoulli), Lagrange, and Laplace appeared on the scene to solve challenging problems related to theory and applications. Several applications were made to mechanics, particularly to astronomy and continuum mechanics.

In the nineteenth century, the general theory was enriched by a better understanding of the nature of solutions, e.g., through existence and uniqueness theorems. This went hand in hand with the clarification of the foundations of analysis through the notions of limit and continuity as we use them today. In the twentieth century, the general theory was developed in many directions, influenced by functional analysis and other mathematical disciplines.

Besides the persons mentioned above, among the mathematicians cum scientists who have contributed significantly in this area are Fourier, Legendre, Bessel, d'Alembert, Cauchy, Riemann, Monge, Poisson, Dirichlet, Gauss, Navier, Stokes, Maxwell, Helmholtz, Korteweg, de Vries, Poincaré, Dieudonné, Sobolev, Kruskal, Lax, Lions, Stampacchia, Nirenberg, Brezis, Browder, Hörmander, Mosco, and Zeidler. Among Indian mathematicians who have notably contributed in this field are recipient King Faisal award and FRS Narasimhan, Adimurti, Kesavan, Vaninathan, and Gowda.

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In this chapter, we present basic methods for computing solutions of differential equations through explicit formula, as well as many examples of how differential equations are used for the modeling of real phenomena.

11.1 Introduction and Basic Notions

How do differential equations arise? As an example, let us study a simple mathematical model for the development (or evolution) of a fish population in a large pond. Let $P(t)$ be the size (in millions) of the population at time t (in years). Its rate of change $P'(t)$ at time t equals the rate of change due to breeding minus the rate of change due to harvesting. Let us assume that the former is equal to 20% of the current population, and that the latter is equal to 10 million fish per year. The function P then satisfies

$$P'(t) = 0.2\,P(t) - 10$$

for each time t. The underlying equation is commonly written as

$$P' = 0.2\,P - 10. \tag{11.1}$$

This is the differential equation that models how the fish population changes. The unknown quantity in the equation is P as a function of the time t. This equation can be used to predict the fish population in the future.

A rough estimate can be made as follows. Suppose that at the initial time $t = 0$, the fish population equals 80 million. At this time, $P'(0) = 0.2 \cdot 80 - 10 = 16 - 10 = 6$, a rate of change of six million fish per year. On this basis, we may estimate the population size after 1 year as $80 + 6 = 86$ million fish. This number, however, will be an underestimate since the population size, and consequently its rate of change, will increase during the year.

An analysis of the differential equation, which will be done later, yields that the family of functions (and no other function)

$$P(t) = 50 + ce^{0.2t}, \tag{11.2}$$

c being an arbitrary constant, solves (11.1). Actually, given this family, one may check directly that for every value of c we obtain a solution of (11.1), as

$$P'(t) - (0.2\,P(t) - 10) = 0.2\,ce^{0.2t} - 0.2 \cdot (50 + ce^{0.2t}) + 10 = 0.$$

The family (11.2) is called the general solution of (11.1). If, as above, we also require the initial condition $P(0) = 80$, we have to set $c = 30$. The corresponding solution $P(t) = 50 + 30e^{0.2t}$ is called a particular solution of (11.1). (It yields the value $P(1) \approx 86.64$.) The combined pair (the differential equation and the initial condition) is called an initial value problem.

Example 11.1 (a) Check that $P(t) = ce^{2t}$ is a solution of the differential equation $P' = 2P$ for arbitrary values of the constant c.
(b) Find the particular solution which satisfies the initial condition $P(0) = 200$.

Solution: (a) Differentiating $P(t) = ce^{2t}$, we obtain $P'(t) = 2ce^{2t} = 2P(t)$, so $P(t) = ce^{2t}$ satisfies $P' = 2P$.
(b) For $t = 0$ and $P(0) = 200$, we must have $200 = ce^0$, so $c = 200$. Hence $P(t) = 200e^{2t}$ is a particular solution which solves the given initial value problem.

It is a natural question posed by many beginners in the field why one should study differential equations. The answer is that many laws or models governing natural phenomena involve the rates at which things change. This leads to mathematical equations in which, besides the unknown function itself, its derivatives appear, that is, we have to solve differential equations. This occurs in problems from fluid mechanics, population growth, circuit design, heat transfer, seismic waves, option trading. The solutions to differential equations are functions. If they can be expressed symbolically, they are given by mathematical formulas; if they are represented graphically, they look like curves (for ordinary differential equations) or surfaces (for partial differential equations).

Since differential equations model many real-world situations, the question of whether a solution exists and is unique, and how it depends upon changes in the equation or its parameters, respectively, can have great practical importance. If we know how the velocity of a satellite is changing, can we know its position and velocity for the future? If we know the initial population of a city, and we know how the population is changing, can we predict the population in the future? Yes we can; if we know the initial value of some quantity and we know how it is changing, we should be able to find the future value of the quantity. In terms of differential equations, an initial value problem (a differential equation with an initial condition) representing a real-life situation which satisfies some natural conditions has a unique solution.

Order and General Form of a Differential Equation

The **order** of a differential equation is the order of the highest derivative in the equation. For example,

$$y' + 20y = e^x, \qquad y'' + 10y = \sin x,$$

are differential equations of order 1 and 2, respectively. In symbols we can express the general form of an nth-order ordinary differential equation in one dependent variable as

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \tag{11.3}$$

Here, F is a real-valued function of $n + 2$ variables, and $y', \ldots, y^{(n)}$ stand for the derivatives of the unknown function y; we will also use the alternative (traditional) notation

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}.$$

A **solution** of (11.3) is a real-valued function y defined on some open interval J which satisfies

$$F(x, y(x), y\prime(x), \ldots, y^{(n)}(x)) = 0$$

for all $x \in J$.

A differential equation of the form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x) \tag{11.4}$$

is called **linear**. Here, $a_0, a_1, a_2, \ldots a_n$ and g are functions of x on some open interval J. If a_n is not the zero function, the equation is of order n. If $g = 0$ (the zero function), it is called **homogeneous**, otherwise it is called **inhomogenous** or nonhomogeneous. When analyzing linear equations, we will mainly be concerned with equations

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

of first and second order, respectively.

Initial and Boundary Value Problems

As we have seen above, the first-order differential equation $y' = y$ has the functions $y(x) = ce^x$ as solutions, where c is an arbitrary constant. In order to fix a particular solution, one has to specify an additional condition. The second-order equation

$$y'' + y = 0$$

is solved by the functions

$$y(x) = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are constants; indeed, differentiating these functions yields

$$y'(x) = -c_1 \sin x + c_2 \cos x, \qquad y''(x) = -c_1 \cos x - c_2 \sin x.$$

In this case, two additional conditions are needed to single out a particular solution.

This is not coincidental. It turns out that, typically, an nth-order ordinary differential equation requires us to specify n additional conditions to obtain a unique solution. The most common ones are initial conditions and boundary conditions.

Definition 11.1 (*Initial Value Problem*) Here, the additional conditions relate to a single x-value. They are called **initial conditions** or **initial values**, and the differential equation together with the initial conditions is called an **initial value problem**. Its order is defined to be the same as the order of the differential equation.

Example 11.2 (i) $y' + y = 3$ with $y(0) = 2$ is an initial value problem of first order, $y(0) = 2$ is the initial condition.

(ii) $y'' + 2y = 0$ with $y(1) = 2$ and $y'(1) = -3$ is an initial value problem of second order. The initial conditions are $y(1) = 2$ and $y'(1) = -3$. In this case, the values of the function y and its derivative are specified at $x = 1$.

Definition 11.2 (*Boundary Value Problem*) Here, the additional conditions relate to two or more x values. They are called **boundary conditions** or **boundary values**, and the differential equation together with the boundary conditions is called a **boundary value problem**. Its order is defined to be the same as the order of the differential equation.

Example 11.3 (i) $y'' - y' + y = x^3$ with $y(0) = 4$ and $y'(1) = -2$ is a boundary value problem of second order. The boundary conditions are specified at two points, namely $x = 0$ and $x = 1$.
(ii) $y'' - y' + y = x^3$ with $y(0) = 4$ and $y(1) = -2$ is also a boundary value problem of second order. In this case, both boundary conditions refer to values of the function y itself, and not to values of the derivative y'.
(iii) $y' - y = 1$ with $y(0) = y(1)$ is a boundary value problem of first order. Such a boundary condition is called a **periodic boundary condition**.

The following questions are relevant, from the theoretical standpoint as well as when one uses boundary and initial value problems as a tool to compute solutions to real-world problems.
Problem 1. When does a solution exist? That is, does an initial value problem or a boundary value problem necessarily have a solution?
Problem 2. Is the solution unique? That is, is there only one solution of a given initial value or boundary value problem?
The following theorem states that under the specified conditions, a first-order initial value problem has a unique solution.

Theorem 11.1 *Let f and $\partial f/\partial y$ be functions which are defined and continuous*[1] *in some rectangle R of the xy-plane, and let (x_0, y_0) be a point in the interior of that rectangle. Then on some interval centered at x_0 there exists a unique solution of the initial value problem*

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Example 11.4 (i) $y(x) = 4e^x$ is the unique solution of the initial value problem

$$y' = y, \quad y(0) = 4$$

on the real line $(-\infty, \infty)$. It is the particular solution of the differential equation $y' = y$ which passes through the point $(0, 4)$.

Verification: We have $y'(x) = 4e^x = y(x)$ and $y(0) = 4e^0 = 4$. Thus, the given function solves the initial value problem. The right hand side $f(x, y) = y$ satisfies

[1] See Definition 8.5.

the assumptions of the theorem, so we conclude from the theorem that no other solution exists.[2]
(ii) Find a solution of the initial value problem $y' = y$, $y(1) = 3$. That is, find a solution of the differential equation $y' = y$ which passes through the point $(1, 3)$. Is it unique?

Solution: As in part (i) one checks that $y(x) = ce^x$ solves the given equation $y' = y$. Imposing the given initial condition we get $3 = y(1) = ce^1$, so $c = 3e^{-1}$. Therefore, $y(x) = 3e^{-1}e^x = 3e^{x-1}$ is a solution of the initial value problem. As in part (i), the theorem is applicable, so the solution is unique.

A corresponding theorem holds for the initial value problem of nth order

$$y^{(n)} = f(x, y, \dots, y^{(n-1)}) \tag{11.5}$$

$$y(x_0) = y_0, y'(x_0) = y_0^{(1)}, \ \dots \ , y^{(n-1)}(0) = y_0^{(n-1)} \tag{11.6}$$

where the real-valued function f depends on $n + 1$ variables and $y_0, y_0^{(1)}, \dots, y_0^{(n-1)}$ are given numbers. In this case, one assumes that besides f, all partial derivatives $\partial f/\partial y$, $\partial f/\partial y'$,...are defined and continuous in some rectangular box in $\mathbb{R}^{n+1}$ containing $(x_0, y_0, \dots, y_0^{(n-1)})$ in its interior.

Remark 11.1 Theorem 11.1 and its generalization to nth-order equations are special cases of the famous Picard-Lindelöf theorem.

The superposition principle. We consider the homogeneous linear equation of order n,

$$a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = 0.$$

If the functions y_1 and y_2 are solutions, then also the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is a solution, for arbitrary values of the constants c_1 and c_2. (One can verify this directly by inserting y into the differential equation.) This property of linear equations is called the **superposition principle**.

General and particular solution. It can be shown that the homogeneous linear equation of order n (note that a_n is set to 1)

$$y^{(n)} + \dots + a_1(x)y' + a_0(x)y = 0 \tag{11.7}$$

[2] Assume that there exists a second solution z with $z(x) \neq y(x)$ for some $x > x_0$. Let $x_M \geq x_0$ be the largest number such that $z(\xi) = y(\xi)$ for all $\xi \in [x_0, x_M]$. Then the functions y and z yield two different solutions of $y' = f(x, y)$ through the point $(x_M, y(x_M))$ in some interval centered at x_M, contradicting the theorem. Therefore, such a second solution cannot exist. In the same manner, one shows that the solution is unique for values $x < x_0$.

has n linearly independent[3] solutions $y_1, \ldots, y_n$. Such a set $\{y_1, \ldots, y_n\}$ of solutions is called a **fundamental set** for (11.7). By the superposition principle,

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$

is a solution, for arbitrary constants $c_1, \ldots, c_n$. It is called the **general solution** of (11.7). Any solution obtained by fixing the constants is called a **particular solution** of (11.7). For example, the constants can be determined from the initial conditions (11.6).

Historical note. Some people say that the study of differential equations began in 1675 when the German mathematician cum philosopher Gottfried Wilhelm von Leibniz (1646–1716) wrote the equation

$$\int x dx = \frac{1}{2}x^2.$$

The search for general methods of finding solution of differential equations started when the English physicist and mathematician Isaac Newton (1643–1727) classified first-order differential equations into three classes, namely

$$(1)\ \frac{dy}{dx} = f(x), \quad (2)\ \frac{dy}{dx} = f(x, y), \quad (3)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

The first two cases contain only ordinary derivatives of one or more dependent variables, with respect to a single independent variable; they are ordinary differential equations. The third case involves partial derivatives of one variable which depends on several independent variables; it represents a partial differential equation.

The simplest method of solving differential equations, namely the separation of variables method, was developed by the joint effort of Leibniz and Jacob Bernoulli (1655–1705) around 1691. Johann Bernoulli (1667–1748), the younger brother of Jacob, also contributed to the development of the separation of variables method. Johann Bernoulli also introduced a linear homogeneous differential equation around 1692.

A differential equation of the type

$$\frac{dy}{dx} + P(x) = f(x)y^n$$

was introduced by Johann Bernoulli in December 1695 and solved by Leibniz in 1696. Nowadays this equation is known as the Bernoulli equation.

Jacopo Riccati (1676–1754) introduced the equation, now bearing his name,

$$y' = P(x) + Q(x)y + R(x)y^2.$$

[3] See Definition 10.3.

Another Bernoulli, Daniel, had already solved special cases of this equation. The Swiss mathematician cum physicist, Leonard Euler (1707–1783) made a significant progress when he posed and solved the problem of reducing a particular class of second order differential equation to that of a first-order differential equation. He interacted with Johann Bernoulli in 1739 on the study of homogeneous linear differential equation with constant coefficients. He also studied non-homogeneous linear equations and applied the method of successive order reduction to solve equations of higher order.

Alexis-Claude Clairaut (1713–1765) studied differential equations of the form

$$y = xy' + f(y')$$

in 1734, nowadays called Clairaut equations. Besides a one-parametric family of solutions as usual, this equation also possesses other solutions which are termed singular; thus Clairaut was one of the first to study initial value problems with nonunique solutions.

The Italian mathematician and astronomer Joseph-Louis Lagrange (1736–1813) had embarked on the problem of integrating factor for the general linear equation. He also invented the method of variation of parameters.

The French mathematician and philosopher Jean-Baptiste le Rond d'Alembert (1717–1783) further developed the theory of ordinary linear differential equations. He provided fundamental contributions to mathematical continuum mechanics; in this context, he introduced partial differential equations into the modeling of vibrating strings.

One might say that all known methods of computing explicit formulas for solutions of first-order differential equations had been obtained during the time period referred to as above.

11.2 Separation of Variables

Definition 11.3 A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y), \tag{11.8}$$

where g and h are functions of x and y only, respectively, is called **separable** or said to have **separable variables**.

The simplest case arises when $h = 1$, that is, the right hand side of (11.8) does not depend on y,

$$\frac{dy}{dx} = g(x).$$

This means that we want to find a function y whose derivative equals the function g. So, we obtain y as

$$y = G + c, \quad y(x) = G(x) + c, \tag{11.9}$$

where G is an antiderivative[4] of g, and c is a constant.

Example 11.5 Solve the following differential equation:

$$\frac{dy}{dx} = \cos 7x.$$

Solution: Here, $g(x) = \cos 7x$. An antiderivative of g is $G(x) = (1/7)\sin 7x$, so the solution is

$$y(x) = \frac{\sin 7x}{7} + c.$$

Remark 11.2 In Chap. 6, we have discussed extensively how to find antiderivatives of a given function.

We now turn to the general case

$$\frac{dy}{dx} = g(x)h(y).$$

The **separation of variables** method works as follows. In the first step, one separates the "x" and the "y" part[5]

$$\frac{1}{h(y)}\,dy = g(x)\,dx\,. \tag{11.10}$$

In the second step, one integrates

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx + c\,;$$

this means, one finds an antiderivative H of the function p defined by $p(y) = 1/h(y)$, an antiderivative G of the function g, and writes

$$H(y) = G(x) + c.$$

From this equation, in the third step one determines y as a function of x. In the fourth step, one checks whether y is a solution (by inserting it and its derivative into the differential equation), and determines its domain.

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[4]See Definition 6.1.

[5]This is a purely formal computation; while the expression "dy/dx" has a clear mathematical meaning—it stands for the derivative of the function y with respect to x—it is not helpful in the present context to try to attach a meaning to the expressions "dx" and "dy".

Example 11.6 Solve the differential equation

$$y' = \frac{y}{x} \tag{11.11}$$

Solution: Here $g(x) = 1/x$, $h(y) = y$ and $p(y) = 1/y$. Thus,

$$H(y) = \ln y, \quad G(x) = \ln x.$$

Hence, the second step of the method yields (we denote the integration constant by b)

$$\ln y = \ln x + b. \tag{11.12}$$

Taking the exponential on both sides, we obtain

$$y = e^{\ln y} = e^{\ln x + b} = e^{\ln x} e^b = x e^b.$$

Thus, the solution has the form

$$y(x) = cx,$$

where c is a constant. We have $y'(x) = c = cx/x$, so this function y indeed solves (11.11). Its domain is the whole real line $(-\infty, \infty)$, and c can be an arbitrary real number. Note that, in our computation, Eq. (11.12) tacitly assumes that x and y (and consequently c) are positive, but this no longer matters once we have arrived at the solution.

Example 11.7 Solve the initial value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = 3.$$

Solution: We have

$$g(x) = x, \quad h(y) = -\frac{1}{y}, \quad p(y) = -y,$$

so

$$H(y) = -\frac{1}{2}y^2, \quad G(x) = \frac{1}{2}x^2.$$

Thus, the second step of the separation of variables method yields

$$-\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

or equivalently

$$x^2 + y^2 = -2c.$$

From the initial condition $y(4) = 3$ we get $-2c = 4^2 + 3^2 = 25$, so we arrive at the implicit equation

$$x^2 + y^2 = 25.$$

This equation has two solutions $y(x) = \pm\sqrt{25 - x^2}$, but only the positive one satisfies the initial condition. Its domain is the interval $(-5, 5)$; for $x = \pm 5$ the derivative of y becomes infinite.

11.3 First-Order Linear Equations

In this section, we discuss the linear equation of first order

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $a_1(x) \neq 0$ for all x considered, we can write this differential equation in the form

$$\frac{dy}{dx} + b(x)y = f(x), \tag{11.13}$$

where $b(x) = a_0(x)/a_1(x)$ and $f(x) = g(x)/a_1(x)$. The Eq. (11.13) is called the **standard form** of a linear differential equation of the first order.

We first consider the special case $f = 0$. Let B be an antiderivative of b, so $B' = b$. Then

$$y(x) = ce^{-B(x)}, \tag{11.14}$$

c being a constant, is a solution of (11.13), because

$$y'(x) + b(x)y(x) = ce^{-B(x)} \cdot (-B'(x)) + b(x)y(x) = y(x) \cdot (-b(x)) + b(x)y(x) = 0.$$

(One can find the solution (11.14) with the separation of variables method.)

Example 11.8 Find the solution of the differential equation $y' = 9y$.

Solution: We have $b(x) = -9$, $B(x) = -9x$, so the solution becomes $y(x) = e^{9x}$. It is defined on $(-\infty, \infty)$.

We now consider (11.13) for an arbitrary function f. Again, let B be an antiderivative of b. The function I is defined by

$$I(x) = e^{B(x)} \tag{11.15}$$

is called **integrating factor**. It satisfies

$$I'(x) = e^{B(x)}B'(x) = b(x)I(x).$$

Let Q be an antiderivative of the function $f \cdot I$, so $Q'(x) = f(x)I(x)$. Then using the product rule (we omit the argument x)

$$\frac{d}{dx}(y \cdot I - Q) = y' \cdot I + I' \cdot y - Q' = (-by + f)I + bI \cdot y - f \cdot I = 0.$$

Therefore,

$$y(x)I(x) = Q(x) + c, \tag{11.16}$$

c being a constant, yields a solution y of (11.13).

We thus obtain the following **procedure of solution** for the linear equation of first order.

Step 1: Put the equation in the standard form (11.13) if it is not given in this form.
Step 2: Identify $b(x)$, find an antiderivative B of b and compute the integrating factor $I(x) = e^{B(x)}$.
Step 3: Find an antiderivative Q of $f \cdot I$.
Step 4: The solution y satisfies the equation

$$y(x) \cdot I(x) = Q(x) + c. \tag{11.17}$$

Example 11.9 Find the solution of the following differential equations:

$$\text{(a)} \quad x\frac{dy}{dx} + 2y = 3, \qquad\qquad \text{(b)} \quad x\frac{dy}{dx} + (3x+1)y = e^{-3x}.$$

Solution: (a) We have $b(x) = 2/x$ and $f(x) = 3/x$. We compute

$$B(x) = 2\ln x, \quad I(x) = e^{B(x)} = x^2, \quad f(x)I(x) = \frac{3}{x} \cdot x^2 = 3x, \quad Q(x) = \frac{3}{2}x^2.$$

The solution y therefore satisfies

$$y(x) \cdot x^2 = \frac{3}{2}x^2 + c.$$

Division by x^2 yields

$$y(x) = \frac{3}{2} + \frac{c}{x^2}.$$

It is valid for $x \in \mathbb{R}$, $x \neq 0$.

(b) The standard form becomes

$$\frac{dy}{dx} + \left(3 + \frac{1}{x}\right) y = \frac{e^{-3x}}{x}$$
$$b(x) = 3 + \frac{1}{x}, \quad f(x) = \frac{e^{-3x}}{x}.$$

We have

$$B(x) = 3x + \ln x, \quad I(x) = e^{B(x)} = xe^{3x}, \quad f(x)I(x) = 1, \quad Q(x) = x.$$

The solution y therefore satisfies

$$y(x) \cdot xe^{3x} = x + c,$$

so

$$y(x) = e^{-3x}\left(1 + \frac{c}{x}\right).$$

It is valid for $x \in \mathbb{R}$, $x \neq 0$.

11.4 Solution by Substitution

Some differential equations of first order can be transformed into a separable differential equation or into a linear differential equation of standard form (Eq. (11.13)) by an appropriate substitution. We discuss here two classes of differential equations, one class comprises homogeneous equations and the other class consists of Bernoulli equations.

11.4.1 Homogeneous Equations

A function f of two variables is called **homogeneous** of degree m, where m is an integer, if $f(tx, ty) = t^m f(x, y)$ for all real numbers t.

A first-order differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{11.18}$$

is called **homogeneous** if both coefficient functions M and N are homogeneous of the same degree.

The differential equation (11.18) is written in a form different from what we used to write so far, namely (11.3). The form (11.18) also has a long tradition; it is natural if one thinks of the solution as a curve $(x(s), y(s))$ in the plane, parametrised by some variable s. When we look for solutions for which y is a function of x, as we

do in the present chapter, then (11.18) is defined to be the same as the first-order equation

$$M(x, y) + N(x, y)y' = 0.$$

To solve a differential equation by substitution means that we replace the dependent variable y by another dependent variable u. This transforms the differential equation for y into one for u, which is hopefully easier to solve. For homogeneous differential equation, the substitution

$$y = ux, \quad \text{that is,} \quad u(x) = \frac{y(x)}{x}$$

works. (Alternatively, one may also use the substitution $x = vy$ for a new dependent variable v.) We then obtain a separable equation for u (v, respectively) of first order.

We illustrate this solution method with two examples. The computations are facilitated by using a symbolic calculus; one can (and should) always check at the end whether the function obtained actually solves the differential equation.

Example 11.10 Solve the following homogeneous equations:

(a) $(x - y)dx + xdy = 0$
(b) $(y^2 + yx)dx + x^2dy = 0.$

Solution: (a) We use $y = ux$ to replace y by u. We formally compute $dy = udx + xdu$, so the given equation becomes

$$(x - ux)dx + x(udx + xdu) = 0.$$

This is further processed as

$$xdx + x^2du = 0, \quad \frac{dx}{x} + du = 0.$$

We have obtained a separable differential equation, already written in the separated form as used in (11.10). Solving this yields

$$\ln|x| + u = c, \quad x\ln|x| + y = cx,$$

so we arrive at the solution

$$y(x) = cx - x\ln|x|.$$

(b) Again, we use $y = ux$ and $dy = udx + xdu$. The given equation takes the form

$$(u^2x^2 + ux^2)dx + x^2(udx + xdu) = 0.$$

Appropriate divisions give

$$(u^2 + 2u)dx + xdu = 0, \quad \frac{dx}{x} + \frac{du}{u(u+2)} = 0.$$

Passing to antiderivatives in this separable equation yields

$$\ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| = c.$$

Multiplying by 2 and taking the exponential gives

$$x^2\left|\frac{u}{u+2}\right| = e^{2c}.$$

Replacing u by y/x and e^{2c} by b, we arrive at

$$x^2|y| = b|y + 2x|. \tag{11.19}$$

This is an implicit equation for the solution of the given equation. When x and y have the same sign, the absolute values can be omitted. One may then check by differentiation that functions y which satisfy (11.19) also satisfy the given equation.

11.4.2 Bernoulli Equations

A differential equation of the form

$$\frac{dy}{dx} + b(x)y = f(x)y^n \tag{11.20}$$

is called a Bernoulli equation. If $n \neq 0$ and $n \neq 1$, it can be reduced to a linear equation of first order by the substitution $v = y^{1-n}$. This linear equation can be solved by the method described above in Sect. 11.3.

Example 11.11 Solve the following differential equations:

$$\text{(a)} \quad \frac{dy}{dx} + \frac{1}{x}y = 3y^3 \qquad \text{(b)} \quad \frac{dy}{dx} - y = e^x y^2.$$

Solution: (a) We have $n = 3$. Let $v = y^{1-n} = y^{-2}$. The chain rule yields

$$\frac{dv}{dx} = -2y^{-3} \cdot \frac{dy}{dx}, \quad \frac{dy}{dx} = -\frac{1}{2}y^3 \cdot \frac{dv}{dx}.$$

Substituting these values into the given differential equation, we get

$$-\frac{1}{2}y^3 \cdot \frac{dv}{dx} + \frac{1}{x}y = 3y^3.$$

Division by $-(1/2)y^3$ gives

$$\frac{dv}{dx} - \frac{2}{x}v = -6. \tag{11.21}$$

This equation is of the standard form (11.13), and therefore the solution procedure of Sect. 11.3 is applicable. We have $b(x) = -2/x$ and $f(x) = -6$, and obtain

$$B(x) = -2\ln x, \quad I(x) = e^{B(x)} = x^{-2}, \quad f(x) \cdot I(x) = -6x^{-2}, \quad Q(x) = 6x^{-1}.$$

According to Sect. 11.3, the solution v of (11.21) satisfies

$$\begin{gathered} v(x) \cdot x^{-2} = 6x^{-1} + c, \\ v(x) = 6x + cx^2. \end{gathered}$$

Since $v = y^{-2}$, we get

$$y(x) = \pm\frac{1}{\sqrt{6x + cx^2}}.$$

(b) We have $n = 2$. Let $w = y^{-1}$, then the equation

$$\frac{dy}{dx} - y = e^x y^2$$

becomes

$$\frac{dw}{dx} + w = -e^x.$$

We have $b(x) = 1$, $f(x) = -e^x$, and obtain

$$B(x) = x, \quad I(x) = e^x, \quad f(x) \cdot I(x) = -e^{2x}, \quad Q(x) = -\frac{1}{2}e^{2x}.$$

According to Sect. 11.3, the solution w satisfies

$$w(x) \cdot e^x = -\frac{1}{2}e^{2x} + c.$$

Since $w = y^{-1}$, we get

$$y(x) = \frac{1}{-\frac{1}{2}e^x + ce^{-x}}.$$

11.4.3 Reduction of Order

Let

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{11.22}$$

be a linear second order homogeneous differential equation. Assume that we already know one solution y_1. We want to find a second solution y_2 of the form

$$y_2(x) = u(x)y(x). \tag{11.23}$$

It turns out that the derivative $w = u'$ of u satisfies a linear first-order equation. The latter can be solved by the procedure of Sect. 11.3. Then, we obtain u as an antiderivative of w and y_2 by inserting u into (11.23).

Remark 11.3 This procedure can be generalized to higher order linear differential equations.

Example 11.12 For the second order equation $y'' - y = 0$, we know that $y_1(x) = e^x$ is a solution on the interval $(-\infty, \infty)$. Use reduction of order to find a second solution y_2.

Solution: Let $y_2(x) = u(x)y_1(x)$. Differentiating this product function we get

$$\begin{aligned} y_2'(x) &= u(x)y_1'(x) + u'(x)y_1(x) \\ y_2''(x) &= u(x)y_1''(x) + 2u'(x)y_1'(x) + u''(x)y_1(x). \end{aligned}$$

In order that y_2 becomes a solution of $y'' - y = 0$, we must have (we omit writing the argument "x")

$$0 = y_2'' - y_2 = uy_1'' + 2u'y_1' + u''y_1 - uy_1 = u(y_1'' - y_1) + 2u'y_1' + u''y_1.$$

Since y_1 solves the given equation, we must have

$$0 = 2u'y_1' + u''y_1 = (2u' + u'')e^x,$$

that is, since $e^x \neq 0$,

$$u'' + 2u' = 0.$$

By substituting $u' = w$ in this equation we obtain

$$w' + 2w = 0.$$

This is a linear first-order differential equation. Here, it is not necessary to go back to the procedure of Sect. 11.3 since we already know that

$$w(x) = ce^{-2x}$$

is the general solution. The function $u(x) = e^{-2x}$ solves $u' = w$ for $c = -2$. Thus, we have arrived at a second solution

$$y_2(x) = u(x)y_1(x) = e^{-2x}e^x = e^{-x}$$

of the equation $y'' - y = 0$. Since y_2 is not a constant multiple of y_1 and both y_1, y_2 are nonzero, these solutions are linearly independent.

11.4.4 Homogeneous Linear Equations with Constant Coefficients

We consider in this subsection equations of the type

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0, \tag{11.24}$$

where the coefficients $a_n, \ldots, a_1, a_0$ are real constants and $a_n \neq 0$. It turns out that all solutions of (11.24) are exponential functions or are constructed from exponential functions.

We try to find a solution of the form $y(x) = e^{\lambda x}$. After substituting $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$ and so on, Eq. (11.24) gives us

$$a_n \lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0.$$

Since $e^{\lambda x} \neq 0$ for all x, we can divide by $e^{\lambda x}$ and obtain

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0. \tag{11.25}$$

Equation (11.25) is called the **auxiliary equation**. Its left side is a polynomial of order n. If λ is a root of this polynomial, the corresponding exponential function $y(x) = e^{\lambda x}$ is a solution of (11.24).

Let us consider the special case $n = 2$ of (11.24), written in the form

$$ay'' + by' + cy = 0 \tag{11.26}$$

with constants a, b, c. The auxiliary equation becomes

$$a\lambda^2 + b\lambda + c = 0, \tag{11.27}$$

The roots of (11.27) are

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$
$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We know that (i) λ_1 and λ_2 are real and distinct if $b^2 - 4ac > 0$, (ii) λ_1 and λ_2 are real and equal if $b^2 - 4ac = 0$, (iii) λ_1 and λ_2 are conjugate complex numbers if $b^2 - 4ac < 0$.

Case (i) Two distinct real roots

Let λ_1 and λ_2 be two distinct real roots of (11.27). We find two solutions

$$y_1(x) = e^{\lambda_1 x}, \quad y_2(x) = e^{\lambda_2 x}.$$

We can check that y_1 and y_2 are linearly independent on every interval and thus form a fundamental set of solutions. Thus,

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{11.28}$$

is the general solution of (11.26) in this case.

Case (ii) A double real root

If $\lambda_1 = \lambda_2$, we obtain only one exponential solution $y_1(x) = e^{\lambda_1 x}$. A second solution is

$$y_2(x) = x e^{\lambda_1 x}.$$

This can be obtained by the method of order reduction, or verified directly by differentiation. The general solution in this case becomes

$$y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}. \tag{11.29}$$

Case (iii) Conjugate complex roots

In this case, $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where α and β are real and $\beta > 0$. Then

$$y_1(x) = e^{(\alpha + i\beta)x}, \quad y_2(x) = e^{(\alpha - i\beta)x}$$

are two linearly independent solutions; they are complex-valued, as the exponentials $e^{\pm i\beta}$ are complex numbers. The general solution of (11.26) becomes

$$y(x) = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}. \tag{11.30}$$

We will obtain real solutions through a suitable choice of the constants c_1 and c_2. From Euler's formula

$$e^{i\theta} = \cos\theta + i \sin\theta$$

where θ is any real number, we obtain

$$e^{i\beta x} = \cos \beta x + i \sin \beta x,$$

$$\cos \beta x = \frac{e^{i\beta x} + e^{-i\beta x}}{2}, \quad \sin \beta x = \frac{e^{i\beta x} - e^{-i\beta x}}{2i}.$$

The choices $c_1 = c_2 = 1$ and $c_1 = 1, c_2 = -1$ respectively give the two solutions

$$z_1(x) = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x},$$
$$z_2(x) = e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}.$$

Therefore

$$z_1(x) = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x} \cos \beta x,$$
$$z_2(x) = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2e^{\alpha x} \sin \beta x.$$

These functions are real-valued and thus the sought-for real solutions of (11.26). Moreover, these solutions form a fundamental set on $(-\infty, \infty)$. Consequently the general solution is

$$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x). \tag{11.31}$$

This solution represents an oscillation whose frequency is determined by β and whose amplitude is constant in the case $\alpha = 0$ (which corresponds to purely imaginary roots) or increasing respectively, decreasing in x in the case $\alpha > 0$ respectively, $\alpha < 0$.[6] Equation (11.26) models the **simple harmonic oscillator** (case $\alpha = 0$) and the **damped harmonic oscillator** (case $\alpha < 0$), respectively.

Example 11.13 Solve the following differential equations:

(i) $2y'' - 5y - 3y = 0$.
(ii) $y'' + 5y' - 6y = 0$.
(iii) $y'' + 8y' + 16y = 0$.
(iv) $y'' + 4y' + 7y = 0$.

Solution: of (i) The auxiliary equation is $2\lambda^2 - 5\lambda - 3 = 0$ which can be written as

$$(2\lambda + 1)(\lambda - 3) = 0.$$

Its two roots are $\lambda_1 = -1/2$, $\lambda_2 = 3$. The solution has the form (11.28), that is,

$$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}.$$

(ii) The auxiliary equation is $\lambda^2 + 5\lambda - 6 = 0$. This can be written in the form

[6] See also Chap. 10 on Fourier methods.

$$(\lambda - 1)(\lambda + 6) = 0.$$

Its roots are $\lambda_1 = 1$, $\lambda_2 = -6$. Again, the solution has the form form (11.28), that is,

$$y = c_1 e^x + c_2 e^{-6x}.$$

(iii) The auxiliary equation is $\lambda^2 + 8\lambda + 16 = 0$. Its roots are $\lambda_1 = \lambda_2 = -4$. The solution is of the form (11.29), that is,

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} = c_1 e^{-4x} + c_2 x e^{-4x}.$$

(iv) The auxiliary equation is $m^2 + 4m + 7 = 0$. Its roots λ_1 and λ_2 are given by $\lambda_1 = -2 + i\sqrt{3}$, $\lambda_2 = -2 - i\sqrt{3}$. The solution is of the form (11.31), that is,

$$y = e^{-2x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x).$$

Example 11.14 Solve the initial value problem

$$\begin{gathered} y'' + 3y' - 2y = 0 \\ y(0) = 1, \quad y'(0) = 2. \end{gathered}$$

Solution: The auxiliary equation is

$$\lambda^2 + 3\lambda + 2 = 0.$$

Its roots λ_1 and λ_2 are

$$\lambda_1 = -1, \quad \lambda_2 = -2.$$

Therefore, the solution is of the form (11.28), that is,

$$y = c_1 e^{-x} + c_2 e^{-2x}.$$

To find c_1 and c_2 we use the initial conditions

$$y(0) = 1, \quad y'(0) = 2.$$

We have

$$1 = y(0) = c_1 e^{-0} + c_2 e^{-0},$$

so

$$c_1 + c_2 = 1.$$

Since $y'(x) = -c_1 e^{-x} - 2c_2 e^{-2x}$, the second initial condition yields

$$2 = y'(0) = -c_1 e^{-0} - 2c_2 e^{-0} = -c_1 - 2c_2,$$

so

$$c_1 + 2c_2 = -2.$$

Solving the linear system $c_1 + c_2 = 1$, $c_1 + 2c_2 = -2$ we obtain $c_2 = -3$ and $c_1 = 4$. Therefore, the solution is

$$y(x) = 4e^{-x} - 3e^{-2x}.$$

Remark 11.4 In general, to solve an nth order differential equation (11.24) we must solve the nth degree polynomial equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

If all roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of this equation are real and distinct, the functions $y_1(x) = e^{\lambda_1 x}, \ldots, y_n(x) = e^{\lambda_n x}$ form a fundamental set, and the general solution of (11.24) is

$$y(x) = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + \cdots + c_ne^{\lambda_n x}.$$

If m is a multiple root, then functions $xe^{\lambda x}$, $x^2e^{\lambda x}$,…appear in the fundamental set. If there are pairs of conjugate complex roots, then corresponding pairs as in (11.31) appear in the fundamental set.

11.5 Modeling with Differential Equations

In this section we discuss the solution of different real world problems represented by differential equations of first order. These problems are
(11.5.1) Growth and decay
(11.5.2) Population growth (population dynamics)
(11.5.3) Pollution of lakes
(11.5.4) Quantity of a drug in the body
(11.5.5) Spread of disease, technologies, and rumor
(11.5.6) Newton's law of heating and cooling
(11.5.7) Timing of death (police investigation in criminal cases)

11.5.1 Growth and Decay

Growth and decay of a nonnegative quantity y can be modeled by considering y as a function of time t and formulating a differential equation

$$\frac{dy}{dt} = f(t, y). \tag{11.32}$$

This means that the rate of change of y at the time t equals the value $f(t, y(t))$. The quantity grows when f is positive, and decays when f is negative.

If the growth rate $f(t, y)$ is equal to a constant k, then

$$y(t) = kt + c$$

with an arbitrary constant c solves (11.32). This describes **linear growth** and decay, respectively. If $k < 0$, this model only makes sense until the time t when $y(t)$ becomes zero.

If $f(t, y) = ky$ with k being constant, that is, the growth rate is linear, then

$$y(t) = ce^{kt}$$

solves (11.32). This describes **exponential growth** and decay, respectively, with a **growth factor** k.

A third variant is $f(t, y) = kt$; then the growth rate is linear with respect to time, but independent from y. The solution in this case becomes

$$y(t) = \frac{k}{2}t^2 + c,$$

that is, we have quadratic growth.

In all variants, c equals $y(0)$, the amount of the quantity at time zero.

Example 11.15 Find the general solution to the following differential equations:

$$\text{(a)} \quad \frac{dy}{dt} = 2t, \qquad\qquad \text{(b)} \frac{dy}{dt} = 2y.$$

Solution: (a) $dy/dt = 2t$, so $y(t) = t^2 + c$.

(b) $dy/dt = 2y$, so $y(t) = ce^{2t}$. This is an example of exponential growth.

11.5.2 Population Growth

One of the earliest attempts to model human population growth by means of mathematics was made by the English economist Thomas Malthus in 1798. Essentially, the idea of his model, now called the Malthusian model,

He assumed that that the rate at which a population of a country grows at a certain time is proportional to the total population of the country at that time. This is nothing else than the model with linear growth rate from the previous subsection; in this context, it is called the **Malthusian model**. If $N(t)$ and k denote the total population at time t and the proportionality constant, respectively, we have

$$\frac{dN}{dt} \propto N, \qquad \frac{dN}{dt} = kN. \tag{11.33}$$

As we know, its general solution is given by

$$N(t) = ce^{kt}.$$

When augmented by an initial condition, the solution of (11.33) will provide the population size at any future time t. This simple model, which does not take into account many factors (immigration and emigration, for example) that influence the growth or decline of human populations, nevertheless turned out to be fairly accurate in predicting the population of the United States during the years 1790–1860. Since it yields exponential growth, which is unbounded when k is positive, it cannot be realistic for large times; nevertheless, (11.33) is still used to model the growth of small populations over short time intervals.

In 1837, the Dutch biologist Verhulst improved the Malthusian model while looking at fish populations in the Adriatic sea. He reasoned that the rate of change of population $N(t)$ with respect to t should be influenced by growth factors such as the population itself, and also factors tending to limit the population, such as limitations of food and space. In his model, he assumes that the growth factors are incorporated into a term $aN(t)$, and limiting factors into a term $-bN(t)^2$, with a and b being positive constants whose values depend on the particular population. From this, he obtained the logistic model of population growth,

$$\frac{dN}{dt} = aN - bN^2. \tag{11.34}$$

If we assume the initial population at time $t = 0$ to be $N(0) = N_0$, we arrive at the initial value problem

$$\frac{dN}{dt} = aN - bN^2 N(0) = N_0.$$

We will see below that its solution is given by

$$N(t) = \frac{aN_0 e^{at}}{a - bN_0 + bN_0 e^{at}}. \tag{11.35}$$

Example 11.16 The population of a community is known to increase at a rate proportional to the number of people present at time t. If the population has doubled in 6 years, how long it will take to triple?

Solution: Let $N(t)$ denote the population at time t. Let $N(0)$ denote the initial population at $t = 0$. The model is

$$\frac{dN}{dt} = kN,$$

where the constant k is unknown. Its solution is is $N(t) = Ae^{kt}$, where $A = N(0)$. By the given data

$$Ae^{6k} = N(6) = 2N(0) = 2A,$$

that is,

$$e^{6k} = 2, \quad k = \frac{1}{6}\ln 2.$$

We want to determine t such that $N(t) = 3A = 3N(0)$. We compute

$$N(0)e^{kt} = 3N(0), \quad e^{kt} = 3, \quad \ln 3 = kt = \tfrac{1}{6}(\ln 2)t,$$
$$t = 6\tfrac{\ln 3}{\ln 2} \approx 9.5$$

The population triples after approximately 9.5 years.

Example 11.17 Let the population of a country be decreasing at a rate proportional to its population. If the population has decreased to 25% in 10 years, how long did it take to decrease to 40%?

Solution: This phenomenon can again be modeled by $dN/dt = kN$. Its solution is

$$N(t) = N(0)e^{kt},$$

where $N(0)$ is the initial population. We compute

$$N(0)e^{10k} = N(10) = \frac{1}{4}N(0), \quad k = \frac{1}{10}\ln\frac{1}{4}.$$

We want to determine t such that $N(t) = (2/5)N(0)$. We compute

$$N(0)e^{kt} = N(t) = \frac{2}{5}N(0), \quad e^{kt} = \frac{2}{5}, \quad kt = \ln\frac{2}{5},$$
$$t = 10\frac{\ln(2/5)}{\ln(1/4)} \approx 6.6$$

The population has been reduced to 40% after approximately 6.6 years.

Example 11.18 Let $N(t)$ be the population at time t and let N_0 denote the initial population, i.e., $N(0) = N_0$. Find the solution to the differential equation

$$\frac{dN}{dt} = aN - bN^2$$

with the initial condition $N(0) = N_0$.

Solution: This is a separable differential equation. We use separation of variables,

$$\frac{dN}{aN - bN^2} = dt. \tag{11.36}$$

In order to find an antiderivative, we decompose the left side into partial fractions. We set

$$\frac{1}{aN - bN^2} = \frac{1}{N(a - bN)} = \frac{A}{N} + \frac{B}{a - bN}.$$

The constants A and B must satisfy $1 = A(a - bN) + BN$ for arbitrary N. This gives $Aa = 1$ and $0 = -bA + B$, so

$$A = \frac{1}{a}, \quad B = \frac{b}{a}.$$

Equation (11.36) becomes

$$\left(\frac{1}{aN} + \frac{b}{a(a - bN)}\right) dN = dt.$$

Integrating both sides we get

$$\frac{1}{a} \ln |N| - \frac{1}{a} \ln |a - bN| = t + c$$

In order to solve this for N, we compute

$$\ln \left| \frac{N}{a - bN} \right| = at + ac, \quad \frac{N}{a - bN} = c_1 e^{at},$$

so

$$N(t) = \frac{ac_1}{bc_1 + e^{-at}}.$$

Using the condition $N(0) = N_0$ gives

$$c_1 = \frac{N_0}{a - bN_0}$$

After substitutions and simplifications the solution becomes

$$N(t) = \frac{aN_0 e^{at}}{a - bN_0 + bN_0 e^{at}}.$$

11.5.3 Pollution of Lakes

Let $Q(t)$ be the volume of some pollutant at time t in a lake of total volume V. Suppose that clean water is flowing into the lake at a constant rate r and that water flows out at the same rate; consequently, the total volume V remains constant. Assume that the pollutant is evenly spread throughout the lake, and that the clean water coming into the lake immediately mixes with the rest of the water.

How does Q vary with time? First, notice that since the pollutant is being taken out of the lake but not added, Q decreases with time, and the water leaving the lake becomes less polluted, so the rate at which the pollutant leaves also decreases with time.

To understand how Q changes with time, we write a differential equation for Q. The rate at which the pollutant leaves at time t is equal to $-Q'(t)$. (The outflow is positive, and Q is decreasing.) The total rate of the outflow equals r, and the pollution is represented in the outflow with the fraction $Q(t)/V$. Thus, the rate at which the pollutant leaves is equal to

$$\frac{Q(t)}{V} \cdot r.$$

So the differential equation is

$$\frac{dQ}{dt} = -\frac{r}{V} Q. \tag{11.37}$$

Its solution is

$$Q(t) = Q_0 e^{-rt/V}, \tag{11.38}$$

where Q_0 is the volume of the pollutant at time zero. This tells us that Q is decreasing. By (11.37), Q' is increasing. Consequently, Q is convex.[7] In addition, the pollutants will never be completely removed from the lake though the quantity remaining will become arbitrarily small. Indeed, $\lim_{t\to\infty} Q(t) = 0$, the t-axis is a horizontal asymptote for the function Q.

Example 11.19 Assume that the lake has a volume of 46 units, and the rate of inflow and outflow equals 17.5 units per year. How long it will take for 90% of the pollutant to be removed from the lake? For 99% to be removed?

Solution: By the given data, $r/V = 17.5/46 = 0.38$, so at time t we have

$$Q(t) = Q_0 e^{-0.38t}.$$

When 90% of the pollution has been removed, 10% remains, so $Q(t) = 0.1 Q_0$. Substituting gives

$$0.1 Q_0 = Q_0 e^{-0.38t}.$$

[7] See Definition 4.4.

Canceling Q_0 and solving for t gives

$$t = \frac{-\ln(0.1)}{0.38} \approx 6.$$

Thus, it takes approximately 6 years to remove 90% of the pollution. Similarly, when 99% of the pollution has been removed, $Q(t) = 0.01 Q_0$, so we solve

$$0.01 Q_0 = Q_0 e^{-0.38t}$$

giving

$$-\frac{\ln(0.01)}{0.38} \approx 12.$$

It takes approximately 12 years to remove 99% of the pollution.

11.5.4 The Quantity of a Drug in the Body

The exponential decay model which we have used above to describe pollutants leaving a lake can also be applied to other contaminants flowing in or out of a fluid system, provided we have complete mixing. Another example is the quantity of a drug in a patient's body. After stopping the administration of a drug, the rate at which the drug leaves the body can be assumed proportional to the quantity of the drug left in the body. If we let A represent the quantity of drug in the body,

$$\frac{dA}{dt} = -kA.$$

The minus sign indicates that the quantity of the drug in the body is decreasing. The solution to this differential equation is $A(t) = A_0 e^{-kt}$; the quantity decreases exponentially. The constant k depends on the drug. A_0 is the amount of drug in the body at time zero. Sometimes physicians convey information about the relative decay rate in form of the *half-life*, which is the time it takes for A to decrease by a factor of $1/2$.

Example 11.20 Valproic acid is a drug used to control epilepsy; its half-life in the human body is about 15 h.
(a) Use the half-life to find the constant k in the differential equation $dA/dt = -kA$
(b) At what time will 10% of the original dose remain?

Solution: (a) Since the half-life is 15 h, we know that the quantity remaining equals $A(t) = 0.5A_0$ when $t = 15$. We substitute this into the solution of the differential equation, $A(t) = A_0 e^{-kt}$

$$0.5A_0 = A(t) = A_0 e^{-15k}.$$

Dividing by A_0 and taking the logarithm yields

$$\ln 0.5 = -15k,$$

so

$$k = -\frac{\ln 0.5}{15} \approx 0.0462.$$

(b) To find the time when 10% of the original dose remains in the body, we substitute $0.10A_0$ for the quantity $A(t)$ remaining, and solve for the time t. We compute

$$0.10A_0 = A_0e^{-0.0462t}, \quad 0.10 = e^{-0.0462t}, \quad \ln 0.10 = -0.0462t,$$

so

$$t = -\frac{\ln 0.10}{0.0462} \approx 49.84$$

There will be 10% of the drug still in the body at $t = 49.84$, or after about 50 h.

11.5.5 Spread of Diseases, Technologies and Rumor

Example 11.21 Suppose some students in a school of 1000 students are carrying a flu virus. Find a differential equation governing the number of people $N(t)$ who have contracted the flu if the rate at which the disease spreads is proportional to the number of interactions between the number of students with flu and the number of students, who have not yet been exposed to it.

Solution: Let N be the number of students with the flu. Then the number of students not infected is $1000 - N$. We assume that the number of interactions between these two groups is proportional to the product of the sizes of these groups. The model then becomes

$$\frac{dN}{dt} = kN(1000 - N)$$

with some proportionality constant $k > 0$.

Example 11.22 A technological innovation is introduced into a community with a fixed population of M people, say at time $t = 0$. Find a differential equation representing the number of people $N(t)$ who have adopted the innovation at time t under the condition that the rate at which the innovation spreads through the community is jointly proportional to the number of people who have adopted it and who have not adopted it.

Solution: At time t, the number of people who have not adopted the invention is equal to $M - N(t)$. If we assume that at time $t = 0$ a single person has adopted the invention, the model becomes

$$\frac{dN}{dt} = kN(M - N), \quad N(0) = 1,$$

with some proportionality constant $k > 0$.

Example 11.23 **(a)** Along the lines of the previous two examples, describe a model for the spread of a rumor and solve it.
(b) Currently the rumor is known to 300 students in a residential university of 45000 students. It will be known to 900 students after one week. Find the number of students who will know the rumor after 4 weeks.

Solution: **(a)** We assume that the rate of spreading is proportional to the number of interactions between students who know, respectively, do not know the rumor, and that this number is again proportional to the product of the sizes of the respective group. Let $N(t)$ be the number of students who know the rumor at time t, and let M be the total number of students. The desired model then is

$$\frac{dN}{dt} = \mu N(M - N),$$

where $\mu > 0$ is some proportionality constant. It can be solved by the separation of variables method,

$$\frac{dN}{N(M-N)} = \mu dt,$$

$$\int \frac{dN}{N(M-N)} = \mu \int dt + c,$$

$$\frac{1}{M} \ln |\frac{N}{M-N}| = \mu t + c,$$

$$\ln \frac{N}{M-N} = M\mu t + Mc,$$

as $N > 0$ and $M - N > 0$.
Taking the exponential on both sides gives

$$\frac{N}{M-N} = e^{M\mu t + Mc} = Ae^{M\mu t}, \quad A = e^{Mc}.$$

Solving this equation for N yields

$$N(t) = \frac{MAe^{M\mu t}}{Ae^{M\mu t} + 1} = \frac{M}{1 + \frac{1}{A}e^{-\mu M t}}.$$

(b) We have $M = 45000$ and $N(0) = 300$. Setting $b = 1/A$, we compute b from

$$300 = N(0) = \frac{M}{1+b} = \frac{45000}{1+b}, \quad b = 149.$$

We then get

$$900 = N(1) = \frac{45000}{1 + 149e^{-M\mu}}, \quad 1 + 149e^{-M\mu} = 50, \quad e^{-M\mu} = \frac{49}{149}.$$

As $e^{-4M\mu} = (e^{-M\mu})^4$, we finally obtain

$$N(4) = \frac{45000}{1 + 149\left(\frac{49}{149}\right)^4} \approx 16400.$$

There are approximately 16400 students who know the rumor after 4 weeks.

11.5.6 Application of Newton's Law of Cooling

Example 11.24 A thermometer is removed from a room whose temperature is 80 °F and is taken outside where the air temperature is 10 °F. After 2 min, the thermometer shows the reading 40 °F. What is the reading of the thermometer at $t = 3$ min? How long will it take for the thermometer to reach 20 °F?

Solution: This phenomenon is modeled by

$$\frac{dT}{dt} = \lambda(T - 10)$$

by Newton's law of cooling. Its solution is $T(t) = 10 + ce^{\lambda t}$.

As $T(0) = 80$, we have $80 = 10 + ce^0$, so $c = 70$. As $T(2) = 40$, we have $40 = 10 + 70e^{2\lambda}$, so

$$2\lambda = \ln\frac{30}{70}, \quad \lambda = \frac{1}{2}\ln\frac{3}{7}.$$

Thus

$$T(3) = 10 + 70e^{3\cdot\frac{1}{2}\ln\frac{3}{7}}.$$

Solving

$$20 = T(t) = 10 + 70e^{t\frac{1}{2}\ln\frac{3}{7}},$$

we get

$$10 = 70e^{t \cdot \frac{1}{2}\ln\frac{3}{7}}, \quad \left(\frac{1}{2}\ln\frac{3}{7}\right)t = \ln\frac{10}{70}, \quad t = \frac{1}{\frac{1}{2}\ln\frac{3}{7}}\ln\frac{1}{7}.$$

Example 11.25 A 4 kg roast, initially at 60 °F is placed in a 375 °F oven at 6 pm. At 7.15 pm, the temperature of the roast is 125 °F. Find the time when the roast will be at 150 °F.

