Linear & Conic Programming Reformulations of Two-Stage Robust Linear Programs

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(special thanks to Samuel Burer)

July 5th, 2017

A Classical Distribution Problem

- Facility location-transportation model

$$\underset{I\in\{0,1\}^n, x, Y\geq 0}{\text{maximize}} \quad \eta \overbrace{\sum_i \sum_j Y_{ij}}^{\textit{sold product}} - \left(\overbrace{c^T x + K^T I}^{\textit{location cost}} + \overbrace{\sum_i \sum_j (p_i + t_{ij}) Y_{ij}}^{\textit{transportation\&production cost}} \right)$$

$$\text{s. t.} \quad \sum_j Y_{ij} \leq x_i \,,\ \forall\, i, \qquad (\textit{Capacity constraint})$$

$$\sum_i Y_{ij} \leq d_j \,,\ \forall\, j, \quad (\textit{Demand constraint})$$

$$x_i \leq M I_i \,,\ \forall\, i, \qquad (\textit{Facility Size constraint})$$

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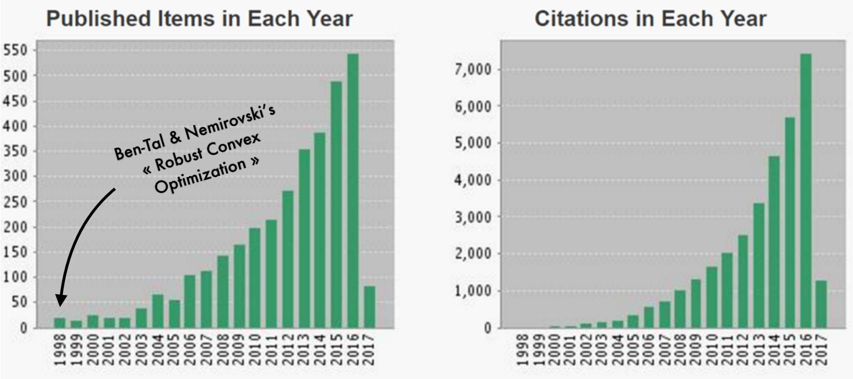
How can one account for demand uncertainty?

Robust Optimization Is Now a Well Established Methodology



A Classical Robust Distribution Problem

- Robust Facility location-transportation model:

$$\underset{I \in \{0,1\}^n, x}{\text{maximize}} \quad \min_{d \in \mathcal{D}} h(I, x, d)$$

$$\text{s. t.} \quad x_i \leq M I_i \ , \forall\, i, \qquad (\textit{Facility Size} \ \ \textit{constraint})$$

where $h(I, x, d)$ is the optimal value of

$$\max_{Y \geq 0} \quad \eta \overbrace{\sum_i \sum_j Y_{ij}}^{\textit{sold product}} - \left(\overbrace{c^T x + K^T I}^{\textit{location cost}} + \overbrace{\sum_i \sum_j (p_i + t_{ij}) Y_{ij}}^{\textit{transportation\&production cost}} \right)$$

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Outline

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Static Robust Linear Program

[Ben-Tal & Nemirovski (2000), 1296 citations !]

- Consider the following static problem:

$$\underset{x\in\mathcal{X},y}{\text{maximize}} \quad c^T x + f^T y \tag{1a}$$

$$\text{s. t.} \quad Ax + By \leq D(x)z \ , \ \forall z \in \mathcal{Z} \tag{1b}$$

where we assume $n_x + n_y$ decision variables, J constraints, and m uncertain parameters.

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  where we assume $n_x + n_y$ decision variables, J constraints, and m uncertain parameters.

- If $\mathcal{Z} := \{z \in \mathbb{R}^m \,|\, z \geq 0,\ Pz = q\}$ is a non-empty polyhedral set defined by K constraints, then

$$\text{Problem } (\textbf{??}) \ \equiv \ \underset{x \in \mathcal{X}, y, \Lambda}{\text{maximize}} \quad c^T x + f^T y$$
$$\text{s. t.} \quad Ax + By + \Lambda q \leq 0$$
$$D(x) + \Lambda P \geq 0 \,,$$

  where $\Lambda \in \mathbb{R}^{J \times K}$.

