

Chapter 6:

Robust Nonlinear Programming

Our robust nonlinear programming formulation

- We focus on the following formulation:

$$g(x, z) \leq 0, \forall z \in \mathcal{Z}$$

where:

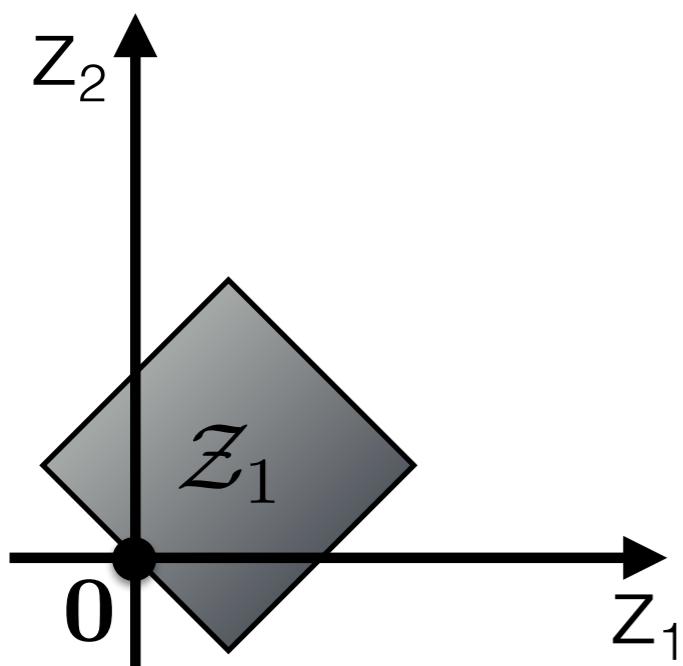
- $g(\cdot, \cdot)$ is a mapping defined over the convex domain $\mathcal{X}_g \times \mathcal{Z}_g$
- $g(x, z)$ is convex in x and concave in z
- $\mathcal{Z} \subset \mathbb{R}^m$ is a given non-empty, convex, and compact set
- there exist a z_0 in the relative interior of \mathcal{Z} intersect \mathcal{Z}_g

Relative interior definition

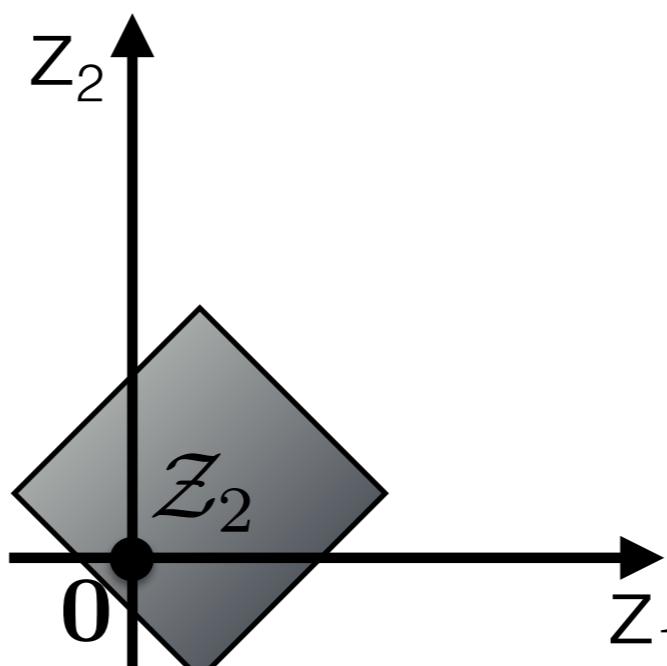
- There exists a ball centered at z_0 and radius $\epsilon > 0$ which projection on the affine space spanned by \mathcal{Z} is included in \mathcal{Z}

For convex uncertainty sets this translates:

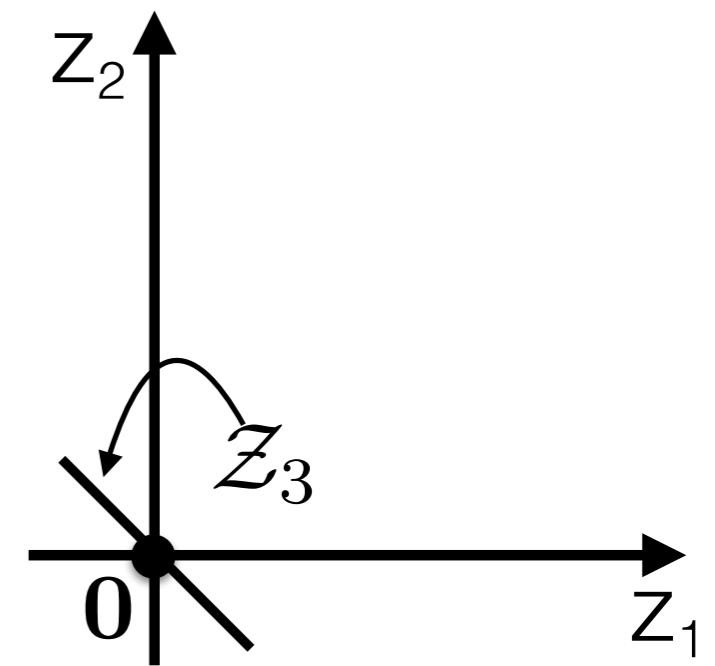
$$\exists \epsilon > 0, \forall z \in \mathcal{Z}, z_0 - \epsilon((z - z_0)/\|z - z_0\|_2) \in \mathcal{Z}\}$$



$0 \notin \text{relint}(\mathcal{Z}_1)$



$0 \in \text{relint}(\mathcal{Z}_2)$



$0 \in \text{relint}(\mathcal{Z}_3)$

Fenchel Robust Counterpart

Assumptions:

- $g(x, z)$ is convex in x and concave in z
- $\mathcal{Z} \subset \mathbb{R}^m$ is a given non-empty, convex, and compact set
- there exist a z_0 in the relative interior of both \mathcal{Z} and \mathcal{Z}_g

 **Theorem 6.2.** : *The vector $x \in \mathcal{X}$ satisfies the robust constraint (6.1) if and only if $x \in \mathcal{X}$ and $v \in \mathbb{R}^m$ satisfy the single inequality*

$$(FRC) \quad \delta^*(v|\mathcal{Z}) - g_*(x, v) \leq 0 , \quad (6.2)$$

where the support function δ^* is defined as

$$\delta^*(v|\mathcal{Z}) := \sup_{z \in \mathcal{Z}} z^T v$$

and the partial concave conjugate function g_* is defined as

$$g_*(x, v) := \inf_{z \in \mathcal{Z}_g} v^T z - g(x, z) .$$

When \mathcal{Z} is bounded strong duality follows from Sion's minimax theorem

Lemma 4.4. : (Sion's minimax theorem [32]) Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex set and $\mathcal{Z} \in \mathbb{R}^m$ be a compact convex set, and let h be a real-valued function on $\mathcal{X} \times \mathcal{Z}$ with

1. $h(x, \cdot)$ lower semicontinuous and quasi-convex on \mathcal{Z} , $\forall x \in \mathcal{X}$
2. $h(\cdot, z)$ upper semicontinuous and quasiconcave on \mathcal{X} , $\forall z \in \mathcal{Z}$

then

$$\sup_{x \in \mathcal{X}} \min_{z \in \mathcal{Z}} h(x, z) = \min_{z \in \mathcal{Z}} \sup_{x \in \mathcal{X}} h(x, z).$$

- Coro #1: If both \mathcal{X} and \mathcal{Z} are convex, one of them is bounded, and $h(x, z)$ is concave in x & convex in z , then

$$\sup_{x \in \mathcal{X}} \inf_{z \in \mathcal{Z}} h(x, z) = \inf_{z \in \mathcal{Z}} \sup_{x \in \mathcal{X}} h(x, z)$$

