Chapter 1 Problems

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1 Homework 1

1.1 Question (1.1)

Evaluate the exercises in Mathematica Notebook 1.1

1.2 Question (1.2)

Given the set of polynomials of degree 3 in variable x, $P_a = a_0+a_1x+a_2x^2+a_3x^3$, where a_0,..., a_3 are real numbers. Let the binary operation $P_a + P_b$ denote ordinary addition. Show that set P_γ constitutes a linear vector space.

1.2.1 Answer

To show that P_γ constitutes a linear vector space, we must fulfill the following requirements:

1. There exists an operation, which we denote by the sign + sign, so that if α, γ are any two members of the vector space, V, then so is the quantity $\alpha + \gamma$.
2. For scalar c, there exists a scalar multiplication operation defined so that if β is a vector in V, then so is $c\beta = \beta c$. If a, b are scalars, $ab\beta = a(b\beta)$
3. multiplication is distributive, i.e. $c(\alpha + \beta) = c\alpha + c\beta$, also for scalar a,b, $(a + b)\alpha = a\alpha + b\alpha$.
4. The + operation is associative, i.e. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
5. The + operation is commutative, i.e. $\alpha + \beta = \beta + \alpha$
6. There exists a null vector 0 which has the property $0 + \alpha = \alpha$ for every vector α in V.
7. For every α in V there exists an inverse vector, $-\alpha$, that has the property

$$\alpha + -\alpha = 0$$

We can prove that P_γ is a vector space by proving that the polynomials of degree three are vector spaces:

1. Let $P_\gamma = P_a + P_b$, where $P_b = b_0 + b_1x + b_2x^2 + b_3x^3$ and b_0, ... b_3 are real numbers. Then

$$P_a + P_b = a_0 + a_1x + a_2x^2 + a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3$$
$$= a_0 + b_0 + a_1x + b_1x + a_2x^2 + b_2x^2 + a_3x^3 + b_3x^3$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3,$$

Since the terms, a_0, ... a_3 and b_0, ... b_3 are both real numbers, the addition of these terms also results in a real number. Thus, polynomials of the third degree, and therefore P_γ, are supported by the first requirement as vector spaces.

2. Let $c \in \mathbb{R}$ be some scalar. Then

$$cP_b = c(b_0 + b_1x + b_2x^2 + b_3x^3)$$
$$= cb_0 + cb_1x + cb_2x^2 + cb_3x^3$$
$$= b_0c + b_1xc + b_2x^2c + b_3x^3c$$
$$= P_bc,$$

Now let a,$d \in \mathbb{R}$ be some scalars. Then

$$adP_b = ad(b_0 + b_1x + b_2x^2 + b_3x^3)$$
$$= adb_0 + adb_1x + adb_2x^2 + adb_3x^3$$
$$= a(db_0) + a(db_1x) + a(db_2x^2) + a(db_3x^3)$$
$$= a(db_0 + db_1x + db_2x^2 + db_3x^3)$$
$$= a(dP_b),$$

Thus, polynomials of the third degree, and therefore P_γ, are supported by the second requirement as vector spaces.

3. Given scalars c, a, $d \in \mathbb{R}$:

$$c(P_a + P_b) = c[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3]$$
$$= c(a_0 + b_0) + c(a_1 + b_1)x + c(a_2 + b_2)x^2 + c(a_3 + b_3)x^3$$
$$= (ca_0 + cb_0) + (ca_1 + cb_1)x + (ca_2 + cb_2)x^2 + (ca_3 + cb_3)x^3$$
$$= cP_a + cP_b,$$

Additionally,

$$(a + d)P_a = (a + d)(a_0 + a_1x + a_2x^2 + a_3x^3)$$
$$= (a + d)a_0 + (a + d)a_1x + (a + d)a_2x^2 + (a + d)a_3x^3$$
$$= aa_0 + aa_1x + aa_2x^2 + aa_3x^3 + da_0 + da_1x + da_2x^2 + da_3x^3$$

$$= aP_a + dP_a,$$

Thus, requirement three supports polynomials of the third degree, and therefore P_γ, as a vector space.

4. Given some polynomial of P_c of the same form as P_a, P_b:

$$P_a+(P_b+P_c) = a_0+a_1x+a_2x^2+a_3x^3+(b_0+b_1x+b_2x^2+b_3x^3+c_0+c_1x+c_2x^2+c_3x^3)$$

$$= (a_0 + a_1x + a_2x^2 + a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3) + c_0 + c_1x + c_2x^2 + c_3x^3$$

$$= (P_a + P_b) + P_c,$$

Thus, requirement four supports polynomials of the third degree, and therefore P_γ, as a vector space.

5.

$$P_a + P_b = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$$

$$= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 + (b_3 + a_3)x^3$$

$$= P_b + P_a,$$

Thus, requirement five supports polynomials of the third degree, and therefore P_γ, as a vector space.

6. Given the third degree polynomial, $P_0 = 0 + 0x + 0x^2 + 0x^3 = 0$:

$$P_a + P_0 = 0 + 0x + 0x^2 + 0x^3 + a_0 + a_1x + a_2x^2 + a_3x^3$$

$$= 0 + a_0 + a_1x + a_2x^2 + a_3x^3$$

$$= a_0 + a_1x + a_2x^2 + a_3x^3$$

$$= P_a,$$

Thus, requirement six supports polynomials of the third degree, and therefore P_γ, as a vector space.

7. Given an "inverse" polynomial, $-P_a = -a_0 + -a_1x + -a_2x^2 + a_3x^3$:

$$P_a + -P_a = a_0 + a_1x + a_2x^2 + a_3x^3 + -a_0 + -a_1x + -a_2x^2 + -a_3x^3$$

$$= (a_0 + -a_0) + (a_1 + -a_1)x + (a_2 + -a_2)x^2 + (a_3 + -a_3)x^3$$

$$= 0 + 0x + 0x^2 + 0x^3$$

$$= P_0,$$

This follows the claim made by requirement six, where $P_0 = 0 + 0x + 0x^2 + 0x^3$. Thus, requirement seven supports polynomials of the third degree, and therefore P_γ, as a vector space.

Every requirement has been fulfilled at this point, so we can confirm that polynomials of the third degree, and therefore P_γ belongs a linear vector space, V.

1.3 Question (1.3)

Answer the exercises in Mathematica Notebook 1.2

1.4 Question (1.4)

The state

$$|\psi\rangle = \frac{1}{\sqrt{6}}|01101\rangle + \sqrt{\frac{2}{3}}|11111\rangle + \frac{1}{\sqrt{6}}|00001\rangle$$

describes a register of five q bulbs. (a) calculate the probability that the first q bulb is in the on position after making a measurement with the device $\mathbb{1} \otimes^{\sim} \mathbb{1} \otimes^{\sim} \mathbb{1} \otimes^{\sim} \mathbb{1} \otimes^{\sim} \mathbf{n}$. (b) What is the probability that all q-bulbs are in the on position, after a measurement with device $\mathbf{n} \otimes^{\sim} \mathbf{n} \otimes^{\sim} \mathbf{n} \otimes^{\sim} \mathbf{n} \otimes^{\sim} \mathbf{n}$. (c) Calculate the probability that at least three qbulbs are found in the off-position. (d) A measurement reveals the occupation configuration 01101. Immediately after that measurement another measurement with $\mathbf{N}$ is made. Calculate the probability that at least three q-bulbs are in the off-position.

