Chapter 4 Problems

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Question 4.1

In the interval $\frac{-L}{2} < x < \frac{L}{2}$, the set of real functions $u_n(x)$, parameterized by integer index n, are said to be orthonormal if

$$\int_{\frac{-L}{2}}^{\frac{L}{2}} u_n(x)u_m(x)dx = \delta_{nm}$$

for all values of n,m. Prove that the functions

$$v_n(x) \equiv \sqrt{\frac{2}{L}}cos(\frac{2\pi nx}{L}) \quad w_m(x) \equiv \sqrt{\frac{2}{L}}sin(\frac{2\pi mx}{L})$$

are orthonormal for positive integer n, m.

Answer (4.1.1)

Suppose $v_n(x)$ is orthonormal to $w_m(x)$, then the two functions are linearly independent and the following equation must be true:

$$v_n(x)w_m(x) = 0$$

for all points of x between $\frac{-L}{2}$ to $\frac{L}{2}$. Now we can evaluate the two functions

$$v_n(x)w_m(x) = \frac{2}{L}cos(\frac{2\pi nx}{L})sin(\frac{2\pi mx}{L})$$

and this must be true for all values of x, so we should integrate from $\frac{-L}{2}$ to $\frac{L}{2}$. The result is:

$$\int_{\frac{-L}{2}}^{\frac{L}{2}} \frac{2}{L}cos(\frac{2\pi nx}{L})sin(\frac{2\pi mx}{L})dx$$

Suppose $n = m$, then we can apply u-substitution to evaluate the integral, which yields:

$$\int_{\frac{-L}{2}}^{\frac{L}{2}} \frac{2}{L}cos(\frac{2\pi nx}{L})sin(\frac{2\pi nx}{L}) = \frac{2}{L}\frac{L}{2\pi m}\int_{\frac{-L}{2}}^{\frac{L}{2}} udu = \frac{1}{2\pi m}[\frac{(\frac{L}{2})^2}{2} - \frac{(\frac{-L}{2})^2}{2}] = 0$$

Which holds true for our original assumption. Now, suppose $n! = m$. Since n and m are positive integers, we know by the coefficient 2π that the result inside cos/sin will *always* be a multiple of 2π, which yields the same result of cos/sin regardless of m, n. Thus, we can infer that the proof for $n = m$ applies to all values of $n! = m$ as well. We can solidify this by making $m = n + i$ for some integer i; then

$$\int_{\frac{-L}{2}}^{\frac{L}{2}} \frac{2}{L}cos(\frac{2\pi nx}{L})sin(\frac{2\pi xn}{L} + \frac{2\pi ix}{L}) \equiv \int_{\frac{-L}{2}}^{\frac{L}{2}} \frac{2}{L}cos(\frac{2\pi nx}{L})sin(\frac{2\pi xn}{L}) = 0$$

Question 4.2

Using the orthonormality conditions proved in problem (4.1), verify identities (4.2).

Answer (4.2.1)

The identities in question are

$$a_0 = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)dx$$

$$a_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)cos(\frac{2\pi nx}{L})dx$$

$$b_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)sin(\frac{2\pi nx}{L})dx$$

I'm not sure how to complete the proof for these identities. Do I take the dot product of a/b??? To prove they are mutually orthonormal?

Question 4.3

Derive relations (4.4). Provide a detailed description of each step in the outline of the proof.

Answer (4.3.1)

Given relations

$$a_0 = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)dx$$

$$a_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)cos(\frac{2\pi nx}{L})dx$$

$$b_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)sin(\frac{2\pi nx}{L})dx,$$

$$exp(\pm 2\pi ix/L) = cos(2\pi x/L) \pm isin(2\pi x/L),$$

and

$$f(x) = \sum_{n=0}^{\infty} a_n cos(\frac{2\pi nx}{L}) + \sum_{n=1}^{\infty} b_n sin(\frac{2\pi nx}{L})$$

We can begin our derivation of expression (4.4). First, we must evaluate a_0 in $f(x)$ to combine coefficients:

$$f(x) = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)dx + \sum_{n=1}^{\infty} a_n cos(\frac{2\pi nx}{L}) + b_n sin(\frac{2\pi nx}{L})$$

Now we must apply Euler's formula, which results in

$$f(x) = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)dx + \sum_{n=1}^{\infty} \frac{1}{2}((a_n + ib_n)exp(-2\pi inx/L) + (a_n - ib_n)exp(2\pi inx/L))$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}((a_n + ib_n)exp(-2\pi inx/L) + (a_n - ib_n)exp(2\pi inx/L))$$

Just like in the book, we can apply the relation $a_n = a_{-n}$ and $b_n = b_{-n}$, which yields

$$f(x) = \sum_{n=0}^{\infty} (a_n + ib_n)exp(-2\pi inx/L))$$

At ths point, it is clear to see that $a_n + ib_n = h_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)cos(\frac{2\pi nx}{L})dx + i\frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)sin(\frac{2\pi nx}{L})dx$ Which is equivalent to

$$h_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)[cos(\frac{2\pi nx}{L}) + isin(\frac{2\pi nx}{L})] = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)exp(i\frac{2\pi nx}{L})$$

Not sure how to 1) Get n to run from $n = -\infty$ to $n = \infty$ in f(x) and how to remove the factor of 2 in h_n.