Solution: We assume that at any instant the temperature $T(t)$ of the roast is uniform throughout. We have $T(t) < 375$ at any time t. By Newton's law of cooling (here, in fact, heating),

$$\frac{dT}{dt} = \lambda(375 - T)$$

with some constant $\lambda > 0$. The general solution of this equation is

$$T(t) = 375 + c^{-\lambda t}.$$

We take $t = 0$ which corresponds to 6 pm. From the initial condition $T(0) = 60$ we obtain $c = -315$. Measuring t in minutes, 7:15 pm corresponds to $t = 75$. We thus have

$$125 = T(75) = 375 - 315e^{-75\lambda}.$$

We compute

$$315e^{-75\lambda} = 250, \quad \frac{315}{250} = e^{75\lambda}, \quad \lambda = \frac{1}{75}\ln\frac{315}{250}.$$

Solving

$$150 = T(t) = 375 - 315e^{-\lambda t}$$

for t yields

$$315 = 225e^{\lambda t}, \quad t = \frac{1}{\lambda}\ln\frac{315}{225}.$$

11.5.7 Application of Newton's Cooling Law for Determining Time of Death

The time of death of a murdered person can be estimated with the help of modeling through differential equations. A police personnel discovers the body of a dead person presumably murdered, and the problem is to estimate the time of death. The body is located in a room that is kept at a constant 70 °F. After the death, the body will radiate heat into the cooler room, causing the body's temperature to decrease, assuming that the victim's temperature was a normal 98.6 °F at the time of death. Forensic experts will try to estimate this time from the body's current temperature and calculate how

long it would have had to lose heat to reach that temperature. According to Newton's law of cooling, the body will radiate heat energy into the room at a rate proportional to the difference in temperature between the body and the room. If $T(t)$ is the body temperature at time t, then for some constant of proportionality k,

$$T^{'}(t) = k(T(t) - 70).$$

The corresponding linear differential equation

$$T' = k(T - 70)$$

we have already solved; its general solution is

$$T(t) = 70 + ce^{kt}.$$

The constants k and c can be determined provided the following information is available: The Time of arrival of the police personnel, the temperature of the body just after his arrival, the temperature of the body after a certain interval of time. Assume that the officer arrived at 10:40 pm and the body temperature was 94.4 °F. This means that if the officer considers 10:40 pm as $t = 0$ then $T(0) = 94.4 = 70 + c$, so $c = 24.4$ giving $T(t) = 70 + 24.4e^{kt}$. Assume that the officer makes another measurement of the temperature after 90 min, that is, at 12:10 am, and temperature was then 89 °F. This means that

$$89 = T(90) = 70 + 24.4e^{90k}.$$

We determine k through the computation

$$e^{90k} = \frac{19}{24.4}, \quad 90k = \ln\left(\frac{19}{24.4}\right), \quad k = \frac{1}{90}\ln\left(\frac{19}{24.4}\right).$$

The officer has now temperature function

$$T(t) = 70 + 24.4e^{\frac{t}{90}\ln(\frac{19}{24.4})}.$$

In order to find at which last time the body was at 98.6° F (presumably the time of death), one has to solve for time the equation

$$T(t) = 98.6 = 70 + 24.4e^{\frac{t}{90}\ln(\frac{19}{24.4})}.$$

This is done by computing

$$\frac{28.6}{24.4} = e^{\frac{t}{90}\ln(\frac{19}{24.4})}, \quad \ln\left(\frac{28.6}{24.4}\right) = \frac{t}{90}\ln\left(\frac{19}{24.4}\right).$$

Therefore, the time of death, according to this mathematical model, was

$$t = \frac{90 \ln(28.6/24.4)}{\ln(19/24.4)},$$

which is approximately -57 min. Thus, the death occurred approximately 57 min before the first measurement at 10:40 pm, that is at 9:43 pm approximately.

11.6 Introduction to Partial Differential Equations

In the previous sections, we have studied some basic ordinary differential equations and used them for modeling various real-world problems. However, since the world around us is three-dimensional, many relevant quantities depend on the three space coordinates, and possibly on time. Consequently, differential equations for the corresponding mathematical functions naturally involve partial derivatives of these functions. Indeed, the basic laws of continuum mechanics and thermodynamics for the conservation of mass, momentum and energy become, when formulated in mathematical terms, partial differential equations. The same applies to the basic laws of continuum electrodynamics, as well as to the equations which describe processes on very small scales (molecular or atomistic level) and on very large scales (e.g., stellar evolution).

Seen from a different angle, partial differential equations play a role in modeling a very broad range of phenomena, including somewhere at first one does not expect them to do so. Let us just present a (somewhat arbitrary) list of such phenomena:

(i) Diffusion of one material within another, smoke particles in air.
(ii) Chemical reactions, such as the Belousov–Zhabotinsky reaction which exhibits fascinating pattern structures.
(iii) Dispersion of populations; individuals move both randomly and to avoid overcrowding.
(iv) Pursuit and evasion in predator–prey systems.
(v) Pattern formation in animal coats, the formation of zebra stripes.
(vi) Dispersion of pollutants in a running stream.
(vii) Appropriate price of an option in a capital market.

When describing phenomena taking place in three-dimensional space, it is often possible to let the unknown functions depend on one space coordinate only. This happens if the actual dependence on the other coordinates is very slight (or nonexistent). Moreover, in order to understand the behavior of a partial differential equation in three space dimensions (3D) it is often helpful if at first one analyses its behavior in one space dimension (1D).

We present some examples of partial differential equations in 1D. The unknown function u is real-valued and depends on x and t.

(a) Heat equation or diffusion equation

The equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

describes the evolution of the temperature u as a function of space x and time t. The constant k denotes the thermal diffusivity. It was introduced by Fourier in his celebrated memoir "Théorie analytique de la chaleur" which appeared in 1822.

(b) Wave equation

The equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

describes the evolution of some quantity u which exhibits wave propagation. The constant c denotes the wave speed. This equation can be applied to model elastic (e.g., vibrations of an elastic rod), to acoustic and to electromagnetic wave propagation. It was introduced and analyzed by d'Alembert in 1752 as a model for a vibrating string.

(c) Linear transport equation

The equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

describes the transport of a spatially distributed quantity u with constant speed c. In this case, u is to be understood as a density function with respect to x, that is, the integral

$$\int_a^b u(x, t)\, dx$$

represents the total amount of the quantity present within the interval $[a, b]$ at time t. For example, u could be the density of cars per unit kilometer on a road (modeled as a subset of the real line) which move at a constant speed in one direction.

(d) Scalar conservation law

The equation

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0,$$

describes nonlinear transport (the linear case is recovered when a is constant). It is used to model various types of flow. In the special case $a(u) = u$ it is called **Burgers' equation**; it arises in particle and fluid flow with zero viscosity. That equation was introduced by Betman in 1915 and later studied by Burgers in 1948.

(e) Telegraph equation

The equation

$$\frac{\partial^2 u}{\partial t^2} + A\frac{\partial u}{\partial t} + Bu = \frac{\partial^2 u}{\partial x^2},$$

where A and B are constants, arises in the study of propagation of electrical signals in a cable transmission line. Both the current I and voltage V satisfy equations of this type. The telegraph equation was introduced by Oliver Heaviside in 1880. It also arises in the propagation of pressure waves in the study of pulsating blood flow in arteries.

(f) Korteweg de Vries (KDV) Equation

The equation

$$\frac{\partial u}{\partial t} + cu\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

models shallow water waves. It was originally discovered by Boussinesq in 1877 and rediscovered by Korteweg and de Vries in 1895.

Let us now pass to examples of partial differential equations in 3D (or 2D). Within these equations, there often appears the **Laplace operator** Δ defined by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad \text{respectively} \qquad \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Since $\Delta u = \operatorname{div}(\nabla u)$, it is also written as $\Delta u = (\nabla \cdot \nabla)u = \nabla^2 u$.

(g) Laplace equation

The Laplace equation[8]

$$-\Delta u = 0,$$

is probably the most studied partial differential equation since it appeared in Laplace's work in 1785. Its inhomogeneous counterpart

$$-\Delta u = f$$

is called the **Poisson equation**. In the latter, f is a function which depends on the space variables and represents some additional influence arising from the specific situation in which the equation is applied. For example, these equations are satisfied by the electrostatic potential in absence (presence, respectively) of charges, by the gravitational potential in the absence (presence, respectively) of mass, by the equilibrium displacement of a membrane in absence (presence, respectively) of distributed forces, by the steady-state temperature in the absence (presence, respectively) of thermal sources or sinks, and by the velocity potential for an inviscid, incompressible,

[8]The minus sign is a convention which is widely adopted in the mathematical theory of the Laplace equation.

irrotational homogeneous fluid in the absence (presence, respectively) of sources and sinks.

The 1D examples (a)–(c) actually arise from their 3D formulation. To obtain the latter, one just has to replace $\partial^2 u/\partial x^2$ by Δu and, for the transport equation, $c\partial u/\partial x$ by $c \cdot \nabla u$ (the scalar constant is replaced by a constant vector).

(h) Helmholtz equation

The equation

$$(\Delta + k^2)u = 0$$

has been found useful in diffraction theory. It was introduced by Helmholtz in 1860. It is obtained from the wave equation when one looks for solutions of the form $u(x,t) = v(x)w(t)$.

(i) Eikonal equation

The eikonal equation

$$\|\nabla u\| = 0$$

models problems of geometric optics. Here, $\|\nabla u(x)\|$ denotes the length of the gradient vector $\nabla u(x)$ at the space point x.

(j) Klein–Gordon equation

The equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u + \mu^2 u = 0$$

arises in quantum field theory, where $\mu = mc/h$, m is the mass, c the speed of light and h the Planck constant. It is named after the physicists Oskar Klein and Walter Gordon who proposed it in 1926.

(k) Schrödinger equation

The equation

$$ih\frac{\partial u}{\partial t} + \frac{h^2}{2m}\Delta u - Vu = 0.$$

is a fundamental equation of quantum mechanics. Here, u is the wave function of a particle with mass m and potential energy V, h is the Planck constant and i denotes the imaginary unit. The Austrian physicist Erwin Schrödinger, who developed this equation in 1926, obtained a Nobel prize in 1933 for this work.

(l) Navier–Stokes equation

The equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = \nu \Delta u$$

is satisfied by the velocity vector $u = (u_1, u_2, u_3)$ of an incompressible fluid, that is, a fluid whose density ρ is constant. The variable p stands for the pressure of the fluid and the constant ν denotes the kinematic viscosity. In the period from 1822 to 1845, several scientists including Navier and Stokes contributed to the formulation of this equation. In the special case $\nu = 0$, it is called the **Euler equation**, published by Euler in 1757.

The study of partial differential equations (PDEs) started in the eighteenth century in the work of Euler, d'Alembert and Laplace, motivated by continuum mechanics. Until the present time, their relevance in all kinds of situations in science and technology and (more recently) economy and even social sciences provided a rather strong incentive to develop their mathematical understanding, besides being intrinsically interesting for mathematicians. A lot of mathematical research in other areas of analysis (e.g., functional analysis) was inspired by the need of tools for solving problems with partial differential equations.

11.7 Applications of Fourier Methods to Partial Differential Equations

Fourier methods can be used to solve boundary value problems for linear partial differential equations. We discuss this briefly for three equations which we have already encountered in Chap. 8, namely the wave equation in one dimension,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{where } c \text{ is a constant,}$$

the heat equation in one dimension

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad \text{where } k > 0 \text{ is a constant,}$$

and the Laplace equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

For problems on a bounded interval one uses Fourier series, while for problems on the real line one uses the Fourier transform.

11.7.1 Fourier Methods for the Wave Equation

A Boundary Value Problem. We consider the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)\,, \quad 0 < x < l\,, \quad t > 0\,, \tag{11.39}$$

on the bounded space interval $(0, l)$ for positive times $t > 0$. Let us mention as one particular application that (11.39) is an approximate model for transverse vibrations of an elastic string of length l. There one assumes that at rest the string occupies the horizontal interval $[0, l]$, that is, any value $x \in [0, l]$ corresponds to a material point of the string. The variable u denotes the transverse displacement with respect to the state of rest $u = 0$, that is, $u(x, t)$ gives the vertical position of the string particle x at time t. We do not derive (11.39) from the laws of continuum mechanics (which is in fact not so easy, since in reality, a string is a three-dimensional object), but only remark that the wave speed c is given by $\sqrt{q\sigma/\rho}$, where q is the cross-sectional area, σ the tensile stress acting on a cross section and ρ the mass density of the spring.

The boundary conditions

$$u(0,t) = u(l,t) = 0\,, \quad t \geq 0\,, \tag{11.40}$$

describe the situation where the string is fixed at both ends $x = 0$ and $x = l$. Let us choose $t = 0$ as an initial time and impose the initial conditions

$$u(x,0) = f(x)\,, \qquad 0 \leq x \leq l\,, \tag{11.41}$$

$$\frac{\partial u}{\partial t}(x,0) = 0\,, \qquad 0 \leq x \leq l\,. \tag{11.42}$$

This means that the string is brought to the position described by the given function f, where it is held at velocity 0, and then suddenly released at time $t = 0$. In order that (11.40) and (11.41) are compatible, we require that $f(0) = f(l) = 0$.

In order to find the solution of the boundary value problem (11.39)–(11.41), one employs as a first step the method of **separation of variables** and as a second step the Fourier series representation. "Separation of variable" here means that we are looking for a solution of the form

$$u(x,t) = U(x)T(t)\,, \quad 0 \leq x \leq l\,, \quad 0 \leq t\,. \tag{11.43}$$

We substitute this into the wave equation and obtain

$$U(x)T''(t) = c^2 U''(x)T(t)\,,$$

where $T' = dT/dt$ and $U' = dU/dx$. Then

$$\frac{U''(x)}{U(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}\,. \tag{11.44}$$

The left side of this equation depends only on x and the right side only on t. Since it has to be satisfied for all values of t and x in the intervals considered, we can fix any value for t for which $T(t) \neq 0$, and see that the left side must be equal to $T''(t)/(c^2T(t))$, no matter which value we choose for x. Thus, considered as a function of x the left-hand side is constant. Let us denote this constant by $-\lambda$ (the negative sign is customary and convenient, but we would arrive at the same final result if we just used λ). Therefore,

$$\frac{U''(x)}{U(x)} = -\lambda = \frac{1}{c^2}\frac{T''(t)}{T(t)} \tag{11.45}$$

for all x and t. In this manner, the wave equation has been separated into the two differential equations

$$U''(x) + \lambda U(x) = 0\,, \tag{11.46}$$

$$T''(t) + \lambda c^2 T(t) = 0\,. \tag{11.47}$$

Let us now determine U. The boundary condition (11.40) becomes

$$U(0) = U(l) = 0\,. \tag{11.48}$$

For $\lambda \leq 0$, the only solution of (11.46) which satisfies (11.48) is the trivial solution $U = 0$ (we will not prove this here) leading to the trivial solution $u = 0$ of the wave equation, which is indeed the solution of the boundary value problem in the trivial case $f = 0$, but not if $f \neq 0$. For $\lambda > 0$, one checks that

$$U(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x) \tag{11.49}$$

is a solution of (11.46)—in fact, it is the general solution—, where a and b are constants. Since $U(0) = a$, the boundary condition $U(0) = 0$ yields $a = 0$. For $b = 0$, we again get the trivial solution $U = 0$. For $b \neq 0$, the boundary condition $U(l) = 0$ implies $\sin(\sqrt{\lambda}l) = 0$, which means that $\sqrt{\lambda}l$ must be a positive integer multiple of π. Therefore, the possible values for λ are

$$\lambda_n = \frac{n^2\pi^2}{l^2}\,, \quad n = 1, 2, 3, \ldots \tag{11.50}$$

Let us remark that the λ_n's are called **eigenvalues**, and the corresponding functions $U_n(x) = \sin(n\pi x/l)$ are called **eigenfunctions** of the boundary value problem (11.46) and (11.48).

The Eq. (11.47) for T is solved analogously. Its general solution is

$$T(t) = a\cos(\sqrt{\lambda}ct) + b\sin(\sqrt{\lambda}ct) \tag{11.51}$$

The initial condition (11.42) for $\partial u/\partial t$ gives $0 = U(x)T'(0)$, so $T'(0) = 0$ for the nontrivial case, and therefore $b = 0$. From (11.50) we get solutions

$$T_n(t) = \cos\left(\frac{n\pi ct}{l}\right). \tag{11.52}$$

Putting together the results above, we have found that the functions

$$u_n(x,t) = \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right), \quad n = 1, 2, 3, \ldots \tag{11.53}$$

are special solutions of the wave Eq. (11.39) which satisfy the boundary conditions (11.40) and (11.42). Moreover, we have

$$u_n(x,0) = \sin\left(\frac{n\pi x}{l}\right).$$

From those functions u_n, we construct a solution u which also satisfies the initial condition $u(x,0) = f(x)$, using Fourier series. We consider the odd extension of f on $[-l, l]$ defined by $f(x) = -f(-x)$ for $x < 0$. Its Fourier coefficients on $[-l, l]$ are (see Sect. 10.2.3)

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\,, \quad a_n = 0\,. \tag{11.54}$$

If f is smooth enough, it is equal to its Fourier series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n u_n(x,0)\,.$$

As the final step, one can show that the series

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)\,,$$

(where u_n and b_n are defined in (11.53) and (11.54)) converges and defines a function u which solves the given boundary value problem.

An Initial Value Problem on an Unbounded Domain. We apply the Fourier transform to solve the wave equation on the real line with given initial conditions at time $t = 0$,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}\,, \quad -\infty < x < \infty, \;\; t > 0\,, \tag{11.55}$$

$$u(x,0) = f(x)\,, \quad \frac{\partial u}{\partial t}(x,0) = 0\,, \quad -\infty < x < \infty\,. \tag{11.56}$$

Because x varies along the entire real line, we can try to compute the unknown function u with the aid of the Fourier transform with respect to the x variable which we denote by $\mathscr{F}_x$, that is,

$$\hat{u}(\xi,t) = (\mathscr{F}_x u)(\xi,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}\,dx\,.$$

Here, t plays the role of a parameter. We want to apply $\mathscr{F}_x$ to both sides of (11.55). For the left side, we obtain

$$\begin{aligned}\left(\mathscr{F}_x\left(\frac{\partial^2 u}{\partial t^2}\right)\right)(\xi) &= \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2}(x,t)e^{-i\xi x}\,dx = \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}\,dx \\ &= \frac{\partial^2}{\partial t^2}\hat{u}(\xi,t)\,.\end{aligned} \tag{11.57}$$

Here we have interchanged the differentiation w.r.t. t with the integration w.r.t. x, see Theorem 8.5.

For the right side of (11.55) we use formula (10.28) and obtain

$$\left(\mathscr{F}_x\left(\frac{\partial^2 u}{\partial x^2}\right)\right)(\xi) = -\xi^2\hat{u}(\xi,t)\,. \tag{11.58}$$

Thus the wave equation takes the form

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi,t) = -c^2\xi^2\hat{u}(\xi,t)$$

or

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi,t) + c^2\xi^2\hat{u}(\xi,t) = 0\,.$$

Since the space derivative has disappeared, we can treat this equation as an ordinary differential equation w.r.t. t for the unknown function $\hat{u}(\xi,t)$, where ξ appears as a parameter. Its general solution has the form

$$\hat{u}(\xi,t) = a_\xi\cos(\xi ct) + b_\xi\sin(\xi ct)$$

for some constants a_ξ and b_ξ depending on ξ. Applying $\mathscr{F}_x$ to the initial conditions (11.56) we get

$$\begin{aligned}a_\xi &= \hat{u}(\xi,0) = \hat{f}(\xi)\,,\\ b_\xi &= \frac{1}{c\xi}\tfrac{\partial}{\partial t}\hat{u}(\xi,0) = \frac{1}{c\xi}\left(\widehat{\frac{\partial u}{\partial t}}\right)(\xi,0) = 0\,.\end{aligned}$$

Thus $\hat{u}(\xi, t) = \hat{f}(\xi)\cos(\xi c t)$. We apply the inverse Fourier transform and finally obtain

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\cos(\xi c t)e^{i\xi x}\, d\xi\,.$$

11.7.2 Fourier Methods for the Heat Equation

A Boundary Value Problem. Let us consider the boundary value problem for the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}\,, && 0 < x < l\,,\ t > 0\,, \\ u(0, t) = u(l, t) &= 0\,, && t \geq 0\,, \\ u(x, 0) &= f(x)\,, && 0 \leq x \leq l\,. \end{aligned} \tag{11.59}$$

We proceed in the same manner as in Sect. 11.7.1, using separation of variables as well as Fourier series. By substituting $u(x, t) = U(x)T(t)$ into the heat equation we get

$$U(x)T'(t) = kU''(x)T(t)\,.$$

After division we see that

$$\frac{T'(t)}{kT(t)} = -\lambda = \frac{U''(x)}{U(x)}$$

holds for some constant λ, since the left side does not depend on t and the right side does not depend on x. Due to the given boundary conditions, U has to satisfy

$$U'' + \lambda U = 0\,, \quad U(0) = U(l) = 0\,.$$

We determine the values of λ for which this boundary value problem has nontrivial solutions U (the eigenfunctions). After some computation we arrive at the solution

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)e^{-n^2\pi^2 kt/l^2}\,,$$

where

$$b_n = \frac{2}{l}\int_0^l f(\xi)\sin\left(\frac{n\pi\xi}{l}\right)d\xi\,,$$

of the boundary value problem (11.59) for the heat equation.

An Initial Value Problem. We consider the initial value problem for the heat equation on the whole real line

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}\,, \quad -\infty < x < \infty\,,\ t > 0\,,\\ u(x,0) &= f(x)\,, \quad -\infty < x < \infty\,.\end{aligned} \tag{11.60}$$

We proceed similarly as in the case of the wave equation. We take the Fourier transform w.r.t. x of both sides of the heat equation,

$$\left(\mathscr{F}_x\left(\frac{\partial u}{\partial t}\right)\right)(\xi) = k\left(\mathscr{F}_x\left(\frac{\partial^2 u}{\partial x^2}\right)\right)(\xi)\,.$$

We evaluate the left side,

$$\begin{aligned}\left(\mathscr{F}_x\left(\frac{\partial u}{\partial t}\right)\right)(\xi) &= \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)e^{-i\xi x}\,dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}\,dx\\ &= \frac{\partial}{\partial t}\hat{u}(\xi,t)\,,\end{aligned} \tag{11.61}$$

and the right side,

$$\left(\mathscr{F}_x\left(\frac{\partial^2 u}{\partial x^2}\right)\right)(\xi) = -\xi^2\hat{u}(\xi,t)\,. \tag{11.62}$$

Thus, the heat equation takes the form

$$\frac{\partial}{\partial t}\hat{u}(\xi,t) + k\xi^2\hat{u}(\xi,t) = 0\,,$$

where ξ appears as a parameter only. We already know that this differential equation has the general solution

$$\hat{u}(\xi,t) = a_\xi e^{-\xi^2 kt}\,.$$

Using the transform of the initial condition we get

$$a_\xi = \hat{u}(\xi,0) = \hat{f}(\xi)\,.$$

Taking the inverse Fourier transform according to (10.34) we arrive at

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\xi^2 kt}e^{i\xi x}\,d\xi\,.$$

11.7.3 Fourier Methods for the Laplace Equation

On a rectangular two-dimensional domain, we can apply Fourier series to solve a boundary value problem for the Laplace equation. Consider as an example

$$\begin{gathered}\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,, \quad 0 < x < l\,, \quad 0 < y < p\,,\\ u(x, 0) = 0\,, \quad 0 \le x \le l\,,\\ u(0, y) = u(l, y) = 0\,, \quad 0 \le y \le p\,,\\ u(x, p) = (l - x)\sin x\,, \quad 0 \le x \le l\,,\end{gathered}$$

where l and p are positive numbers. Using separation of variables and Fourier series, one can obtain the solution in form of the infinite series

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4l^2}{\sinh(n\pi p/l)} \frac{n\pi[1 - (-1)^n \cos l]}{(l^2 - n^2\pi^2)^2} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi y}{l}\right).$$

For the boundary value problem in the upper half planes

$$\begin{gathered}\Delta u(x, y) = 0\,, \quad -\infty < x < \infty\,, \quad y > 0\,,\\ u(x, 0) = f(x)\,, \quad -\infty < x < \infty\,,\end{gathered}$$

one can obtain with the use of the Fourier transform the solution

$$\begin{aligned} u(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-|\xi| y} e^{-i\xi(\eta - x)}\, d\xi \right) f(\eta)\, d\eta \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\eta)}{y^2 + (\eta - x)^2}\, d\eta\,. \end{aligned}$$

11.8 Exercises

11.11.1 Find the general solution of the following differential equations:

$$\text{(a)}\quad \frac{dy}{dt} = 0.03y, \quad \text{(b)}\quad \frac{dy}{dt} = ky,\ k > 0, \quad \text{(c)}\quad \frac{dy}{dt} = \sin 5t.$$

11.11.2 A bank account earns interest continuously at a rate of 5% of the current balance per year. Assume that the initial value deposition is Rs 10,000 and that no deposits or withdrawals are made.
(a) Write the differential equation modeling the balance in the account.
(b) Solve the differential equation and draw the graph of the solution.

11.11.3 Solve the following differential equations:

$$\text{(a)}\quad x^2dy + y^2dx = 0, \quad \text{(b)}\quad dx - x^2dy = 0, \quad \text{(c)}\quad \frac{dy}{dx} + 5xy = 0.$$

11.11.4 Solve the following differential equations:

$$\text{(a)} \quad x^3 \frac{dy}{dx} + 3x^2 y = \cos x, \quad \text{(b)} \quad x \frac{dy}{dx} + y = \frac{1}{x^2}.$$

11.11.5 Solve the following initial value problem:

$$\frac{dy}{dx} + 5y = 20, \quad y(0) = 2.$$

11.11.6 Solve the following initial value problem

$$x \frac{dy}{dx} - 2y = 3\frac{y^4}{x}, \quad y(1) = \frac{1}{2}.$$

11.11.7 Solve the initial value problem

$$\frac{dy}{dx} = xy^2, \quad y(1) = -\frac{2}{3}.$$

11.11.8 In a city of India the rate at which the population grows at any time is proportional to the size of the population. If the population was 125,000 in 1970 and 140,000 in 1990, what is the expected population in 2020?

11.11.9 A representative of a pharmaceutical company recommends that a new drug of his company be given every T hours in doses of quantity y_0 for an extended period of time. Find the saturation level of the drug in the patient's body.

11.11.10 A simple model for the shape of a tsunami or a tidal wave is given by

$$\frac{dH}{dx} = H\sqrt{4 - 2H},$$

where $H(x) \geq 0$ is the height of the wave expressed as a function of its position relative to a point off shore.
(a) Find all constant solution of the differential equation by inspection.
(b) Solve the tsunami model by separating variables.
(c) Draw the graph of the solution that satisfies the condition $H(0) = 2$.

11.11.11 A lady was found murdered in her home. Police arrived at 11 am. The temperature of the body at that time was 31 °C, and one hour later 30 °C. The temperature of the room where the body was discovered was 22 °C. Estimate the time at which the murder was committed.

11.11.12 Examine whether the functions $f_1(x) = e^x$ and $f_2(x) = \sin x$ are linearly independent.

11.11.13 Solve the following differential equations:

$$\text{(a)} \ 4y'' - 10y' + 25y = 0, \quad \text{(b)} \ y'' - 16y' + 64y = 0, \quad \text{(c)} \ y'' + 2y' + 2y = 0.$$

11.11.14 Solve the initial value problem

$$4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5.$$

11.11.15 Show that $u(x, y) = \ln(x^2 + y^2)$ is a solution of the Laplace equation in two dimensions, that is,

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

11.11.16 Show that $u(x, t) = \sin x \cos t$ is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

11.11.17 Show that $u(x, t) = t^{-1/2} e^{-x^2/t}$ is a solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4} \frac{\partial^2 u}{\partial x^2}.$$

11.11.18 Find the general solution of

$$\frac{\partial^2 u}{\partial x^2}(x, y) = 0.$$

11.11.19 Solve the following boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \ t > 0$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = h(x), \quad 0 < x < l$$

where h is a given function with $h(0) = h(\ell) = 0$ and the constant c is the wave speed.

11.11.20 Solve the following boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1$$

subject to

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t \geq 0,$$
$$u(x, 0) = x(1 - x), \quad 0 < x < 1.$$

11.11.21 Solve the following boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\pi < x < \pi,$$

subject to

$$u(x,0) = x, \quad -\pi < x < \pi,$$
$$u(-\pi,t) = u(\pi,t), \quad \frac{\partial u}{\partial x}(-\pi,t) = \frac{\partial u}{\partial x}(\pi,t), \quad t \geq 0.$$

11.11.22 Solve the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < a,\ 0 < y < b$$

$$u(x,0) = 0, \quad 0x \leq a$$

$$u(0,y) = u(a,y) = 0, \quad 0 \leq y \leq b$$

$$u(x,b) = x, \quad 0 < x < a$$

Chapter 12
Calculus with MATLAB

12.1 Introduction

The Mathworks Corp. is the world's leading developer of technical computing software MATLAB for engineers and scientists in industries, government, and education; it is the language of technical computing which is high-level language and interactive environment that enables us to perform computationally intensive tasks faster than with traditionally programming languages such as C, C++, and FORTRAN. The main objective of this chapter is to introduce the MATLAB and highlight its role for better understanding of concepts and methods of calculus. We focus our attention on visualization of scalar- and vector-valued function, interpretation and illustration of concepts such as limits, differentiation, integration, sequences, and series including Taylor and Fourier series, ordinary differential equation (ODEs), and optimization.

MATLAB stands for MATrix LABoratory developed by Mathworks Corporation, United States. MATLAB is a high-performance language for technical computing. It integrates computation, visualization, and programming in an easy-to-use environment where problems and solutions are expressed in familiar mathematical notation. Today, MATLAB has evolved as tool which finds its role in almost every field of science and engineering. This book deals with some fundamentals of MATLAB.

MATLAB is very versatile software. It finds its application in almost all areas of engineering and technology. Basically, MATLAB was developed to solve mathematical problems later it was modified to find its role in various other areas of engineering and technologies such as signal processing, image processing, control systems, fuzzy logic, neural networks, and many more. To see other areas, one can go to MATLAB help and toolboxes.

MATLAB has evolved over a period of years with input from many users. In university environments, it is the standard instructional tool for introductory and advanced courses of mathematics, engineering, and sciences. In industry, MATLAB is the tool of choice for high productivity, research, development, and analysis.

M. Brokate et al., *Calculus for Scientists and Engineers*, Industrial and Applied Mathematics, https://doi.org/10.1007/978-981-13-8464-6_12

12.2 Important Elements of MATLAB

12.2.1 Advantages of MATLAB

MATLAB has many advantages as compared with conventional computer languages or programs for solving technical problems which can be summarized as follows:

1. **Easy to use**: MATLAB is an interpreted language. It is very easy to use. There are number of inbuilt functions which are optimized for particular problem. It is really very easy to evaluate certain mathematical expressions just by passing the values in these functions. With the help of these functions, large mathematical problems can easily be solved in very few lines of code while to solve same mathematical problems the lines of code written in other languages may extend to several hundreds.
2. **Platform Independence**: The MATLAB software is supported by various operating systems such as Win XP/Vista, UNIX, and MAC OS X. The program written in one platform can run on other platform.
3. **Powerful Graphics**: This is one of the important features of MATLAB which makes it great for technical data analysis and interpretation. MATLAB supports wide variety of plotting data so that it can be interpreted well. It supports colored 2D, 3D plots, animation, and videos that make it unique.
4. **Graphical User Interface**: MATLAB includes tools that allow a programmer to interactively develop a graphical user interface (GUI) for his program. With this capability, programmer can design sophisticated data analysis program that can be operated by relatively inexperienced users.
5. **MATLAB Help**: The MATLAB help system is powerful and user-friendly. It gives very good help on almost every topics and commands. User can get help both offline and online (through Internet). MATLAB has hundreds of examples (demo) on various problems that give an idea to write good program to the users.
6. **MATLAB Compiler**: MATLABs flexibility and platform independence is achieved by compiling MATLAB programs into a device independent p-code and then interpreting the p-code instructions at run time. But this is causing slow execution of programs. A separate MATLAB compiler is also available, which can compile MATLAB program directly into executables (exe) files that run faster.

12.2.2 How to Run MATLAB?

Let us first start with MATLAB environment. We can open MATLAB window by clicking MATLABs icon on desktop.

On double clicking the icon, we will see the window Fig. 12.1.

Here, we can see that, MATLAB main window is divided into three parts, i.e., Workspace/Current Directory, Command History, and Command Window.

The workspace shows the variables, which are in use currently with their properties such as size, type, maximum value, and minimum value. If you press the current directory tab, it will show you the contents of the current directory. The command window is a place, where we can write our MATLAB command to be executed.
MATLAB file formats: MATLAB can read or write various types of files. However, there are mainly five types of files for storing data or programs that one will go through them frequently. They are

(a) **M-Files**: These are standard ASCII text files with .m extension. They are written in MATLAB editor. Our main program files are written in M-file formats. Later, we will show one example of M-file.
(b) **MAT Files**: These are binary files with .mat extension. These files are created when we save variables of MATLAB workspace.
(c) **Fig Files**: These are binary files with .fig extension to store graphics (i.e., curves obtained by plotting the data).
(d) **P Files**: These are compiled M-files with .p extension. Mainly, these files are used for distribution purpose of MATLAB programs with hidden MATLAB code.
(e) **MEX Files**: These are MATLAB callable C and FORTRAN programs with .mex extension. These files are used in interfacing MATLAB with C or FORTRAN.

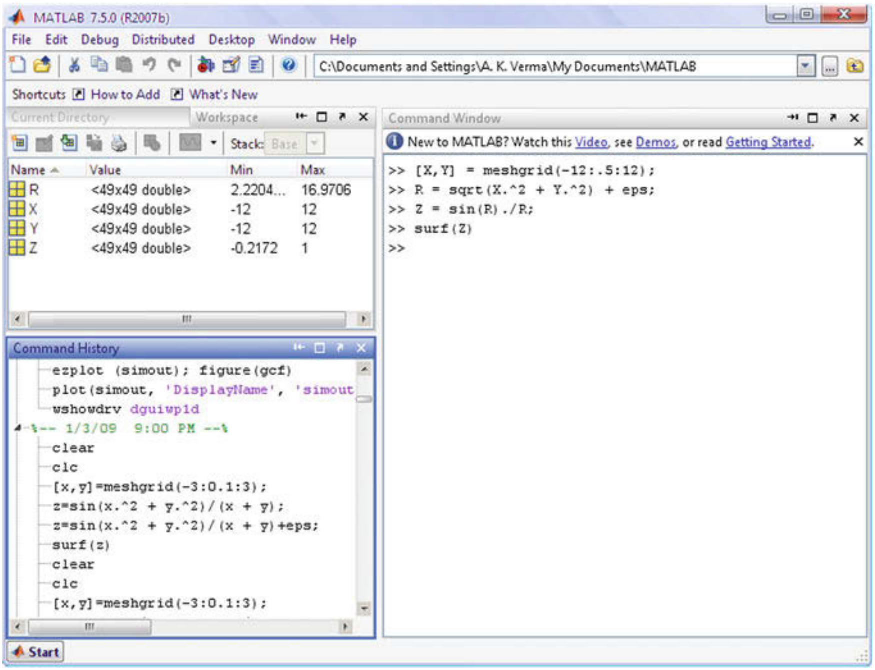


Fig. 12.1 Main window

Starting MATLAB:

Variables, Operators, and Matrices. Let us start with simple MATLAB commands related to matrices. Now, how to enter a Matrix?
Simply write the following on the command prompt in the command window:
$>> a = [1\ \ 2\ \ 3;\ \ 5\ \ 7\ \ 4;\ \ 9\ \ 8\ \ 6]$
and press Enter. We will see following in the command window:

$$
\begin{array}{l}
>> a = \\
\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 5 & 7 & 4 \\ 9 & 8 & 6 \end{array}
\end{array}
$$

It shows that our 3×3 matrix is stored in variable a. This variable can be seen in the workspace window. Here we can see that a semicolon is used to separate the row. Now let us do some basic operations on this matrix. Simply type a on command prompt ($>> a$) and press enter. We get

$$
\begin{array}{l}
\text{ans } = \\
\qquad \begin{array}{ccc} 1 & 5 & 9 \\ 2 & 7 & 8 \\ 3 & 4 & 6 \end{array}
\end{array}
$$

Obviously, it is the transpose of matrix a. Here ans is the default variable provided by MATLAB, if we don't specify our own variable. Similarly, we can do other operations. Students are advised to try the following commands on the command prompt:

1. $>>$**Inv(a)** [It will give the inverse of matrix a]
2. $>>$**det(a)** [Determinant of matrix a]
3. $>>$**eig(a)** [Eigen Value & Eigen Vector of a]
4. $>>$**diag(a)** [This gives elements of main diagonal]
5. $>>$**rank(a)** [Finds rank of matrix a]

Now let us do some operations on two matrices such as summation, subtraction, multiplication, etc.

Matrix a is already stored in memory, now enter another matrix b as given below:
$>> b = [2\ \ 4\ \ 5;\ \ 1\ \ 6\ \ 8;\ \ 3\ \ 4\ \ 6]$
Here, we can notice that we have put a semicolon at the end of matrix b. The function of the semicolon is to suppress output in workspace window, which becomes useful in the cases when we generate very large matrices.

Students are advised to try the following operations on two matrices a and b:

1. $>> c = a + b$ [Summation of matrix a and b and result is stored in c]
2. $>> c = a - b$ [Subtraction of matrix b from a]
3. $>> c = a * b$ [Multiplication of matrix a and b]

4. $>> c = a. * b$ [Element-by-element multiplication of matrix a and b]
5. $>> c = a./b$ [Element-by-element division of matrix a and b]

There is another operator called left division ('\'), which is used in solving systems of linear algebraic equations (see the MATLAB help for detail).

Besides these arithmetic operators, MATLAB has relational and logical operators also. Please see the MATLAB help for this.

Writing script files (M-files): In the previous section, we have seen some basic operations performed by writing commands on the command prompt in the command window. It was easy, when we have to perform a single and easy operation on a variable. But in the cases, when we have to perform a several complicated and interdependent operations, then it is not possible to use the command prompt. MATLAB has given an editor in which we can write programs called Script file which can be compiled and run and results are displayed in the command window or figure window, if some graphs are requested in program. Here we are giving one very simple example of writing an M-file to solve a system of linear algebraic equations. The equations are given below:

$$x + 2y - z = 10, 4x + 6y + z = 20, x - 8y + 3z = 8.$$

These three equations can be written in matrix form as $[A].[x] = [B]$

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 6 & 1 \\ 1 & -8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 8 \end{bmatrix}$$

and solution can be obtained by $[x] = [A]^{-1}.[B]$ (in MATLAB syntax it is $x = \text{inv}(A) * B$).

The solution can be found by writing M-file in editor and executing it. To open the editor window, go to File menu of MATLAB window and select New > M-file and write the following code:

% Solution of Linear Algebraic Equations.
$A = [12 - 1; 461; 1 - 83];$
$B = [10; 20; 8];$
$xyz = \text{inv}(A) * B;$
xyz
After executing it, we get following result:

$$xyz =$$

$$\begin{array}{r} 8.6667 \\ -1.6667 \\ -4.6667 \end{array}$$

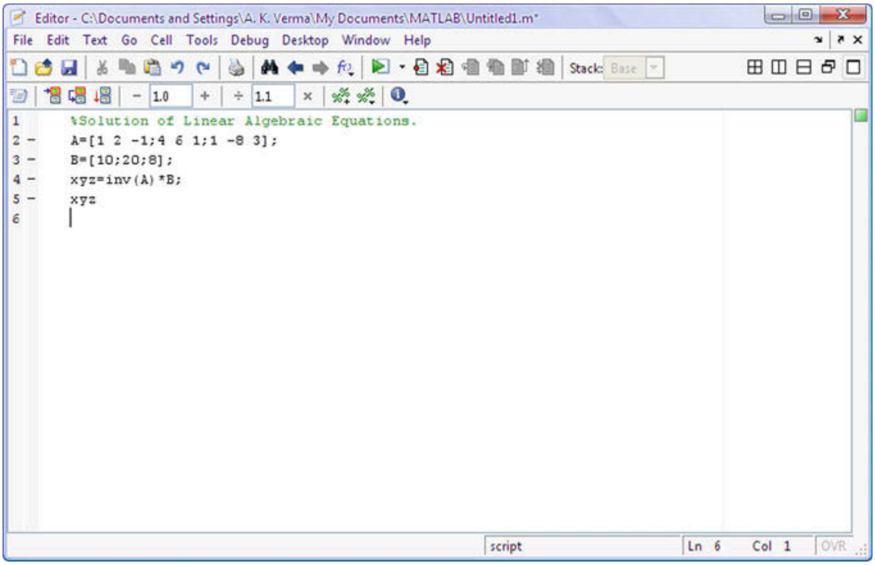


Fig. 12.2 Editor window

The first line which starts with % is comment line and it is not executed.
The editor window is shown in Fig. 12.2.
Now how to run the program from editor window? It is easy. Just press the Run button. MATLAB will ask to save the program, first save it and if we are not in the current directory, then MATLAB will ask us to change the directory, allow the directory change, and now program is run and result is seen in command window. Second crude method is copy whole program and paste it at command prompt in command window and press enter, program will be run without saving it.

12.2.3 MATLAB Functions

MATLAB has rich library of inbuilt functions to perform various tasks of various fields of science and engineering. Here, we are giving use of some very basic MATLAB functions. The one, we have already used, that is inv(a). Here, inv () is a function, in which parameter *a* is passed and result is returned by this function. Here, reader is encouraged to explore the MATLAB help for various functions for different applications (Fig. 12.3).

Besides MATLAB's inbuilt functions, user can write his own function. The user-defined function is written in separate M-file and saved with the name of that function (i.e., functionname.m).

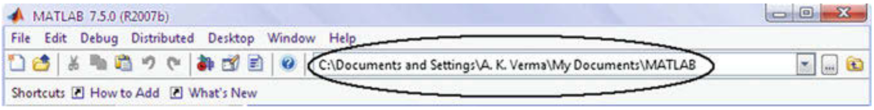


Fig. 12.3 MATLAB's Current Directory

Here we are giving one example of how user-defined functions can be created. We have written one function, which will solve quadratic equation ($ax^2 + bx + c = 0$)

$$\text{function } [x1, x2] = \text{eqsolve2}(a, b, c)$$
$$\text{discr } = b^2 - 4 * a * c \ \% \text{ finding discriminant}$$
$$x1 = (-b + \text{ sqrt(discr)})/(2 * a);$$
$$x2 = (-b - \text{ sqrt(discr)})/(2 * a);$$

This function file is first saved with name "eqsolve2.m" in some directory. Now this function can be used to calculate two values of x (roots) simply by passing values of "a", "b", and "c" on command prompt as given below:

$$>> \quad [x1, x2] = \text{ eqsolve2}(2, 5, 1)$$
$$x1 =$$
$$-0.2192$$
$$x2 =$$
$$-2.2808$$

And another example is given below:

$$[x1, x2] = \text{ eqsolve2}(2, 4, 5)$$
$$x1 =$$
$$-1.0000 + 1.2247i$$
$$x2 =$$
$$-1.0000 - 1.2247i$$

This M-file is called function file. We can also call functions in main file. These main M-files are called script files. To run above program from command prompt, one should ensure that the directory in which this function file saved is same what is written in MATLAB's current directory field. Also, when we are using a main script file with several function files, all must be in same directory.

12.3 Visualization of Scalar- and Vector-Valued Function

Let us discuss an important feature of MATLAB that is handling powerful graphics. MATLAB has capability of presenting our output and helping us in interpreting the data graphically with help of various types of curves and plots.

12.3.1 Plotting Scalar Functions with MATLAB

2D dimensional plots. To plot a function, we have to create two arrays (vectors), one containing abscissa, and the other corresponding function values. Let us plot $f(x) = \sin(x)$. Type following commands on command prompt:

$$>> x = -2 * \mathrm{pi} : \mathrm{pi}/100 : 2 * \mathrm{pi}; \text{\% range of } x \text{ from } -2p \text{ to } +2p \text{ in steps of } p/100.$$
$$>> fx = \sin(x); \text{\%function sine() to compute sine of all } x.$$
$$>> \mathrm{plot}(x, fx) \text{ \%plot function, which plot } fx \text{ versus } x.$$
$$>> \mathrm{grid} \text{ \% creates grid in plot.}$$

After giving all these commands, and pressing Enter Key, we get following curve window as shown in Fig. 12.4:

Let us plot now $f(t) = e^{-t/10} \sin t$. Type the following commands on command prompt:
$>> t = 0 : 0.01 : 50;$
$>> ft = \exp(-t/10). * \sin(t);$
$>> \mathrm{plot}(t, ft)$
$>>$ grid

In above two examples, we have used plot() function for plotting. Now, we will see that how we can combine two or more plots in one window. Just for simplicity, let us combine plot of function $f_1(t) = e^{-t/10} \sin t$ (Fig. 12.5) with another function $f_2(t) = e^{-t/10}$ in same figure window (Fig. 12.6). Type following commands:

$>> t = 0 : 0.01 : 50;$
$>> ft1 = \exp(-t/10);$
$>> ft2 = \exp(-t/10). * sin(t);$
$>> \mathrm{plot}(t, ft1, t, ft2)$ % This is how we can combine two plots.
$>>$ grid
Similarly, we can add several graphs in same window with the command plot $(x, fx, y, fy, z, fz, ..)$. Please see MATLAB help for more detail.

We can plot several other types of 2D plots in MATLAB with different attributes such as colors, styles, etc. here we are taking some more examples.

Output graphs of these examples are shown in Fig. 12.7.

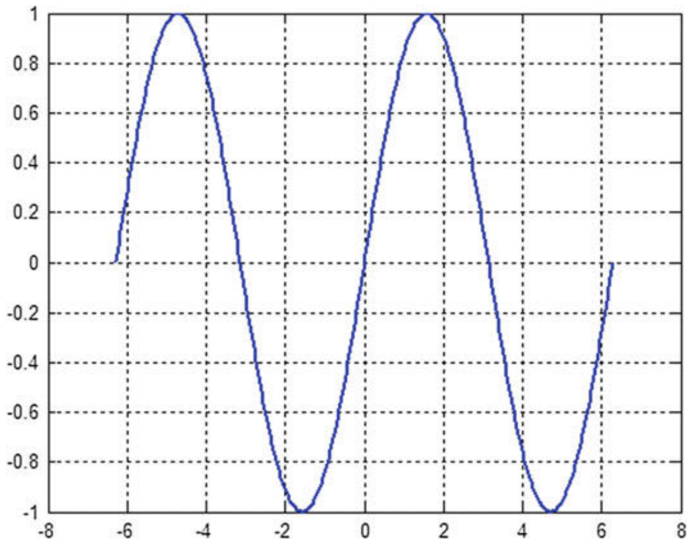


Fig. 12.4 Plot of a curve

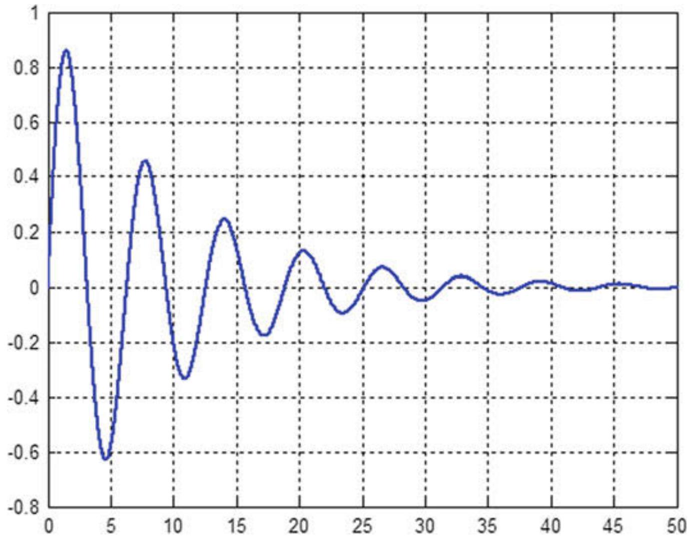


Fig. 12.5 Plot of $f(t) = e^{-t/10} \sin t$

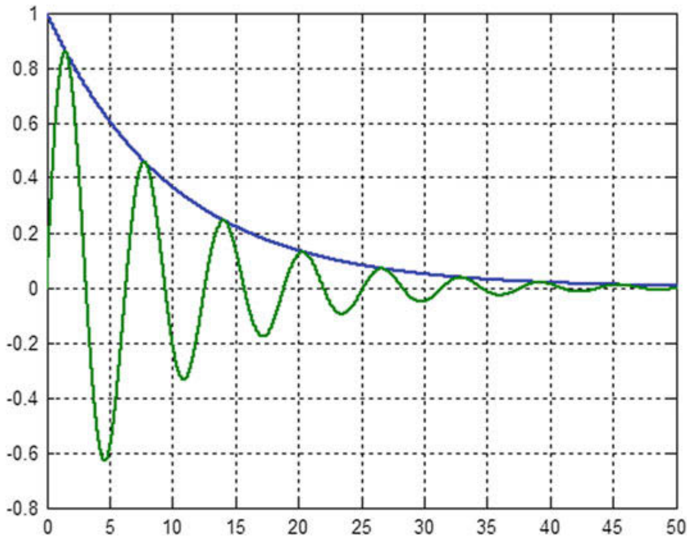


Fig. 12.6 Combining two curves in a single plot

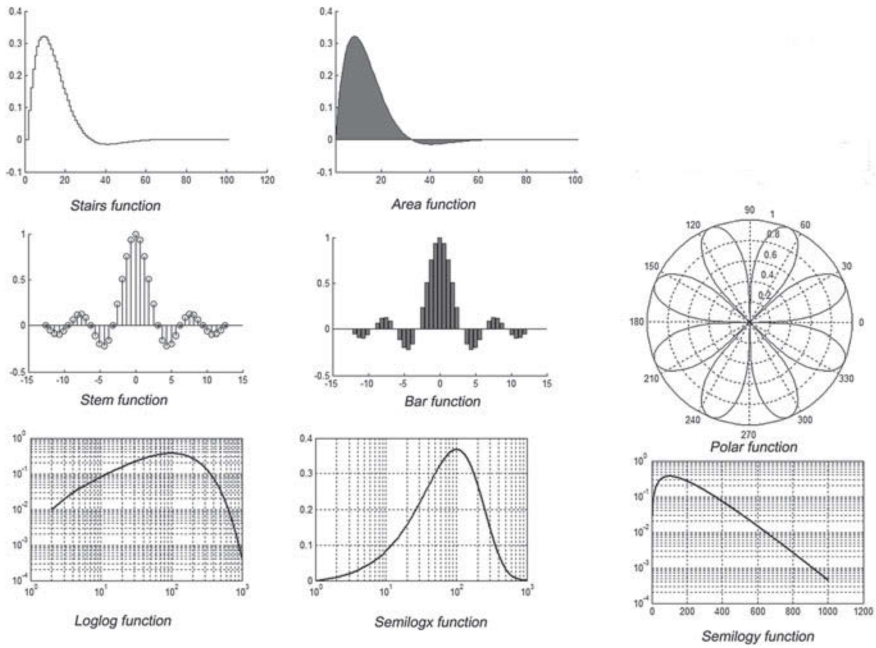


Fig. 12.7 Various types of MATLAB plots

Table 12.1 Some MATLAB functions and their code

Function	MATLAB Code
Stairs()	$>> x = 0 : 0.1 : 10;$
	$>> y = \exp(-x). * \sin(x)$
	$>>$ stairs(x, y)
Area()	$>> x = 0 : 0.1 : 10;$
	$>> y = \exp. * \sin(x)$
	$>>$ area(x, y)
Stem()	$>> x = -4 * \text{pi} : \text{pi}/5 : 4 * \text{pi};$
	$>> y = \sin(x)./x;$
	$>> y((\text{length}(y) - 1)/2 + 1) = 1;$
	$>>$stem(x, y)
Bar()	$>> x = -4 * \text{pi} : \text{pi}/5 : 4 * \text{pi};$
	$>> y = \sin(x)./x;$
	$>> y((\text{length}(y) - 1)/2 + 1) = 1;$
	$>>$bar(x, y)
Semilogx()	$>> x = 0 : 0.1 : 10;$
	$>> y = x. * \exp(-x);$
	$>>$semilogx(x, y)
	$>>$grid
Semilogy	$>> x = 0 : 0.1 : 10;$
	$>> y = x. * \exp(-x);$
	$>>$semilogy(x, y)
	$>>$grid
Loglog()	$>> x = 0 : 0.1 : 10;$
	$>> y = x. * \exp(-x);$
	$>>$loglog(x, y)
	$>>$grid
Polar()	$>>$ theta $= 0 : \text{pi}/100 : 2 * \text{pi};$
	$>> r = \text{sqrt}(\text{abs}(\sin(4 * \text{theta})));$
	$>>$polar(theta, r)

There are several other types of plots also, one can go through MATLAB's help (Table 12.1).

Three-dimensional (3D) plots. Let us have a look on 3D plots. Three-dimensional graphs, we can plot for functions of two variables such as $z = f(x, y)$. Here, we will plot $x(t) = e^{t/10}. \sin(t)$ verses $y(t) = e^{t/10}. \cos(t)$ along with "t" axis (see Fig. 12.8). Type the following commands:

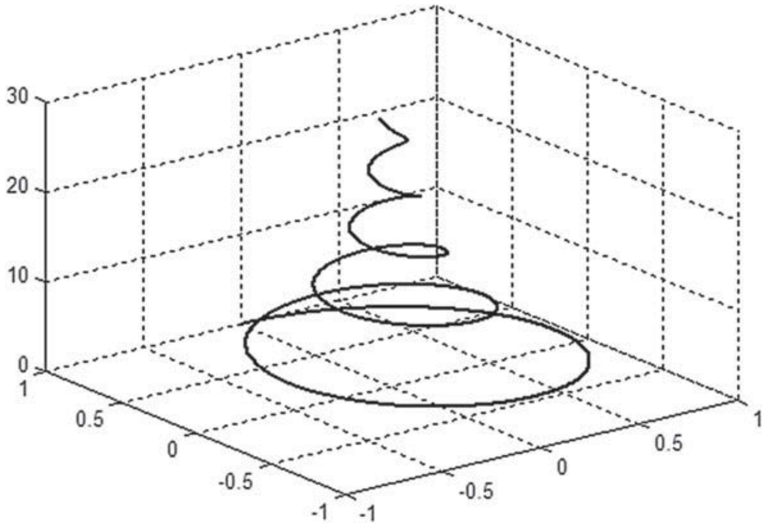


Fig. 12.8 3D plot of a spiraling curve

$>> t = 0 : 0.01 : 30;$
$>> x = \exp(-0.1 * t). * \sin(t);$
$>> y = \exp(-0.1 * t). * \cos(t);$
>> plot3(x, y, t) % 3d plot command.
>> grid
We will get the following figure:

MATLAB has several specialized 3D plots. Just try this one
>> [x, y] = meshgrid($-8 : 0.5 : 8$);
>> r = sqrt(x.^2+y.^2) + eps;
>> $z = \sin(r)./r;$
>> mesh(x, y, z)
This will give, following result (Fig. 12.9):

One can try following plot commands on same function code and see the results:
>> surf(x, y, z)
>> contour(z)
>> surfc(x, y, z)
>> surf1(x, y, z)
>> meshz(x, y, z)
>> waterfall(z)
Please see help on each command in MATLAB help.

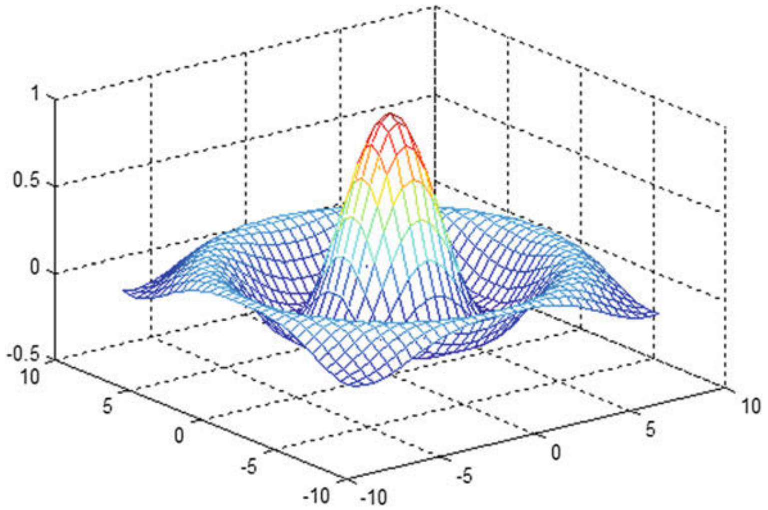


Fig. 12.9 3D plot of a surface

12.3.2 Plots for Vector-Valued Functions in 2D and 3D

In variousenlargethispage8pt applications, we need to visualize the vector-valued functions (vector fields). MATLAB has several functions to visualize vector field in 2D and 3D such as quiver, quiver3, stream, stream3, streamline, streamline3, streamslice, streamtube, streamribbon, streamparticles, coneplot, divergence, curl, etc. Here we plot for the function $z = xe^{-(x^2+y^2)}$.

