

# Chapter 2:

## Robust Counterpart of Linear Programs

# General robust LP

(Robust counterpart)

$$\begin{aligned} & \underset{x}{\text{maximize}} && \min_{z \in \mathcal{Z}} h(x, z) \\ & \text{subject to} && g_j(x, z) \leq 0, \forall z \in \mathcal{Z}, \forall j = 1, \dots, J. \end{aligned}$$

We assume that the nominal problem is an LP

$$\begin{aligned} h(x, z) &:= c(z)^T x + d(z) \\ g_j(x, z) &:= a_j(z)^T x - b_j(z), \end{aligned}$$

And that all functions are affine in «  $Z$  »

$$\begin{aligned} c(z) &:= (P_0 z + p_0) & \& d(z) &= q_0^T z + r_0, \\ a_j(z) &:= (P_j z + p_j) & \& b_j(z) &= q_j^T z + r_j, \end{aligned}$$

In other words, we are left with the following LP-RC

$$(\text{LP-RC}) \quad \underset{x}{\text{maximize}} \quad \min_{z \in \mathcal{Z}} z^T P_0^T x + q_0^T z + p_0^T x + r_0 \quad (2.1a)$$

$$\text{subject to} \quad z^T P_j^T x + p_j^T x \leq q_j^T z + r_j, \forall z \in \mathcal{Z}, \forall j = 1, \dots, J. \quad (2.1b)$$

# NP-completeness for general uncertainty sets

- Take the robust counterpart optimization problem

$$\begin{aligned} & \underset{x}{\text{maximize}} && c^T x \\ & \text{subject to} && (a + z)^T x \leq b, \forall z \in \mathcal{Z}, \end{aligned}$$

- Verifying for a fixed «  $x$  » whether the following claim is true is NP-complete in general, and in particular when the uncertain vector contains integer variables

$$z^T x \leq b - a^T x, \forall z \in \mathcal{Z} \Leftrightarrow \max_{z \in \mathcal{Z}} z^T x \leq b - a^T x$$

# Scenario based uncertainty

- Consider the following robust counterpart

$$\begin{aligned} & \underset{x}{\text{maximize}} && c^T x \\ & \text{subject to} && (a + z)^T x \leq b, \forall z \in \mathcal{Z}, \end{aligned}$$

with scenario based uncertainty

$$\mathcal{Z} := \{\bar{z}_1, \bar{z}_2, \dots, \bar{z}_K\}$$

- Then, one can reduce the problem to

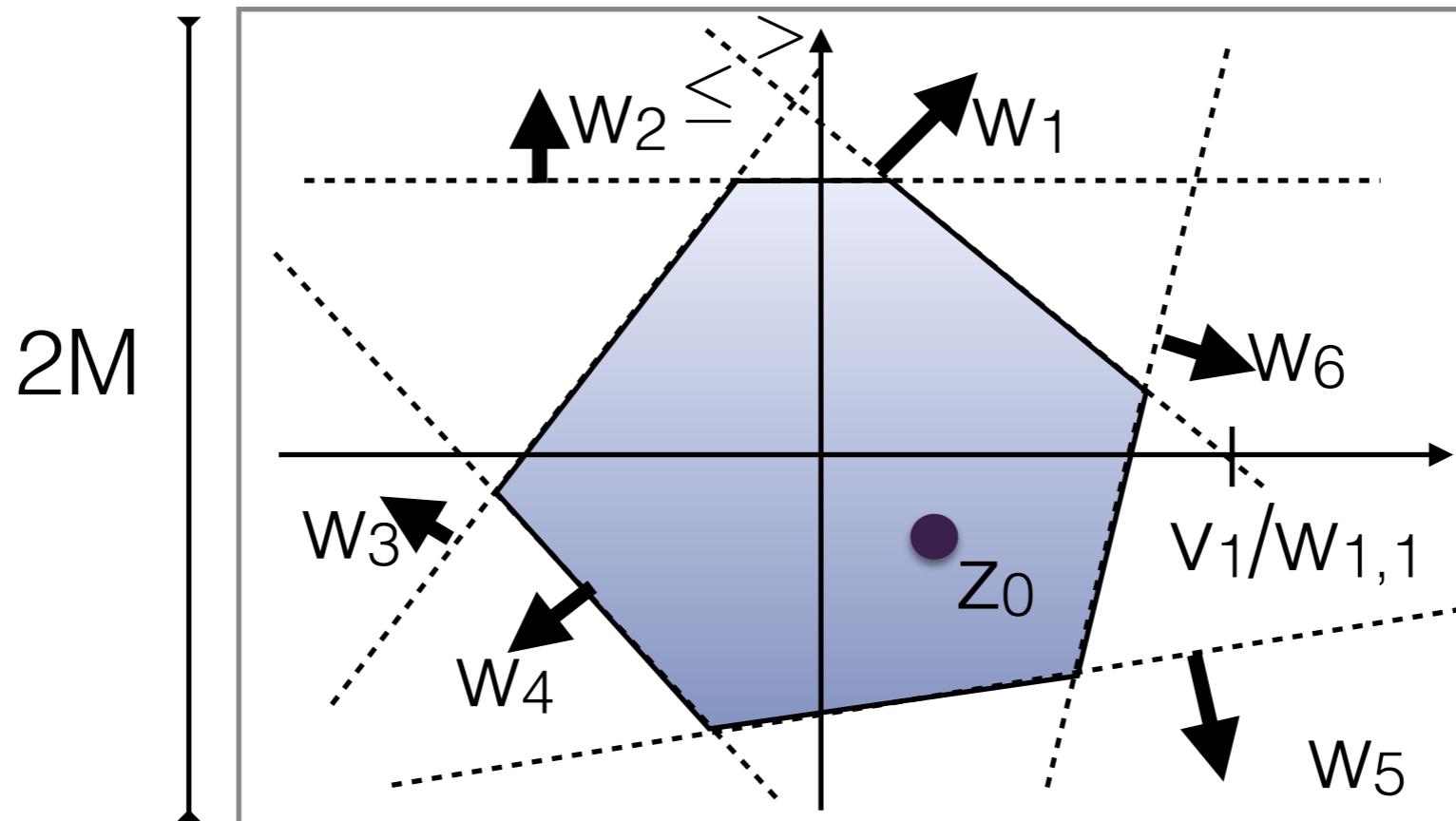
$$\begin{aligned} & \underset{x}{\text{maximize}} && c^T x \\ & \text{subject to} && (a + \bar{z}_i)^T x \leq b, \forall i = 1, \dots, K \end{aligned}$$

