From Examples to Student Responses in an Interactive Linear Algebra Textbook: Conceptions of Spanning Sets

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RTMUS RESEARCH ON TEACHING MATHEMATICS IN UNDERGRADUATE SETTINGS UNIVERSITY OF MICHIGAN

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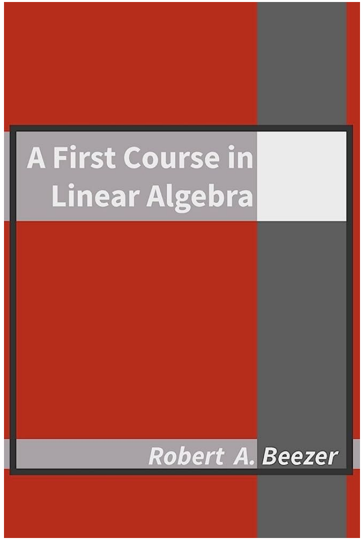
Undergraduate Teaching and learning Mathematics with Open Source Textbooks (UTMOST)

Investigate the affordances and challenges of developing and using open access, open source, and interactive textbooks in the teaching and learning of undergraduate mathematics

- How do **students** and **instructors** use textbooks?
- How can we develop open textbooks that improve **teaching** and **learning** of undergraduate mathematics?

Context



- Textbook: *A First Course in Linear Algebra* (Beezer, 2021)
  - For sophomores and juniors
  - Proof-based

- Topic: Spanning Sets (SS)
  - Substantial research base
  - Difficult for students to learn

(Rasmussen & Wawro, 2017; Stewart, Andrews-Larson & Zandieh, 2019)

Interactive Feature Studied: Reading Questions

Students

- Prior to class, **read** textbook section(s) assigned
- **Answer** the questions at the end of the section
- **Refer** back if necessary

1. Let S be the set of three vectors below.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Let $W = \langle S \rangle$ be the span of S. Is the vector $\begin{bmatrix} -1 \\ 8 \\ -4 \end{bmatrix}$ in W? Give an explanation of the reason for your answer.

My answer $\rightarrow$ Yes, this vector is in W. This is because the RREF is consistent. (Sample response from RQ1, #59)

Interactive Feature Studied: Reading Questions

Teachers

- Access **real-time data** on how students engage with the content
- Gain insights on **students' thinking** → **Plan lessons** accordingly

51102801 06 Oct 22:18

Yes. We would put this into an augmented matrix. We then get the row-reduced echelon form. Since we do have solutions for it, the vector w is in the span of S.

51102802 07 Oct 23:47

The vector [-1,8,-4] is in the span W, the reason is because when the matrix is in RREF, there are infinite solutions, this indicates that this vector is apart of the span.

51102804 25 Oct 15:27

The vector [-1,8,-4] is in W because by the Theorem SLSLC then S can be equal to the vector [-1,8,-4] therefore, I can convert into an augmented matrix which I can aply RREF to find the scalar/solutions: [1,-2,1]. Then I check to see if the scalars make the statement true in which it does:1[1,2,-1]+-2{3,-4,2]+1[4,-2,1]=[-1,8,-4].

51102806 05 Oct 20:42

Yes it is. The three vectors in S and the fourth vector represent a linear combination. This last vector is a solution to the three.

51102807 07 Oct 01:42

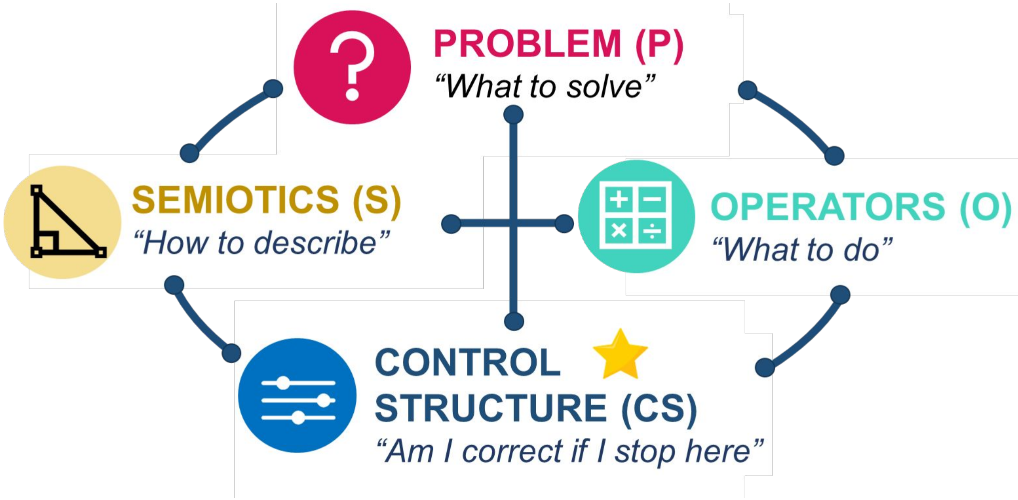
Yes, it would be in the span as when the rref of the augmented matrix made up of all the vectors is found, the system/answer is consistent.