First, we take quiver plot. Just write following code on command prompt:
>> $[x, y]$ = meshgrid$(-2 : 0.1 : 2)$;
>> $z = x. * \exp(-x.\hat{}2 - y.\hat{}2)$;
>> $[dx, dy]$ = gradient(z);
>> quiver (x, y, dx, dy)
We get following plot: (Fig. 12.10)

Now we use streamline function to plot a vector $ui + vj$ where $u = f(x, y)$ and $v = g(x, y)$ are given as

$$u = x + y - x(x^2 + y^2) \text{ and } v = -x + y - y(x^2 + y^2).$$

This, we can achieve by writing this code on command prompt as given below:
>> $[x, y]$ = meshgrid$(-2 : 0.1 : 2)$;
>> $u = x + y - x. * (x.\hat{}2\text{+y.}\hat{}2)$;
>> $v = -x + y - y. * (x.\hat{}2\text{+}y.\hat{}2)$;
>> $x0 = [-2 - 2 - 2 - 2 - 0.5 - 0.5.5.52222 - 0.1 - 0.01.01]$;
>> $y0 = [-2 - 0.5.52 - 22 - 22 - 2 - 0.5.52 - 0.01.01 - 0.01.01]$;
>> streamline$(x, y, u, v, x0, y0)$
>> axis square

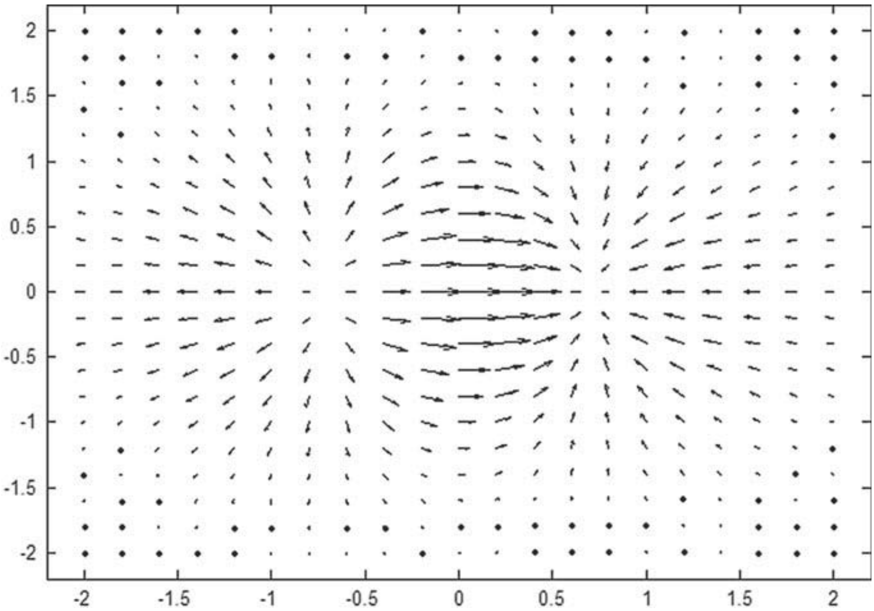
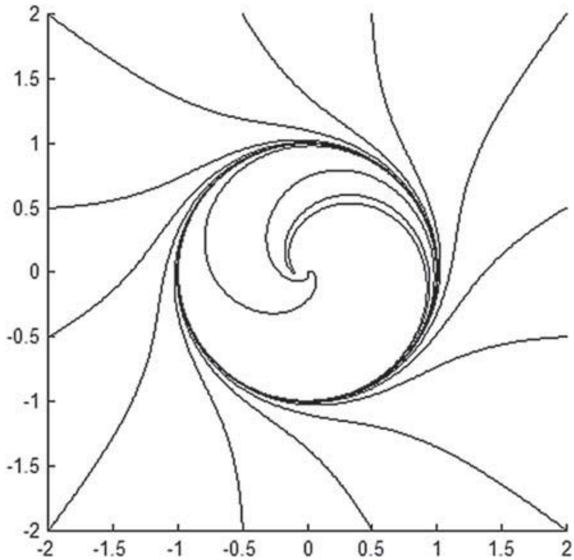


Fig. 12.10 Visualization of a 2D vector field

Fig. 12.11 Streamlines



We get this result (Fig. 12.11)

For other types of plots, kindly go through the MATLAB help.

12.4 Certain Topics of Calculus with MATLAB

There are several areas of mathematics where MATLAB has been of utmost value, but here we confine ourselves to the interaction of MATLAB and Calculus. More precisely, in this section, we demonstrate relevance of MATLAB to concepts of calculus such as differentiation and integration, limits of functions, series and sequences, ordinary differential equation (ODEs), optimization (finding minima and maxima), and Fourier analysis.

12.4.1 Differentiation and Integration

Differentiation and integration can be done in two ways either numerically or symbolically. To understand symbolic differentiation and integration, we need to go through Symbolic Math Toolbox which incorporates symbolic computation into the numeric environment of MATLAB.

Symbolic Integration. First of all, we have to define symbolic variables by syms command as given below:
$>>$ syms x, t;
Then write function of x or t and int() function as given below:
$>>$ $fx = 1/(1$+xˆ2);
$>>$ $y = int(fx)$ % computes integration of expression $\sin x$.
We get
$y =$
$a\tan(x)$ % which is $tan^{-1}(x)$
We can visualize these results also with ezplot() function, just add following code in above code of integration:
$>>$ ezplot(fx)
$>>$ hold on
$>>$ ezplot(y)
We get Fig. 12.12

This was an example of indefinite integral, we can do definite integral also, see the code given below:
$>>$ $ft = t * \log(1+t)$;
$>>$ $y =$ int(ft, 0, 1) % function fx is integrated from 0 to 1.
$y =$
$1/4$

Symbolic Differentiation. Similarly, for expression t^2, we will find differentiation as
$>>$ tˆ2;
$>>$ $y =$ diff(ft)
$y =$
$2 * t$

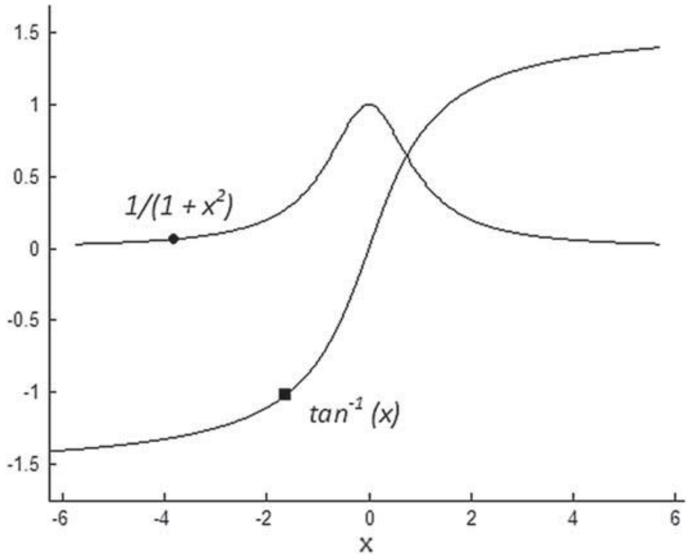


Fig. 12.12 Symbolic integration

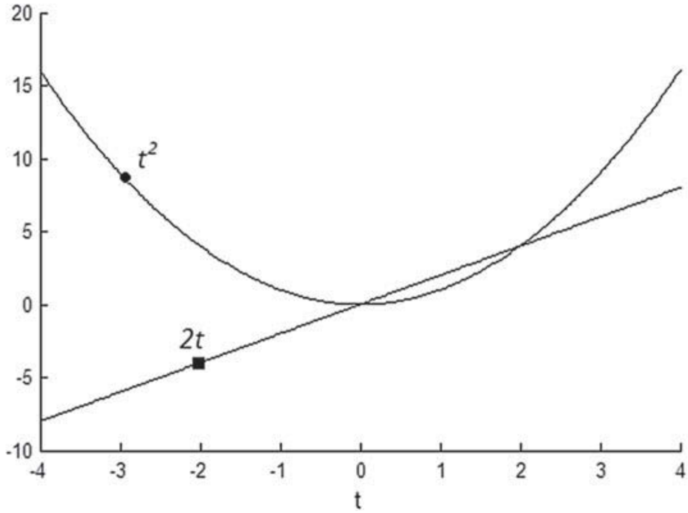


Fig. 12.13 Symbolic differentiation

We can visualize the result also with ezplot() (see Fig. 12.13) as given below:

We can obtain second differentiation by diff(ft, 2) command.

$>> y1 = $ diff(ft, 2)
$>> y1 =$
2

Similarly, we can have nth differentiation by diff(fx, n).

Numerical Integration. In MATLAB, we can do integration numerically. For this, MATLAB has several functions such as quad, quad1, quadl, quadgk, quadv, dblquad, and triplequad. Here we will introduce only quad function which numerically integrates based on adaptive Simpson quadrature rule.

The general syntax is given as $>> y =$ quad("fx'', a, b), which evaluates $y = \int_a^b f(x)\,dx$. Let us evaluate $\int_0^2 e^{-x^2}\,dx$ using quad function. First, we have to write function file which will describe input function, then on command prompt, we can use that file to evaluate integral for any limit combinations. The code for function file is given below which is written as M-file in editor.

% function file for describing function $f(x)$ to be used in quad()
function y = myfunction(x)
y = exp(-x.^2);
(*the above function file has to be saved with name myfunction.m)
To evaluate integral write following on command prompt:
$>> y =$ quad('myfunction', 0, 2)
$y =$
0.8821
There is another simple way in which M-file is not required. That is for command prompt. The code is given below:
$>> fx =$ inline('exp(-x.^2)');
$>> y =$ quad(fx, 0, 2)
$y =$
0.8821
The double integration can also be found by using dblquad function. The general syntax is $>> y =$ dblquad(fxy, $x$min, $x$max, $y$min, $y$max) which evaluates $y = \displaystyle\int_{x\min}^{x\max}\int_{y\min}^{y\max} f(x, y)\,dx\,dy$. Let us evaluate $f(x, y) = \int_\pi^{2\pi}\int_0^{2\pi}(y\sin x - x\sin y)\,dx\,dy$ using dblquad function. The code is given below:
$>> fxy =$ inline('$y.*\sin(x) - x.*\sin(y)$');
$>> q =$ dblquad(fxy, pi, $2*pi$, 0, pi)
$q =$
-39.4784
For details about other functions, please see MATLAB's help.

12.4.2 Finding Limits of Functions

The fundamental idea in calculus is to make calculations on functions as a variable "gets close to" or approaches a certain value. Recall that the definition of the derivative is given by a limit $\lim_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$, provided this limit exists. The general syntax is $>>$ limit(fx, x, limit value). We can find limit as given below:

$>>$ syms$x$;
$>>$ limit$(\sin(x)/x, x, 0)$ % here $fx = \sin(x)/x$ and $x \to 0$
ans =
1
We can find one-sided limit also such as left sided or right sided. The general syntax is
$>>$ limit$(fx, x,$ limit value, right) or $>>$ limit$(fx, x,$ limit value, left). Let us find right and left limits of the function $f(x) = x/|x|$ as x approaches to 0. The code is given below:
$>>$ limit$(x/\text{abs}(x), x, 0,'$ right$')$
ans =
1
And
$>>$ limit$(x/\text{abs}(x), x, 0,'$ left$')$
ans =
-1
Now, we will see one application of limits in drawing tangent line at a point on a given curve. Let the given curve is $y = x^2 + 1$ and we want to plot a tangent line at point (2, 5). If $P(x_0, y_0)$ is a point on the graph of a function f, then the tangent line to the graph of f at P, also called the tangent line to the graph of f at x_0,

$$m_{\tan} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists. If the limit does not exist, then by agreement the graph has no tangent line at P.

The program code is given below:

```
% Program to draw the tangent line at a point on a curve.
syms xh;
x0 = 2; % At point (2, 5), tangent is to be drawn
y0 = 5;
y = x^2 + 1; % Function declaration
y1 = subs(y, (h + x0));
y2 = subs(y, x0);
y3 = (y1 − y2)/h;
m = limit(y3, h, 0);
line = (m(x − x0) + y0);
ezplot(y)
hold on
ezplot(line)
plot(x0, y0, 'ro', 'MarkerFaceColor', 'g')
v = [−8 8 − 10 40];
axis(v)
grid
```

we get following plot (Fig. 12.14)

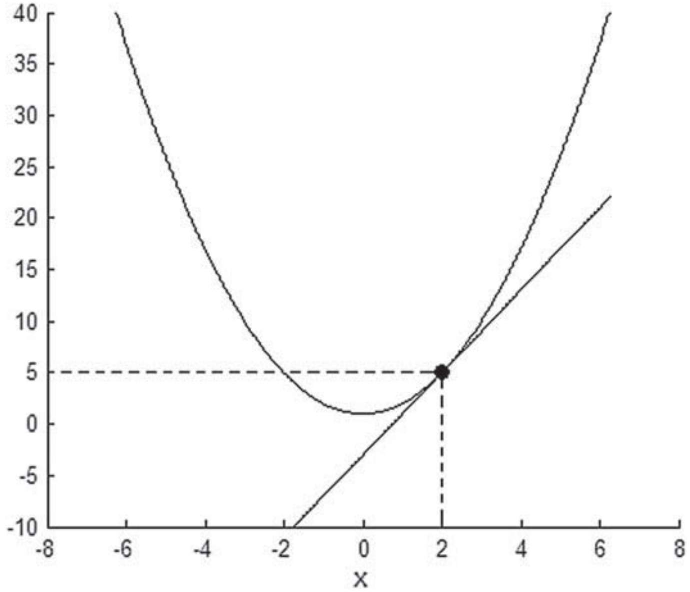


Fig. 12.14 Tangent line at a point on a curve

12.4.3 Sequences and Series

We can compute the summation of finite and infinite series using symsum() function, if they exist. To find sum of $\sum_{k=l}^{m} f(k)$. The general syntax is $>>$ symsum(fk, l, m).

Let us have one example of $\sum_{k=1}^{\infty}(1/k^2) = 1 + 1/2^2 + 1/3^2 + 1/4^2 + \ldots\infty$. The code is given below:

$>>$ syms xk
$>>$ $s1 =$ symsum$(1/k$^2, 1, inf)
$s1 = 1/6 *$ pi^2

If series is given as $1 + x^2 + x^3 + x^4 + x^5 + \ldots\infty$. Then code will be as follows:

$>>$ syms xk
$>>$ $s2 =$ symsum$(x$^$k, k, 0, inf)$
$s2 =$
$-1/(x - 1)$

In symsum, the default variable is k.

Now we will see that how Taylor series can be obtained. The expression of Taylor series is $\sum_{n=0}^{\infty}(x-a)^n \frac{f^{(n)}}{n!}$. Which is expansion about a. The n terms about a can be found by following command:

$>>$ taylor(fx, n, a),
where, fx: function of x,
n: gives $(n - 1)$st-order polynomial and

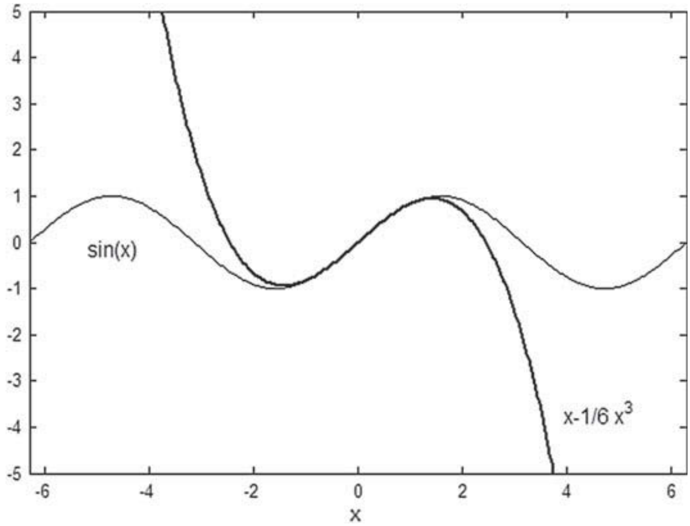


Fig. 12.15 Taylor approximation of the sine function

a: a point around which polynomial will be calculated.
Let us graphically compare $f(x) = \sin(x)$, $-4 < x < 4$ with its Taylors polynomial of fourth order computed around $a = 0$ (see Fig. 12.15). The code is given below:
$>>$ syms x;
$>>$ $fx = \sin(x)$;
$>>$ $fxx =$ taylor$(fx, 5, 0)$ % we can write taylor$(fx, 5)$ also
$fxx =$
$x - 1/6 * x\hat{}3$
For plotting the function, write the following code:
$>>$ ezplot(fx)
$>>$ hold on
$>>$ ezplot(fxx)

12.4.4 Solving Ordinary Differential Equations (ODEs)

Differential equation is very important part of calculus as it has role in applied areas of mathematics; such one of them is modeling of physical systems. In MATLAB, we can solve differential equations in two ways, either symbolically or numerically. First, we start with symbolic solution of ODEs.

Symbolic Solution. The function dsolve computes symbolic solutions to ordinary differential equations. The equations are specified by symbolic expressions containing the letter D to denote differentiation. The symbols D2, D3…DN correspond to the

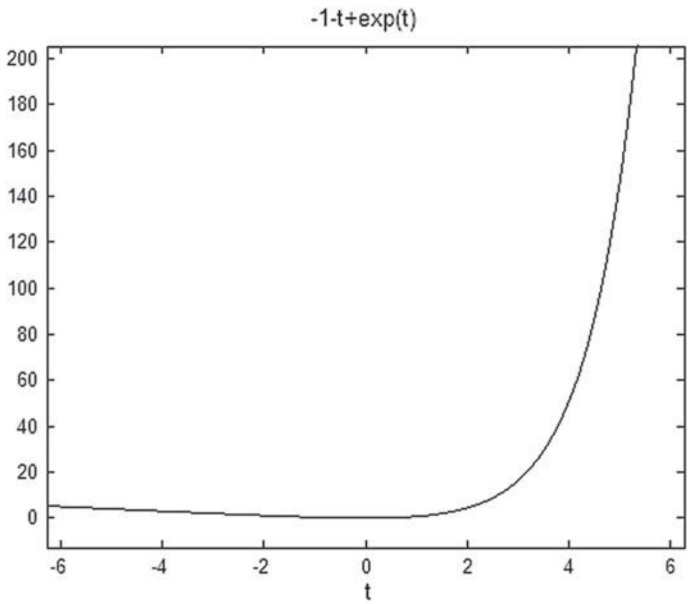


Fig. 12.16 Solution of a differential equation

second, third, ..., Nth derivative, respectively. Thus, D2y is equivalent of d^2y/dt^2. We can specify the initial conditions also. For example, let us solve $\dfrac{dx}{dt} = x + t$, with initial condition $x(0) = 0$. Then write following code:
$>> x = \text{dsolve}('Dx = x + t',' x(0) = 0')$
$x =$
$-1 - t + \exp(t)$
With ezplot function, we can plot this solution also (Fig. 12.16).

We can solve higher order differential equations also. Let the differential equation to be solved is $\dfrac{d^2y}{dt^2} = \cos 2t - y$, with initial conditions $y(0) = 1$ and $\dfrac{dy(0)}{dt} = 0$. To solve it write the following code (Table 12.2):
$>> y = \text{dsolve}(`D2y = \cos(2 * t) - y',' y(0) = 1',' Dy(0) = 0')$
$y =$
$4/3 * \cos(t) - 1/3 * \cos(2 * t)$

Numerical Solution. MATLAB has several inbuilt functions to solve various types of ODEs numerically. The detail is given below.

Here, we are taking one simple example of ode45(). The general syntax is given below $>>$ [time, solution] = ode45(myfunction, tspan, xo), where myfunction is the ODE, tspan is the time interval in which we want solution, and xo is initial condition. Let us same ode, which we have taken above for symbolic solution. This

Table 12.2 ODE solvers in MATLAB

Solver	Solves these kinds of problems	Method
ode45	Nonstiff differential equations	Runge–Kutta
ode23	Nonstiff differential equations	Runge–Kutta
ode113	Nonstiff differential equations	Adams
ode15s	Stiff differential equations and DAEs	NDFs (BDFs)
ode23s	Stiff differential equations	Rosenbrock
ode23t	Moderately stiff differential equations and DAEs	Trapezoidal rule
ode23tb	Stiff differential equations	TR-BDF2
ode15i	Fully implicit differential equations	BDFs

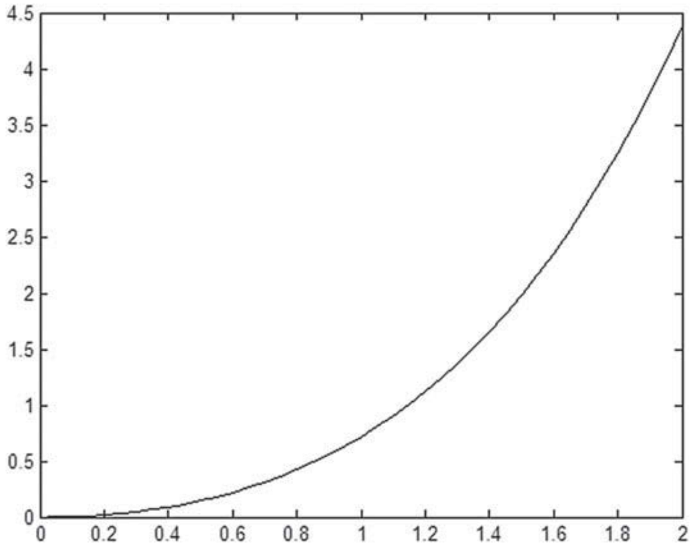


Fig. 12.17 Numerical solution of an ODE

ode can be solved in two ways, either writing function file describing ODE or by using inline function. The code is given below:
>> myode=inline('$x + t$');
>> $[t, x] =$ ode45(myode, [02], 0);
>> plot(t, x) (see Fig. 12.17).

We can code in alternate form also by writing function file. The code is given below:
% function file saved as myode.m
function $dxdt =$ myode(t, x)
$dxdt = x + t$;
On command prompt, we will write the following:

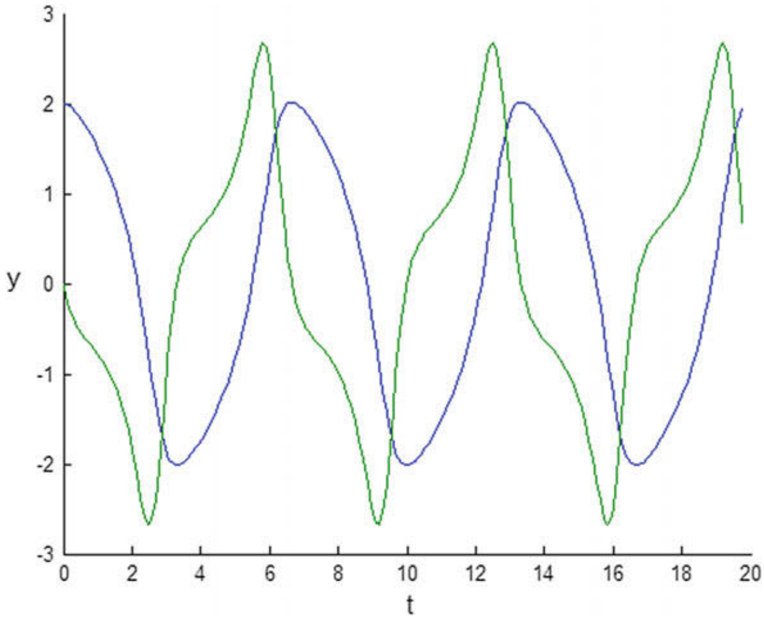


Fig. 12.18 Solution of an ODE of higher order

$>>$ $[t, x] =$ ode45('myode', [02], 0);
$>>$ plot(t, x)
We can solve higher order ODEs also. For this, we have to represent them as a set of first-order ODEs. For example, let us solve $\dfrac{d^2y}{dt^2} - (1 - y^2)\dfrac{dy}{dt} + y = 0$ with initial condition $y(0) = 2$ and $\dfrac{dy(0)}{dt} = 0$.

Before going code writing, we have to express second-order ode into set of 1^{st} order ode. The ode can also be written as $\ddot{y} = (1 - y^2)\dot{y} - y$...........(1)

Let $y = x1$ and $x2 = \dot{x}1$............(2)

Then equation in y can be written as $\dot{x}2 = (1 - x1^2)x2 - x1$.............(3)

The equation (2) and (3) can be written as $\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \end{bmatrix} = \begin{bmatrix} x2 \\ (1 - x1^2)x2 - x1 \end{bmatrix}$............ (4), the equation (4) is of the form $\dot{x} = f(x)$ (Fig. 12.18).

Now we write function file which will describe equation (4). The code is given below function $x$dot = myode(t, x) $x$dot $= [x(2); (1 - x(1)^2) * x(2) - x(1)]$.

Then write following commands on command prompt:
$>>$ $[t, x] =$ ode45('myode', [020], [2; 0]);
$>>$plot(t, x)

12.4.5 *Animated Phase Portraits of Nonlinear and Chaotic Dynamical Systems**

Introduction. The aim of this section is to present programs allowing to highlight the *slow–fast* evolution of the solutions of nonlinear and chaotic dynamical systems such as Van der Pol, Chua, and Lorenz models. These programs provide animated phase portraits in dimension two and three, i.e., "integration step-by-step" which are useful tools enabling to understand the dynamic of such systems.

Van der Pol model. The oscillator of Van der Pol [7] is a second-order system with nonlinear frictions which can be written as

$$\frac{d^2y}{dt^2} - \alpha(1 - y^2)\frac{dy}{dt} + y = 0.$$

The particular form of the friction which can be carried out by an electric circuit causes a decrease of the amplitude of the great oscillations and an increase of the small. This equation constitutes the "paradigm of relaxation-oscillations". According to d'Alembert transformation [1], any single nth-order differential equation may be transformed into a system of n simultaneous first-order equations and conversely.

Let us consider $\alpha(1 - y^2)\dot{y} = \frac{d}{dt}\alpha\left(y - \frac{y^3}{3}\right)$ and let us pose: $x_1 = y$ and $y = \alpha\dot{x}_2$. Thus, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{1}{\alpha}\begin{bmatrix} x_1 + x_2 - \frac{x_1^3}{3} \\ -x_1 \end{bmatrix},$$

when α becomes very large, x_1 becomes a "fast" variable and x_2 a "slow" variable. In order to analyze the limit $\alpha \to \infty$, we introduce a small parameter $\varepsilon = 1/\alpha^2$ and "slow time" $t' = t/\alpha = \sqrt{\varepsilon}t$. Thus, the system can be written as

$$\begin{bmatrix} \varepsilon\dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - \frac{x_1^3}{3} \\ -x_1 \end{bmatrix},$$

with ε a small positive real parameter $\varepsilon = 0.05$. System (1) which has been extensively studied since nearly one century is called a *slow–fast dynamical system or a singularly perturbed dynamical system.* Although it has been established that system (1) cannot be integrated by quadratures (closed-form) it is well known that it admits a solution of *limit-cycle* type. The program presented here enables to emphasize the *slow–fast* evolutions of the solution on this *limit cycle*.

First, copy the filed named "vanderpol" and "vanderpolpp" into the "current folder" of MATLAB. Then, open the *M*-file called "vanderpolpp" (see Fig. 12.19) and press the green button (red circle on Fig. 12.19) to provide an animated plot 2D. On Fig. 12.20, the solution materialized by a green point (green circle on Fig. 12.20)

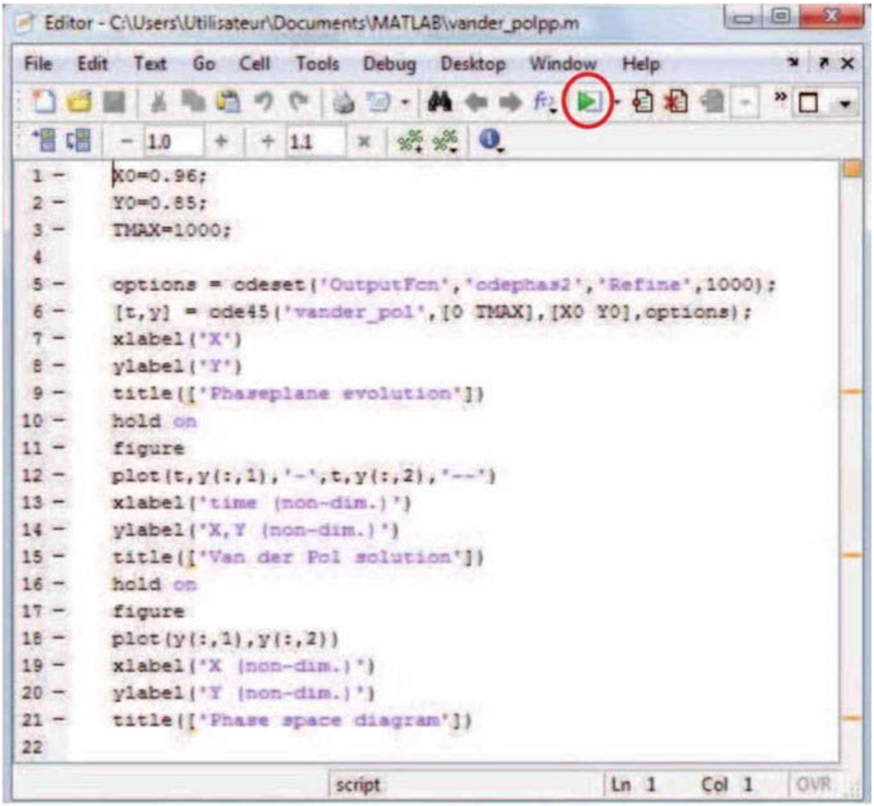


Fig. 12.19 Program for 2D animated phase portait

Table 12.3 Description of the main functions

Property	Description
OutputFcn	A function for the solver to call after every successful integration step.
odephas2	2D Phase plane
Refine	Increase the number of output points by a factor of Refine

which evolves on the limit cycle, i.e., slowly on the nearly vertical parts and fast on the nearly horizontal parts.

All the main functions are described in Table 12.3. The program provides the animated phase portrait corresponding to the solution of system (1) and the time series (Fig. 12.20).

Chua's model. The L.O. Chua's circuit [2] is a relaxation oscillator with a cubic nonlinear characteristic elaborated from a circuit comprising a harmonic oscillator of which operation is based on a field-effect transistor, coupled to a relaxation oscillator

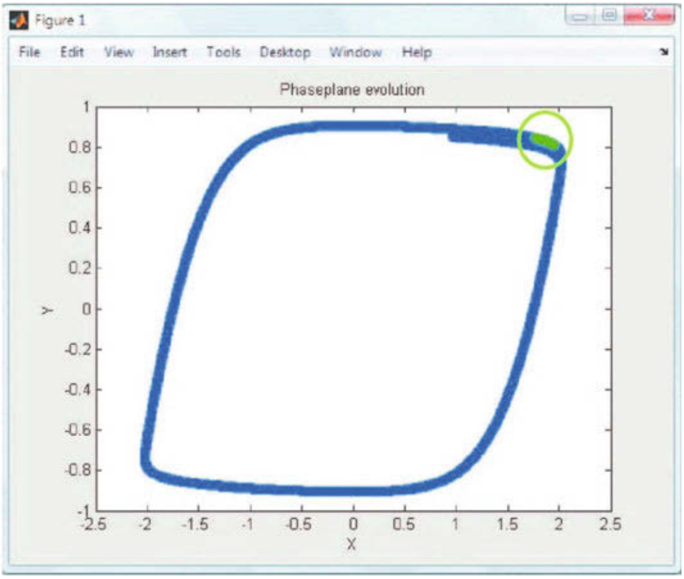


Fig. 12.20 Animated phase portrait in 2D

composed of a tunnel diode. The modeling of the circuit uses a capacity which will prevent from abrupt voltage drops and will make it possible to describe the fast motion of this oscillator by the following equations which also constitute a *slow–fast dynamical system* or a *singularly perturbed dynamical system*.

$$\begin{bmatrix} \varepsilon \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_3 - \frac{44}{3}x_1^3 - \frac{41}{2}x_1^2 - \mu x_1 \\ -x_3 \\ -0.7x_1 + x_2 + 0.24x_3 \end{bmatrix},$$

with ε and μ are real parameters $\varepsilon = 0.05$, $\mu = 2$. The system (2) which cannot be integrated by quadratures (closed-form) exhibits a solution evolving on "chaotic attractor" in the shape of a "double-scroll". The program presented here enables to emphasize the *slow–fast* evolutions of the solutions on the "chaotic attractor".
First, copy the files named "chua" and "chua1" into the "current folder" of MATLAB. Then, open the M-File called "chua1" and press the green button to provide an animated plot 3D. The solution materialized by a green point evolves on the attractor according to slow and fast motion. The function "odephas2" is simply replaced by "odephas3".

Lorenz model. The purpose of the model established by Edward Lorenz [5] was in the beginning to analyze the unpredictable behavior of weather. After having developed nonlinear partial derivative equations starting from the thermal equation

Table 12.4 Description of the main functions

Name	Description
Vanderpol	Dynamical system (1) with $\varepsilon = 1/20$
Vanderpolpp	Animated phase portrait 2D
Chua	Dynamical system (2) with $\varepsilon = 1/20$, $\mu = 2$
Chua1	Animated phase portrait 3D
Lorenz	Dynamical system (3) with $\sigma = 10$, $\beta = 8/3$, $r = 28$
Lorenz1	Animated phase portrait 3D

and Navier–Stokes equations, Lorenz truncated them to retain only three modes. The most widespread form of the Lorenz model is as follows:

$$\begin{bmatrix} \varepsilon \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \sigma(x_2 - x_1) \\ -x_1 x_3 + r x_1 - x_2 \\ x_1 x_2 - \beta x_3 \end{bmatrix}$$

with σ, r, and β are real parameters: $\sigma = 10$, $\beta = \frac{8}{3}$, $and\ r = 28$.

Although this system is not singularly perturbed since it does not contain any small multiplicative parameter, it is a slow–fast dynamical system. Its solution exhibits a solution evolving on “chaotic attractor” in the shape of a “butterfly”. The program presented here enables to emphasize the slow–fast evolutions of the solution in the “chaotic attractor” (Table 12.4).

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2. Chua LO, Komuro M, Matsumoto T (1986) The double scroll family. IEEE Trans Circuits Syst 33(11):1072–1097
3. Ginoux M (2009) Differential geometry applied to dynamical systems, vol 66. World scientific series on nonlinear science, Series A. World Scientific, Singapore
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6. Van Der Pol B (1926) On Relaxation-Oscillations. Philos Mag 7(2):978–992

12.4.6 Finding Minima and Maxima

Using MATLAB, we can find unconstrained minima and maxima of a nonlinear function within a specified range. For this purpose, fminbnd() and fminsearch() functions are used. The general syntax of fminbnd is given below $>> x = f\text{minbnd}(fx, x1, x2)$, where fx is a function of x, $x1$, and $x2$ define range in which minima is to be found.

Let us take one example function $f(x) = \dfrac{1}{10 + 5\cos\left(\frac{x}{2} + 2\right)}$ to find its minima, the example code is given below:

$>> fx =' 1/(10 + 5 * \cos((x/2) + 2))';$
$>> x = f\text{minbnd}(fx, -6, -2)$
$x =$
-4
$>> y = \text{eval}(fx, x)$
$y =$
0.0667
$>> \text{ezplot}(fx)$
$>>$ hold on
$>> \text{plot}(x, y,' \text{ro}')$
We get the result (Fig. 12.21).

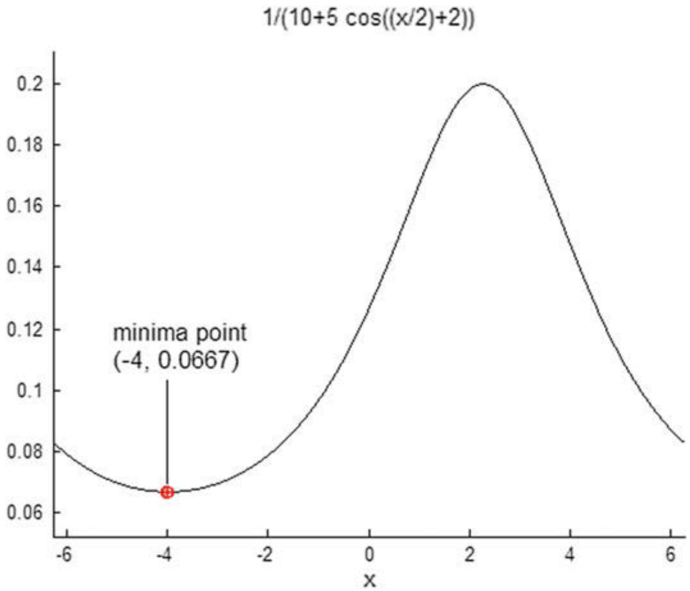

Fig. 12.21 Minimum of a given function

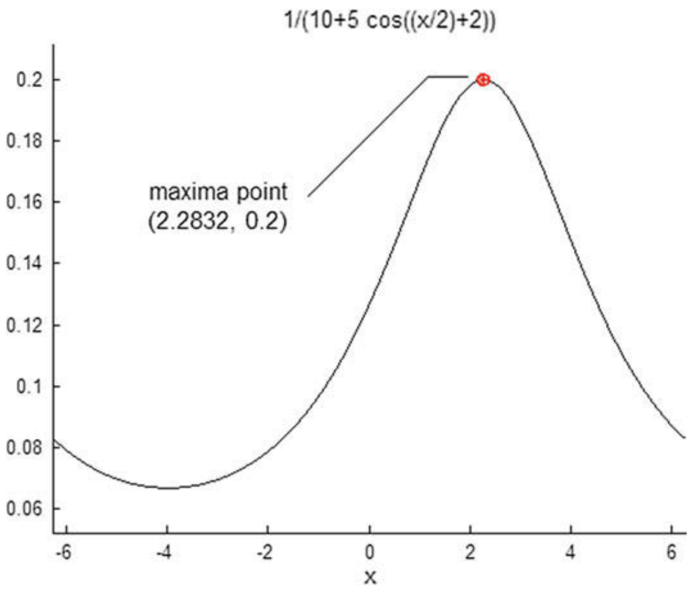

Fig. 12.22 Maximum of the same function

We can find maxima also for this function. But for that we will find minima of inverted function, i.e., $f(x)$ that obviously will be the maxima of $f(x)$. The code is given below:
$>> \ fx =' 1/(10 + 5 * \cos((x/2) + 2))';$
$>> \ fx\text{inv} =' -1/(10 + 5 * \cos((x/2) + 2))';$ % Inverted function
$>> \ x\text{inv} \ = f\text{minbnd}(fx\text{inv}, 0, 4)$
$x\text{inv} =$
2.2832
$>> \ y\text{inv} = \text{eval}(fx\text{inv}, x\text{inv})$
$y\text{inv} =$
-0.2000
$>>$ ezplot(fx)
$>>$ hold on
$>> \ \text{plot}(x\text{inv}, -y\text{inv},' \text{ro}')$
We get the result (Fig. 12.22).

12.4.7 Fourier Analysis

We have described some of the basic properties of Fourier series in Chapter 10. MATLAB helps us to understand these properties in much better way by plotting some functions along with several of the partial sums of their Fourier series. This will illustrate that the partial sum of Fourier series of a function is not equal to the function itself. Through MATLAB, one can experiment to observe the shape of partial sums $S_n(x)$, when n increases.

Here we present one very versatile program written in MATLAB to compute Fourier series of the given function. This program can be used to compute Fourier series of any function, just by doing small modification in main program body.

The program code is given below to find Fourier series of the square function which is defined as $f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 < x \le \pi \end{cases}$ The program code is given below:

```
% Fourier Analysis of square functions for first 15 harmonics.
t0 = −pi; % initial time
t0_ T = pi; % final time
mp = 0; % mid point
T = t0_ T−t0; % time period
syms t; % sym variable declaration
ft = −diff(t); % −1 part of function
ftt = diff(t); % 1 part of function
w0 = 2 ∗ pi/T; % frequency
n = 1 : 15; % number of Harmonics
% computation of Trigonometric Fourier series Coefficients
a0 = 1/T ∗ (int(ft, −pi, 0) + int(ftt, 0, pi));
an = 2/T ∗ (int(ft ∗ cos(n ∗ w0 ∗ t), −pi, 0)+ int(ftt ∗ cos(n ∗ w0 ∗ t), 0, pi));
bn = 2/T ∗ (int(ft ∗ sin(n ∗ w0 ∗ t), −pi, 0) + int(ftt ∗ sin(n ∗ w0 ∗ t), 0, pi));
ann = an. ∗ cos(n ∗ w0 ∗ t);
bnn = bn. ∗ sin(n ∗ w0 ∗ t);
avg = double(a0);% converting sym variable to value
t = −pi : pi/100 : pi;
suma = 0; sumb = 0;
for j = 1 : 15 % taking 15 harmonics
sumb = sumb + bnn(j);
suma = suma + ann(j);
end
bnsum = eval(sumb);
ansum = eval(suma);
plot(t, avg + bnsum + ansum) % plot of truncated harmonics function
hold on
% plotting actual function
t1 = −pi : pi/1000 : mp;
plot(t1, −1,′r′)
t2 = mp : pi/1000 : pi;
plot(t2, 1,′r′)
grid on
% formatting plot
xlabel(‘Time’)
ylabel(‘Amplitude’)
title(‘Fourier approximation plot for 15 harmonics for square function’)
legend(‘Fourier Approximation’, ‘Actual Function’)
```

The result is shown below (Fig. 12.23)

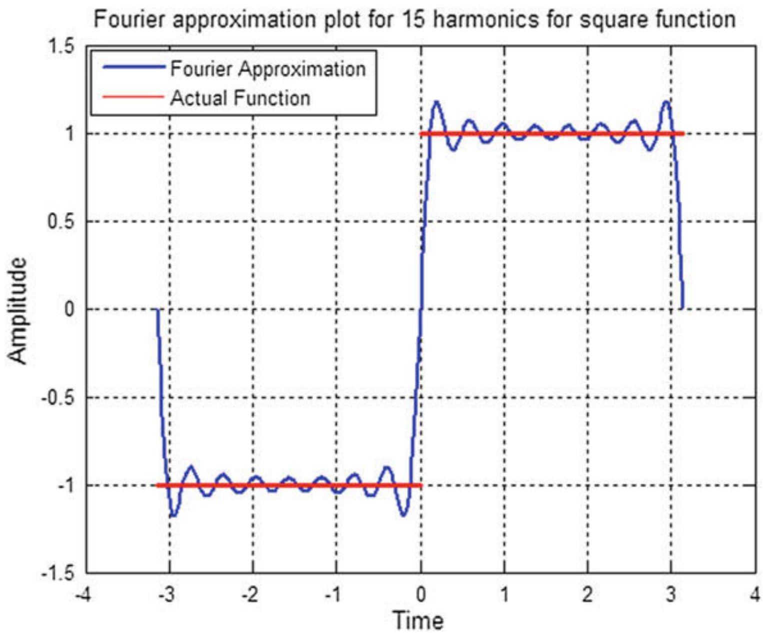

Fig. 12.23 Fourier approximation of the square function

Here is another program to plot harmonics of the ramp function which is defined as
$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$
One can see the required modifications in this program as compared with previous one.

% Fourier Analysis of ramp function for first 5 harmonics.
$t0 = -1$; % initial time
t0_ T=1; % final time
$mp = 0$; % mid point
T = t0_ T−t0; % time period
syms t; % sym variable declaration
$ft = \mathrm{diff}(\mathrm{diff}(t))$; % zero part of function
$ftt = t$; % t part of function
$w0 = 2 * \mathrm{pi}/T$; % frequency
$n = 1:5$; % number of Harmonics
% computation of Trigonometric Fourier Series Coefficients
$a0 = 1/T * (\mathrm{int}(ft, -1, 0) + \mathrm{int}(ftt, 0, 1))$;
$an = 2/T * (\mathrm{int}(ft * \cos(n * w0 * t), -1, 0) + \mathrm{int}(ftt * \cos(n * w0 * t), 0, 1))$;
$bn = 2/T * (\mathrm{int}(ft * \sin(n * w0 * t), -1, 0) + \mathrm{int}(ftt * \sin(n * w0 * t), 0, 1))$;
$ann = an. * \cos(n * w0 * t)$;
$bnn = bn. * \sin(n * w0 * t)$;
avg = double($a0$);% converting sym variable to value

$t = -1.5 : 0.010 : 1.5;$
suma = 0; sumb = 0;
for $j = 1 : 5$ % First 5 harmonics
sumb = sumb + bnn(j);
suma = suma + ann(j);
end
bnsum = eval(sumb);
ansum = eval(suma);
plot(t, avg + bnsum + ansum) % plot of truncated harmonics function
hold on
% plotting actual function
$t1 = -1 : 0.010 : 0;$
plot($t1$, 0,$'r'$)
$t2 = 0 : 0.010 : 1;$
plot($t2$, $t2$,$'r'$)
grid on
% formatting plot
$x$label('Time')
$y$label('Amplitude')
title('Fourier approximation plot for 5 harmonics for ramp function')
legend('Fourier Approximation', 'Actual Function')

The result is shown in Fig. 12.24.

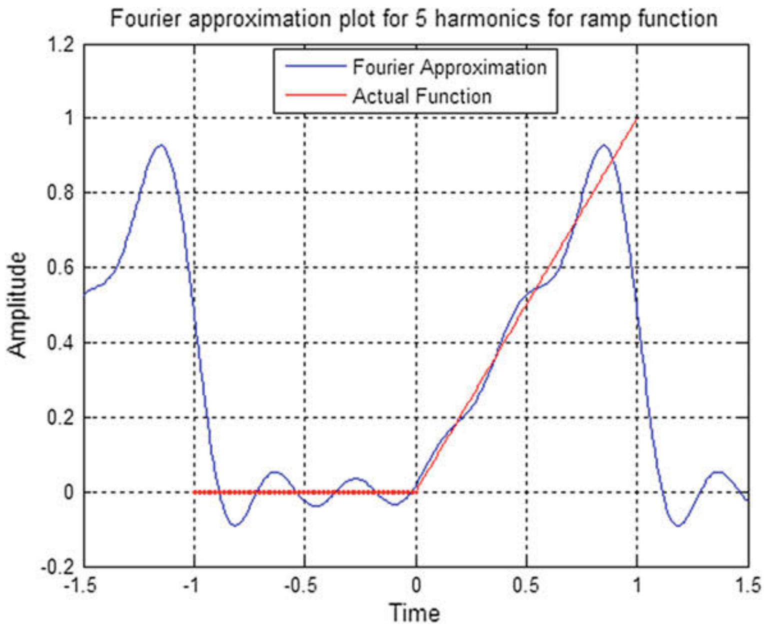

Fig. 12.24 Fourier approximation of the ramp function

12.5 Exercises

12.5.1 Write the MATLAB program to obtain 2D plot of following function. The type of plot is specified with the functions.

(a) Simple line plot for $f(x) = \begin{cases} e^x & x < 0 \\ (x-1)^2 & x \geq 0 \end{cases}$.
(b) Stem plot of $f(x) = x \sin x + e^{-x/5} \cos x, 0 \leq x \leq 10$.
(c) Quiver plot of $f(x, y) = xi + (x^2 + y^2)j$ over the rectangle $[-2, 2] \times [-2, 2]$.
(d) Polar plot of $r = 2e^{(-\frac{\theta}{10})}, 0 \leq \theta \leq 10\pi$.

12.5.2 Write the MATLAB program to obtain 3D plot of following functions. The type of plot is specified with the functions.

(a) Simple line plot for $x = t \sin t$, $y = t \cos t$, $z = t$ for $0 \leq t \leq 10\pi$.
(b) Quiver plot for $f(x, y, z) = yi + xj + (x^2 + z)k$ over the rectangle $[-2, 2] \times [-2, 2] \times [-1, 1]$.
(c) Surf plot for the figures 8.1.3, 8.1.6 of Chapter 8 of this book.
(d) Mesh plot for the function $z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$, $-3 \leq x \leq 3, -3 \leq y \leq 3$.

12.5.3 Write a MATLAB program to compute the first 200 partial sums of the series:
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln^2(n)}{n^{1.5}}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \ln(n+1)}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

12.5.4 Find the limits of the following functions:
(a) $\displaystyle\lim_{x\to 0} \frac{1}{5 + 4\cos x}$, (b) $\displaystyle\lim_{x\to\infty} \frac{2x}{\sqrt{x^2 - 1}}$, (c) $\displaystyle\lim_{x\to 0} \frac{e^{\cos^{-1} x}}{\sqrt{1 - x^2}}$,
(d) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{9 + 4x^2}}$.

12.5.5 Find symbolic and numerical integration of $\displaystyle\int_0^5 e^x \sin x \, dx$ and compare the results.

12.5.6 Find symbolic differentiation of following functions and plot them:
(a) $(4x^2 - 1)(7x^3 + x)$, (b) $\dfrac{(x^2 - 1)}{(x^4 + 1)}$, (c) $\dfrac{\sin x}{(1 + \cos x)}$,
(d) $e^x \sin x$.

12.5.7 Write a MATLAB program to solve following differential equations numerically and plot the output variables:

(a) $\ddot{y} + \sin y = 0$ with initial conditions $y(0) = 1, \dot{y}(0) = 0$.
(b) $\dot{x} = 3x - 4\cos t$ with initial condition $x(0) = 1$.

12.5.8 Write a program in MATLAB to compute Fourier series of following function:

$$f(x) = \begin{cases} x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

and plot partial sums of first 5, 10, and 30 harmonics and show the result from -3π to $+\pi$.

Appendix A
Real Numbers and Inequalities

A.1 The Number System

The simplest numbers are the **natural numbers** $1, 2, 3, 4, 5, \ldots$. The numbers $\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$ are called **integer numbers** or simply **integers**. We denote the set of all natural numbers by $\mathbb{N}$ and the set of all integers by $\mathbb{Z}$. It is clear that $\mathbb{N} \subset \mathbb{Z}$, that is, the set of natural numbers is a subset of the set of integers.

The numbers of the form $\frac{p}{q}$, where p and q are integers with $q \neq 0$, are called **rational numbers**. For example, $\frac{3}{4}$, $\frac{-2}{5}$, $\frac{11}{2}$ are rational numbers. We denote the set of all rational numbers by $\mathbb{Q}$. We have $\mathbb{Z} \subset \mathbb{Q}$, since any integer p can be written as the ratio $\frac{p}{1}$.

Numbers which cannot be written as the ratio of two integers are called **irrational**. For example $\sqrt{2}$, $1 + \sqrt{2}$, $\sqrt{5}$ and π are irrational numbers. (In Sect. 1.6 we present Euclid's proof that $\sqrt{2}$ is irrational.) Together, rational and irrational numbers form what is called the **real number system**. Thus, a real number is either rational or irrational. The set of all real numbers is denoted by $\mathbb{R}$, and we have that $\mathbb{Q} \subset \mathbb{R}$.

We assume that the reader is familiar with the algebraic operations (addition, subtraction, multiplication, and division) for numbers. Recall that division by zero is not allowed.

Since the square x^2 of a real number x is nonnegative, there cannot be any real number which satisfies the equation $x^2 = -1$, or $x^2 + 1 = 0$. Nevertheless, it has been found very useful in mathematics to introduce the number $i = \sqrt{-1}$, so that $i^2 = -1$. i is called the imaginary unit. Numbers of the form $a + bi$, where a and b are real numbers, are called **complex numbers**. For example, $2 + 5i$ and $\sqrt{3} - 6i$ are complex numbers. A complex number $a + bi$ is called **imaginary** (or **pure imaginary**) if $a = 0$, so $9i$ is an imaginary number. Since $a = a + 0i$, every real number a can be interpreted as a complex number, as for example the number $\frac{2}{3}$,

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thus $\mathbb{R} \subset \mathbb{C}$. The algebraic operations are defined for complex numbers in the natural way, for example

$$\begin{aligned}
(3+5i)+(4-\sqrt{2}i) &= 3+4+5i-\sqrt{2}i = 7+(5-\sqrt{2})i\,,\\
(3+5i)-(4-\sqrt{2}i) &= 3-4+5i-(-\sqrt{2}i) = -1+(5+\sqrt{2})i\,,\\
(3+5i)\cdot(4-\sqrt{2}i) &= 3\cdot 4+5i\cdot 4-3\sqrt{2}i-5i\cdot\sqrt{2}i\\
&= (12+5\sqrt{2})+(20-3\sqrt{2})i\,.
\end{aligned}$$

The division of two complex numbers is a bit more complicated, in general one has

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i\,.$$

Rational and irrational numbers can be distinguished by their decimal representations. Decimals like $12.345 = 12.3450000\ldots$, where only zeroes appear from some point onwards, are called **terminating decimals**. They correspond to those rational numbers p/q whose denominator q has only 2's and 5's as prime factors. Every other rational number has a periodic decimal representation, that is, a representation where a certain string of numbers is repeated from some point onwards, for example

$$\begin{aligned}
\frac{31}{60} &= 0.51666666\ldots\,,\\
\frac{13}{17} &= 0.7647058823529411\,76470\ldots
\end{aligned}$$

Irrational numbers have nonperiodic decimal representations. For example, the decimal representations

$$\begin{aligned}
\pi &= 3.141592653\ldots\,,\\
\sqrt{2} &= 1.414213562\ldots\,,
\end{aligned}$$

do not exhibit any periodic repetition. Moreover, let us remark that if we truncate a nonterminating decimal representation (periodic or nonperiodic) of a real number x at some point, the resulting terminating decimal will only be an approximation to x.

A.2 Intervals, Absolute Value and Inequalities

We say that a real number x is less than another real number y, and write $x < y$, if $x - y$ is less than zero, that is, $x - y$ is negative or $y - x$ is positive. Writing "$x \le y$" (read "x is less than or equal to y") means that $x - y$ is less than or equal to 0, that is, $x - y$ is negative or it is equal to 0.

The set of all real numbers x such that $a \leq x \leq b$ is called the **closed interval** from a to b and denoted as $[a, b]$. The set of all real numbers x such that $a < x < b$ is called the **open interval** from a to b and denoted as (a, b).

Definition A.1 The **absolute value** or **magnitude** of a real number x is denoted by $|x|$ and is defined as

$$|x| = \begin{cases} x\,, & x > 0\,, \\ 0\,, & x = 0\,, \\ -x\,, & x < 0\,. \end{cases}$$

Properties of the absolute value. For any real numbers x and y, the number $|x - y|$ represents the distance of x and y on the real line. (Thus $|x|$ equals the distance of x from 0.) Moreover, the following properties hold.

(i) $\sqrt{x^2} = |x|$.
(ii) $|-x| = |x|$. (A number and its negative have the same absolute value.)
(iii) $|xy| = |x||y|$. (The absolute value of a product is the product of the absolute values.)
(iv) $|x + y| \leq |x| + |y|$. (This is called the triangle inequality.)
(v) $||x| - |y|| \leq |x - y|$. (This is called the inverse triangle inequality.)