1.4.1 Answer (a)

We note that the outer product on $|\psi\rangle$, defined by the Kronecker product, $(\mathbb{1} \otimes^{\sim} \mathbb{1} \otimes^{\sim} \mathbb{1} \otimes^{\sim} \mathbb{1} \otimes^{\sim} \mathbf{n})\,|\psi\rangle$, is equivalent to:

$$(\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbf{n})\frac{1}{\sqrt{6}}|01101\rangle+(\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbf{n})\sqrt{\frac{2}{3}}|11111\rangle+(\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbb{1}\otimes^{\sim}\mathbf{n})\frac{1}{\sqrt{6}}|00001\rangle,$$

Given that $\mathbf{n} = |1\rangle\langle 1|$, it is clear that the first term of $|\psi\rangle$, $\frac{1}{\sqrt{6}}|01101\rangle$, is unchanged. This is because $\mathbf{n}$ operates on the first q-bulb (the right-most number) of the first term, $|1\rangle$, and does not change the value; the other operators of our measurement device are identity operators and do not affect the probability. Thus, we must use the coefficient of the first term, $\frac{1}{\sqrt{6}}$, to calculate the probability using the born rule. Similarly, the second and third terms, $\sqrt{\frac{2}{3}}|11111\rangle$ and $\frac{1}{\sqrt{6}}|00001\rangle$, are unaffected by our operator because the first q-bulb in both states has a value of one. When measured, the probability that the first q-bulb is in the on position is:

$$\begin{aligned} |\,|\psi\rangle\,|^2 &= \left(\sqrt{\sqrt{\frac{2}{3}}^2 + \frac{1}{\sqrt{6}}^2 + \frac{1}{\sqrt{6}}^2}\right)^2 \\ &= 1 \end{aligned}$$

.

1.4.2 Answer (b)

Applying the same logic as part (a), the probability that all q-bulbs are found in the on position with our measurement device is:

$$(\sqrt{(\sqrt{\frac{2}{3}})^2})^2$$
$$= \frac{2}{3}$$

1.4.3 Answer (c)

Without an explicit measurement device, the probability that at least three q-bulbs of $|\psi\rangle$ are found in the off position is:

$$|\frac{1}{\sqrt{6}}|^2 = (\sqrt{\frac{1}{\sqrt{6}}^2})^2 = \frac{1}{6},$$

This because only one of the three states, $|00001\rangle$, has at least three q-bulbs in the "off" state or in a state 0.

1.4.4 Answer (d)

By the collapse hypothesis, the equation has been modified as such:

$$|\psi\rangle = |01101\rangle$$

. When we measure the result with N, the probability that three q-bulbs are off is zero.

1.5 Question (1.5)

Given the states

$$|\psi\rangle = \frac{1}{\sqrt{6}}|01101\rangle + \sqrt{\frac{2}{3}}|11111\rangle + \frac{i}{\sqrt{6}}|00001\rangle$$

and

$$|\phi\rangle = \frac{1}{\sqrt{2}}|01101\rangle + \frac{1}{\sqrt{2}}|01111\rangle$$

. Evaluate $\langle\phi|\psi\rangle, |\psi\rangle\langle\phi|, |\phi\rangle\langle\psi|$:

1.5.1 Answer $\langle\phi|\psi\rangle$

The dual of $|\phi\rangle$ is $\langle\phi| = \frac{1}{\sqrt{2}}\langle 10110| + \frac{1}{\sqrt{2}}\langle 11110|$. Thus, $\langle\phi|\psi\rangle$ is:

$$(\frac{1}{\sqrt{2}}\langle 10110| + \frac{1}{\sqrt{2}}\langle 11110|)(\frac{1}{\sqrt{6}}|01101\rangle + \sqrt{\frac{2}{3}}|11111\rangle + \frac{i}{\sqrt{6}}|00001\rangle)$$
$$= \frac{1}{\sqrt{12}}$$

1.5.2 Answer $|\psi\rangle\langle\phi|$

We have already derived the dual of $|\phi\rangle$, so the result is:

$$|\psi\rangle\langle\phi| = \frac{1}{\sqrt{12}}|01101\rangle\langle 10110| + \frac{1}{\sqrt{3}}|11111\rangle\langle 10110| + \frac{i}{\sqrt{12}}|00001\rangle\langle 10110|$$
$$+\frac{1}{\sqrt{12}}|01101\rangle\langle 11110| + \frac{1}{\sqrt{3}}|11111\rangle\langle 11110| + \frac{i}{\sqrt{12}}|00001\rangle\langle 11110|$$

1.5.3 Answer $|\phi\rangle\langle\psi|$

Now we derive the dual of $|\psi\rangle$, which is: $\langle\psi| = \frac{1}{\sqrt{6}}\langle 10110| + \sqrt{\frac{2}{3}}\langle 11111| + \frac{-i}{\sqrt{6}}\langle 10000|$. The outer product , $|\phi\rangle\langle\psi|$, is:

$$|\phi\rangle\langle\psi| = \frac{1}{\sqrt{12}}|01101\rangle\langle 10110| + \sqrt{\frac{2}{6}}|01101\rangle\langle 11111| + \frac{-i}{\sqrt{12}}|01101\rangle\langle 10000|$$
$$+\frac{1}{\sqrt{12}}|01111\rangle\langle 10110| + \sqrt{\frac{2}{6}}|01111\rangle\langle 11111| + \frac{-i}{\sqrt{12}}|01111\rangle\langle 10000|$$

1.6 Question (1.6)

Consider operator $\mathbf{X} = |\phi\rangle\langle\psi|$, show that $\mathbf{X}^\dagger = |\psi\rangle\langle\phi|$. Hint: use this expression for $\mathbf{X}^\dagger$ and operate it on bra $\langle\Gamma|$. Compare the result with that obtained by $\mathbf{X}|\Gamma\rangle$.

1.6.1 Answer

The operation of $\mathbf{X}$ on $|\Gamma\rangle$ results in:

$$\mathbf{X}|\Gamma\rangle = |\phi\rangle\langle\psi||\Gamma\rangle = |\phi\rangle\langle\psi|\Gamma\rangle = c|\phi\rangle,$$

for some complex number $c = \langle\psi|\Gamma\rangle$. On the other hand, the operation of $\mathbf{X}^\dagger$ on $\langle\Gamma|$ results in:

$$\langle\Gamma| \mathbf{X}^\dagger = \langle\Gamma| |\psi\rangle \langle\phi| = \langle\Gamma|\psi\rangle \langle\phi| = c^* \langle\phi| \,,$$

for some complex number $c^* = \langle\Gamma|\psi\rangle$. It is clear that first result is the dual of the second result. This proves that $\mathbf{X}^\dagger$ must equal $|\psi\rangle \langle\phi|$ as stated by Definition 1.1.