Objective

Wouldn’t it be nice if the instructors knew *what to look for* in the responses and *how to look at them* to help them with lesson planning?

… other than simply marking them as correct / incorrect

Proposal: Balacheff's *cK¢* Model of Conceptions



(Balacheff & Gaudin, 2009)

⭐ This study focused on **control structures**

Research Questions

- What conceptions of spanning sets are promoted in the examples in the textbook?
- What control structures do students use when responding to reading questions?
- How to evaluate correctness of students' responses using the idea of conceptions/control structures?

Exploratory Investigation

X 76

- Collected all students' responses from six instructors to **two reading questions** (RQ1 and RQ2) in the Spanning Sets section

- Analyzed **textbook examples** related to RQs (same problem)
  - addressing "whether a vector is in the span of the spanning set"
- Identified semiotic systems, operators, and control structures

Example 1. A basic span.

Example 2. Span of the columns of Archetype A.

Example 3. Span of the columns of Archetype B.

Control Structures Analysis

- Identified **control structures observed** in student responses
- 2-3 coders coded control structures in textbook examples and student responses
- Disagreements were resolved via consensus

Example:

"Yes, this vector is in W. This is because the RREF is consistent." **(RQ1, response #46)**

Control structure observed: Consistency of the system of equations

Correctness Analysis

- 2-3 coders classified the responses by a **2-dimensional correctness analysis** (criteria to apply control structures and link by conception)
- Disagreements resolved via consensus

Correctness Categories:
Pair = (criteria, link by conception)

✔✔	✘✔
✔✘	✘✘

Example: In RQ1, the system is consistent and the vector is in span

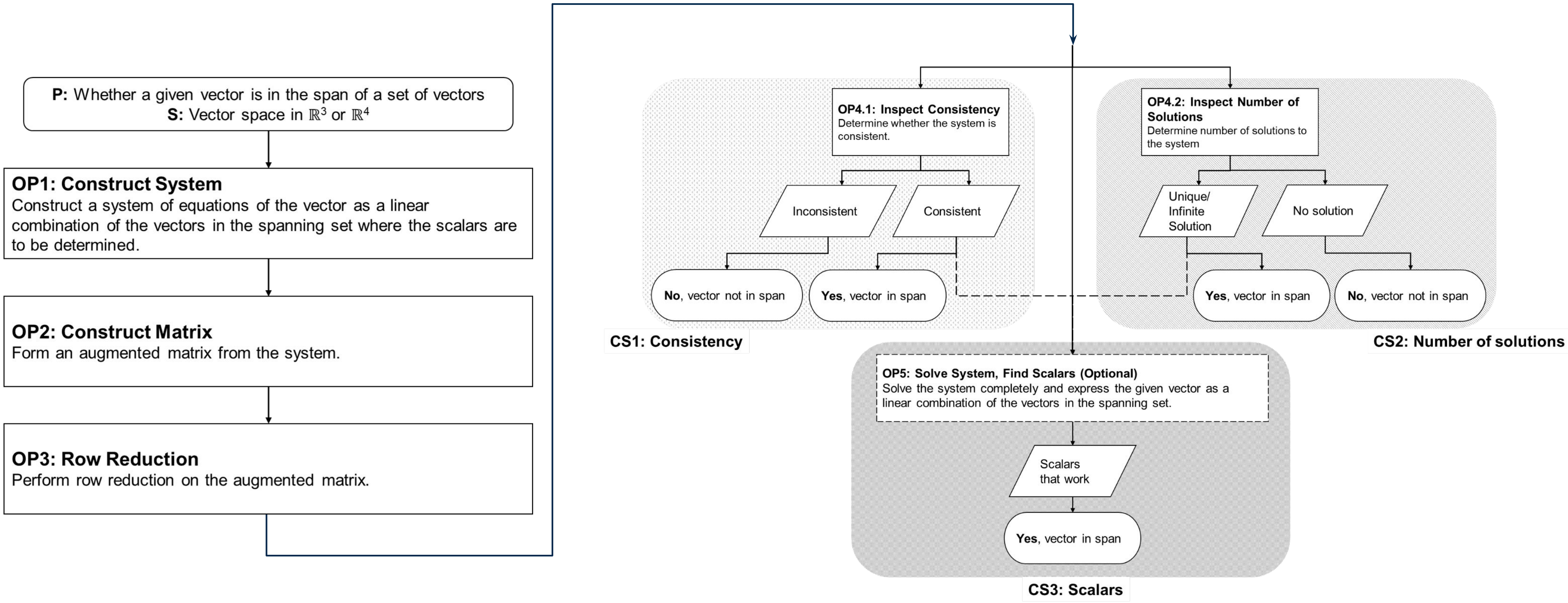
"consistent → yes, vector in span" (✔✔)

