Chapter 2 Problems

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Homework 2

Question (2.1)

Do the exercises in Mathematica Notebook 2.1

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Answer (2.1.1)

Question (2.2)

Give the matrix representations of the states $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + exp(i\delta)\,|1\rangle$, and $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - exp(i\beta)\,|1\rangle)$, and their dual.

Answer (2.2.1)

For $|\psi\rangle$, the matrix representation is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ exp(i\delta) \end{bmatrix}$$

and for $\langle\psi|$, the matrix representation is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & exp(-i\delta) \end{bmatrix}$$

The matrix representation for $|\phi\rangle$ is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -exp(i\beta) \end{bmatrix}$$

and for $\langle\phi|$, the matrix representation is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -exp(-i\beta) \end{bmatrix}$$

Question (2.3)

Using the matrices obtained in problem 2.2, evaluate $\langle\phi|\psi\rangle, \langle\psi|\phi\rangle$. Compare your results with that obtained using the methods discussed in Chapter 1.

Answer (2.3.1)

We know that $\langle\phi| = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -exp(-i\beta)\end{bmatrix}$ and $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ exp(i\delta)\end{bmatrix}$. Then, $\langle\phi|\psi\rangle =$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -exp(-i\beta)\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ exp(i\delta)\end{bmatrix} = 1/2(1 - exp(i(\delta - \beta)))$$

Meanwhile $\langle\psi|\phi\rangle =$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & exp(-i\delta)\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ -exp(i\beta)\end{bmatrix} = 1/2(1 - exp(i(\beta - \delta)))$$

The result is exactly the same as using the braket notation method in chapter one. Because of the 1-to-1 correspondence previously explained in this chapter, matrix multiplication can evaluate an inner product described in braket notation.

Question (2.4)

Find the matrix representation for $|\phi\rangle\langle\psi|, |\psi\rangle\langle\phi|$, where $|\psi\rangle, |\phi\rangle$ are defined in problem (2.2).

Answer (2.4.1)

Given $|\phi\rangle\langle\psi|$, the matrix representation is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ -exp(i\beta)\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & exp(-i\delta)\end{bmatrix} = \begin{bmatrix}1 & exp(-i\delta) \\ -exp(i\beta) & -exp(i(\beta - \delta))\end{bmatrix}$$

For the operator $|\psi\rangle\langle\phi|$, the matrix representation is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ exp(i\delta)\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -exp(-i\beta)\end{bmatrix} = \begin{bmatrix}1 & -exp(-i\beta) \\ exp(i\delta) & -exp(i(\delta - \beta))\end{bmatrix}$$

Question (2.5)

Consider the operator

$$\mathbf{O} \equiv |0\rangle\langle 0| + i\,|1\rangle\langle 0| - i\,|0\rangle\langle 1| - |1\rangle\langle 1|$$

(a) Evaluate, using Dirac's method disucssued in chapter 1, $\mathbf{O}|\psi\rangle$ where $|\psi\rangle$ is defined in problem 2.2. (b) Evaluate by re-rexpressing $\mathbf{O}$ and $|\psi\rangle$ as matrices. Show that the results obtained in both pictures are isomorphic to each other.

Answer (2.5.1)

In Dirac's notation the operation

$$(|0\rangle\langle 0| + i|1\rangle\langle 0| - i|0\rangle\langle 1| - |1\rangle\langle 1|)|\psi\rangle = \frac{1}{\sqrt{2}}((1 - iexp(i\delta))|0\rangle + (-exp(i\delta) + i)|1\rangle)$$

Answer (2.5.2)

In matrix notation, $\mathbf{0} \equiv \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ and we will use the matrix interpretation derived in (2.2). Then

$$\mathbf{O}|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 + iexp(i\delta) \\ -i - exp(i\delta) \end{bmatrix}$$

Question (2.6)

Identify the following states on the Bloch sphere surface:

1. $|\psi_1\rangle = \frac{i}{\sqrt{10}}|0\rangle - \frac{3}{\sqrt{10}}|1\rangle$
2. $|\psi_2\rangle = exp(i\pi/4)|0\rangle$
3. $|\psi_3\rangle = \frac{i}{\sqrt{2}}(|0\rangle - |1\rangle)$

