KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)

Basics of Counting

Ahmad Muklason, Ph.D

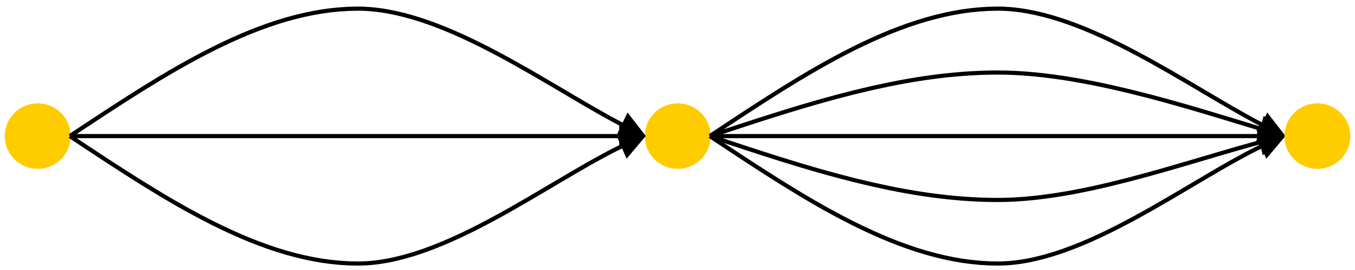
Outline

1. Product Rule
2. The Sum Rule
3. More Complex Counting Problem
4. The pigeonhole principle
5. Permutation
6. Combination

The product rule

If there are n_1 ways to do task 1, and n_2 ways to do task 2

- Then there are $n_1 n_2$ ways to do both tasks in sequence
- This applies when doing the "procedure" is made up of separate tasks
- We must make one choice **AND** a second choice



Product rule example

There are 18 math majors and 325 CS majors

How many ways are there to pick one math major and one CS major?

Total is 18 * 325 = 5850 ways

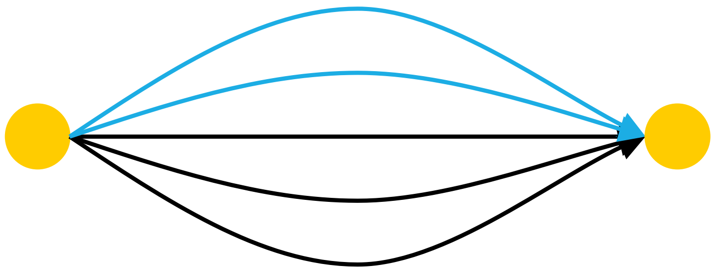
Product rule example

- How many strings of 4 decimal digits…
- Do not contain the same digit twice?
  - We want to chose a digit, then another that is not the same, then another…
    - First digit: 10 possibilities
    - Second digit: 9 possibilities (all but first digit)
    - Third digit: 8 possibilities
    - Fourth digit: 7 possibilities
  - Total = 10*9*8*7 = 5040
- End with an even digit?
  - First three digits have 10 possibilities
  - Last digit has 5 possibilities
  - Total = 10*10*10*5 = 5000

The sum rule

If there are n_1 ways to do task 1, and n_2 ways to do task 2

- If these tasks can be done at the same time, then…
- Then there are n_1+n_2 ways to do one of the two tasks
- We must make one choice **OR** a second choice



Sum rule example

There are 18 math majors and 325 CS majors

How many ways are there to pick one math major or one CS major?