Criteria error
"inconsistent → no, vector not in span" (✘✔)

"consistent → no, vector not in span" (✔✘)
Link error

Criteria error
"inconsistent → yes, vector in span" (✘✘)
Link error

Findings- Conceptions in the Textbook



Findings- Control Structures in RQ1 and Their Correctness

Correctness Categories:

Pair = (criteria, link by conception)

✔✔ ✘✔

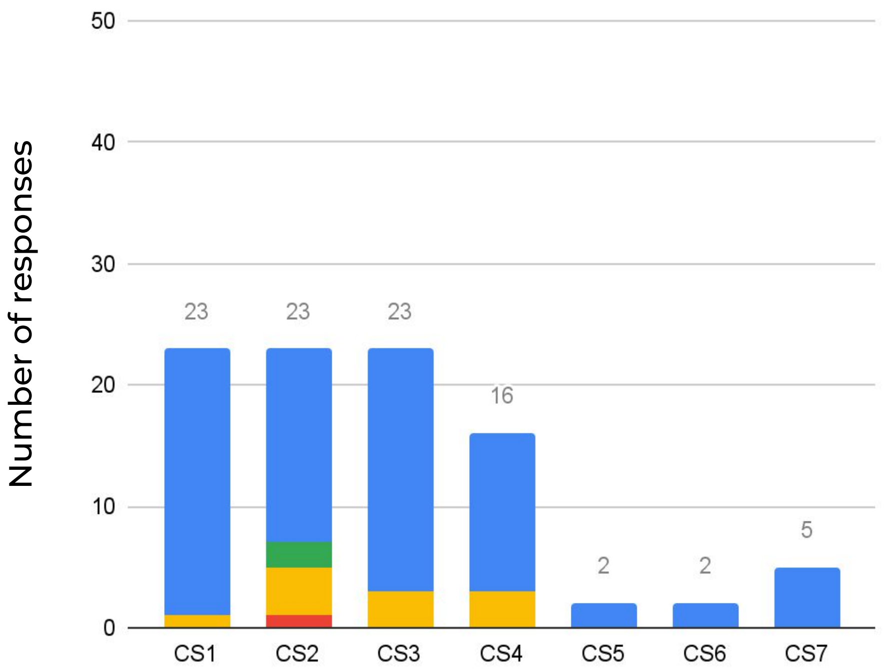
✔✘ ✘✘

Control Structures:

bold = also in textbook

1. Consistency of the system
2. Number of solutions to the system
3. Scalars that work as a linear combination

4. Existence of linear combination
5. Utilizing SAGE*
6. Justification with pivots
7. Justification with number of free variables



* SAGE is a Computer Algebra System (CAS) embedded in the textbook.

Observation 1: Responses showed a **wider variety of control structures** than in the textbook

- May rely on **existing knowledge** rather than reading the text
- May **selectively read** certain parts of the text
- **Mode of thinking** may differ from those in the textbook (e.g., analytic-arithmetic vs. analytic-structural)

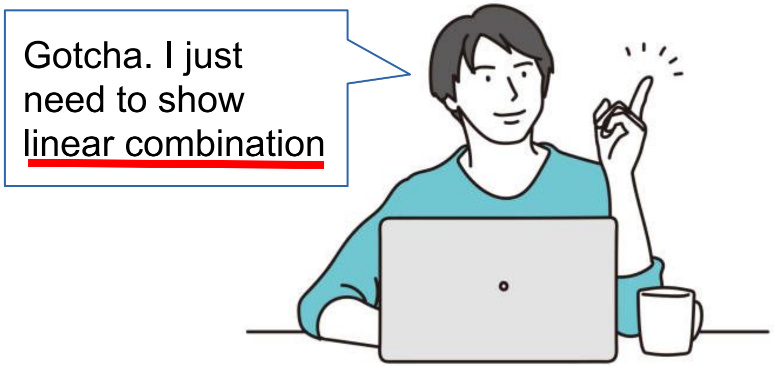
Observation 1: Responses had a **wider variety of control structures** than in the textbook

It is possible that the students

- Use **existing knowledge**
- **Selectively read** certain parts of the textbook

Definition SSCV. Span of a Set of Column Vectors. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, their ***span***, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. Symbolically,

$$\begin{aligned}\langle S \rangle &= \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_p\mathbf{u}_p \,|\, \alpha_i \in \mathbb{C},\ 1 \le i \le p\} \\ &= \left\{ \sum_{i=1}^{p} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\ 1 \le i \le p \right\}.\end{aligned}$$



Findings- Control Structures in RQ1 and Their Correctness

Correctness Categories:
Pair = (criteria, link by conception)