Answer (2.6.1)

Let us define c_1 and c_2 as the complex coefficients for the states $|0\rangle$ and $|1\rangle$. Then **QUESTION: Do I just identify X, Y, and Z coordinates on Bloch Sphere?**

$$c_1 = 0 + \frac{i}{\sqrt{10}}$$

and

$$c_2 = -\frac{3}{\sqrt{10}} + i0$$

Using the Hopf map we described in section (2.1.2), we can define the X, Y, and Z coordinates on the Bloch sphere to be:

$$X = 0$$

$$Y = -\frac{6}{10},$$

and

$$Z = \frac{1}{10} - \frac{9}{10} = -\frac{8}{10}$$

Answer (2.6.2)

Using the same logic as the previous answer, let us define c_1 and c_2 as the complex coefficients for the states $|0\rangle$ and $|1\rangle$. Then

$$c_1 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

and

$$c_2 = 0$$

Then we convert using the equations described in (2.16)

$$X = 0,$$

$$Y = 0,$$

and

$$Z = 1$$

Answer (2.6.3)

Using the same logic as the previous answer, let us define c_1 and c_2 as the complex coefficients for the states $|0\rangle$ and $|1\rangle$. Then

$$c_1 = \frac{i}{\sqrt{2}}$$

and

$$c_2 = -\frac{i}{\sqrt{2}}$$

Then we convert using the equations described in (2.16)

$$X = 1,$$

$$Y = 0,$$

and

$$Z = 0$$

This concludes the solutions to (2.6)

Question (2.7)

Using the matrix representations for the Pauli matrices, verify identities (2.25).

Answer (2.7.1)

We understand that the pauli matrices are

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

and

$$\sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We also understand that the identities in (2.25) are

$$\sigma_X\sigma_Y - \sigma_Y\sigma_X = 2i\sigma_Z,$$

$$\sigma_Y\sigma_Z - \sigma_Z\sigma_Y = 2i\sigma_X,$$

and

$$\sigma_Z\sigma_X - \sigma_X\sigma_Z = 2i\sigma_Y,$$

Now we begin by operating each equation on the left-hand side and simplifying it to match the right-hand side.

Answer (2.7.1.1)

The operation $\sigma_X\sigma_Y$ is described below as:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

meanwhile, $\sigma_Y\sigma_X$ is described below as:

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

So the result is

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$$

The right-hand side of the equation is

$$2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It is easy to see that equation 1 holds.

Answer (2.7.1.2)

Using similar logic as the previous equation, we will evaluate the left hand side of the equation, $\sigma_Y \sigma_Z - \sigma_Z \sigma_Y$:

The operation $\sigma_Y \sigma_Z$ is

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

and the operation $\sigma_Z \sigma_Y$ is:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

Thus, the equation $\sigma_Y \sigma_Z - \sigma_Z \sigma_Y$ results in:

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = 2i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2i\sigma_X$$

Thus, equation 2 holds.

Answer (2.7.1.3)

Finally, we have equation 3, $\sigma_X \sigma_Z - \sigma_Z \sigma_X$, which we can evaluate:

The operation $\sigma_X \sigma_Z$ is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and the operation $\sigma_Z \sigma_X$ is:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Thus, equation 3 holds. This concludes the solution for (2.7)

Question (2.8)

Given the matrix

$$\begin{bmatrix} 4 & -i\pi \\ 2exp(i\pi/4) & 3 \end{bmatrix}$$

show that it can be expressed in the form (2.24), by identifying the values of the parameters α, β, b, c

Answer (2.8.1)

