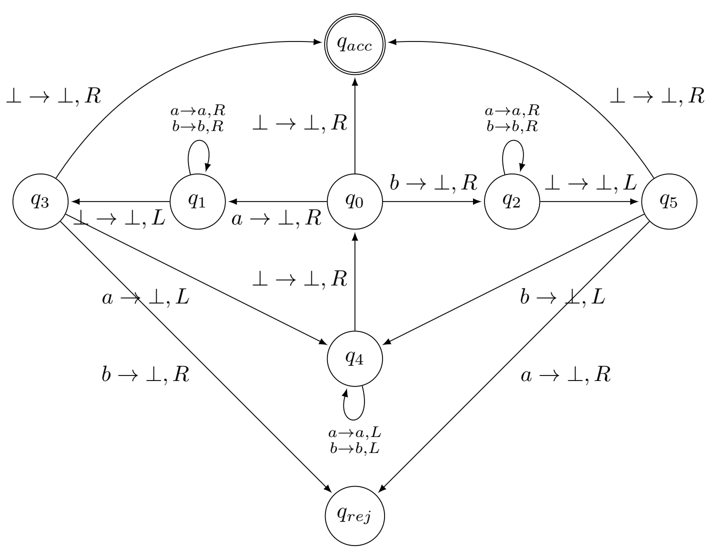
EECS 376: Foundations of Computer Science
Cook Levin Theorem Building Blocks

1 Understanding The Tableau

A *tableau* represents the computational history of a Turing machine on a given input. Each row in the tableau corresponds to a configuration– a "snapshot"– of the TM's executation at a particular step.
In each configuration,

- The contents of the tape are shown in order from the leftmost cell.
- The active state is inserted directly **to the left** of the symbol where the tape head is positioned.
- Each row is bordered by # symbols to mark the beginning and end of the tape.

Consider the following Turing machine M_{pal} that decides the language $L_{pal} = \{s \in \{a, b\}^* : s \text{ is a palindrome}\}$, where q_0 is the start state. Given the input $s =$ "$abba$", complete the tableau on the right to show the step-by-step computational history of this TM as M_{pal} processes s.



#	q_0	a	b	b	a	$\perp$	...	$\perp$	#
#	$\perp$					$\perp$	...	$\perp$	#
#	$\perp$	b	q_1	b	a	$\perp$	...	$\perp$	#
#	$\perp$	b				$\perp$	...	$\perp$	#
#	$\perp$	b	b	a	q_1	$\perp$	...	$\perp$	#
#	$\perp$	b	b				...	$\perp$	#
#	$\perp$	b	q_4	b	$\perp$	$\perp$	...	$\perp$	#
#	$\perp$	q_4	b				...	$\perp$	#
#				b	$\perp$	$\perp$	...	$\perp$	#
#	$\perp$	q_0	b	b	$\perp$	$\perp$	...	$\perp$	#
#	$\perp$	$\perp$			$\perp$	$\perp$	...	$\perp$	#
#	$\perp$	$\perp$			$\perp$	$\perp$	...	$\perp$	#
#	$\perp$	$\perp$			$\perp$	$\perp$	...	$\perp$	#
#	$\perp$	q_4	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	#
#	$\perp$	$\perp$	q_0	$\perp$	$\perp$	$\perp$	...	$\perp$	#
#	$\perp$				$\perp$	$\perp$	...	$\perp$	#

2 Identifying Valid Windows

Recall from lecture that one critical ingredient in the proof of the Cook-Levin theorem is to specify which contents of the 2×3 "windows" of the verifier's computation tableau are valid.
A 2×3 "window" is *valid* if it represents a configuration that could appear in a legitimate tableau, essentially ensuring the proper execution of the Turing machine. Determine whether each of the following is a valid window in the tableau of M_{pal} *for any input.*

(a)

a	#	b
a	#	b

(b)

q_2	a	b
a	q_2	b

(c)

a	q_1	b
a	b	q_5

(d)

a	a	q_3
a	a	b

(e)

a	a	a
a	a	q_4

(f)

$\perp$	b	q_2
$\perp$	q_5	b

(g)

b	q_3	a
q_4	b	$\perp$

(h)

a	a	a
b	b	a

3 From Tableau to Boolean Formulas

A key part in the Cook Levin Theorem proof is to convert the constraints on the tableau into Boolean formulas. We will derive some of them here.
First, for some symbol $s \in S = \Gamma \cup Q \cup \{\#\}$, we define the variable

$$t_{i,j,s} = \begin{cases} T & \text{if cell } (i,j) \text{ on the tableau contains symbol } s \\ F & \text{otherwise} \end{cases}$$

Since we are only interested in efficient verifiers, we consider tableau of size $n^k \times n^k$ for some constant k.

(a) Derive a Boolean formula that evaluates to true if and only if the TM enters the accept state within n^k steps. *Hint: This is equivalent to having* <u>*at least one*</u> q_{acc} *on the tableau.*

(b) Derive a Boolean formula that evaluates to true if and only if every cell on the tableau contains *exactly one* symbol in S. *Hint: For each cell* (i,j) *ensure (1)* <u>*at least one*</u> $t_{i,j,s}$ *is true and (2)* <u>*no more than one*</u> $t_{i,j,s}$ *is true.*

(c) Now consider the first row of the tableau. Derive a Boolean formula that evaluates to be true if and only if all of the constraints below are met:

- The first cell contains the '#' symbol.
- The second cell contains 'q_0'.
- The next n cells contains the string $x_1 x_2 \ldots x_n$. You may assume all $x_i \in \Sigma$.
- Immediately following x, the next cell contains the '\$' symbol.
- For the $(n+4)$-th up to the $(n^k - 1)$-th cell: The cells contain a certificate string over the alphabet Σ, followed by '$\perp$' symbols in any remaining cells up to the $(n^k - 1)$-th cell.
- The n^k-th cell contains the symbol '#'.