Total is 18 + 325 = 343

Sum rule example

- How many strings of 4 decimal digits…
  - Have exactly three digits that are 9s?
    - The string can have:
      - The non-9 as the first digit
      - OR the non-9 as the second digit
      - OR the non-9 as the third digit
      - OR the non-9 as the fourth digit
      - Thus, we use the sum rule
    - For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
    - Thus, the answer is 9+9+9+9 = 36

More complex counting problems

We combining the product rule and the sum rule

Thus we can solve more interesting and complex problems

Wedding pictures example

Consider a wedding picture of 6 people

- There are 10 people, including the bride and groom

How many possibilities are there if the bride must be in the picture

- Product rule: place the bride AND then place the rest of the party
- First place the bride
  - She can be in one of 6 positions
- Next, place the other five people via the product rule
  - There are 9 people to choose for the second person, 8 for the third, etc.
  - Total = 9*8*7*6*5 = 15120
- Product rule yields 6 * 15120 = 90,720 possibilities

Wedding pictures example

Consider a wedding picture of 6 people
- There are 10 people, including the bride and groom

How many possibilities are there if the bride and groom must both be in the picture
- Product rule: place the bride/groom AND then place the rest of the party
- First place the bride and groom
  - She can be in one of 6 positions
  - He can be in one 5 remaining positions
  - Total of 30 possibilities
- Next, place the other four people via the product rule
  - There are 8 people to choose for the third person, 7 for the fourth, etc.
  - Total = 8*7*6*5 = 1680
- Product rule yields 30 * 1680 = 50,400 possibilities

Wedding pictures example

Consider a wedding picture of 6 people
- There are 10 people, including the bride and groom

How many possibilities are there if only one of the bride and groom are in the picture
- Sum rule: place only the bride
  - Product rule: place the bride AND then place the rest of the party
  - First place the bride
    - She can be in one of 6 positions
  - Next, place the other five people via the product rule
    - There are 8 people to choose for the second person, 7 for the third, etc.
      - We can’t choose the groom!
  - Total = 8*7*6*5*4 = 6720
  - Product rule yields 6 * 6720 = 40,320 possibilities
- OR place only the groom
  - Same possibilities as for bride: 40,320
- Sum rule yields 40,320 + 40,320 = 80,640 possibilities

Wedding pictures example

Consider a wedding picture of 6 people

- There are 10 people, including the bride and groom

Alternative means to get the answer

How many possibilities are there if only one of the bride and groom are in the picture

- Total ways to place the bride (with or without groom): 90,720 → From part (a)
- Total ways for both the bride and groom: 50,400 → From part (b)
- Total ways to place ONLY the bride: 90,720 – 50,400 = 40,320
- Same number for the groom
- Total = 40,320 + 40,320 = 80,640

The inclusion-exclusion principle

When counting the possibilities, we can't include a given outcome more than once!

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

- Let A_1 have 5 elements, A_2 have 3 elements, and 1 element be both in A_1 and A_2
- Total in the union is 5 + 3 – 1 = 7, not 8

Inclusion-exclusion example

How many bit strings of length eight start with 1 or end with 00?

Count bit strings that start with 1

- Rest of bits can be anything: $2^7 = 128$
- This is $|A_1|$

Count bit strings that end with 00

- Rest of bits can be anything: $2^6 = 64$
- This is $|A_2|$

Count bit strings that both start with 1 and end with 00

- Rest of the bits can be anything: $2^5 = 32$
- This is This is $|A_1 \cap A_2|$

Use formula $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Total is $128 + 64 – 32 = 160$

Bit string possibilities

How many bit strings of length 10 contain either 5 consecutive 0s or 5 consecutive 1s?

Consider 5 consecutive 0s first

Sum rule: the 5 consecutive 0's can start at position 1, 2, 3, 4, 5, or 6

- Starting at position 1: Remaining 5 bits can be anything: $2^5 = 32$
- Starting at position 2
  - First bit must be a 1: Otherwise, we are including possibilities from the previous case!
  - Remaining bits can be anything: $2^4 = 16$

Bit string possibilities (cont'd)

- Starting at position 3
  - Second bit must be a 1 (same reason as above)
  - First bit and last 3 bits can be anything: $2^4 = 16$
- Starting at positions 4 and 5 and 6
  - Same as starting at positions 2 or 3: 16 each
- Total = 32 + 16 + 16 + 16 + 16 + 16 = 112