A.3 The Binomial Theorem

Let x and y be real or complex numbers, and let n be any nonnegative integer. Then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\,,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}\,, \quad n! = n(n-1)(n-2)\cdots 2 \cdot 1\,.$$

Appendix B
Analytic Geometry

Rectangular coordinates. The real line consists of all real numbers; each point on the real line is associated with a real number. Let us now consider the plane. A **rectangular coordinate system** (also called **Cartesian coordinate system**) consists of two perpendicular coordinate lines, called coordinate axes. The intersection of the two axes is called the **origin** of the coordinate system and denoted by O. Usually, the coordinate axes are chosen along the horizontal and vertical direction; the horizontal axis is called the x-axis, the vertical axis is called the y-axis. In this case, the plane and the axis together are referred to as the xy-plane (Fig. B.1). Any point P in the plane is represented by a pair (x_1, y_1) of real numbers, x_1 is called the x-coordinate of P and y_1 is called the y-coordinate of P. Points on the x-axis have the form $(x_1, 0)$, while points on the y-axis have the form $(0, y_1)$. The origin O has the coordinates $(0, 0)$. The vertical line through a point $(x_1, 0)$ and the horizontal line through $(0, y_1)$ are represented, respectively, by the equations

$$x = x_1 \qquad \text{and} \qquad y = y_1 .$$

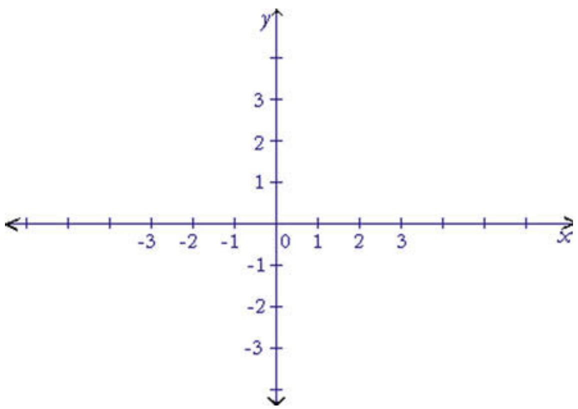

Fig. B.1 Rectangular coordinate system

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Lines, distances, circles. Let P_1 and P_2 be two points in the xy-plane, having coordinates (x_1, y_1) and (x_2, y_2) respectively. Consider the straight line passing through P_1 and P_2. Whenever $x_1 \neq x_2$, the number

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

is called the slope of this line; the line itself is described by the equation

$$y = m(x - x_1) + y_1 .$$

The distance between P_1 and P_2 is defined by

$$d(P_1, P_2) = d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

Indeed, this number yields the length of the line segment which connects P_1 and P_2, by the theorem of Pythagoras. As a particular case, the distance of $P_1 = (x_1, 0)$ and $P_2 = (x_2, 0)$ equals

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1| .$$

The equation of a circle with center (x_0, y_0) and radius r is given by

$$(x - x_0)^2 + (y - y_0)^2 = r^2 .$$

If $(x_0, y_0) = (0, 0)$ then this equation becomes $x^2 + y^2 = r^2$, it describes the circle centered at the origin with radius r.

Polar coordinates. Instead of Cartesian coordinates, it is often more convenient to use polar coordinates which we usually denote by (r, θ). The Cartesian coordinates (x, y) and the polar coordinates (r, θ) of a given point P in the plane are related by

$$x = r\cos\theta , \quad y = r\sin\theta . \tag{B.1}$$

The number r equals the length of the line segment joining the origin O and the point P, and θ equals the angle between the line OP and the x-axis (Fig. B.2). Note that θ is usually measured in radians, that is, an angle of 90° has $\theta = \pi/2$, the full angle of 360° has $\theta = 2\pi$ and so on. Formula (B.1) expresses the Cartesian coordinates of P in terms of the polar coordinates of P Conversely, we obtain the polar coordinates of P from its Cartesian coordinates by the formula

$$r = \sqrt{x^2 + y^2} , \quad \theta = \tan^{-1}\frac{y}{x} . \tag{B.2}$$

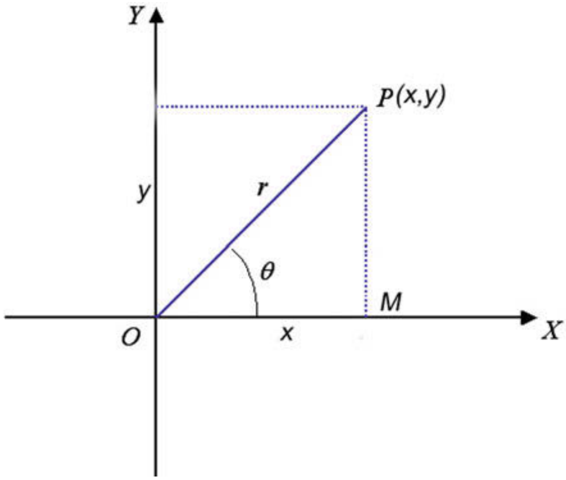

Fig. B.2 Polar coordinates

Appendix C
Trigonometry

C.1 Trigonometric Functions

The trigonometric functions are directly related to the geometry of the circle. Consider the point P with Cartesian coordinates (x, y) and polar coordinates (r, θ) as in Fig. C.1. The standard trigonometric functions are defined as follows.

$$\begin{aligned}\sin\theta &= \frac{y}{r} \qquad \text{(read as “sine of } \theta\text{”)},\\ \cos\theta &= \frac{x}{r} \qquad \text{(read as “cosine of } \theta\text{”)},\end{aligned}$$

and analogously

$$\tan\theta = \frac{y}{x}\,, \quad \cot\theta = \frac{x}{y}\,, \quad \sec\theta = \frac{r}{x}\,, \quad \csc\theta = \frac{r}{y}\,,$$

called the tangent, the cotangent, the secant and the cosecant, respectively. From elementary geometry, we see that the definitions above do not depend on the chosen value of r, as long as $r > 0$, so it suffices to consider the case $r = 1$ (the unit circle).

Recall that θ is usually measured in radians ($\theta = \pi/2$ corresponds to an angle of $90°$ and so on.)

C.2 Trigonometric Identities

A trigonometric identity is an equation involving trigonometric functions that is true for all angles for which both sides of the equation are defined. In the following, we state a few useful trigonometric identities.

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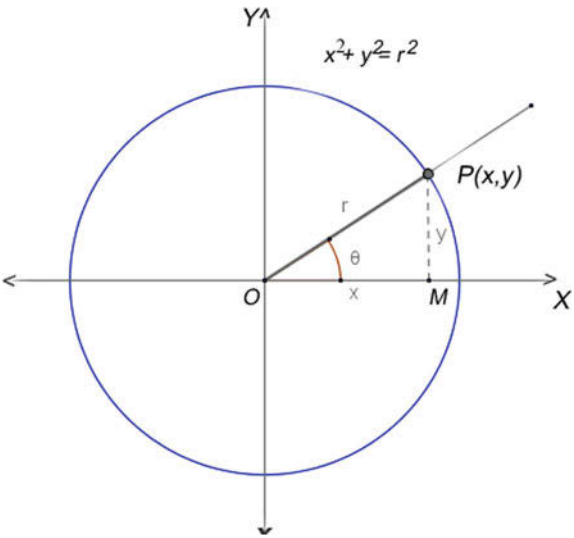

Fig. C.1 Trigonometric functions and the circle

$$\cos^2\theta + \sin^2\theta = 1\,. \tag{C.1}$$

$$1 + \tan^2\theta = \sec^2\theta\,. \tag{C.2}$$

$$1 + \cot^2\theta = \csc^2\theta\,. \tag{C.3}$$

$$\sin(\theta + 2\pi) = \sin(\theta - 2\pi) = \sin\theta\,. \tag{C.4}$$

$$\cos(\theta + 2\pi) = \cos(\theta - 2\pi) = \cos\theta\,. \tag{C.5}$$

In general,

$$\begin{aligned} \sin(\theta \pm 2n\pi) &= \sin\theta\,, \quad \text{for } n = 0, 1, 2, 3, \ldots, \\ \cos(\theta \pm 2n\pi) &= \cos\theta\,, \quad \text{for } n = 0, 1, 2, 3, \ldots. \end{aligned} \tag{C.6}$$

$$\tan(\theta + \pi) = \tan\theta\,, \quad \tan(\theta - \pi) = \tan\theta\,. \tag{C.7}$$

Next we give formulas which involve sums or multiples of angles.

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,. \tag{C.8}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\,. \tag{C.9}$$

$$\sin 2\theta = 2\sin\theta\cos\theta\,. \tag{C.10}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta\,. \tag{C.11}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}\,. \tag{C.12}$$

$$\cos 2\theta = 2\cos^2\theta - 1\,. \tag{C.13}$$

$$\cos 2\theta = 1 - 2\sin^2\theta\,. \tag{C.14}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} . \tag{C.15}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} . \tag{C.16}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} . \tag{C.17}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} . \tag{C.18}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} . \tag{C.19}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} . \tag{C.20}$$

Theorem C.1 *We have*

$$\cos x < \frac{\sin x}{x} < 1 , \quad \textit{for any } x \textit{ satisfying } 0 < |x| \le \frac{\pi}{2} . \tag{C.21}$$

A geometric proof. Assume first that $x > 0$. The proof is based on the relations between certain areas in Fig. C.2.

For a circle of radius 1, the figure shows a sector of angle x with corners O, A and P, and additional auxiliary points B and Q, connected to P resp. A by vertical lines of length $\sin x$ resp. $\tan x$. The distance between O and B equals $\cos x$. We then have

$$\begin{aligned} \text{area of triangle OAP } &= \frac{1}{2}\cdot 1 \cdot \sin x , \\ \text{area of sector OAP } &= \frac{1}{2}x , \\ \text{area of triangle OAQ } &= \frac{1}{2}\cdot 1 \cdot \tan x . \end{aligned}$$

We see that

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x = \frac{1}{2}\frac{\sin x}{\cos x} ,$$

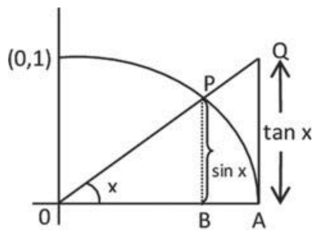

Fig. C.2 Geometric proof of Theorem C.1

since the triangle OAP is contained in the sector OAP, which in turn is contained in the triangle OAQ. Multiplying with 2 and dividing by $\sin x$ results in

$$\frac{1}{\sin x} < \frac{x}{\sin x} < \frac{1}{\cos x}\,.$$

Taking reciprocals yields the assertion (C.21) for $x > 0$. For $x < 0$ it is enough to observe that

$$\cos(-x) = \cos x\,, \quad \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}\,.$$

An alternative proof, for small values of x. This proof uses the power series representation of the sine and the cosine, as discussed in Chap. 5. We have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\,,$$

therefore

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\,. \tag{C.22}$$

On the other hand,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\,, \tag{C.23}$$

If we consider just the first two terms of the series, we get for $x > 0$

$$1 - \frac{x^2}{2!} < 1 - \frac{x^2}{3!} < 1\,. \tag{C.24}$$

If x is small enough, inequality (C.24) continues to hold even after we add the remaining terms of the series (C.22) and (C.23), because those remaining terms have exponents of x greater than 2, and hence the remaining sums have the form $x^2 r(x)$ for some function r with $\lim_{x\to 0} r(x) = 0$.

C.3 Inverse Trigonometric Functions

The **inverse trigonometric functions** or **cyclometric functions** are obtained as inverse functions of the trigonometric functions. However, in order that this works out correctly in the sense of Definition 1.12 (the general definition of an inverse function), one has to restrict the domain of the original function to an interval where the latter is increasing resp. decreasing. For example, the sine function is increasing on the interval $[-\pi/2, \pi/2]$, thus by Theorem 1.2 it has an inverse defined on its range $[-1, 1]$ with values in $[-\pi/2, \pi/2]$. This inverse function is called **arcsine**;

more precisely, it is called the **principal branch** of the arcsine. Indeed, on the interval $[\pi/2, 3\pi/2]$ the sine is decreasing (again with range $[-1, 1]$), so we could also define an inverse on $[-1, 1]$ with range $[\pi/2, 3\pi/2]$, and similarly on other intervals.

In the following table, we list the principal branches of the 6 standard trigonometric functions. Their names are obtained by putting the syllable "arc" in front of the original function.

Name	**Standard notation**	**Domain**	**Range in radians**	**Range in degrees**
arcsine	$y = \arcsin(x)$	$x \in [-1, 1]$	$-\pi/2 \le y \le \pi/2$	$-90° \le y \le 90°$
arccosine	$y = \arccos(x)$	$x \in [-1, 1]$	$0 \le y \le \pi$	$0° \le y \le 180°$
arctangent	$y = \arctan(x)$	$x \in \mathbb{R}$	$-\pi/2 < y < \pi/2$	$-90° < y < 90°$
arccotangent	$y = \operatorname{arccot}(x)$	$x \in \mathbb{R}$	$0 < y < \pi$	$0° < y < 180°$
arcsecant	$y = \operatorname{arcsec}(x)$	$x \ge 1$ or $x \le -1$	$0 \le y < \pi/2$ or $\pi/2 < y \le \pi$	$0° \le y < 90°$ or $90° < y \le 180°$
arccosecant	$y = \operatorname{arccsc}(x)$	$x \le -1$ or $x \ge 1$	$-\pi/2 \le y < 0$ or $0 < y \le \pi/2$	$-90° \le y < 0°$ or $0° < y \le 90°$

The "inverse function" notations $\sin^{-1}$, $\cos^{-1}$ etc. for arcsin, arccos etc. are natural, but one must be aware of the following notational conflict: Since we commonly write $\sin^2 x$ instead of $(\sin x)^2$, an unspecified use of "$\sin^{-1}(x)$" might mean "$\arcsin(x)$" as well as $1/(\sin x)$".

C.4 Inverse Hyperbolic Functions

The **inverse hyperbolic functions** are the inverses of the hyperbolic functions sinh, cosh, tanh, coth, sech and csch. Similarly as in the case of trigonometric functions, one has to consider suitable domains and ranges in order to define inverses in the sense of Definition 1.12. (In contrast to the trigonometric functions, this is not always necessary; sinh is invertible on its entire domain $\mathbb{R}$.) They are called **area hyperbolic functions**, and for the mathematical notation the prefix "ar" precedes the name of the original function, for example, "arsinh" denotes the inverse of sinh.

Table C.1 lists the principal branches of the inverse hyperbolic functions.

The inverse hyperbolic functions are expressible in terms of natural logarithms. The formulas in the following Table C.2 hold for all x in the domains of the inverse hyperbolic functions.

Table C.1 Principal branches of the inverse hyperbolic functions

Function	Domain	Range
$y = \operatorname{arsinh}(x)$	$x \in (-\infty, \infty)$	$y \in (-\infty, \infty)$
$y = \operatorname{arcosh}(x)$	$x \in [1, \infty)$	$y \in [0, \infty)$
$y = \operatorname{artanh}(x)$	$x \in (-1, 1)$	$y \in (-\infty, \infty)$
$y = \operatorname{arcoth}(x)$	$x \in (-\infty, -1) \cup (1, \infty)$	$y \in (-\infty, 0) \cup (0, \infty)$
$y = \operatorname{arsech}(x)$	$x \in (0, 1]$	$y \in [0, \infty)$
$y = \operatorname{arcsch}(x)$	$x \in (-\infty, 0) \cup (0, \infty)$	$y \in (-\infty, 0) \cup (0, \infty)$

Table C.2 Relations between the area hyperbolic functions and the logarithm

$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$	$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$
$\operatorname{artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$	$\operatorname{arcoth}(x) = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$
$\operatorname{arsech}(x) = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$	$\operatorname{arcsch}(x) = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{\lvert x \rvert}\right)$

Appendix D

D.1 The Sandwich Theorem

We present the proof of the sandwich theorem, Theorem 2.5. First, we treat the case of the right-hand limit, that is, we assume that $\lim_{x\to c^+} g(x) = \lim_{x\to c^+} h(x) = L$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality $c < x < c + \delta$ implies

$$L - \varepsilon < g(x) < L + \varepsilon \quad \text{and} \quad L - \varepsilon < h(x) < L + \varepsilon. \tag{D.1}$$

Since $g(x) \le f(x) \le h(x)$ by assumption, we get

$$L - \varepsilon < g(x) \le f(x) \le h(x) < L + \varepsilon\,,$$

so $-\varepsilon < f(x) - L < \varepsilon$. Therefore, the inequality $c < x < c + \delta$ implies that $|f(x) - L| < \varepsilon$. This concludes the proof for the right-hand limit. For the left-hand limit, suppose that $\lim_{x\to c^-} g(x) = \lim_{x\to c^-} h(x) = L$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality $c - \delta < x < c$ implies (D.1). We proceed as before and obtain that $c - \delta < x < c$ implies that $|f(x) - L| < \varepsilon$. Finally, let $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. By what we just have proved, $\lim_{x\to c^+} f(x) = L = \lim_{x\to c^-} f(x)$. Hence, $\lim_{x\to c} f(x)$ exists and equals L.

D.2 Rolle's Theorem

Proof of Theorem 4.5. We have to find a $c \in (a, b)$ such that $f'(c) = 0$. We distinguish two cases. If f is constant in $[a, b]$, then $f'(x) = 0$ for all x in (a, b), so for c we can choose an arbitrary point in (a, b). For the second case, assume that f is not constant in $[a, b]$. Set $r = f(a) = f(b)$. There must be either a point x in (a, b) where $f(x) > r$ or a point x in (a, b) where $f(x) < r$. Assume that the first situation occurs (the proof for the second situation is analogous). Since f is

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continuous on $[a, b]$, it follows from the Extreme-Value Theorem 2.8 that f has a maximum value at some point c in $[a, b]$. The point c cannot be an endpoint, since $f(a) = f(b) = r < f(x)$. By hypothesis f is differentiable everywhere on (a, b), thus $f'(c)$ exists. By Theorem 4.1 we get $f'(c) = 0$.

D.3 The Mean Value Theorem

Proof of Theorem 4.4. The two-point form of the equation of the secant line $y = s(x)$ joining $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or equivalently

$$y = s(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)\,. \tag{D.2}$$

The difference v between the values of the function f and of the secant line s equals

$$v(x) = f(x) - s(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

Since f is continuous on $[a, b]$ and differentiable on (a, b), so is v. Because $f(a) = s(a)$ and $f(b) = s(b)$ we have

$$v(a) = v(b) = 0\,,$$

so that the function v satisfies the hypothesis of Rolle's Theorem on the interval $[a, b]$. Thus there is a point c in (a, b) such that $v'(c) = 0$. Using (D.2) we compute

$$v'(x) = f'(x) - s'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}\,,$$

so in particular

$$v'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}\,.$$

Thus, at the point c in (a, b), where $v'(c) = 0$, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

D.4 Taylor's Theorem

For convenience of the reader we first restate the theorem.

Theorem 5.15. Let f be differentiable up to order $n+1$ in an open interval I containing a point a. Then for each x in I, there exists a number c between x and a, that is, $x < c < a$ or $c < a < x$, such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{k}(a)}{k}(x-a)^k + R_n(x) = P_n(x) + R_n(x) \tag{D.3}$$

holds with

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\,. \tag{D.4}$$

Proof of Theorem 5.15. We fix $x \in I$, $x \neq a$, and consider the auxiliary function

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k\,. \tag{D.5}$$

We note that $g(x) = f(x) - f(x) = 0$ and that

$$g(a) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x) - P_n(x)\,. \tag{D.6}$$

From (D.5) we see that g is differentiable on (a, x) due to our assumptions on f. Using the product rule we compute its derivative (which is taken with respect to t, not to x!) as

$$\begin{aligned} g'(t) &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}k(x-t)^{k-1} \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n\,, \end{aligned} \tag{D.7}$$

because all other terms cancel out. We define a second auxiliary function

$$h(t) = g(t) - g(a)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}\,. \tag{D.8}$$

We see that

$$h(a) = 0\,, \quad h(x) = g(x) = f(x) - f(x) = 0\,.$$

Since h is differentiable between a and x, Rolle's theorem asserts that there exists a c between a and x such that

$$h'(c) = 0\,.$$

From (D.8) and (D.7) we obtain

$$\begin{aligned} 0 = h'(c) &= g'(c) + g(a) \cdot (n+1)\frac{(x-c)^n}{(x-a)^{n+1}} \\ &= -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + g(a) \cdot (n+1)\frac{(x-c)^n}{(x-a)^{n+1}}\,. \end{aligned} \tag{D.9}$$

Since $x \neq c$, we can divide by $(x-c)^n$. Rearranging then gives

$$g(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\,. \tag{D.10}$$

But we have already seen in (D.6) that $g(a) = f(x) - P_n(x)$, so (D.10) yields

$$f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

which was to be proved.

D.5 The Fundamental Theorem of Calculus

In this section, we present the precise definition of the integral and the proofs of the fundamental theorem of calculus and of related results.

Definition of the integral. As explained in the beginning of Chap. 6, the integral

$$\int_a^b f(x)\,dx$$

of the function f (which we assume to be bounded) over the interval $[a, b]$ is defined through an approximation procedure which involves Riemannian sums

$$s_\Delta = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})\,. \tag{D.11}$$

Here, Δ is a partition of the interval $[a, b]$ of the form $a = x_0 < x_1 < \cdots < x_n$, and each point ξ_k lies somewhere in the subinterval $[x_{k-1}, x_k]$. Thus, the value of s_Δ depends on the choice of Δ as well as on the choice of the points ξ_k. One can estimate the influence of the latter choice through the notion of **oscillation** of a bounded function. If I is a subset of the domain of f (here, of the interval $[a, b]$), the

oscillation of f on I is defined as the maximum possible difference of two function values on I, that is,

$$\mathrm{osc}_I(f) = \max_{x,z \in I} |f(x) - f(z)| \,. \tag{D.12}$$

We remark that it may happen that this maximum does not exist (that is, there are no points $x, z \in I$ where a maximum value is attained). One then replaces the maximum by the so-called **supremum**, which in this case is equal to the smallest number η such that $|f(x) - f(z)| \leq \eta$ for all $x, z \in I$.

Let us now consider two different Riemannian sums for the same partition Δ,

$$s_\Delta = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \,, \quad \tilde{s}_\Delta = \sum_{k=1}^{n} f(\tilde{\xi}_k)(x_k - x_{k-1}) \,. \tag{D.13}$$

We estimate their difference as

$$\begin{aligned} |s_\Delta - \tilde{s}_\Delta| &= \left| \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - \sum_{k=1}^{n} f(\tilde{\xi}_k)(x_k - x_{k-1}) \right| \\ &\leq \sum_{k=1}^{n} |f(\xi_k) - f(\tilde{\xi}_k)|(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} O_k(f)(x_k - x_{k-1}) =: V_\Delta(f) \,, \end{aligned} \tag{D.14}$$

where $O_k(f)$ denotes the oscillation of f on the subinterval $I = [x_{k-1}, x_k]$. The number $V_\Delta(f)$ defined in (D.14) is called the **oscillation sum** of f for the partition Δ.

Definition D.1 (*Integrable Function*) A bounded function $f : [a, b] \to \mathbb{R}$ is said to be **integrable on** $[a, b]$, if for every $\varepsilon > 0$ there exists a partition Δ of $[a, b]$ such that $V_\Delta(f) \leq \varepsilon$.

The latter condition means that, in view of (D.14), we can enforce the difference between different Riemannian sums for the same partition Δ to become as small as we want, if we choose Δ fine enough.

Let now $f : [a, b] \to \mathbb{R}$ be integrable. In order to define its integral, one goes through the following three steps.

1. One proves that
$$|s_\Delta - s_{\tilde{\Delta}}| < V_\Delta(f) \,,$$
whenever the partition $\tilde{\Delta}$ is a refinement of the partition Δ (that is, $\tilde{\Delta}$ is obtained from Δ by adding partition points), for all Riemannian sums s_Δ and $s_{\tilde{\Delta}}$ for those partitions.
2. One proves that
$$|s_\Delta - s_{\tilde{\Delta}}| < V_\Delta(f) + V_{\tilde{\Delta}}(f) \,,$$

for arbitrary partitions Δ and $\tilde{\Delta}$ and all Riemannian sums for those partitions.
3. One proves that there exists a unique number I such that

$$|I - s_\Delta| \leq V_\Delta(f) \tag{D.15}$$

holds for all partitions and all corresponding Riemannian sums.

We then define the **integral of f on** $[a, b]$ as

$$\int_a^b f(x)\,dx = I\,,$$

where I is the number obtained in step 3 above.

It was stated in Theorem 6.2 that every continuous function on a closed interval $[a, b]$ is integrable. Its proof uses the notion of uniform continuity. A function $f : [a, b] \to \mathbb{R}$ is called **uniformly continuous on** $[a, b]$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(z)| < \varepsilon$ holds for all $x, z \in [a, b]$ with $|x - z| < \delta$. This is similar to, but not the same as the definition of continuity. It is a theorem (not treated in this book) that every continuous function defined on a closed and bounded interval $[a, b]$ is uniformly continuous on that interval.

Proof of Theorem 6.2. Let $f : [a, b] \to \mathbb{R}$ be continuous. According to Definition D.1, for an arbitrarily given $\varepsilon > 0$ we have to find a partition Δ such that $V_\Delta(f) < \varepsilon$. Choose $\delta > 0$ small enough such that $|f(x) - f(z)| < \varepsilon/(b - a)$ whenever $|x - z| < \delta$. (This is possible since, by what we said just above, f is also uniformly continuous.) Next, choose a natural number n large enough such that $(b - a)/n < \delta$. Take as partition Δ the equidistant partition of $[a, b]$ with $x_k - x_{k-1} = (b - a)/n$. Then the oscillation $O_k(f)$ of f on $[x_{k-1}, x_k]$ satisfies $O_k(f) \leq \varepsilon/(b - a)$ for every k, hence

$$V_\Delta(f) = \sum_{k=1}^{n} O_k(f)(x_k - x_{k-1}) \leq n \frac{\varepsilon}{b - a} \cdot \frac{b - a}{n} = \varepsilon\,.$$

Therefore, f is integrable.

Let us remark that there are different ways of defining the integral. We have chosen a method which can be generalized conveniently to the case of double and triple integrals, see Sect. D.10 below.

The fundamental theorem of calculus. We present the proofs of both parts of this theorem, as well as that of the mean value theorem for integrals, Theorem 6.7. We begin with the latter.

Proof of Theorem 6.7. Since f is continuous, it attains its maximum and minimum on $[a, b]$ by Theorem 2.8. Let

$$M = \max_{x\in[a,b]} f(x)\,, \quad m = \min_{x\in[a,b]} f(x)\,.$$

Since $m \le f(x) \le M$ for all $x \in [a, b]$, due to (6.25) we have

$$m(b-a) = \int_a^b m\, dx \le \int_a^b f(x)\, dx \le \int_a^b M\, dx = M(b-a)\,.$$

We divide by $b - a$ and obtain

$$m \le \frac{1}{b-a} \int_a^b f(x)\, dx \le M\,.$$

By the intermediate value Theorem 2.7, f attains every value between m and M. Therefore there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx\,.$$

The theorem is proved.

Proof of Part 1, Theorem 6.3. To establish this theorem, we must show that if x is an arbitrary point in $[a, b]$, then $F'(x) = f(x)$, that is,

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)\,.$$

In order to prove this, fix $x \in [a, b]$ and let h be any number such that $x + h \in [a, b]$. Using the definition of F together with property (6.26) yields

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \\ &= \int_a^x f(t)\, dt + \int_x^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \\ &= \int_x^{x+h} f(t)\, dt\,. \end{aligned}$$

Consequently, if $h \ne 0$, then

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\, dt\,.$$

In the case $h > 0$, by the mean value Theorem 6.7, just proved above, we can find a number $z = z(h)$ in the open interval $(x, x + h)$ such that

$$\int_x^{x+h} f(t)\, dt = f(z(h)) \cdot (x + h - x) = f(z(h)) \cdot h$$

and, therefore,

$$\frac{F(x+h)-F(x)}{h} = f(z(h))\,. \tag{D.16}$$

Since $x < z < x+h$, we have $\lim_{h\to 0^+} z(h) = x$. It then follows from the continuity of f that

$$\lim_{h\to 0^+} f(z(h)) = f(x)\,,$$

and we conclude from (D.16) that

$$\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = f(x)\,.$$

If $h < 0$, we may prove in a similar way that

$$\lim_{h\to 0^-} \frac{F(x+h)-F(x)}{h} = f(x)\,.$$

The two preceding one-sided limits imply that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)\,.$$

This completes the proof.

Proof of Part 2, Theorem 6.4. Let F be any antiderivative of f and let

$$G(x) = \int_a^x f(t)\,dt\,. \tag{D.17}$$

From Theorem 6.3 we know that G is an antiderivative of G, and from Theorem 6.1 we know that there is a constant C such that

$$G(x) = F(x) + C$$

for every x in $[a, b]$. Together with (D.17) this implies that

$$\int_a^x f(t)\,dt = F(x) + C$$

for every x in $[a, b]$. If we let $x = a$ and use the fact that $\int_a^a f(t)\,dt = 0$, we obtain $0 = F(a) + C$, or $C = -F(a)$. If we let $x = b$, we arrive at

$$\int_a^b f(t)\,dt = F(b) + C = F(b) - F(a)\,.$$

The theorem is proved. (Recall that it is irrelevant whether the integration variable is denoted by x or by t.)

The integral test. Finally we prove Theorem 6.8 which says that for continuous positive nonincreasing functions the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Proof of Theorem 6.8. First, assume that $\int_1^{\infty} f(x)\,dx$ converges. We define a function $\varphi : [1,\infty) \to \mathbb{R}$ by setting $\varphi(x) = f(n)$ if $n-1 \le x < n$. Since f is nonincreasing, we have $\varphi(x) \le f(x)$ for all $x \ge 1$. For every natural number $N \ge 2$ we therefore obtain, using (6.25),

$$\begin{aligned} 0 \le \sum_{n=2}^{N} f(n) &= \sum_{n=2}^{N} \int_{n-1}^{n} \varphi(x)\,dx = \int_1^N \varphi(x)\,dx \le \int_1^N f(x)\,dx \\ &\le \int_1^{\infty} f(x)\,dx < \infty\,. \end{aligned}$$

Thus, the partial sums of the series $\sum_{n=1}^{\infty} f(n)$ are bounded by $\int_1^{\infty} f(x)\,dx$, and hence the series converges. To prove the converse, assume that $\sum_{n=1}^{\infty} f(n)$ converges. We define a function $\psi : [1,\infty) \to \mathbb{R}$ by setting $\psi(x) = f(n-1)$ if $n-1 \le x < n$. Since f is nonincreasing, we have $\psi(x) \ge f(x)$ for all $x \ge 1$. For every natural number $N \ge 2$ we therefore obtain, using (6.25),

$$\begin{aligned} 0 \le \int_1^N f(x)\,dx \le \int_1^N \psi(x)\,dx &= \sum_{n=2}^{N} \int_{n-1}^{n} \psi(x)\,dx = \sum_{n=2}^{N} f(n-1) \\ &\le \sum_{n=1}^{\infty} f(n) < \infty\,. \end{aligned} \tag{D.18}$$

The function $F : [1,\infty) \to \mathbb{R}$ defined by

$$F(t) = \int_1^t f(x)\,dx$$

is nondecreasing and, due to (D.18), bounded by the finite number $\sum_{n=1}^{\infty} f(n)$. Therefore, the improper limit $\lim_{t\to\infty} F(t)$ exists which means that the improper integral converges.

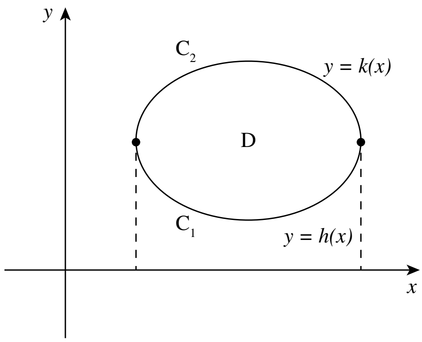

Fig. D.1 Special form of the region D

D.6 The Green–Ostrogradski Theorem

For convenience of the reader, we restate the theorem.

Theorem 9.8. Let D be a bounded domain in the plane whose boundary C is a closed, simple and positively oriented curve. Let $\mathbf{F} = (f, g)$ be a vector field whose components are continuously differentiable. Then

$$\oint_C f(x, y)\, dx + g(x, y)\, dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA\,. \tag{D.19}$$

Proof of Theorem 9.8. In this proof we restrict ourselves to the case where the region D has the form indicated in Fig. D.1. First, we consider the case $g = 0$. The boundary C of D consists of a lower part C_1 which is the graph of some function $y = h(x)$, and of an upper part C_2 which is the graph of some function $y = k(x)$; again we refer to Fig. D.1. The line integral becomes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}\,,$$

where, due to the counterclockwise orientation, C_1 is traversed from left to right, and C_2 from right to left. We already have computed (see formula (9.89)) that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^b f(x, h(x))\, dx\,,$$

and analogously we obtain (note the reversal of the integration limits)

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_b^a f(x, k(x))\, dx\,.$$

Taken together the formulas above yield

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b f(x, h(x)) - f(x, k(x))\, dx\,. \tag{D.20}$$

Now we consider the double integral. The region D has the form

$$D = \{(x, y) : a \leq x \leq b\,,\ h(x) \leq y \leq k(x)\}\,.$$

According to (8.46), we compute the double integral on the right-hand side of (D.19) for $g = 0$, using the fundamental theorem of calculus,

$$\begin{aligned} \iint_D -\frac{\partial f}{\partial y}(x, y)\, dA &= -\int_a^b \int_{h(x)}^{k(x)} \frac{\partial f}{\partial y}(x, y)\, dy\, dx \\ &= -\int_a^b f(x, k(x)) - f(x, h(x))\, dx\,. \end{aligned}$$

From (D.20) we see that in the case $g = 0$ indeed

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f(x, y)\, dx = \iint_D -\frac{\partial f}{\partial y}(x, y)\, dA\,. \tag{D.21}$$

An analogous proof for the case $f = 0$, taking $\tilde{F} = (0, g)$, shows that

$$\oint_C \tilde{\mathbf{F}} \cdot d\mathbf{r} = \oint_C g(x, y)\, dy = \iint_D \frac{\partial g}{\partial x}(x, y)\, dA\,. \tag{D.22}$$

(In that proof, we decompose the boundary C of D into a left part and a right part, described by some functions $x = \tilde{h}(y)$ and $x = \tilde{k}(y)$, respectively.) For the general case, we decompose $(f, g) = (f, 0) + (0, g)$, and add (D.21) and (D.22). The theorem is proved for domains D of the form in Fig. D.1.

For more general domains D, the two-dimensional variant of the divergence theorem of Gauss can be used conveniently to prove Theorem 9.8.

D.7 The Divergence Theorem of Gauss

The Gauss divergence theorem, Theorem 9.9, asserts that

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \operatorname{div} \mathbf{F}\, dV\,. \tag{D.23}$$

Here, D is a domain in $\mathbb{R}^3$ with boundary Σ and outer unit normal field $\mathbf{n}$. We present a proof of Theorem 9.9 for the special case where D and Σ can be represented, with respect to the coordinate directions x, y and z, in the following way. With respect to

the z-coordinate, we assume that D has the form

$$D = \{(x, y, z) : k(x, y) < z < h(x, y)\,,\ (x, y) \in D_z\}\,, \tag{D.24}$$

where $D_z = p_z(D)$ with $p_z(x, y, z) = (x, y, 0)$ being the projection of D onto the xy-plane, and k and h are certain functions. Moreover, we assume that the boundary surface Σ of D consists of an upper part Σ_+ described as $z = h(x, y)$, and a lower part Σ_- described as $z = k(x, y)$, and possibly a part Σ_0 which is vertical to the xy-plane (that is, the normals to Σ_0 are perpendicular to the z-direction).

Before proceeding further, let us illustrate this situation with two examples. If D is a ball bounded by a sphere Σ, the surfaces Σ_+ and Σ_- are the upper and lower half sphere, respectively, while Σ_0 is empty. If D is a rectangular box parallel to the coordinate axes, Σ_+ and Σ_- are the rectangles forming the top and the bottom, respectively, while Σ_0 consists of the four vertical sides.

We assume moreover that D admits analogous representations with respect to the x- and the y-coordinate.

Proof of Theorem 9.9. Recall that for $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$, its divergence is given by

$$\operatorname{div}\mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\,.$$

Due to the form of D described above, we can decompose a volume integral over D into an outer integral over the two-dimensional projection D_z and an integral over intervals in the z direction. We compute (using Fubini's theorem, and then the fundamental theorem of calculus)

$$\begin{aligned}\iiint_D \frac{\partial f_3}{\partial z}\,dV &= \iint_{D_z}\left[\int_{k(x,y)}^{h(x,y)} \frac{\partial f_3}{\partial z}(x, y, z)\,dz\right]dx\,dy\\ &= \iint_{D_z}\Big[f_3(x, y, h(x, y)) - f_3(x, y, k(x, y))\Big]\,dx\,dy\\ &= \iint_{D_z} f_3(x, y, h(x, y))\,dx\,dy - \iint_{D_z} f_3(x, y, k(x, y))\,dx\,dy\,.\end{aligned} \tag{D.25}$$

The surface integral of $\mathbf{F}\cdot\mathbf{n} = f_1n_1 + f_2n_2 + f_3n_3$ decomposes into

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_\Sigma f_1n_1\,d\sigma + \iint_\Sigma f_2n_2\,d\sigma + \iint_\Sigma f_3n_3\,d\sigma\,. \tag{D.26}$$

For the third term on the right-hand side, we consider the partition of Σ as described above,

$$\iint_\Sigma f_3n_3\,d\sigma = \iint_{\Sigma_+} f_3n_3\,d\sigma + \iint_{\Sigma_-} f_3n_3\,d\sigma + \iint_{\Sigma_0} f_3n_3\,d\sigma\,. \tag{D.27}$$

On $\varSigma_0$, the unit outer normal $\mathbf{n}$ is perpendicular to the z-direction, so $n_3 = 0$ on $\varSigma_0$ and the corresponding integral vanishes. We consider $\varSigma_+$. Since $\varSigma_+$ is given as $z = h(x, y)$ and D lies below $\varSigma_+$ the unit outer normal $\mathbf{n}$ points upward. In the point, (x, y, z) with $z = h(x, y)$ it is given by

$$\mathbf{n}(x, y, z) = \frac{1}{\nu}\big(- \partial_x h\mathbf{i} - \partial_y h\mathbf{j} + \mathbf{k}\big), \quad \nu = \sqrt{1 + (\partial_x h)^2 + (\partial_y h)^2}, \tag{D.28}$$

where the partial derivatives $\partial_x h$ and $\partial_y h$ are evaluated at (x, y). We thus obtain

$$n_3 = \frac{1}{\nu} \tag{D.29}$$

for its third component. We transform the surface integral over $\varSigma_+$ into a two-dimensional integral over the region D_z, according to Definition 9.20,

$$\begin{aligned}\iint_{\varSigma_+} f_3 n_3\, d\sigma = \iint_{\varSigma_+} \frac{f_3}{\nu}\, d\sigma = \iint_{D_z} \frac{f_3}{\nu} \cdot \sqrt{1 + (\partial_x h)^2 + (\partial_y h)^2}\, dA \\ = \iint_{D_z} f_3(x, y, h(x, y))\, dx\, dy\,.\end{aligned} \tag{D.30}$$

The surface integral over $\varSigma_-$ is treated analogously; there, however, the outer normal vector points downward, so we have $n_3 = -1/\nu$ instead of (D.29). Consequently,

$$\iint_{\varSigma_-} f_3 n_3\, d\sigma = - \iint_{D_z} f_3(x, y, k(x, y))\, dx\, dy\,. \tag{D.31}$$

Putting together (D.25), (D.27), (D.30) and (D.31) we arrive at

$$\iint_{\varSigma} f_3 n_3\, d\sigma = \iiint_D \frac{\partial f_3}{\partial z}\, dV\,. \tag{D.32}$$

Working with the representations of D with respect to the x- and the y-coordinate, one obtains in an analogous manner that

$$\iint_{\varSigma} f_1 n_1\, d\sigma = \iiint_D \frac{\partial f_1}{\partial x}\, dV\,, \quad \iint_{\varSigma} f_2 n_2\, d\sigma = \iiint_D \frac{\partial f_2}{\partial y}\, dV\,. \tag{D.33}$$

Adding the three equation in (D.32) and (D.33) yields (D.23). This completes the proof for the special form of D as considered.

To prove the divergence theorem for domains D of general form, one employs so-called **partitions of unity** of the vector field $\mathbf{F}$, which reduces the general situation to a situation where similar computations can be done as in the proof presented above. In addition, let us remark that, while we have treated the situation in three-dimensional space, more or less the same proof works for the case of an n-dimensional region D

bounded by an $n-1$-dimensional surface Σ, where n is an arbitrary number greater or equal to 2. Both these developments are, however, outside the scope of this book.

D.8 Stokes' Theorem

This section is devoted to the proof of the theorem of Stokes, Theorem 9.12. It states that under suitable assumptions, introduced prior to this theorem in Sect. 9.5.3,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_\Sigma (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,d\sigma\,, \tag{D.34}$$

where $\mathbf{F}$ is a vector field, Σ is a surface with unit normal field $\mathbf{n}$ and boundary curve C, suitably oriented.

Proof of Theorem 9.12. The strategy of the proof is to transform the situation to the xy-plane, in order to apply the Green–Ostrogradski theorem. Let us first consider the surface integral on the right-hand side of (D.34). The surface Σ is described as $z = S(x, y)$, that is, as the graph of a function S defined on D, the projection of Σ onto the xy-plane. The unit normal at a point $(x, y, S(x, y))$ of Σ is given by

$$\mathbf{n} = \frac{1}{\nu}\big(-\partial_x S\mathbf{i} - \partial_y S\mathbf{j} + \mathbf{k}\big)\,, \quad \nu = \sqrt{1+(\partial_x S)^2+(\partial_y S)^2}\,, \tag{D.35}$$

where the right-hand side is evaluated at (x, y). Since

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix} \tag{D.36}$$

$$= (\partial_y f_3 - \partial_z f_2)\mathbf{i} + (\partial_z f_1 - \partial_x f_3)\mathbf{j} + (\partial_x f_2 - \partial_y f_1)\mathbf{k}\,, \tag{D.37}$$

the integrand of the surface integral becomes the scalar function

$$(\operatorname{curl}\mathbf{F})\cdot\mathbf{n} = \frac{1}{\nu}\Big((\partial_y f_3 - \partial_z f_2)\cdot(-\partial_x S) + (\partial_z f_1 - \partial_x f_3)\cdot(-\partial_y S) + (\partial_x f_2 - \partial_y f_1)\Big)\,, \tag{D.38}$$

to be evaluated at points of Σ. Using Definition 9.20 we can transform the surface integral into a double integral over the domain D,

$$\begin{aligned}\iint_\Sigma (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,d\sigma &= \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\cdot\nu\,dA\\ &= \iint_D \Big((\partial_y f_3-\partial_z f_2)\cdot(-\partial_x S)+(\partial_z f_1-\partial_x f_3)\cdot(-\partial_y S)\\ &\qquad +(\partial_x f_2-\partial_y f_1)\Big)\,dA\end{aligned} \tag{D.39}$$

Let us now consider the line integral on the left side of (D.34). If C is parametrized by $\mathbf{r}:[a,b]\to\mathbb{R}^3$, then by the definition of the line integral we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt\,. \tag{D.40}$$

Since Σ is the graph of S defined on D, C is the graph of S restricted to the boundary Γ of D. Let Γ be positively oriented by the parametrization $\mathbf{q}:[a,b]\to\mathbb{R}^2$, set

$$\mathbf{r}(t) = q_1(t)\mathbf{i}+q_2(t)\mathbf{j}+S(q_1(t),q_2(t))\mathbf{k}\,. \tag{D.41}$$

Using the chain rule we obtain

$$\mathbf{r}'(t) = q_1'(t)\mathbf{i}+q_2'(t)\mathbf{j}+\Big(\partial_x S\cdot q_1'(t)+\partial_y S\cdot q_2'(t)\Big)\mathbf{k}\,, \tag{D.42}$$

where $\partial_x S$ and $\partial_y S$ are evaluated at $\mathbf{q}(t)=(q_1(t),q_2(t))$. Let us now define the plane vector field $\tilde{\mathbf{F}}$ by

$$\begin{aligned}\tilde{\mathbf{F}}_1(x,y) &= f_1(x,y,S(x,y))+f_3(x,y,S(x,y))\partial_x S(x,y)\\ \tilde{\mathbf{F}}_2(x,y) &= f_2(x,y,S(x,y))+f_3(x,y,S(x,y))\partial_y S(x,y)\,.\end{aligned} \tag{D.43}$$

Setting $x=q_1(t)$ and $y=q_2(t)$ we see in view of (D.41) and (D.42) that

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = \tilde{\mathbf{F}}(\mathbf{q}(t))\cdot\mathbf{q}'(t) \tag{D.44}$$

for all $t\in[a,b]$. (This is the reason for defining $\tilde{\mathbf{F}}$ by (D.43).)

The theorem of Green and Ostrogradski, Theorem 9.8, asserts that

$$\oint_\Gamma \tilde{\mathbf{F}}\cdot d\mathbf{r} = \iint_D \Big(\partial_x\tilde{\mathbf{F}}_2-\partial_y\tilde{\mathbf{F}}_1\Big)\,dA\,. \tag{D.45}$$

We compute the partial derivatives from (D.43) with the aid of the chain rule,

$$\begin{aligned}\partial_y \tilde{\mathbf{F}}_1 &= \partial_y f_1 + \partial_z f_1 \cdot \partial_y S + (\partial_y f_3 + \partial_z f_3 \cdot \partial_y S)\partial_x S \\ &\quad + f_3 \cdot \partial_y \partial_x S\,, \\ \partial_x \tilde{\mathbf{F}}_2 &= \partial_x f_2 + \partial_z f_2 \cdot \partial_x S + (\partial_x f_3 + \partial_z f_3 \cdot \partial_x S)\partial_y S \\ &\quad + f_3 \cdot \partial_x \partial_y S\,.\end{aligned} \tag{D.46}$$

Since the function S was assumed to have continuous second partial derivatives, we can interchange their order, so we have $\partial_x \partial_y S = \partial_y \partial_x S$. Thus we obtain from (D.46) that

$$\partial_x \tilde{\mathbf{F}}_2 - \partial_y \tilde{\mathbf{F}}_1 = (\partial_z f_2 - \partial_y f_3) \cdot \partial_x S + (\partial_x f_3 - \partial_z f_1) \cdot \partial_y S + (\partial_x f_2 - \partial_y f_1)\,. \tag{D.47}$$

We compare this expression with the corresponding one in (D.39) and find that

$$\iint_D \left(\partial_x \tilde{\mathbf{F}}_2 - \partial_y \tilde{\mathbf{F}}_1\right) dA = \iint_\Sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma\,. \tag{D.48}$$

We now put together the previous calculations and finally conclude that

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b \tilde{\mathbf{F}}(\mathbf{q}(t)) \cdot \mathbf{q}'(t)\, dt \\ &= \oint_\Gamma \tilde{\mathbf{F}} \cdot d\mathbf{r} = \iint_D \left(\partial_x \tilde{\mathbf{F}}_2 - \partial_y \tilde{\mathbf{F}}_1\right) dA \\ &= \iint_\Sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma\,.\end{aligned}$$

The proof is complete.

D.9 Conservative fields

In this section, we prove the two Theorems 9.4 and 9.5 concerning conservative vector fields.

Proof of Theorem 9.4. Let $\mathbf{r} : [a, b] \to \mathbb{R}^3$ be a parametrization of C, so $\mathbf{A} = \mathbf{r}(a)$ and $\mathbf{B} = \mathbf{r}(b)$. We set $g(t) = \psi(\mathbf{r}(t))$. From the chain rule we get

$$g'(t) = \nabla\psi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,.$$

We compute

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_a^b \nabla\psi(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt\\ &= \int_a^b g'(t)\,dt = g(b)-g(a) = \psi(\mathbf{r}(b))-\psi(\mathbf{r}(a))\\ &= \psi(\mathbf{B})-\psi(\mathbf{A})\,.\end{aligned}$$

In the middle line of this computation, we have used the fundamental theorem of calculus. Indeed, one may view Theorem 9.4 as a generalization of the fundamental theorem of calculus to line integrals.

Proof of Theorem 9.5. Since we already know that every conservative vector field is circulation free, it remains to prove that every vector field, which is circulation free, is conservative. Let $\mathbf{F}$ be circulation free. We first show that $\mathbf{F}$ has the property of path independence. If C_1 and C_2 are two curves with initial point A and end point B, let C denote the curve which first connects A to B via C_1 and then B to A via C_2 in the opposite direction (the latter curve we denote by $-C_2$). Since $\mathbf{F}$ is circulation free,

$$0 = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} - \int_{C_2}\mathbf{F}\cdot d\mathbf{r}\,,$$

thus

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r}\,.$$

Therefore, the line integral is path independent. We now fix a point P in D and define a function ψ by

$$\psi(\mathbf{x}) = \int_{C_x}\mathbf{F}\cdot d\mathbf{r}\,,\quad \mathbf{x}\in D\,, \tag{D.49}$$

where C_x is a curve which connects P to $\mathbf{x}$. (Since the line integral is path independent, it does not matter which curve we choose.) We claim that ψ is a potential for $\mathbf{F}$ in D. To this end, fix $\mathbf{x}\in D$ and choose $h>0$ so small that the line segment L from $\mathbf{x}$ to $\mathbf{x}+h\mathbf{e}_1$ lies in D, where $\mathbf{e}_1$ denotes the unit vector in the x-direction. Since we obtain a curve from P to $\mathbf{x}+h\mathbf{e}_1$ by first traversing C_x and then L, we see from the definition of ψ that

$$\psi(\mathbf{x}+h\mathbf{e}_1) = \psi(\mathbf{x}) + \int_L \mathbf{F}\cdot d\mathbf{r}\,. \tag{D.50}$$

We parametrize L by $\mathbf{r}(t) = \mathbf{x}+t\mathbf{e}_1$ with $0\le t\le h$, then $\mathbf{r}'(t)=\mathbf{e}_1$ and

$$\int_L \mathbf{F}\cdot d\mathbf{r} = \int_0^h \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{e}_1\,dt = \int_0^h f_1(\mathbf{r}(t))\,dt\,,$$

therefore

$$\frac{\psi(\mathbf{x}+h\mathbf{e}_1)-\psi(\mathbf{x})}{h}=\frac{1}{h}\int_0^h f_1(\mathbf{r}(t))\,dt\,.$$

Since the limit of the right-hand side exists as $h \to 0$ and is equal to $f_1(\mathbf{r}(0)) = f_1(\mathbf{x})$, we obtain

$$\frac{\partial\psi}{\partial x}(\mathbf{x}) = f_1(\mathbf{x})\,,\quad \mathbf{x}\in D\,.$$

An analogous argument works for the other coordinate directions, so we finally conclude that $\nabla\psi = \mathbf{F}$. The proof is complete.

D.10 Definition of Multiple Integrals

In this section, we present the definition of double and triple integrals,

$$\iint_Q f(x,y)\,dA\,,\quad \text{respectively}\quad \iiint_Q f(x,y,z)\,dV\,,$$

over rectangular regions Q. The exposition closely parallels that given in Appendix D.5 for the case of a single variable. We, therefore, shorten it here somewhat and refer the reader to Sect. D.5 for some more details.

The double integral. Let the function f be defined in a rectangular region $Q = [a,b]\times[c,d]$ of the xy-plane. We partition Q into rectangular subregions as follows. Choose a partition of $[a,b]$ of the form $a = x_0 < x_1 < \cdots < x_n = b$, and a partition of $[c,d]$ of the form $c = y_0 < y_1 < \cdots < y_m = d$. This gives us a partition Δ of Q into nm rectangles

$$Q_{ij} = \{(x,y) : x_{i-1}\le x\le x_i\,,\ y_{j-1}\le y\le x_j\}\,,\quad 1\le i\le n\,,\ 1\le j\le m\,.$$

A Riemannian sum for the partition Δ is defined as

$$s_\Delta = \sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_i,\eta_j)(x_i-x_{i-1})(y_j-y_{j-1})\,. \tag{D.51}$$

Here, each point (ξ_i,η_j) lies somewhere in the rectangle Q_{ij}. We define the oscillation of f on the rectangles Q_{ij} as

$$O_{ij}(f) = \mathrm{osc}_{Q_{ij}}(f) = \max_{x,z\in Q_{ij}} |f(x)-f(z)|\,. \tag{D.52}$$

Again, we have to replace the maximum by the supremum if the former does not exist. Next, we consider two different Riemannian sums for the same partition Δ,

$$
\begin{aligned}
s_\Delta &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}) \,, \\
\tilde{s}_\Delta &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(\tilde{\xi}_i, \tilde{\eta}_j)(x_i - x_{i-1})(y_j - y_{j-1}) \,.
\end{aligned}
\tag{D.53}
$$

Their difference can be estimated as

$$
|s_\Delta - \tilde{s}_\Delta| = \sum_{i=1}^{n}\sum_{j=1}^{m} O_{ij}(f)(x_i - x_{i-1})(y_j - y_{j-1}) =: V_\Delta(f) \,. \tag{D.54}
$$

The number $V_\Delta(f)$ defined in (D.54) is called the **oscillation sum** of f for the partition Δ.

Definition D.2 (Integrable Function) A bounded function $f : Q \to \mathbb{R}$ is said to be **integrable on** Q, if for every $\varepsilon > 0$ there exists a partition Δ of Q such that $V_\Delta(f) \leq \varepsilon$.

The latter condition means that, in view of (D.54), we can enforce the difference between different Riemannian sums for the same partition Δ to become as small as we want, if we choose Δ fine enough.

Let now $f : Q \to \mathbb{R}$ be integrable. In order to define its integral, one goes through the same three steps as we did in Sect. D.5.

1. One proves that
$$
|s_\Delta - s_{\tilde{\Delta}}| < V_\Delta(f) \,,
$$
whenever the partition $\tilde{\Delta}$ is a refinement of the partition Δ (that is, $\tilde{\Delta}$ is obtained from Δ by adding partition points), for all Riemannian sums s_Δ and $s_{\tilde{\Delta}}$ for those partitions.
2. One proves that
$$
|s_\Delta - s_{\tilde{\Delta}}| < V_\Delta(f) + V_{\tilde{\Delta}}(f) \,,
$$
for arbitrary partitions Δ and $\tilde{\Delta}$ and all Riemannian sums for those partitions.
3. One proves that there exists a unique number I such that
$$
|I - s_\Delta| \leq V_\Delta(f) \tag{D.55}
$$
holds for all partitions and all corresponding Riemannian sums.

We then define the **integral of** f **on** Q as

$$
\iint_Q f(x, y)\, dA = I \,,
$$

where I is the number obtained in step 3 above.

Theorem .1 *Let $f : Q \to \mathbb{R}$ be continuous. Then f is integrable on Q.*

The proof of this theorem is the same as that given in Sect. D.5 for a function of a single variable, except that one has to replace the intervals by rectangles.

The triple integral. Let the function f be defined in a rectangular region $Q = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ of three-dimensional space. We partition each of those intervals,

$$a_1 = x_0 < \cdots x_n = b_1 \,, \quad a_2 = y_0 < \cdots y_m = b_2 \,, \quad a_3 = z_0 < \cdots z_l = b_3 \,.$$

The partition Δ now consists of rectangular regions

$$Q_{ijk} = [x_i, x_{i-1}] \times [y_j, y_{j-1}] \times [z_k, z_{k-1}] \,.$$

The Riemannian sums have the form

$$s_\Delta = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} f(\xi_i, \eta_j, \zeta_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) \,,$$

where (ξ, η_j, ζ_k) lies somewhere in Q_{ijk}. From this point onwards, the definition of

$$\iiint_Q f(x, y, z)\, dV$$

proceeds in a completely analogous manner as in the case of the double integral.

Solutions of Selected Exercises

Solutions to the Exercises of Chap. 1

1.8.1 $f(x) = \dfrac{1}{x},\ f(\dfrac{3}{5}) = \dfrac{5}{3},\ f(-\dfrac{2}{7}) = -\dfrac{7}{2}.$

1.8.2 $f(x) = |x| - x,\ f(2) = 0,\ f(-2) = |-2| - (-2) = 2 + 2 = 4,\ f(50) = 0,\ f(-40) = 80.$

1.8.3 $f(x) = \dfrac{1}{x^2 - 3}$ is defined if $x^2 - 3 \neq 0$ or equivalently $x^2 \neq 3$, that is, $x \neq \pm\sqrt{3}$. The domain of f is the set of all real numbers different from $-\sqrt{3}$ and $\sqrt{3}$, that is, it is the union of three intervals, $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$. For $x = 5$ we get $f(x) = \dfrac{1}{5^2 - 3} = \dfrac{1}{25 - 3} = \dfrac{1}{22}$.