Question 4.4

Evaluate the Fourier sums in the exercises given in Mathematica Notebook 4.1

Answer (4.4.1)

Don't need to do this

Question 4.5

Evaluate the discrete Fourier transforms (DFT) in the exercises given in Mathematica Notebook 4.2.

Answer (4.5.1)

Don't need to do this

Question 4.6

Using Mathematica, construct the matrix representation of operator **QFT** for a register containing eight qubits. Using that matrix, demonstrate explicitly, that

$$\mathbf{QFT}^{\dagger}\mathbf{QFT} = \mathbf{QFT}\mathbf{QFT}^{\dagger} = \mathbb{1}$$

Answer 4.6.1

Don't need to do this

Question 4.7

Using Mathematica, explicitly verify identities (4.26), for a four-qubit gate.

Answer (4.7.1)

Don't need to do this

Question 4.8

Write a Mathematica script that implements the code, for an arbitrary input state $|j\rangle_2$, illustrated in panel (a) of Fig. 4.2. Construct a two-qubit swap operator, and calculate the action of the swap operator on the output of this gate. Compare your result with that of $\mathbf{QFT}\,|j\rangle_2$.

Answer 4.8.1

Don't need to do this

Question 4.9

Do the exercises in Mathematica Notebook 4.3

Answer (4.9.1)

Don't need to do this

Question 4.10

Write a Mathematica script that implements the oracle function $\mathbf{U}_f$, defined in (4.44), for a five-qubit control register. The oracle function is defined so that

$$f(21) = 1$$

and is zero otherwise. Construct a matrix representation of the oracle function and show that it is unitary.

Answer (4.10.1)

Don't need to do this

Question 4.11

With operator $\mathbf{U}_f$ obtained in problem (4.10), show, explicitly, that identity (4.46) is satisfied.

Answer (4.11.1)

Don't have to do

Question 4.12

For the system defined in problem (4.10), construct the matrix representation of operator $\mathbf{V}$ defined in (4.47). Verify (4.48).

Answer (4.12.1)

Don't have to do

Question 4.13

Construct the matrix representation of operator $\mathbf{H}^{\otimes n}$, where $n = 5$

Answer (4.13.1)

We know that matrix representation of H is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and applying the direct product against itself once results in

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

which we denote H^2. To get H^4, all we have to do is change to operate $H^2 H^2$ on one another, which is

$$H^4 = \begin{pmatrix} H^2 & H^2 & H^2 & H^2 \\ H^2 & -H^2 & H^2 & -H^2 \\ H^2 & H^2 & H^2 & H^2 \\ H^2 & -H^2 & H^2 & -H^2 \end{pmatrix}$$

and finally, we can add another H^5 to create

$$H^5 = H^4 \otimes H = \begin{pmatrix} H^2 & H^2 & H^2 & H^2 & H^2 & H^2 & H^2 & H^2 \\ H^2 & -H^2 & H^2 & -H^2 & H^2 & -H^2 & H^2 & -H^2 \\ H^2 & H^2 & H^2 & H^2 & H^2 & H^2 & H^2 & H^2 \\ H^2 & -H^2 & H^2 & -H^2 & H^2 & -H^2 & H^2 & -H^2 \\ H^2 & H^2 & H^2 & H^2 & -H^2 & -H^2 & -H^2 & -H^2 \\ H^2 & -H^2 & H^2 & -H^2 & -H^2 & H^2 & -H^2 & H^2 \\ H^2 & H^2 & H^2 & H^2 & -H^2 & -H^2 & -H^2 & -H^2 \\ H^2 & -H^2 & H^2 & -H^2 & -H^2 & H^2 & -H^2 & H^2 \end{pmatrix}$$

$$= \begin{pmatrix} H^4 & H^4 \\ H^4 & -H^4 \end{pmatrix}$$

Question 4.14

For the system defined in problem (4.10), construct the ket $|\phi\rangle$ defined in (4.52). Using your previous results for $\mathbf{U}_f$ and $\mathbf{V}$ verify the equality

$$\mathbf{U}_f(\mathbf{H}^{\otimes n} |0\rangle \otimes \mathbf{H}) |1\rangle = \mathbf{V} |\phi\rangle \otimes \mathbf{H} |1\rangle$$

Answer (4.14.1)

Don't need to do

Question 4.15

For the system defined in problem (4.10). Construct the diffusion gate $\mathbf{W}$, and the Grover operator $\mathbf{WV}$.

Answer (4.15.1)

Don't need to do

Question 4.16

With the Grover operator constructed in problem (4.15), verify identities (4.58)

Answer (4.16.1)

Don't need to do

Question 4.17

Using the Grover operator obtained in problem (4.16), construct a table of the probablities $p(\epsilon) = |\langle\epsilon|\,\mathbf{G}\,|\phi\rangle|^2$, and make a plot of $p(\epsilon)$ for all integer $0 \leq \epsilon \leq 2^5$.

Answer (4.17.1)

Don't need to do

Question 4.18

Repeat problem (4.17) for operator $\mathbf{G}^p$ where $p = 1, 2, 3...$ Comment on the trend observed with this data. Is there an optimum choice for p?

Answer (4.18.1)

Don't need to do

Question 4.19

Write a Mathematica script that simulates Grover's algorithm for a ten-qubit register.

Answer (4.19.1)

Don't need to do