The 5 consecutive 1's follow the same pattern, and have 112 possibilities

There are two cases counted twice (that we thus need to exclude): 0000011111 and 1111100000

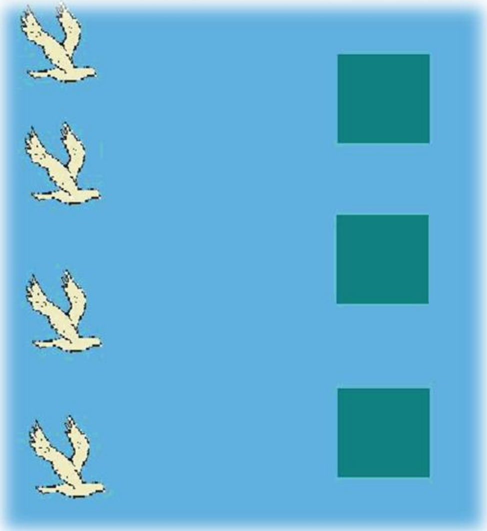
Total = 112 + 112 – 2 = 222

The pigeonhole principle

Suppose a flock of pigeons fly into a set of pigeonholes to roost

If there are more pigeons than pigeonholes, then there must be at least one pigeonhole that has more than one pigeon in it

If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects



Pigeonhole principle examples

In a group of 367 people, there must be two people with the same birthday

As there are 366 possible birthdays

In a group of 27 English words, at least two words must start with the same letter

As there are only 26 letters

Generalized pigeonhole principle

If N objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ objects

Generalized pigeonhole principle

Among 100 people, there are at least

$\lceil 100/12 \rceil = 9$ born on the same month

How many students in a class must there be to ensure that 6 students get the same grade (one of A, B, C, D, or F)?

The "boxes" are the grades. Thus, $k = 5$

Thus, we set $\lceil N/5 \rceil = 6$

Lowest possible value for N is 26

Example

A bowl contains 10 red and 10 yellow balls

How many balls must be selected to ensure 3 balls of the same color?

- One solution: consider the “worst” case
  - Consider 2 balls of each color
  - You can’t take another ball without hitting 3, Thus, the answer is 5
- Via generalized pigeonhole principle
  - How many balls are required if there are 2 colors, and one color must have 3 balls?
  - How many pigeons are required if there are 2 pigeon holes, and one must have 3 pigeons?
  - number of boxes: $k = 2$, We want $\lceil N/k \rceil = 3$
  - What is the minimum N? $N = 5$

Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether order matters or not
- Consider a poker hand:
  - A♦, 5♥, 7♣, 10♠, K♠
- Is that the same hand as:
  - K♠, 10♠, 7♣, 5♥, A♦
- Does the order of the cards are handed out matter?
  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations

Permutations

A permutation is an ordered arrangement of the elements of some set *S*.

- Let $S = \{a, b, c\}$
- c, b, a is a permutation of S
- b, c, a is a *different* permutation of S

An *r*-permutation is an ordered arrangement of *r* elements of the set.

- A♦, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards

The notation for the number of *r*-permutations: $P(n, r)$

- The poker hand is one of P(52, 5) permutations

Permutations

r-permutation notation: $P(n, r)$

The poker hand is one of P(52, 5) permutations

Number of all possible poker hands (consist of 5 cards):

$P(52, 5) = 52*51*50*49*48 = 311{,}875{,}200$

$$P(n, r) = n\,(n-1)(n-2)\ldots(n-r+1)$$

$$= n!\,/\,(n-r)!$$

r-permutations example

How many ways are there to select a 1st, 2nd, and 3rd winner from 100 different people who have entered a contest?

- $P(100, 3) = 100*99*98 = 970,200$

How many ways are there for 5 people in the class to give presentations if there are 27 people in the class?

- There are 27 students in the class
- $P(27,5) = 27*26*25*24*23 = 9,687,600$
- Note that the order they go in does matter in this example!