1.8.4 (a) $f(x) = \sqrt{x} + 5$. Since $\sqrt{x}$ is a real number iff $x \geq 0$, the domain of f is $[0, \infty)$ and its range is $[5, \infty)$.

(b) $f(x) = \sqrt{x - 3}$. Since $\sqrt{x - 3}$ is a real number iff $x - 3 \geq 0$, we must have $x \geq 3$. The domain of f is $[3, \infty)$ and its range is $[0, \infty)$.

(c) $f(x) = \dfrac{1}{\sqrt{9 - x^2}}$. The number $\sqrt{9 - x^2}$ is a real number iff $9 - x^2 \geq 0$, for which $x^2 \leq 9$, that is, $-3 \leq x \leq 3$. But at $x = \pm 3$, $\sqrt{9 - x^2} = 0$ and its reciprocal is not defined. Therefore, the domain of f is $(-3, 3)$ and its range is $[\frac{1}{3}, \infty)$.

(d) $f(x) = |x - 2|$. The domain of f is $(-\infty, \infty)$, and its range is $[0, \infty)$.

(e) $f(x) = x^2 - 6$. The domain of f is $(-\infty, \infty)$, its range is $[-6, \infty)$.

1.8.5 Let h denote the length of the box and x the edge length of the cross section, which is a square. The girth (the perimeter of this square) has length $4x$. We must have $4x + h = 108$, Therefore $h = 108 - 4x$. The volume V of the box equals $x^2 h$. As a function of x only, it becomes

$$V(x) = x^2(108 - 4x) = 108x^2 - 4x^3.$$

M. Brokate et al., *Calculus for Scientists and Engineers*, Industrial and Applied Mathematics, https://doi.org/10.1007/978-981-13-8464-6

Since edge length of the square and the length of the box cannot be negative, we must have $x \geq 0$ and $h = 108 - 4x \geq 0$, that is, $x \leq 27$. Therefore, $0 \leq x \leq 27$, and the domain of the function V is [0, 27].

1.8.6 Let the radius of the can be r and the height be h. Its total surface area S is given by

$$S = 2\pi r^2 + 2\pi r h\,.$$

Its volume V has to satisfy $V = \pi r^2 h = 22$, therefore $h = \dfrac{22}{\pi r^2}$. We thus can write S as a function of r only,

$$S(r) = 2\pi r^2 + 2\pi r \cdot \frac{22}{\pi r^2} = 2\pi r^2 + \frac{44}{r}\,.$$

Since r can take any positive value, the domain of S is $(0, \infty)$.

1.8.7

$$\begin{aligned}
&\text{For } a = 2 \text{ and } x = 4 \text{ we obtain } a^x = 2^4 = 16,\\
&\text{for } a = 2 \text{ and } x = -1 \text{ we obtain } a^x = 5^{-1} = 1/5,\\
&\text{for } a = 2 \text{ and } x = \sqrt{2} \text{ we obtain } a^x = 2^{\sqrt{2}},\\
&\text{for } a = e \text{ and } x = \sqrt{2} \text{ we obtain } a^x = e^{\sqrt{2}},\\
&\text{for } a = e \text{ and } x = \sqrt{\pi} \text{ we obtain } a^x = e^{\sqrt{\pi}}.
\end{aligned}$$

1.8.8 (a) $f(x) = x^{-5}$ is an odd function.
(b) $f(x) = x^4 + 3x^2 - 1$ is an even function.
(c) $f(x) = \dfrac{x}{x^2 - 1}$ is an odd function.
(d) $f(t) = |t^3|$ is an even function.
(e) $h(t) = \sqrt{t^4 + 3}$ is an even function.

1.8.9 Given $f(x) = x^2 + x + 1$, we get

$$f(x-4) = (x-4)^2 + (x-4) + 1 = x^2 + 16 - 8x + x - 4 + 1 = x^2 - 7x + 13\,.$$
$$f(x+4) = (x+4)^2 + (x+4) + 1 = x^2 + 16 + 8x + x + 4 + 1 = x^2 + 9x + 21\,.$$
$$f\left(\frac{1}{2}x\right) = \left(\frac{x}{2}\right)^2 + \frac{x}{2} + 1 = \frac{x^2}{4} + \frac{x}{2} + 1\,.$$
$$\begin{aligned}
f(2x-4) &= (2x-4)^2 + (2x-4) + 1 = 4x^2 - 16x + 16 + 2x - 4 + 1\\
&= 4x^2 - 14x + 13\,.
\end{aligned}$$

1.8.10 For $f(x) = x + 6$, $g(x) = x^2 - 4$ we obtain

(a) $f(g(x)) = f(x^2 - 4) = x^2 - 4 + 6 = x^2 + 2.$
(b) $f(f(2)) = f(2 + 6) = f(8) = 8 + 6 = 14.$
(c) $g(g(3)) = g(9 - 4) = g(5) = 5^2 - 4 = 21.$

(d) $f(f(x)) = f(x+6) = x + 12.$
(e) $g(f(x)) = g(x+6) = (x+6)^2 - 4 = x^2 + 12x + 32.$

1.8.11 For $f(x) = x + 1$, $g(x) = \dfrac{1}{x+1}$ we obtain

(a) $g(f(\frac{1}{2})) = g(\frac{3}{2}) = \dfrac{1}{5/2} = \dfrac{2}{5}.$
(b) $f(f(x)) = f(x+1) = x + 2.$
(c) $g(g(x)) = g(\dfrac{1}{x+1}) = \dfrac{1}{\frac{1}{x+1} + 1} = \dfrac{x+1}{x+2}.$
(d) $f(g(\frac{1}{3})) = f(\dfrac{1}{\frac{1}{3}+1}) = f(\frac{3}{4}) = \dfrac{3}{4} + 1 = \dfrac{7}{4}.$

1.8.12 (a) $(f \circ g)(x) = x$; $g(x) = \frac{1}{x}$. We compute

$$f(g(x)) = x\,, \quad \text{therefore } f\left(\frac{1}{x}\right) = x\,, \quad \text{therefore } f(x) = \frac{1}{x}\,.$$

(b) $f(x) = \dfrac{x}{x-1}$, $g(x) = \dfrac{x}{x-1}$. We compute

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{x}{x-1} \cdot \frac{x-1}{1}$$
$$= x\,.$$

(c) Inserting $f(x) = \dfrac{x}{x-1}$ into $(f \circ g)(x) = x$, we get

$$\frac{g(x)}{g(x) - 1} = x\,.$$

In order to solve for $g(x)$ we transform as $g(x) = x[g(x) - 1]$ and moreover

$$g(x)(x-1) = x\,, \quad g(x) = \frac{x}{x-1}\,.$$

1.8.16 Let $W(t)$ be the amount of solid waste at time t (in years). Then $W(1960) = 82.3$ and $W(1980) = 139.1$. The slope of the linear function W becomes

$$m = \frac{\Delta W}{\Delta t} = \frac{139.1 - 82.3}{1980 - 1960} = \frac{56.8}{20} = 2.84 \quad \text{millions of tons/year.}$$

Let the linear function be $W(t) = b + mt$. Then

$$82.3 = b + 2.84 \cdot 1960\,, \quad \text{therefore } b = -5484.1\,.$$

The equation of the function is $W(t) = -5484.1 + 2.84t$. At $t = 2020$, we get $W(t) = -5484.1 + 2.84 \cdot 2020 = 252.7$ million tons.

1.8.17 (a) Let the equation of the line be $y = f(x) = mx + b$. To find the slope m we use any two points from the table, say the first two, and obtain

$$m = \frac{\Delta y}{\Delta x} = \frac{180 - 200}{20 - 10} = \frac{-20}{10} = -2\,.$$

To find b use any point, say $x = 20$, $y = 180$,

$$180 = -2 \cdot 20 + b\,, \quad b = 220\,.$$

Hence $f(x) = 220 - 2x$.

(b)

t	40	60	80	100
f(t)	2.4	2.2	2	1.8

Table 1.8.2(a)

Since $f(t)$ increases by 0.2 for every increase of 20 in t, the table could represent a linear function with slope $\frac{-0.2}{20} = -0.01$.

x	0	2	4	6
g(x)	10	16	26	40

Table 1.8.2(b)

Between $x = 0$ and $x = 2$ the value of $g(x)$ increases by 6 as x goes up by 2. Between $x = 2$ and $x = 4$, the value of $g(x)$ increases by 10 as x goes up by 2. Since the slope of g is not constant, g cannot be a linear function (the table cannot represent a linear equation).

(c)

t	5	10	15	20
h(t)	100	90	80	70

Table 1.8.2(c)

Since $h(t)$ decreases by 10 for every increase of 5 in t, the table can represent a linear equation, arising from a linear function h with slope = $-\frac{10}{5} = -2$.

1.8.18

a (advertisement)	6	8	10	12
S (sales)	200	240	280	320

Since the increase in sales S is 40 (in 1000 Euro) for every increase of 2 (in 1000 Euro) of advertisement a, S could be a linear function

$$S(a) = ma + b\,. \qquad (*)$$

To find the slope m, choose any two points, say $(6, 200)$ and $(8, 240)$, and compute

$$m = \frac{\Delta S}{\Delta a} = \frac{240 - 200}{8 - 6} = \frac{40}{2} = 20\,.$$

To find b, we use the point $(6, 200)$ in (*),

$$200 = 20 \cdot 6 + b\,, \quad \text{so } b = 200 - 120 = 80\,.$$

The function S is therefore

$$S(a) = 20a + 80\,.$$

If 3500 Euro is spent on advertising, i.e. $a = 3.5$, we obtain the sales

$$S(3.5) = 20 \cdot 3.5 + 80 = 70 + 80 = 150$$

in thousands of Euro, or 150,000 Euro.

1.8.19 We have the relationship

$$F = mC + b\,.$$

The freezing point of water is $F = 32°$ or $C = 0°$, whereas the boiling point is $F = 212°$ or $C = 100°$. This gives the equations

$$32 = m \cdot 0 + b \quad \text{and} \quad 212 = m \cdot 100 + b\,,$$

so $b = 32$ and $m = \dfrac{(212 - 32)}{100} = \dfrac{9}{5}$. We thus can transform F into C and vice versa by the formulas

$$F = \frac{9}{5}C + 32\,, \quad C = \frac{5}{9}(F - 32)\,.$$

The Celsius equivalent of 90 °F is $C = \dfrac{5}{9}(90 - 32) = 32.2°$. The Fahrenheit equivalent of -5 °C is $F = \dfrac{9}{5}(-5) + 32 = 23°$.

1.8.20 The height of the box is x, its length is $30 - 2x$, and its width is $20 - 2x$. Since the volume V is the product of length, width and height, we obtain

$$V(x) = (30 - 2x)(20 - 2x)x = 4x^3 - 100x^2 + 600x\,.$$

1.8.21 The requirement for the volume gives $6 = 1.5yx$.

(a) $y = \dfrac{6}{1.5x} = \dfrac{4}{x}$.

(b)

$$
\begin{aligned}
S &= xy + 2\cdot 1.5x + 2\cdot 1.5y = x\cdot\frac{4}{x} + 2\cdot 1.5x + 2\cdot 1.5\cdot\frac{4}{x}\\
&= 4 + 3x + \frac{12}{x}\,.
\end{aligned}
$$

1.8.22 (a) Since the corresponding triangles (see Fig. 1.28) are similar,

$$\frac{y}{b} = \frac{y+h}{a}\,, \quad \text{or} \quad ya = by + bh\,.$$

Solving for y we get the function

$$y(h) = \frac{bh}{a-b}\,.$$

(b) The volume V becomes

$$
\begin{aligned}
V &= \frac{1}{3}\pi a^2(y+h) - \frac{1}{3}\pi b^2 y = \frac{\pi}{3}[(a^2-b^2)y + a^2h]\\
&= \frac{\pi}{3}\left[(a^2-b^2)\frac{bh}{a-b} + a^2h\right] = \frac{\pi}{3}[(a+b)bh + a^2h]\\
&= \frac{\pi}{3}h(a^2+ab+b^2)
\end{aligned}
$$

(c) We have $600 = \frac{\pi}{3}h(6^2 + 6\cdot 3 + 3^2)$, therefore

$$h = \frac{1800}{63\pi} = \frac{200}{7\pi} = \frac{200\cdot 7}{7\cdot 22} = \frac{100}{11} \approx 9.1\,ft\,.$$

1.8.24 Figure E.1 gives the graph of the function

$$y = 3\sin 2t\,.$$

The waves have a maximum of 3 and a minimum of -3. So the amplitude is 3. The graph completes one full cycle between $t = 0$ and $t = \pi$. Therefore the period is π.

1.8.25 (a) Figure E.2 shows the graph of the function

$$y = 4.9\cos\left(\frac{\pi}{6}t\right) + 5$$

(b) At $t = 12$, $y = 4.9\cos(2\pi) + 5 = 9.9$ ft.
The water level at high tide was 9.9 ft.

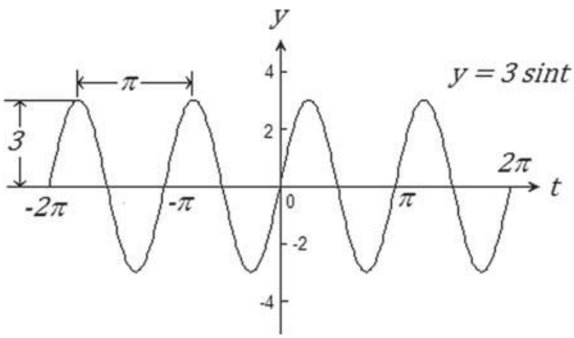

Fig. E.1 Graph of the function $y = 3\sin(2t)$

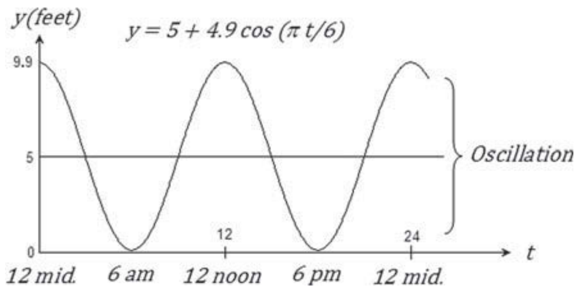

Fig. E.2 Graph of the function $y = 5 + 4.9\cos(\pi t/6)$

(c) Low tide occurs at $t = 6$ (the minimum value of $\cos\left(\frac{\pi}{6}t\right)$ equals -1) and at $t = 18$ (at 6 p.m.). The water level at this time is 0.1 ft.

(d) The period is 12 h and represents the interval between successive high tides or successive low tides. Here we have assumed the period to be 12 h. If this were correct, the high tide would always be at noon or midnight, but actually it progresses through the day. The interval between successive high tides actually averages about 12 h 24 min, and one should take this into account in a more precise mathematical model.

(e) The maximum height is 9.9 ft and the minimum is 0.1 ft, so the amplitude is

$$\frac{9.9 - 0.1}{2} = 4.9\,.$$

This is half the difference between the depths at high and low tide.

Solutions to the Exercises of Chap. 2

2.5.1

$$f(x) = \begin{cases} 1 - x^2\,, & x < 0\,, \\ \frac{1}{3}\,, & x = 0\,, \\ 1 - x\,, & x > 0\,. \end{cases}$$

Considering the first and the last line in the case distinction, we get $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = 1$. Therefore $\lim_{x\to 0} f(x)$ exists and is equal to 1. Let us remark that these results do not depend on the value of $f(0)$, or even on whether or not f is defined at $x = 0$.

2.5.2 (a) We have

$$\lim_{x\to 4} \frac{x^2 - x - 12}{x^2 - 4x} = \lim_{x\to 4} \frac{(x-4)(x+3)}{x(x-4)} = \lim_{x\to 4} \frac{x+3}{x} = \frac{7}{4}\,.$$

(b)

$$\begin{aligned}\lim_{x\to -4} \frac{x^3 + 64}{x + 4} &= \lim_{x\to -4} \frac{(x+4)(x^2 + 16 - 4x)}{(x+4)} \\ &= \lim_{x\to -4} x^2 + 16 - 4x = (-4)^2 + 16 + 16 = 48\,.\end{aligned}$$

(c) The quotient rule does not apply immediately since both numerator and denominator increase without bound as $x \to \infty$. As a first step, we divide both numerator and denominator by x^2,

$$\frac{3x^2 + 7x - 6}{4x^2 - 3x + 6} = \frac{3 + \frac{7}{x} - \frac{6}{x^2}}{4 - \frac{3}{x} + \frac{6}{x^2}}\,.$$

Since on the right-hand side we can pass to the limit as $x \to \infty$, we get

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2 + 7x - 6}{4x^2 - 3x + 6} &= \frac{\lim_{x\to\infty}\left[3 + \frac{7}{x} - \frac{6}{x^2}\right]}{\lim_{x\to\infty}\left[4 - \frac{3}{x} + \frac{6}{x^2}\right]} \\ &= \frac{\lim_{x\to\infty} 3 + 7\lim_{x\to\infty} \frac{1}{x} - 6\lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 4 - 3\lim_{x\to\infty}\left(\frac{1}{x}\right) + 6\lim_{x\to\infty} \frac{1}{x^2}} \\ &= \frac{3 + 7\cdot 0 - 6\cdot 0}{4 - 3\cdot 0 + 6\cdot 0} = \frac{3}{4}\,.\end{aligned}$$

(In general, the limit as $x \to \infty$ or x as $x \to -\infty$ of any rational function for which the degree of the numerator is less than or equal to the degree of the denominator can be found as in this example.)

(d) We know that

$$\cos x < \frac{\sin x}{x} < 1\,, \quad \text{(see Appendix C2)},$$

and that $\lim_{x\to 0} \cos x = \cos 0 = 1$ and $\lim_{x\to 0} 1 = 1$. By the Sandwich theorem,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,.$$

(e)

$$\lim_{x\to 0}(x+\cos x)=\lim_{x\to 0}x+\lim_{x\to 0}\cos x=0+1=1\,.$$

(f)

$$\lim_{x\to 0}\left(e^x+\frac{\sin x}{x}\right)=\lim_{x\to 0}e^x+\lim_{x\to 0}\frac{\sin x}{x}=1+1=2\,.$$

2.5.3

$$f(x)=\begin{cases}8\,, & x \text{ rational,}\\ 3\,, & x \text{ irrational.}\end{cases}$$

$\lim_{x\to c} f(x)$ does not exist. Intuitively, as x approaches c, x passes through both rational and irrational numbers, and $f(x)$ therefore jumps back and forth between 8 and 3. In view of the formal definition of the limit we see that no matter how small we choose an interval $I=(c-\delta, c+\delta)$ around c, there will always be rational and irrational numbers in it with function values 8 and 3, respectively. Thus, for $\varepsilon=1$ there can be no number L such that $|f(x)-L|<\varepsilon=1$ for all x in the interval I.

2.5.4 We have

$$\begin{aligned}\frac{1-\cos x}{x}&=\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}=\frac{1-\cos^2 x}{x(1+\cos x)}\\&=\frac{\sin^2 x}{x(1+\cos x)}=\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\,,\end{aligned}$$

and therefore

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right)=1\cdot 0=0\,.$$

2.5.5 We consider $f(x)=\cos(\frac{\pi}{2}x)-x^2$ on [0, 1]. Since f is the difference of two continuous functions, it is continuous. We have

$$f(0)=\cos 0-0^2=1>0\,,\quad f(1)=\cos\frac{\pi}{2}-1^2=-1<0\,.$$

Therefore by intermediate value theorem there exists at least one number $c\in(0,1)$ such that $f(c)=0$.

Solutions to the Exercises of Chap. 3

3.9.1 We have $s(t)=4t^2+3t$. The velocity v at time t is given by $v(t)=s'(t)=8t+3$. The velocities at time $t=0$ and $t=3$ are $v(0)=3$ and $v(3)=24+3=27$, respectively.

3.9.2 (i) Change in area: $A(r+h)-A(h)=\pi(r+h)^2-\pi r^2=2\pi rh+\pi h^2$.
(ii) Change in circumference: $C(r+h)-C(h)=2\pi(r+h)-2\pi r=2\pi h$.

3.9.3 Hint: Use the method of induction (see Chap. 1).

3.9.4 (i) For $f(x) = (2x^5 - x)(x^3 + 1)$ we get

$$f'(x) = (2x^5 - x) \cdot 3x^2 + (10x^4 - 1)(x^3 + 1) = 16x^7 + 10x^4 - 4x^3 - 1\,.$$

(ii) For $f(x) = 10x^{-4} + 3x^{-2}$ we get $f'(x) = -40x^{-5} - 6x^{-3}$.

(iii) For $f(x) = \dfrac{-3x^3 - 1}{2x^2 + 1}$ we get

$$\begin{aligned} f'(x) &= \frac{(2x^2+1)(-9x^2) - (-3x^3 - 1) \cdot 4x}{(2x^2+1)^2} \\ &= \frac{-18x^4 - 9x^2 + 12x^4 + 4x}{(2x^2+1)^2} = \frac{-6x^4 - 9x^2 + 4x}{(2x^2+1)^2} \\ &= \frac{-x(6x^3 + 9x - 4)}{(2x^2+1)^2}\,. \end{aligned}$$

(iv) For $f(x) = (x^2 + 1)(x - 1)(x + 5)$ we get

$$\begin{aligned} f'(x) &= (x^2+1)(x - 1 + x + 5) + (x - 1)(x + 5)2x \\ &= (x^2+1)(2x + 4) + (x^2 + 4x - 5)2x \\ &= 2x^3 + 4x^2 + 2x + 4 + 2x^3 + 8x^2 - 10x \\ &= 4x^3 + 12x^2 - 8x + 4 = 4(x^3 + 3x^2 - 2x + 1)\,. \end{aligned}$$

(v) For $f(x) = (1 + x^2)x^3 e^x \ \ln x = (x^3 + x^5)e^x \ \ln x$ we get

$$f'(x) = (3x^2 + 5x^4)e^x \ \ln x + (x^3 + x^5)\left(\frac{e^x}{x} + e^x \ \ln x\right)\,.$$

(vi) For $f(x) = \ln(1 + 3x^2)$ we get $f'(x) = \dfrac{6x}{1 + 3x^2}$.

(vii) For $f(x) = e^{x^2}$ we get $f'(x) = e^{x^2} \cdot 2x$.

3.9.5 (a) For $y = x \ln x - x$ we get

$$y' = \frac{x}{x} + \ln x - 1 = \ln x\,, \quad y'' = \frac{1}{x}\,, \quad y''' = -\frac{1}{x^2}\,.$$

(b) For $y = \dfrac{1}{\sqrt{x^2 + 4}} = (x^2 + 4)^{-\frac{1}{2}}$ we get

$$\begin{aligned} y' &= -\frac{1}{2} 2x(x^2 + 4)^{-\frac{3}{2}} = -x(x^2 + 4)^{-\frac{3}{2}} \\ y'' &= -(x^2 + 4)^{-\frac{3}{2}} + \frac{3}{2} x(x^2 + 4)^{-\frac{5}{2}} \cdot 2x \end{aligned}$$

$$= -(x^2+4)^{-\frac{3}{2}} + 3x^2(x^2+4)^{-\frac{5}{2}}$$
$$y''' = \frac{3}{2}(x^2+4)^{-\frac{5}{2}}2x - 3x^2 \cdot \frac{5}{2}(x^2+4)^{-\frac{7}{2}} \cdot 2x + 6x(x^2+4)^{-\frac{5}{2}}\,.$$

(c) For $y = e^{2x}(e^{2x} - e^{-2x}) = e^{4x} - 1$, we get

$$y' = 4e^{4x}\,, \quad y'' = 16e^{4x}\,, \quad y''' = 64e^{4x}\,.$$

3.9.6 (a) For $f(t) = t^{100} + t^{40} + t^2$ we get

$$f'(t) = 100t^{99} + 40t^{39} + 2t\,, \quad f''(t) = 9900t^{98} + 40 \times 39t^{38} + 2\,,$$
$$f'''(t) = 9900 \cdot 98t^{97} + 40 \cdot 39 \cdot 38t^{37}\,.$$

(b) For $f(t) = (3t+5)^2$ we get

$$f'(t) = 2(3t+5)\cdot 3\,, \quad f''(t) = 6\cdot 3 = 18\,, \quad f'''(t) = 0\,.$$

(c) For $f(t) = t^5$ we get

$$f'(t) = 5t^4\,, \quad f''(t) = 20t^3\,, \quad f'''(t) = 60t^2\,.$$

3.9.7 We have $f(t) = g(t)^2$ for all t, so $f'(t) = 2g(t)g'(t)$ for all t and therefore $f'(1) = 2g(1)g'(1)$.

3.9.8 We have $y^5 + xy + x^2 = 3$. Differentiating with respect to x, considering y as a function of x, we get

$$5y^4y' + xy' + y + 2x = 0\,,$$
$$(5y^4 + x)y' = -2x - y\,,$$
$$y' = \frac{-2x - y}{5y^4 + x}\,.$$

3.9.9 For $y = (3x+5)^2$ we get $y' = 2(3x+5)\cdot 3 = 6(3x+5)$.
For $y = (-5x^2 + x - 1)^2$ we get $y' = 2(-5x^2 + x - 1)(-10x + 1)$.

3.9.10 (a) For $f(x) = \dfrac{1}{(x^5 - x + 1)^3} = (x^5 - x + 1)^{-3}$ we get

$$f'(x) = -3(x^5 - x + 1)^{-4}\frac{d}{dx}(x^5 - x + 1)$$
$$= -3(x^5 - x + 1)^{-4}(5x^4 - 1) = -\frac{3(5x^4 - 1)}{(x^5 - x + 1)^4}\,.$$

(b) For $f(x) = \sin^3 x$ we get

$$f'(x) = 3\sin^2 x \cdot \frac{d}{dx}(\sin x) = 3\sin^2 x \cdot \cos x\,.$$

(c) For $f(x) = \dfrac{x}{\sqrt{1-x^2}}$ we get

$$f'(x) = \frac{(\sqrt{1-x^2})(1) - x\frac{1}{2}\frac{(-2x)}{\sqrt{1-x^2}}}{1-x^2} = \frac{1}{(1-x^2)^{3/2}}\,.$$

3.9.11 (a) For $y = x^3 \sin^2(5x)$ we get

$$\begin{aligned}\frac{dy}{dx} &= x^3 \cdot 2\sin 5x \cdot \frac{d}{dx}(\sin 5x) + 3x^2 \sin^2(5x)\\ &= 10x^3 \sin 5x \cdot \cos 5x + 3x^2 \sin^2(5x)\,.\end{aligned}$$

(b) For $y = \dfrac{\sin x}{\sec(3x+1)}$ we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sec(3x+1)\cdot\cos x - \sin x \cdot \sec(3x+1)\cdot\tan(3x+1)\cdot 3}{[\sec(3x+1)]^2}\\ &= \frac{\cos x - 3\sin x \cdot \tan(3x+1)}{\sec(3x+1)}\,.\end{aligned}$$

(c) For $y = \cos^3(\sin 2x)$ we get

$$\begin{aligned}\frac{dy}{dx} &= 3\cos^2(\sin 2x)\cdot\frac{d}{dx}[\cos(\sin 2x)]\\ &= 3\cos^2(\sin 2x)\cdot\left[-\sin(\sin 2x)\cdot\frac{d}{dx}(\sin 2x)\right]\\ &= -6\cos^2(\sin 2x)\cdot\sin(\sin 2x)\cdot\cos 2x\,.\end{aligned}$$

3.9.13 For the surface area we have $S(r) = 4\pi r^2$. The differential at r for a change of an amount h is given by $dS = S'(r)h = 8\pi r h$. Since $r = 1$ and $h = 0.01$, we get

$$\Delta S \approx dS = 8\pi \cdot 1 \cdot 0.01 = 0.08\pi \approx 0.251\,m^2\,.$$

3.9.14 If we differentiate the equation $y^3 = 2x^2 + c$ with respect to x, we get $3y^2\frac{dy}{dx} = 4x$. Therefore, the given differential equation is satisfied.

3.9.15 Differentiating $xy = c$ with respect to x, we get $xy'(x) + y = 0$, so the given differential equation is satisfied.

3.9.16 (a) $f'(x) = \dfrac{1}{\frac{x}{1+x^2}}[\dfrac{(1+x^2) - x\cdot 2x}{(1+x^2)^2}] = \dfrac{1-x^2}{x(1+x^2)}\,.$

(b) $f'(x) = \dfrac{1}{x\ln x}$.

(c) $f'(x) = \dfrac{1}{2\sqrt{1+\ln^2 x}} \dfrac{2\ln x}{x} = \dfrac{\ln x}{x\sqrt{1+\ln^2 x}}$.

(d) $f'(x) = -\dfrac{1}{x} \sin(\ln x)$.

3.9.17 (a) $f'(x) = e^{1/x} \dfrac{d}{dx}(\dfrac{1}{x}) = -\dfrac{1}{x^2} e^{1/x}$.

(b)

$$f'(x) = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$
$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} .$$

(c) $f'(x) = e^x \cos(e^x)$.

(d) $f'(x) = \dfrac{1}{\cos(e^x)}(-\sin e^x) \cdot e^x = -e^x \tan(e^x)$.

3.9.18 (a) Differentiating $y + \ln(xy) - 2 = 0$, we get

$$y' + \frac{1}{xy}\left(xy' + y\right) = 0\,, \quad \left(1 + \frac{1}{y}\right) y' + \frac{1}{x} = 0\,,$$

therefore

$$y' = -\frac{y}{x(1+y)} .$$

(b)

$$\begin{aligned} y &= \ln(x \tan y) \\ y' &= \frac{1}{x \tan y}\left(x(\sec^2 y)y' + \tan y\right) \\ y'\left(1 - \frac{\sec^2 y}{\tan y}\right) &= \frac{1}{x} \\ y' &= \frac{\tan y}{x(\tan y - \sec^2 y)} . \end{aligned}$$

3.9.19 (a) $f(x) = \sin^{-1}(\frac{1}{5}x)$

$$f'(x) = \frac{1}{\sqrt{1 - \frac{x^2}{25}}} . \frac{1}{5} = \frac{1}{\sqrt{25 - x^2}} .$$

(b) $f(x) = (\tan x)^{-1} = \dfrac{1}{\tan x} = \cot x$

$f'(x) = -\csc^2 x$.

(c) $f(x) = \sin^{-1} x + \cos^{-1} x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0\ .$$

3.9.20 Let us define

$$y(x) = (1+x)^{1/x}\ .$$

Taking the natural logarithm on both sides, we get

$$\ln y(x) = \ln\left((1+x)^{1/x}\right) = \frac{1}{x}\ln(1+x)\,,$$

$$\lim_{x\to 0}\ln y(x) = \lim_{x\to 0}\frac{\ln(1+x)}{x}\ .$$

This is an indeterminate form of type $0/0$. By l'Hôpital's rule

$$\lim_{x\to 0}\ln y(x) = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1\,,$$

therefore $\ln y(x) \to 1$ as $x \to 0$. By the continuity of the exponential function

$$y(x) = e^{\ln y(x)} \to e^1\,,\quad \text{as } x \to 0,$$

that is, $y(x) \to e$ as $x \to 0$. Thus $\lim_{x\to 0}(1+x)^{1/x} = e$.

3.9.21 (a) Let $y(x) = x^{\frac{(\ln a)}{(1+\ln x)}}$. After taking the logarithm on both sides, we pass to the limit as $x \to 0^+$,

$$\begin{aligned}\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\left[\frac{\ln a}{1+\ln x}\ln x\right] &= \ln a \cdot \lim_{x\to 0^+}\frac{\ln x}{1+\ln x}\\ &= \ln a \cdot \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{1}{x}} \qquad \text{(by l'Hòpital's rule)}\\ &= \ln a\ .\end{aligned}$$

Therefore, $\lim_{x\to 0^+} y(x) = e^{\ln a} = a$.

(b) Let $y(x) = x^{\frac{(\ln a)}{(1+\ln x)}}$. After taking the logarithm on both sides,

$$\begin{aligned}\lim_{x\to +\infty}\ln y(x) = \lim_{x\to +\infty}\left[\frac{\ln a}{1+\ln x}\ln x\right] &= \ln a \cdot \lim_{x\to +\infty}\frac{\ln x}{1+\ln x}\\ &= \ln a \cdot \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{1}{x}} \qquad \text{(by l'Hòpital's rule)}\\ &= \ln a\ .\end{aligned}$$

Therefore $\lim_{x\to +\infty} y(x) = e^{\ln a} = a$.

(c) Let $y(x) = (x+1)^{(\ln a)/x}$. Taking the logarithm on both sides and passing to the limit, we get

$$\begin{aligned}\ln y(x) &= \frac{\ln a}{x}\ln(x+1)\,,\\ \lim_{x\to 0}\ln y(x) &= \lim_{x\to 0}\left[\ln a\frac{\ln(x+1)}{x}\right]\\ &= \ln a\cdot\lim_{x\to 0}\frac{1}{x+1} = \ln a\,.\end{aligned}$$

Therefore, $\lim_{x\to 0} y(x) = e^{\ln a} = a$.

3.9.22 We consider the identity

$$\frac{\sin x}{x}\cdot\frac{x\sin\left(\frac{1}{x}\right)}{\sin x} = \sin\left(\frac{1}{x}\right).$$

We know that $\lim_{x\to 0^+}\frac{\sin x}{x} = 1$. If $\lim_{x\to 0^+}\frac{x\sin(\frac{1}{x})}{\sin x}$ would exist, then also $\lim_{x\to 0^+}\sin(\frac{1}{x})$ would exist. But we know that the latter does not exist because $\sin(\frac{1}{x})$ oscillates between -1 and 1 as $x\to 0^+$. So $\lim_{x\to 0^+}\frac{x\sin(\frac{1}{x})}{\sin x}$ does not exist.

3.9.23 (a) Let $y(x) = e^{ax}\sin bx$. We compute

$$\begin{aligned}y' &= ae^{ax}\sin bx + be^{ax}\cos bx\\ y'' &= a^2e^{ax}\sin bx + abe^{ax}\cos bx + abe^{ax}\cos bx - b^2e^{ax}\sin bx\\ &= (a^2-b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx\,,\end{aligned}$$

and furthermore

$$\begin{aligned}&y'' - 2ay' + (a^2+b^2)y\\ &\quad= (a^2-b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx\\ &\qquad - 2a(ae^{ax}\sin bx + be^{ax}\cos bx)\\ &\qquad + (a^2+b^2)e^{ax}\sin bx\\ &\quad= 0\,.\end{aligned}$$

(b) Let $y = \tan^{-1}x$. We get

$$y' = \frac{1}{1+x^2}\,,\quad y'' = \frac{-2x}{(1+x^2)^2}\,.$$

The following computations yield expressions for $\cos y$ and $\sin y$. From the definition of y, we get

$$x = \tan y = \sqrt{\sec^2 y - 1}\,, \quad \sec^2 y = 1 + x^2\,,$$
$$\cos^2 y = \frac{1}{1+x^2}\,, \quad \cos y = \frac{1}{\sqrt{1+x^2}}\,,$$
$$\sin y = \sqrt{1-\cos^2 y} = \sqrt{1 - \frac{1}{1+x^2}} = \sqrt{\frac{x^2}{1+x^2}}\,.$$

From these formulas, we conclude that

$$\begin{aligned} -2y'' + 2\sin y \cos^3 y &= -\frac{2x}{(1+x^2)^2} + 2\frac{x}{\sqrt{1+x^2}}\frac{1}{(1+x^2)^{3/2}} \\ &= 0\,. \end{aligned}$$

3.9.24 Let $y(x) = 3^{2x}5^{7x}$. Then

$$\ln y(x) = 2x \ln 3 + 7x \ln 5\,.$$

Differentiating with respect to x, we get

$$\frac{1}{y}y' = 2\ln 3 + 7\ln 5\,, \quad y' = (2\ln 3 + 7\ln 5)y\,,$$

so y' is proportional to y.

Solutions to Exercises of Chap. 4

4.6.1 (i) $f(x) = \sec\left(\frac{x}{2}\right)$

$f'(x) = \frac{1}{2}\sec\left(\frac{x}{2}\right)\cdot\tan\left(\frac{x}{2}\right) = \frac{1}{2}\frac{\sin(x/2)}{\cos^2(x/2)}$

$f'(x) = 0$ gives $\sin(x/2) = 0$ which is true for $x = 0$ (in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

$f''(x) = \frac{1}{2}\left[\frac{1}{2}\sec^3\left(\frac{x}{2}\right) + \frac{1}{2}\tan^2\left(\frac{x}{2}\right)\cdot\sec\left(\frac{x}{2}\right)\right]$

$f''(0) = \frac{1}{4} > 0.$

Therefore f has a local minimum at $x = 0$.

(ii) $f(x) = \tan x - 2\sec x$, $f'(x) = \sec^2 x - 2\sec x\cdot\tan x$.

$f'(x) = 0$ gives $\sec x\cdot(\sec x - 2\tan x) = 0$,

i.e. $\sec x = 0$ or $\sec x - 2\tan x = 0$.

This gives

$$\sec x - 2\tan x = 0\,, \quad \frac{1}{\cos x} - 2\frac{\sin x}{\cos x} = 0\,,$$

so $\sin x = \dfrac{1}{2}$, thus $x = \dfrac{\pi}{6}$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$.
Therefore $x = \pi/6$ is a critical point for f.
$f''(x) = 2\sec^2 x \cdot \tan x - 2\tan^2 x \cdot \sec x - 2\sec^3 x$
$f''\left(\dfrac{\pi}{3}\right) = 2 \cdot 4 \cdot \dfrac{1}{\sqrt{3}} - 2 \cdot \dfrac{1}{3} \cdot \dfrac{2}{\sqrt{3}} - \dfrac{8}{3\sqrt{3}} = -\dfrac{4}{3\sqrt{3}} < 0.$
Therefore, $x = \pi/3$ is a local maximum for f.

(iii) $f(x) = \sin x - \cos x$
$f'(x) = \cos x + \sin x$
$f'(x) = 0$ holds for $x = \frac{3\pi}{4}$ in $[0, \pi]$. Moreover,

$$f(0) = -1\,, \quad f\left(\frac{3\pi}{4}\right) = \sqrt{2}\,, \quad f(\pi) = 1\,,$$

so the maximum value equals $\sqrt{2}$ at $x = \frac{3\pi}{4}$, and the minimum value equals -1 at $x = 0$. (We have $f''(\dfrac{3\pi}{4}) < 0$.)

(iv) $f(x) = |6 - 4x| = \begin{cases} 6 - 4x\,, & x < 3/2\,, \\ -6 + 4x\,, & x > 3/2\,. \end{cases}$
$f'(x) = \begin{cases} -4\,, & x < 3/2\,, \\ 4\,, & x > 3/2\,. \end{cases}$
$f'(x)$ does not exist when $x = 3/2$. Since otherwise $f'(x) \neq 0$, the point $3/2$ is the only candidate for an extremum in $(-3, 3)$. We have

$$f(-3) = 18\,, \quad f(3/2) = 0\,, \quad f(3) = 6\,,$$

so the maximum value is 18 at $x = -3$ and the minimum value is 0 at $x = 3/2$.

4.6.2 (i) For $f(x) = 3 - 4x - 2x^2$ we get $f'(x) = -4 - 4x$; the condition $f'(x) = 0$ gives $x = -1$, so f has a unique stationary point at $x = -1$. Since $f''(x) = -4$, we get $f''(-1) = -4 < 0$, so $f(x)$ has a local maximum at $x = -1$. It has no local minimum. Since moreover $f(x)$ tends to $-\infty$ for $x \to -\infty$ and for $x \to +\infty$, f must have a global maximum. Since every global maximum is also a local maximum, it has to be at $x = -1$.

(ii) $f(x) = x^3 - 3x - 2$ is a polynomial of odd degree. Therefore, $\lim\limits_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ have opposite signs (one is $+\infty$ and the other is $-\infty$) so there can be no global extremum.

4.6.3 Let the length of the sides of the rectangle be x and y, respectively. Its area A equals $A = xy$. We have $p = 2x + 2y$ for its perimeter, so we may express y as $y = \frac{p}{2} - x$ and obtain

$$A(x) = \frac{px}{2} - x^2\,, \quad \text{where } x \in \left[0, \frac{p}{2}\right].$$

Now

$$A'(x) = \frac{p}{2} - 2x\,,$$

so $A'(x) = 0$ if and only if $x = \frac{p}{4}$. In this case, $y = \frac{p}{4}$. Moreover, $A'' = -2 < 0$ everywhere. Therefore, the area becomes maximal when $x = \frac{p}{4}$ and $y = \frac{p}{4}$ which means that the rectangle is a square.

4.6.4 Let the length of the sides of the rectangle be x and y, respectively. Its perimeter equals $p = 2x + 2y$ We have $A = xy$ for its area, so we may express y as $y = A/x$ and consequently

$$p(x) = 2x + 2\frac{A}{x}\,, \quad p'(x) = 2 - \frac{2A}{x^2}\,.$$

We have $p'(x) = 0$ if and only if $x^2 - A = 0$ or $x = \sqrt{A}$, since x cannot be negative. Then $y = \sqrt{A}$. Moreover, we have $p''(x) = 4A/x^3$ which is positive for $x = \sqrt{A}$.

Thus p has a minimum at $x = \sqrt{A}$. Since now $x = y = \sqrt{A}$, the rectangle of minimal perimeter is a square.

4.6.7 Let D be the point opposite to A such that $\angle ADB = 90°$. Then the distance between D and B equals $\sqrt{3^2 - 1^2} = \sqrt{8}\,\text{km}$. Let x denote the distance between the point C and the point D, so $0 \le x \le \sqrt{8}$. Let k be the cost per km of the pipe above the ground. Then the cost per km of the pipe under the water equals $4k$. The pipe length above ground is $\sqrt{8} - x$, its cost is $k(\sqrt{8} - x)$. The pipe length under water is $\sqrt{1 + x^2}$, its cost is $4k\sqrt{1 + x^2}$. For the total cost P we obtain

$$P(x) = k(\sqrt{8} - x) + 4k\sqrt{1 + x^2}\,, \quad P'(x) = -k + \frac{4kx}{\sqrt{1 + x^2}}\,.$$

We have $P'(x) = 0$ if and only if $x = \sqrt{1/15} \simeq 0.26\,\text{km}$. This is a minimum since

$$P''(x) = \frac{4k}{(1 + x^2)^{3/2}} > 0$$

for all x, in particular for x as computed above. The distance from C to B becomes $\sqrt{8} - \sqrt{\frac{1}{15}} \simeq 2.57\,\text{km}$. Note that the solution does not depend on the actual amount of the cost k, as long as the proportion between the cost above ground and under water remains fixed.

4.6.8 We have $f(x) = x^2 + px + q$, so $f'(x) = 2x + p$. In order that 1 is an extremum of f we must have $f'(1) = 0$, that is, $2 + p = 0$. This gives $p = -2$, so

$$f(x) = x^2 - 2x + q\,.$$

Since moreover $f(1) = 3$ is assumed, we get $3 = 1 - 2 + q$ or $q = 4$. Therefore,

$$f(x) = x^2 - 2x + 4\,.$$

Since $f''(x) = 2$ for all x, we have $f''(1) > 0$, and therefore f has a local minimum at $x = 1$.

4.6.9 We have

$$\begin{aligned} f(x) &= \frac{64}{\sin x} + \frac{27}{\cos x} = 64 \csc x + 27 \sec x\,, \\ f'(x) &= -64 \csc x \cdot \cot x + 27 \sec x \cdot \tan x \end{aligned}$$

We get $f'(x) = 0$ when $-64 \csc x \cdot \cot x + 27 \sec x \cdot \tan x = 0$, and thus compute

$$\frac{\csc x \cdot \cot x}{\sec x \cdot \tan x} = \frac{27}{64}\,, \quad \frac{\cos x \cdot \cos x \cdot \cos x}{\sin x \cdot \sin x \cdot \sin x} = \frac{27}{64}\,,$$
$$(\cot x)^3 = \left(\frac{3}{4}\right)^3\,, \quad \tan x = \frac{4}{3}\,.$$

This gives

$$\cos x = \frac{3}{5}\,, \quad x = \cos^{-1}\left(\frac{3}{5}\right)\,.$$

To obtain the second derivative, we start from

$$f'(x) = -64 \frac{\cos x}{\sin^2 x} + 27 \frac{\sin x}{\cos^2 x}$$

and compute

$$\begin{aligned} f''(x) &= -\frac{64}{\sin^4 x} \cdot (-\sin^3 x - 2\cos^2 x \cdot \sin x) + \frac{27}{\cos^4 x} \cdot (-\cos^3 x + 2\sin^2 x \cdot \cos x) \\ &= \frac{64}{\sin x} + 128 \frac{\cos^2 x}{\sin^3 x} - \frac{27}{\cos x} + 54 \frac{\sin^2 x}{\cos^3 x}\,. \end{aligned}$$

At $x = \cos^{-1}(\frac{3}{5})$ we have $\cos x = \frac{3}{5}$ and $\sin x = \frac{4}{5}$. For this particular x we, therefore, get

$$\begin{aligned} f''(x) &= \frac{64}{4/5} + 128 \frac{9}{25} \frac{125}{64} - \frac{27}{3/5} + 54 \frac{16}{25} \frac{125}{27} \\ &= 16 \cdot 5 + 90 - 45 + 160 > 0\,. \end{aligned}$$

Therefore, f has a local minimum at $x = \cos^{-1}\frac{3}{5}$ which lies in the interval $(0, \dfrac{\pi}{2})$. From the formula

$$f'(x) = \frac{27\cos^3 x \cdot \left(\tan^3 x - \frac{64}{27}\right)}{\sin^2 x \cdot \cos^2 x} = \frac{27\cos x \cdot \left(\tan^3 x - \frac{64}{27}\right)}{\sin^2 x}$$

we see that besides the point x for which $\tan x = \frac{4}{3}$ there is no other critical point in the interval $(0, \dfrac{\pi}{2})$. Since moreover

$$\lim_{x\to 0^+} f(x) = +\infty = \lim_{x\to(\pi/2)^-} f(x)\,,$$

there are no local maxima, and the point x as computed above must be a global minimum.

4.6.10 (a) For $f(x) = -2x^3 - 6x^2 + 5$ we get

$$f'(x) = -6x^2 - 12x = -6x(x+2)\,.$$

We have $f'(x) = 0$ for $x = 0$ and $x = -2$, both of which lie within $[-3, 1]$. We compute

$$f''(x) = -12x - 12\,, \quad f''(0) = -12 < 0\,, \quad f''(-2) = 12 > 0\,.$$

Therefore, f has a local maximum at $x = 0$ with maximum value $f(0) = 5$, and a local minimum at $x = -2$ with minimum value $f(-2) = -3$.

(b) For $f(x) = x^4 - 5x^2 + 4$ we get

$$f'(x) = 4x^3 - 10x = 2x(2x^2 - 5)\,,$$

so $f'(x) = 0$ holds for $x = 0$ and $x = \pm\sqrt{\dfrac{5}{2}}$, however, $-\sqrt{\dfrac{5}{2}} \notin [0, 2]$. We compute

$$f''(x) = 12x^2 - 10\,, \quad f''(0) = -10 < 0\,, \quad f''\left(\sqrt{\frac{5}{2}}\right) = 20 > 0\,.$$

Therefore, f has a local maximum at $x = 0$ (which happens to lie at the boundary of the interval) with maximum value $f(0) = 4$, and a local minimum at $x = \sqrt{\dfrac{5}{2}}$ with minimum value $f(\sqrt{\dfrac{5}{2}}) = -\dfrac{9}{4}$.

4.6.11 Let y be the height of the rectangle and x the radius of the semicircle. The circular arc has length πx, and the perimeter of the window is $\pi x + 2y + 2x$ which has to be equal to 15, so $2y(x) = 15 - (2+\pi)x$. The area A thus

becomes

$$A(x) = 2xy(x) + \frac{1}{2}\pi x^2 = 15x - \left(2 + \frac{\pi}{2}\right)x^2 .$$

To maximize the area, we compute x from

$$0 = A'(x) = 15 - (4+\pi)x\,, \quad \text{which gives } x = \frac{15}{4+\pi}\,.$$

Since $A''(x) = -(4+\pi) < 0$ for all x, this is indeed a maximum. The corresponding value for y becomes

$$y = \frac{1}{2}\left[15 - \frac{(2+\pi)15}{4+\pi}\right] = \frac{15}{4+\pi}\,.$$

4.6.12 Let x denote the width of the field, y its length, and k the length of the barn, so $0 < k < y$. The amount of fence needed is $2x + y + (y-k)$ which has to be equal to 500, so

$$y = \frac{500+k}{2} - x\,.$$

For the area A of the field we have

$$A(x) = x \cdot y(x) = \left(\frac{500+k}{2}\right)x - x^2\,.$$

In order to maximize A, we need

$$0 = A'(x) = \frac{500+k}{2} - 2x\,, \quad \text{thus} \quad x = \frac{500+k}{4}\,,$$

with the corresponding value $y = \dfrac{500+k}{4} = x$. Since $A''(x) = -2 < 0$ for all x, we have a maximum. Since $x = y$, the rectangle is, in fact, a square. Note that the result does not depend on the size of the barn.

4.6.13 Let the semicircle be given as $x^2 + y^2 = a^2$ with $y \geq 0$. We consider rectangles with corners $(x,0)$, $(-x,0)$, (x,y) and $(-x,y)$ with $0 \leq x \leq a$ and $y = \sqrt{a^2 - x^2}$. Their area A satisfies $A(x) = 2x\sqrt{a^2-x^2}$, so

$$A'(x) = 2\sqrt{a^2-x^2} - \frac{1}{2}\frac{2x \cdot 2x}{\sqrt{a^2-x^2}} = \frac{2(a^2-2x^2)}{\sqrt{a^2-x^2}}\,.$$

For maximum area, $A'(x) = 0$, so $x = \dfrac{1}{2}\sqrt{2}a$. Then

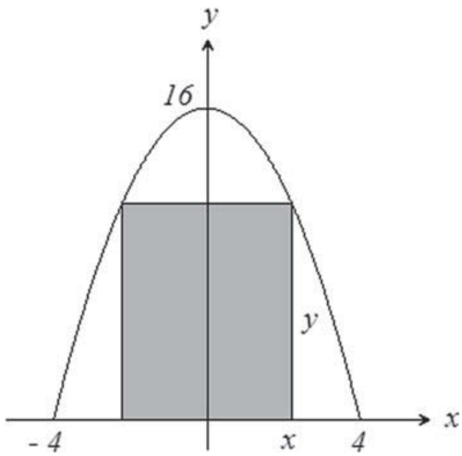

Fig. E.3 Maximizing the area of a rectangle

$$y = \sqrt{a^2 - x^2} = \sqrt{a^2 - \frac{a^2}{2}} = \frac{1}{2}\sqrt{2}a\,.$$

Therefore, the length of rectangle equals $\sqrt{2}a$, its width equals $\frac{1}{2}\sqrt{2}a$.

4.6.14 The situation is depicted in Fig. E.3. We have $A = 2xy$ for the area, and $y = 16 - x^2$. We must have $0 \leq x \leq 4$. We thus get

$$A(x) = 2x(16 - x^2) = 32x - 2x^3\,, \quad A'(x) = 32 - 6x^2\,.$$

For maximum area, $A'(x) = 0$, so $x = \frac{4}{\sqrt{3}}$. We have $A''(x) = -12x$, thus $A''(\frac{4}{\sqrt{3}}) < 0$. Therefore, $x = \frac{4}{\sqrt{3}}$ maximizes the area, and we have $y = \frac{32}{3}$. The dimensions of the rectangle with the largest area are $\frac{8}{\sqrt{3}}$ and $\frac{32}{3}$.

4.6.15 Let the length of the rectangle be x and its width be y, let the circle be given by $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 10^2$, that is, $x^2 + y^2 = 400$ (Fig. E.4). So $y(x) = \sqrt{400 - x^2}$ and the area A satisfies

$$A(x) = x\sqrt{400 - x^2}\,, \quad A'(x) = \frac{2(200 - x^2)}{\sqrt{400 - x^2}}\,,$$

where $0 \leq x \leq 20$. We get $A'(x) = 0$ when $x = \sqrt{200} = 10\sqrt{2}$. For $x = 10\sqrt{2}$ we have $A(x) = 200$, whereas for $x = 0$ and $x = 20$ we get $A(x) = 0$. So the area is maximal when $x = 10\sqrt{2}$ and $y = \sqrt{400 - 200} = 10\sqrt{2}$. (One may check that $A''(x) < 0$ for $x = 10\sqrt{2}$.)

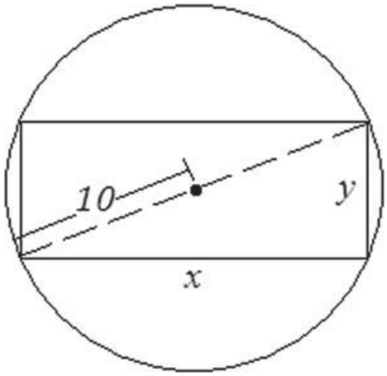

Fig. E.4 Rectangle inscribed in a circle

4.6.16 (a) We have

$$N(t) = 5000(25 + te^{-t/20}),$$
$$N'(t) = 5000\left(e^{-t/20} - \frac{1}{20}te^{-t/20}\right) = 250(20 - t)e^{-t/20}.$$

Therefore, $N'(t) = 0$ at $t = 20$. We have

$$N(0) = 125000, \quad N(20) \approx 161788, \quad N(100) \approx 128369.$$

The global maximum is $N = 161788$ at $t = 20$. (Indeed $N''(t) < 0$ for $t = 20$.) The global minimum is $N = 125000$ at $t = 0$.
(b) A minimum of N' occurs when $0 = N''(t) = 12.5(t - 40)e^{-t/20} = 0$, that is, when $t = 40$. (One may check that $N'''(t) > 0$ at this point.)

4.6.17 Let r and h be the radius and height of the cylinder. The volume of the inscribed cylinder is $V = \pi r^2 h$. We must have $r^2 + \left(\frac{h}{2}\right)^2 = R^2$, thus $r^2 = R^2 - \frac{h^2}{4}$. Moreover, $0 \le h \le 2R$. So

$$V(h) = \pi\left(R^2 - \frac{h^2}{4}\right)h = \pi\left(R^2 h - \frac{h^3}{4}\right),$$
$$V'(h) = \pi\left(R^2 - \frac{3}{4}h^2\right).$$

We get $V'(h) = 0$ when $h = \frac{2R}{\sqrt{3}}$. For $h = 0$ and $h = 2R$ we have $V(h) = 0$, for $h = \frac{2R}{\sqrt{3}}$ we have $V = \frac{4\pi}{3\sqrt{3}}R^3$. One checks that $V''(h) < 0$ at the latter value of h. So the volume is largest when $h = \frac{2R}{\sqrt{3}}$ and $r = \sqrt{\frac{2}{3}}R$.

Solutions to the Exercises of Chap. 5

5.5.3 (a) The given series can be written as

$$2\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}+\cdots\right)=2\frac{1}{1-\frac{1}{3}}=6$$

as $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}+\cdots$ is a geometric series with common ratio $r=\frac{1}{3}$ and first term $a=1$.

(b) It can be written into a geometric series with common ratio $r=\dfrac{1}{100}$ and first term $a=\frac{1}{100}$. Therefore the sum of the series is $\dfrac{\frac{1}{100}}{1-\frac{1}{100}}=\dfrac{9}{99}=\dfrac{1}{11}$.

(c) It is a geometric series with $a=1$ and $r=-\dfrac{1}{2}$ and so its sum is $\dfrac{1}{1+\frac{1}{2}}=\dfrac{2}{3}$.

(d) It is a geometric series with $a=1$ and $r=-2$. Since $|r|=2>1$, the series diverges.