Two-Stage Robust Linear Programs

[Ben-Tal et al. (2004), 824 citations !]

- Consider the following two-stage problem:

$$(TSRLP) \qquad \underset{x\in\mathcal{X},y(\cdot)}{\text{maximize}} \quad \min_{z\in\mathcal{Z}} c^T x + f^T y(z)$$
$$\text{s. t.} \quad Ax + By(z) \leq D(x)z \ \forall z \in \mathcal{Z}$$

where $y : \mathbb{R}^m \to \mathbb{R}^{n_y}$

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- This problem can also be represented as

$$(TSRLP) \qquad \underset{x\in\mathcal{X}}{\text{maximize}} \qquad \min_{z\in\mathcal{Z}} h(x,z)$$

where

$$h(x,z) := \max_{y} \qquad c^T x + f^T y$$
$$\text{s. t.} \qquad Ax + By \le D(x)z\,.$$

Complexity of Two-Stage Robust Linear Programs

- Unfortunately, the two-stage robust linear program is known to be intractable in general [Ben-Tal et al. (2004)].

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- Conservative approximation obtained by using affine adjustment functions :

$$y(z) := y + Yz$$

The two-stage robust problem reduces to

$$(AARC) \quad \underset{x\in\mathcal{X},y,Y}{\text{maximize}} \quad \min_{z\in\mathcal{Z}} c^T x + f^T(y+Yz)$$
$$\text{s. t.} \quad Ax + B(y+Yz) \leq D(x)z \;\forall z \in \mathcal{Z}$$

COMPLEXITY OF TWO-STAGE ROBUST LINEAR PROGRAMS

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- Some exact methods have been proposed but without polynomial time convergence guarantees [Zeng & Zhao (2013)]

Outline

Copositive Programming Reformulation I

- Assumptions
  1. $\mathcal{Z}$ is a non-empty and bounded polyhedral set
  2. The TSRLP problem is bounded above, i.e.
     $\forall x \in \mathcal{X}, \exists z \in \mathcal{Z}, h(x,z) < \infty$.

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- Let our robust optimization problem take the form

$$\underset{x\in\mathcal{X}}{\text{maximize}} \quad \psi(x)\ ,$$

where

$$\psi(x) \ := \ \min_{z\in\mathcal{Z}} \quad \max_{y} \quad c^T x + f^T y \tag{2a}$$

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- Since (**??**) is bounded, strong LP duality applies

$$\psi(x) \;=\; \min_{z\in\mathcal{Z},\lambda\ge 0} \qquad c^T x + z^T D(x)^T \lambda - (Ax)^T \lambda$$

$$B^T \lambda = f$$

Copositive Programming Reformulation I

- The function $\psi(x)$ minimizes a non-convex quadratic function over a polyhedron in the non-negative orthant

$$\psi(x) \;=\; \min_{\tilde{z}\geq 0} \quad c^T x + \tilde{z}^T \tilde{Q}(x)\tilde{z} - \tilde{c}(x)^T \tilde{z}$$
$$\tilde{A}\tilde{z} = \tilde{b}\,,$$

where $\tilde{z} := [\lambda^T \quad z^T] \in \mathbb{R}^{J+m}$ and where

$$\tilde{Q}(x) := \begin{bmatrix} 0 & (1/2)D(x) \\ (1/2)D(x)^T & 0 \end{bmatrix} \quad \tilde{c}(x) := \begin{bmatrix} -(1/2)Ax \\ 0 \end{bmatrix}$$
$$\tilde{A} := \begin{bmatrix} B^T & 0 \\ 0 & P \end{bmatrix} \qquad \tilde{b} := \begin{bmatrix} d \\ q \end{bmatrix}$$