1.7 Question (1.7)

Re-express the following states $|17\rangle_5$, $|5\rangle_5$,$|12\rangle_5$ in binary notation, i.e., $|k_4k_3k_2k_1k_0\rangle$, $k_i \in$ 0,1:

1.7.1 Answer

$$|17\rangle_5 = |10001\rangle \,,$$
$$|5\rangle_5 = |00101\rangle \,,$$
$$|12\rangle_5 = |01100\rangle \,.$$

1.8 Question (1.8)

Consider the operator $\mathbf{X} \equiv exp(i\alpha) |00110\rangle \langle 00100| + |00111\rangle \langle 11111|$. Evaluate $\mathbf{X}|\phi\rangle$ where $|\phi\rangle$ is given by $\sum_i^{32} |i\rangle_5$.

1.8.1 Answer

The result of this operation gives only two vectors. The vectors $|00100\rangle$ and $|11111\rangle$ are operated on by $\mathbf{X}$ with d_{ij} value of 1. This leaves the remaining vectors $exp(i\alpha) |00110\rangle$ and $exp(i\alpha) |00111\rangle$. Thus the final result is:

$$\mathbf{X} |\phi\rangle = exp(i\alpha) |00110\rangle + |00111\rangle$$

1.9 Question (1.9)

Show that operator, in the Hilbert space of a single qubit, $\mathbf{X} \equiv |0\rangle \langle 1| + |1\rangle \langle 0|$ is Hermitian. Solve the following equation:

$$\mathbf{X} |\psi\rangle = \lambda |\psi\rangle$$

1.9.1 Answer (Part one)

We can prove that $\mathbf{X} = |0\rangle \langle 1| + |1\rangle \langle 0|$ is hermitian by finding the dual of our operator. The dual of $\mathbf{X}$ is:

$$|1\rangle \langle 0| + |0\rangle \langle 1| = \mathbf{X}^\dagger$$

It is clear at this point that $\mathbf{X}^\dagger$ is equivalent to $\mathbf{X}$. Thus, $\mathbf{X}$ is hermitian.

1.9.2 Answer (Part Two)

The equation has the same form as equation (1.24); thus, we are expected to solve an eigenvalue and eigenvector problem. Let $|\psi\rangle = c_1|0\rangle + c_2|1\rangle$. Now we can solve for our expected eigenvalues. Let us place every term on the left-hand side of the equation:

$$\mathbf{X}|\psi\rangle - \lambda|\psi\rangle = 0,$$

We can further simplify the components of the equation:

$$\mathbf{X}|\psi\rangle = c_2|0\rangle + c_1|1\rangle,$$

$$\lambda|\psi\rangle = c_1\lambda|0\rangle + c_2\lambda|1\rangle,$$

and when we place them together, the result is:

$$\mathbf{X}|\psi\rangle - \lambda|\psi\rangle = (c_2 - c_1\lambda)|0\rangle + (c_1 - c_2\lambda)|1\rangle$$

$|0\rangle$ and $|1\rangle$ are linearly independent, which means that the coefficients, $(c_2 - c_1\lambda)$ and $(c_1 - c_2\lambda)$ must equal zero to satisfy the simplified equation above. This means we can split our single equation into two:

$$(c_2 - c_1\lambda) = 0,$$

and

$$(c_1 - c_2\lambda) = 0$$

Let us use the first equation, $(c_2 - c_1\lambda) = 0$, to solve for c_2, then $c_2 = c_1\lambda$. Using substitution in our second equation, we get $c_1(1 - \lambda^2) = 0$. This means that either $c_1 = c_2 = 0$, which is an undesirable, trivial solution, or $1 - \lambda^2 = 0$. Thus, we must solve for the appropriate λ(s), also known as the eigenvalue(s).

Clearly, $\lambda = +1, -1$ and we have solved half of the problem. Now, we must find the eigenstates for each eigenvalue. Given $\lambda = 1$:

$$\mathbf{X}|\psi\rangle - \lambda|\psi\rangle = (c_2 - c_1)|0\rangle + (c_1 - c_2)|1\rangle = 0$$

Clearly, $c_2 = c_1$, and we are able to choose any arbitrary, non-zero number. Let $c_2 = c_1 = 3$, then the eigenstate, $|\psi\rangle$, is:

$$|\psi\rangle = 3|0\rangle + 3|1\rangle$$

We have solved for $\lambda = 1$. Given $\lambda = -1$:

$$\mathbf{X}|\psi\rangle - \lambda|\psi\rangle = (c_2 + c_1)|0\rangle + (c_1 + c_2)|1\rangle = 0$$

Clearly, $c_2 = -c_1$, and we are able to choose any arbitrary, non-zero number. Let $c_2 = -c_1 = 3$, then the eigenstate, $|\psi\rangle$, is:

$$|\psi\rangle = -3|0\rangle + 3|1\rangle$$

We have solved for the corresponding eigenstates of $\lambda = -1, +1$; this concludes the solution for the problem.

1.10 Question (1.10)

In the Hilbert space of three qubits, consider the operator:

$A \equiv |000\rangle \langle 000| + 2|001\rangle \langle 100| - 2|010\rangle \langle 010| + 3|100\rangle \langle 001| + |011\rangle \langle 110| - |101\rangle \langle 101|$

Find all the eigenvalues and eigenvectors of A. Identify the degenerate eigenvalues and show that any linear combination of the corresponding eigenvectors are also eigenstates of A. Now

1.10.1 Answer

Let us use the same logic as in question (1.9) to solve for eigenvalues and eigenstates. We first have to apply our three qubit operator, A, on a three qubit vector, $|\psi\rangle$ in a superposition of every value; in this case, there are eight possible states in a three qubit vector. Thus,

$|\psi\rangle = c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + c_3|011\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + c_7|111\rangle$

Now we place the equation, $A|\psi\rangle = \lambda|\psi\rangle$ on the left-hand side and solve for the eigenvalues, λ. We know that

$$A|\psi\rangle = c_0|000\rangle + 2c_1|001\rangle - 2c_2|010\rangle + c_3|011\rangle + 3c_4|100\rangle - c_5|101\rangle,$$

and

$$\lambda|\psi\rangle = c_0\lambda|000\rangle + ... + c_6\lambda|110\rangle + c_7\lambda|111\rangle$$

When we place the equations together, we get:

$$\begin{gathered} A|\psi\rangle - \lambda|\psi\rangle = \\ c_0(1-\lambda)|000\rangle + c_1(2-\lambda)|001\rangle - c_2(2+\lambda)|010\rangle + c_3(1-\lambda)|011\rangle \\ +c_4(3-\lambda)|100\rangle - c_5(1+\lambda)|101\rangle - c_6\lambda|110\rangle - c_7\lambda|111\rangle \\ = 0 \end{gathered}$$

Once again, the trivial solution is $c_0 = c_1 = ... = c_7 = 0$. However, this is undesirable; we want to find a solution that is non-trivial. The eigenvalues are clearly $\lambda = 1, 2, -2, 1, 3, -1, 0, 0$ and the eigenvalues $\lambda = 1$ and $\lambda = 0$ are degenerate with a multiplicity of two. Now we solve for each eigenvalue.

