KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)

MATRICES

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Outline

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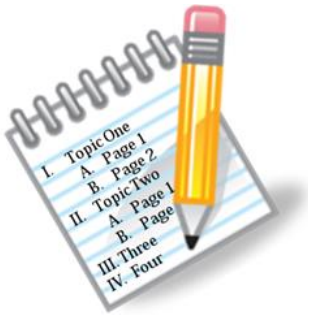

Introduction: What & Why?

A matrix is a two dimensional array of objects.

A vector is a special case of a matrix with only one dimension.

They are important because they can be manipulated in a number of ways in a computer and can be very useful in certain applications.

The programming environment Matlab^{TM} uses matrices and vectors solely to do mathematical type programming.

Examples

$$\begin{bmatrix} -4 & 3 & 5 & 7 & 8 \\ 0 & 5 & 7 & 2 & 4 \\ 3 & 5 & -1 & 6 & 2 \end{bmatrix}$$

- This is a 3 by 5 matrix.
- That is, it has 3 rows and 5 columns.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Note that this is a special matrix - the identity matrix.

$$\begin{bmatrix} \textit{Botham} \\ \textit{Pieterson} \\ \textit{Sobers} \\ \textit{Bradman} \end{bmatrix}$$

- This is a vector

Notation

We usually denote a matrix by a capital letter and the elements by lower case letters in the following way:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}$$

BTW this is a square matrix.

This can be shortened to $P = [p_{ij}]$ where $i = 1 \ldots 4$ and $j = 1 \ldots 4$

Operation: Addition (and Subtraction)

To add/subtract two matrices together they must be the same size, that is have the same number of rows and columns. You simply add the elements together.

$$\begin{bmatrix} -4 & 3 & 5 & 7 & 8 \\ 0 & 5 & 7 & 2 & 4 \\ 3 & 5 & -1 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 9 & 3 & 4 \\ -2 & 5 & 3 & 5 & 1 \\ 2 & 2 & -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 14 & 10 & 12 \\ -2 & 10 & 10 & 7 & 5 \\ 5 & 7 & -2 & 10 & 3 \end{bmatrix}$$

Operation: Multiplication

This is more complicated ☺

To add and subtract matrices have to be same size but not for multiplication.

There are some differences:

- The number of columns in first matrix must be same as rows in second. So if we are multiplying A (n x m) with B then B must have m rows.
- Order of multiplication matters

Operation: Multiplication

$$\begin{bmatrix} -4 & 3 & 5 & 7 & 8 \\ 0 & 5 & 7 & 2 & 4 \\ 3 & 5 & -1 & 6 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ -2 & 5 \\ 2 & 2 \\ 3 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 89 \\ 14 & 67 \\ 14 & 60 \end{bmatrix}$$

Let's look at how this was arrived at.

So how do we multiply?

Suppose we have:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \\ p_{41} & p_{42} & p_{43} \end{bmatrix} \text{ and } Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{bmatrix}$$

If S = P x Q what will be dimensions of S?

For s_{11} we use the first row of P and the the first column of Q.

$s_{11} = p_{11} * q_{11} + p_{12} * q_{21} + p_{13} * q_{31}$

What will be s_{32}? Clue: work out which rows and columns to use

Example

$$\begin{bmatrix} -4 & 5 & 4 \\ 0 & 4 & 7 \\ 3 & 4 & -1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & 6 & 4 \\ -1 & 3 & 6 & -3 \\ 2 & -2 & 4 & -1 \end{bmatrix} = ?$$

Any ideas how we might program a computer to multiply matrices?

Operation: Transpose

Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $\mathbf{A}$, denoted by $\mathbf{A}^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$. In other words, if $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Example:

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Operation: Transpose

Matrices that do not change when their rows and columns are interchanged are often important.

A square matrix $\mathbf{A}$ is called *symmetric* if $\mathbf{A} = \mathbf{A}^t$. Thus $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

Example:

The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is symmetric.

Operation: Meet and Join

A matrix all of whose entries are either 0 or 1 is called a zero–one matrix.

Zero–one matrices are often used to represent discrete structures.

