# Stat 157: Simultaneous Wagers Kelly System

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We now expand the Kelly system to allow for the bettor to have $d \geq 1$ different opportunities available to bet on at each coup. Similar to before, we let $X_i : \Omega \to [-1, \infty)$ represent the gambler's profit per unit bet on the $i$th betting opportunity, which are not necessarily independent. We also assume that at least one of the opportunities is superfair:

$$\max_{1 \leq 1 \leq d} \mathbb{E}[X_i] > 0$$

Write $\mathbf{X} = (X_1, \ldots, X_n)$ and let $\mathbf{X}_1, \ldots, \mathbf{X}_n \overset{\text{iid}}{\sim} \mathbf{X}$, where $\mathbf{X}_i = (X_{i1}, \ldots, X_{in})$. These $X_{ij}$ represent the profit per unit bet for the $i$th coup on the $j$th betting opportunity. Also consider the betting system $\mathbf{B}_1, \ldots, \mathbf{B}_n$, where $\mathbf{B}_i = (B_{i1}, \ldots, B_{in})$. Here, $B_{ij}$ represents the bet on the $i$th coup on the $j$th betting opportunity. We let $\mathbf{B}_1 = \mathbf{b}_1 \geq \mathbf{0}$, a non-negative constant vector and have that $\mathbf{B}_n = \mathbf{b}_n(\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}) \geq \mathbf{0}$. That is, for $n \geq 2$, we have the $n$th bet only depend on the result of the first $n-1$ coups where $\mathbf{b}_n$ is a non-random vector function. We also require that the gambler's fortune at time $n$ satisfies

$$F_n = F_{n-1} + \mathbf{B}_n \cdot \mathbf{X}_n = F_{n-1} + \sum_{j=1}^{d} B_{nj} X_{nj} = F_0 + \sum_{i=1}^{n} \sum_{j=1}^{d} B_{ij} X_{ij}$$

We also require that the gambler cannot bet more than his current fortune, $B_{n,1} + \ldots + B_{n,d} \leq F_{n-1}$. Since at least one of the opportunities is superfair, we expect that betting everything on the most favorable opportunity might maximize profits, but we saw last time that this may lead to an unsatisfactory betting system. Instead, we define the set of betting proportions

$$\Delta = \left\{ \mathbf{f} = (f_1, \ldots, f_d) : \forall i, f_i \geq 0, \sum_{i=1}^{d} f_i \leq 1 \right\}$$

(From a financial viewpoint, where we are interested in investing in $d$ different stocks, $\Delta$ represents the set of asset allocations, where if our total capital is $C$, we put $f_i C$ into stock $i$, and if $\sum_{i=1}^{d} f_i < 1$, we put $\left(1 - \sum_{i=1}^{d} f_i\right) C$ into some sort of fixed income investment.)

Instead of betting all of our money on the opportunity with the greatest return, it is a better idea to spread out our money and bet on multiple opportunities with some strategy $\mathbf{f} \in \Delta$. When doing so, our bet will be

$$\mathbf{B}_n = \mathbf{f} F_{n-1}$$

And our fortune will be

$$F_n = F_{n-1}(1 + \mathbf{f} \cdot \mathbf{X}_n) = F_0 \prod_{j=1}^{n} (1 + \mathbf{f} \cdot \mathbf{X}_j)$$

And as before, we define $r_n(\mathbf{f}) = n^{-1} \log(F_n / F_0)$ and $\mu(\mathbf{f}) = \mathbb{E}[\log(1 + \mathbf{f} \cdot \mathbf{X})]$ on $\Delta_0 = \{\mathbf{f} \in \Delta : \mathbb{P}(\mathbf{f} \cdot \mathbf{X} = -1) = 0\}$ ($\mu$ must be defined on $\Delta_0$ so that we don't have almost sure ruin). And with the exact same argument as last time using the SLLN, we have $r_n(\mathbf{f}) \overset{\text{as}}{\to} \mu(\mathbf{f})$. So we can view $\mu(\mathbf{f})$ as the long term geometric rate of growth. And our aim is to find an optimal allocation $\mathbf{f}^*$ to maximize $\mu$.