Let us solve for $\lambda = 1$, which implies:

$$\begin{gathered} A|\psi\rangle - \lambda|\psi\rangle = \\ c_1|001\rangle - 3c_2|010\rangle + 2c_4|100\rangle - 2c_5|101\rangle - c_6|110\rangle - c_7|111\rangle \end{gathered}$$

The values, c_1, c_2, c_4, c_5, c_6, c_7 must equal zero. The values c_0 and c_3 on the other hand, are arbitrary. Let $c_0 = 0$ and $c_3 = -1$. One eigenstate for $\lambda = 1$ is:

$$|\psi\rangle = -|011\rangle,$$

and another eigenstate might be:

$$|\psi\rangle = 2\,|000\rangle\,,$$

where $c_0 = 2$ and $c_3 = 0$. Remember, these values are arbitrarily chosen. Another eigenstate might be a linear combination $|\psi\rangle = 2\,|000\rangle - |011\rangle$. This is possible by manipulating c_0 and c_3 to non-zero values; at least one number, c_0 or c_3 in this case, must be non-zero to provide a non-trivial solution; a linear combination implies that there is more than one non-zero number, or c_0 and c_3 are non-zero in this case.

We have found the eigenstates for $\lambda = 1$. Now let $\lambda = 2$, then $c_0 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$ and c_1 is an arbitrary value. Let $c_1 = 1$, then the eigenstate for $\lambda = 2$ is:

$$|\psi\rangle = |001\rangle$$

Let $\lambda = -2$. Then, $c_0 = c_1 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$ and c_2 is an arbitrary value. Let $c_2 = 1$, then the eigenstate for $\lambda = -2$ is:

$$|\psi\rangle = |010\rangle$$

Let $\lambda = 3$. Then, $c_0 = c_1 = c_2 = c_3 = c_5 = c_6 = c_7 = 0$ and c_4 is an arbitrary value. Let $c_4 = 1$, then the eigenstate for $\lambda = 3$ is:

$$|\psi\rangle = |100\rangle$$

Let $\lambda = -1$. Then, $c_0 = c_1 = c_2 = c_3 = c_4 = c_6 = c_7 = 0$ and c_5 is an arbitrary value. Let $c_5 = 1$, then the eigenstate for $\lambda = -1$ is:

$$|\psi\rangle = |101\rangle$$

Finally, let $\lambda = 0$. Then, $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0$ and c_6 and c_7 are arbitrary values. Let $c_6 = 1$ and $c_7 = 0$, then the eigenstate for $\lambda = 0$ is:

$$|\psi\rangle = |110\rangle$$

and another eigenstate might be:

$$|\psi\rangle = -\,|111\rangle$$

where $c_6 = 0$ and $c_7 = -1$. Another eigenstate might be $|\psi\rangle = |110\rangle - |111\rangle$ as a linear combination of the previous eigenstates. This is possible by manipulating the coefficients c_6 and c_7; at least one number must be non-zero, and a linear combination implies that both numbers are non-zero.

1.11 Question (1.11)

In a two-qubit Hilbert space, consider the operator

$$A = |00\rangle\langle 01| + |10\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01|$$

Find the eigenvalues and eigenstates of A.

1.11.1 Answer

Let us follow the form of equation (1.24), $X|\Phi\rangle = \phi|\Phi\rangle$, where we substitute X for A, ϕ for λ and $|\Phi\rangle$ for $|\psi\rangle$. Let $|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$. Now we can solve for our eigenvalues, λ.

We know that $A|\psi\rangle = c_0|10\rangle + c_1|01\rangle + c_2|10\rangle + c_2|00\rangle$ and $\lambda|\psi\rangle = c_0\lambda|00\rangle + c_1\lambda|01\rangle + c_2\lambda|10\rangle + c_3\lambda|11\rangle$. Thus,

$$A|\psi\rangle - \lambda|\psi\rangle = (c_2 - c_0\lambda)|00\rangle + c_1(1-\lambda)|01\rangle + [c_0 + c_2(1-\lambda)]|10\rangle - c_3\lambda|11\rangle$$

Here, c_1 and c_3 are only dependent on λ, so the corresponding eigenvalues are $\lambda = 1, 0$. Now we must solve for the other eigenvalue(s) using the other coefficients:

$$c_2 - c_0\lambda = 0$$

and

$$c_0 + c_2(1-\lambda) = 0$$

Using the first equation, $c_2 = c_0\lambda$. This implies that the second equation could be represented as:

$$c_0 + c_0(\lambda - \lambda^2) = c_0(1 + \lambda - \lambda^2) = 0,$$

and the eigenvalues in this case are $\lambda = \frac{-1+\sqrt{5}}{-2}, -(\frac{1+\sqrt{5}}{2})$. Thus, the eigenvalues for matrix A are $\lambda = 1, 0, \frac{-1+\sqrt{5}}{-2}, -(\frac{1+\sqrt{5}}{2})$. Now we solve for each eigenvalue.

Let $\lambda = 1$, then $c_0 = c_2 = c_3 = 0$ and c_1 is arbitrary. Let $c_1 = -.5$ then the corresponding eigenstate is:

$$|\psi\rangle = -\frac{1}{2}|01\rangle$$

Let $\lambda = 0$, then $c_0 = c_1 = c_2 = 0$ and c_3 is arbitrary. Let $c_3 = -2$ then the corresponding eigenstate is:

$$|\psi\rangle = -2|01\rangle$$

Let $\lambda = \frac{-1+\sqrt{5}}{-2}$, then $c_1 = c_3 = 0$ and $c_0 = \frac{-2c_2}{-1+\sqrt{5}}$. Let $c_2 = -1+\sqrt{5}$, then $c_0 = 2$ and the corresponding eigenstate is:

$$|\psi\rangle = 2|00\rangle + (-1+\sqrt{5})|10\rangle$$

Let $\lambda = -\frac{1+\sqrt{5}}{2}$, then $c_1 = c_3 = 0$ and $c_0 = -\frac{2c_2}{1+\sqrt{5}}$. Let $c_2 = 1+\sqrt{5}$, then $c_0 = -2$ and the corresponding eigenstate is:

$$|\psi\rangle = -2|00\rangle + (1+\sqrt{5})|10\rangle$$

This concludes the solution for all eigenvalues and eigenstates in (1.11).