Permutation formula proof

There are n ways to choose the first element

n-1 ways to choose the second

n-2 ways to choose the third

…

n-r+1 ways to choose the r^{th} element

By the product rule, that gives us:

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$$

Example

How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

- Note that the set has 7 elements

The last character must be a

- The rest can be in any order

Thus, we want a 6-permutation on the set $\{b, c, d, e, f, g\}$

$P(6,6) = 6! = 720$

Combinations

What if order doesn’t matter?

In poker, the following two hands are equivalent:

A♦, 5♥, 7♣, 10♠, K♠

K♠, 10♠, 7♣, 5♥, A♦

The number of *r*-combinations of a set with *n* elements, where *n* is non-negative integer and $0 \leq r \leq n$ is:

$$C(n, r) = n! / (r! (n-r)!)$$

Ex.: How many different poker hands are there (5 cards)?

$C(52, 5) = 2,598,960$

Combination formula proof

Let $C(52, 5)$ be the number of ways to generate unordered poker hands

The number of ordered poker hands is $P(52, 5) = 311{,}875{,}200$

The number of ways to order a single poker hand is $P(5, 5) = 5! = 120$

The total number of unordered poker hands is the total number of ordered hands divided by the number of ways to order each hand

Thus, $C(52, 5) = P(52, 5)/P(5, 5)$

Combination formula proof

Let $C(n, r)$ be the number of ways to generate unordered combinations

The number of ordered combinations (i.e. r-permutations) is $P(n, r)$

The number of ways to order a single one of those r-permutations $P(r, r)$

The total number of unordered combinations is the total number of ordered combinations (i.e. r-permutations) divided by the number of ways to order each combination

Thus, $C(n, r) = P(n, r)/P(r, r)$

Combination formula proof

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

Examples

How many bit strings of length 10 contain:

- exactly four 1's?
  - Find the positions of the four 1's
  - Does the order of these positions matter? Nope!
  - Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
  - Thus, the answer is $C(10, 4) = 210$

- at most four 1's?
  - There can be 0, 1, 2, 3, or 4 occurrences of 1
  - Thus, the answer is:
  - $C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3) + C(10, 4)$
  - $= 1 + 10 + 45 + 120 + 210 = 386$

Examples

How many bit strings of length 10 contain:

- at least four 1's?
  - There can be 4, 5, 6, 7, 8, 9, or 10 occurrences of 1
  - Thus, the answer is:
  - $C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10)$
    - $= 210+252+210+120+45+10+1$
    - $= 848$
  - Alternative answer: subtract from 2^{10} the number of strings with 0, 1, 2, or 3 occurrences of 1

- an equal number of 1's and 0's?
  - Thus, there must be five 0's and five 1's
  - Find the positions of the five 1's
  - Thus, the answer is $C(10,5) = 252$

Combinatorial proof

A *combinatorial proof* is a proof that uses counting arguments to prove a theorem

Rather than some other method such as algebraic techniques

Essentially, show that both sides of the proof manage to count the same objects

Most of the questions in this section are phrased as, "find out how many possibilities there are if …"

Instead, we could phrase each question as a theorem:

"Prove there are *x* possibilities if …"

Corollary

Let n and r be non-negative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$

Proof: $$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

and

$$C(n, n-r) = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!}$$

Corollary example

There are C(52, 5) ways to pick a 5-card poker hand

There are C(52, 47) ways to pick a 47-card hand

c(52, 5) = 2,598,960 = C(52, 47)

When dealing 47 cards, you are picking 5 cards to not deal

- As opposed to picking 5 card to deal

Example

How many ways are there to sit 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?

First, place the first person in the north-most chair

Only one possibility

Then place the other 5 people

There are P(5, 5) = 5! = 120 ways to do that

By the product rule, we get 1*120 =120

Alternative means to answer this:

There are P(6, 6) = 720 ways to seat the 6 people around the table

For each seating, there are 6 “rotations” of the seating

Thus, the final answer is 720/6 = 120