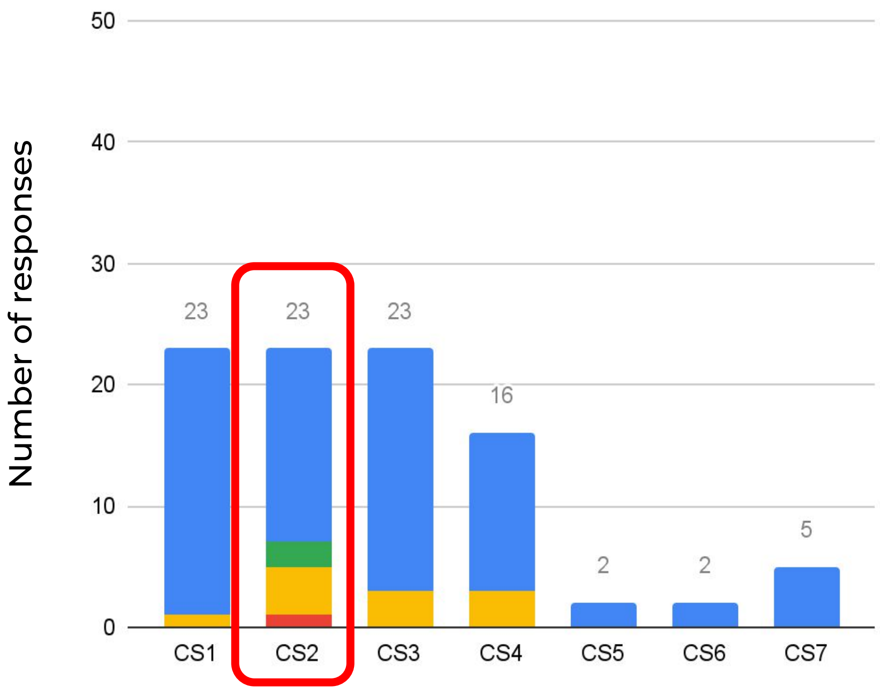
✔✔ ✘✔

✔✘ ✘✘

Control Structures:
bold = also in textbook

1. Consistency of the system
2. Number of solutions to the system
3. Scalars that work as a linear combination

4. Existence of linear combination
5. Utilizing SAGE*
6. Justification with pivots
7. Justification with number of free variables

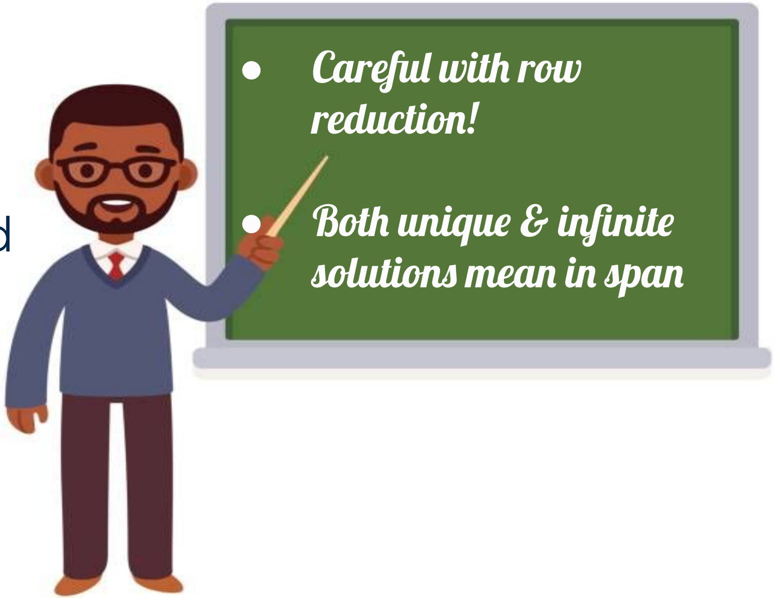


* SAGE is a Computer Algebra System (CAS) embedded in the textbook.

Observation 2: **Variation in the level of correctness** when applying the control structures

Some control structures may be more **complex to apply** or more **abstract to understand**

- Spend time going over the criteria and the links for those



Observation 2: **Variation in the level of correctness** when applying the control structures

- Some control structures may be more **complex to apply** or more **abstract to understand**
  - Remind the students in class based on their mistakes

Be careful when doing row reduction!

Both unique and infinite solutions mean in span!