1.12 Question (1.12)

Given the single qubit operators

$$\mathbf{A} = |0\rangle \langle 0| + |1\rangle \langle 1| , \mathbf{B} = i |0\rangle \langle 1| - i |1\rangle \langle 0|$$

Show that $[\mathbf{A},\mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA} = 0$. Find the eigenstates for the operator **B** and show that they are also eigenstates of operator **A**. What are the eigenvalues associated with operator **A**?

1.12.1 Answer (prove AB-BA=0)

To answer the first part of the question, we will evaluate **AB** and **BA**:

$$\mathbf{AB} = i |0\rangle \langle 1| - i |1\rangle \langle 0|$$

and

$$\mathbf{BA} = i |0\rangle \langle 1| - i |1\rangle \langle 0|$$

Clearly, **AB** equals **BA**, so the statement $\mathbf{AB} - \mathbf{BA} = 0$ is true.

1.12.2 Answer(Eigenstates of B)

Let us follow the form of equation (1.24). Additionally, substitute $|\Phi\rangle$ for $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$ and substitute ϕ for λ. Then

$$B |\psi\rangle = -ic_0 |1\rangle + ic_1 |0\rangle$$

and

$$\lambda |\psi\rangle = c_0\lambda |0\rangle + c_1\lambda |1\rangle$$

This implies that $b |\psi\rangle - \lambda |\psi\rangle$ is

$$(ic_1 - \lambda c_0) |0\rangle - (ic_0 + \lambda c_1) |1\rangle = 0$$

We now separate the coefficients into two separate equations:

$$ic_1 - c_0\lambda = 0, ic_0 + c_1\lambda = 0$$

Using the first equation, $c_1 = -ic_0\lambda$, and we can substitute into the second equation, $ic_0 - ic_0\lambda^2 = ic_0(1 - \lambda^2) = 0$. Thus, the eigenvalues are $\lambda = 1, -1$.

For $\lambda = 1$, $c_0 = ic_1$. Let $c_1 = 2$, then $c_0 = 2i$ and the eigenstate is:

$$|\psi\rangle = 2i |0\rangle + 2 |1\rangle$$

For $\lambda = -1$, $c_0 = -ic_1$. Let $c_1 = 2$, then $c_2 = -2i$ and the eigenstate is:

$$|\psi\rangle = -2i |0\rangle + 2 |1\rangle$$

1.12.3 Eigenstates/Eigenvalues of A

We can prove that the eigenstates we identified in **B** are the eigenstates in **A** by applying equation 1.24. Given $A\left|\psi\right\rangle = \lambda\left|\psi\right\rangle$, where $\left|\psi\right\rangle = 2\left|0\right\rangle + 2i\left|1\right\rangle$:

$$A\left|\psi\right\rangle = 2\left|0\right\rangle + 2i\left|1\right\rangle = \left|\psi\right\rangle = \lambda\left|\psi\right\rangle$$

This only works if the eigenvalue of **A** is $\lambda = 1$. We can also do the same for $\left|\psi\right\rangle = 2\left|0\right\rangle - 2i\left|1\right\rangle$:

$$A\left|\psi\right\rangle = 2\left|0\right\rangle - 2i\left|1\right\rangle = \left|\psi\right\rangle = \lambda\left|\psi\right\rangle$$

Similarly this only works if the eigenvalue of **A** is $\lambda = 1$. Thus, the eigenstates of **B** are the same as **A**. There is only one eigenvalue in **A**, which is $\lambda = 1$. We can verify this by searching for the eigenvalues in a traditional fashion:

$$A\left|\psi\right\rangle - \lambda\left|\psi\right\rangle = c_0(1-\lambda)\left|0\right\rangle + c_1(1-\lambda)\left|1\right\rangle$$

Clearly $\lambda = 1$ for the matrix **A**.

1.13 Question (1.13)

Prove the following: for two Hilbert space operators, **A, B,** so that $[\mathbf{A},\mathbf{B}] = 0$, show that, if **A** has non-degenerate eigenstates, $\left|a_1\right\rangle, \left|a_2\right\rangle, ...$, then the kets $\left|a_i\right\rangle$ are also eigenstates of operator **B**.

1.13.1 Answer

If $[\mathbf{A},\mathbf{B}] = 0$ and **A** has non-degenerate eigenstates, this implies the form $A\left|a_i\right\rangle = \lambda_i\left|a_i\right\rangle$. Additionally, the equation, $[\mathbf{A},\mathbf{B}]\left|a_i\right\rangle = 0$, implies that $\mathbf{AB}\left|a_i\right\rangle = \mathbf{BA}\left|a_i\right\rangle$. Note that we can substitute the two equations we have derived thus far:

$$\mathbf{AB}\left|a_i\right\rangle = \mathbf{BA}\left|a_i\right\rangle = \mathbf{B}\lambda\left|a_i\right\rangle = \lambda\mathbf{B}\left|a_i\right\rangle$$

Clearly, $\mathbf{B}\left|a_i\right\rangle$ and $\left|a_i\right\rangle$ are eigenstates of A, but because we assume non-degenerate eigenstates, $\mathbf{B}\left|a_i\right\rangle$ can not be a different state from $\left|a_i\right\rangle$; otherwise, the associated eigenvalue would be degenerate. Thus, the equation

$$\mathbf{B}\left|a_i\right\rangle = \phi\left|a_i\right\rangle$$

for some real scalar ϕ, holds true, and we prove that any eigenstate of A, $\left|a_i\right\rangle$, is an eigenstate of B.

1.14 Question (1.14)

Prove Theorems 1.1 and 1.2:

Theorem 1.1: The eigenvalues of a Hermitian operators are real numbers

Theorem 1.2: If the eigenvalues of Hermitian operator are distinct, then the corresponding eigenvectors are mutually orthogonal. If some of the eigenvalues are not distinct, or degenerate, then a linear combination of that subset of eigenvectors can be made to be mutually orthogonal

1.14.1 Answer (Theorem 1.1)

Recall the equation given in (1.24), then for a given matrix, A:

$$A|\psi\rangle = \lambda|\psi\rangle$$

for some eigenstate $|\psi\rangle$ and an eigenvalue λ. The dual of the equation is:

$$\langle\psi| A^\dagger = \lambda^* \langle\psi|$$

Note that $A^\dagger = A$, and if we know apply the dual onto $|\psi\rangle$, our equation becomes:

$$\langle\psi| A |\psi\rangle = \lambda^* \langle\psi|\psi\rangle$$

Substituting $A|\psi\rangle$ with $\lambda|\psi\rangle$, our final equation is:

$$\langle\psi| A |\psi\rangle = \lambda \langle\psi|\psi\rangle = \lambda^* \langle\psi|\psi\rangle$$

Thus, $\lambda = \lambda^*$, which is only possible if λ is a real number.

