# Chapter 5 Problems

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## Question 5.1

For Scenario $(A)$ introduced in Sect. 5.2, evaluate the following expectation values

$$< \sigma_X >, < \sigma_Y >, < \sigma_Z >, < \sigma_X \sigma_Z >$$

### Answer 5.1.1

We know that the expectation value can be found by

$$Tr[\rho A] =$$

In scenario A, the scientists contain half of the states of a qubit in $|1\rangle$ and half in $|0\rangle$. Thus,$\rho$ is

$$\rho = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We acknowledge that $\rho \equiv I$, so we can simply take the trace of the matrices we are evaluating. For our equation, $< \sigma_X >$:

$$< \sigma_X >= Tr[\sigma_X] = 0 + 0 = 0$$

for $< \sigma_Y >$:

$$< \sigma_Y >= Tr[\sigma_Y] = 0 + 0 = 0$$

for $< \sigma_Z >$, we have

$$< \sigma_Z >= Tr[\sigma_Z] = 1 - 1 = 0$$

and, finally, for $< \sigma_X \sigma_Z >=< \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} >$

$$Tr[\sigma_X \sigma_Z] = Tr[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}] = 0 + 0 = 0$$

## Question 5.2

For scenario (B) introduced in Sect 5.2 evaluate the following expectation values

$$< \sigma_X >, < \sigma_Y >, < \sigma_Z >, < \sigma_X \sigma_Z >$$

### Answer 5.2.1

We know that the expectation value can be found by

$$Tr[\rho A] =$$

where $A$ is the matrix of which we hope to find the expectation value, and $\rho$ is characterized by scenario $B$ as

$$\rho = |U\rangle \langle U| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

We are now able to solve for all portions of the question. For $< \sigma_X >$, our result is

$$< \sigma_X >= Tr[\rho\sigma_X] = Tr[\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] = 2\frac{1}{2} = 1$$

For $< \sigma_Y >$,

$$< \sigma_Y >= Tr[\rho\sigma_Y] = Tr[\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}] = i - i\frac{1}{2} = 0$$

For $< \sigma_Z >$

$$< \sigma_Z >= Tr[\rho\sigma_Z] = Tr[\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}] = 1 - 1\frac{1}{2} = 0$$

and, finally, for $< \sigma_X\sigma_Z >= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$< \sigma_X\sigma_Z >= Tr[\rho\sigma_X\sigma_Z] = \frac{1}{2}Tr[\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}] = \frac{1}{2}(1-1) = 0$$

## Question 5.3

Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - exp(-i\frac{\pi}{2})\,|1\rangle)$$

(a) Construct the density operator expressed in bra-ket notation.

(b) Using the computational basis, construct the density matrix for this state.

### Answer 5.3.a

To construct the density operator we simply take the outer product of each component of $|\psi\rangle$ such that

$$\rho = p_1\,|0\rangle\,\langle 0| + p_2\,|1\rangle\,\langle 1|$$

Where $p_1 + p_2 = 1$. To find the probabilities $p_i$, we simply take the absolute value squared, which evaluates to

$$\rho = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1|$$

### Answer 5.3.b

The matrix form of our density matrix is simply

$$\begin{pmatrix} \langle 0| \rho |0\rangle & \langle 0| \rho |1\rangle \\ \langle 1| \rho |0\rangle & \langle 1| \rho |1\rangle \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## Question 5.4

Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - exp(-i\frac{\pi}{2}) |1\rangle)$$

(a) Construct the density operator $\rho$ expressed in bra-ket notation and evaluate $\rho\rho$.

(b)Using the computation basis construct the density matrix $\rho$ and evaluate the matrix product $\rho\rho$. Evaluate the trace of $\rho^2$.

### Answer 5.4.a

The density operator is

$$\rho = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|)$$

and $\rho\rho$ is

$$\rho\rho = \frac{1}{4}[(|0\rangle \langle 0| + |1\rangle \langle 1|)(|0\rangle \langle 0| + |1\rangle \langle 1|)] = \frac{1}{4}(|0\rangle \langle 0| + |1\rangle \langle 1|) = \frac{1}{2}\rho$$

### Answer 5.4.b

The matrix version of $\rho$ is

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $pp = p^2$ is

$$\rho^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The trace of $\rho^2$ is

$$Tr[\rho^2] = \frac{1}{4} Tr[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}] = \frac{1}{4}(1 + 1) = \frac{1}{2}$$

## Question 5.5

Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - exp(-i\frac{\pi}{2})\,|1\rangle)$$

Using basis $|u\rangle$ $|v\rangle$ vectors, defined in (2.20), construct the density matrix $\rho$ and evaluate the matrix product $\rho\rho$. Evaluate the trace of $\rho^2$.

