Chapter 3 Problems

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Question (3.1)

Using the four-bit adder circuit given in Notebook 3.1, evaluate the binary sum 0101 + 0001. Does this circuit predict the correct answer for 1000 + 1000? If not, generalize it so that the correct result is obtained.

Answer (3.1.1)

Don't need to do

Question (3.2)

Itemize all possible functions for the mapping $f : 0, 1^2 \longrightarrow 0, 1^1$. Show that the mapping $f : 0, 1^n \longrightarrow 0, 1^m$ allows $2^{2^n m}$ unique functions f.

Answer (3.2.1)

The possible functions for the mapping $f : 0, 1^2 \longrightarrow 0, 1^1$ are $2^{4(1)} = 16$:

xy	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
00	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
01	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
10	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

In general if there are n bits allowed as input, there are 2^n possible inputs that need to be mapped. Each of these inputs is mapped into m outputs, and for each input there are 2^m possible outputs. Another way of wording this is that the 2^m outputs are repeated 2^n times, or there are $2^{2^n m}$ functions available to us.

Answer (3.2.2)

For a mapping $f : 0, 1^n \longrightarrow 0, 1^m$, we know that the possible inputs for an n qubit system is 2^n. Additionally, for every qubit value

Question (3.3)

The state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is input to a σ_X gate, find the output state. Repeat for the σ_Y and σ_Z gates.

Answer (3.3.1)

The output state for $\sigma_X\,|\psi\rangle$ is:

$$\sigma_X\,|\psi\rangle = \beta\,|0\rangle + \alpha\,|1\rangle$$

Answer (3.3.2)

For $\sigma_Y\,|\psi\rangle$:

$$\sigma_Y\,|\psi\rangle = -i\beta\,|0\rangle + i\alpha\,|1\rangle$$

Answer (3.3.3)

For $\sigma_Z\,|\psi\rangle$:

$$\sigma_Z\,|\psi\rangle = \alpha\,|0\rangle - \beta\,|1\rangle$$

Question (3.4)

Find the output of the two qubit gate $\sigma_X \otimes \mathbb{1}$ for the following inputs (a) $|00\rangle$, (b) $|01\rangle$, (c) $|10\rangle$, and (d) $|11\rangle$. Repeat with the gate $\mathbb{1} \otimes \sigma_X$. Are the truth tables for these two gates identical? Comment.

Answer (3.4.1)

We will evaluate all basis kets $|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle$ with the gate $\sigma_X \otimes \mathbb{1}$:

- $(\sigma_X \otimes \mathbb{1})\,|00\rangle = |10\rangle$
- $(\sigma_X \otimes \mathbb{1})\,|01\rangle = |11\rangle$
- $(\sigma_X \otimes \mathbb{1})\,|10\rangle = |00\rangle$
- $(\sigma_X \otimes \mathbb{1})\,|11\rangle = |01\rangle$

Answer (3.4.2)

No we will evaluate the basis kets with $\mathbb{1} \otimes \sigma_X$:

- $(\mathbb{1} \otimes \sigma_X)\,|00\rangle = |01\rangle$
- $(\mathbb{1} \otimes \sigma_X)\,|01\rangle = |00\rangle$
- $(\mathbb{1} \otimes \sigma_X)\,|10\rangle = |11\rangle$
- $(\mathbb{1} \otimes \sigma_X)\,|11\rangle = |10\rangle$

Clearly, the results are exactly the same as in (3.4.1). This means that the truth tables for these two gates are identical.

Question (3.5)

Below, in Fig (3.11) panel (a), the symbol σ_i, for $i = X, Y, Z$ refers to the pauli gates. Evaluate the truth tables for this circuit for each of the pauli gates. Compare your result for the σ_X gate with the truth table for the **CNOT** gate shown in Fig. (3.7). Comment.

Adding Quantum Circuits? Should I add?

Answer (3.5.1)

For the Pauli X gate, we have The following truth table

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

This is clearly equivalent to the $CNOT$ gate truth table.

Answer (3.5.2)

For the Pauli Z gate, we have the following truth table

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	-$\|11\rangle$

Answer (3.5.3)

For the Pauli Y gate, we have the following truth table

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$i\|11\rangle$
$\|11\rangle$	$-i\|10\rangle$

Question (3.6)

Repeat problem (3.5) for the gate shown in panel (b) of fig (3.11)

Answer (3.6.1)

For the Pauli-X gate, we have

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|11\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	$\|01\rangle$

Which, again, resembles the operation of the CNOT gate in the opposite direction.

Answer (3.6.2)

For the Pauli-Z gate, we have

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	$-\|11\rangle$

Answer (3.6.3)

For the Pauli-Y gate, we have

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$-i\|11\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	$i\|01\rangle$

Question (3.7)

Construct the truth table for the circuit shown in Fig. (3.12). Compare your result with that obtained in problem (3.6) for the case $\sigma_i = \sigma_X$.