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ zero–one matrices. Then the *join* of $\mathbf{A}$ and $\mathbf{B}$ is the zero–one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$. The join of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \vee \mathbf{B}$. The *meet* of $\mathbf{A}$ and $\mathbf{B}$ is the zero–one matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$. The meet of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \wedge \mathbf{B}$.

This arithmetic is based on the Boolean operations ∧ and ∨, which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example

Find the join and meet of the zero–one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: We find that the join of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Boolean Product

Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero–one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then the *Boolean product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ matrix with (i, j)th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Find the Boolean product of A and B, where:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Boolean Product

Solution:

The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Boolean Product

Let $\mathbf{A}$ be a square zero–one matrix and let r be a positive integer. The rth *Boolean power* of $\mathbf{A}$ is the Boolean product of r factors of $\mathbf{A}$. The rth Boolean product of $\mathbf{A}$ is denoted by $\mathbf{A}^{[r]}$. Hence

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}}_{r \text{ times}}.$$

(This is well defined because the Boolean product of matrices is associative.) We also define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

Example

Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^{[n]}$ for all positive integers n.

Solution: We find that

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We also find that

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Additional computation shows that

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The reader can now see that $\mathbf{A}^{[n]} = \mathbf{A}^{[5]}$ for all positive integers n with $n \geq 5$.

Operation: Inverse

The inverse of a matrix can be important in some applications.

It only works for square matrices.

B is the inverse of A if $A * B = B * A$.

The inverse of A is denoted by A^{-1}

If we multiply A by A^{-1} we get the identity matrix I.

Note that $I * A = A * I = A$

Calculating The Inverse

Not all matrices have an inverse.

First thing to do is to check whether it has . . . essentially matrix A only has an inverse if rows in the matrix are not multiples of each other.

Inverse rule

A matrix P has an inverse if the determinant of P is not equal to zero.

$\det P \neq 0$

Determinant

Suppose $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ Then the determinant is *det P = ps − qr*.

If this is non-zero we can find the inverse.

For larger matrices we do something very straightforward.

We use the 2x2 matrices contained in the matrix (Cofactor expansion method).

Suppose $P = \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix}$ then

$$det\ P = p * det \begin{bmatrix} t & u \\ w & x \end{bmatrix} + q * det \begin{bmatrix} s & u \\ v & x \end{bmatrix} + r * det \begin{bmatrix} s & t \\ v & w \end{bmatrix}$$

Example

Cofactor expansion along the 2nd column.

Det(A)= $a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32}+a_{42}C_{42}$

Since a_{12}, a_{32}, and a_{42} are 0 then:

Det(A)= $a_{22}C_{22}$

$C_{22} = (-1)^{2+2} M_{22}$

Cofactor expansion of M_{22} using the second column.

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$\det(A) = 1 \cdot -2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2(1+2)$$

$$= -6$$

Theorem 1

If $A(n \times n)$ is an upper/lower triangular matrix, then Det(A) is the **product of all elements in its main diagonal.**

Example:

$$A = \begin{pmatrix} \mathbf{2} & 7 & -3 \\ 0 & \mathbf{-3} & 7 \\ 0 & 0 & \mathbf{6} \end{pmatrix}$$

Det$(A) = 2\times(-3)\times 6 = -36$

Proof:

$$\left(\begin{array}{ccc|cc} 2 & 7 & -3 & 2 & 7 \\ 0 & -3 & 7 & \mathbf{0} & -3 \\ \mathbf{0} & \mathbf{0} & 6 & \mathbf{0} & \mathbf{0} \end{array}\right)$$

Det$(A) = 2\times(-3)\times 6 = -36$

Calculating the inverse

Once we have calculated the determinant and we know the inverse exists there are various ways to calculate the inverse.

But we will use the Gaussian elimination approach.

We start with a matrix A and **augment** it with the identity matrix and then manipulate this.

Suppose $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ we augment it to give $\left[\begin{array}{cc|cc} p & q & 1 & 0 \\ r & s & 0 & 1 \end{array}\right]$

We then manipulate this by swopping rows, multiplying rows by a constant or adding multiples of rows to put the identity matrix on the left hand side.