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\begin{aligned}
\psi(x) \;=\; \min_{\tilde{z}\geq 0} \quad & c^T x + trace(\tilde{Q}(x)^T \tilde{z}\tilde{z}^T) - \tilde{c}(x)^T \tilde{z} \\
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Copositive Programming Reformulation I

- The function $\psi(x)$ has an equivalent convex optimization reformulation ($\tilde{Z} := \tilde{z}\tilde{z}^T$) [Burer (2009)]

$$\psi(x) \;=\; \min_{\tilde{Z},\tilde{z}} \qquad c^T x + trace(\tilde{Q}(x)^T \tilde{Z}) - \tilde{c}(x)^T \tilde{z}$$

$$\tilde{A}\tilde{z} = \tilde{b}$$

$$\tilde{A}\tilde{Z} = \tilde{b}\tilde{z}^T$$

$$\begin{bmatrix} \tilde{Z} & \tilde{z} \\ \tilde{z}^T & 1 \end{bmatrix} \in \mathcal{K}_{\mathrm{CP}} \;\; \& \;\; \mathrm{rank}\left(\begin{bmatrix} \tilde{Z} & \tilde{z} \\ \tilde{z}^T & 1 \end{bmatrix}\right) = 1$$

where $\mathcal{K}_{\mathrm{CP}}$ is the cone of completely positive matrices, i.e.

$$\mathcal{K}_{\mathrm{CP}} := \left\{ M \,\middle|\, M = \sum_{k\in K} \tilde{z}_k \tilde{z}_k^T \text{ for some } \{\tilde{z}_k\}_{k\in K} \subset \mathbb{R}_+^{J+m+1} \right\}.$$

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Copositive Programming Reformulation I

- By conic duality we get

$$
\begin{aligned}
\psi(x) \geq \max_{\tilde{W},\tilde{w},\tilde{v},t} \quad & \tilde{c}(x)^T x + \tilde{b}^T \tilde{w} - t \\
\text{s. t.} \quad & \tilde{v} = \tilde{c}(x) - (1/2)(\tilde{A}^T \tilde{w} - \tilde{W}^T \tilde{b}) \\
& \begin{bmatrix} \tilde{Q}(x) - (1/2)(\tilde{W}^T \tilde{A} + \tilde{A}^T \tilde{W}) & \tilde{v} \\ \tilde{v}^T & t \end{bmatrix} \in \mathcal{K}_{\mathrm{Cop}} \,,
\end{aligned}
$$

where $\mathcal{K}_{\mathrm{Cop}}$ is the cone of copositive matrices, i.e.

$$
\mathcal{K}_{\mathrm{Cop}} := \left\{ M \,\middle|\, M = M^T,\ z^T M z \geq 0 \,,\, \forall\, z \in \mathbb{R}_+^{J+m+1} \right\} .
$$

Copositive Programming Reformulation I

Theorem 1 *[Xu & Burer (2016), Hanasusanto & Kuhn (2016)]*
If the TSRLP problem has "complete recourse", i.e.

$$\exists\, y \in \mathbb{R}^{n_y}\,,\; By < 0\,,$$

then the copositive program

$$(Copos_1) \quad \underset{x\in\mathcal{X},\tilde{W},\tilde{w},\tilde{v},t}{\text{maximize}} \quad c^T x + \tilde{b}^T\tilde{w} - t$$

$$\text{s. t.} \quad \tilde{v} = \tilde{c}(x) - (1/2)(\tilde{A}^T\tilde{w} - \tilde{W}^T\tilde{b})$$

$$\begin{bmatrix} \tilde{Q}(x) - (1/2)(\tilde{W}^T\tilde{A} + \tilde{A}^T\tilde{W}) & \tilde{v} \\ \tilde{v}^T & t \end{bmatrix} \in \mathcal{K}_{Cop}\,,$$

provides an exact reformulation of the TSRLP problem. Otherwise, $Copos_1$ only provides a conservative approximation.