- Coro #2: If both \mathcal{X} and \mathcal{Z} are convex, one of them is bounded, and $h(x, z)$ is convex in x & concave in z , then

$$\inf_{x \in \mathcal{X}} \sup_{z \in \mathcal{Z}} h(x, z) = \sup_{z \in \mathcal{Z}} \inf_{x \in \mathcal{X}} h(x, z)$$

Example : Quadratic Programming

 **Example 6.3.** : Consider the following robust optimization constraint:

$$p(x)^T z + s(x) - z^T P(x)z \leq 0, \quad \forall z \in \mathcal{Z},$$

where $p : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an affine function of x , $s : \mathbb{R}^n \rightarrow \mathbb{R}$, and $P(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$, and finally where

$$\mathcal{Z} := \{z \in \mathbb{R}^m \mid z^T Q z \leq r\},$$

with $Q \in \mathbb{R}^{m \times m}$ a symmetric matrix and $r \in \mathbb{R}$.

After describing $g(x, z)$ as $g(x, z) := p(x)^T z + s(x) - z^T P(x)z$ and letting $z_0 = 0$, one needs to make the following assumptions in order to apply theorem 6.2:

- Impose that $x \in \mathcal{X}_g$ with $\mathcal{X}_g := \{x \mid P(x) \succeq 0\}$, namely that we have the guarantee that $P(x)$ is positive semi-definite in order to make $g(x, z)$ concave in z .
- Impose that $Q \succ 0$ and that $r > 0$, namely that Q is positive definite to ensure that \mathcal{Z} is convex and bounded, and that $0 \in \text{relint}(\mathcal{Z})$.

When applying theorem 6.2, we obtain that the constraint is equivalent to

$$\exists v \in \mathbb{R}^m \quad \delta^*(v|\mathcal{Z}) - g_*(x, v) \leq 0,$$

Example : Quadratic Programming under Polyhedron

- It's really rather simple to obtain the robust counterpart under a different uncertainty set. Simply replace the conjugate of the support function.
- Example: using $Bz \leq b$ instead of ellipsoid

Ellipsoidal set

$$\delta^*(v|\mathcal{Z}) = \sqrt{r}\|Q^{-1/2}v\|_2 \rightarrow \begin{aligned} & \sqrt{r}\|Q^{-1/2}v\|_2 - t \leq 0 \\ & \begin{bmatrix} P(x) & (v - p(x))/2 \\ (v - p(x))^T/2 & -s(z) - t \end{bmatrix} \succeq 0 \end{aligned}$$

Example : Quadratic Programming under Polyhedron

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- Example: using $Bz \leq b$ instead of ellipsoid

Polyhedral set

$$\delta^*(v|\mathcal{Z}) = \inf_{\lambda: \lambda \geq 0, B^T \lambda = v} b^T \lambda \quad \rightarrow$$

$$b^T \lambda - t \leq 0$$

$$B^T \lambda = v$$

$$\lambda \geq 0$$

$$\begin{bmatrix} P(x) & (v - p(x))/2 \\ (v - p(x))^T/2 & -s(z) - t \end{bmatrix} \succeq 0$$

Some tractable reformulation are beyond the reach of FRC



Example 6.4. : Consider the following robust optimization constraint:

$$p(x)^T z + s(x) - z^T P(x)z \leq 0, \forall z \in \mathcal{Z},$$

where $p : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an affine function of x , $s : \mathbb{R}^n \rightarrow \mathbb{R}$, and $P(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$, and finally where

$$\mathcal{Z} := \{z \in \mathbb{R}^m \mid z^T Q z \leq r\},$$

with $Q \in \mathbb{R}^{m \times m}$, $q \in \mathbb{R}^m$, and $r \in \mathbb{R}$.

- S-lemma can be used to demonstrate that this RC is equivalent to

$$\exists \lambda \geq 0 \quad \begin{bmatrix} P(x) + \lambda Q & -p(x)/2 \\ -p(x)^T/2 & -s(x) - r\lambda \end{bmatrix} \succeq 0$$

even when $P(x)$ is not PSD as long as $\bar{z}^T Q \bar{z} < r$ for some \bar{z} .