### Answer 5.5.1

We know by question question (3) that

$$\rho = \frac{1}{2}\,|0\rangle\,\langle 0| + \frac{1}{2}\,|1\rangle\,\langle 1|$$

and we also know that $|u\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|v\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Thus $\frac{1}{2}(|u\rangle\,\langle u| + |v\rangle\,\langle v|)$ is a good choice:

$$\rho = \frac{1}{2}(|u\rangle\,\langle u| + |v\rangle\,\langle v|) = \frac{1}{4}(2\,|0\rangle\,\langle 0| + 2\,|1\rangle\,\langle 1|) = \frac{1}{2}\,|0\rangle\,\langle 0| + \frac{1}{2}\,|1\rangle\,\langle 1|$$

Evaluating the trace $\rho^2$ simply results in

$$Tr[\rho^2] = Tr[\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}] = Tr[\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}] = \frac{1}{2}$$

## Question 5.6

Given a qubit in state $|\psi(t_0)\rangle = |0\rangle$ at time $t_0 = 0$, and the Hamiltonian operator $\mathbf{H} = \hat{h}\sigma_X$, find $|\psi(t)\rangle$ for $t > 0$. Construct the time dependent density operator.

$$\rho(t) = |\psi(t)\rangle\,\langle\psi(t)|$$

Find the density matrix with respect to basis $|0\rangle$, $|1\rangle$.

### Answer 5.6.1

We understand that, for $t > 0$, $|\psi(t)\rangle = \mathbf{U}(t, t_0)\,|\psi(t_0)\rangle = \mathbf{U}(t, t_0)\,|0\rangle$, where $t$ is the time at $t > (t_0 = 0)$. Given that the Hamiltonian is time-independent (due to the cancellation of $\hbar$), then

$$U(t, t_0) = exp(-i(t - t_0)\sigma_x)$$
$$= \begin{pmatrix} 0 & exp(-i(t - t_0)) \\ exp(-i(t - t_0)) & 0 \end{pmatrix}$$

This implies that

$$|\psi(t)\rangle = \begin{pmatrix} 0 & exp(-i(t-t_0)) \\ exp(-i(t-t_0)) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ exp(-i\Delta t) \end{pmatrix}$$

,

where $\Delta t = t - t_0$ The time-dependent density operator then looks like

$$\rho(t) = exp(-i\Delta t) |1\rangle \langle 1| exp(i\Delta t) = |1\rangle \langle 1|$$

# Question 5.7

Given the Hamiltonian $\mathbf{H} = \hat{h}\sigma_Z$, and state $|\psi(t_0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, evaluate, in the computational basis, density matrix

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0)$$

where $U(t, t_0)$ is the time development operator. You can assume that $t_0 = 0$.

## Answer 5.7.1

We know that, since the Hamiltonian is time-independent:

$$U(t, t_0) = exp(-i(t-t_0)\sigma_Z) = \begin{pmatrix} exp(-i\tau) & 0 \\ 0 & -exp(-i\tau) \end{pmatrix}$$

Where $\tau = t - t_0$. We also know that the density operator at $t_0$ is

$$\rho(t_0) = |\psi(t_0)\rangle \langle \psi(t_0)| = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) = \frac{1}{2}(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|)$$

When we plug these two values into the density matrix at $t > t_0$, then

$$\rho(t) = \begin{pmatrix} exp(-i\tau) & 0 \\ 0 & -exp(-i\tau) \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} exp(i\tau) & 0 \\ 0 & -exp(i\tau) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} exp(-i\tau) & 0 \\ 0 & -exp(-i\tau) \end{pmatrix} \begin{pmatrix} exp(i\tau) & 0 \\ 0 & -exp(i\tau) \end{pmatrix} = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|)$$