**Lemma 10.2.1:** With our assumptions on $\mathbf{X}$ above, the function $\mu : \Delta_0 \to (-\infty, \infty)$ is concave and achieves a positive maximum. This maximum is not necessarily unique, but if $\mu$ achieves maxima at $\mathbf{f}_0^*$ and $\mathbf{f}_1^*$, then $\mathbb{P}(\mathbf{f}_0^* \cdot \mathbf{X} = \mathbf{f}_1^* \cdot \mathbf{X}) = 1$.
*Proof:* To show concavity, note that log is a concave function and let $\mathbf{f}_0, \mathbf{f}_1 \in \Delta_0$ and $\lambda \in (0,1)$. Then,

$$\begin{aligned}\mu(\lambda\mathbf{f}_0 + (1-\lambda)\mathbf{f}_1) &= \mathbb{E}[\log(1 + \lambda\mathbf{f}_0 \cdot \mathbf{X} + (1-\lambda)\mathbf{f}_1 \cdot \mathbf{X})] \\ &\geq \mathbb{E}[\lambda \log(1 + \mathbf{f}_0 \cdot \mathbf{X}) + (1-\lambda)\log(1 + \mathbf{f}_1 \cdot \mathbf{X})] \\ &= \lambda\mathbb{E}[\log(1 + \mathbf{f}_0 \cdot \mathbf{X})] + (1-\lambda)\mathbb{E}[\log(1 + \mathbf{f}_1 \cdot \mathbf{X})] \\ &= \lambda\mu(\mathbf{f}_0) + (1-\lambda)\mu(\mathbf{f}_1)\end{aligned}$$

To show that $\mu$ achieves a maximum, first define $h : (-1, \infty) \to (-\infty, \infty)$, $u \mapsto \log(1+u)$. This is a continuous function, hence its integral is also continuous. In particular, $\mu(\mathbf{f}) = \mathbb{E}[h(\mathbf{f} \cdot \mathbf{X})]$ is continuous on $\Delta_0$. In particular, it is continuous on every compact subset of the form $\{\mathbf{f} \in \Delta_0 : \mu(\mathbf{f}) \geq -C\} \subset \Delta_0$. Thus by the extreme value theorem, $\mu$ will achieve a maximum on each of these subsets, hence $\mu$ will achieve a maximum. This is a positive maximum since if we look at $\mathbf{f}$ with only one nonzero coordinate, then we can apply Lemma 10.1.1 to get $\mu(\mathbf{f}^*) \geq \mu(\mathbf{f}) > 0$. To prove the last assertion, notice that when we have the maximum achieved at both $\mathbf{f}_0^*$ and $\mathbf{f}_1^*$, then we must have equality above in the proof of concavity, however we can only have concavity if and only if $\mathbf{f}_0^* \cdot \mathbf{X} \overset{\text{as}}{=} \mathbf{f}_1^* \cdot \mathbf{X}$. ∎

The following theorem, with a slight modification in parts (d) and (e) is a vector form of Theorem 10.1.2. The proof is identical (except with vectors where appropriate).

Define $\sigma^2(\mathbf{f}) = \text{Var}(\log(1 + \mathbf{f} \cdot X))$ and denote $F_n(\mathbf{f})$ when we want to emphasize the dependence on $\mathbf{f}$.

**Theorem 10.2.2:** With the same assumptions on $\mathbf{X}$ with $\mathbf{X}_1, \mathbf{X}_2, \ldots$ i.i.d as $\mathbf{X}$ and $\mathbf{f} \in \Delta_0$, we have (a) $\lim_{n\to\infty}(F_n(\mathbf{f})/F_0)^{1/n} = \exp(\mu(\mathbf{f}))$ a.s.
(b) If $\mu(\mathbf{f}) > 0$, then $\lim_{n\to\infty} F_n(\mathbf{f}) = \infty$ a.s.
(c) If $\mu(\mathbf{f}) < 0$, then $\lim_{n\to\infty} F_n(\mathbf{f}) = 0$ a.s.
(d) If $\mu(\mathbf{f}) = 0$ and $\sigma(\mathbf{f}) > 0$, then $\limsup_{n\to\infty} F_n(\mathbf{f}) = \infty$ a.s. and $\liminf_{n\to\infty} F_n(\mathbf{f}) = 0$ a.s.
(e) If $\mu(\mathbf{f}) < \mu(\mathbf{f}^*)$, then $\lim_{n\to\infty} F_n(\mathbf{f}^*)/F_n(\mathbf{f}) = \infty$ a.s.
(f) If $\sigma(\mathbf{f}) > 0$, then