Some useful theorems

Theorem 6.5. : If $\mathcal{Z} := \{z \in \mathbb{R}^m \mid 0 \leq z \leq 1, \sum_i z_i \leq \rho\}$, then

$$\begin{aligned}\delta^*(v|\mathcal{Z}) := & \inf_{\omega \in \mathbb{R}^m, \lambda \in \mathbb{R}} \sum_i \omega_i + \rho\lambda \\ \text{subject to } & \lambda \geq v_i - \omega_i, \forall i \\ & \lambda \geq 0, \omega \geq 0.\end{aligned}$$

Theorem 6.7. : If $\mathcal{Z} \subset \mathbb{R}^m$ is an affine projection of $\mathcal{Z}_1 \subset \mathbb{R}^{m_1}$, namely that $\mathcal{Z} := \{z \in \mathbb{R}^m \mid \exists z' \in \mathcal{Z}_1, z = Az' + a_0\}$ for some $A \in \mathbb{R}^{m \times m_1}$ and $a_0 \in \mathbb{R}^m$, then $\delta^*(v|\mathcal{Z}) = a_0^T v + \delta^*(A^T v|\mathcal{Z}_1)$.

Theorem 6.9. : If $g(x, z)$ is a positive affine mapping of $g'(x, z)$, namely that $g(x, z) := \alpha g'(x, z) + \beta$ for some $\alpha > 0$, then $g_*(x, v) = \alpha g'_*(x, v/\alpha) - \beta$.

Other useful theorems

Corollary 6.8. : Consider using the budgeted uncertainty set $\mathcal{Z} := \{z \in \mathbb{R}^m \mid -1 \leq z \leq 1, \sum_i |z_i| \leq \Gamma\}$, then

$$\delta^*(v|\mathcal{Z}) := \inf_{\omega^+ \in \mathbb{R}^m, \omega^- \in \mathbb{R}^m, \lambda \in \mathbb{R}} \quad \sum_i \omega_i^+ + \sum_i \omega_i^- + \rho \lambda$$

subject to

$$\begin{aligned} \lambda &\geq v_i - \omega_i^+, \quad \forall i \\ \lambda &\geq -v_i - \omega_i^-, \quad \forall i \\ \omega &\geq 0, \quad \lambda \geq 0 . \end{aligned}$$

Hence, the robust counterpart takes the form:

$$\exists \omega^+ \in \mathbb{R}^m, \omega^- \in \mathbb{R}^m, \lambda \in \mathbb{R}, v \in \mathbb{R}^m, \quad \left\{ \begin{array}{l} \sum_i \omega_i^+ + \sum_i \omega_i^- + \rho \lambda - g_*(x, v) \leq 0 \\ \lambda \geq v_i - \omega_i^+, \quad \forall i \\ \lambda \geq -v_i - \omega_i^-, \quad \forall i \\ \omega^+ \geq 0, \quad \omega^- \geq 0, \quad \lambda \geq 0 \end{array} \right..$$

Table 6.1: Table of reformulations for uncertainty sets (Table 1 in [8])