1.14.2 Answer (Theorem 1.2)

Suppose that, if the eigenvalues of a hermitian operator are distinct, then the corresponding eigenvectors are mutually orthogonal. Let $A|x_1\rangle = \lambda_1|x_1\rangle$ and $A|x_2\rangle = \lambda_2|x_2\rangle$, be distinct eigenvalues of a hermitian operator. It follows that the equation:

$$\lambda_1 - \lambda_2 \neq 0$$

is true. Since the associated eigenvectors, $|x_1\rangle$ and $|x_2\rangle$, are mutually orthogonal, the inner product would be zero. We can use fact in the equation:

$$(\lambda_1 - \lambda_2)\langle x_1|x_2\rangle = 0,$$

where the $\langle x_1|x_2\rangle$ must be zero for the equation to hold true. Simplifying the equation yields:

$$\lambda_1 \langle x_1| |x_2\rangle = \lambda_2 \langle x_1| |x_2\rangle$$

From our previous equations, we know that $\langle x_1| \lambda_1 = \langle x_1| A^\dagger$ and $\lambda_2 |x_2\rangle = A|x_1\rangle$. Then, our equation becomes:

$$\langle x_1| A^\dagger |x_2\rangle = \langle x_1| A |x_2\rangle$$

It is obvious at this point that we must prove $A^\dagger = A$; this is proven by the hermitian property of A, and we now conclude our proof.

1.15 Question (1.15)

Given operator $\mathbf{A}$ in an n-dimensional Hilbert space with orthonormal eigenvectors $|a_1\rangle, |a_2\rangle, ... |a_n\rangle$, prove that $\mathbf{A} = \sum_i^n a_i |a_i\rangle \langle a_i|$, where a_i is the eigenvalue associated with $|a_i\rangle$.

1.15.1 Answer

Suppose the equation $\mathbf{A} = \sum_i^n a_i |a_i\rangle \langle a_i|$ is true. If operator $\mathbf{A}$ acts on an arbitrary eigenvector, $|a_j\rangle$, the result is:

$$\mathbf{A} |a_j\rangle = \sum_i^n a_i |a_i\rangle \langle a_i|a_j\rangle$$

which is non-zero when $i = j$, so

$$\mathbf{A} |a_j\rangle = a_j |a_j\rangle \langle a_j|a_j\rangle = a_j |a_j\rangle$$

Clearly, the operator yields the eigenvector and associated eigenvalue we set to find. Thus, we solidify that $\mathbf{A} = \sum_i^n a_i |a_i\rangle \langle a_i|$ is true.

1.16 Question (1.16)

Show that the operator

$$\mathbf{U} = cos\theta |0\rangle \langle 0| + exp(i\phi)sin\theta |0\rangle \langle 1| + exp(-i\phi)sin\theta |1\rangle \langle 0| - cos\theta |1\rangle \langle 1|$$

is unitary.

1.16.1 Answer

If $\mathbf{U}$ is unitary, than $\mathbf{U}\mathbf{U}^\dagger = \mathbb{1}$, where $\mathbb{1} = |0\rangle \langle 0| + |1\rangle \langle 1|$. Let us solve for $\mathbf{U}^\dagger$:

$$\mathbf{U}^\dagger = cos(\theta) |0\rangle \langle 0| + exp(-i\phi)sin(\theta) |1\rangle \langle 0| + exp(i\phi)sin(\theta) |0\rangle \langle 1| - cos(\theta) |1\rangle \langle 1|$$

Clearly, $\mathbf{U} = \mathbf{U}^\dagger$, so we can apply $\mathbf{U}$ onto itself and prove that it is unitary. For convenience, let us substitute $\exp(\phi)$, for some arbitrary ϕ, for e^ϕ:

$$\mathbf{U}\mathbf{U} = [cos\theta |0\rangle \langle 0| + exp(i\phi)sin\theta |0\rangle \langle 1| +$$
$$exp(-i\phi)sin\theta |1\rangle \langle 0| - cos\theta |1\rangle \langle 1|][cos\theta |0\rangle \langle 0| +$$
$$exp(i\phi)sin\theta |0\rangle \langle 1| + exp(-i\phi)sin\theta |1\rangle \langle 0| - cos\theta |1\rangle \langle 1|]$$

which simplifies to:

$$\mathbf{U}\mathbf{U} = cos^2(\theta) |0\rangle \langle 0| + cos(\theta)sin(\theta)e^{i\phi} |0\rangle \langle 1|$$
$$sin^2(\theta)e^{i(\phi-\phi)} |0\rangle \langle 0| - e^{i\phi}sin(\theta)cos(\theta) |0\rangle \langle 1|$$

$$e^{-i\phi}sin(\theta)cos(\theta)\left|1\right\rangle\left\langle 0line\right| + e^{i(\phi-\phi)}sin^2(\theta)\left|1\right\rangle\left\langle 1\right|$$
$$-cos(\theta)e^{-i\phi}sin(\theta)\left|1\right\rangle\left\langle 0\right| + cos^2(\theta)\left|1\right\rangle\left\langle 1\right|$$
$$= (cos^2(\theta) + sin^2(\theta))\left|0\right\rangle\left\langle 0\right| + 0 + 0 + (sin^2(\theta) + cos^2(\theta))\left|1\right\rangle\left\langle 1\right|$$
$$= \left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|$$

Thus, $\mathbf{UU}^\dagger = \mathbb{1}$.

1.17 Question (1.17)

Consider the operator $\mathbf{X} = \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|$, evaluate:

$$\mathbf{X}^\sim = \mathbf{UXU}^\dagger$$

where $\mathbf{U}$ is given in problem (1.16)

1.17.1 Answer

We know that $\mathbf{U} = cos\theta\left|0\right\rangle\left\langle 0\right| + exp(i\phi)sin\theta\left|0\right\rangle\left\langle 1\right| + exp(-i\phi)sin\theta\left|1\right\rangle\left\langle 0\right| - cos\theta\left|1\right\rangle\left\langle 1\right|$. We also know from problem (1.16) that $\mathbf{U}$ is hermitian, so $\mathbf{U} = \mathbf{U}^\dagger$; the operation $\mathbf{XU}$ is:

$$\mathbf{XU} = cos\theta\left|1\right\rangle\left\langle 0\right| + exp(i\phi)sin\theta\left|1\right\rangle\left\langle 1\right| + exp(-i\phi)sin\theta\left|0\right\rangle\left\langle 0\right| - cos\theta\left|0\right\rangle\left\langle 1\right|$$

and the operation $\mathbf{U(XU)}$ is:

$$\mathbf{U}(\mathbf{XU}) = (cos\theta\left|0\right\rangle\left\langle 0\right| + exp(i\phi)sin\theta\left|0\right\rangle\left\langle 1\right| + exp(-i\phi)sin\theta\left|1\right\rangle\left\langle 0\right| - cos\theta\left|1\right\rangle\left\langle 1\right|)$$
$$(cos\theta\left|1\right\rangle\left\langle 0\right| + exp(i\phi)sin\theta\left|1\right\rangle\left\langle 1\right| + exp(-i\phi)sin\theta\left|0\right\rangle\left\langle 0\right| - cos\theta\left|0\right\rangle\left\langle 1\right|) \longrightarrow$$
$$\mathbf{U}(\mathbf{XU}) = cos(\theta)exp(-i\phi)sin(\theta)\left|0\right\rangle\left\langle 0\right| - cos^2(\theta)\left|0\right\rangle\left\langle 1\right|$$
$$+exp(i\phi)sin(\theta)cos(\theta)\left|0\right\rangle\left\langle 0\right| + exp(2i\phi)sin^2(\theta)\left|0\right\rangle\left\langle 1\right|$$
$$+exp(-2i\phi)sin^2(\theta)\left|1\right\rangle\left\langle 0\right| - exp(-i\phi)sin(\theta)cos(\theta)\left|1\right\rangle\left\langle 1\right|$$
$$-cos(\theta)sin(\theta)exp(i\phi)\left|1\right\rangle\left\langle 1\right| - cos^2(\theta)\left|1\right\rangle\left\langle 0\right| \longrightarrow$$

$$\mathbf{U}(\mathbf{XU}) = (sin(\theta)cos(\theta)(exp(i\phi)+exp(-i\phi)))\left|0\right\rangle\left\langle 0\right| + (-cos^2(\theta)+sin^2(\theta)exp(2i\phi))\left|0\right\rangle\left\langle 1\right|$$
$$+(-cos^2(\theta)+sin^2(\theta)exp(-2i\phi))\left|1\right\rangle\left\langle 0\right| - (sin(\theta)cos(\theta)(exp(i\phi)+exp(-i\phi)))\left|1\right\rangle\left\langle 1\right|$$

Thus, our evaluation of $\mathbf{X}^\sim$ is:

$$\mathbf{X}^\sim = \alpha\left|0\right\rangle\left\langle 0\right| + \beta\left|0\right\rangle\left\langle 1\right| + \gamma\left|1\right\rangle\left\langle 0\right| - \alpha\left|1\right\rangle\left\langle 1\right|,$$

where $\alpha = sin(\theta)cos(\theta)(2cos(\phi))$, $\beta = -cos^2(\theta) + sin^2(\theta)exp(2i\phi)$, and $\gamma = -cos^2(\theta) + sin^2(\theta)exp(-2i\phi)$

1.18 Question (1.18)

Find the eigenvalues and eigenstates for operator $\mathbf{X}^{\sim}$ given in problem (1.17) $cos(\phi) + isin(\phi) + cos(-\phi) + isin(-\phi) = 2cos(\phi)$

1.18.1 Answer

Given that our equation is:
$\mathbf{X}^{\sim} = \alpha |0\rangle \langle 0| + \beta |0\rangle \langle 1| + \gamma |1\rangle \langle 0| - \alpha |1\rangle \langle 1|$,
where $\alpha = sin(\theta)cos(\theta)(2cos(\phi))$, $\beta = -cos^2(\theta) + sin^2(\theta)exp(2i\phi)$, and $\gamma = -cos^2(\theta) + sin^2(\theta)exp(-2i\phi)$. To find the eigenvalues and eigenstates we must follow the form in equation (1.24):

$$\mathbf{X}^{\sim} |\psi\rangle = \lambda |\psi\rangle ,$$

where $|\psi\rangle = c_1 |0\rangle + c_2 |1\rangle$. Then the left-hand-side of the equation is

$$\mathbf{X}^{\sim} |\psi\rangle = (\alpha c_1 + \beta c_2) |0\rangle + (\gamma c_1 - \alpha c_2) |1\rangle$$

and the right-hand-side of the equation is

$$\lambda |\psi\rangle = c_1 \lambda |0\rangle + c_2 \lambda |1\rangle$$

Thus, the equation $\mathbf{X}^{\sim} |\psi\rangle - \lambda |\psi\rangle = 0$ is:

$$\mathbf{X}^{\sim} |\psi\rangle - \lambda |\psi\rangle = (\alpha c_1 + \beta c_2 - \lambda c_1) |0\rangle + (\gamma c_1 - \alpha c_2 - \lambda c_2) |1\rangle$$

Since $|1\rangle$ and $|0\rangle$ are linearly independent, we can separate our equation into two such that:

$$(\beta c_2 + (\alpha - \lambda) c_1) = 0$$

and

$$(\gamma c_1 - (\alpha + \lambda) c_2) = 0$$

Since we can not use the trivial solution, $c_1 = c_2 = 0$; however, we can substitute c_2 into the first equation and solve for a proper eigenvalue:

$$\beta(\frac{\gamma c_2}{\alpha + \lambda}) + (\alpha + \lambda) c_2 = (\beta\gamma + (\alpha + \lambda)^2) c_2 = 0$$

Which is only true if our eigenvalue is

$$\lambda = \sqrt{-\beta\gamma} - \alpha$$

Meanwhile c_2 is now arbitrary as the result will remain zero. Choose $c_2 = 1$ for our given eigenvalue; then $c_1 = -\frac{-\beta}{2\alpha - \sqrt{-\beta\gamma}}$ as our second equation has determined. Thus, our eigenstate, $|\psi\rangle$ corresponding to our eigenvalue, $\lambda = \sqrt{-\beta\gamma} - \alpha$, is:

$$|\psi\rangle = -\frac{-\beta}{2\alpha - \sqrt{-\beta\gamma}} |0\rangle + |1\rangle$$

Thus, we conclude our search for the eigenvalues and eigenstates of $\mathbf{X}^{\sim}$

1.19 Question (1.19)

Evaluate $\mathbf{Y}^{\sim} = \mathbf{U}\mathbf{Y}\mathbf{U}^{\dagger}$ where, $\mathbf{Y} = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$ and $\mathbf{U}$ is defined in problem (1.17). Demonstrate that:

$$[\mathbf{X}^{\sim}, \mathbf{Y}^{\sim}] = 2i\mathbf{Z}^{\sim}$$

where $\mathbf{Z}^{\sim} = \mathbf{U}(|0\rangle\langle 0| - |1\rangle\langle 1|)\mathbf{U}^{\dagger}$, and $\mathbf{X}^{\sim}$ is defined in problem (1.17).