# Question 5.8

A system's time evolution is governed by the time-independent Hamiltonian $\mathcal{H}$. Show that the density operator $\rho(t)$ obeys the following first order differential equation

$$\frac{d\rho(t)}{dt} = \frac{1}{i\hat{h}}[\mathcal{H}, \rho(t)]$$

### Answer 5.8.1

We understand that $\rho(t)$ is represented as

$$\rho(t) = \mathbf{U}(t, t_0) |\psi(t)\rangle \langle\psi(t)| \mathbf{U}^\dagger(t, t_0)$$

with a simple substitution into the first order differential equation, we have

$$\frac{d\rho(t)}{dt} = \frac{d\mathbf{U}(t, t_0)}{dt} |\psi(t)\rangle \langle\psi(t)| \mathbf{U}^\dagger(t, t_0)$$

Postulate V claims that $i\hbar \frac{d\mathbf{U}(t,t_0)}{dt} = \mathcal{H}\mathbf{U}(t, t_0)$, so our equation can change to

$$\frac{1}{i\hbar}\mathcal{H}\mathbf{U}(t, t_0) |\psi(t)\rangle \langle\psi(t)| \mathbf{U}^\dagger(t, t_0) = \frac{1}{i\hbar}\mathcal{H}\rho(t)$$

This final answer is equivalent to our expected answer.

## Question 5.9

Generalize the time evolution equation given in problem (5.8), if Hamiltonian $\mathbf{H}$(t) is a function of time.

### Answer 5.9.1

When the Hamiltonian operates as a function of time, the unitary operator $U(t, t_0) = Texp(-i \int_{t_0}^t dt'\mathcal{H}(t')/\hbar)$. Given that our density operator is

$$\rho(t) = \mathbf{U}(t, t_0) |\psi(t)\rangle \langle\psi(t)| \mathbf{U}^\dagger(t, t_0)$$

we understand that we can extend our equation to

$$\rho(t) = Texp(-i \int_{t_0}^t dt'\mathcal{H}(t')/\hbar) |\psi(t)\rangle \langle\psi(t)| Texp(i \int_{t_0}^t dt'\mathcal{H}(t')/\hbar)$$

So the resulting function $\frac{d\rho(t)}{dt}$ is

$$\begin{aligned}\frac{d\rho(t)}{dt} &= \frac{d[Texp(-i \int_{t_0}^t dt'\mathcal{H}(t')/\hbar) |\psi(t)\rangle \langle\psi(t)| Texp(i \int_{t_0}^t dt'\mathcal{H}(t')/\hbar)]}{dt} \\ &= Texp(-i\mathcal{H}(t)/\hbar) |\psi(t')\rangle \langle\psi(t')| Texp(i\mathcal{H}(t)/\hbar)\end{aligned}$$

**Not sure if this is correct.**

# Question 5.10

Using $\rho(t)$ obtained in problem (5.7), find the time dependent expectation value

$$\sigma_Z(t) \equiv \langle\psi(t)|\,\sigma_Z\,|\psi(t)\rangle$$

Do the same for $\sigma_X(t)$ and $\sigma_Y(t)$.

## Answer 5.10.1

Given $\rho(t) = U(t,t_0)\rho(t_0)U^{\dagger}(t,t_0) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|$,
we can find the time-dependent expectation value by evaluating

$$Tr[\rho(t)\sigma_z(t)] = Tr[\frac{1}{2}\sigma_z(t)] = 0$$

## Answer 5.10.2

Given the same value of $\rho(t)$, we evaluate expectation value with $\sigma_x$

$$Tr[\rho(t)\sigma_x(t)] = Tr[\frac{1}{2}\sigma_x(t)] = 0$$

## Answer 5.10.3

Given the same value of $\rho(t)$, we evaluate expectation value with $\sigma_y$

$$Tr[\rho(t)\sigma_y(t)] = Tr[\frac{1}{2}\sigma_y(t)] = 0$$

# Question 5.11

For each single-qubit density matrix itemized below, determine whether it represents a pure or mixed state.

$$(a)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad (b)\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix} \quad (c)\begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{3}{4}\end{pmatrix} \quad (d)\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

## Answer 5.11.1

The matrix

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

is a pure state because $Tr[\rho^2] = 1$

### Answer 5.11.2

The matrix

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

is a mixed state because $Tr[\rho^2] = \frac{1}{2} < 1$.

### Answer 5.11.3

The matrix

$$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$

is a mixed state because $Tr[\rho^2] = \frac{10}{16} < 1$.