(e) We have

$$\begin{aligned}&\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n+1)(n+2)}\\&\quad=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\cdots\\&\quad=\frac{1}{2}-\frac{1}{n+2}=\frac{1}{2}\qquad\text{as } n\to\infty\end{aligned}$$

(f) The series is

$$\begin{aligned}&5\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots\right)\\&\quad=5\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\cdots\right]\\&\quad=5\left[1-\frac{1}{n+1}+\cdots\right]=5.\end{aligned}$$

5.5.4 (a) It is a geometric series with $a=1$, $r=-\dfrac{1}{4}$. Hence, it is convergent and its sum equals $\dfrac{1}{1-(-\frac{1}{4})}=\dfrac{4}{5}$.

(b) It is a convergent series whose sum equals $7\cdot\dfrac{\frac{1}{4}}{1-\frac{1}{4}}=\dfrac{7}{3}$.

(c) It is the sum of two convergent series, and

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)=\sum_{n=0}^{\infty}\frac{5}{2^n}+\sum_{n=0}^{\infty}\frac{1}{3^n}=\frac{5\cdot\frac{1}{2}}{1-\frac{1}{2}}+\frac{\frac{1}{3}}{1-\frac{1}{3}}=5+\frac{1}{2}=\frac{11}{2}\,.$$

(d) It is a difference of two convergent series, and

$$\sum_{n=0}^{\infty}\frac{5}{2^n}-\sum_{n=0}^{\infty}\frac{1}{3^n}=5\cdot\frac{1}{1-\frac{1}{2}}-\frac{1}{1-\frac{1}{3}}=10-\frac{3}{2}=\frac{17}{2}\,.$$

(f) Here $a=2$ and $r=\dfrac{2^{n+1}}{5^n}\Big/\dfrac{2^{n+2}}{5^{n+1}}=\dfrac{5}{2}$. Since $|r|>1$ the series diverges.

5.5.5 (a) It is a geometric series with $a=\dfrac{1}{\sqrt{2}}$ and $r=\dfrac{1}{\sqrt{2}}$ and hence it is convergent. Its sum is $\dfrac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}=\dfrac{1}{\sqrt{2}-1}$.

(b) Since the sequence $s_n=(\sqrt{2})^n$ does not converge to 0, the series is divergent. Alternatively, since it is a geometric series with $r=\sqrt{2}>1$, it is divergent.

(c) It is a geometric series with $a=\dfrac{3}{2}$ and $r=-\dfrac{1}{2}$, hence it is convergent and its sum is $\dfrac{\frac{3}{2}}{1+\frac{1}{2}}=1$.

5.5.6 See Sect. 6.9.

5.5.7 (a) Let $s_n=\dfrac{1}{\sqrt{n+1}+\sqrt{n}}$. We have

$$s_n=\frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})\cdot(\sqrt{n+1}-\sqrt{n})}=\sqrt{n+1}-\sqrt{n}\,.$$

By the Mean Value Theorem, applied to $f(x)=\sqrt{x}$ on the interval $[n,n+1]$ we have, since $f'(x)\geq f'(n+1)$ for $x\leq n+1$,

$$s_n=\sqrt{n+1}-\sqrt{n}\geq\frac{1}{2\sqrt{n+1}}\,.$$

Let $t_n=\dfrac{1}{n+1}$. We get

$$\frac{s_n}{t_n}=\frac{\sqrt{n+1}-\sqrt{n}}{n}\geq\frac{\sqrt{n+1}}{2}\,,\quad\lim_{n\to\infty}\frac{s_n}{t_n}=+\infty\,.$$

Since we know that $\sum_n t_n$ diverges, the limit comparison test implies that $\sum_n s_n$ diverges, too.

(b) We have $s_n = \dfrac{1}{x^n + x^{-n}}$. Consider first the case $x < 1$. Set $t_n = x^n$, then

$$\frac{s_n}{t_n} = \frac{1}{x^n + x^{-n}} \cdot \frac{1}{x^n} = \frac{1}{x^{2n}+1}, \quad \lim_{n\to\infty} \frac{s_n}{t_n} = 1 .$$

Since $\sum t_n$ is convergent, the limit comparison test implies that $\sum s_n$ is also convergent.

In the case $x > 1$, we consider $u_n = x^{-n}$ and get

$$\frac{s_n}{u_n} = \frac{1}{x^n + x^{-n}} \cdot x^n = \frac{1}{1+x^{-2n}}, \quad \lim_{n\to\infty} \frac{s_n}{t_n} = 1 .$$

But $\sum u_n$ is convergent, so $\sum s_n$ is also convergent.

When $x = 1$, $\sum s_n = \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \cdots$ which is divergent. Hence, $\sum s_n$ converges for $x < 1$ and $x > 1$ but diverges for $x = 1$.

5.5.8 (a) We have $s_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $s_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{n+1}}$. The ratio becomes

$$\begin{aligned} \frac{s_{n+1}}{s_n} &= \frac{x^{2n}}{(n+2)\sqrt{n+1}} \cdot \frac{(n+1)\sqrt{n}}{x^{2n-2}} \\ &= \left[\frac{n+1}{n+2}\left(\frac{n}{n+1}\right)^{1/2}\right] x^2 = \lim_{n\to\infty}\left[\frac{1+1/n}{1+2/n} . \frac{1}{\sqrt{1+1/n}}\right] x^2 \\ &= x^2 . \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = x^2 .$$

By the ratio test, $\sum s_n$ converges if $x^2 < 1$ and diverges if $x^2 > 1$. If $x^2 = 1$, then

$$s_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}} \cdot \frac{1}{1+1/n} .$$

Taking $t_n = \frac{1}{n^{3/2}}$, we get $\lim_{n\to\infty} \dfrac{s_n}{t_n} = \lim_{n\to\infty} \dfrac{1}{1+1/n} = 1$. Since $\sum t_n = \sum \frac{1}{n^{3/2}}$ is a convergent series, the limit comparison test implies that $\sum s_n$ is also convergent. Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

(b) Here we have

$$\frac{s_{n+1}}{s_n} = \frac{2^{n+1}-2}{2^{n+1}+1}x^n \cdot \frac{2^n+1}{2^n-2} \cdot \frac{1}{x^{n-1}} = \frac{2-\frac{2}{2^n}}{2+\frac{1}{2^n}} \cdot \frac{1+\frac{1}{2^n}}{1-\frac{2}{2^n}}x\,,$$

therefore $\lim\limits_{n\to\infty} \frac{s_{n+1}}{s_n} = \frac{2-0}{2+0} \cdot \frac{1+0}{1-0}x = x$. Thus by the ratio test, $\sum s_n$ converges for $x < 1$ and diverges for $x > 1$. But the ratio test is not applicable for $x = 1$. In that case,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n-2}{2^n+1} = \lim_{n\to\infty} \frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}} = 1 \neq 0.$$

Therefore, $\sum s_n$ diverges for $x = 1$. Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

5.5.10 (a) We have $\left(\frac{n^2}{2^n}\right)^{1/n} = \frac{(n^2)^{1/n}}{(2^n)^{1/n}} = \frac{n^{2/n}}{2}$ and therefore

$$\lim_{n\to\infty} \left(\frac{n^2}{2^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{2/n}}{2} = \frac{1}{2} < 1\,.$$

Thus, the root test implies that $\sum \frac{n^2}{2^n}$ converges.

(b) We have

$$\left[\left(\frac{1}{1+n}\right)^n\right]^{1/n} = \frac{1}{1+n}\,, \quad \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1\,.$$

Therefore the root test implies that $\sum \left(\frac{1}{1+n}\right)^n$ converges.

5.5.11 We have

$$(s_n)^{1/n} = \begin{cases} \frac{n^{1/n}}{2}\,, & n \text{ is odd,} \\ \frac{1}{2}\,, & n \text{ is even.} \end{cases}$$

Therefore

$$\frac{1}{2} \leq (s_n)^{1/n} \leq \frac{n^{1/n}}{2}$$

holds for all n. Since $n^{1/n} \to 1$ as $n \to \infty$, we have $\lim_{n\to\infty} s_n^{1/n} = \frac{1}{2} < 1$ (by the Sandwich theorem), that is, $\lim\limits_{n\to\infty} s_n^{1/n} < 1$. Therefore, the series converges by the root test.

5.5.12 For $f(x) = \frac{1}{1+x}$ we get

$$f'(x) = -\frac{1}{(1+x)^2}\,, \quad f''(x) = \frac{2}{(1+x)^3}\,, \quad f'''(x) = -\frac{2\cdot 3}{(1+x)^4}\,, \ldots$$

$$f^n(x) = \frac{n!}{(1+x)^{n+1}}(-1)^n\,,$$

$$f(2) = \frac{1}{1+2} = \frac{1}{3}\,, \quad f'(2) = -\frac{1}{3^2}\,, \quad f''(2) = \frac{2}{3^3}\,,$$

$$f'''(2) = \frac{-6}{3^4}\,, \ldots, f^n(2) = \frac{(-1)^n n!}{3^{n+1}}\,.$$

The required Taylor series is

$$\begin{aligned}\frac{1}{1+x} &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots \\ &= \frac{1}{3} - \frac{1}{3^2}(x-2) + \frac{1}{3^3}(x-2)^2 - \frac{1}{3^4}(x-2)^3 + \ldots + \frac{(-1)^n}{3^{n+1}}(x-2)^n + \cdots\,.\end{aligned}$$

5.5.13 The series expansion gives

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1)$$

with

$$R_n(1) = e^c\frac{1}{(n+1)!}$$

for some c between 0 and 1. For the purpose of this example, we assume that we know that $e < 3$. Hence we are certain that

$$\frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}\,,$$

because $1 < e^c < 3$ for $0 < c < 1$.
Since $9! = 362880$, we have $1/9! > 10^{-6}$, whereas $3/10! < 10^{-6}$. Thus, we should take $(n+1)$ to be at least 10 or n to be at least 9. With an error of less than 10^{-6},

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{9!} \approx 2.718282\,.$$

5.5.14 (a) The central angle θ of a circle of radius R subtended by an arc of length s is $\theta = \dfrac{s}{R}$. From the figure in the text
$\cos\theta = \dfrac{R}{R+C}$ or $\sec\theta = \dfrac{R+C}{R}$
or $R\sec\theta = R + C$ or $C = R(\sec\theta - 1)$
or $C = R\left[\sec\left(\frac{s}{R}\right) - 1\right]$.

(b) $\sec x = f(x)$
$f'(x) = \sec x \tan x$
$f''(x) = \sec^3 x + \sec x \tan^2 x$
$f'''(x) = 5\sec^3 x \tan x + \sec x \tan^3 x$
$f^{iv}(x) = 18\sec^3 x \tan^2 x + 5\sec^5 x + \sec x \tan^4 x$
$f(0) = 1,\ f'(0) = 0,\ f''(0) = 1,\ f'''(0) = 0,\ f^{iv}(0) = 1.$
$f(x) = \sec x \approx 1 + \frac{1}{2!}x^2 + \frac{5}{4!}x^4$ (using Maclaurin series)

$$C = R\left[\sec\left(\frac{s}{R}\right) - 1\right] \approx R\left[\left(1 + \frac{s^2}{2R^2} + \frac{5s^4}{24R^4}\right) - 1\right]$$
$$= \frac{s^2}{2R} + \frac{5s^4}{24R^3}.$$

Solutions to the Exercises of Chap. 6

6.11.1 $\int (4x+5)\,dx = 4\frac{x^2}{2} + 5x + C = 2x^2 + 5x + C.$

6.11.2 $\int (9t^2 - 5t + 9)\,dt = 9\frac{t^3}{3} - 5\frac{t^2}{2} + 9t + C = 3t^3 - \frac{5}{2}t^2 + 9t + C.$

6.11.3 $\int (3\sqrt{u} + \frac{2}{\sqrt{u}})\,du = 3\cdot\frac{u^{3/2}}{3/2} + 2\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2u^{3/2} + 4\sqrt{u} + C.$

6.11.4 $\int (5z^{-7} + 7z^{-3} - z)\,dz = 5\frac{z^{-6}}{-6} + 7\frac{z^{-2}}{-2} - \frac{z^2}{2} + C = -\frac{5}{6z^6} - \frac{7}{2z^2} - \frac{z^2}{2} + C.$

6.11.5

$$\int (x^{2/3} - 4x^{-1/5} + 4)\,dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} - 4\frac{x^{-\frac{1}{5}+1}}{-\frac{1}{5}+1} + 4x + C$$
$$= \frac{3}{5}x^{5/3} - 5x^{4/5} + 4x + C\,.$$

6.11.6 $\int u(1+u^2)\,du = \int (u + u^3)\,du = \frac{u^2}{2} + \frac{u^4}{4} + C.$

6.11.7 $\int (2x^{-1} - \sqrt{2}e^x)\,dx = 2\ln x - \sqrt{2}e^x + C.$

6.11.8 $\int (x^{2/3} - \sin x)\,dx = \frac{x^{\frac{2}{3}} + 1}{\frac{2}{3}+1} + \cos x + C = \frac{3}{5}x^{5/3} + \cos x + C.$

6.11.9 $\int \frac{\sec x}{\cos x}\,dx = \int \sec^2 x\,dx = \tan x + C.$

6.11.10 $\int \frac{\sec u \sin u}{\cos u}\,du = \int \sec u \tan u\,du = \sec u + C.$

6.11.11 $\int (1+\sin^2\theta\csc\theta)\,d\theta = \int (1+\sin\theta)\,d\theta = \theta - \cos\theta + C.$

6.11.12 $\int \frac{\sin 2\theta}{\cos\theta}\,d\theta = \int \frac{2\sin\theta\cos\theta}{\cos\theta}\,d\theta = 2\int \sin\theta\,d\theta = -2\cos\theta + C.$

6.11.13

$$\int \frac{(1+\cot^2 x)\cot x}{\csc x}\,dx = \int \frac{\csc^2 x\cot x}{\csc x}dx = \int \csc x\cot x\,dx$$
$$= -\csc x + C\,.$$

6.11.14 $\frac{d}{dx}(\sin^2 x) = 2\sin x\cos x;$ $\frac{d}{dx}(-\cos^2 x) = -2\cos x(-\sin x) = 2\sin x\cos x;$
$\frac{d}{dx}(-\frac{1}{2}\cos 2x) = -\frac{1}{2}(-2\sin 2x) = \sin 2x = 2\sin x\cos x.$

6.11.15 $u = x^2+1, du = 2xdx,$

$$\int 2x(x^2+1)^9\,dx = \int u^9\,du = \frac{u^{10}}{10} + C = \frac{(x^2+1)^{10}}{10} + C\,.$$

6.11.16 $u = x^2+6; du = 2x\,dx; x\,dx = \frac{1}{2}\,du,$

$$\int \frac{x}{x^2+6}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+6) + C\,.$$

6.11.17 $u = 1+\cos 3x; du = -3\sin 3x\,dx,$

$$\int \frac{\sin 3x}{(1+\cos 3x)}\,dx = -\frac{1}{3}\int \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|1+\cos 3x| + C\,.$$

6.11.18 $u = 1+e^{2x}; du = 2e^{2x}\,dx,$

$$\int e^{2x}\sqrt{1+e^{2x}}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2} = \frac{1}{3}(1+e^{2x})^{3/2} + C\,.$$

6.11.19 $u = 1-\cos 2x, \frac{1}{2}\,du = \sin 2x\,dx,$

$$\int \frac{\sin 2x}{\sqrt{1-\cos 2x}}\,dx = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C = \sqrt{1-\cos 2x} + C\,.$$

6.11.20 $u = 1+\cos x; -du = \sin x\,dx,$

$$\int \sin x(1+\cos x)^2\,dx = -\int u^2\,du = -\frac{1}{3}u^3 + C$$
$$= -\frac{1}{3}(1+\cos x)^3 + C = -\frac{1}{3}(1+\cos^3 x + 3\cos^2 x + 3\cos x) + C$$

$$= -\cos x - \cos^2 x - \frac{1}{3}\cos^3 x + D\,, \quad \text{where } D = C - \frac{1}{3}.$$

6.11.21 $u = \sin 3\theta$; $du = 3\cos 3\theta\, d\theta$,

$$\int \sin^5 3\theta \cos 3\theta\, d\theta = \frac{1}{3}\int u^5\, du = \frac{1}{3}\frac{u^6}{6} + C = \frac{1}{18}\sin^6 3\theta + C\,.$$

6.11.22 (a) $\displaystyle\int_{-2}^{6} 12\, dx = \left[12x\right]_{-2}^{6} = 12\cdot 6 - 12\cdot(-2) = 72 + 24 = 96.$

(b)

$$\int_{-1}^{4}(4-6x)\, dx = \left[4x - 6\frac{x^2}{2}\right]_{-1}^{4} = \left(4\cdot 4 - 6\cdot\frac{4^2}{2}\right) - \left(4\cdot(-1) - 6\cdot\frac{1}{2}\right)$$
$$= (16-48) - (-4-3) = -32 + 7 = -25\,.$$

(c) $\displaystyle\int_{3}^{3} 900\, dx = 0.$

(d) $\displaystyle\int_{-4}^{4} 2\, dx = \left[2x\right]_{-4}^{4} = 2\cdot 4 - 2\cdot(-4) = 8 + 8 = 16.$

(e)

$$\begin{aligned}\int_{2}^{8}(2x^2+5x+2)\, dx &= \left[2\frac{x^3}{3} + 5\frac{x^2}{2} + 2x\right]_{2}^{8}\\ &= \left(\frac{2}{3}\cdot 8^3 + 5\cdot\frac{8^2}{2} + 16\right) - \left(2\cdot\frac{2^3}{3} + 5\cdot\frac{2^2}{2} + 4\right)\\ &= \left(\frac{1024}{3} + 160 + 16\right) - \left(\frac{16}{3} + 10 + 4\right)\\ &= \left(\frac{1024}{3} - \frac{16}{3}\right) + 162\\ &= \frac{1008}{3} + 162 = 336 + 162 = 498\,.\end{aligned}$$

(f)

$$\int_{0}^{2}\sqrt{4-x^2}\, dx = \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{0}^{2} \quad (\text{substituting } x = 2\sin\theta)$$
$$= 0 + 2\sin^{-1}1 - 2\sin^{-1}0 = 2\cdot\frac{\pi}{2} = \pi\,.$$

(g)

$$\int_{-\pi/3}^{\pi/3} \sin x\, dx = \big[-\cos x\big]_{-\pi/3}^{\pi/3} = -\left(\cos\frac{\pi}{3} - \cos\left(-\frac{\pi}{3}\right)\right)$$
$$= -\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{3}\right) = 0\,.$$

Alternatively, since the sine is an odd function ($\sin(-x) = -\sin x$), we conclude directly that

$$\int_{-\pi/3}^{\pi/3} \sin x\, dx = 0\,.$$

6.11.23 (a)

$$\int_6^1 f(x)\,dx + \int_{-3}^6 f(x)\,dx = -\int_1^6 f(x)\,dx + \int_{-3}^1 f(x)\,dx + \int_1^6 f(x)\,dx$$
$$= \int_{-3}^1 f(x)\,dx\,.$$

(b)

$$\int_{-2}^6 f(x)\,dx - \int_{-2}^2 f(x)\,dx = \left(\int_{-2}^2 f(x)\,dx + \int_2^6 f(x)\,dx\right) - \int_{-2}^2 f(x)\,dx$$
$$= \int_2^{-2} f(x)\,dx + \int_{-2}^6 f(x)\,dx = \int_2^6 f(x)\,dx\,.$$

(c) $\displaystyle\int_d^h f(x)\,dx - \int_g^h f(x)\,dx = \int_d^h f(x)\,dx + \int_h^g f(x)\,dx = \int_d^g f(x)\,dx.$

6.11.24 (a)

$$\int_{-3}^7 f(x)\,dx = \int_{-3}^4 -(x-4)\,dx + \int_4^7 (x-4)\,dx$$
$$= \left[4x - \frac{x^2}{2}\right]_{-3}^4 + \left[\frac{1}{2}x^2 - 4x\right]_4^7$$
$$= \left[(16-8) - \left(-12 - \frac{9}{2}\right)\right] + \left[\left(\frac{49}{2} - 28\right) - (8-16)\right]$$
$$= \left[8 - \left(-\frac{33}{2}\right)\right] + \left[-\frac{7}{2} + 8\right] = 16 + 13 = 29\,.$$

(b)

$$\int_{-2}^{8} g(x)\,dx = \int_{-2}^{3} 1\,dx + \int_{3}^{8} x\,dx = [x]_{-2}^{3} + \left[\frac{x^2}{2}\right]_3^8$$
$$= (3-(-2)) + \left(\frac{64}{2} - \frac{9}{2}\right) = 5 + \frac{55}{2} = \frac{65}{2}\,.$$

6.11.25 (a) $\int \ln x\,dx = \int 1\cdot\ln x\,dx = x\cdot\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - x + C.$

(b)

$$\int (\cos x)^2\,dx = \int (\cos x)(\cos x)\,dx = \sin x\cos x - \int \sin x(-\sin x)\,dx$$
$$= \sin x\cos x + \int \sin^2 x\,dx$$
$$= \sin x\cos x + \int \frac{(1-\cos 2x)}{2}\,dx$$
$$= \sin x\cos x + \frac{x}{2} - \frac{\sin 2x}{4} + C$$
$$= \frac{\sin 2x}{2} + \frac{x}{2} - \frac{\sin 2x}{4} + C = \frac{1}{4}\sin 2x + \frac{x}{2} + C\,.$$

(c)

$$\int \tan^{-1} x\,dx = \int 1\cdot\tan^{-1} x\,dx = x\tan^{-1} x - \int x\frac{1}{1+x^2}\,dx$$
$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$
$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$
$$= x\tan^{-1} x - \ln\sqrt{1+x^2} + C\,.$$

(d)

$$\int x^3 e^x\,dx = e^x\cdot x^3 - \int 3x^2 e^x\,dx$$
$$= x^3e^x - 3\left[x^2e^x - \int 2xe^x\,dx\right]$$
$$= x^3e^x - 3x^2e^x + 6\left[xe^x - \int e^x\,dx\right]$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C\,.$$

(e)

$$\begin{aligned}\int_0^1 xe^{-3x}\,dx &= \left[x\frac{e^{-3x}}{-3}\right]_0^1 + \frac{1}{3}\int_0^1 e^{-3x}\,dx\\ &= -\frac{1}{3}e^{-3} + \frac{1}{3}\left[\frac{e^{-3x}}{-3}\right]_0^1 = -\frac{e^{-3}}{3} - \frac{1}{9}(e^{-3}-1)\\ &= -\frac{4}{9}e^{-3} + \frac{1}{9}\,.\end{aligned}$$

(f)

$$\begin{aligned}\int_0^4 \ln(x^2+1)\,dx &= \int_0^4 1.\ln(x^2+1)\,dx\\ &= \big[x\ln(x^2+1)\big]_0^4 - \int_0^4 \frac{x}{x^2+1}\cdot 2x\,dx\\ &= 4\ln 17 - 2\int_0^4 \frac{x^2+1-1}{x^2+1}\,dx\\ &= 4\ln 17 - 2\left[\int_0^4 1\,dx - \int_0^4 \frac{1}{x^2+1}\,dx\right]\\ &= 4\ln 17 - 2\big[x\big]_0^4 - \big[\tan^{-1}x\big]_0^4\\ &= 4\ln 17 - 8 - \tan^{-1}4\,.\end{aligned}$$

6.11.26 (a)

$$\begin{aligned}\int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\big[-e^{-x}\big]_0^b\\ &= \lim_{b\to\infty}(-e^{-b}+1) = 1\,.\end{aligned}$$

(b)

$$\begin{aligned}\int_{-1}^\infty \frac{x}{1+x^2}\,dx &= \lim_{b\to\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_{-1}^b\\ &= \lim_{l\to\infty}\frac{1}{2}[\ln(1+l^2) - \ln 2] = +\infty\,,\end{aligned}$$

so this integral is divergent.

(c)

$$\int_5^\infty \frac{2}{x^2-1}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx = \lim_{b\to\infty}\left[\ln\left(\frac{x-1}{x+1}\right)\right]_5^b$$
$$= \lim_{b\to\infty}\left(\ln\left(\frac{b-1}{b+1}\right)-\ln\frac{4}{6}\right) = -\ln\frac{2}{3} = \ln\frac{3}{2}\,.$$

(d)

$$\int_{-\infty}^0 \frac{e^x}{3-2e^x}\,dx = \lim_{b\to-\infty}\frac{1}{2}\int_b^0 \frac{2e^x}{3-2e^x}\,dx = \lim_{b\to-\infty}\left[-\frac{1}{2}\ln|3-2e^x|\right]_b^0$$
$$= \lim_{b\to-\infty}\frac{1}{2}\ln(3-2e^b) = \frac{1}{2}\ln 3\,.$$

(e)

$$\int_1^\infty \frac{1}{x^4}\,dx = \lim_{b\to\infty}\left[\frac{x^{-3}}{-3}\right]_1^b = \lim_{b\to\infty}-\frac{1}{3}\left(\frac{1}{b^3}-1\right) = \frac{1}{3}\,.$$

6.11.27 Let

$$\frac{6x^2+13x+6}{(x+2)(x+1)^2} = \frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\,.$$

This implies

$$6x^2+13x+6 = A(x+1)^2+B(x+2)(x+1)+C(x+2)\,.$$

Putting $x=-1$ gives $C=-1$, putting $x=-2$ gives $A=4$, putting $x=0$ gives $6=A+2B+2C$ which implies $B=2$. Therefore,

$$\int \frac{6x^2+13x+6}{(x+2)(x+1)}\,dx = 4\int\frac{1}{x+2}\,dx + 2\int\frac{1}{x+1}\,dx - \int\frac{1}{(x+1)^2}\,dx$$
$$= 4\ln|x+2| + 2\ln|x+1| + \frac{1}{x+1} + D\,.$$

6.11.28 (a)

$$\int\cos^5\theta\,d\theta = \int(1-\sin^2\theta)^2\cos\theta\,d\theta$$
$$= \int(1-2\sin^2\theta+\sin^4\theta)\cos\theta\,d\theta$$
$$= \int\cos\theta\,d\theta - 2\int\sin^2\theta\cos\theta\,d\theta + \int\sin^4\theta\cos\theta\,d\theta$$
$$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C\,.$$

(b)

$$\begin{aligned}\int \sin^3 x \cos^3 x\, dx &= \int \sin^3 x(1-\sin^2 x)\cos x\, dx\\ &= \int \sin^3 x \cos x\, dx - \int \sin^5 x \cos x\, dx\\ &= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C\,.\end{aligned}$$

(c)

$$\begin{aligned}\int_0^{\pi/6} \sec^3\theta \tan\theta\, d\theta &= \int_0^{\pi/6} \sec^2\theta \cdot \sec\theta\tan\theta\, d\theta\\ &= \left[\frac{1}{3}\sec^3\theta\right]_0^{\pi/6} = \frac{1}{3}\left[\left(\frac{2}{\sqrt{3}}\right)^3 - 1\right].\end{aligned}$$

(d)

$$\begin{aligned}&\int_0^{\pi/3} (\sin^4 3x)(\cos^3 3x)\, dx = \int_0^{\pi/3} (\sin^4 3x)(1-\sin^2 3x)\cos 3x\, dx\\ &= \frac{1}{3}\int_0^{\pi/3} (\sin^4 3x)\cdot 3\cos 3x\, dx - \frac{1}{3}\int_0^{\pi/3} 3(\sin^6 3x)\cos 3x\, dx\\ &= \left[\frac{1}{15}\sin^5 3x - \frac{1}{21}\sin^7 3x\right]_0^{\pi/3} = 0\,.\end{aligned}$$

(e) Let $u = \sin\theta$, then $du = \cos\theta\, d\theta$,

$$\begin{aligned}\int \frac{\cos\theta}{\sqrt{2-\sin^2\theta}}\, d\theta &= \int \frac{1}{\sqrt{2-u^2}}\, du = \sin^{-1}\frac{u}{\sqrt{2}} + C\\ &= \sin^{-1}\left(\frac{\sin\theta}{\sqrt{2}}\right) + C\,.\end{aligned}$$

(f) Let $x = \sqrt{3}\tan\theta$, then $dx = \sqrt{3}\sec^2\theta\, d\theta$ and $3 + x^2 = 3\sec^2\theta$,

$$\begin{aligned}&\int_0^3 \frac{x^3}{(3+x^2)^{5/2}}\, dx = \int_0^{\pi/3} \frac{3^{3/2}\tan^3\theta}{3^{5/2}\sec^5\theta}\sqrt{3}\sec^2\theta\, d\theta\\ &= \frac{1}{\sqrt{3}}\int_0^{\pi/3} \sin^3\theta\, d\theta = \frac{1}{\sqrt{3}}\int_0^{\pi/3} (1-\cos^2\theta)\sin\theta\, d\theta\\ &= \frac{1}{\sqrt{3}}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi/3} = \frac{1}{\sqrt{3}}\left[\left(-\frac{1}{2}+\frac{1}{24}\right) - \left(-1+\frac{1}{3}\right)\right]\\ &= \frac{5\sqrt{3}}{72}\,.\end{aligned}$$

Solutions to the Exercises of Chap. 7

7.7.1 (a) The region can be described as (see Fig. E.5)

$$\left\{(x, y) : 0 \le x \le \pi,\ \sin 4x \le y \le 1 + \cos \frac{x}{3}\right\}.$$

The area of this region becomes

$$A = \int_0^{\pi} \left[\left(1 + \cos \frac{x}{3}\right) - \sin 4x\right] dx = \left[x + 3\sin \frac{x}{3} + \frac{1}{4}\cos 4x\right]_0^{\pi}$$
$$= \pi + \frac{3\sqrt{3}}{2} + \frac{1}{4} - \frac{1}{4} = \pi + \frac{3}{2}\sqrt{3} \approx 5.74\,.$$

(b) The region can be described as

$$\left\{(x, y) : 0 \le x \le \pi,\ 3\sin \frac{x}{2} \le y \le 4 + \cos 2x\right\}.$$

Its area is

$$A = \int_0^{\pi} \left[(4 + \cos 2x) - 3\sin \frac{x}{2}\right] dx = \left[4x + \frac{1}{2}\sin 2x + 6\cos \frac{x}{2}\right]_0^{\pi}$$
$$= 4\pi - 6 \approx 6.57\,.$$

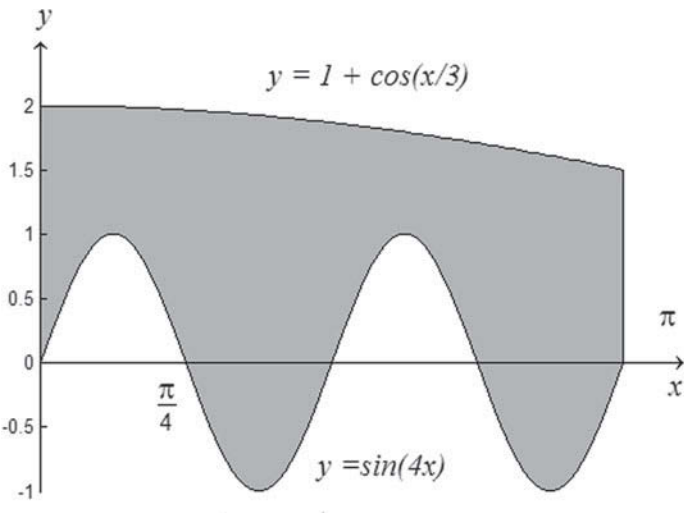

Fig. E.5 Region whose area has to be computed

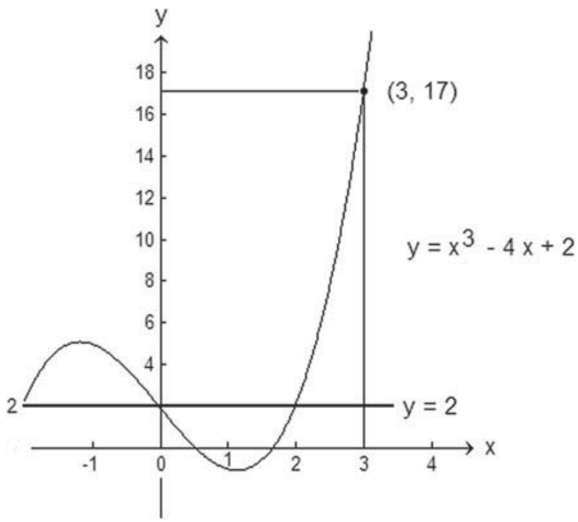

Fig. E.6 Area between two curves

7.7.2 (a) On $[-1, 3]$, $f(x) = g(x)$ at $x = 0$ and $x = 2$, and the graphs of f and g intersect at $(0, 2)$ and $(2, 2)$, see Fig. E.6. We have $f(x) \geq g(x)$ on $[-1, 0]$ and on $[2, 3]$, while $f(x) \leq g(x)$ on $[0, 2]$. We obtain the area as

$$
\begin{aligned}
A &= \int_{-1}^{0} [(x^3 - 4x + 2) - 2]\, dx + \int_{0}^{2} [2 - (x^3 - 4x + 2)]\, dx \\
&\quad + \int_{2}^{3} [(x^3 - 4x + 2) - 2]\, dx \\
&= \frac{7}{4} + 4 + \frac{25}{4} = 12\,.
\end{aligned}
$$

(b) On $[0, 2\pi]$, $f(x) = g(x)$ at $x = \dfrac{\pi}{4}$ and $x = \dfrac{5\pi}{4}$, and the graphs of f and g intersect at the corresponding points, see Fig. E.7. We have $f(x) \geq g(x)$ on $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$ and $f(x) \leq g(x)$ on $\left[0, \dfrac{\pi}{4}\right] \cup \left[\dfrac{5\pi}{4}, 2\pi\right]$. The required area becomes

$$
\begin{aligned}
A &= \int_{0}^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx \\
&\quad + \int_{5\pi/4}^{2\pi} (\cos x - \sin x)\, dx \\
&= (\sqrt{2} - 1) + 2\sqrt{2} + (1 + \sqrt{2}) = 4\sqrt{2}\,.
\end{aligned}
$$

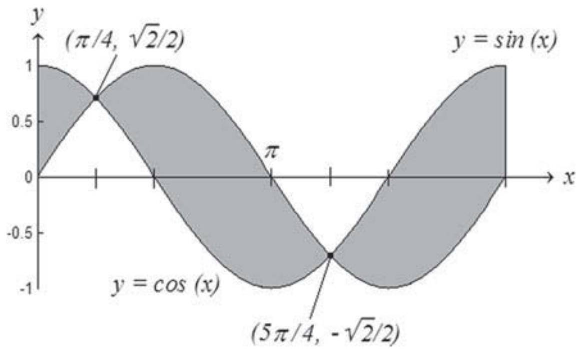

Fig. E.7 Area between two curves

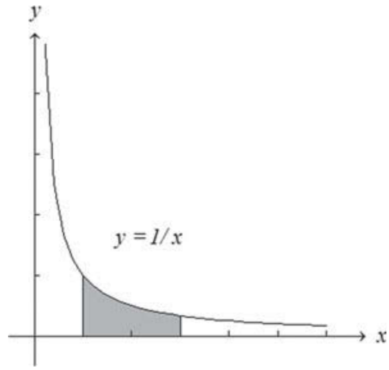

Fig. E.8 Region defining a solid of revolution

7.7.3 (a) Let $f(x) = \frac{1}{x}$. The region D is sketched in Fig. E.8. The volume of the solid of revolution generated by D becomes

$$V = \pi \int_1^3 \left(\frac{1}{x}\right)^2 dx = \pi \left[-\frac{1}{x}\right]_1^3 = \pi\left(-\frac{1}{3} - (-1)\right) = \frac{2\pi}{3}\,.$$

(b) The graphs of $x = \frac{1}{2}y$ and $x = \frac{1}{2}\sqrt{y}$ intersect for $y = 0$ and $y = 1$, and $\frac{1}{2}\sqrt{y} \geq \frac{1}{2}y$ on [0, 1], see Fig. E.9. The volume of the corresponding solid of revolution becomes

$$\begin{aligned} V &= \pi \int_0^1 \left(\frac{1}{2}\sqrt{y}\right)^2 - \left(\frac{1}{2}y\right)^2 dy = \frac{\pi}{4}\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 \\ &= \frac{\pi}{4}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{24}\,. \end{aligned}$$

7.7.4 The upper half of the ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is described by the function

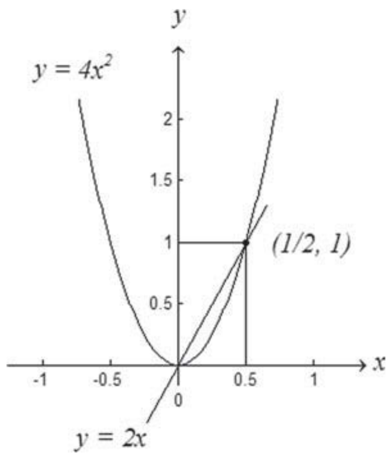

Fig. E.9 Graphs of the functions in Exercise 7.7.3(b)

$$y = f(x) = \frac{b}{a}\sqrt{a^2 - x^2}\,, \quad \text{and} \quad f'(x) = -\frac{bx}{a}(a^2 - x^2)^{-1/2}\,.$$

The circumference C equals 4 times the length of the graph of f for $x \geq 0$,

$$C = 4\int_0^a \sqrt{1 + f'(x)^2}\,dx = 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}\,dx\,.$$

To evaluate the integral, substitute $x = a\sin\theta$. Then $dx = a\cos\theta\,d\theta$, and the integration limits for θ are 0 and $\pi/2$. We compute

$$\begin{aligned} C &= 4\int_0^{\pi/2} \sqrt{1 + \frac{b^2a^2\sin^2\theta}{a^2\cdot a^2\cos^2\theta}}\,a\cos\theta\,d\theta \\ &= 4a\int_0^{\pi/2} \sqrt{\cos^2\theta + \frac{b^2}{a^2}\sin^2\theta}\,d\theta \\ &= 4a\int_0^{\pi/2} \sqrt{\cos^2\theta + \sin^2\theta - e^2\sin^2\theta}\,d\theta, \end{aligned}$$

(note that $b^2 = a^2(1 - e^2)$, where e is the eccentricity)

$$= 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\,d\theta\,.$$

7.7.5 By Hooke's law, $F(x) = kx$ for some constant k where F is the force and x is the change in length relative to the natural length, so

$$25 = k\frac{0.08 - 0.75}{0.08} = k\frac{1}{16}, \quad k = 400\,.$$

The work done becomes

$$W = \int_0^{0.80-0.70} 400x\,dx = \int_0^{0.1} 400x\,dx = 200 \cdot (0.1)^2 = 2\ \text{Nm}.$$

7.7.6 Let the rope be placed along the y-axis, and let $y = 0$ be the initial position of the bottom of the rope. If the rope is pulled y m then $(60 - y)$ m of the rope is suspended with a weight of $\frac{1}{4}(60 - y)$ kg. Therefore, the work done in lifting the rope equals $g\int_0^{60}(60 - y)\frac{1}{4}\,dy$ Newton meter. The work done in lifting the motor through 60 m is $60 \cdot 50g$ Nm = 29430 Nm. The total work becomes

$$W = 29430 + \frac{g}{4}\int_0^{60}(60 - y)\,dy = 29430 + 9.81 \cdot \left[15y - \frac{1}{8}y^2\right]_0^{60}$$
$$= 33844.5\ \text{Nm}.$$

7.7.7 Let $F = \dfrac{k}{d^2}$, where d is the distance between the two electrons and k is the proportionality constant.

(a) The distance between the two electrons is $5 - x$ m, where x is the x-coordinate of the second electron, so $F(x) = k/(5 - x)^2$ N and the work done becomes

$$W = \int_0^3 \frac{k}{(5 - x)^2}\,dx = k\left[\frac{1}{5 - x}\right]_0^3 = \frac{3k}{10}\ \text{Nm}.$$

(b) The distance between the third (moving) electron and the electron at $(-5, 0)$ is $x - (-5) = x + 5$ m. The distance between the moving electron and the electron at $(5, 0)$ is $x - 5$ m. The repelling forces are $\dfrac{k}{(5 - x)^2}$ and $\dfrac{k}{(5 + x)^2}$, respectively. Since the latter acts in the direction of the movement while the former acts against it, the net force is the difference of these two forces, that is, $F(x) = \dfrac{k}{(5 - x)^2} - \dfrac{k}{(5 + x)^2}$. The total work becomes

$$W = \int_0^3 \left[\frac{k}{(5 - x)^2} - \frac{k}{(5 + x)^2}\right] dx = k\left[\frac{1}{5 - x} + \frac{1}{5 + x}\right]_0^3 = \frac{9k}{40}\ \text{Nm}.$$

7.7.8 The amount of gasoline used after 2 h is

$$\int_0^2 t\sqrt{9-t^2}\,dt = \left[-\frac{1}{2}\frac{(9-t^2)^{3/2}}{3/2}\right]_0^2 = 9 - \frac{1}{3}5\sqrt{5} \approx 5.27 \text{ gal.}$$

7.7.9 The average flow rate $\overline{F}$ is computed as

$$\begin{aligned}\overline{F} &= \frac{1}{T-0}\int_0^T F(t)\,dt = \frac{1}{T}\int_0^T \frac{F_1}{(1+\alpha t)^2}\,dt \\ &= \frac{F_1}{\alpha T}\int_0^T (1+\alpha t)^{-2}\alpha\,dt = \frac{F_1}{\alpha T}\left[\frac{(1+\alpha t)^{-1}}{-1}\right]_0^T \\ &= \frac{F_1}{\alpha T}\left[-\frac{1}{1+\alpha T}+1\right] = \frac{F_1}{\alpha T}\left[\frac{-1+1+\alpha T}{1+\alpha T}\right] = \frac{F_1}{\alpha T}\left[\frac{\alpha T}{1+\alpha T}\right] \\ &= \frac{F_1}{1+\alpha T}\,.\end{aligned}$$

7.7.10 The average cost $\overline{C}$ becomes

$$\begin{aligned}\overline{C} &= \frac{1}{400}\int_{100}^{500}(4000+10q+0.1q^2)\,dq = \frac{1}{400}\left[4000q + \frac{10q^2}{2} + \frac{0.1q^3}{3}\right]_{100}^{500} \\ &= \frac{1}{400}\left[4000\cdot 400 + 5(250000-10000) + \frac{0.1}{3}\left((500)^3-(100)^3\right)\right] \\ &= \frac{1}{400}\left[1600000 + 5\cdot 240000 + \frac{0.1}{3}\cdot 24000000\right] \\ &= \frac{1}{4}[16000+12000+8000] = \frac{1}{4}\cdot 36000 = 900 \text{ (Euro).}\end{aligned}$$

7.7.11 The average concentration $\overline{C}$ on $[0, T]$ is

$$\begin{aligned}\overline{C} &= \frac{1}{T}\int_0^T \frac{R}{F_1}(1+\alpha t)^2\,dt = \frac{R}{TF_1}\left[\frac{(1+\alpha t)^3}{3\alpha}\right]_0^T \\ &= \frac{R}{3F_1\alpha T}\left[1+\alpha^3T^3+3\alpha^2T^2+3\alpha T-1\right] \\ &= \frac{R}{3F_1\alpha T}[\alpha^3T^3+3\alpha^2T^2+3\alpha T] = \frac{R}{F_1}\left(1+\alpha T+\frac{1}{3}\alpha^2T^2\right).\end{aligned}$$

7.7.12 (a) The probability of eventual breakdown is

$$\begin{aligned}P(0 \le X < \infty) &= \int_0^\infty p(x)\,dx = \frac{1}{4}\int_0^\infty e^{-x/4}\,dx = -\frac{4}{4}\left[e^{-x/4}\right]_0^\infty . \\ &= -(0-1) = 1\,.\end{aligned}$$

(b) The probability of breakdown within the first 12 years is

$$P(0 \le X \le 12) = \int_0^{12} p(x)\,dx = \frac{1}{4}\int_0^{12} e^{-x/4}\,dx = \frac{1}{4}\left[\frac{e^{-x/4}}{\left(-\frac{1}{4}\right)}\right]_0^{12}$$
$$= -(e^{-12/4} - e^0) = (1 - e^{-3}) \simeq 0.9502\,.$$

7.7.13 (a) The density function (see Fig. 7.25) is zero for all $t > 3$, so no one waits more than 3 h. The longest time anyone has to wait is 3 h.

(b) The fraction of patients who wait between 1 and 2 h is equal to the area under the density function between $t = 1$ and $t = 2$. We estimate this area by counting squares: about 7.5 squares in this region, each of area $0.5 \cdot 0.1 = 0.05$. The approximate area thus becomes $7.5 \cdot 0.05 = 0.375$. Therefore 37.5% of patients wait between 1 and 2 h.

(c) The fraction of patients waiting less than 1 h is equal to the area under the density function for $t < 1$. There are about 12 squares in this area. Thus, the approximate area is $12 \cdot 0.05) = 0.60$. Therefore about 60% of patients see the doctor in less than one hour.

7.7.14 The value $P(t)$ of the cumulative distribution function P equals the fraction of days on which the catch is less than t tons of fish. Since the catch is never less than 2 tons, we have

$$P(t) = 0\,, \quad \text{for } t \le 2.$$

Since the catch is always less than 8 tons, we have

$$P(t) = 1\,, \quad \text{for } t \le 8.$$

For t in the range $2 < t < 8$, the value of $P(t)$ is given by the integral

$$P(t) = \int_{-\infty}^{t} p(x)\,dx = \int_2^t p(x)\,dx\,,$$

which is equal to the area under the graph of p between $x = 2$ and $x = t$. We count the grid squares in the given figure, each square has area 0.04. For example,

$$P(3) = \int_2^3 p(x)\,dx \approx \text{ Area of 2.5 squares } = 2.5 \cdot 0.04 = 0.10\,.$$

The following table contains the values of $P(t)$:

t(tons of fish)	P(t) (fraction of fishing days)
2	0
3	0.10
4	0.24
5	0.42
6	0.64
7	0.85
8	1

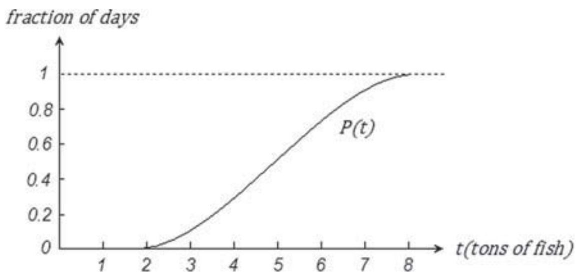

Fig. E.10 Cumulative distribution function

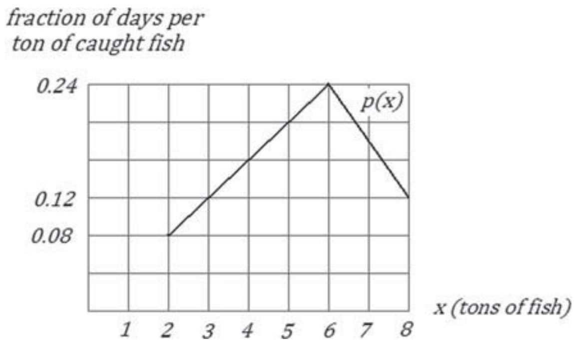

Fig. E.11 Density function

(b) The probability that the catch is between 5 and 7 tons can be found using either the density function p or the cumulative distribution function P (Fig. E.10). Using the density function, this probability is represented by the shaded area in Fig. E.11 which is about 10.75 squares, so the required probability is

$$\int_5^7 p(x)\,dx \;\approx\; \text{Area of 10.75 squares} = 10.75 \cdot 0.04 = 0.43\,.$$

From the cumulative distribution function, this probability can be found as the difference

$$P(7) - P(5) = 0.85 - 0.42 = 0.43\,.$$

7.7.15 (a) The fraction of calls lasting between 1 and 2 min is given as

$$P(1 \le X \le 2) = 0.4 \int_1^2 e^{-0.4x}\,dx = \frac{0.4}{-(0.4)}\Big[e^{-0.4x}\Big]_1^2$$
$$= e^{-0.4} - e^{-0.8} = 0.221\,.$$

(b) $P(X \le 1) = 0.4 \displaystyle\int_0^1 e^{-0.4x}\,dx = -\left[e^{-0.4x}\right]_0^1 = 1 - e^{-0.4} = 0.3297.$

(c)

$$P(0 < X \le 2) = 0.4 \int_0^2 e^{-0.4x}\,dx = -\left[e^{-0.4x}\right]_0^2$$
$$= (1 - e^{-.8}) = 0.5507\,.$$

(d) $P(X \ge 3) = 0.4 \displaystyle\int_3^\infty e^{-0.4x}\,dx = -\left[e^{-0.4}x\right]_3^\infty = e^{-1.2} = 0.3012.$

For percentages, multiply by 100.

7.7.16 We get

$$P(t) = \int_{-\infty}^t p(x)\,dx = \int_{-\infty}^t 0.4e^{-0.4x}\,dx = 0.4 \int_0^t e^{-0.4x}\,dx$$
$$= -\left[e^{-0.4x}\right]_0^t = 1 - e^{-0.4t}\,.$$

7.7.17 The probability having to wait no more than 7 min is

$$P(0 \le X \le 7) = \int_0^7 f(x)\,dx = \frac{1}{10}\int_0^7 1\,dx = \left[\frac{x}{10}\right]_0^7 = \frac{7}{10}\,.$$

The average waiting time is

$$\int_{-\infty}^\infty x f(x)\,dx = \frac{1}{10}\int_0^{10} x\,dx = \frac{1}{10}\left[\frac{x^2}{2}\right]_0^{10} = 5 \text{ min.}$$

7.7.18 Let us recall that the exponential density function f is given by

$$f(x) = \begin{cases} ke^{-kx}\,, & \text{if } x \ge 0, \\ 0\,, & \text{if } x < 0. \end{cases}$$

Here $k = 1/6$. We obtain the required probabilities as

$$P(0 \le X \le 4) = \frac{1}{6}\int_0^4 e^{-x/6}\,dx = -\frac{6}{6}\left[e^{-x/6}\right]_0^4 = 1 - e^{-2/3} = 0.4866\,,$$
$$P(X \ge 6) = \frac{1}{6}\int_6^\infty e^{-x/6}\,dx = -\left[e^{-x/6}\right]_6^\infty = e^{-1} = 0.3679\,.$$

7.7.19 We have $f(x) = 4e^{-4x}$ and

$$P(X \ge 2) = \int_2^\infty f(x)\,dx = 4\int_2^\infty e^{-4x}\,dx = -\left[e^{-4x}\right]_2^\infty = e^{-8}$$
$$= 0.000335\,.$$

7.7.20 Let $v'(t) = a$ be the constant acceleration. Integrating, we get $v(t) = at + C$. Since $v(0) = 0$ we have $C = 0$, thus $v(t) = at$. The distance traveled satisfies $s'(t) = v(t)$. Integrating, we get

$$s(t) = \frac{at^2}{2} + D\,.$$

We have $D = 0$ since $s(0) = 0$, so

$$s(t) = \frac{1}{2}at^2\,, \quad s(10) = \frac{1}{2}a \cdot 100 = 50a\,.$$

The condition $s(10) = 500$ gives $a = 10\,\mathrm{m/s^2}$.

7.7.21 As in the previous exercise, $v(t) = at + C$. Since $v(0) = 60\,\mathrm{km/h} = \frac{50}{3}$ m/s, we get

$$v(t) = at + \frac{50}{3}\,, \quad 0 = v(9) = 9a + \frac{50}{3}\,, \quad a = -\frac{50}{27}\,\frac{\mathrm{m}}{\mathrm{s}^2}\,.$$

The minus sign means deceleration (braking).

7.7.22 Since $A'(t) = 5 + 0.01t$, we obtain the amount $A(t)$ of gas consumed until time t by integration,

$$A(t) = 5t + 0.01\frac{t^2}{2} + C\,.$$

Since $A(0) = 0$ we have $C = 0$. We determine t from

$$100 \cdot 10^9 = 5t + \frac{1}{200}t^2\,,$$

which is equivalent to $t^2 + 10^3 t - 2 \cdot 10^{13} = 0$. Its solutions are

$$t = \frac{-10^3 \pm \sqrt{10^6 + 8 \cdot 10^{13}}}{2} = \frac{-10^3 \pm 10^3\sqrt{1 + 8 \cdot 10^7}}{2}\,,$$

so the positive solution becomes $t = 500(-1 + \sqrt{1 + 8 \cdot 10^7})$ years.

7.7.23 The average rate $\overline{P}$ of heat production is

$$\begin{aligned}
\overline{P} &= \frac{1}{\frac{2\pi}{\omega} - 0} \int_0^{2\pi/\omega} (I_M^2 \sin^2 \omega t) R\, dt \\
&= \left(\frac{\omega R I_M^2}{2\pi}\right) \int_0^{2\pi/\omega} \frac{1}{2}(1 - \cos 2\omega t)\, dt \\
&= \frac{\omega R I_M^2}{4\pi} \left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^{2\pi/\omega} = \frac{\omega R I_M^2}{4\pi}\left(\frac{2\pi}{\omega} - 0\right) \\
&= \frac{1}{2} R I_M^2 .
\end{aligned}$$

Solutions to the Exercises of Chap. 8

8.9.1 (a) If interest is compounded annually, m increases by a factor of 1.05 every year, so

$$m = f(b, t) = b \cdot 1.05^t .$$

(b) If interest is compounded continuously, m increases according to the function e^{kt} with $k = 0.05$, so

$$m = f(b, t) = be^{0.05t} .$$

8.9.2 (a) Keeping x fixed at 8 means that one considers an injection of 8 mg of the drug. The expression $f(8, t)$ describes the concentration of the drug in the blood resulting from an 8 mg injection as a function of time t. Figure E.12 shows the graph of $f(8, t) = te^{-t}$. We notice that the concentration in the blood attains a maximum at 1 h after injection and it eventually approaches zero.

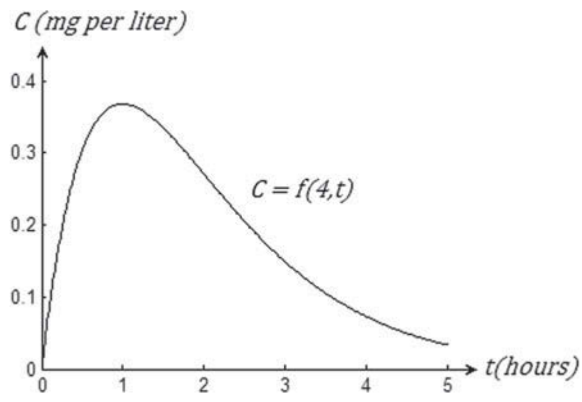

Fig. E.12 The function $f(4, t)$ shows the concentration in the blood resulting from a 4 mg injection

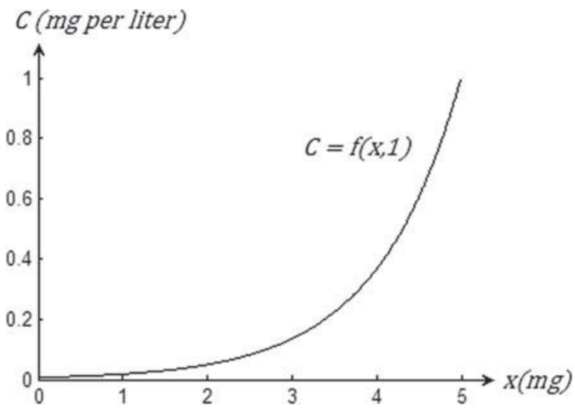

Fig. E.13 The function $f(x, 1)$ shows the concentration in the blood 1 h after the injection

(b) Keeping t fixed at 1 means that we are focusing on the concentration which is present 1 h after the injection. The expression $f(x, 1)$ gives the concentration of the drug in the blood 1 h after the injection as a function of the amount x injected. Figure E.13 shows the graph of $f(x, 1) = e^{-(9-x)} = e^{x-9}$. This function is an increasing function of x.

8.9.3 For $f(x) = x \sin x$ we have

(a) $f(x - y) = (x - y) \sin(x - y)$.
(b) $f(\frac{x}{y}) = \frac{x}{y} \sin(\frac{x}{y})$.
(c) $f(xy) = xy \sin(xy)$.