1.19.1 Answer (Part one)

We know that $\mathbf{U} = cos\theta\,|0\rangle\langle 0| + exp(i\phi)sin\theta\,|0\rangle\langle 1| + exp(-i\phi)sin\theta\,|1\rangle\langle 0| - cos\theta\,|1\rangle\langle 1|$. We also know from problem (1.16) that $\mathbf{U}$ is hermitian, so we consider $\mathbf{UYU}$ instead; the operation $\mathbf{YU}$ is:

$$\mathbf{YU} = -iexp(-i\phi)sin(\theta)\,|0\rangle\langle 0| + icos(\theta)\,|0\rangle\langle 1| + icos(\theta)\,|1\rangle\langle 0| + iexp(i\phi)sin(\theta)\,|1\rangle\langle 1|$$

And the operation $\mathbf{U}(\mathbf{YU})$ is:

$$\begin{gathered}\mathbf{UYU} = -icos(\theta)sin(\theta)exp(-i\phi)\,|0\rangle\langle 0| + icos^2(\theta)\,|0\rangle\langle 1| \\ +icos(\theta)sin(\theta)exp(i\phi)\,|0\rangle\langle 0| + iexp(2i\phi)sin^2(\theta)\,|0\rangle\langle 1| \\ -iexp(-2i\phi)sin^2(\theta)\,|1\rangle\langle 0| + icos(\theta)sin(\theta)exp(-i\phi)\,|1\rangle\langle 1| \\ -icos^2(\theta)\,|1\rangle\langle 0| - icos(\theta)sin(\theta)exp(i\phi)\,|1\rangle\langle 1|,\end{gathered}$$

which can be simplified to

$$\mathbf{UYU} = 2icos(\theta)sin(\theta)[sin(\phi)]\,|0\rangle\langle 0| + i(cos^2(\theta) + exp(2i\phi)sin^2(\theta))\,|0\rangle\langle 1|$$

$$-i(cos^2(\theta) + exp(-2i\phi)sin^2(\theta))\,|1\rangle\langle 0| + 2icos(\theta)sin(\theta)[sin(\phi)]\,|1\rangle\langle 1|$$

Thus, our final answer is

$$\mathbf{Y}^{\sim} = \kappa\,|0\rangle\langle 0| + i\mu\,|0\rangle\langle 1| - i\Gamma\,|1\rangle\langle 0| + \kappa\,|1\rangle\langle 1|,$$

where $\kappa = 2icos(\theta)sin(\theta)[sin(\phi)]$, $\mu = (cos^2(\theta) + exp(2i\phi)sin^2(\theta))$, and $\Gamma = (cos^2(\theta) + exp(-2i\phi)sin^2(\theta))$.

1.19.2 Answer (Part Two)

From problem (1.17), we know that

$$\mathbf{U} = cos\theta\,|0\rangle\langle 0| + exp(i\phi)sin\theta\,|0\rangle\langle 1| + exp(-i\phi)sin\theta\,|1\rangle\langle 0| - cos\theta\,|1\rangle\langle 1|$$

and

$$\mathbf{X}^{\sim} = \alpha\,|0\rangle\langle 0| + \beta\,|0\rangle\langle 1| + \gamma\,|1\rangle\langle 0| - \alpha\,|1\rangle\langle 1|,$$

for $\alpha = sin(\theta)cos(\theta)(2cos(\phi))$, $\beta = -cos^2(\theta) + sin^2(\theta)exp(2i\phi)$, and $\gamma = -cos^2(\theta) + sin^2(\theta)exp(-2i\phi)$. We also know that $\mathbf{Z} = \mathbf{U}(|0\rangle\langle 0| - |1\rangle\langle 1|)\mathbf{U}^\dagger$. We first evaluate the left hand side of our equation:

$$[\mathbf{X}^\sim, \mathbf{Y}^\sim] = X^\sim Y^\sim - Y^\sim X^\sim$$

$$= (\kappa\alpha - i\beta\Gamma)|0\rangle\langle 0| + (i\alpha\mu + \beta\kappa)|0\rangle\langle 1| + (\gamma\kappa + i\alpha\Gamma)|1\rangle\langle 0| + (i\gamma\mu - \alpha\kappa)|1\rangle\langle 1|$$

$$-[(\kappa\alpha + i\mu\gamma)|0\rangle\langle 0| + (-i\mu\alpha + \kappa\beta)|0\rangle\langle 1| + (\kappa\gamma - i\Gamma\alpha)|1\rangle\langle 0| - (i\Gamma\beta + \kappa\alpha)|1\rangle\langle 1|]$$

thus, the left hand side of our equation is:

$$[\mathbf{X}^\sim, \mathbf{Y}^\sim] = -i(\mu\gamma + \beta\Gamma)|0\rangle\langle 0| - 2i\mu\alpha|0\rangle\langle 1| + 2i\Gamma\alpha|1\rangle\langle 0| + i(\mu\gamma + \beta\Gamma)|1\rangle\langle 1|$$

Meanwhile, the right hand-side of the equation is:

$$2i\mathbf{Z}^\sim = 2i\mathbf{U}(|0\rangle\langle 0| - |1\rangle\langle 1|)\mathbf{U}^\dagger$$

$$= 2i((cos^2(\theta) - sin^2(\theta))|0\rangle\langle 0| + 2exp(i\phi)sin(\theta)cos(\theta)|0\rangle\langle 1| +$$

$$2exp(-i\phi)sin(\theta)cos(\theta)|1\rangle\langle 0| + (sin^2(\theta) - cos^2(\theta))|1\rangle\langle 1|)$$

At this point, it is easy to verify that the coefficients of both sides are equivalent to one another. Thus, $[\mathbf{X}^\sim, \mathbf{Y}^\sim] = 2i\mathbf{Z}^\sim$.

1.20 (Question 1.20)

Consider the operator:

$$P = 2\mathbf{n} \otimes^\sim \mathbf{n} \otimes^\sim \mathbf{n} + \mathbb{1} \otimes^\sim \mathbb{1} \otimes^\sim \mathbf{n} - \mathbf{n} \otimes^\sim \mathbb{1} \otimes^\sim \mathbf{n} - \mathbb{1} \otimes^\sim \mathbf{n} \otimes^\sim \mathbf{n}$$

Show that $\mathbf{P}$ is a projection operator

1.20.1 Answer

If $\mathbf{P}$ is a projection operator, then it has the property $PP = P$. Then:

$$PP = (2\mathbf{n} \otimes^\sim \mathbf{n} \otimes^\sim \mathbf{n} + \mathbb{1} \otimes^\sim \mathbb{1} \otimes^\sim \mathbf{n} - \mathbf{n} \otimes^\sim \mathbb{1} \otimes^\sim \mathbf{n} - \mathbb{1} \otimes^\sim \mathbf{n} \otimes^\sim \mathbf{n})$$

$$2\mathbf{n} \otimes^\sim \mathbf{n} \otimes^\sim \mathbf{n} + \mathbb{1} \otimes^\sim \mathbb{1} \otimes^\sim \mathbf{n} - \mathbf{n} \otimes^\sim \mathbb{1} \otimes^\sim \mathbf{n} - \mathbb{1} \otimes^\sim \mathbf{n} \otimes^\sim \mathbf{n}$$

$$= 2\mathbf{n} \otimes^\sim \mathbf{n} \otimes^\sim \mathbf{n} + \mathbb{1} \otimes^\sim \mathbb{1} \otimes^\sim \mathbf{n} - \mathbf{n} \otimes^\sim \mathbb{1} \otimes^\sim \mathbf{n} - \mathbb{1} \otimes^\sim \mathbf{n} \otimes^\sim \mathbf{n}$$

Since $\mathbf{n} \otimes \mathbf{n} = \mathbf{n}$ and the identity operator acting on another operator results in the original operator.