We understand that equation (2.24) is

$$b\sigma_X + c\sigma_Y + \alpha\sigma_Z + \beta\mathbb{1},$$

where $\alpha = (a-d)/2$, $\beta = (a+d)/2$ and, $\mathbb{1}$ is the 2×2 identity matrix. This equation stems from the 2×2 matrix of the form

$$\begin{bmatrix} a & b-ic \\ b+ic & d \end{bmatrix}$$

Clearly, $a = 4$ and $d = 3$, so $\alpha = \frac{1}{2}$ and $\beta = \frac{7}{2}$. Now we can solve for b and c by equating the matrix above with our original values as such:

$$2exp(i\pi/4) = b + ic$$

and

$$-i\pi = b - ic$$

From the second equation, $b = i(c-\pi)$. Simplifying the first equation and using substitution, we get
$2exp(i\pi/4) = \sqrt{2} + i\sqrt{2} = i(c-\pi) + ic = i(2c-\pi)$
Now we can solve for c:

$$\sqrt{2} + i\sqrt{2} = i(2c-\pi) \longrightarrow i\sqrt{2} - \sqrt{2} = 2c - \pi \longrightarrow c = \frac{-\sqrt{2}+\pi}{2} - i\frac{\sqrt{2}}{2}$$

As a result, b is:

$$b = i(c-\pi) = i\frac{-\sqrt{2}+\pi}{2} + \frac{\sqrt{2}}{2} - i\pi = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}+\pi}{2}$$

We understand at this point that equation (2.24) is equivalent to the simplified matrix containing variables a, b,c, and d. From our results, we know that our given matrix can be represented by the components of the simplified matrix. Thus, our given matrix can be represented as:

$$(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}+\pi}{2})\sigma_X + (\frac{-\sqrt{2}+\pi}{2} - i\frac{\sqrt{2}}{2})\sigma_Y + \frac{1}{2}\sigma_Z + \frac{7}{2}\mathbb{1}$$

It is easily verifiable that the equation above and our original matrix are equal. This concludes problem (2.8).

Question (2.9)

Find the conjugate transpose $\mathbf{U}^\dagger$ of expression (2.27). Evaluate the matrix product $\mathbf{U}\mathbf{U}^\dagger$ to confirm that $\mathbf{U}$ is unitary.

Answer (2.9.1)

From expression (2.27), we know that $\mathbf{U} = exp(i\gamma)\begin{bmatrix} exp(i\phi)cos(\theta) & exp(i\beta)sin(\theta) \\ -exp(-i\beta)sin(\theta) & exp(-i\phi)cos(\theta) \end{bmatrix}$, where γ, ϕ, β, θ are real numbers. The conjugate transpose (or the adjoint) of $\mathbf{U}$ is:

$$\mathbf{U}^\dagger = exp(-i\gamma)\begin{bmatrix} exp(-i\phi)cos(\theta) & -exp(i\beta)sin(\theta) \\ exp(-i\beta)sin(\theta) & exp(i\phi)cos(\theta) \end{bmatrix}$$

Now, we evaluate $\mathbf{U}\mathbf{U}^\dagger$ so confirm that $\mathbf{U}$ is unitary:

$$\mathbf{U}\mathbf{U}^\dagger = \begin{bmatrix} exp(i\phi)cos(\theta) & exp(i\beta)sin(\theta) \\ -exp(-i\beta)sin(\theta) & exp(-i\phi)cos(\theta) \end{bmatrix}\begin{bmatrix} exp(-i\phi)cos(\theta) & -exp(i\beta)sin(\theta) \\ exp(-i\beta)sin(\theta) & exp(i\phi)cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & exp(i(\phi+\beta))[cos(\theta)sin(\theta) - sin(\theta)cos(\theta)] \\ exp(-i(\beta+\phi))[sin(\theta)cos(\theta) - cos(\theta)sin(\theta)] & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}$$

Thus, $\mathbf{U}$ from expression (2.27) is a unitary matrix.

Question (2.10)

Use mathematica Notebook 2.3 to exponentiate the operators $\sigma_x, \sigma_y, \sigma_z$ as defined in Eqs. (2.31) and (2.32). Using these results to confirm relation (2.33).

Answer (2.10.1)

Question (2.11)

Use mathematica Notebook 2.3 to construct the operator

$$\mathbf{W} = \mathbf{U}_Z(\phi/2)\mathbf{U}_Y(\theta/2)\mathbf{U}_Z(-\phi/2)$$

Demonstrate that $\mathbf{W}$ is unitary.