# Polyhedral uncertainty

**Assumption 2.2.** : The uncertainty set  $\mathcal{Z}$  is a non-empty and bounded polyhedron that can be defined according to

$$\mathcal{Z} := \{z \in \mathbb{R}^m \mid w_i^T z \leq v_i, \forall i = 1, \dots, s\},$$

where for each  $i = 1, \dots, s$ , we have that  $w_i \in \mathbb{R}^{1 \times m}$  and  $v_i \in \mathbb{R}$  capture a facet of the polyhedron through the expression  $w_i^T z = v_i$ . Moreover, since  $\mathcal{Z}$  is non-empty, there must exist a  $z_0 \in \mathcal{Z}$  and since it is bounded there must exist some  $M > 0$  such that  $\mathcal{Z} = \mathcal{Z} \cap \{z \in \mathbb{R}^m \mid -M \leq z \leq M\}$ .



# LP reformulation for LP-RC with polyhedral set

Verifying whether  $\forall z \in \mathcal{Z}, z^T x \leq b - a^T x$  is equivalent to evaluating the optimal value of the following problem

$$(\Psi :=) \quad \underset{z}{\text{maximize}} \quad x^T z \quad (2.3a)$$

$$\text{subject to} \quad Wz \leq v \quad (2.3b)$$

**Theorem 2.3.** : (LP Duality see Chapter 4 of [16]) Under assumption 2.2, the optimal value of linear program (2.3) is equal to the optimal value of the following dual problem

$$(\Upsilon^* :=) \quad \underset{\lambda}{\text{minimize}} \quad v^T \lambda \quad (2.4a)$$

$$\text{subject to} \quad W^T \lambda = x \quad (2.4b)$$

$$\lambda \geq 0 \quad (2.4c)$$

where  $\lambda \in \mathbb{R}^s$ . Moreover, problem (2.4) has a feasible solution.

# Weak vs. Strong duality

- Weak duality :  $\Psi \leq \Upsilon^*$

- Proof of weak duality:

$$\begin{aligned}\Psi &:= \max_{z: Wz \leq v} x^T z \\ &= \max_z \min_{\lambda: \lambda \geq 0} x^T z + \lambda^T (v - Wz) \\ &\leq \min_{\lambda: \lambda \geq 0} \max_z x^T z + \lambda^T (v - Wz) \\ &= \min_{\lambda: \lambda \geq 0, x = W^T \lambda} v^T \lambda = \Upsilon^*\end{aligned}$$

- The challenge of Theorem 2.3 is to prove strong duality
- Strong duality does not apply for most non-convex  $\mathcal{Z}$ .