$$\frac{\sqrt{n}}{\sigma(\mathbf{f})}\left(\frac{1}{n}\log\left(\frac{F_n(\mathbf{f})}{F_0}\right) - \mu(\mathbf{f})\right) \xrightarrow{\text{d}} \mathcal{N}(0,1)$$

As we did last time, we can use 10.2.2(f) to define a $1-\alpha$ confidence interval:

$$\lim_{n\to\infty} \mathbb{P}(F_n(\mathbf{f}) \in [F_0 \exp\{\mu(\mathbf{f})n - z_{1-\alpha/2}\sigma(\mathbf{f})\sqrt{n}\}, F_0 \exp\{\mu(\mathbf{f})n + z_{1-\alpha/2}\sigma(\mathbf{f})\sqrt{n}\}]) = 1-\alpha$$

$$\lim_{n\to\infty} \mathbb{P}(F_n(\mathbf{f}) \in [F_0 \exp\{\mu(\mathbf{f})n - z_{1-\alpha}\sigma(\mathbf{f})\sqrt{n}\}, \infty]) = 1-\alpha$$

**Example 10.2.3 (Horse Race Betting)** *$d \geq 2$ simultaneous betting opportunities with only one winner*
Suppose we have $d \geq 2$ horses we can bet on denoted $1, 2, \ldots, d$. Denote the probability that horse $i$ will win as $p_i$ for $i \in \{1, \ldots, d\}$ where $p_1 > 0, p_2 > 0, \ldots, p_d > 0$ and $\sum_{i=1}^d p_i = 1$ (there will be no ties). Suppose for each $i$, a bet on horse $i$ pays $a_i$ to 1, where $a_i > 0$ and

$$\max_{1\leq i\leq d} \mathbb{E}[X_i] = \max_{1\leq i\leq d}\{(a_i+1)p_i - 1\} > 0$$

Define the basic strategies $\mathbf{e}_1, \ldots, \mathbf{e}_d \in \Delta$ as the strategy that bets everything on horse $i$ and nothing on the rest, i.e. $\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, etc. Define also $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$ as the all ones vector. Then the distribution of $\mathbf{X}$ is given by

$$\mathbb{P}(\mathbf{X} = (a_i+1)\mathbf{e}_i - \mathbf{1} = (-1, \ldots, -1, a_i, -1, \ldots, -1)) = p_i \quad \forall i \in \{1, \ldots, d\}$$

We want to find a unique $\mathbf{f}^* \in \Delta_0$ that maximizes

$$\mu(\mathbf{f}) = \mathbb{E}[\log(1+\mathbf{f}\cdot\mathbf{X})] = \sum_{i=1}^{d} p_i \log\left(1-\sum_{j=1}^{d} f_i + (a_i+1)f_i\right) = \sum_{i=1}^{d} p_i \log\left(w+(a_i+1)f_i\right)$$

Where $w = 1-\sum_{i=1}^{d} f_i$ be the proportion of withheld capital (money we don't bet). Here, $\Delta_0 = \Delta\backslash\{f\in\Delta : \sum_{j=1}^{d} f_j = 1, f_i = 0 \text{ for some i}\}$. $\mathbf{f}^*$ need not be unique, but (we state without proof) a necessary and sufficient condition for uniqueness is

$$\sum_{i=1}^{d}(a_i+1)^{-1} \neq 1$$

If we let $(w,\mathbf{f}) \in [0,1]\times\Delta_0$, we can view this as a nonlinear optimization problem

| | |
|---|---|
| Maximize | $\mu(w,\mathbf{f}) = \sum_{i=1}^{d} p_i \log\left(w+(a_i+1)f_i\right)$ |
| s.t | $w \geq 0,\ f_1 \geq 0,\ldots,f_d \geq 0$ |
| and | $1-w-f_1-\ldots-f_d = 0$ |