Findings- Distribution of Control Structures in RQ1 and Correctness

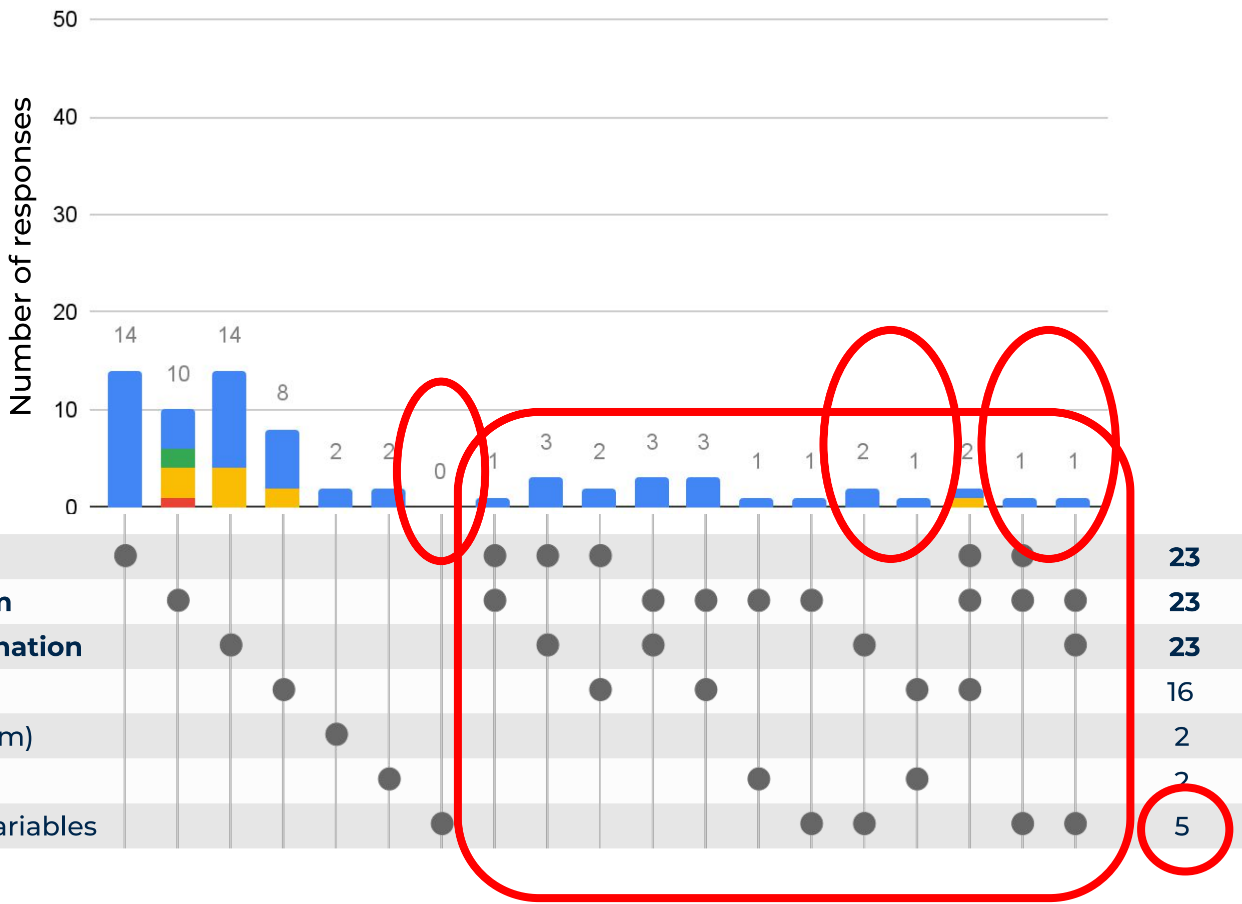
Correctness Categories:
Pair = (criteria, link by conception)

✔✔ ✘✔

✔✘ ✘✘

Control Structures:
bold = also in textbook

1. **Consistency of the system**
2. **Number of solutions to the system**
3. **Scalars that work as linear combination**
4. Existence of linear combination
5. Utilize SAGE (embedded CAS system)
6. Justification with pivots
7. Justification with number of free variables