Answer (2.11.1)

Question (2.12)

Use the operator that you obtained in problem 2.11, to evaluate the following, (a)$\mathbf{W}^\dagger\sigma_X\mathbf{W}$, (b) $\mathbf{W}^\dagger\sigma_Y\mathbf{W}$, (c) $\mathbf{W}^\dagger\sigma_Z\mathbf{W}$. Comment on your results.

Answer (2.12.1)

Given the matrix $\mathbf{W}$ is:

$$\begin{bmatrix} cos(\theta/2) & exp(i\phi)sin(\theta/2) \\ -exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix}$$

we can solve for each portion of the question:

Answer (2.12.1.1)

The equation $\mathbf{W}^\dagger \sigma_X \mathbf{W}$ is represented as:

$$\mathbf{W}^\dagger \sigma_X \mathbf{W} = \begin{bmatrix} cos(\theta/2) & -exp(i\phi)sin(\theta/2) \\ exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} cos(\theta/2) & exp(i\phi)sin(\theta/2) \\ -exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix}$$

which is equivalent to:

$$\mathbf{W}^\dagger \sigma_X \mathbf{W} = \begin{bmatrix} -[exp(i\phi) + exp(-i\phi)]sin(\theta/2)cos(\theta/2) & cos^2(\theta/2) - exp(2i\phi)sin^2(\theta/2) \\ cos^2(\theta/2) - exp(-2i\phi)sin^2(\theta/2) & [exp(i\phi) + exp(-i\phi)]sin(\theta/2)cos(\theta/2) \end{bmatrix}$$

Answer (2.12.1.2)

The equation $\mathbf{W}^\dagger \sigma_Y \mathbf{W}$ is represented as:

$$\mathbf{W}^\dagger \sigma_Y \mathbf{W} = \begin{bmatrix} cos(\theta/2) & -exp(i\phi)sin(\theta/2) \\ exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} cos(\theta/2) & exp(i\phi)sin(\theta/2) \\ -exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix}$$

Which is equivalent to:

$$\mathbf{W}^\dagger \sigma_Y \mathbf{W} = \begin{bmatrix} icos(\theta/2)sin(\theta/2)[exp(-i\phi) - exp(i\phi)] & -iexp(2i\phi)sin^2(\theta/2) - icos^2(\theta) \\ iexp(-2i\phi)sin^2(\theta/2) + icos^2(\theta) & icos(\theta/2)sin(\theta/2)[exp(i\phi) - exp(-i\phi)] \end{bmatrix}$$

Answer (2.12.1.3)

The equation $\mathbf{W}^\dagger \sigma_Z \mathbf{W}$ is represented as:

$$\mathbf{W}^\dagger \sigma_Z \mathbf{W} = \begin{bmatrix} cos(\theta/2) & -exp(i\phi)sin(\theta/2) \\ exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} cos(\theta/2) & exp(i\phi)sin(\theta/2) \\ -exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix}$$

Which is equivalent to

$$\mathbf{W}^\dagger \sigma_Z \mathbf{W} = \begin{bmatrix} cos^2(\theta/2) + sin^2(\theta) & 2exp(i\phi)cos(\theta/2)sin(\theta/2) \\ 2exp(-i\phi)cos(\theta/2)sin(\theta/2) & -cos^2(\theta/2) + sin^2(\theta) \end{bmatrix}$$

Question (2.13)

Consider the operator $\mathbf{A} = \mathbf{W}\sigma_\mathbf{X}\mathbf{W}^\dagger$, where $\mathbf{W}$ is the operator defined in problem (2.11), find the eigenvalues and eigenstates of $\mathbf{A}$.