### Answer 5.11.4

The matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

is a pure state because $Tr[\rho^2] = 1$

## Question 5.12

Which of the following is not a valid density matrix

$$(a)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (b)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (c)\frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \quad (d)\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

### Answer 5.12.1

The matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is invalid because the state is not normalized.

### Answer 5.12.2

The matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is valid because the density matrix $\sum_i p_i = 1$ holds true and the matrix is normalized.

### Answer 5.12.3

The matrix

$$\frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

is valid because, the matrix is normalized and $\sum_i p_i = 1$.

### Answer 5.12.4

The matrix

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

is invalid because there exists a probability $p_i$ less than 0 (negative).

## Question 5.13

Construct the density matrices for the following two-qubit states in the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

$$(a)\frac{1}{4}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$(b)\frac{1}{4}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$(c)\frac{1}{4}(|00\rangle + i\,|10\rangle + i\,|01\rangle - |11\rangle)$$

$$(d)\frac{1}{\sqrt{2}}(|01\rangle - i\,|10\rangle)$$

### Answer 5.13.1

For this state, the density operator is

$$\begin{gathered}\rho = \frac{1}{8}([|00\rangle\langle 00| - |01\rangle\langle 00| + |10\rangle\langle 00| - |11\rangle\langle 00|] \\ +[-|00\rangle\langle 10| + |01\rangle\langle 10| - |10\rangle\langle 10| + |11\rangle\langle 10|] \\ +[|00\rangle\langle 01| - |01\rangle\langle 01| + |10\rangle\langle 01| - |11\rangle\langle 01|] \\ +[-|00\rangle\langle 11| + |01\rangle\langle 11| - |10\rangle\langle 11| + |11\rangle\langle 11|])\end{gathered}$$

### Answer 5.13.2

For this state, the density operator is

$$\rho = \frac{1}{8}([|00\rangle\langle 00| + |01\rangle\langle 00| + |10\rangle\langle 00| + |11\rangle\langle 00|]$$
$$+[|00\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 10| + |11\rangle\langle 10|]$$
$$+[|00\rangle\langle 01| + |01\rangle\langle 01| + |10\rangle\langle 01| + |11\rangle\langle 01|]$$
$$+[|00\rangle\langle 11| + |01\rangle\langle 11| + |10\rangle\langle 11| + |11\rangle\langle 11|])$$

### Answer 5.13.3

For this state, the density operator is

$$\rho = \frac{1}{8}([|00\rangle\langle 00| - i\,|01\rangle\langle 00| - i\,|10\rangle\langle 00| - |11\rangle\langle 00|]$$
$$+[i\,|00\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 10| - i\,|11\rangle\langle 10|]$$
$$+[i\,|00\rangle\langle 01| + |01\rangle\langle 01| + |10\rangle\langle 01| - i\,|11\rangle\langle 01|]$$
$$+[-|00\rangle\langle 11| + i\,|01\rangle\langle 11| + i\,|10\rangle\langle 11| + |11\rangle\langle 11|])$$

### Answer 5.13.4

For this state, the density operator is

$$\rho = \frac{1}{2}(|01\rangle\langle 10| - i\,|01\rangle\langle 01| - i\,|10\rangle\langle 10| - |10\rangle\langle 01|)$$

## Question 5.14

For each two-qubit density matrix $\rho_{AB}$ obtained in problem (5.13), evaluate the partial traces

$$Tr_A\rho_{AB}, \quad Tr_B\rho_{AB}$$

Which of the states in problem (5.13) are entangled states?

### Answer 5.14.1

We will evaluate the density matrix

$$\rho_{AB} = \frac{1}{8}([|00\rangle\langle 00| - |01\rangle\langle 00| + |10\rangle\langle 00| - |11\rangle\langle 00|]$$
$$+[-|00\rangle\langle 10| + |01\rangle\langle 10| - |10\rangle\langle 10| + |11\rangle\langle 10|]$$
$$+[|00\rangle\langle 01| - |01\rangle\langle 01| + |10\rangle\langle 01| - |11\rangle\langle 01|]$$
$$+[-|00\rangle\langle 11| + |01\rangle\langle 11| - |10\rangle\langle 11| + |11\rangle\langle 11|])$$