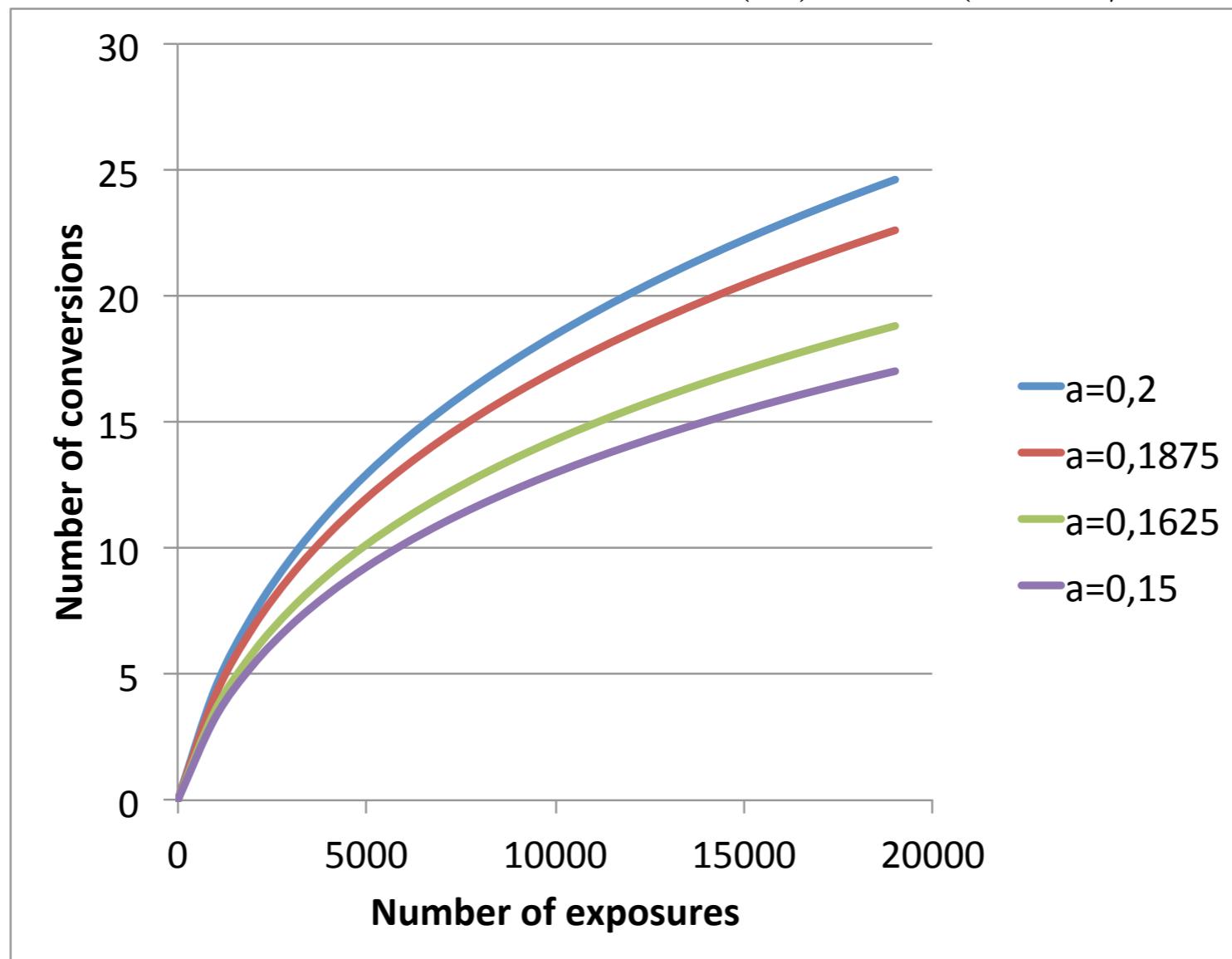
Uncertainty region \mathcal{Z}	Support function $\delta^*(v \mathcal{Z})$
Box	$\rho\ v\ _1$
Ball	$\rho\ v\ _2$
Polyhedral	$\inf_{w \geq 0: B^T w = v} b^T w$
Cone	$\inf_{w \in C^*: B^T w = v} b^T w$
KL-Divergence	$\inf_{u \geq 0} \sum_l z_l^0 u e^{(v_l/u)-1} + \rho u$
Geometric prog.	$\inf_{u \geq 0, w \geq 0: \sum_i d_i w_i = v} \sum_i \{w_i \ln \left(\frac{w_i}{\alpha_i u} \right) - w_i\} + \rho u$
Intersection	$\inf_{\{w_i\}: \sum_i w_i = v} \sum_i \delta^*(w^i \mathcal{Z}_i)$
Example	$\inf_{(w^1, w^2): w^1 + w^2 = v} \rho_1 \ w^1\ _\infty + \rho_2 \ w^2\ _2$
Minkowski sum	$\sum_i \delta^*(v \mathcal{Z}_i)$
Example	$\rho_\infty \ v\ _1 + \rho_2 \ v\ _2$
Convex hull	$\max_i \delta^*(v \mathcal{Z}_i)$
Example	$\max\{\rho_\infty \ v\ _1, (z^0)^T v + \rho_2 \ v\ _2\}$