8.9.4 For $h(x, y, z) = xy^2z^3 + 4$ we have

(a) $h(a + b, a - b, b) = (a + b)(a - b)^2b^3 + 4$.
(b) $h(0, 0, 0) = 4$.
(c) $h(t, t^2, -t) = -t \cdot t^4t^3 + 4 = -t^8 + 4$.
(d) $h(-6, 4, 2) = (-6) \cdot 16 \cdot 8 + 4 = -764$.

8.9.5 (i) The domain of $f(x, y) = xe^{-\sqrt{y+2}}$ is the set all of points (x, y) above or on the line $y = -2$.
(ii) The domain of $f(x, y, z) = e^{xyz}$ is the set of all points in3-space.
(iii) The domain of $f(x, y, z) = \frac{xyz}{x + y + z}$ is the set of all points (x, y, z) which do not lie on the plane $x + y + z = 0$.

8.9.6 (i) See Fig. E.14.
(ii) See Fig. E.15.
(iii) See Fig. E.16.

8.9.7 (a) (i) See Fig. E.17.
(ii) See Fig. E.18.
(b) (i) See Fig. E.19.
(ii) See Fig. E.20.

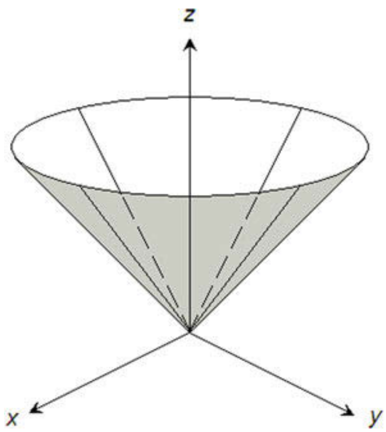

Fig. E.14 Solution of Exercise 8.9.6(a)

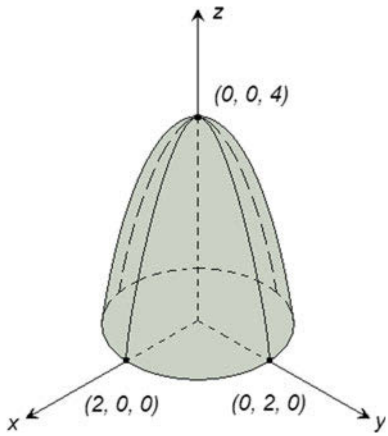

Fig. E.15 Solution of Exercise 8.9.6(b)

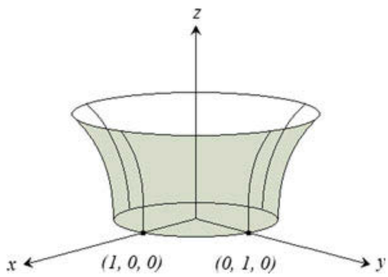

Fig. E.16 Solution of Exercise 8.9.6(c)

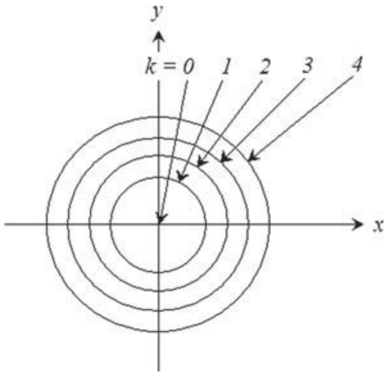

Fig. E.17 Solution of Exercise 8.9.7(a)(i)

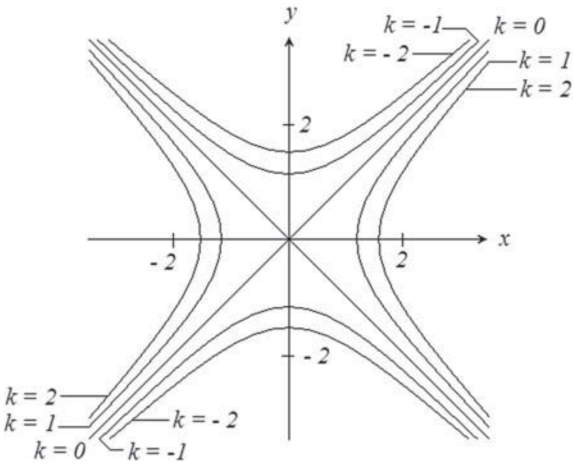

Fig. E.18 Solution of Exercise 8.9.7(a)(ii)

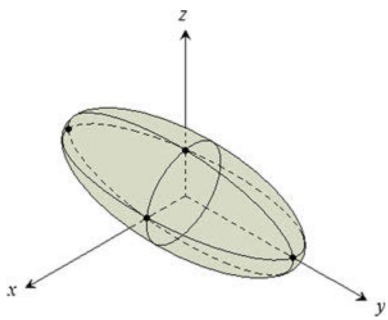

Fig. E.19 Solution of Exercise 8.9.7(b)(i)

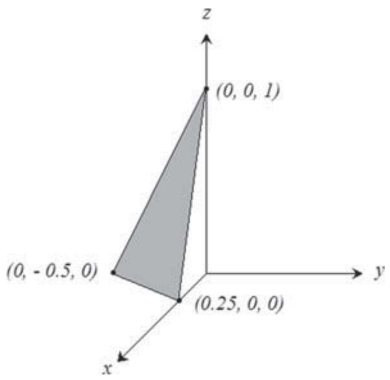

Fig. E.20 Solution of Exercise 8.9.7(b)(ii)

8.9.8 We have $T(x, y) = \dfrac{c}{\sqrt{x^2 + y^2}}$, where c is a constant.

(a) The isothermal curves $T(x, y) = k$ are given by $x^2 + y^2 = \dfrac{c^2}{k^2}$. These are circles with center $(0, 0)$ and radius c/k. To sketch them for $k = 1, 2, 3$ requires knowledge of the value of the proportionality constant c.

(b) If $T = 40\,°\mathrm{C}$ at $(x, y) = (4, 3)$, we can compute c from

$$40 = \frac{c}{\sqrt{4^2 + 3^2}},$$

so $c = 200$. For $T = 20\,°\mathrm{C}$ the equation is

$$x^2 + y^2 = \left(\frac{200}{20}\right)^2 = 100.$$

The isothermal curve for a temperature of $20\,°\mathrm{C}$ is $x^2 + y^2 = 100$.

8.9.9 We obtain $V(x, y) = \dfrac{8}{\sqrt{16 + x^2 + y^2}}$. The equipotential curves are circles.

8.9.10 $PV = nkT$ implies $P(V, T) = \dfrac{nkT}{V}$. The level curves are the curves where $\dfrac{nkT}{V} = c$ holds for some constant c. In this case, $V = \dfrac{nk}{c}T$ gives lines in the VT-plane passing through the origin. Physical Significance: If the state of the system moves along one of those level curves, the pressure remains constant, while volume and temperature change.

8.9.11 (a) We have $P(A, v) = kAv^3$, where $k > 0$ is a constant.

(b) For level curves $P(A, v) = kAv^3 = c$ we get $A = \dfrac{c}{kv^3}$. These level curves show the combinations of area and wind velocity that result in a fixed power $P = c$.

(c) We have $v = 20$ and $A = 5^2\pi$, since the radius equals 5. The given data yield $3000 = k5^2\pi \cdot 20^3$, so $k = \dfrac{3}{200\pi}$. The level curve at $P = 4000$ is described by

$$4000 = \frac{3}{200\pi} A v^3\,, \qquad A v^3 = \frac{8}{3}\pi \times 10^5\,.$$

8.9.12 (a) The isobars are curves of constant pressure, so they are given by $ax^2 + by^2 + c = k$ for some constant $k \geq c$. They are ellipses, alternatively written as

$$\frac{x^2}{\frac{k-c}{a}} + \frac{y^2}{\frac{k-c}{b}} = 1\,.$$

(b) A region of low pressure occurs when its atmospheric pressure p is less than that in the surrounding area. Here, the minimum value $p(0, 0) = c$ of the atmospheric pressure occurs at the origin. As we move away from origin, the atmospheric pressure increases. Hence there is an area of low pressure near the origin.

8.9.13 (a) The function $f(x, y) = \dfrac{xy^3}{x + y}$ is continuous at $(-1, 2)$. Therefore, its limit at this point exists and equals $f(-1, 2)$, so

$$\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x + y} = \frac{(-1)\cdot 2^3}{(-1) + 2} = -8.$$

(b) Along $y = 0$, $\dfrac{x - y}{x^2 + y^2} = \dfrac{1}{x}$, therefore the limit as $x \to 0$ does not exist because $\dfrac{1}{x} \to \infty$ as $x \to 0$. So $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x - y}{x^2 + y^2}$ does not exist.

8.9.14 We have

$$f(x, y) = \begin{cases} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}\,, & (x, y) \neq (0, 0)\,, \\ 1\,, & (x, y) = (0, 0)\,. \end{cases}$$

Take $z = g(x, y) = x^2 + y^2$. Then $\displaystyle\lim_{z\to 0^+} \frac{\sin z}{z} = 1 = f(0, 0)$. Since moreover g is continuous, f is continuous at 0.

8.9.15 (a) We have $f(x, y) = \ln(x + y - 1)$. The logarithm is continuous for arguments in $(0, \infty)$, and $x + y - 1 > 0$ holds if and only if $x + y > 1$. Therefore, f is continuous on the set $\{(x, y) : x + y > 1\}$.

(b) We have $f(x, y) = \sqrt{x}e^{\sqrt{1-y^2}}$. The square root is defined and continuous for nonnegative arguments. In order that f is defined in a neighborhood of (x, y), we must have $x > 0$ and moreover $1 - y^2 > 0$

which is equivalent to $|y| < 1$. Therefore, f is continuous on the set $\{(x, y) : x > 0 \text{ and } |y| < 1\}$.

8.9.16 (a) For $z(x, y) = y^2 e^{x^2} + \dfrac{1}{x^2 y^3}$ we get

$$\begin{aligned}
\frac{\partial z}{\partial x}(x, y) &= 2xy^2 e^{x^2} - 2x^{-3}y^{-3}\,, \\
\frac{\partial^2 z}{\partial y \partial x}(x, y) &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)(x, y) = 4xye^{x^2} + 6x^{-3}y^{-4}\,, \\
\frac{\partial z}{\partial y}(x, y) &= 2ye^{x^2} - 3x^{-2}y^{-4}\,, \\
\frac{\partial^2 z}{\partial x \partial y}(x, y) &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)(x, y) = 4xye^{x^2} + 6x^{-3}y^{-4}\,.
\end{aligned}$$

(b) For $w(x, y) = \dfrac{x^2}{y^2 + z^2}$ we get

$$\begin{aligned}
\frac{\partial w}{\partial y}(x, y) &= -2x^2 y (y^2 + z^2)^{-2}\,, \\
\frac{\partial^2 w}{\partial y^2}(x, y) &= -2x^2 (y^2 + z^2)^{-2} + 8x^2 y^2 (y^2 + z^2)^{-3}\,, \\
\frac{\partial^2 w}{\partial z \partial y^2}(x, y) &= 8x^2 z (y^2 + z^2)^{-3} - 48x^2 y^2 z (y^2 + z^2)^{-4} \\
&= 8x^2 z (y^2 + z^2)^{-4}[(y^2 + z^2) - 6y^2] \\
&= \frac{8x^2 z (z^2 - 5y^2)}{(y^2 + z^2)^4}\,.
\end{aligned}$$

8.9.17 (a) For $f(x, y) = \arctan \dfrac{y}{x}$ we get

$$\begin{aligned}
\frac{\partial f}{\partial x}(x, y) &= -\frac{y}{x^2}\frac{1}{\left[1 + \left(\frac{y}{x}\right)^2\right]} = -\frac{y}{x^2 + y^2}\,, \quad \frac{\partial^2 f}{\partial x^2}(x, y) = \frac{2xy}{(x^2 + y^2)^2}\,, \\
\frac{\partial f}{\partial y}(x, y) &= \frac{1}{x}\frac{1}{\left[1 + \frac{y}{x}\right]^2} = \frac{x}{x^2 + y^2}\,, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = -\frac{2xy}{(x^2 + y^2)^2}\,.
\end{aligned}$$

Hence $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

(b) For $f(x, t) = (x - at)^4 + \cos(x + at)$ we get

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,t) &= 4(x-at)^3 - \sin(x+at)\,,\\
\frac{\partial^2 f}{\partial x^2}(x,t) &= 12(x-at)^2 - \cos(x+at)\,,\\
\frac{\partial f}{\partial t}(x,t) &= -4a(x-at)^3 - a\sin(x+at)\,,\\
\frac{\partial^2 f}{\partial t^2}(x,t) &= 12a^2(x-at)^2 - a^2\cos(x+at) = a^2\frac{\partial^2 f}{\partial x^2}(x,t)\,.
\end{aligned}$$

8.9.18 (a) For $u = x^2 - y^2$ and $v = 2xy$ we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y\,.$$

(b) For $u = e^x \cos y$ and $v = e^x \sin y$ we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x\cos y\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x\sin y\,.$$

(c) For $u = \ln(x^2+y^2)$ and $v = 2\tan^{-1}(\frac{x}{y})$ we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{2x}{x^2+y^2}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{2y}{x^2+y^2}\,.$$

8.9.19 For $z = f(u)$ and $u = g(x,y)$ we have

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}\,, \quad \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}\,.$$

Therefore

$$\begin{aligned}
&\text{(i)} \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{dz}{du}\frac{\partial u}{\partial x}\right) = \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{dz}{du}\frac{\partial^2 u}{\partial x^2},\\
&\text{(ii)} \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{dz}{du}\frac{\partial u}{\partial y}\right) = \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{dz}{du}\frac{\partial^2 u}{\partial y^2},\\
&\text{(iii)} \quad \frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{dz}{du}\frac{\partial u}{\partial x}\right) = \frac{d^2 z}{du^2}\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{dz}{du}\frac{\partial^2 u}{\partial y\partial x}.
\end{aligned}$$

8.9.20 Let a, b be the length and width of parallelogram and α be the angle between the two sides, see Fig. E.21. We must have $a > 0$, $b > 0$ and $0 < \alpha < \pi$. Since $2a + 2b = l$ for l given, $b = \dfrac{l-2a}{2}$. The area A of the parallelogram is equal to $ab\sin\alpha$, so

$$A(a,\alpha) = \frac{a}{2}(l-2a)\sin\alpha\,.$$

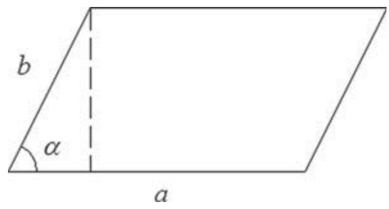

Fig. E.21 Maximizing the area of a parallelogram

We have

$$\frac{\partial A}{\partial \alpha}(a,\alpha) = \frac{a}{2}(l-2a)\cos\alpha\,, \quad \frac{\partial A}{\partial a}(a,\alpha) = \frac{1}{2}(l-4a)\sin\alpha\,.$$

At the maximum, the partial derivatives are zero.
$\frac{\partial A}{\partial a} = 0$ gives $a = \frac{l}{4}$, since $\sin\alpha \neq 0$.
$\frac{\partial A}{\partial \alpha} = 0$ gives $\cos\alpha = 0$, that is, $\alpha = \frac{\pi}{2}$. The second derivatives become

$$A_{aa} = -2\sin\alpha\,, \quad A_{\alpha\alpha} = -\frac{a}{2}(l-2a)\sin\alpha\,, \quad A_{a\alpha} = \frac{1}{2}(l-4a)\cos\alpha\,.$$

For $a = \frac{l}{4}$ and $\alpha = \frac{\pi}{2}$ we obtain

$$A_{aa}A_{\alpha\alpha} - A_{a\alpha}^2 = 2\cdot\frac{l}{8}\cdot\frac{l}{2} = \frac{l^2}{8} > 0\,, \quad A_{aa} = -2 < 0\,.$$

Therefore the area is maximal for $a = \frac{l}{4}$ and $\alpha = \frac{\pi}{4}$, that is, for a square with side length $\frac{l}{4}$.

8.9.21 Let $x > 0$, $y > 0$ and $z > 0$ be the length, width and height respectively of the box. Assume that the amount of material needed is proportional to the area S to be covered, which is given by $S = xy + 2xz + 2yz$. Since the volume $V = xyz$ is a given fixed number, we can eliminate $z = \frac{V}{xy}$, so

$$S(x,y) = xy + 2\frac{V}{y} + 2\frac{V}{x}\,.$$

Setting $S_x = y - \frac{2V}{x^2} = 0$, $S_y = x - \frac{2V}{y^2} = 0$ and solving for x, y, we obtain $x = (2V)^{1/3}$, $y = (2V)^{1/3}$. Therefore, the critical point is $((2V)^{1/3}, (2V)^{1/3})$. The second partial derivatives are

$$S_{xx} = \frac{4V}{x^3}\,, \quad S_{yy} = \frac{4V}{y^3}\,, \quad S_{xy} = 1\,.$$

For $x = (2V)^{1/3}$ and $y = (2V)^{1/3}$, we get $S_{xx}S_{yy} - S_{xy}^2 = 2 \cdot 2 - 1 = 3 > 0$ and $S_{xx} = 2 > 0$. So at the point $((2V)^{1/3}, (2V)^{1/3})$ there is a relative minimum. The corresponding length and width is $x = y = (2V)^{1/3}$, the height is $z = \dfrac{(2V)^{1/3}}{2}$.

8.9.22 Let x, y and z be the length, width and height of the box, respectively. We have to minimize $C = 2 \cdot 2xy + 2xz + 2yz$, given $8 = xyz$. Taking $z = \dfrac{8}{xy}$, the cost becomes

$$C(x, y) = 4xy + \frac{16}{y} + \frac{16}{x}\,.$$

Setting $C_x = 4y - \dfrac{16}{x^2} = 0$ and $C_y = 4x - \dfrac{16}{y^2} = 0$, we get $yx^2 = 4 = xy^2$ and hence $x = y = 4^{1/3}$ ft, since $x > 0$ and $y > 0$). Moreover, $z = \dfrac{8}{xy} = 2 \cdot 4^{1/3}$ ft. The second derivatives are

$$C_{xx} = \frac{32}{x^3}\,, \quad C_{yy} = \frac{32}{y^3}\,, \quad C_{xy} = 4\,.$$

For $x = y = 4^{1/3}$ we get $C_{xx}C_{yy} - C_{xy}^2 = 8 \cdot 8 - 16 > 0$ and $C_{xx} = 8 > 0$. Hence the cost is minimum for $x = y = 4^{1/3}$ and $z = 2 \cdot 4^{1/3}$ ft.

8.9.23 The perimeter of the window is $P = x + 2y + x \sec\theta$, its area is

$$A = xy + \frac{1}{2}x \cdot \frac{1}{2}x \tan\theta = xy + \frac{1}{4}x^2 \tan\theta\,.$$

Since $P = 4$, we can solve for y to obtain $y = \frac{1}{2}(4 - x - x\sec\theta)$. Thus

$$A(x, \theta) == 2x - \frac{1}{4}x^2(2 + 2\sec\theta - \tan\theta)\,.$$

In order to maximize A, we first set

$$0 = A_\theta = -\frac{1}{4}x^2(2\sec\theta\tan\theta - \sec^2\theta)\,,$$

which gives $2\tan\theta = \sec\theta$, since $x^2 \neq 0$ and $\sec\theta \neq 0$, so $2\sin\theta = 1$ and $\theta = \frac{\pi}{6}$. Next, we set

$$0 = A_x = 2 - \frac{1}{2}x(2 + 2\sec\theta - \tan\theta) = 2 - \frac{1}{2}x\left(2 + \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\,,$$

which gives $2 - \dfrac{1}{2}x(2+\sqrt{3}) = 0$, hence

$$x = \frac{4}{2+\sqrt{3}} = 8 - 4\sqrt{3}\,, \quad y = 2 - \frac{2}{3}\sqrt{3}\,.$$

8.9.24 We will fit the data to a line $y = ax + b$ using the method of least squares. We get

$$\sum_{k=1}^{10} x_k^2 = 54,785\,, \quad \sum_{k=1}^{10} x_k = 723\,,$$
$$\sum_{k=1}^{10} x_k y_k = 54,277\,, \quad \sum_{k=1}^{10} y_k = 708\,.$$

We have to solve

$$\begin{aligned} 54785a + 723b &= 54277 \\ 723a + 10b &= 708 \end{aligned}$$

This gives $a = \dfrac{30,886}{25,121} \approx 1.23$ and $b = -\dfrac{454,491}{25,121} \approx -18.09$. Thus $y \approx 1.23x - 18.09$. For $x = 70$ we obtain $y \approx 68$.

8.9.26 (a) We have $F(x, y, \lambda) = xy + \lambda(4x^2 + 8y^2 - 16)$. Setting

$$0 = F_x = y + 8x\lambda\,, \quad 0 = F_y = x + 16y\lambda = 0\,,$$

we get $\dfrac{y}{8x} = \dfrac{x}{16y}$, so $x^2 = 2y^2$, and thus $4 \cdot 2y^2 + 8y^2 = 16$. This gives $y^2 = 1$ and therefore $y = \pm 1$. We thus have four critical points, namely $(\pm\sqrt{2}, -1)$ and $(\pm\sqrt{2}, 1)$. Since

$$f(-\sqrt{2}, -1) = f(\sqrt{2}, 1) = \sqrt{2}\,,$$
$$f(-\sqrt{2}, 1) = f(\sqrt{2}, -1) = -\sqrt{2}\,,$$

and since there exists a global minimum and a global maximum, the maximum value equals $\sqrt{2}$ and is attained at $(-\sqrt{2}, -1)$ and $(\sqrt{2}, 1)$, while the minimum value equals $-\sqrt{2}$ and is attained at $(-\sqrt{2}, 1)$ and $(\sqrt{2}, -1)$.

(b) We have $F(x, y, \lambda) = x - 3y - 1 + \lambda(x^2 + 3y^2 - 16)$. Setting

$$0 = F_x = 1 + 2\lambda x\,, \quad 0 = F_y = -3 + 6\lambda y\,,$$

we get $\dfrac{1}{2x} = -\dfrac{1}{2y}$, so $y = -x$, hence $x^2 + 3x^2 = 16$ and thus $x = \pm 2$. We obtain the critical points $(-2, 2)$ and $(2, -2)$. Since a maximum and a minimum exists and $f(-2, 2) = -9$, $f(2, -2) = 7$, we have a maximum at $(2, -2)$ with value 7, and a minimum $(-2, 2)$ with value -9.

8.9.27 We have to minimize the distance $f(x, y) = x^2 + y^2$ of (x, y) from the origin, subject to the constraint $2x - 4y = 3$. We define $F(x, y, \lambda) = x^2 + y^2 + \lambda(2x - 4y - 3)$ and set

$$0 = F_x = 2x + 2\lambda\,, \quad 0 = F_y = 2y - 4\lambda\,.$$

The equations $2x = -2\lambda, 2y = 4\lambda$ give $y = -2x$, so $2x + 8x = 3$ and therefore $x = \dfrac{3}{10}$. Therefore, the point we are looking for is $\left(\dfrac{3}{10}, -\dfrac{3}{5}\right)$.

8.9.29 (a)

$$\int_0^3 \int_{-2}^{-1} (4xy^3 + y)\,dx\,dy = \int_0^3 \Big[2x^2y^3 + xy\Big]_{x=-2}^{x=-1} dy$$
$$= \int_0^3 (-6y^3 + y)\,dy = \left[-\frac{6}{4}y^4 + \frac{y^2}{2}\right]_0^3 = -117\,.$$

(b)

$$\int_0^{\pi/6} \int_0^{\pi/2} (x\cos y - y\cos x)\,dy\,dx = \int_0^{\pi/6} \left[x\sin y - \frac{y^2}{2}\cos x\right]_{y=0}^{y=\pi/2} dx$$
$$= \int_0^{\pi/6} \left(x - \frac{\pi^2}{8}\cos x\right) dx = \left[\frac{x^2}{2} - \frac{\pi^2}{8}\sin x\right]_0^{\pi/6} = \frac{\pi^2}{72} - \frac{\pi^2}{8}.\frac{1}{2}$$
$$= -\frac{7}{144}\pi^2 \approx -0.48\,.$$

8.9.30 A rough sketch is given in the Fig. E.22. The area A becomes

$$A = \int_{-3}^4 \int_{2-\frac{x}{2}}^{8-\frac{x^2}{2}} dy\,dx = \int_{-3}^4 \left[8 - \frac{x^2}{2} - 2 + \frac{x}{2}\right] dx$$
$$= \int_{-3}^4 \left(6 - \frac{x^2}{2} + \frac{x}{2}\right) dx = \left[6x - \frac{x^3}{6} + \frac{x^2}{4}\right]_{-3}^4 = \frac{343}{12}\,.$$

8.9.31 The region D is shown as the shaded portion in Fig. E.23.
Solving $r = 100$ and $r = 200\sin\theta$, the points of intersection are obtained as $\sin\theta = \frac{1}{2}$. So θ varies from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$. Due to symmetry, it suffices to let θ vary from $\frac{\pi}{6}$ to $\frac{\pi}{2}$ and multiply by 2, so the area becomes

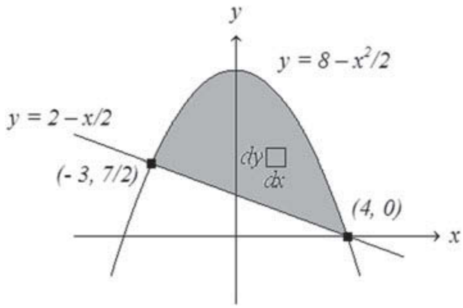

Fig. E.22 Lake bordered by a straight dam

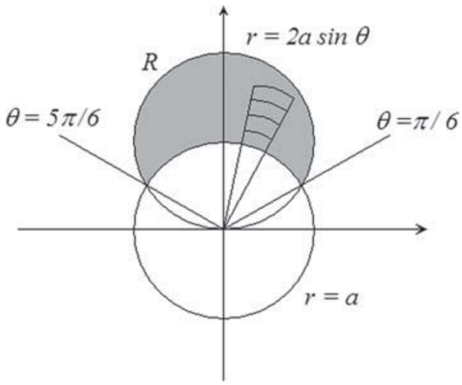

Fig. E.23 Area outside a circle and inside another circle

$$
\begin{aligned}
A &= \iint_D dA = 2\int_{\pi/6}^{\pi/2}\int_{100}^{200\sin\theta} r\,dr\,d\theta \\
&= 2\int_{\pi/6}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=100}^{r=200\sin\theta} d\theta = 10^4\int_{\pi/6}^{\pi/2}(4\sin^2\theta-1)\,d\theta \\
&= 10^4\int_{\pi/6}^{\pi/2}\left[4\frac{(1-\cos 2\theta)}{2}-1\right]d\theta = 10^4\int_{\pi/6}^{\pi/2}(1-2\cos 2\theta)\,d\theta \\
&= 10^4\Big[\theta-\sin 2\theta\Big]_{\pi/6}^{\pi/2} = 10^4\left[\frac{\pi}{2}-\frac{\pi}{6}+\frac{\sqrt{3}}{2}\right] \\
&= 10^4\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right).
\end{aligned}
$$

8.9.32 For the lemniscate $r^2 = a^2\sin 2\theta$, the area A of the region R becomes (Fig. E.24)

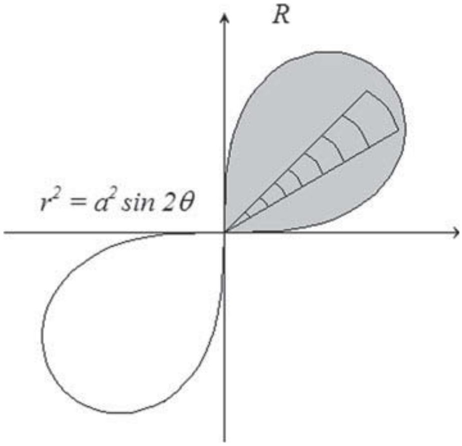

Fig. E.24 One loop of a lemniscate

$$\begin{aligned} A &= \iint_R dA = \int_0^{\pi/2} \int_0^{a\sqrt{\sin 2\theta}} r\, dr\, d\theta = \frac{1}{2}\int_0^{\pi/2} a^2 \sin 2\theta\, d\theta \\ &= -\frac{1}{4}a^2\Big[\cos 2\theta\Big]_0^{\pi/2} = -\frac{1}{4}a^2(-1-1) = \frac{1}{2}a^2 . \end{aligned}$$

8.9.33 (a) Transforming into polar coordinates, we get

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dy\, dx = \int_0^{\pi/2} \int_0^a e^{-r^2} r\, dr\, d\theta =: I .$$

To evaluate $\int_0^a e^{-r^2} r\, dr$, we substitute $t = r^2, dt = 2r dr$. We then obtain

$$\int_0^a e^{-r^2} r\, dr = \frac{1}{2}\int_0^{a^2} e^{-t}\, dt = -\frac{1}{2}e^{-t}|_0^{a^2} = -\frac{1}{2}(e^{-a^2}-1) = \frac{1-e^{-a^2}}{2} ,$$

$$I = \frac{1}{2}(1-e^{-a^2})\int_0^{\pi/2} d\theta = \frac{\pi}{4}(1-e^{-a^2}) .$$

(b) The region of integration is bounded by $y = 0$ and the semicircle $y = \sqrt{a^2-x^2}$. We obtain

$$\begin{aligned} \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\, dy\, dx &= \int_0^{\pi} \int_0^a r^3 r\, dr\, d\theta \\ &= \int_0^{\pi} \left[\frac{r^5}{5}\right]_{r=0}^{r=a} d\theta = \frac{1}{5}a^5 \int_0^{\pi} d\theta = \frac{1}{5}\pi a^5 . \end{aligned}$$

Solutions to the Exercises of Chap. 9

9.7.1 (a) Let $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$
Then

$$\begin{aligned}(\mathbf{x}+\mathbf{y})+\mathbf{z} &= [(x_1, x_2, x_3) + (y_1, y_2, y_3)] + (z_1, z_2, z_3)\\ &= (x_1+y_1, x_2+y_2, x_3+y_3) + (z_1, z_2, z_3)\\ &= [(x_1+y_1)+z_1, (x_2+y_2)+z_2, (x_3+y_3)+z_3]\\ &= [x_1+(y_1+z_1), x_2+(y_2+z_2), x_3+(y_3+z_3)]\\ &= (x_1, x_2, x_3) + (y_1+z_1, y_2+z_2, y_3+z_3)\\ &= \mathbf{x}+(\mathbf{y}+\mathbf{z})\,.\end{aligned}$$

(d) Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Then

$$\begin{aligned}\lambda(\mathbf{x}\cdot\mathbf{y}) &= \lambda(x_1y_1 + x_2y_2 + x_3y_3) = (\lambda x_1)y_1 + (\lambda x_2)y_2 + (\lambda x_3)y_3\\ &= (\lambda\mathbf{x})\cdot\mathbf{y}\,.\end{aligned}$$

Analogously $\lambda(\mathbf{x}\cdot\mathbf{y}) = \mathbf{x}\cdot(\lambda\mathbf{y})$.

(e) Let $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$. Then

$$\mathbf{x}\cdot(\mathbf{x}\times\mathbf{y}) = \begin{vmatrix} x_1 & x_2 & x_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix} = 0\,,$$

since the determinant is zero when two rows are equal.

(f) For the same reason

$$\mathbf{y}\cdot(\mathbf{x}\times\mathbf{y}) = \begin{vmatrix} y_1 & y_2 & y_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix} = 0\,.$$

(g) Let $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $\mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$, $\mathbf{z} = z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$. Then

$$\begin{aligned}\mathbf{x}\times(\mathbf{y}+\mathbf{z}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1 & x_2 & x_3\\ y_1+z_1 & y_2+z_2 & y_3+z_3\end{vmatrix}\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1 & x_2 & x_3\\ z_1 & z_2 & z_3\end{vmatrix}\\ &= (\mathbf{x}\times\mathbf{y}) + (\mathbf{x}\times\mathbf{z})\,.\end{aligned}$$

(h) $\mathbf{x}\times(\mathbf{y}\times\mathbf{z})$ is parallel to the plane spanned by $\mathbf{y}$ and $\mathbf{z}$, and orthogonal to $\mathbf{x}$. Therefore $\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = \alpha\mathbf{y} + \beta\mathbf{z}$ for some scalars α and β, and

$$0 = \mathbf{x} \cdot \big[\mathbf{x} \times (\mathbf{y} \times \mathbf{z})\big] = \alpha(\mathbf{x} \cdot \mathbf{y}) + \beta(\mathbf{x} \cdot \mathbf{z})\,.$$

Thus $\alpha = \lambda(\mathbf{x} \cdot \mathbf{z})$ and $\beta = -\lambda(\mathbf{x} \cdot \mathbf{y})$ for some scalar λ. This gives

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \lambda\big[(\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}\big]\,. \tag{*}$$

Assume now that $\mathbf{z} = a_3\mathbf{k}$, $\mathbf{y} = b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{x} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Substituting these for $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in (*) we have

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{x} \times (b_2a_3\mathbf{i}) = -c_2b_2a_3\mathbf{k} + b_2a_3c_3\mathbf{j}\,.$$

Moreover,

$$\begin{aligned}\lambda\big[(\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}\big] &= \lambda\big[(c_3a_3)(b_2\mathbf{j} + b_3k) - (c_2b_2 + b_3c_3)(a_3\mathbf{k})\big] \\ &= \lambda\big[c_3a_3b_2\mathbf{j} - c_2b_2a_3\mathbf{k}\big]\,,\end{aligned}$$

so $\lambda = 1$ and $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$. The case of general $\mathbf{y}$ and $\mathbf{z}$ can be treated either by rotating the standard unit vectors into a suitable orthogonal triple of unit vectors, or by doing an analogous computation for other choices of $\mathbf{y}$ and $\mathbf{z}$, and employing linearity.

(i) Let $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}, \mathbf{z} = z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$. Then

$$\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}, \quad \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \end{vmatrix},$$

$$\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Each determinant can be obtained from the others by two row interchanges. Therefore, by the properties of determinants, they are equal, so

$$\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})\,.$$

(j)

$$\mathbf{x} \times \mathbf{x} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0$$

9.7.2 We want to show that

$$(\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{z} \times \mathbf{w}) = (\mathbf{x} \cdot \mathbf{z})(\mathbf{y} \cdot \mathbf{w}) - (\mathbf{y} \cdot \mathbf{z})(\mathbf{x} \cdot \mathbf{w}) \tag{*}$$

holds for arbitrary vectors.

(a) If $\mathbf{x} = \mathbf{y}$, both sides of (*) are zero.

(b) For $\mathbf{x} = \mathbf{k}$ and $\mathbf{y} = \alpha\mathbf{i} + \beta\mathbf{j}$, we compute

$$\begin{gathered}
\mathbf{x} \times \mathbf{y} = -\beta\mathbf{i} + \alpha\mathbf{j}\,, \\
\mathbf{z} \times \mathbf{w} = (z_2w_3 - z_3w_2)\mathbf{i} + (z_3w_1 - z_1w_3)\mathbf{j} + (z_1w_2 - z_2w_1)\mathbf{k}\,, \\
\mathbf{x} \times \mathbf{y} \cdot \mathbf{z} \times \mathbf{w} = -\beta(z_2w_3 - z_3w_2) + \alpha(z_3w_1 - z_1w_3)\,, \\
\mathbf{x} \cdot \mathbf{z} = z_3\,, \quad \mathbf{y} \cdot \mathbf{w} = \alpha w_1 + \beta w_2\,, \\
\mathbf{x} \cdot \mathbf{w} = w_3\,, \quad \mathbf{y} \cdot \mathbf{z} = \alpha z_1 + \beta z_2\,,
\end{gathered}$$

$$\begin{aligned}
(\mathbf{x} \cdot \mathbf{z}) \cdot (\mathbf{y} \cdot \mathbf{w}) - (\mathbf{y} \cdot \mathbf{z}) \cdot (\mathbf{x} \cdot \mathbf{w}) = \\
&= z_3(\alpha w_1 + \beta w_2) - (\alpha z_1 + \beta z_2)w_3 \\
&= \alpha(z_3w_1 - z_1w_3) - \beta(z_2w_3 - z_3w_2)\,.
\end{aligned}$$

(c) Decompose an arbitrary $\mathbf{y}$ into $\mathbf{y} = (\alpha\mathbf{i} + \beta\mathbf{j}) + \gamma\mathbf{k}$, apply (a) and (b) and add the resulting equations.

(d) Perform steps (a)–(c) analogously for $\mathbf{x} = \mathbf{i}$ and $\mathbf{x} = \mathbf{j}$, instead of $\mathbf{x} = \mathbf{k}$. Decompose an arbitrary $\mathbf{x}$ as $\mathbf{x} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, and apply the previous results.

9.7.3 (a) (i) We have $x_0 = 4$, $y_0 = 2$, $a = -1$ and $b = 5$, therefore the parametric equations are $x = 4 - t$, $y = 2 + 5t$.

(ii) $x = 1 + 4t$, $y = 2 + 5t$, $z = -3 - 7t$.

(b) (i) $x = 2 + t$, $y = -3 - 4t$,

(ii) $x = -1 - t$, $y = 3t$, $z = 2$.

9.7.4 The equation is $(x - 2) + 4(y - 6) + 2(z - 1) = 0$, or $x + 4y + 2z = 28$.

9.7.5 (c) Taking $\mathbf{F} = \mathbf{G} = \mathbf{r}$ in (a) yields

$$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t)$$

$$\text{so} \qquad \frac{d}{dt}\left[\|\mathbf{r}(t)\|^2\right] = 2\mathbf{r}(t) \cdot \mathbf{r}'(t)\,.$$

Since $\|\mathbf{r}(t)\|$ is constant, its derivative is zero, thus $2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$. Hence, $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$.

9.7.6 (a) (i) For $\mathbf{F} = \sinh(x - z)\mathbf{i} + 2y\mathbf{j} + (z - y^2)\mathbf{k}$ we get

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sinh(x - z) & 2y & z - y^2 \end{vmatrix} = -2y\mathbf{i} - \cosh(x - z)\mathbf{j}\,,$$

$$\operatorname{div}(\mathbf{curl}\,\mathbf{F}) = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(-\cosh(x - z)) + \frac{\partial}{\partial z}(0) = 0\,.$$

(ii) For $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ we obtain

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}\,,$$

$$\operatorname{div}(\mathbf{curl}\,\mathbf{F}) = 0\,.$$

(b) (i) For $\varphi(x, y, z) = -2x^3yz^2$ we obtain

$$\frac{\partial\varphi}{\partial x} = -6x^2yz^2\,, \quad \frac{\partial\varphi}{\partial y} = -2x^3z^2\,, \quad \frac{\partial\varphi}{\partial z} = -4x^3yz\,,$$

therefore

$$\nabla\varphi = -6x^2yz^2\mathbf{i} - 2x^3z^2\mathbf{j} - 4x^3yz\mathbf{k}\,,$$

$$\begin{aligned}\mathbf{curl}\,\nabla\varphi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -6x^2yz^2 & -2x^3z^2 & -4x^3yz \end{vmatrix} \\ &= (-4x^3z + 4x^3z)\mathbf{i} + (-12x^2yz + 12x^2yz)\mathbf{j} \\ &\quad + (-6x^2z^2 + 6x^2z^2)\mathbf{k} = \mathbf{0}\,.\end{aligned}$$

(ii) For $\varphi(x, y, z) = e^{x+y+z}$ we obtain

$$\nabla\varphi = e^{x+y+z}(\mathbf{i} + \mathbf{j} + \mathbf{k})\,,$$

$$\begin{aligned}\mathbf{curl}\,\nabla\varphi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y+z} & e^{x+y+z} & e^{x+y+z} \end{vmatrix} \\ &= (e^{x+y+z} - e^{x+y+z})\mathbf{i} + (e^{x+y+z} - e^{x+y+z})\mathbf{j} \\ &\quad + (e^{x+y+z} - e^{x+y+z})\mathbf{k} = \mathbf{0}\,.\end{aligned}$$

9.7.7 (a) Let $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$, $\mathbf{G} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$. Then

$$\mathbf{F} \times \mathbf{G} = [f_2g_3 - f_3g_2]\mathbf{i} + [f_3g_1 - f_1g_3]\mathbf{j} + [f_1g_2 - f_2g_1]\mathbf{k}\,,$$

and moreover

$$\begin{aligned}\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= \frac{\partial}{\partial x}[f_2g_3 - f_3g_2] + \frac{\partial}{\partial y}[f_3g_1 - f_1g_3] + \frac{\partial}{\partial z}[f_1g_2 - f_2g_1] \\ &= g_1\left[\frac{\partial f_3}{\partial y}i - \frac{\partial f_2}{\partial z}\right] + g_2\left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right] + g_3\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right] \\ &\quad + f_1\left[\frac{\partial g_2}{\partial z} - \frac{\partial g_3}{\partial y}\right] + f_2\left[\frac{\partial g_3}{\partial x} - \frac{\partial g_1}{\partial z}\right] + f_3\left[\frac{\partial g_1}{\partial y} - \frac{\partial g_2}{\partial x}\right] \\ &= \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})\,.\end{aligned}$$

(b) From (a) we know that $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$. Take $\mathbf{F} = \nabla\varphi$, $\mathbf{G} = \nabla\psi$. Using $\nabla \times \nabla\varphi = 0$, $\nabla \times \nabla\psi = 0$, we get

$$\nabla \cdot (\nabla\varphi \times \nabla\psi) = \nabla\psi \cdot \mathbf{0} - \nabla\varphi \cdot \mathbf{0} = \mathbf{0}\,.$$

9.7.8 Let $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(a) Since $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, We get $(\mathbf{C} \cdot \mathbf{r}) = c_1x + c_2y + c_3z$, so $\nabla(\mathbf{C} \cdot \mathbf{r}) = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} = \mathbf{C}$.

(b) Since $\mathbf{C}$ is a constant function, $\operatorname{div}\mathbf{C} = 0$ and thus

$$\operatorname{div}(\mathbf{r} - \mathbf{C}) = \operatorname{div}\mathbf{r} - \operatorname{div}\mathbf{C} = \operatorname{div}\mathbf{r} = 1 + 1 + 1 = 3\,.$$

(c)

$$\nabla \times (\mathbf{r} - \mathbf{C}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - k_1 & y - k_2 & z - k_3 \end{vmatrix} = 0\,.$$

9.7.11 The work done is equal to $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Let D be the disk enclosed by the circle C, let $A(D)$ denote its area. We compute, using the theorem of Green–Ostrogradski,

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C (e^x - y + x\cosh x)\,dx + (y^{3/2} + x)\,dy \\ &= \iint_D \left[\frac{\partial}{\partial x}(y^{3/2} + x) - \frac{\partial}{\partial y}(e^x - y + x\cosh x)\right] dA \\ &= \iint_D (1 + 1)\,dA = 2A(D) = 2\pi(12)^2 = 288\pi\,.\end{aligned}$$

9.7.12 Let D denote the region enclosed by the curve C.

(a)

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right] dA = \iint_D (-y^2 - x^2)\,dA \\ &= \int_0^{\pi/2}\int_0^2 (-r^2)r\,dr\,d\theta \qquad \text{(using polar coordinates)} \\ &= \frac{\pi}{2}\int_0^2 -r^3\,dr = \frac{\pi}{2}\left[-\frac{r^4}{4}\right]_0^2 = -2\pi\,.\end{aligned}$$

(b)

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_D \left[\frac{\partial}{\partial x}(\sinh y^3 - 4x) - \frac{\partial}{\partial y}(e^{\sin x} - y)\right] dA \\ &= \iint_D (-4+1)\,dA = -3A(D) = -3\pi 4^2 \\ &= -48\pi\,.\end{aligned}$$

(c)

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint \left[\frac{\partial}{\partial x}(x^2 - y^2) - \frac{\partial}{\partial y}(x^2 + y^2)\right] dA \\ &= \iint_D (2x - 2y)\,dA = 2\iint_D (x-y)\,dA = 0\,,\end{aligned}$$

since the integrand $f(x, y) = x - y$ satisfies $f(-x, -y) = -f(x, y)$ and the region is symmetric with respect to the origin.

9.7.13 (a) For $\mathbf{F} = 4x\mathbf{i} - 6y\mathbf{j} + \mathbf{k}$ we get

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(6y) = 4 - 6 = -2\,.$$

Let D be the solid cylinder defined by $x^2 + y^2 \le 4$ and $0 \le z \le 6$. It has radius 2 and height 6, so its volume equals $6\pi 2^2 = 24\pi$. We get

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D (-2)\,dV = -48\pi\,.$$

(b) We have $\mathbf{F} = 2yz\mathbf{i} - 4xz\mathbf{j} + xy\mathbf{k}$ and therefore $\operatorname{div}\mathbf{F} = 0$, hence $\iiint_D \operatorname{div}\mathbf{F}\,dV = 0$ for the ball D bounded by the sphere Σ.

9.7.14 For $\mathbf{F} = 3xy\mathbf{i} + z^2\mathbf{k}$ we have $\operatorname{div}\mathbf{F} = 3y + 2z$. By the divergence theorem,

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D (3y + 2z)\,dV\,.$$

The integral is zero because the integrand $f(x, y, z) = 3y + 2z$ satisfies $f(-x, -y, -z) = -f(x, y, z)$ and the region D is symmetric with respect to the origin. Therefore

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,d\sigma = 0\,.$$

9.7.15 By Gauss' divergence theorem

$$\iint_\Sigma \mathbf{C}\cdot\mathbf{n}\,d\sigma = \iiint_D \operatorname{div}\mathbf{C}\,dV = \iiint_D 0\,dV = 0\,,$$

since $\mathbf{C}$ is constant and therefore $\operatorname{div}\mathbf{C} = 0$.

9.7.16 For $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$ we have

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xy \end{vmatrix} = (x-y)\mathbf{i} - y\mathbf{j} - x\mathbf{k}\,.$$

The surface Σ is parametrized by $z = S(x, y) = 8 - 2x - 4y$ with domain $D = \{(x, y) : x \geq 0\,,\ y \geq 0\,,\ x + 2y \leq 4\}$. The partial derivatives $\partial_x S = -2$ and $\partial_y S = -4$ are constant, an outer normal is given by $\mathbf{N} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, the corresponding outer unit normal is

$$\mathbf{n} = \frac{1}{\sqrt{21}}(2\mathbf{i} + 4\mathbf{j} + \mathbf{k})\,.$$

Since $\sqrt{1 + (\partial_x S)^2 + (\partial_y S)^2} = \sqrt{21}$, the surface integral is computed as

$$\begin{aligned}
\iint_\Sigma (\mathbf{curl}\,\mathbf{F})\cdot\mathbf{n}\,d\sigma &= \iint_D (2x - 2y - 4y - x)\,dA \\
&= \int_0^2 \int_0^{4-2y} (x - 6y)\,dx\,dy = \int_0^2 \left[\frac{x^2}{2} - 6xy\right]_{x=0}^{x=4-2y} dy \\
&= \int_0^2 \left[\frac{(4-2y)^2}{2} - 6y(4-2y)\right] dy \\
&= \int_0^2 2(4 + y^2 - 4y - 12y + 6y^2)\,dy \\
&= 2\int_0^2 (4 + 7y^2 - 16y)\,dy = 2\left[4y + 7\frac{y^3}{3} - 16\frac{y^2}{2}\right]_{y=0}^{y=2} \\
&= 2\left[8 + \frac{56}{3} - 32\right] = -\frac{32}{3}\,.
\end{aligned}$$

9.7.17 By Stokes' theorem, $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_\Sigma (\mathbf{curl}\,\mathbf{F})\cdot\mathbf{n}\,d\sigma$, where $\mathbf{n}$ is a suitably oriented normal to the disk $x^2 + y^2 \leq 1$. We have

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y & x^2 y & xza \end{vmatrix} = -za\mathbf{j} + (2xy + 1)\mathbf{k}\,.$$

Since the disk is horizontal, it is parametrized by $z = S(x, y) = 0$, and $\mathbf{n} = \mathbf{k}$. Therefore, $\mathbf{curl\,F} \cdot \mathbf{n} = 2xy + 1$ and $\sqrt{1 + (\partial_x S)^2 + (\partial_y S)^2} = \sqrt{1} = 1$. We now compute

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\Sigma (\mathbf{curl\,F}) \cdot \mathbf{n}\, d\sigma \iint_\Sigma 1\, d\sigma = \iint_D 1\, dA = \pi\,,$$

since $x = y = 0$ on Σ and the area of the disk D equals π.

9.7.18 Since in all three cases, the vector fields are defined on all of $\mathbb{R}^3$, by Remark 9.7, in order to see whether $\mathbf{F}$ is conservative it is equivalent to check whether we have $\mathbf{curl\,F} = \mathbf{0}$ everywhere.

(a) For $\mathbf{F} = \cosh(x + y)(\mathbf{i} + \mathbf{j} - \mathbf{k})$ we have

$$\begin{aligned}\mathbf{curl\,F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cosh(x+y) & \cosh(x+y) & \cosh(x+y) \end{vmatrix} \\ &= -\sinh(x+y)\mathbf{i} + \sinh(x+y)\mathbf{j}\,.\end{aligned}$$

Since $\mathbf{curl\,F}$ is not everywhere zero, $\mathbf{F}$ is not conservative.

(b) For $\mathbf{F} = 2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k}$ we have

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -2y & 2z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}\,.$$

Hence $\mathbf{F}$ is conservative.

(c) The vector field $\mathbf{F}(x, y, z) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is constant, hence all its partial derivatives are zero and therefore $\mathbf{curl\,F}$ is everywhere zero. Hence $\mathbf{F}$ is conservative.

9.7.19 The given vector field is $\mathbf{F} = (x^2y - z^2)\mathbf{i} + (y^3 - x)\mathbf{j} + (2x + 3z - 1)\mathbf{k}$. The parametric equations of the curve C are $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, thus $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ and $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$. We compute

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \cos^2 t \sin t(-\sin t) + (\sin^3 t - \cos t)\cos t \\ &= -\cos^2 t \sin^2 t + \sin^3 t \cos t - \cos^2 t) \\ &= -\frac{\sin^2 2t}{4} + \sin^3 t \cos t - \frac{1}{2}(1 + \cos 2t)\,.\end{aligned}$$

The line integral becomes

$$\begin{aligned}\oint \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt\\ &= \int_0^{2\pi}\left[-\frac{1}{8}(1-\cos 4t)+\sin^3 t\cos t-\frac{1}{2}-\frac{1}{2}\cos 2t\right]dt\\ &= -\frac{1}{8}\cdot 2\pi-\frac{1}{2}\cdot 2\pi = -\frac{\pi}{4}-\pi=-\frac{5}{4}\pi\,.\end{aligned}$$

The surface Σ is described by $z = S(x, y)$, where $S(x, y) = 1 - x^2 - y^2$ with domain $D = \{(x, y) : x^2 + y^2 \le 1\}$. Since $\partial_x S = -2x$ and $\partial_y S = -2y$, the vector field of unit normals for Σ with the correct orientation is

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|}\,, \quad \mathbf{N} = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}\,.$$

Furthermore, we have

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y - z^2 & y^3 - x & 2x + 3z - 1\end{vmatrix} = (-2z-2)\mathbf{j} + (-1-x^2)\mathbf{k}\,.$$

Since $\sqrt{1+(\partial_x S)^2+(\partial_y S)^2} = \|\mathbf{N}\|$, the surface integral becomes

$$\begin{aligned}&\iint_\Sigma (\mathbf{curl}\,\mathbf{F})\cdot\mathbf{n}\,d\sigma\\ &\quad= \iint_\Sigma \frac{1}{\|\mathbf{N}\|}((-2z-2)\mathbf{j}+(-1-x^2)\mathbf{k})\cdot(2x\mathbf{i}+2y\mathbf{j}+\mathbf{k})\,d\sigma\\ &\quad= \iint_D \left[2y(2x^2+2y^2-4)-1-x^2\right]dA\\ &\quad= \int_0^{2\pi}\int_0^1\left[2r\sin\theta(2r^2-4)-1-r^2\cos^2\theta\right]r\,dr\,d\theta\\ &\quad= \cdots = -\frac{5}{4}\pi\,.\end{aligned}$$

Solutions to the Exercises of Chap. 10

10.5.1 A finite or infinite set of vectors $x_1, x_2, \ldots,$ is said to be linearly independent if for all n and all scalars λ_i

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$$

implies that $\lambda_i = 0$ for every $i = 1, 2, \ldots, n$.
The set $\{x_i\}$ is said to be orthonormal if $\|x_i\| = 1$ for all i and $x_i \perp x_j$ when-

ever $i \neq j$, that is, it consists of mutually orthogonal unit vectors.
An orthogonal set can be converted into an orthonormal set by dividing each vector by its magnitude.

10.5.2 (a) We have

$$\int_{\pi/4}^{5\pi/4} e^x \sin x \, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x \Big|_{\pi/4}^{5\pi/4} = 0 .$$

(b) For $m \neq n$ we have

$$\begin{aligned}
&\int_0^{\pi/2} \cos(2n+1)x \cos(2m+1)x \, dx \\
&= \frac{1}{2} \int_0^{\pi/2} [\cos 2(n-m)x + \cos 2(n+m+1)x] \, dx \\
&= \frac{1}{4(n-m)} \sin 2(n-m)x \Big|_0^{\pi/2} + \frac{1}{4(n+m+1)} \sin 2(n+m+1)x \Big|_0^{\pi/2} \\
&= 0 .
\end{aligned}$$

10.5.3 (a) The Fourier coefficients for $f(x) = x + \pi$ are

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) \, dx = 2\pi , \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) \cos nx \, dx = 0 , \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) \sin nx \, dx = \frac{2}{n} (-1)^{n+1} .
\end{aligned}$$

Therefore $f(x) = \pi + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$.