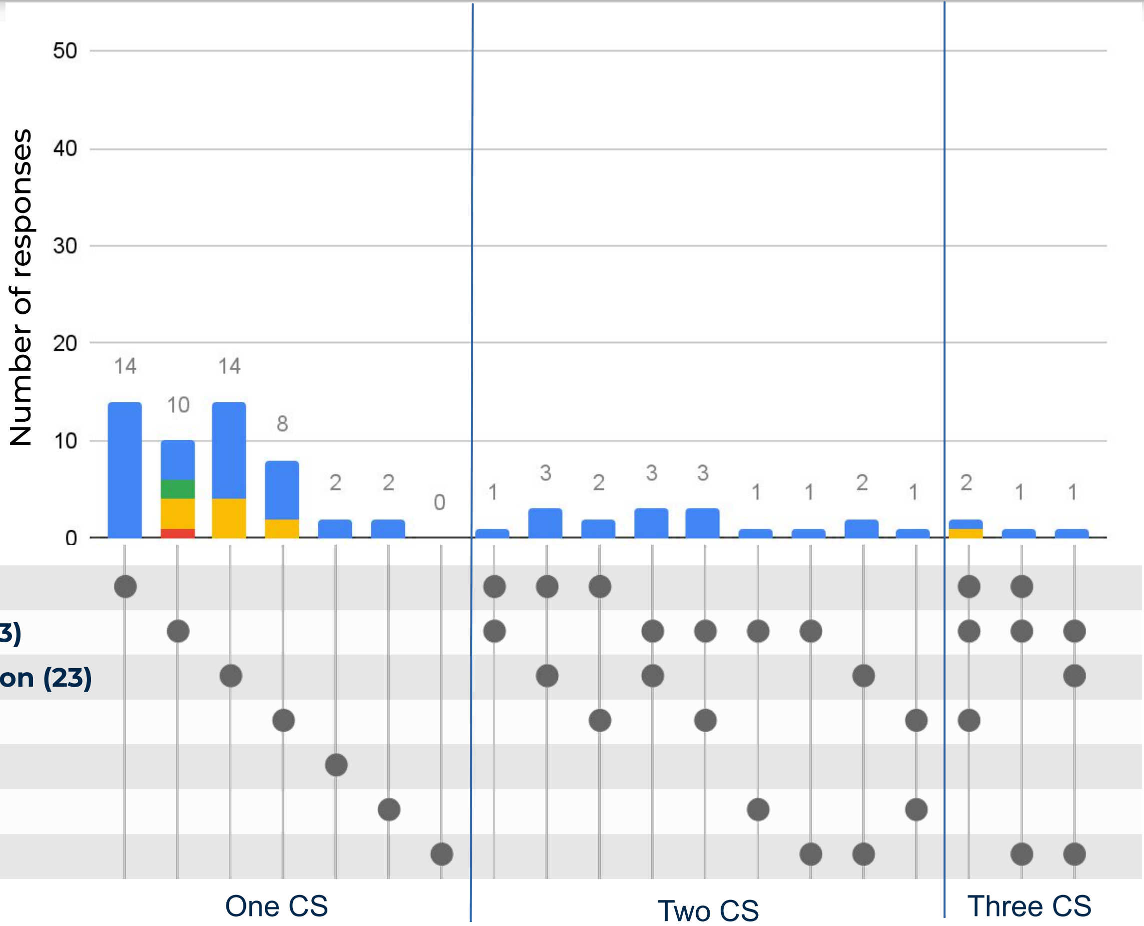
Findings- Distribution of Control Structures in RQ1 and Correctness

Correctness Categories:
Pair = (criteria, link by conception)

- Blue: ✔ ✔
- Yellow: ✘ ✔
- Green: ✔ ✘
- Red: ✘ ✘

Control Structures:
bold = also in textbook

1. Consistency of the system (23)
2. Number of solutions to the system (23)
3. Scalars that work as linear combination (23)
4. Existence of linear combination (16)
5. SAGE (2)
6. Pivots (2)
7. Number of free variables (5)



Observation 3: Some responses had **multiple control structures**

I'll just write down **everything I know** and hopefully prof thinks it's correct!

- Why are there multiple control structures?
- Do students see them as **distinct**?
- When and how do they see the connections?

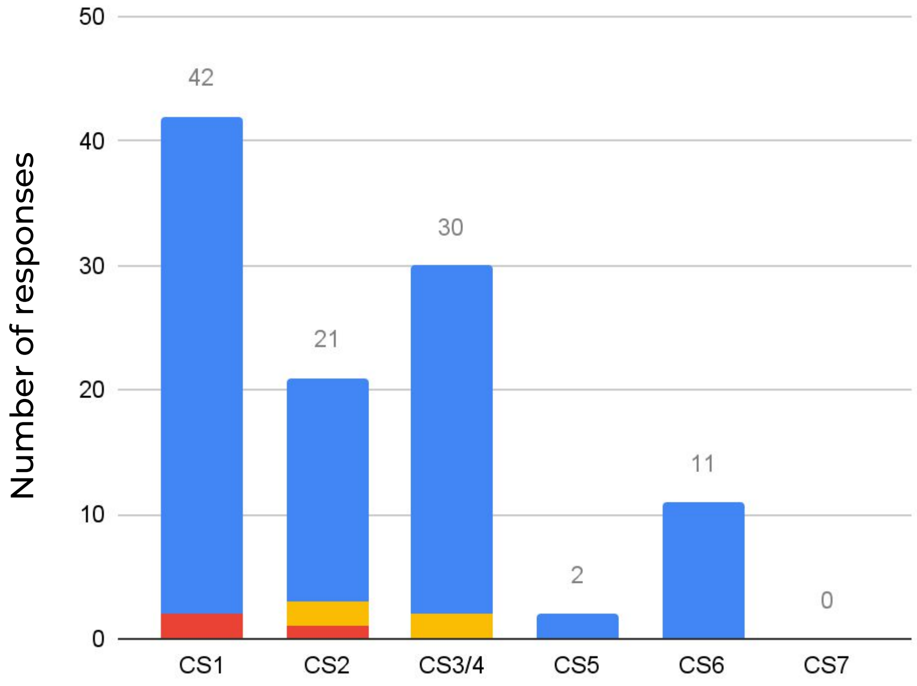
Finding 2.2a: Control Structures in RQ2 and Their Correctness