Table 6.2: Table of reformulations for constraint functions (Table 2 in [8])

Constraint function	$g(x, z)$	Partial concave conjugate $g_*(x, v)$
Linear in z	$z^T g(x)$	$\begin{cases} 0 & \text{if } v = g(x) \\ -\infty & \text{otherwise} \end{cases}$
Concave in z , separable in z and x	$g(z)^T x$	$\sup_{\{s^i\}_{i=1}^n : \sum_{i=1}^n s^i = v} \sum_i x_i (g_i)_*(s^i/x_i)$
Example	$-\sum_i \frac{1}{2}(z^T Q_i z)x_i$	$\sup_{\{s^i\}_{i=1}^n : \sum_{i=1}^n s^i = v} -\frac{1}{2} \sum_{i=1}^n \frac{(s^i)^T Q_i^{-1} s^i}{x_i}$
Sum of functions	$\sum_i g_i(x, z)$	$\sup_{\{s^i\}_{i=1}^n : \sum_i s^i = v} \sum_i (g_i)_*(x, s^i)$
Sum of separable functions	$\sum_i g_i(x, z_i)$	$\sum_{i=1}^n (g_i)_*(x, v_i)$
Example	$-\sum_{i=1}^m x_i^{z_i},$ $x_i > 1, 0 \leq z \leq 1$	$\begin{cases} \sum_{i=1}^m \left(\frac{v_i}{\ln x_i} \ln \frac{-v_i}{\ln x_i} - \frac{v_i}{\ln x_i} \right) & \text{if } v \leq 0 \\ -\infty & \text{otherwise} \end{cases}$
Exponential in z	$g(x)e^z$ with $g(x) > 0, \forall x$	$\begin{cases} v \ln(-v/g(x)) - v & \text{if } v \leq 0 \\ -\infty & \text{otherwise} \end{cases}$

Planning an ad campaign with exposure rate uncertainty

Figure of the converted number of customers per ad displayed on a website according to $h_i(x_i) := 30(1 + x_i/1000)^{a_i} - 30$



Planning an ad campaign with exposure rate uncertainty

- Derive a tractable reformulation for the robust counterpart of this problem:

$$\begin{aligned} & \underset{x}{\text{maximize}} && \sum_i h_i(x) \\ & \text{subject to} && \sum_i p_i x_i \leq B \\ & && x \geq 0 , \end{aligned}$$

with

$$h_i(x_i) := c_i(1+x_i/d_i)^{a_i} - c_i$$

$$\mathcal{U}_1 := \{a \in \mathbb{R}^n \mid \exists z \in \mathbb{R}^n, 0 \leq z \leq 1, \sum_i z_i \leq \Gamma, a_i = \bar{a}_i(1 - 0.25z_i), \forall i\}$$

where $\bar{a} \in [0, 1]^n$

Exercise

Exercise 6.3. (*More robust non-linear reformulations*)

Consider the robust optimization problem:

$$\begin{aligned} & \underset{x}{\text{maximize}} && \min_{z \in \mathcal{Z}} \sum_i x_i \exp(z_i) \\ & \text{subject to} && \sum_i x_i \leq 1 \\ & && x \geq 0 , \end{aligned}$$

where

$$\mathcal{Z} := \{z \in \mathbb{R}^n \mid \exists v \in [-1, 1]^n, w \in [-1, 1], z = \mu + Q(v + w), \|v\|_1 \leq \Gamma\} .$$

Question: Derive a tractable reformulation of this problem as a convex optimization problem of finite dimension ?