# Example: box uncertainty

Consider the robust optimization problem:

$$\begin{aligned} & \underset{x}{\text{maximize}} && c^T x \\ & \text{subject to} && (a + z)^T x \leq b, \forall z \in \mathcal{Z} \\ & && 0 \leq x \leq 1, \end{aligned}$$

with  $\mathcal{Z} := \{z \in \mathbb{R}^n \mid -\hat{z} \leq z \leq \hat{z}\}$

- Formulate an equivalent finite dimensional linear program
- Implement this linear program (a.k.a. the reduced form of the model) using RSOME (incomplete Colab file)

# Implementation in RSOME

(see complete Colab file)

- Robust counterpart:
- Reduced form:

```
#Create model
model = ro.Model('simpleExample_rawrobust')
x=model.dvar(n)

#Create uncertain vector
z= model.rvar(n)
#Create uncertainty set
boxSet= (z>=zBarMinus, z <= zBarPlus)

model.max(c@x)
#Robustify the constraint
model.st((a+z)@x<=b).forall(boxSet))
model.st(x>=0)
model.st(x<=1)
model.solve(my_solver)
```

```
#Create model
model = ro.Model('simpleExample_redrobust')
x=model.dvar(n)
#Create auxiliary variables
lambdaPlus=model.dvar(n)
lambdaMinus=model.dvar(n)

model.max(c@x)
#Modify the deterministic constraint
model.st(a@x + zBarPlus@lambdaPlus -zBarMinus@lambdaMinus <=b)
#Add constraints from dual representation of worst-case optimization
model.st(lambdaPlus-lambdaMinus == x)
model.st(lambdaPlus>=0)
model.st(lambdaMinus>=0)

model.st(x>= 0)
model.st(x<=1)

model.solve(my_solver)
```

# Example: box uncertainty (reformulation #2)

Consider the robust optimization problem:

$$\begin{aligned} & \underset{x}{\text{maximize}} && c^T x \\ & \text{subject to} && (a + z)^T x \leq b, \forall z \in \mathcal{Z} \\ & && 0 \leq x \leq 1, \end{aligned}$$

with  $\mathcal{Z} := \{z \in \mathbb{R}^n \mid -\hat{z} \leq z \leq \hat{z}\}$

- Formulate an equivalent finite dimensional linear program using the equivalent uncertainty set definition:

$$\mathcal{Z} = \left\{ z \in \mathbb{R}^n \mid \exists \Delta^+ \in \mathbb{R}^n, \Delta^- \in \mathbb{R}^n, \quad \begin{array}{l} \Delta^+ \geq 0, \Delta^- \geq 0, \\ z = \Delta^+ - \Delta^-, \\ \Delta^+ + \Delta^- \leq \hat{z} \end{array} \right\}$$

# Equivalent LP reformulation for LP-RC

**Theorem 2.7.** : *The LP-RC problem, with a polyhedral  $\mathcal{Z}$  described through  $Wz \leq v$  (as in assumption 2.2), is equivalent to the following linear program*

$$\begin{aligned} & \underset{x, \{\lambda^{(j)}\}_{j=0}^J}{\text{maximize}} && p_0^T x + r_0 - v^T \lambda^{(0)} \\ & \text{subject to} && W^T \lambda^{(0)} = -P_0^T x - q_0 \\ & && p_j^T x + v^T \lambda^{(j)} \leq r_j, \forall j = 1, \dots, J \\ & && W^T \lambda^{(j)} = P_j^T x - q_j, \forall j = 1, \dots, J \\ & && \lambda^{(j)} \geq 0, \forall j = 0, \dots, J \end{aligned}$$

where  $\lambda^{(j)} \in \mathbb{R}^s$  are additional certificates that need to be optimized jointly with  $x$ .

# SOCPr reformulation for LP-RC with ellipsoidal uncertainty

Verifying whether  $\forall z \in \mathcal{Z}, z^T x \leq b - a^T x$  with

$$\mathcal{Z} := \{z \in \mathbb{R}^m | z^T \Sigma^{-1} z \leq \gamma^2\}$$

and  $\Sigma \succ 0$  is equivalent to evaluating the optimal value of the following problem

$$\Psi := \max_{z: z^T \Sigma^{-1} z \leq \gamma^2} x^T z$$

One can demonstrate using Cauchy-Schwartz inequality

$$a^T b \leq \|a\|_2 \|b\|_2$$

that this is equivalent to

$$\Psi = \gamma \sqrt{x^T \Sigma x} = \gamma \|\Sigma^{1/2} x\|_2$$

# SOCP reformulation for LP-RC with polyhedral set ellipsoidal uncertainty

**Theorem.** *The LP-RC problem, with ellipsoidal set  $\mathcal{Z}$  described is equivalent to the following second order cone program*

$$\begin{aligned} & \underset{x}{\text{maximize}} && p_0^T x + r_0 - \gamma \|\Sigma^{1/2}(P_0^T x + q_0)\|_2 \\ & \text{subject to} && p_j^T x + \gamma \|\Sigma^{1/2}(P_j^T x - q_j)\|_2 \leq r_j, \forall j = 1, \dots, J. \end{aligned}$$