(b) The Fourier coefficients for $f(x) = e^{-8x}$ are

$$\begin{aligned}
a_0 &= \frac{1}{2} \int_{-4}^{4} e^{-8x} \, dx = \frac{1}{16} (e^{32} - e^{-32}) , \\
a_n &= \frac{1}{2} \int_{-4}^{4} e^{-8x} \cos\left(\frac{n\pi x}{4}\right) dx = (e^{32} - e^{-32}) \frac{64(-1)^n}{1024 + n^2\pi^2} , \\
b_n &= \frac{1}{2} \int_{-2}^{2} e^{-8x} \sin\left(\frac{n\pi x}{4}\right) dx = (e^{32} - e^{-32}) \frac{2n\pi(-1)^n}{1024 + n^2\pi^2} .
\end{aligned}$$

The Fourier series is

$$\frac{1}{32}(e^{32}-e^{-32})+(e^{32}-e^{-32})\sum_{n=1}^{\infty}\left(\frac{64(-1)^n}{1024+n^2\pi^2}\cos\frac{n\pi x}{4}+\frac{2n\pi(-1)^n}{1024+n^2\pi^2}\sin\frac{n\pi x}{4}\right).$$

(c) The Fourier coefficients for

$$f(x)=\begin{cases}0 & -\pi<x<0\\ x^2 & 0\le x<\pi\end{cases}$$

are

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{\pi}x^2\,dx=\frac{1}{3}\pi^2,$$

$$\begin{aligned}a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}x^2\cos nx\,dx\\&=\frac{1}{\pi}\frac{x^2}{n}\sin nx\Big|_0^{\pi}-\frac{2}{n\pi}\int_0^{\pi}x\sin nx\,dx\\&=-\frac{2}{n\pi}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]=2\frac{(-1)^n}{n^2}.\end{aligned}$$

$$\begin{aligned}b_n&=\frac{1}{\pi}\int_0^{\pi}x^2\sin nx\,dx=\frac{1}{\pi}\left[-\frac{x^2}{n}\cos nx\right]_0^{\pi}+\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx\\&=\frac{\pi}{n}(-1)^{n+1}+\frac{2}{n^2\pi}[(-1)^n-1].\end{aligned}$$

As Fourier series we obtain

$$f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx+\left(\frac{\pi}{n}(-1)^{n+1}+\frac{2}{n^2\pi}[(-1)^n-1]\right)\sin nx\right].$$

10.5.4 (a) The function $f(x)=x+\pi$, $-\pi<x<\pi$ is continuous at $x=\frac{\pi}{2}$, so

$$\begin{aligned}\frac{3\pi}{2}&=f\left(\frac{\pi}{2}\right)=\pi+\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin\frac{n\pi}{2}\\&=\pi+2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right),\\ \text{so}\quad\frac{\pi}{4}&=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\end{aligned}$$

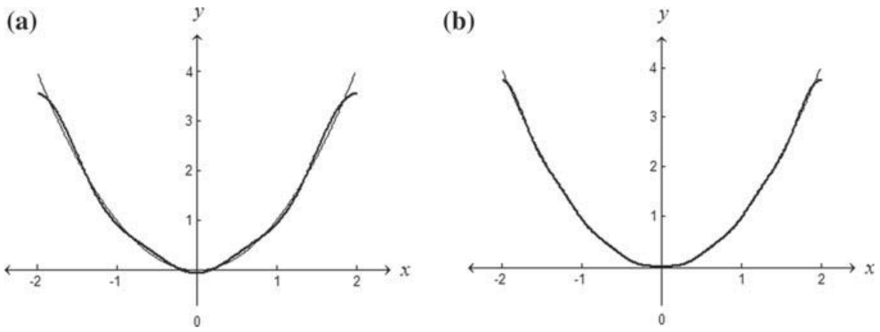

Fig. E.25 **a** Third partial sum of the Fourier series of $f(x) = x^2$ on $[-2, 2]$, **b** Sixth partial sum of the Fourier series of $f(x) = x^2$ on $[-2, 2]$

(b) The function f defined in 10.5.3 (c) is discontinuous at $x = \pi$.
The Fourier series converges to $\frac{1}{2}[f(\pi - 0) + f(-\pi + 0)] = \frac{\pi^2}{2}$ at $x = \pi$.
That is,

$$\begin{aligned}\frac{\pi^2}{2} &= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos n\pi + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^2\pi}[(-1)^n - 1]\right)\sin n\pi\right] \\ &= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2} \\ &= \frac{\pi^2}{6} + 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right),\end{aligned}$$

therefore

$$\frac{\pi^2}{6} = \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots .$$

10.5.5 See Fig. E.25.
10.5.6 (a) The complex Fourier series of f is

$$\sum_{n=-\infty}^{\infty}\frac{2n\pi i[\cos(1) - 1] + \sin 1}{1 - 4n^2\pi^2}e^{2n\pi i x},$$

it converges to

$$g(x) = \begin{cases}\frac{1+\cos(1)}{2} & x = 0 \text{ or } x = 1 \\ \cos x & 0 < x < 1 .\end{cases}$$

(b) The complex Fourier series of f is

$$\frac{3}{4} - \frac{1}{2\pi} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n} \left\{ \sin \frac{n\pi}{2} + i \left[\cos \frac{n\pi}{2} - 1 \right] \right\} e^{\frac{n\pi i x}{2}} ,$$

it converges to

$$g(x) = \begin{cases} \frac{1}{2} & x = 0 \text{ or } x = 1 \text{ or } x = 4 \\ 0 & 0 < x < 1 \\ 1 & 1 < x < 4 \end{cases}$$

10.5.7 (a) The function $f(x) = x^3$, is an odd function, we expand it in a sine series on the interval $-\pi < x < \pi$. We get

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx = \frac{2}{\pi} \left(-\frac{x^3}{n} \cos nx \Big|_0^{\pi} + \frac{3}{n} \int_0^{\pi} x^2 \cos nx \, dx \right) \\ &= \frac{2\pi^2}{n} (-1)^{n+1} - \frac{12}{n^2 \pi} \int_0^{\pi} x \sin nx \, dx \\ &= \frac{2\pi^2}{n} (-1)^{n+1} - \frac{12}{n^2 \pi} \left(-\frac{x}{n} \cos nx \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right) \\ &= \frac{2\pi^2}{n} (-1)^{n+1} + \frac{12}{n^2} (-1)^n . \end{aligned}$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2\pi^2}{n} (-1)^{n+1} + \frac{12}{n^2} (-1)^n \right) \sin nx .$$

(b) The function

$$f(x) = \begin{cases} x - 1 & -\pi < x < 0 \\ x + 1 & 0 \le x < \pi \end{cases}$$

is an odd function. We expand f in a sine series,

$$b_n = \frac{2}{\pi} \int_0^{\pi} (x+1) \sin nx \, dx = \frac{2(\pi + 1)}{n\pi} (-1)^{n+1} + \frac{2}{n\pi} .$$

Therefore

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2(\pi + 1)}{n\pi} (-1)^{n+1} + \frac{2}{n\pi} \right) \sin nx .$$

10.5.8 Using $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$, we have

$$
\begin{aligned}
c_n &= \frac{1}{\pi}\int_0^{\pi} f(x)e^{-\frac{2inx}{\pi}}\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} (\sin x)e^{-\frac{2inx}{\pi}}\,dx \\
&= \frac{1}{\pi 2i}\int_0^{\pi} (e^{ix}-e^{-ix})e^{-\frac{2inx}{\pi}}\,dx \\
&= \frac{1}{2\pi i}\int_0^{\pi} \left(e^{\left(1-\frac{2n}{\pi}\right)ix} - e^{-\left(1+\frac{2n}{\pi}\right)ix}\right)dx \\
&= \frac{1}{2\pi i}\left[\frac{1}{i\left(1-\frac{2n}{\pi}\right)}e^{\left(1-\frac{2n}{\pi}\right)ix} + \frac{1}{i\left(1+\frac{2n}{\pi}\right)}e^{-\left(1+\frac{2n}{\pi}\right)ix}\right]_0^{\pi} \\
&= \frac{\pi(1+e^{-2in})}{\pi^2-4n^2}\,.
\end{aligned}
$$

The fundamental period is $T=\pi$ so $\omega=\frac{2\pi}{\pi}=2$ and the values of $n\omega$ are $0, \pm 2, \pm 4, \pm 6, \ldots$. The values of $|c_n|$ for $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ are shown in the table.

n	–5	–4	–3	–2	–1	0	1	2	3	4	5
c_n	0.0198	0.0759	0.2380	0.4265	0.5784	0.6366	0.5784	0.4265	0.2380	0.0759	0.0198

10.5.9 (a) The Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\,,$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx\,, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\,.$$

The Nth partial sum of the series is

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{l}\,.$$

We compute

$$
\begin{aligned}
0 &\le \int_{-l}^{l} (f(x)-S_N(x))^2\,dx \\
&= \int_{-l}^{l} f^2(x)\,dx - 2\int_{-l}^{l} f(x)S_N(x)\,dx + \int_{-l}^{l} S_N^2(x)\,dx
\end{aligned}
$$

$$= \int_{-l}^{l} f^2(x)\,dx - 2\int_{-l}^{l} f(x)\left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{l}\right) dx +$$
$$\int_{-l}^{l}\left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{l}\right)\cdot\left(\frac{a_0}{2} + \sum_{m=1}^{N} a_m \cos\frac{m\pi x}{l} +\right.$$
$$\left. + \sum_{m=1}^{N} b_m \sin\frac{m\pi x}{l}\right) dx$$

$$= \int_{-l}^{l} f^2(x)dx - a_0\int_{-l}^{l} f(x)\,dx - 2\sum_{n=1}^{N} a_n \int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx -$$
$$- 2\sum_{n=1}^{N} b_n \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx + \frac{a_0^2}{4}\int_{-l}^{l} dx + 2\frac{a_0}{2}\left(\sum_{n=1}^{N} a_n \int_{-l}^{l}\cos\frac{n\pi x}{l}\,dx +\right.$$
$$\left. + \sum_{n=1}^{N} b_n \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right)$$

$$+ \sum_{n=1}^{N}\sum_{m=1}^{N} a_m b_n \int_{-l}^{l}\cos\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx + \sum_{n=1}^{N}\sum_{m=1}^{N} a_n b_m \int_{-l}^{l}\cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx$$
$$+ \sum_{n=1}^{N}\sum_{m=1}^{N} a_m a_n \int_{-l}^{l}\cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx + \sum_{n=1}^{N}\sum_{m=1}^{N} b_m b_n \int_{-l}^{l}\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$= \int_{-l}^{l} f^2(x)\,dx - a_0^2 l + \frac{a_0^2 2l}{4} - 2\sum_{n=1}^{N} l a_n^2 - 2\sum_{n=1}^{N} b_n^2 + \sum a_n^2 l + \sum b_n^2 l\,.$$

In view of

$$\int_{-l}^{l}\cos\frac{n\pi x}{l}\,dx = \int_{-l}^{l}\sin\frac{n\pi x}{l}\,dx = \int_{-l}^{l}\cos\frac{n\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0$$

and of

$$\int_{-l}^{l}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_{-l}^{l}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \begin{cases} 0 & n \neq m \\ l & n = m \end{cases},$$

we now conclude that

$$0 \le \int_{-l}^{l} f^2(x)\,dx - a_0^2 l + \frac{a_0^2}{2} l - 2l\sum_{n=1}^{N} a_n^2 + l\sum_{n=1}^{N} a_n^2 - 2\sum_{n=1}^{N} l b_n^2 + \sum_{n=1}^{N} b_n^2 l\,,$$

so

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) \le \frac{1}{l}\int_{-l}^{l} f^2(x)\,dx\,.$$

Letting $N \to \infty$ we finally arrive at

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \le \frac{1}{l}\int_{-l}^{l} f^2(x)\,dx\,.$$

(b) Under the given conditions the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

converges uniformly to $f(x)$. Now a uniformly convergent series of continuous functions can be integrated term by term, so

$$\begin{aligned}\int_{-l}^{l}[f(x)]^2\,dx &= \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx \\ &\quad + \sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx + b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right].\end{aligned}$$

Since

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx\,, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\,,$$

we get

$$\int_{-l}^{l}|f(x)|^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right].$$

(c) By (b), $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is convergent. Due to the properties of convergent series, $a_n^2 + b_n^2 \to 0$ and therefore $a_n \to 0$ and $b_n \to 0$.

10.5.10 (a) The Fourier transform of $f(t) = te^{-|t|}$ is

$$\hat{f}(\xi) = \int_{-\infty}^{0} te^{t}e^{-i\xi t}\,dt + \int_{0}^{\infty} te^{-t}e^{-i\xi t}\,dt = \frac{4i\xi}{(\xi^2+1)^2}\,.$$

(b) The Fourier transform is

$$\hat{f}(\xi) = \int_{-5}^{5} \sin(\pi t) e^{-i\xi t}\, dt = i\left[\frac{\sin 5(\xi+\pi)}{\xi+\pi} - \frac{\sin 5(\xi-\pi)}{\xi-\pi}\right].$$

(c) The Fourier transform is

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-k}^{0} e^{-i\xi t}\, dt + \int_{0}^{k} e^{-i\xi t}\, dt = \int_{k}^{0} e^{i\xi t}\, dt + \int_{0}^{k} e^{-i\xi t}\, dt \\
&= -2i \int_{0}^{k} \frac{e^{i\xi t} - e^{-i\xi t}}{2i}\, dt = -2i \int_{0}^{k} \sin(\xi t)\, dt \\
&= \frac{2i}{\xi}[\cos(\xi k) - 1]
\end{aligned}$$

10.5.11 We have

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{i\xi_0 t} f(t) e^{-i\xi t}\, dt = \int_{-\infty}^{\infty} f(t) e^{-i(\xi-\xi_0)t}\, dt = \hat{f}(\xi - \xi_0).$$

10.5.12 (a) We get

$$\begin{aligned}
\hat{g}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi t} f(-t)\, dt = \int_{-\infty}^{\infty} e^{-i\xi(-\tau)} f(\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} e^{-i(-\xi)\tau} f(\tau)\, d\tau = \hat{f}(-\xi).
\end{aligned}$$

(b) We start from

$$g(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i\xi t}\, dt.$$

We interchange ξ and t and take the Fourier transform

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} \hat{f}(t)\, dt = \int_{-\infty}^{\infty} e^{it(-\xi)} \hat{f}(t)\, dt = 2\pi f(-\xi),$$

by the formula for the inverse Fourier transform.

10.5.15 We find $\hat{f}$ for $f = \chi_{[-\frac{1}{2},\frac{1}{2}]}$.

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi t} f(t)\, dt = \int_{-1/2}^{1/2} e^{-i\xi t}\, dt = \left.\frac{e^{-i\xi t}}{-i\xi}\right|_{t=-1/2}^{t=1/2} \\
&= -\frac{1}{i\xi}[e^{-\frac{1}{2}\xi i} - e^{\frac{1}{2}\xi i}] = \frac{2}{\xi}\left[\frac{e^{\frac{\xi i}{2}} - e^{-\frac{\xi i}{2}}}{2i}\right] = \frac{2}{\xi}\sin\frac{\xi}{2}.
\end{aligned}$$

10.5.16 We want to prove that $f * g$ is a continuous function on $\mathbb{R}$ if f and g are square integrable on $\mathbb{R}$. Let $\varepsilon > 0$ be given. Then given $x, y \in \mathbb{R}$

$$|(f * g)(x) - (f * g)(y)| = \left| \int_{\mathbb{R}} f(t)g(x-t)\,dt - \int_{\mathbb{R}} f(t)g(y-t)\,dt \right|$$
$$\leq \int_{\mathbb{R}} |f(t)||g(x-t) - g(y-t)|\,dt \tag{E.1}$$

and, using the Cauchy–Schwarz inequality for functions,

$$\leq \left(\int_R |f(t)|^2\,dt \right)^{1/2} \left(\int_{\mathbb{R}} |g(x-t) - g(y-t)|^2\,dt \right)^{1/2}$$
$$= \left(\int_{\mathbb{R}} |f(t)|^2\,dt \right)^{1/2} \left(\int_{\mathbb{R}} |g(t-(x-y)) - g(t)|^2\,dt \right)^{1/2}. \tag{*}$$

By a result stated below,

$$\lim_{x \to y} \left(\int_{\mathbb{R}} |g((t) - (x-y)) - g(t)|^2\,dt \right)^{1/2} = 0\,.$$

Therefore, taking the limit $y \to x$ in (*) we see that $f * g$ is continuous in x, and thus on $\mathbb{R}$, since x is arbitrary.
The result on the continuity of the translation needed here is the following: Suppose that f is square integrable on $\mathbb{R}$. Then

$$\lim_{t \to 0} \int_{\mathbb{R}} |f(x) - f(x-t)|^2\,dx = 0\,.$$

Solutions to Exercises of Chap. 11

11.11.1 (a) Use separation of variables.

$$\frac{dy}{y} = 0.03dt, \quad \ln y = 0.03t + \ln c, \quad \frac{y}{c} = e^{0.03t}, \quad \text{so} \;\; y(t) = ce^{0.03t}.$$

(b) $y(t) = ce^{kt}$, (c) $y(t) = (1/5)\cos 5t + c$.

11.11.2 (a) Let A be the balance account in rupees as a function of t. The interest is being added continuously to the account at a rate of 5% of the balance at that moment, so the rate at which balance is increasing equals 5% of the current balance. The following differential equation describes the process:

$$\frac{dA}{dt} = 0.05A.$$

It may be noted that initial deposit does not come into the picture as it does not affect the process.

(b) The initial value of A at time 0 is $A(0) = 10000$. Thus the solution becomes $A(t) = A(0)e^{0.05t} = 10000e^{0.5t}$.

11.11.3 (a)

$$x^2dy + y^2dx = 0, \quad \frac{dy}{y^2} = -\frac{dx}{x^2}, \quad -\frac{1}{y} = \frac{1}{x} + c.$$

Solving for y yields

$$y(x) = -\frac{x}{1+cx}.$$

(b)

$$dx - x^2dy = 0, \quad dy = \frac{dx}{x^2}, \quad y = -\frac{1}{x} + c.$$

(c)

$$\frac{dy}{dx} = -5xy, \quad \frac{dy}{y} = -5x, \quad \ln y = -\frac{5}{2}x^2 + \ln c, \quad \frac{y}{c} = e^{-\frac{5}{2}x^2}.$$

11.11.4 (a) $b(x) = 3x^2/x^3 = 3/x$, $f(x) = (\cos x)/x^3$. We compute the integrating factor I as

$$B(x) = 3\ln x, \quad I(x) = e^{B(x)} = e^{3\ln x} = x^3.$$

$$f(x)I(x) = \frac{\cos x}{x^3} \cdot x^3 = \cos x, \quad Q(x) = \sin x.$$

The solution y satisfies

$$y(x) \cdot x^3 = Q(x) + c = \sin x + c, \quad y(x) = \frac{\sin x + c}{x^3}.$$

(b) $b(x) = 1/x$, $f(x) = 1/x^3$, $B(x) = \ln x$, $I(x) = e^{\ln x} = x$, $f(x)I(x) = 1/x^2$, $Q(x) = -1/x$. The solution y satisfies

$$y(x)I(x) = y(x) \cdot x = Q(x) + c = -\frac{1}{x} + c, \quad y(x) = -\frac{1}{x^2} + \frac{c}{x}.$$

11.11.5 $b(x) = 5$, $B(x) = 5x$, $I(x) = e^{5x}$, $f(x) = 20$, $f(x)I(x) = 20e^{5x}$, $Q(x) = 4e^{5x}$. The solution y satisfies

$$y(x)I(x) = y(x)e^{5x} = Q(x) + c = 4e^{5x} + c, \quad y(x) = 4 + ce^{5x},$$

where c is to be determined from the initial condition $y(0) = 2$. This gives $c = -2$ so

$$y(x) = 4 - 2e^{5x}.$$

11.11.6 From the given differential equation we get

$$\frac{dy}{dx} = -\frac{2}{x}y = \frac{3}{x^2}y^4.$$

This is a Bernoulli equation with $n = 4$, $b(x) = -2/x$ and $f(x) = 3/x^2$. Substituting $v = y^{-3}$ we get

$$\frac{dv}{dx} + \frac{6}{x}v = -\frac{9}{x^2}.$$

This is a linear equation for v with $b(x) = 6/x$. We get $B(x) = 6 \ln x$ and the integrating factor $I(x) = e^{6 \ln x} = 6x$. The solution v satisfies

$$x^6 v(x) = -\frac{9}{5}x^5 + c, \quad v(x) = -\frac{9}{5}x^{-1} + cx^{-6}.$$

As $v = y^{-3}$, the initial condition $y(1) = 1/2$ gives $c = 49/5$, so

$$y(x)^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{r}x^{-6}.$$

11.11.7 We apply the separation of variables method and compute

$$\frac{dy}{dx} = xy^2, \quad \frac{dy}{y^2} = xdx, \quad -\frac{1}{y} = \frac{x^2}{2} + c, \quad y(x) = -\frac{2}{x^2 + c}.$$

Inserting the initial condition $y(1) = -2/3$ yields $c = 2$.

11.11.8 The model is $dN/dt = kN$. It has the solution $N(t) = ce^{kt}$. Choosing for $t = 0$ the year 1970, we get $c = N(0) = 125000$. We compute k from

$$140000 = N(20) = 125000e^{20k}, \quad 20k = \ln\left(\frac{140}{125}\right), \quad k = \frac{1}{20}\ln\left(\frac{140}{125}\right),$$

so $k \approx 0.00567$. This yields the estimate for the population in the year 2020

$$N(50) = 125000e^{50k} \approx 166000.$$

11.11.9 Since the initial dose is y_0, the drug concentration at any time $0 < t < T$ is found from the equation $y = y_0 e^{-kt}$, the solution of the equation $dy/dt = -ky$. At $t = T$ the second dose of y_0 is taken, which increases the drug level to $y(T) = y_0 + y_0 e^{-kT} = y_0(1 + e^{-kT})$. For $t > T$, the drug level immediately begins to decrease. To find its mathematical expression we solve the initial value problem:

$$\frac{dy}{dt} = -ky, \quad y(T) = y_0(1 + e^{-kT}).$$

Solving this initial value problem we get

$$y = y_0(1 + e^{-kT})e^{-k(t-T)}.$$

This equation gives the drug level for $T < t < 2T$. The third dose of y_0 is to be taken at $t = 2T$, and the drug level just before this dose is taken is given by

$$y = y_0(1 + e^{-kT})e^{-k(2T-T)} = y_0(1 + e^{-kT})e^{-kT}.$$

The dosage y_0 taken at $t = 2T$ raises the drug level to

$$y(2T) = y_0 + y_0(1 + e^{-kT})e^{-kT} = y_0(1 + e^{-kT} + e^{-2kT}).$$

Continuing in this way, we find that, after the $(n + 1)$th dose is taken, the drug level is

$$y(nT) = y_0(1 + e^{-kT} + e^{-2kT} + \cdots + e^{-nkT}).$$

We notice that the drug level after the $(n + 1)$th dose is the sum of the first $(n + 1)$ terms of a geometric series, with the first term y_0 and the common ratio e^{-kT}. The sum can be written as

$$y(nT) = \frac{y_0(1 - e^{-(n+1)kT})}{1 - e^{-kT}}.$$

As n becomes large, the drug level at the discrete sequence $0, T, 2T, \ldots$ of times approaches a saturation value y_s given by

$$y_s = \lim_{n\to\infty} y(nT) = \frac{y_0}{1 - e^{-kT}}.$$

11.11.10 (a) The constant solutions belong to those values of H for which $dH/dx = 0$. As $dH/dx = H\sqrt{4 - 2H}$ this happens when $H = 0$ or when $0 = \sqrt{4 - 2H}$, that is, $H = 2$.
(b) Using the separation of variables method, start from

$$\frac{dH}{H\sqrt{4 - 2H}} = dx.$$

Integrating both sides we get

$$-\tanh^{-1}\left(\frac{1}{2}\sqrt{4-2H}\right) = x + c.$$

Using the fact that the hyperbolic tangent is an odd function we have

$$\frac{1}{2}\sqrt{4-2H} = \tanh(-x-c) = -\tanh(x+c)$$
$$\frac{1}{4}(4-2H) = \tanh^2(x+c)$$

Solving for H we finally obtain, using the identity $1 - \tanh^2 x = \text{sech}^2 x$,

$$H(x) = 2\text{sech}^2(x+c).$$

(c) Inserting $H(0) = 2$ we find that $\text{sech}^2(c) = 1$ and $c = 0$.

11.11.11 Let t be the number of hours after the body was discovered, and $T(t)$ be the temperature (in degrees Celsius) of the body at time t. We want to find the value of t for which $T = 37$ (normal body temperature). This value of t will, of course, be negative. By Newton's law of cooling,

$$\frac{dT}{dt} = k(T-a),$$

where k is a constant and a (the ambient temperature) is 22. Thus,

$$\frac{dT}{dt} = k(T-22).$$

Separating the variables and integrating we get

$$\frac{dT}{T-22} = kdt, \quad \ln|T-22| = kt + c.$$

Because $T - 22 > 0$, we obtain $\ln(T-22) = kt + c$. The condition $T(0) = 31$ yields $c = \ln 9$. The condition $T(1) = 30$ gives $\ln 8 = k + \ln 9$, so $k = \ln(8/9)$. Thus,

$$\ln\frac{T-22}{9} = t\ln\frac{8}{9}.$$

Now we determine t from the condition $T(t) = 37$:

$$\ln\frac{37-22}{9} = t\ln\frac{8}{9}, \quad t = \frac{\ln(15/9)}{\ln(8/9)} \approx -4.34.$$

Accordingly, the murder occurred about 4.34 h before the time of discovery of body at 11:00 am. Since 4.34 h is approximately 4 h and 20 min, the time of the murder is estimated as 6:40 am.

11.11.12 The functions e^x and $\sin x$ are linearly independent as none of them is a constant multiple of the other.

11.11.13 (a) The auxiliary equation is $4\lambda^2 - 10\lambda + 25 = 0$. We have $a = 4$, $b = -10$, $c = 25$, so $b^2 - 4ac = 0$. The auxiliary equation has the double root

$$\lambda_1 = \lambda_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} = \frac{5}{4}.$$

The general solution of the differential equation is

$$y(x) = c_1 e^{\frac{5}{4}x} + c_2 x e^{\frac{5}{4}x}.$$

(b) The auxiliary equation is $\lambda^2 - 16\lambda + 64 = 0$. We have $a = 1, b = -16$ and $c = 64$, so $b^2 - 4ac = 0$, and the auxiliary equation has the double root

$$\lambda_1 = \lambda_2 = -\frac{b}{2a} = 8.$$

The general solution is

$$y(x) = c_1 e^{8x} + c_2 x e^{8x}.$$

(c) The auxiliary equation is $\lambda^2 + 2\lambda + 2 = 0$. We have $a = 1, b = 2$ and $c = 2$, so $b^2 - 4ac = -4 < 0$. The auxiliary equation has two conjugate complex roots

$$\lambda_1 = \frac{-2 + \sqrt{4 - 8}}{2} = -1 + i, \quad \lambda_2 = \frac{-2 - \sqrt{4 - 8}}{2} = -1 - i.$$

We have $\alpha = -1$ and $\beta = 1$. The general solution of the differential equation is

$$y(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x = e^{-x}(c_1 \cos x + c_2 \sin x).$$

11.11.14 The auxiliary equation is $4\lambda^2 - 4\lambda - 3 = 0$. We have $a = 4, b = -4$ and $c = -3$, so $b^2 - 4ac = 16 + 48 = 64 > 0$. We have to distinct real roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{4 + 8}{8} = \frac{3}{2},$$
$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{4 - 8}{8} = -\frac{1}{2}.$$

The general solution is

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 e^{\frac{3}{2}x}$$

with the derivative

$$y'(x) = -c_1 \frac{1}{2} e^{-\frac{1}{2}x} + \frac{3}{2} c_2 e^{\frac{3}{2}x}.$$

Inserting the initial condition gives

$$1 = y(0) = c_1 + c_2, \quad 5 = y'(0) = -\frac{1}{2}c_1 + \frac{3}{2}c_2.$$

From the equations we get $c_1 = -\frac{7}{4}$ and $c_2 = \frac{11}{4}$, so the solution is

$$y(x) = -\frac{7}{4} e^{-\frac{1}{2}x} + \frac{11}{4} e^{\frac{3}{2}x}.$$

11.11.15 We have

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} 2x, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2x \cdot 2x - 2(x^2 + y^2)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2},$$

and in the same manner

$$\frac{\partial^2 u}{\partial y^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}.$$

Thus $\Delta u(x, y) = 0$, that is, $u(x, y) = \ln(x^2 + y^2)$ solves the Laplace equation.

11.11.16 For $u(x, t) = \sin x \cos t$ we get

$$\frac{\partial u}{\partial x} = \cos x \cos t, \quad \frac{\partial^2 u}{\partial x^2} = -\sin x \cos t,$$
$$\frac{\partial u}{\partial t} = -\sin x \sin t, \quad \frac{\partial^2 u}{\partial t^2} = -\sin x \cos t.$$

Thus $u(x, t) = \sin x \cos t$ satisfies the wave equation.

11.11.17 For $u(x, t) = t^{-1/2} e^{-x^2/t}$ we get

$$\frac{\partial u}{\partial t} = -\frac{1}{2} t^{-3/2} e^{-x^2/t} + t^{-5/2} x^2 e^{-x^2/t},$$
$$\frac{\partial u}{\partial x} = -2x t^{-3/2} e^{-x^2/t}, \quad \frac{\partial^2 u}{\partial x^2} = -2t^{-3/2} e^{-x^2/t} + 4x^2 t^{-5/2} e^{-x^2/t},$$

so

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}.$$

11.11.18 The equation $(\partial^2/\partial x^2)u = 0$ means that the function $\partial u/\partial x$ does not depend on x. Therefore,

$$\frac{\partial u}{\partial x}(x, y) = A(y),$$

where A is an arbitrary function. Integrating both sides of this equation with respect to x while keeping y fixed we obtain

$$u(x, y) = A(y)x + B(y),$$

where A and B are arbitrary functions of y.

11.11.19 We look for solutions in the form $u(x, t) = U(x)T(t)$. Inserting u into the differential equation gives $U(x)T''(t) = c^2U''(x)T(t)$. Separation of variables yields that T''/T and U''/U are constant and satisfy

$$U''(x) + \lambda U(x) = 0, \quad T''(t) + \lambda c^2 T(t) = 0$$

for some constant λ. The functions $U_n(x) = \sin(n\pi x/\ell)$ solve the first equation and satisfy $U_n(0) = U_n(\ell) = 0$. The functions $T_n(t) = \sin(n\pi ct/\ell)$ solve the second equation and satisfy $T_n(0) = 0$. So the functions $u_n(x, t) = U_n(x)T_n(t)$ solve the given problem except possibly for the condition $(\partial u/\partial t)(x, 0) = h(x)$. We have

$$T_n'(t) = \frac{n\pi c}{\ell}\cos\left(\frac{n\pi ct}{\ell}\right)$$

and thus

$$\frac{\partial}{\partial t}u_n(x, 0) = U_n(x)T_n'(0) = \frac{n\pi c}{\ell}\sin\left(\frac{n\pi x}{\ell}\right).$$

Let the odd extension of h possess the Fourier expansion

$$h(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right), \quad b_n = \frac{2}{\ell}\int_0^{\ell} h(x)\sin\left(\frac{n\pi x}{\ell}\right)dx\,.$$

We set

$$u(x, t) = \sum_{n=1}^{\infty} d_n u_n(x, t).$$

In order to satisfy the condition $(\partial u/\partial t)(x, 0) = h(x)$, we choose

$$d_n = \left(\frac{n\pi c}{\ell}\right)^{-1} b_n = \frac{2}{n\pi c}\int_0^\ell h(x)\sin\left(\frac{n\pi x}{\ell}\right) dx\,.$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}u(x,0) &= \sum_{n=1}^{\infty} d_n \frac{\partial}{\partial t}u_n(x,0) = \sum_{n=1}^{\infty} d_n \frac{n\pi c}{\ell}\sin\left(\frac{n\pi x}{\ell}\right)\\ &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) = h(x).\end{aligned}$$

As u also solves all the other equations of the boundary value problem by the superposition principle, it solves the given problem.

11.11.20 Assume the solution to be of form $u(x,t) = X(x)T(t)$. Substituting u into the given heat equation we get $X(x)T'(t) = T(t)X''(x)$, so $T'(t)/T(t) = X''(x)/X(x)$. Thus, both sides define constant functions of t and x, respectively, so

$$X''(x) + \lambda X(x) = 0, \quad T'(t) + \lambda T(t) = 0$$

holds for some constant λ. The first equation has the general solution $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. (In order that it can satisfy the boundary conditions $X(0) = X(1) = 0$ and that it is not identically zero, we must have $\lambda > 0$.) The boundary condition $0 = u(0,t) = X(0)T(t)$ gives $a = 0$. The boundary condition $0 = u(1,t) = X(1)T(t)$ gives $\lambda = n^2\pi^2$ where $n = 1, 2, \ldots$. The second equation is solved by $T(t) = e^{-\lambda t}$. Therefore, the functions

$$u_n(x,t) = e^{-n^2\pi^2 t}\sin(n\pi x)$$

solve the given heat equation and the boundary conditions at $x = 0$ and $x = 1$. In order to satisfy the initial condition $u(x,0) = x(1-x)$, we look for a solution u of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t}\sin(n\pi x).$$

Since

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x),$$

it suffices to choose c_n as the Fourier coefficients of the odd extension of $f(x) = x(1-x)$. These are given by the numbers

$$c_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx$$

which may be explicitly computed by evaluating this integral.

11.11.22 We solve this problem by separation of variables. Let $u(x, y) = X(x)Y(y)$ and substitute into the given Laplace equation to obtain $X''(x)Y(y) + X(x)Y''(y) = 0$. This gives $X''(x)/X(x) = Y''(y)Y(y)$, so both sides must be equal to a constant function. Therefore, the functions X and Y must satisfy the differential equations

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

for some constant λ. The boundary condition $0 = u(0, y) = X(0)Y(y)$ and $0 = u(a, y) = X(a)Y(y)$ require $X(0) = X(a) = 0$ since otherwise Y is identically zero which would result in u being identically zero. The boundary value problem for X,

$$X'' + \lambda X = 0, \quad X(0) = X(a) = 0,$$

has the nontrivial solutions

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \quad X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, \ldots$$

For any n, the corresponding function Y_n thus has to satisfy

$$Y'' - \frac{n^2\pi^2}{a^2}Y = 0.$$

This equation has the general solution

$$Y_n(x) = c_n e^{\frac{n\pi y}{a}} + d_n e^{-\frac{n\pi y}{a}}.$$

The boundary condition $0 = u(x, 0) = X(x)Y(0)$ yields as above that we must have $Y_n(0) = 0$. Therefore, $d_n = -c_n$, and Y_n is of the form

$$Y_n(x) = 2c_n \sinh\left(\frac{n\pi y}{a}\right), n = 1, 2, \ldots$$

Therefore, the functions

$$u_n(x, y) = X_n(x)Y_n(y) = \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right), n = 1, 2, \ldots$$

satisfy for $n = 1, 2, \ldots$ the Laplace equation in the rectangle and the homogeneous boundary conditions on the lower side and on the vertical sides. To find a solution which satisfies the condition on the upper side $y = b$, we use a linear superposition of the u_n in form of an infinite series,

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right). \tag{E.2}$$

We need to choose b_n such that

$$x = u(x, b) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right).$$

The rightmost part coincides with the Fourier expansion of the odd extension of $f(x) = x$ on $[0, b]$. We therefore have

$$b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx$$

and thus

$$b_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx.$$

With this choice of b_n, (E.2) gives the solution of the given boundary value problem (a so-called Dirichlet problem) on the rectangle.

Solutions to Exercises of Chap. 12

12.5.1 (a) >> $x$minus = −3 : 0.01 : 0;
>> $x$plus = 0 : 0.01 : 3;
>> y1 = exp($x$minus);
>> y2 = ($x$plus-1).^2;
>> plot($x$minus, y1, $x$plus, y2) (Fig. E.26)

(b) >> $x = 0 : 0.1 : 10$;
>> $fx = (x.{*} \sin(x) + \exp(-x/5).{*} \cos(x))$;
>> stem (x, fx) (Fig. E.27)

(c) >> $[x, y]$ = meshgrid(−2 : 0.4 : 2);
>> $u = x$;
>> $v = (x.\hat{}2 + y.\hat{}2)$;
>> quiver (x, y, u, v) (Fig. E.28)

(d) >> theta = 0 : pi/100 : 10* pi;
>> r = 2* exp(-theta/10)
>> polar(theta, r) (Fig. E.29)

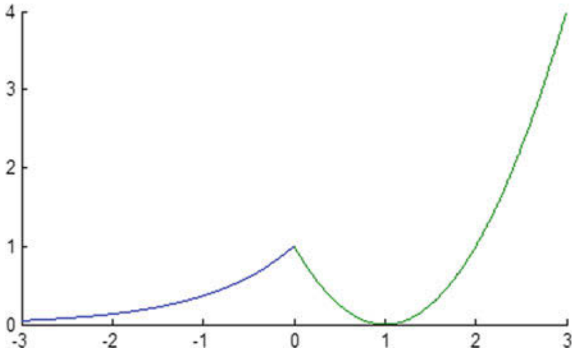

Fig. E.26 Solution of Exercise 12.5.1(a)

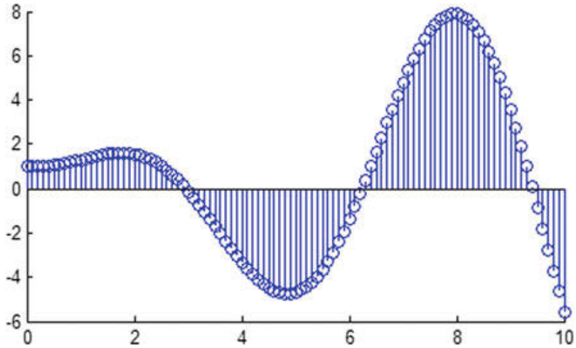

Fig. E.27 Solution of Exercise 12.5.2(b)

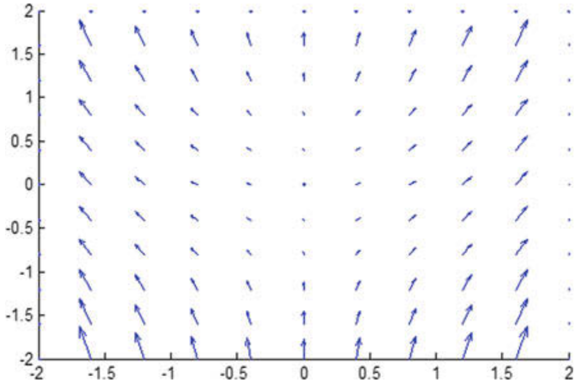

Fig. E.28 Solution of Exercise 12.5.1(c)

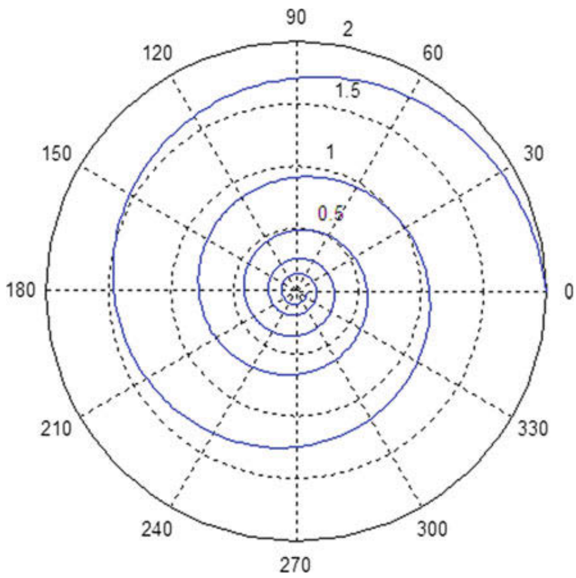

Fig. E.29 Solution of Exercise 12.5.1(d)

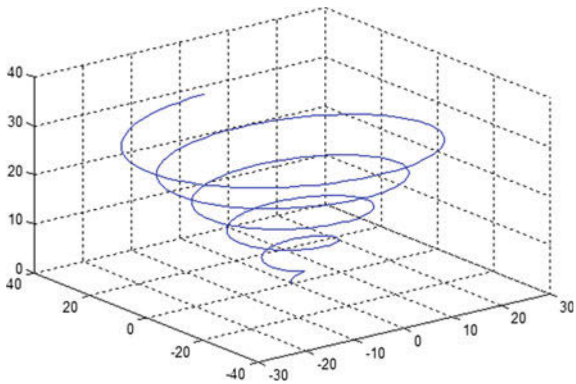

Fig. E.30 Solution of Exercise 12.5.2(a)

12.5.2 (a) $>> t = 0 : \text{pi}/50 : 10^* \text{ pi};$
$>>$ plot3(t.*sin (t), t.*cos (t), t)
$>>$ grid on
$>>$ axis square (Fig. E.30)

(b) $>> [x, y, z] = \text{meshgrid}(-2 : 0.4 : 2, -2 : 0.4 : 2, -1 : 0.4 : 1);$
$>> u = y;$
$>> v = x;$
$>> w = x.\hat{}2 + z;$
$>>$ quiver3(x, y, z, u, v, w) (Fig. E.31)

(c) **Program for Fig. 8.3 of Chap. 8**
$>> [x, y] = \text{meshgrid}(-2 : 0.1 : 2);$

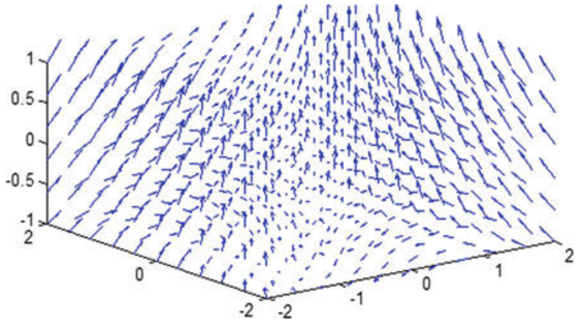

Fig. E.31 Solution of Exercise 12.5.2(b)

>> $c = x.*y$;
>> surf$c(x, y, c)$ % this can be omitted for color plot (Fig. E.32) **Program for Fig. 8.6 of Chap. 8**
>> $[x, y] =$ meshgrid$(-6 : 0.2 : 6)$;
>> $fxy = \sin(\text{sqrt}((x.\hat{}2) + (y.\hat{}2)))$;
>> surf$c(x, y, fxy)$% this can be omitted for color plot (Fig. E.33)

(d) >> $[x, y] =$ meshgrid$(-3 : 0.1 : 3)$;
>> $z = (x.*y.*(x.\hat{}2 - y.\hat{}2))./(x.\hat{}2 + y.\hat{}2)$;
>> mesh(x, y, z) (Fig. E.34)

12.5.3 (a) >> syms k% n is represented by k here
>> sum1 = symsum$(((\log(k))\hat{}2)/k\hat{}1.5, 1, 200)$;
>> double(sum1)

ans =

7.8960

(b) >> syms k% n is represented by k here
>> sum2 = symsum$(((-1)\hat{}(k+1)/(k*\log(k+1)), 1, 200)$;
>> double(sum2) (Fig. E.34)

ans =

1.1360

(c) >> syms k% n is represented by k here
>> $s =$ symsum$(1/k\hat{}3, 1, 200)$;
>> double(s)

ans =

1.2020

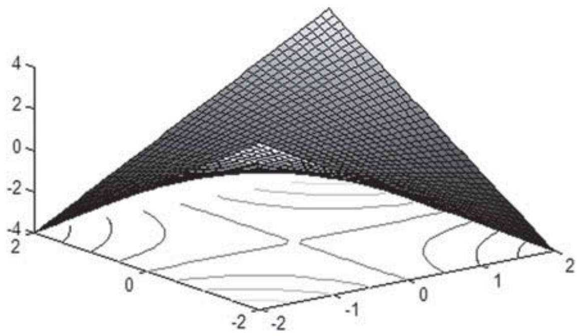

Fig. E.32 Solution of Exercise 12.5.2(c), first figure

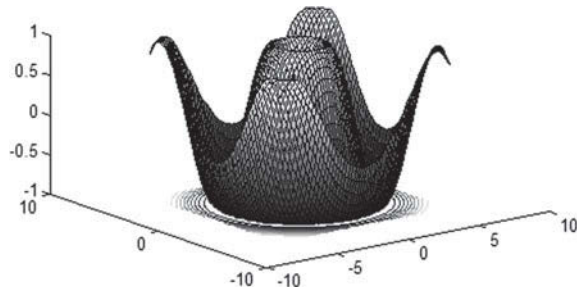

Fig. E.33 Solution of Exercise 12.5.2(c), second figure

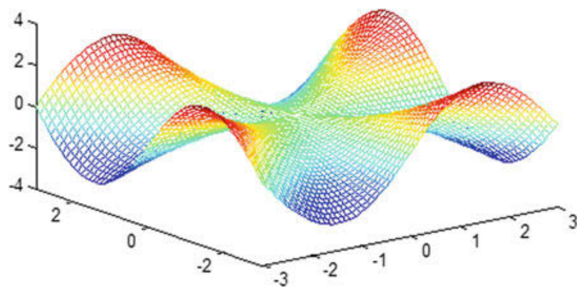

Fig. E.34 Solution of Exercise 12.5.2(d)

12.5.4 (a) $>>$ limit$(1/(5+4{*}\cos(x)), x, 0)$

ans=

1/9

(b) $>>$limit$((2{*}x)/$sqrt$(x\hat{}-1), x,$ inf)

ans =

2

(c) >> limit((exp(a cos (x))./sqrt(1 − x.^2)), x, 0)

ans =

exp(1/2* pi)

>> double(ans)

ans =

4.8105

(d) >> limit(x/sqrt(9 + 4*x^2), x, inf)

ans =

1/2

12.5.5 **Numerical Integration**:

>> myfunc = inline('exp(x).* sin (x)');
>> x = quad(myfunc, 0, 5)
x =

−91.7081

Symbolic Integration:

>> syms x;
>> fx = exp(x)* sin (x);
>> x = int(fx, 0, 5);
>> x = double(x)
ans =

−91.7081

Here, we can see that both the methods are giving same answer.

12.5.6 (a) >> syms x;
>> ya = diff((4*x^2 − 1)(7*x^3 + x))

ya =

8*x*(7x^3 + x) + (4*x^2 − 1)*(21*x^2 + 1)

>> ezplot(ya) (Fig. E.35)

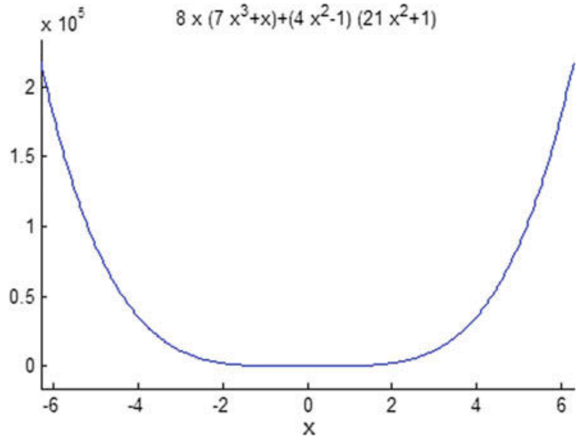

Fig. E.35 Solution of Exercise 12.5.6(a)

(b) $>>$ syms x;
$>>$ $yb = \text{diff}((x\hat{}2 - 1)/(x\hat{}4 + 1))$

yb =

$2{*}x/(x\hat{}4 + 1) - 4{*}(x\hat{}4 + 1)\hat{}2{*}x\hat{}3$ (Fig. E.36)

(c) $>>$ syms x;
$>>$ $yc = \text{diff}(\sin(x)/(1 + \cos(x)))$

yc =

$\cos(x)/(1 + \cos(x)) + \sin(x)\hat{}2/(1 + \cos(x))\hat{}2$ (Fig. E.37)

(d) $>>$ syms x;
$>>$ $yd = \text{diff}(\exp(x){*}\sin(x))$

yd =

$\exp(x){*}\sin(x) + \exp(x){*}\cos(x)$ (Fig. E.38)

12.5.7 (a) **Function file**:
% Save this file with name ‘myode.m’
function $x$dot = myode(t, x)
$x$dot = $[x(2); -\sin(x(1))]$;
On command prompt, write following:
$>>$ $[t, x]$ = ode45('myode', [0, 20], [1; 0]);
$>>$ plot(t, x) (Fig. E.39)

(b) **Function file**:
% Save this file with name ‘myode.m’

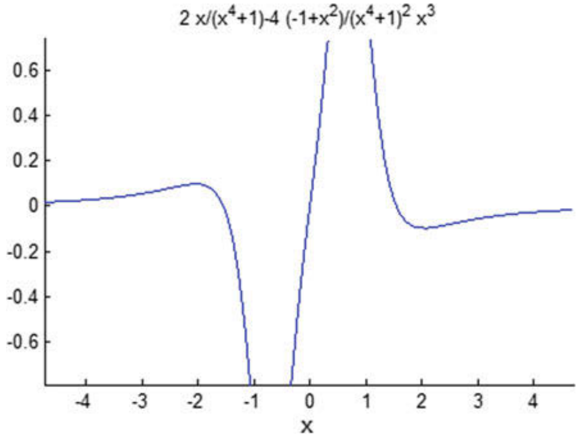

Fig. E.36 Solution of Exercise 12.5.6(b)

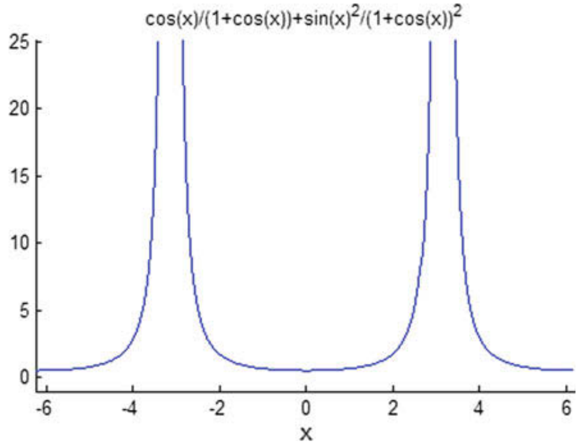

Fig. E.37 Solution of Exercise 12.5.6(c)

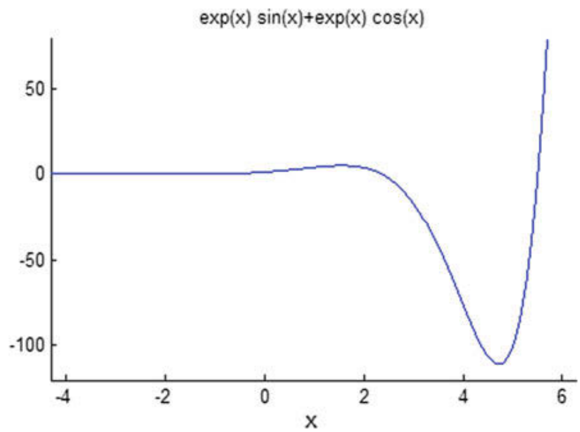

Fig. E.38 Solution of Exercise 12.5.6(d)

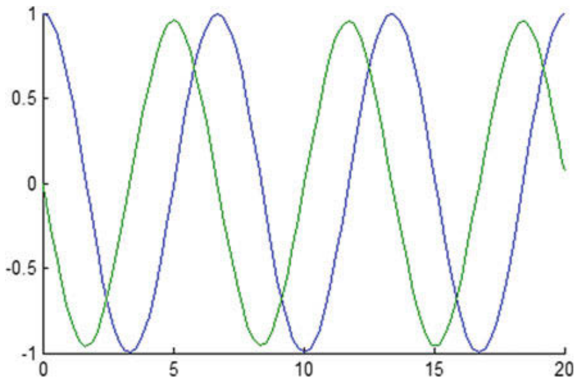

Fig. E.39 Solution of Exercise 12.5.7(a)

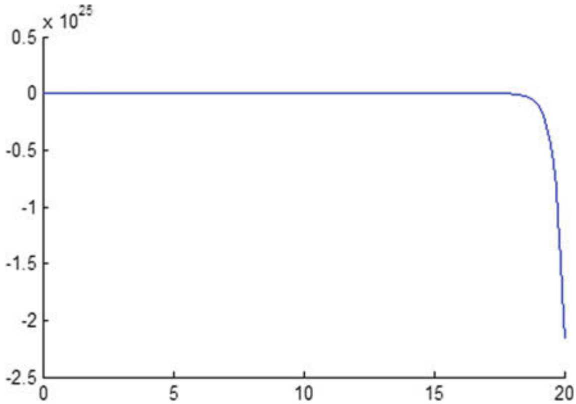

Fig. E.40 Solution of Exercise 12.5.7(b)

function $x$dot = myode(t, x)
$x$dot = 3* − 4* cos (t);
On command prompt, write following:
>> [t, x] = ode45('myode', [0, 20], [1; 0]);
>> plot(t, x) (Fig. E.40)

12.5.8

```
% Fourier Analysis of ramp function for Fig. 12.24
t0 = −pi; % initial time
t0_T = pi; % final time
mp = 0; % mid point
T = t0_T − t0; % time period
syms t; % sym variable declaration
ft = t; % function declaration
w0 = 2*pi/T; % frequency
n = 1.5; % number of Harmonics
% computation of Trigonometric Fourier Series Coefficients
a0 = 1/T*(int(ft, −pi, pi));
an = 2/T*(int(ft* cos (n*w0*t), −pi, pi))
```

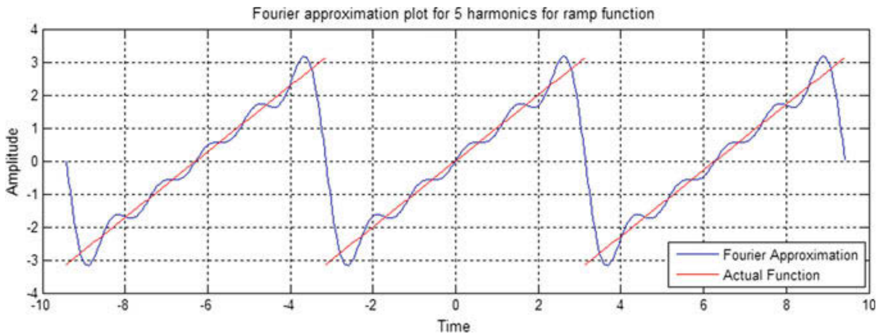

Fig. E.41 Fourier approximation of the ramp function for 5 harmonics

```
bn = 2/T*(int(ft* sin (n*w0*t), −pi, pi))
ann = an.* cos (n*w0*t);
bnn = bn.* sin (n*w0*t);
avg = double(a0); % converting sym variable to value
t = −3*pi : pi/100 : 3*pi; %plotting Fourier series for 3 periods
suma = 0; sumb = 0
for j = 1 : 5
    sumb = sumb + bnn(j);
    suma = suma + ann(j);
end
bnsum = eval(sumb);
ansum = eval(suma);
plot(t, avg+bnsum+ansum)% plot of truncated harmonics function
hold on
% plotting actual function for 3 periods
tx1 = −3* pi: pi/100 : −pi;
tx2 = −pi: pi/100 : pi;
tx3 = pi : pi/100 : 3*pi;
plot(tx1,tx1 + 2*pi,‘r’,tx2,tx2,‘r’,tx3,tx3 – 2*pi,‘r’)
% formatting plot
xlabel(‘Time’)
ylabel(‘Amplitude’)
title(‘Fourier approximation plot for 5 harmonics for ramp function’)
legend(‘Fourier Approximation’, ‘Actual Function’)
```

For above program, we get following result- (Fig. E.41)

Similarly we can draw for 10 and 30 harmonics also just by changing the

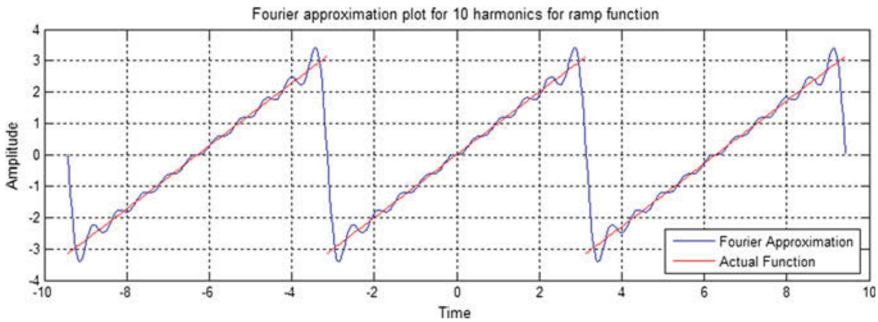

Fig. E.42 Fourier approximation of the ramp function for 10 harmonics

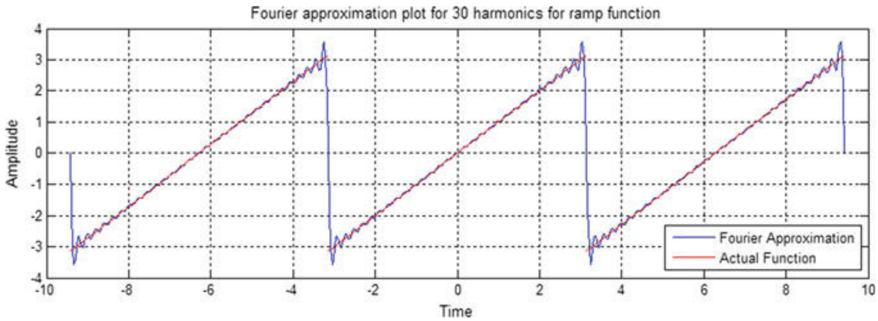

Fig. E.43 Fourier approximation of the ramp function for 30 harmonics

limit of for loop in line number 09 and 19. For example, for 30 harmonics, we have to replace 5 by 30. For 10 and 30 harmonics, see Figs. E.42 and E.43.

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