Answer (3.7.1)

We will construct the truth table for all input values of the qubit for the circuit described. The result is:

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$-i\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$i\|10\rangle$

Clearly, this circuit is an equivalent construction of the $CNOT$, which has the same truth table. In this circuit, the first qubit, q_0, is the control qubit, and anytime $q_0 = 1$, we flip the value of the second qubit q_1.

Question (3.8)

Construct the matrix representation for the gate shown in Fig. (3.12)

Answer (3.8.1)

The matrix can be described as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Question (3.9)

In panel (a) of Fig. (3.13) the state $|000\rangle$ is input to the Hadamard gates shown in that figure. Find the output of this three qubit gate.

Answer (3.9.1)

Clearly, the output of this circuit will be a superposition of every basis ket. The result is:

$$(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

Which is equivalent to $|\psi\rangle$ such that:

$$|\psi\rangle = \frac{1}{4}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

Question (3.10)

Using Born's rule find the probability that the measurement apparatuses shown in panel (b) of Fig. (3.13) detects the state $|010\rangle$.

Answer (3.10.1)

The apparatus in figure (3.13) (b) is the same as in Question (3.9.1) before the apparatus. Thus, our three qubit state is:

$$|\psi\rangle = \frac{1}{4}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

Using the Born rule, we find that there is a $|\langle 010|\psi\rangle|^2 = |\frac{1}{4}|^2 = \frac{1}{8}$ probability of collapsing the system to $|010\rangle$.

Question (3.11)

Using Born's rule find the probability that the measurement apparatus at the bottom-most wire of panel (c) in Fig. (3.13) measures the value 1.

Answer (3.11.1)

Using the all the states from answer (3.11.1), we have $|\psi\rangle$ such that:

$$|\psi\rangle = \frac{1}{4}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

The probability that the bottom measurement apparatus is 1 is:

$$|\langle 001|\psi\rangle|^2 + |\langle 011|\psi\rangle|^2 + |\langle 101|\psi\rangle|^2 + |\langle 111|\psi\rangle|^2 = \frac{1}{8} * 4 = \frac{1}{2}$$

Question (3.12)

Consider the function f defined by truth Table (3.10). Is this function (a) constant (b) balanced (c) neither of those options?

Answer (3.12.1)

Based on the protocol $U_f|xy\rangle \equiv |x\rangle \otimes |y \bigoplus f(x)\rangle$, and the given truth table, we can derive our answer. When $x = 0$, $f(0) = 1$ the result is $|0\rangle \otimes |y \bigoplus 1\rangle$, and when $x = 1$, $f(1) = 0$ and the result is $|0\rangle \otimes |y\rangle$.

Clearly, $f(0) \neq f(1)$ at any point, so the function is balanced.

Question (3.13)

Construct a classical gate, composed of Boolean gates, that evaluates the function defined in Table (3.10)

Answer (3.13.1)

The gate in table (3.10) is clearly the same as the XOR gate. When the input $x = y$, the output is zero. Otherwise, the output is one.

Question (3.14)

Using protocol (3.18), construct a quantum gate that evaluates the function defined in Table (3.10). You should present this gate in the form of a table similar to given by Table (3.8).

Answer (3.14.1)

Protocol (3.18) is:

$$\mathbf{U}_f \left|x\right\rangle_n \otimes \left|y\right\rangle_m = \left|x\right\rangle_n \otimes \left|y + f(x)\right\rangle_m$$

Additionally, since the table is two qubits long, we know our gate will be a 16×16 matrix (extremely large!); to find out what each value of the matrix is, we can use a table to determine all the possible outcomes:

$\left\|xy\right\rangle$	$\left\|x\right\rangle \otimes \left\|y \bigoplus f(x)\right\rangle$
00	**00**
00	01
00	10
00	11
01	00
01	**01**
01	10
01	11
10	00
10	**01**
10	10
10	11
11	00
11	01
11	**10**
11	11

Thus our gate is represented as the table above.

Question (3.15)

Write a Mathematica code that expresses the gate obtained in problem (3.14) as a unitary matrix $\mathbf{U}_f$. Using that matrix, find the output for the following input states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. Find the output for the input state:

$$|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle$$

Answer (3.15.1)

Don't need to do

Question (3.16)

Repeat problem (3.15) for the input state given by $|\psi\rangle = \mathbf{H} \otimes \mathbf{H} |00\rangle$, where $\mathbf{H}$ is the Hadamard gate.

Answer (3.16.1)

Don't need to do

Question (3.17)

Consider the mapping $f : 0, 1^3 \longrightarrow 0, 1^1$ so that $f(x) = 0$ for all x in $0, 1^3$, except for $x = 010$ in which case $f(010) = 1$. Write a Mathematica code for a quantum gate, $\mathbf{U}_f$, that evaluates this function.

Answer (3.17.1)

Don't need to do

Question (3.18)

For the function f defined in problem (3.17), use $\mathbf{U}_f$ to evaluate $\mathbf{U}_f\mathbf{H} \otimes \mathbf{H} \otimes \mathbf{H} |000\rangle$. Estimate the probability that a measurement of the last qubit in the output register gives the value 1.