Denote $(w^*,\mathbf{f}^*)$ as the maximizer. Now by the Karush-Kuhn-Tucker Conditions, there exist $\kappa_0 \geq 0, \kappa_1 \geq 0,\ldots,\kappa_d \geq 0$ and $\lambda$ real such that

$$\frac{d\mu}{dw} + \kappa_0 - \lambda = 0 \text{ and } \kappa_0 w^* = 0$$

$$\frac{d\mu}{df_j} + \kappa_j - \lambda = 0 \text{ and } \kappa_j f_j^* = 0 \quad \forall j \in \{1,\ldots,d\}$$

Now define $I = \{j\in\{1,\ldots,d\} : f_j^* > 0\}$ (the set of horses we bet on), $I^c = \{j\in\{1,\ldots,d\} : f_j^* = 0\}$ (the set of horses we do not bet on).

*Case 1: $w^* > 0$ (partial capital allocated to bets)*
Notice that we must have

$$\sum_{i=1}^{d}(a_i+1)^{-1} > 1$$

If this inequality were <, then we can just distribute our withheld capital proportional to $(a_i+1)^{-1}$ to bet $i$ and be assured of profit from those additional bets, contradicting the optimality of $(w^*,\mathbf{f}^*)$. So at $(w^*,\mathbf{f}^*)$, we have from KKT

$$(1)\quad \frac{\partial\mu}{\partial w} - \lambda = 0 \Rightarrow \sum_{i=1}^{d}\frac{p_i}{w^*+(a_i+1)f_i^*} = \lambda$$

$$(2)\quad \frac{\partial\mu}{\partial f_j} - \lambda = 0 \Rightarrow \frac{(a_j+1)p_j}{w^*+(a_j+1)f_j^*} = \lambda,\ \ j\in I$$

$$(3)\quad \frac{\partial\mu}{\partial f_j} - \lambda = -\kappa_j \leq 0 \Rightarrow \frac{(a_j+1)p_j}{w^*} \leq \lambda,\ \ j\in I^c$$

$$(4)\quad w^* + \sum_{i=1}^{d} f_i^* = 1$$

Where in (1) and (2), since $w^* > 0$, we have $\kappa_0 = 0/w^* = 0$ and $\kappa_1 = 0/f_j^* = 0$.

Now define $q = \sum_{j\in I} p_j$ and $b = \sum_{j\in I}(a_j+1)^{-1}$.
If we rewrite (2) and sum over $j\in I$, we get

$$p_j = \lambda(a_j+1)^{-1}w^* + \lambda f_j^* \Rightarrow q = \lambda b w^* + \lambda(1-w^*)$$

If we multiply (1) by $w^*$ and substitute (2), we get

$$\lambda w^* = \sum_{j\in I} \frac{p_j w^*}{w^* + (a_j+1)f_j^*} + \sum_{j\in I^c} \frac{p_j w^*}{w^*} = \sum_{j\in I} \frac{\lambda w^*}{a_j+1} + 1 - \sum_{j\in I} p_j = \lambda b w^* + 1 - q \Rightarrow q = \lambda b w^* + 1 - \lambda w^*$$

Setting the two $q$'s equal, we have

$$\lambda b w^* + \lambda(1-w^*) = \lambda b w^* + 1 - \lambda w^* \Rightarrow \lambda = 1$$

Thus our conditions of $q$ are equivalent to $w^*(1-b) = 1-q$. Since $w^* > 0$, we must have $q < 1$ and hence $b < 1$. If $q = 1$, then $q = \sum_{j\in I} p_i = 1$, which only happens if $I = \{1, \dots, d\}$ and $w^*(1-b) = 1-q$ will give $b = 1$, contradicting our assumption that $\sum_{i=1}^d (a_i+1)^{-1} > 1$. Thus we have that $w^* = \frac{1-q}{1-b}$, and from (2), we get a full characterization of our optimal solution

$$w^* = \frac{1-q}{1-b}, \qquad f_j^* = \begin{cases} p_j - (a_j+1)^{-1} w^* & \text{if } j \in I; \\ 0 & \text{if } j \in I^c. \end{cases}$$

So if we can identify the set $I$ (the horses we want to bet on), then can solve for the explicit solution.