Relation to AARC

Theorem 2 *[Xu & Burer (2016)]*
When $\mathcal{K}_{Cop}$ is replaced with $\mathcal{N} := \mathbb{R}_+^{J+m+1 \times J+m+1} \subset \mathcal{K}_{Cop}$ the copositive programming reformulation is equivalent to AARC.

Relation to AARC

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When $\mathcal{K}_{Cop}$ is replaced with $\mathcal{N} := \mathbb{R}_+^{J+m+1 \times J+m+1} \subset \mathcal{K}_{Cop}$ the copositive programming reformulation is equivalent to AARC.

- Hence, for any cone $\mathcal{K}$ such that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{K}_{\text{Cop}}$, $Copos_1$ with $\mathcal{K}$ provides a tighter approximation than AARC
- There exists a hierarchy of semidefinite and polyhedral cones $\{\mathcal{K}_i\}_{i=1}^{\infty}$, with $\mathcal{N} \subseteq \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}_{\text{Cop}}$, such that for all $M \in \mathcal{K}_{\text{Cop}}$, there is a i^*, $M \in \mathcal{K}_{i^*}$
  [Parrilo (2000), Bomze & de Klerk (2002)]

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  [Parrilo (2000), Bomze & de Klerk (2002)]

This is valuable for complete recourse problems but what about relatively complete recourse problems ?

How to Fix Relatively Complete Recourse

- Assumption : The TSRLP problem has relatively complete recourse, i.e.

$$\forall\, x \in \mathcal{X}\, \forall\, z \in \mathcal{Z},\ \exists y, Ax + By \le D(x)z$$

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- Assumption : The TSRLP problem has relatively complete recourse, i.e.

$$\forall x \in \mathcal{X}\, \forall z \in \mathcal{Z},\ \exists y, Ax + By \leq D(x)z$$

- This ensures that :

$$\begin{aligned} h(x,z) \ = \ \min_{\lambda} \quad & c^T x + z^T D(x)^T \lambda - (Ax)^T \lambda \qquad \in \mathbb{R} \\ \text{s. t.} \quad & \lambda \in \mathcal{P} := \{\lambda \mid \lambda \geq 0,\ B^T \lambda = f\} \end{aligned}$$

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- Hence, always an optimal solution $\lambda^*(x,z)$ at a vertex of $\mathcal{P}$

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- Hence, always an optimal solution $\lambda^*(x,z)$ at a vertex of $\mathcal{P}$
- Since number of vertices is finite, there exists $u \in \mathbb{R}^J_+$:

$$\begin{aligned} \psi(x) \;=\; \min_{z \in \mathcal{Z}} h(x,z) \;=\; \min_{z \in \mathcal{Z}, \lambda \in \mathcal{P}} \quad & c^T x + z^T D(x)^T \lambda - (Ax)^T \lambda \\ \text{s. t.} \quad & \lambda \leq u \end{aligned}$$

Copositive Programming Reformulation II

Theorem 3 *[revised AJ&D (2016b)]*
If the TSRLP problem has relatively complete recourse, then the copositive program

$$
\begin{array}{lll}
(Copos_2) & \displaystyle\max_{x\in\mathcal{X},\bar{W},\bar{w},\bar{v},t} & c^T x + \bar{b}^T \bar{w} - t \\
& \text{s. t.} & \bar{v} = \bar{c}(x) - (1/2)(\bar{A}^T \bar{w} - \bar{W}^T \bar{b}) \\
& & \begin{bmatrix} \bar{Q}(x) - (1/2)(\bar{W}^T \bar{A} + \bar{A}^T \bar{W}) & \bar{v} \\ \bar{v}^T & t \end{bmatrix} \in \mathcal{K}_{Cop} ,
\end{array}
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provides an exact reformulation of the TSRLP problem.