Answer (2.13.1)

Given that $\mathbf{W} = \begin{bmatrix} cos(\theta/2) & exp(i\phi)sin(\theta/2) \\ -exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix}$, the expression, $\mathbf{W}\sigma_{\mathbf{X}}\mathbf{W}^{\dagger}$ is:

$$\mathbf{A} = \begin{bmatrix} cos(\theta/2) & exp(i\phi)sin(\theta/2) \\ -exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} cos(\theta/2) & -exp(i\phi)sin(\theta/2) \\ exp(-i\phi)sin(\theta/2) & cos(\theta/2) \end{bmatrix}$$

Which is equivalent to:

$$\mathbf{A} = \begin{bmatrix} sin(\theta/2)cos(\theta/2)[exp(i\phi) + exp(-i\phi)] & cos^2(\theta/2) - exp(2i\phi)sin^2(\theta/2) \\ cos^2(\theta/2) - exp(-2i\phi)sin^2(\theta/2) & -sin(\theta/2)cos(\theta/2)[exp(i\phi) + exp(-i\phi)] \end{bmatrix}$$

At this point, we can solve for the eigenvalues and eigenstates of the operator. Let us begin with the equation $Ax = \lambda x$, where x is a 2×1 vector and λ is a real number. Let $x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and, for simplicity, let $\mathbf{A} = \begin{bmatrix} a & b \\ b^* & -a \end{bmatrix}$, where $a = sin(\theta/2)cos(\theta/2)[exp(i\phi) + exp(-i\phi)]$ and $b = cos^2(\theta/2) - exp(2i\phi)sin^2(\theta/2)$. Then

$$Ax - \lambda x = \begin{bmatrix} c_1 a + c_2 b \\ c_1 b^* - c_2 a \end{bmatrix} - \begin{bmatrix} c_1\lambda \\ c_2\lambda \end{bmatrix} = \begin{bmatrix} c_1(a-\lambda) + c_2 b \\ c_1 b^* - c_2(a-\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We know the states $c_1(a-\lambda) + c_2 b$ and $c_1 b^* - c_2(a-\lambda)$ are linearly independent, which means we can separate them and solve for λ:

$$c_1(a-\lambda) + c_2 b = 0$$

and

$$c_1 b^* - c_2(a-\lambda) = 0$$

Using substitution in the first equation, we find that $c_2 = \frac{-c_1(a-\lambda)}{b}$. Then the second equation becomes

$$c_1 b^* + c_1(a-\lambda)^2 = 0$$

Clearly, $\lambda = a - \sqrt{b^*}, a + \sqrt{b^*}$, and we have found our eigenvalues. For $\lambda = a - \sqrt{b^*}$ we get:

$$0 = c_1 b^* - c_2(a - (a - \sqrt{b^*})) = c_1 b^* - c_2\sqrt{b^*} \longrightarrow c_2 = c_1\sqrt{b^*}$$

Choose $c_1 = \sqrt{b^*}$, then $c_2 = b^*$ and our eigenstate is:

$$x = \begin{bmatrix} \sqrt{b^*} \\ b^* \end{bmatrix}$$

for $\lambda = a - \sqrt{b^*}$. Meanwhile $\lambda = a + \sqrt{b^*}$ yields the equation:

$$c_1 b^* + c_2(\sqrt{b^*}) \longrightarrow c_2 = -c_1\sqrt{b^*}$$

Choose $c_1 = -1$, then $c_2 = \sqrt{b^*}$ and the eigenstate is:

$$x = \begin{bmatrix} -1 \\ \sqrt{b^*} \end{bmatrix}$$

for $\lambda = a + \sqrt{b^*}$.

Question (2.14)

Find the eigenvalues and eigenstates of operator

$$\mathbf{A} = \begin{bmatrix} a & \sqrt{2} + i\sqrt{2} \\ \sqrt{2} - i\sqrt{2} & a \end{bmatrix}$$

where a is a real number.