Correctness Categories:
Pair = (criteria, link by conception)

- (blue) ✔✔
- (yellow) ✘✔
- (green) ✔✘
- (red) ✘✘

Control Structures:
bold = also in textbook

1. Consistency of the system
2. Number of solutions to the system
3/4. Mention linear combination
5. Utilizing SAGE*
6. Justification with pivots
7. Justification with number of free variables



* SAGE is a Computer Algebra System (CAS) embedded in the textbook.

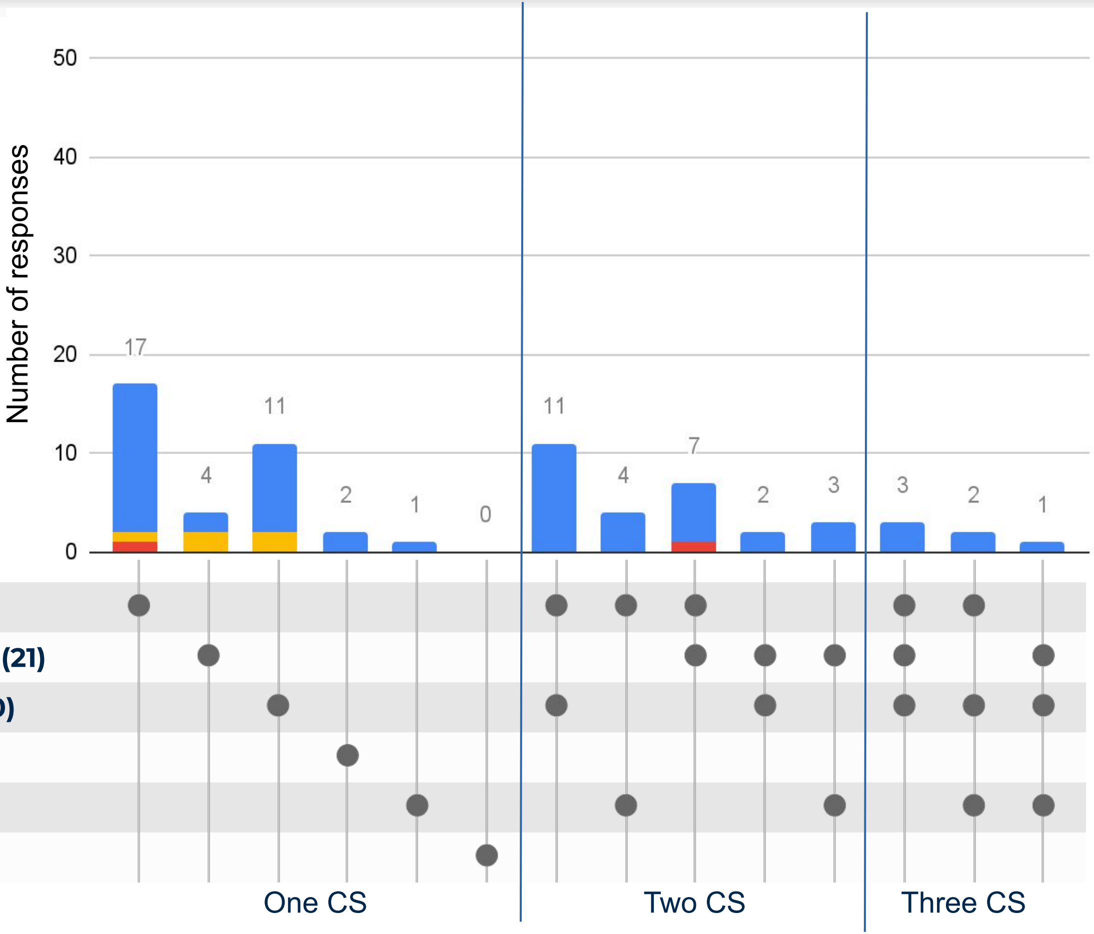
Finding 2.2b: Distribution of Control Structures in RQ2 and Correctness

Correctness Categories:
Pair = (criteria, link by conception)