The Penalized AARC model

Theorem 4 *[revised AJ&D (2016b)]*

When $\mathcal{K}_{Cop}$ is replaced with $\mathcal{N}$ the Copos$_2$ reformulation is equivalent to applying affine adjustments to:

$$
(TSRLP_2) \qquad \underset{x\in\mathcal{X},y(\cdot),\theta(\cdot)}{\text{maximize}} \quad \min_{z\in\mathcal{Z}} c^T x + f^T y(z) - u^T \theta(z)
$$
$$
\text{s. t.} \quad Ax + By(z) \leq D(x)z + \theta(z) \ \forall z \in \mathcal{Z} \,.
$$

Moreover, affine (and static) adjustments are always feasible in $TSRLP_2$.

THE PENALIZED AARC MODEL

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Moreover, affine (and static) adjustments are always feasible in $TSRLP_2$.

- u can be interpreted as a marginal penalty for violating constraints
- $TSRLP_2 \equiv TSRLP$ since u is such that there always exists an optimal solution triplet with $\theta(z) := 0$.

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- u can be interpreted as a marginal penalty for violating constraints
- $TSRLP_2 \equiv TSRLP$ since u is such that there always exists an optimal solution triplet with $\theta(z) := 0$.
- Method for converting a relatively complete recourse multi-stage linear program into a complete recourse one

Outline

Robust Facility Location-Transportation Problem

- In AJ&D (2016b), we identify an instance for which

	AARC model	Penalized AARC (a.k.a. $Copos_2(\mathcal{N})$)	Exact model
Bound on w.-c. profit	0	6600	6600
W.-c. profit of x^*	0	6600	6600

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- We recently randomly generated 10 000 problem instances, 5 facilities & 10 customer locations.

Optimality gap	Proportion of instances	
	AARC	Penalized AARC
$= 0\%$	20.6%	23.8%
$\leq 0.1\%$	20.9%	27.4%
$\leq 1\%$	28.4%	56.3%
Avg. Gap	10.5%	1.6%
Max Gap	50.0%	13.3%

WHAT SIZE PROBLEMS CAN WE SOLVE ? [AJ&D (2017)]

(T,L,N)	Γ	Penalized AARC		Exact
		Full form	Row generation	C&CG
(1,50,100)	10	-	3 241 sec	8 465 sec
	30	-	4 563 sec	-
	50	-	8 460 sec	-
	70	-	3 781 sec	7 682 sec
	90	-	1 382 sec	7 sec
	100	-	< 1 sec	2 sec
	Avg.	-	3 572 sec	-
(20,15,30)	60	-	3 781 sec	184 sec
	180	-	5 646 sec	-
	300	-	10 567 sec	-
	420	-	4 445 sec	-
	540	-	663 sec	-
	600	-	1 sec	<1 sec
	Avg.	-	4 184 sec	-

(– stands for more than two days of computation)

Robust Multi-item Newsvendor

- In AJ&D (2016a): the robust multi-item newsvendor problem with uncorrelated demand is solved optimally by AARC/$Copos(\mathcal{N})$ when using budgeted uncertainty set with integer Γ.

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- In AJ&D (2016b): if demand is correlated than solution improves using *Copos* with $\mathcal{K}^1_{\text{SDP}} \supset \mathcal{N}$:

	AARC	*Copos*($\mathcal{K}^4_{\text{LP}}$)	*Copos*($\mathcal{K}^1_{\text{SDP}}$)	Exact
W.-c. profit bound	41.83	41.83	411.08	825.83
Actual w.-c. profit	41.83	41.83	664.76	825.83

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Conclusion & Open Questions

1. Copositive programming is a useful tool for generating conservative approximations for TSRLP
   - $\mathit{Copos}(\mathcal{K})$ with $\mathcal{K} \supset \mathcal{N}$ always improves on AARC

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1. Copositive programming is a useful tool for generating conservative approximations for TSRLP
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   - What is a hierarchy of polyhedral cones that performs well?
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   - Is it possible to generalize this approach to robust multi-stage & non-linear problems?

Bibliography

Questions & Comments ...

... Thank you!