Answer (2.14.1)

Similar to question (2.13) we will evoke the equation $Ax = \lambda x$, where $x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and λ is a real number. Then

$$Ax - \lambda x = \begin{bmatrix} ac_1 + (\sqrt{2} + i\sqrt{2})c_2 \\ (\sqrt{2} - i\sqrt{2})c_1 + ac_2 \end{bmatrix} - \begin{bmatrix} c_1\lambda \\ c_2\lambda \end{bmatrix} = \begin{bmatrix} c_1(a-\lambda) + (\sqrt{2} + i\sqrt{2})c_2 \\ (\sqrt{2} - i\sqrt{2})c_1 + c_2(a-\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This simplified matrix implies the equations

$$c_1(a-\lambda) + (\sqrt{2} + i\sqrt{2})c_2 = 0$$

and

$$(\sqrt{2} - i\sqrt{2})c_1 + c_2(a-\lambda) = 0$$

which means we can now solve for λ. Solving for c_2 in the first equation, we get $c_2 = \frac{-c_1(a-\lambda)}{\sqrt{2}+i\sqrt{2}}$. Using substitution in the second equation, our equation yields:

$$(\sqrt{2} - i\sqrt{2})c_1 + \frac{-c_1(a-\lambda)^2}{\sqrt{2}+i\sqrt{2}} \longrightarrow 4 = (a-\lambda)^2 \longrightarrow \lambda = a-2, a+2$$

Now that we have our eigenvalues let us find the corresponding eigenstates. For $\lambda = a - 2$, our equation becomes

$$2c_1 + (\sqrt{2} + i\sqrt{2})c_2 = 0$$

Choose $c_2 = \sqrt{2}$, then $c_1 = -1 - i$ and our eigenstate is:

$$x = \begin{bmatrix} -1-i \\ \sqrt{2} \end{bmatrix}$$

For $\lambda = a - 2$. The eigenstate for $\lambda = a + 2$ derived below:

$$-2c_1 + (\sqrt{2} + i\sqrt{2})c_2 = 0$$

Choose $c_2 = 1$, then $c_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$. The eigenstate for $\lambda = a + 2$ is:

$$x = \begin{bmatrix} \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

Question (2.15)

Use mathematica Notebook 2.4 to plot, as a function of time, the electric field given by expression (2.37), for values of the parameters (a) $E_0 = 1, \delta = 0, \delta_0 = \pi, \theta = \pi/2$ (b) $E_0 = 1, \delta = 0, \delta_0 = 0, \theta = 0$, (c) $E_0 = 1, \delta = 0, \delta_0 = 0, \theta = \pi/4$

Answer (2.15.1)

Question (2.16)

Given the state $|\psi\rangle = \sqrt{\frac{3}{8}}\,|0\rangle + \sqrt{\frac{5}{8}}exp(i\pi/4)\,|1\rangle$. Find the standard deviation of measurements with the operators (a) σ_X (b) σ_Y (c) σ_Z

Answer (2.16.1)

Recall that the standard deviation equation is $\sigma = \sqrt{\bar{x^2} - \bar{x}^2}$, where $\bar{x} = \sum_i p_i x_i$. Now we may proceed to answer each part.

Answer (2.16.1.1)

The measurement of $|\psi\rangle$ with σ_X yields:

$$\sigma_X\,|\psi\rangle = \begin{bmatrix} \sqrt{\frac{5}{8}}exp(i\pi/4) \\ \sqrt{\frac{3}{8}} \end{bmatrix}$$

We need to know the eigenvalues of this matrix, so we assume the form:

$$\sigma_X\,|\psi\rangle - \lambda\,|\psi\rangle = \begin{bmatrix} \sqrt{\frac{5}{8}}exp(i\pi/4) - \lambda\sqrt{\frac{3}{8}} \\ \sqrt{\frac{3}{8}} - \lambda\sqrt{\frac{5}{8}}exp(i\pi/4) \end{bmatrix} = 0$$

Clearly $\lambda = \sqrt{\frac{5}{3}}exp(i\pi/4) = \sqrt{\frac{5}{6}} + i\sqrt{\frac{5}{6}}$ and $\lambda = \sqrt{\frac{3}{5exp(i\pi/4)}} = \sqrt{\frac{3}{\frac{5}{\sqrt{2}} + i\frac{5}{\sqrt{2}}}}$

For $\lambda = \sqrt{\frac{5}{6}} + i\sqrt{\frac{5}{6}}$, our eigenstate is:

$$\begin{bmatrix} \sqrt{\frac{5}{8}}exp(i\pi/4) - \sqrt{\frac{5}{8}}exp(i\pi/4) \\ \sqrt{\frac{3}{8}} - \sqrt{\frac{25}{24}}exp(i\pi/4) \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{\frac{3}{8}} - \sqrt{\frac{7}{48}} + i\sqrt{\frac{25}{48}} \end{bmatrix}$$

This means that $\bar{x} = \frac{5}{8}$

Question (2.17)

Find the matrix representation for the following multi-qubit kets.