For convenience, relabel the outcomes as

$$(a_1+1)p_1 \geq (a_2+1)p_2 \geq \dots (a_d+1)p_d$$

So that $I$ will take the form $\{1, \dots, k\}$.

To determine the optimal $k$, define

$$q_k = \sum_{j=1}^k p_j, \quad b_k = \sum_{j=1}^k (a_j+1)^{-1}, \quad w_k = \frac{1-q_k}{1-b_k}$$

Where empty sums are defined to be 0. From (2) and (3), we have that for $j \in I$, $(a_j+1)p_j = w^* + (a_j+1)f_j^* > w^*$ and for $j \in I^c$, we have $(a_j+1)p_j \leq w^*$. Now using the fact that $(a_j+1)p_j > w^*$ for $j \in I$ and $(a_j+1)p_j \leq w^*$ for $j \in I^c$, it suffices to find a unique $k \in \{1, \dots, d-1\}$ so that $b_k < 1$ (i.e. our original assumption holds) and

$$(a_j+1)p_j > w^* \text{ for } j \in \{1, \dots, k\}$$

$$(a_j+1)p_j \leq w^* \text{ for } j \in \{k+1, \dots, d\}$$

Or equivalently, a unique $k$ such that $b_k < 1$ and $(a_{k+1}+1)p_{k+1} \leq w_j < (a_k+1)p_k$.

For $k = 1, \dots, d$, define

$$r_k = q_k + (1-b_k)(a_k+1)p_k = q_{k-1} + (1-b_{k-1})(a_k+1)p_k$$

With this notation, it is sufficient to find a unique $k \in \{1, \dots, d-1\}$ such that $b_k < 1$ and $r_{k+1} \leq 1 < r_k$. Uniqueness follows from the relabeling $(a_1+1)p_1 \geq (a_2+1)p_2 \geq \dots (a_d+1)p_d$

$$r_{k+1} = q_k + (1-b_k)(a_{k+1}+1)p_{k+1} \leq q_k + (1-b_k)(a_k+1)p_k = r_k \quad \forall k \in \{1, \dots, d-1\}$$

For existence, define $k_0 = \max\{k \in \{1, \dots, d-1\} : r_k > 1\}$. Since we have $\max_{1\leq i\leq d}\{(a_1+1)p_i - 1\} > 0$ and $(a_1+1)p_1 \geq (a_2+1)p_2 \geq \dots (a_d+1)p_d$, we get that $r_1 = (a_1+1)p_1 > 1$ and $r_d \leq 1$ since $1-b_d < 0$ by assumption and $r_d = q_d + (1-b_d)(a_d+1)p_d = 1 + (1-b_d)(a_d+1)p_d \leq 1 + 0 = 1$. Hence $r_{k_0+1} \leq 1 < r_{k_0}$ and we must have $b_{k_0} < 1$ otherwise $r_{k_0} > 1$ would fail.

*Case 2:* $w^* = 0$ *(full capital allocated to bets)*
Here, we must have $I = \{1, 2, \dots, d\}$ otherwise ruin would occur with positive probability. By KKT, we have

$$\begin{aligned}
&(5) \quad \frac{\partial \mu}{\partial w} - \lambda = -\kappa_0 \le 0 \Rightarrow \sum_{i=1}^{d} \frac{p_i}{(a_i+1) f_i^*} \le \lambda \\
&(6) \quad \frac{\partial \mu}{\partial f_j} - \lambda = 0 \Rightarrow \frac{p_j}{f_j^*} = \lambda, \quad j \in \{1, \dots, d\} \\
&(7) \quad \sum_{i=1}^{d} f_i^* = 1
\end{aligned}$$

If we rewrite (6) as $p_j = \lambda f_j^*$ and sum over $j$ then we have

$$1 = \sum_{i=1}^{d} p_j = \lambda \sum_{i=1}^{d} f_j^* = \lambda$$

Hence $\lambda = 1$ and $f_j^* = p_j$ for all $j = 1, \dots, d$. (5) implies that $\sum_{i=1}^{d} (a_i + 1)^{-1} < 1$. So from the notation from the end of case 1, we have $b_d < 1, w_d = 0, r_d > 1$ ensuring we have existence and uniqueness all around.