Answer (3.18.1)

If we begin by applying the Hadamard gates only, we get, $|\psi\rangle$ such that:

$$(\mathbf{U}_f \otimes \mathbb{1} \otimes \mathbb{1}) |\psi\rangle = |000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle$$

Since $\mathbf{U}_f$ is ..... **Don't need to do**

Question (3.19)

An electron is subjected to a magnetic field, pointing along the z-axis, of magnitude B_0. At $t_0 = 0$, the electron is in the state $\mathbf{H}|0\rangle$. Estimate the probabilities to find the electron in its ground state $|0\rangle$. At the following time intervals:

$$(a) t = \frac{\hbar}{\mu_0 B_0}\frac{\pi}{4}$$
$$(b) t = \frac{\hbar}{\mu_0 B_0}\frac{\pi}{2}$$
$$(c) t = \frac{\hbar}{\mu_0 B_0}\pi$$

Answer (3.19.1)

If

$$t = \frac{\hbar}{\mu_0 B_0}\frac{\pi}{4}$$

and we know $t_0 = 0$,then our equation $\mathbf{U}(t, t_0)$ is

$$\mathbf{U}(t, t_0) = exp(-i\mu_0 B_0 \sigma_Z \tau/\hbar) = exp(-i\sigma_Z \frac{\pi}{4}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$$

And our original electron evolves to:

$$\mathbf{U}(t, t_0)\mathbf{H}|0\rangle = \mathbf{U}(t, t_0)(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \frac{1}{2}\begin{bmatrix} 1-i \\ 1+i \end{bmatrix} = \frac{1}{2}((1-i)|0\rangle + (1+i)|1\rangle)$$

Answer (3.19.2)

If

$$t = \frac{\hbar}{\mu_0 B_0}\frac{\pi}{2}$$

and we know $t_0 = 0$,then our equation $\mathbf{U}(t, t_0)$ is

$$\mathbf{U}(t, t_0) = exp(-i\mu_0 B_0 \sigma_Z \tau/\hbar) = exp(-i\sigma_Z \frac{\pi}{2}) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

And our original electron evolves to:

$$\mathbf{U}(t, t_0)\mathbf{H}|0\rangle = \mathbf{U}(t, t_0)(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ i \end{bmatrix} = \frac{i}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

(3.19.3)

If

$$t = \frac{\hbar}{\mu_0 B_0}\pi$$

and we know $t_0 = 0$,then our equation $\mathbf{U}(t, t_0)$ is

$$\mathbf{U}(t, t_0) = exp(-i\mu_0 B_0 \sigma_Z \tau/\hbar) = exp(-i\sigma_Z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

And our original electron evolves to:

$$\mathbf{U}(t, t_0)\mathbf{H}\left|0\right\rangle = \mathbf{U}(t, t_0)(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{H}\left|1\right\rangle$$

Question (3.20)

Repeat problem (3.19) for the input state $\mathbf{H}\left|1\right\rangle$

Answer (3.20.1)

If

$$t = \frac{\hbar}{\mu_0 B_0}\frac{\pi}{4}$$

and we know $t_0 = 0$,then our equation $\mathbf{U}(t, t_0)$ is

$$\mathbf{U}(t, t_0) = exp(-i\mu_0 B_0 \sigma_Z \tau/\hbar) = exp(-i\sigma_Z \frac{\pi}{4}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$$

And our original electron evolves to:

$$\mathbf{U}(t, t_0)\mathbf{H}\left|1\right\rangle = \mathbf{U}(t, t_0)(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \frac{1}{2}\begin{bmatrix} 1-i \\ -1-i \end{bmatrix} = \frac{1}{2}((1-i)\left|0\right\rangle - (1+i)\left|1\right\rangle)$$

Answer (3.20.2)

If

$$t = \frac{\hbar}{\mu_0 B_0}\frac{\pi}{2}$$

and we know $t_0 = 0$,then our equation $\mathbf{U}(t, t_0)$ is

$$\mathbf{U}(t, t_0) = exp(-i\mu_0 B_0 \sigma_Z \tau/\hbar) = exp(-i\sigma_Z \frac{\pi}{2}) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

And our original electron evolves to:

$$\mathbf{U}(t, t_0)\mathbf{H}\left|1\right\rangle = \mathbf{U}(t, t_0)(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ -i \end{bmatrix} = -i\mathbf{H}\left|0\right\rangle$$

Answer (3.20.3)

If

$$t = \frac{\hbar}{\mu_0 B_0}\pi$$

and we know $t_0 = 0$,then our equation $\mathbf{U}(t, t_0)$ is

$$\mathbf{U}(t, t_0) = exp(-i\mu_0 B_0 \sigma_Z \tau/\hbar) = exp(-i\sigma_Z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

And our original electron evolves to:

$$\mathbf{U}(t, t_0)\mathbf{H}\left|1\right\rangle = \mathbf{U}(t, t_0)(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{H}\left|0\right\rangle$$