1. $|1\rangle \otimes |1\rangle \otimes |0\rangle$
2. $|1\rangle \otimes |0\rangle \otimes |0\rangle$
3. $|1\rangle \otimes (|1\rangle - |0\rangle) \otimes |0\rangle$
4. $|1\rangle \otimes (|1\rangle - |0\rangle) \otimes (|1\rangle + |0\rangle)$

Answer (2.17.1)

By the rule stated in expression (2.54), $|1\rangle \otimes |1\rangle \otimes |0\rangle$ is:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Answer (2.17.2)

Similarly to part 1, we use expression (2.54) to find $|1\rangle \otimes |0\rangle \otimes |0\rangle$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer (2.17.3)

Use expression (2.54) to find $|1\rangle \otimes (|1\rangle - |0\rangle) \otimes |0\rangle$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We use expression (2.54) twice – once for $-|0\rangle$ and once for $|1\rangle$ – to solve for our answer.

Answer (2.17.4)

Use expression (2.54) to find $|1\rangle \otimes (|1\rangle - |0\rangle) \otimes (|1\rangle + |0\rangle)$. Similarly to Part 3. we must use the expression four times:

$$\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\-1\\-1\\1\\1\end{bmatrix}$$

Question (2.18)

Find the matrix representation of the following operators. (a) $\sigma_X \otimes \mathbb{1}$, (b) $\mathbb{1} \otimes \sigma_X$, (c) σ_X, (d) $\sigma_X \sigma_X$

Answer (2.18.1)

This time we use expression (2.57) to solve for the operator direct product:

Answer (2.18.1.1)

The expression $\sigma_X \otimes \mathbb{1}$ is represented in matrix form as:

$$\sigma_X \otimes \mathbb{1} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}$$

Answer (2.18.1.2)

The expression $\mathbb{1} \otimes \sigma_X$ is represented in matrix form as:

$$\mathbb{1} \otimes \sigma_X = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \otimes \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}$$

Answer (2.18.1.3)

The expression σ_X is expressed in matrix form as:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Answer (2.18.1.4)

The expression $\sigma_X \sigma_X$ is expressed in matrix notation as:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Question (2.19)

Find the matrix representation of the operator,

$$\frac{1}{2}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2}\sigma_Z \otimes \mathbb{1} + \frac{1}{2}\mathbb{1} \otimes \sigma_X - \frac{1}{2}\sigma_Z \otimes \sigma_X$$

Answer (2.19.1)

We will begin solving for the matrix representation by solving for each component of the operator. Let us begin with $\frac{1}{2}\mathbb{1} \otimes \mathbb{1}$:

$$\frac{1}{2}\mathbb{1} \otimes \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The next component is $\frac{1}{2}\sigma_Z \otimes \mathbb{1}$, which is represented as:

$$\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The next component is $\frac{1}{2}\mathbb{1} \otimes \sigma_X$, which is represented as:

$$\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Finally, the last component, $-\frac{1}{2}\sigma_Z \otimes \sigma_X$, is represented as:

$$-\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

The sum of every component is:

$$\frac{1}{2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{bmatrix}=\frac{1}{2}\mathbb{1}\otimes\mathbb{1}+\frac{1}{2}\sigma_Z\otimes\mathbb{1}+\frac{1}{2}\mathbb{1}\otimes\sigma_X-\frac{1}{2}\sigma_Z\otimes\sigma_X$$

Question (2.20)