First, we reformat the density matrix such that

$$\rho_{AB} = \frac{1}{8}([|0\rangle\langle 0| \otimes |0\rangle\langle 0| - |0\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 0| - |1\rangle\langle 0| \otimes |1\rangle\langle 0|]$$

$$+[-|0\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| - |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 1|]$$

$$+[|0\rangle\langle 1| \otimes |0\rangle\langle 0| - |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| - |1\rangle\langle 1| \otimes |1\rangle\langle 0|]$$

$$+[-|0\rangle\langle 1| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 1| - |1\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|]$$

This is slightly easier to read when evaluating the trace of the density matrix. However, we can make it even easier with the matrix adaptation:

$$\rho_{AB} = \frac{1}{8}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

This will help us find the partial trace with respect to qubit $A$ and $B$. Thus,

$$Tr_A[\rho_{AB}] = \frac{2}{8}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{4}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$Tr_B[\rho_{AB}] = \frac{2}{8}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

In addition, we know the state is not entangled because it can be separated into a direct product (i.e. $Tr_A[\rho_{AB}] \otimes Tr_B[\rho_{AB}] = \rho_{AB}$ )

## Answer 5.14.2

Similarly to the previous answer, we will simplify our density operator of (5.13.2) to

$$\rho_{AB} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Taking the trace with respect to qubit A/B results in

$$Tr_A[\rho_{AB}] = \frac{2}{8}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$Tr_B[\rho_{AB}] = \frac{2}{8}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

In addition, we know the state is not entangled because it can be separated into a direct product (i.e. $Tr_A[\rho_{AB}] \otimes Tr_B[\rho_{AB}] = \rho_{AB}$ )

### Answer 5.14.3

Similarly to the previous answer, we will simplify our density operator of 5.13.3 to

$$\rho_{AB} = \frac{1}{8}\begin{pmatrix} 1 & -i & -i & -1 \\ i & 1 & 1 & -i \\ i & 1 & 1 & -i \\ -1 & i & i & 1 \end{pmatrix}$$

so our trace with respect to qubit A/B is

$$Tr_A[\rho_{AB}] = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{4}(|0\rangle\langle 0| - i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$Tr_B[\rho_{AB}] = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{4}(|0\rangle\langle 0| - i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| + |1\rangle\langle 1|)$$

In addition, we know the state is not entangled because it can be separated into a direct product (i.e. $Tr_A[\rho_{AB}] \otimes Tr_B[\rho_{AB}] = \rho_{AB}$ )

### Answer 5.14.4

Similarly to the previous answer, we will simplify our density operator of 5.13.3 to

$$\rho_{AB} = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so our trace with respect to qubit A/B is

$$Tr_A[\rho_{AB}] = \frac{1}{8}\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} = \frac{-i}{8}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$Tr_B[\rho_{AB}] = \frac{1}{8}\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} = \frac{-i}{8}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

In addition, we know the state is entangled because it can not be separated into a direct product (i.e. $Tr_A[\rho_{AB}] \otimes Tr_B[\rho_{AB}] = \rho_{AB}$ ). Our new direct product results in a different matrix than $\rho_{AB}$.

## Question 5.15

Consider the measurement operator $\mathbf{M} = \sigma_X \otimes \sigma_Y$.(a) Find the expectation value of $\mathbf{M}$ for states $(a), (b), (c), (d)$ in problem (5.13). (b) Define the $\rho_A = Tr_B \rho_{AB}$, obtained in problem (5.14), and use it to evaluate

$$Tr(\rho_A \sigma_X).$$

Compare the results obtained in parts (a) and (b). Comment.

## Answer 5.15.a

Recall that to find the expectation value for a density operator and a matrix, we can solve the equation

$$Tr\rho M$$

where $M$ is an operator. Our density operator is charectarized as

$$\begin{aligned}\rho = \frac{1}{8}(&[|00\rangle\langle 00| - |01\rangle\langle 00| + |10\rangle\langle 00| - |11\rangle\langle 00|]\\ &+[-|00\rangle\langle 10| + |01\rangle\langle 10| - |10\rangle\langle 10| + |11\rangle\langle 10|]\\ &+[|00\rangle\langle 01| - |01\rangle\langle 01| + |10\rangle\langle 01| - |11\rangle\langle 01|]\\ &+[-|00\rangle\langle 11| + |01\rangle\langle 11| - |10\rangle\langle 11| + |11\rangle\langle 11|])\end{aligned}$$

which is equivalent to

$$\rho_{AB} = \frac{1}{8}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

as characterized by answer (5.14.1). Thus, our equation becomes

$$Tr[\rho M] = Tr[\frac{1}{8}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}(\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix})]$$

$$= Tr[\frac{1}{8}\begin{pmatrix} -i & -i & -i & -i \\ i & i & i & i \\ -i & -i & -i & -i \\ i & i & i & i \end{pmatrix}] = \frac{1}{8}(-1+1-1+1) = 0$$