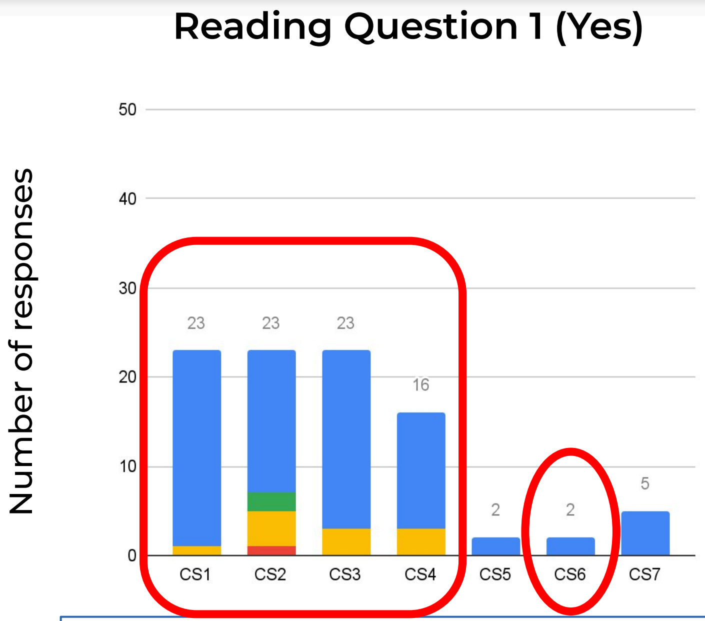
✔✔ ✘✔
✔✘ ✘✘

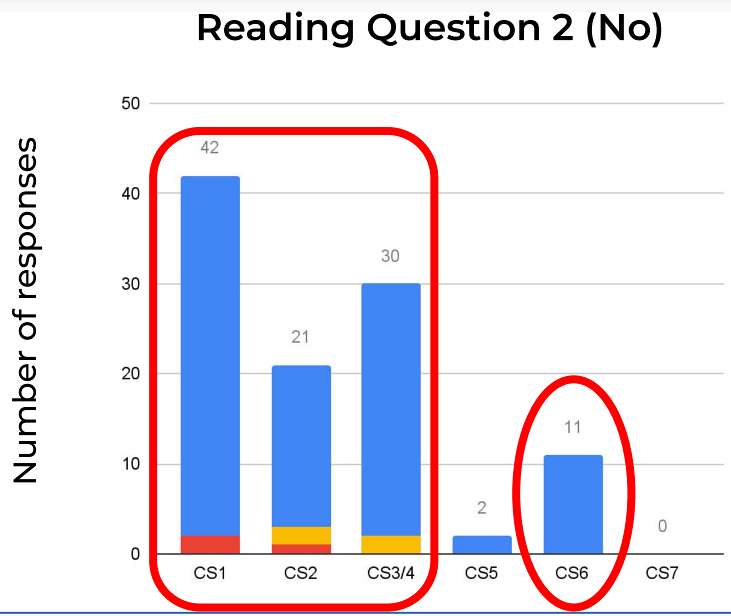
Control Structures:
bold = also in textbook

1. Consistency of the system (42)
2. Number of solutions to the system (21)
4. Existence of linear combination (30)
5. SAGE (2)
6. Pivots (11)
7. Number of free variables (0)



Findings- Comparison: Reading Question 1 vs Reading Question 2





Control Structures:
bold = also in textbook

1. Consistency of the system
2. Number of solutions to the system
3. Scalars that work as a linear combination
4. Existence of linear combination

3/4. Justification with linear combination*
5. Utilizing SAGE
6. Justification with pivots
7. Justification with number of free variables

*Combined in RQ2 because the answer is "No"

Observation 4: Use **different combination of control structures** in the two reading questions

- Both reading questions address the **same problem** “whether the vector is in span of a spanning set”
  - Some students use a different CS in each question
- Correct answer: RQ1: “Yes, vector in span”; RQ2: “No, vector not in span”
  - Might suggest that the choice of control structure is also **dependent on the correct answer**

Observation 4: **Different combination of control structures** across the two reading questions

- The **choice** of control structures might depend on
  - whether vector is **in the span or not**
  - what students **observe** when applying the operators

Row reduction in progress...

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 2 & 5 \\ 0 & 3 & -3 & 0 & 2 \\ 0 & 0 & 0 & 7/3 & 3 \\ 0 & 0 & 0 & 2 & 5 \end{array}\right]$$

Observation 5: Most responses had **both correct** criteria and link by conception

- Reading questions fulfill their purpose of **motivating pre-class reading**
- Reading questions too easy? It could be answered by **inspection**
- Textbook authors should take note on this

Conclusion

- Our contributions:
  - A **lens** to analyze students' engagement with the reading material
  - A **fine-grained analysis method** of correctness of the answer based on the control structure

- Students' **choice** of control structures and the **correctness** in applying them can help instructors to **tailor lessons** to their needs

Future Directions

- More **comprehensive** data collection
  - Operators
  - Interviews
- **Longitudinal analysis** across sections
  - See how conceptions evolve
- Integration of large language models to **automate** the process

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Questions?

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Thank you!