Using the definition of the Kronecker product of matrices, verify (2.58) for arbitrary one-qubit operators, **A**,**B** and state $|\psi\rangle, |\phi\rangle$

Answer (2.20.1)

For operator $\mathbf{A}=\begin{bmatrix}a_0&a_1\\a_2&a_3\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}b_0&b_1\\b_2&b_3\end{bmatrix}$ the kronecker product between the two is:

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix}a_0\mathbf{B}&a_1\mathbf{B}\\a_2\mathbf{B}&a_2\mathbf{B}\end{bmatrix}=\begin{bmatrix}a_0b_0&a_0b_1&a_1b_0&a_1b_1\\a_0b_2&a_0b_3&a_1b_2&a_1b_2\\a_2b_0&a_2b_1&a_3b_0&a_3b_1\\a_2b_2&a_2b_3&a_3b_2&a_3b_3\end{bmatrix}$$

For the one-qubit operators, $|\psi\rangle=\begin{bmatrix}c_0\\c_1\end{bmatrix}$ and $|\phi\rangle=\begin{bmatrix}d_0\\d_1\end{bmatrix}$, the kronecker product is:

$$|\psi\rangle\otimes|\phi\rangle=\begin{bmatrix}c_0d_0\\c_0d_1\\c_1d_0\\c_1d_1\end{bmatrix}$$

So the equation, $(\mathbf{A}\otimes\mathbf{B})(|\psi\rangle\otimes|\phi\rangle)$ is:

$$\begin{bmatrix}a_0b_0&a_0b_1&a_1b_0&a_1b_1\\a_0b_2&a_0b_3&a_1b_2&a_1b_2\\a_2b_0&a_2b_1&a_3b_0&a_3b_1\\a_2b_2&a_2b_3&a_3b_2&a_3b_3\end{bmatrix}\begin{bmatrix}c_0d_0\\c_0d_1\\c_1d_0\\c_1d_1\end{bmatrix}=\begin{bmatrix}c_0d_0a_0b_0+c_0d_1a_0b_1+c_1d_0a_1b_0+c_1d_1a_1b_1\\c_0d_0a_0b_2+c_0d_1a_0b_3+c_1d_0a_1b_2+c_1d_1a_1b_2\\c_0d_0a_2b_0+c_0d_1a_2b_1+c_1d_0a_3b_0+c_1d_1a_3b_1\\c_0d_0a_2b_2+c_0d_1a_2b_3+c_1d_0a_3b_2c_1d_1a_3b_3\end{bmatrix}$$

Now we solve the right hand side of the equation. It is easy to see that

$$\mathbf{A}|\psi\rangle=\begin{bmatrix}a_0c_0+a_1c_1\\a_2c_0+a_3c_1\end{bmatrix}$$

and

$$\mathbf{B}\left|\phi\right\rangle = \begin{bmatrix} b_0d_0 + b_1d_1 \\ b_2d_0 + b_3d_1 \end{bmatrix}$$

So the kronecker product is:

$$\mathbf{A}\left|\psi\right\rangle \otimes \mathbf{b}\left|\phi\right\rangle = \begin{bmatrix} a_0c_0\mathbf{B}\left|\phi\right\rangle + a_1c_1\mathbf{B}\left|\phi\right\rangle \\ a_2c_0\mathbf{B}\left|\phi\right\rangle + a_3c_1\mathbf{B}\left|\phi\right\rangle \end{bmatrix} = \begin{bmatrix} c_0d_0a_0b_0 + c_0d_1a_0b_1 + c_1d_0a_1b_0 + c_1d_1a_1b_1 \\ c_0d_0a_0b_2 + c_0d_1a_0b_3 + c_1d_0a_1b_2 + c_1d_1a_1b_2 \\ c_0d_0a_2b_0 + c_0d_1a_2b_1 + c_1d_0a_3b_0 + c_1d_1a_3b_1 \\ c_0d_0a_2b_2 + c_0d_1a_2b_3 + c_1d_0a_3b_2c_1d_1a_3b_3 \end{bmatrix}$$

Thus, the equation (2.58) holds true.