Thus, our expectation value for $Tr[\rho M]$ is 0. For the second density matrix, we have:

$$\rho_{AB} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

so the expectation value for this is:

$$Tr[\rho M] = \frac{1}{8}Tr[\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}]$$

$$= \frac{1}{8}Tr[(\begin{pmatrix} i & -i & i & -i \\ i & -i & i & -i \\ i & -i & i & -i \\ i & -i & i & -i \end{pmatrix})] = \frac{1}{8}(1-1+1-1) = 0$$

Thus, our expectation value for the second density matrix under operator $M$ is 0. For the third density matrix, we have:

$$\rho_{AB} = \frac{1}{8}\begin{pmatrix} 1 & -i & -i & -1 \\ i & 1 & 1 & -i \\ i & 1 & 1 & -i \\ -1 & i & i & 1 \end{pmatrix}$$

so the expectation value for this is:

$$Tr[\rho M] = \frac{1}{8}Tr[\begin{pmatrix} 1 & -i & -i & -1 \\ i & 1 & 1 & -i \\ i & 1 & 1 & -i \\ -1 & i & i & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}]$$

$$= \frac{1}{8}Tr[(\begin{pmatrix} -i & -1 & 1 & -i \\ 1 & -i & i & 1 \\ 1 & -i & i & 1 \\ i & 1 & -1 & i \end{pmatrix})] = \frac{1}{8}(-1-1+1+1) = 0$$

Thus, our third density matrix with operator $M$ yields an expectation value of 0. Our last density matrix is:

$$\frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the expectation value for this is:

$$Tr[\rho M] = \frac{1}{8}Tr[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}]$$

$$= \frac{1}{8}Tr[(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & -1 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix})] = \frac{1}{8}(0+1+1+0) = \frac{1}{4}$$

Thus, our expectation value for the entangled state under the final density matrix $\rho_{AB}$ and operator $M$ is $\frac{1}{4}$.

## Answer 5.15.2

Recall for from 5.14.1 that $\rho_A$ is

$$\rho_A = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Evaluating the expectation value of the smaller matrix yields:

$$\frac{1}{4}Tr[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] = \frac{1}{4}Tr[\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}] = \frac{1}{4}(-2) = -\frac{1}{2}$$

Thus the expectation value for the individual state defined by $\rho_A$ and the operator $\sigma_X$ is $\frac{-1}{2}$. For problem 5.14.2 the state $\rho_A$ is:

$$\rho_A = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Evaluating the expectation value of the smaller matrix yields:

$$\frac{1}{4}Tr[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] = \frac{1}{4}Tr[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}] = \frac{1}{4}(2) = \frac{1}{2}$$

Thus the expectation value for the individual state defined by $\rho_A$ in 5.14.2 and the operator $\sigma_X$ is $\frac{1}{2}$. For problem 5.14.3 the state $\rho_A$ is:

$$\rho_A = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

Evaluating the expectation value of the smaller matrix yields:

$$\frac{1}{4}Tr[\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] = \frac{1}{4}Tr[\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}] = \frac{1}{4}(-1+1) = 0$$

Thus the expectation value for the individual state defined by $\rho_A$ in 5.14.3 and the operator $\sigma_X$ is 0. For problem 5.14.4 the state $\rho_A$ is:

$$\rho_A = \frac{1}{8}\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$$

Evaluating the expectation value of the smaller matrix yields:

$$\frac{1}{8}Tr[\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] = \frac{1}{4}Tr[\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}] = \frac{1}{8}(0) = 0$$

Thus the expectation value for the individual state defined by $\rho_A$ in 5.14.4 and the operator $\sigma_X$ is 0.

It becomes clear that the expectation value for a multi-qubit, or $n$ qubit, operation does not imply that its sub-operators − namely operations smaller than $n$ − will yield the same expectation value. For part (a), our entangled state yielded a non-zero expectation value; this was not the case for part (b).