Elementary Row Operations

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

$$\left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1 \quad 2 \quad 3}}^{\text{Matrix A}}} & & & \\ 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

By using ERO we have:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \\ & & & \multicolumn{3}{c}{\underbrace{\hphantom{-40 \quad 16 \quad 9}}_{\text{invers of A}}} \end{array}\right]$$

Elementary Row Operations

$$\overbrace{\begin{matrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{matrix}}^{\text{matrix } A} \quad \Bigg| \quad \overbrace{\begin{matrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{matrix}}^{\text{invers of } A}$$

Multiplication of these matrices yield:

$$\begin{pmatrix} -40+26+15 & 16-10-6 & 9-6-3 \\ -80+65+15 & 32-25-6 & 18-15-3 \\ -40+0+40 & 16-0-16 & 9-0-8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{AA^{-1} = I}$$

Calculating the inverse

Find the invers of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

R2 = R2 – 2R1
R3 = R3 – R1

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$$

R3 = R3 + 2R2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

R3 = (-1) x R3

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

R2 = R2 + 3R3
R1 = R1 – 3R3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

R1 = R1 – 2R2

So,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Gaussian elimination

We often have systems of equations that need solving for a number of variables.

This is easy for equations with only two variables:

Suppose we have:

$2x + 4y = 22$ (Eq 1)

$x - y = -1$ (Eq. 2)

What do we do to find x and y?

- Multiply (2) by 2.

  $2x - 2y = -2$ (Eq. 3)

  (1) - (3) → $6y = 24$ so $y = 4$.
- Substitute in, say, (1).

  $2x + 16 = 22$

  $2x = 22 - 16$; $x = 3$

Gaussian elimination

Suppose we have more than two variables and limit ourselves to *k* equations and *k* unknowns.

Lets work with an example

$$2x + 4y + 3z = 26$$

$$x - y - z = -1$$

$$3x + 2y - z = 16$$

We have the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 4 & 3 & 26 \\ 1 & -1 & -1 & -1 \\ 3 & 2 & -1 & 16 \end{array}\right)$$

Gaussian elimination

We wish to manipulate the augmented matrix so that we have a normalized matrix. That is, where the numbers below the diagonal become zero.

We do this the elementary row operations.

First, lets get rid of the 1 in position (2,1). Multiply second row by -2, then add row 1 to row 2

$$\left(\begin{array}{ccc|c} 2 & 4 & 3 & 26 \\ 1 & -1 & -1 & -1 \\ 3 & 2 & -1 & 16 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 4 & 3 & 26 \\ -2 & 2 & 2 & 2 \\ 3 & 2 & -1 & 16 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 4 & 3 & 26 \\ 0 & 6 & 5 & 28 \\ 3 & 2 & -1 & 16 \end{array}\right)$$

R2 = -2R2

R2 = R1 + R2

Gaussian elimination

Now to get rid of the 3 in (3,1). 2*row3 - 3*row1

$$\left(\begin{array}{ccc|c} 2 & 4 & 3 & 26 \\ 0 & 6 & 5 & 28 \\ 3 & 2 & -1 & 16 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 4 & 3 & 26 \\ 0 & 6 & 5 & 28 \\ 0 & -8 & -11 & -46 \end{array}\right)$$

R3 = 2R3 - 3R1

Now get rid of the -8

$$\left(\begin{array}{ccc|c} -6 & -12 & -9 & -78 \\ 0 & 6 & 5 & 28 \\ 0 & 0 & -13 & -26 \end{array}\right)$$

So the last row says: $-13z = -26$, so $z = 2$
Substitute in the previous row: $6y + 5z = 28$
So $6y = 18$ and $y = 3$ and then substitute both in the first row to give $-6x = -24$ and $x = 4$

R3 = 3R3 + 4R2

Operation: multiplication

Suppose A is a matrix with rows M and columns P ($M * P$) and B is a matrix $P * N$.

The pseudo code for $C = A * B$ is:

```
for i = 1 to M
      for j = 1 to N
            c(i,j) = 0
            for k = 1 to P
                  c(i,j) = c(i,j) + a(i,k)*b(k,j)
            end
      end
end
```

The *c(i, j)* is the inner product of the *i*th row of A and the *j*th column of B.