# Question 5.16

Consider the state

$$\frac{1}{4}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

Express it in terms of a Schmidt decomposition. What is its Schmidt number?

## Answer 5.16.1

Recall that, to express a state in terms of a Schmidt decomposition, we apply the formula:

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{p_i}\, |\rho_i\rangle \otimes |\lambda_i\rangle$$

Applying this to our original state, our Schmidt decomposition yields:

$$|\psi\rangle = \frac{1}{\sqrt{4}}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)$$

Our schimidt number can be solved by discovering the density operator of $|\psi\rangle$:

$$\rho_\psi = \frac{1}{\sqrt{4}}\frac{1}{\sqrt{4}}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)(\langle 0| + \langle 1|) \otimes (\langle 0| - \langle 1|)$$

$$= \frac{1}{4}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| - |1\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 1|)$$

Because this is a two-qubit state, there are 16 unique combinations in which $p_i$ is non-zero. Thus, the Schmidt number is 16.

# Question 5.17

Repeat problem (5.16) for states

$$\frac{1}{2}(|01\rangle - |10\rangle)$$

## Answer 5.17.1

Using the same logic as in problem 5.16, our Schmidt decomposition is expressed by

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{p_i}\, |\rho_i\rangle \otimes |\lambda_i\rangle$$

Applying this to our original state, our Schmidt decomposition yields:

$$|\psi\rangle = \frac{1}{\sqrt{2}}[(|0\rangle) \otimes (|1\rangle) - (|1\rangle) \otimes (|0\rangle)]$$

The density operator of this expression is:

$$\begin{aligned} \rho &= \frac{1}{2}[(|0\rangle) \otimes (|1\rangle) - (|1\rangle) \otimes (|0\rangle)][(\langle 1|) \otimes (\langle 0|) - (\langle 0|) \otimes (\langle 1|)] \\ &= (|01\rangle \langle 10| - |10\rangle \langle 10| - |01\rangle \langle 01| + |10\rangle \langle 01|) \end{aligned}$$

Thus, our Schmidt number is 4 for this entangled state.

# Question 5.18

Give a general proof of relations (5.7) and (5.18).

## Answer 5.18.1

Relation (5.7) claims that

$$Tr[\rho A] = \langle \psi | A | \psi \rangle$$

And this is proven on equation (5.8). Relation (5.18) claims that the probability of an eigenvalue being measured, $m_i$, is represented by

$$p(m_i) = Tr[\rho M]$$

We can prove the original relation by recalling our definition of the born rule, which claims that the probability of finding yielding state $|i\rangle$ from the state $|\psi\rangle$ is characterized by

**Not sure what this question is asking...**

# Question 5.19

Give a proof of theorem (5.1) for a bipartite Hilbert space of dimension $n, m$, respectively.

## Answer 5.19.1

Firstly, we assume that $n \neq m$. In the case that $n < m$, then we can express the Schmidt decomposition as

$$|\psi_{AB}\rangle = \sum_{i=1}^{n} \sqrt{p_i}\, |\rho_i\rangle \langle \lambda_i|$$

Where we do not consider the remaining subspaces $n-m$. Similarly if $n > m$, then the Schmidt decomposition is written as:

$$|\psi_{AB}\rangle = \sum_{i=1}^{m} \sqrt{p_i}\, |\rho_i\rangle \langle\lambda_i|$$

Where we do not consider the remaining subspaces $m - n$.

At this point, we apply the same proof as with equal subspaces (after equation (5.49)) and we prove that each subspace of dimension $n \neq m$ can still be expressed in a Schmidt decomposition. **Not sure if this is correct since it feels under-explained**

# Question 5.20

Find the von Neumann entropy for

$$\rho = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \quad \rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

## Answer 5.20.1

For the matrix

$$\rho = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$

the von Neumann entropy is

$$S = -Tr[\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} ln(\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix})] = -Tr[(\begin{pmatrix} \frac{1}{4}ln(\frac{1}{4}) & 0 \\ 0 & \frac{3}{4}ln(\frac{3}{4}) \end{pmatrix})]$$

$$= -(\frac{1}{4}ln(\frac{1}{4}) + \frac{3}{4}ln(\frac{3}{4})) = 0.71$$