Refreshment Quiz

1. Determine the truth value of the statement $\forall x \exists y\ (xy = 1)$ if the domain for the variables consists of:
   a) the nonzero integers.
   b) the positive real numbers.
2. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

Answer

1. The truth values are:
   a) False
   b) True

2. Let w be "Randy works hard," let d be "Randy is a dull boy," and let j be "Randy will get the job." The hypotheses are w, $w \to d$, and $d \to \neg j$.
   a) w 1st hypothesis
   b) $w \to d$ 2nd hypothesis
   c) d modus ponen from (a) and (b)
   d) $d \to \neg j$ 3rd hypothesis
   e) $\neg j$ modus ponen from (c) and (d)

Refreshment Quiz

1. Find a **counterexample**, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
   a) $\forall x \forall y\ (x^2 = y^2 \rightarrow x = y)$
   b) $\forall x \exists y\ (y^2 = x)$
   c) $\forall x \forall y\ (xy \geq x)$

2. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

Answer

1. The counterexamples are:
   a) $x = 2, y = -2$; b) $x = -4$; c) $x = 17, y = -1$

2. Let *w* be "Randy works hard," let *d* be "Randy is a dull boy," and let *j* be "Randy will get the job." The hypotheses are w, $w \to d$, and $d \to \neg j$.

a)	w	1st hypothesis
b)	$w \to d$	2nd hypothesis
c)	d	modus ponen from (a) and (b)
d)	$d \to \neg j$	3rd hypothesis
e)	$\neg j$	modus ponen from (c) and (d)

Refreshment Quiz

1. Determine the truth value of each of these statements if the domain for all variables consists of all integers.
   a) $\forall n \exists m\ (n^2 < m)$
   b) $\exists n \forall m\ (nm = m)$
   c) $\forall n \exists m\ (n + m = 0)$
   d) $\exists n \exists m\ (n^2 + m^2 = 6)$

2. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

Answer

1. The counterexamples are:
   a) True; b) True; c) True; d) False

2. Let *w* be "Randy works hard," let *d* be "Randy is a dull boy," and let *j* be "Randy will get the job." The hypotheses are w, $w \to d$, and $d \to \neg j$.

a)	w	1st hypothesis
b)	$w \to d$	2nd hypothesis
c)	d	modus ponen from (a) and (b)
d)	$d \to \neg j$	3rd hypothesis
e)	$\neg j$	modus ponen from (c) and (d)

KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)

INTRODUCTION TO PROOF

Eko Wahyu Tyas Darmaningrat, S.Kom, MBA.
Email. tyas.darmaningrat@gmail.com
Phone. 082331106699
Office. TC-217

Outline

1) Direct Proofs (Bukti Langsung)
2) Indirect Proofs (Bukti Tidak Langsung)
3) Proof by Contradiction (Bukti dengan Kontradiksi)
4) Vacuous Proofs (Bukti Hampa)
5) Trivial Proofs (Bukti Mudah)
6) Proof by Cases (Bukti per Kasus)

7. Proofs of Equivalences (Bukti Ekuivalensi)
8. Existence Proofs
9. Uniqueness Proofs
10. Counterexamples
11. Mistakes in Proofs
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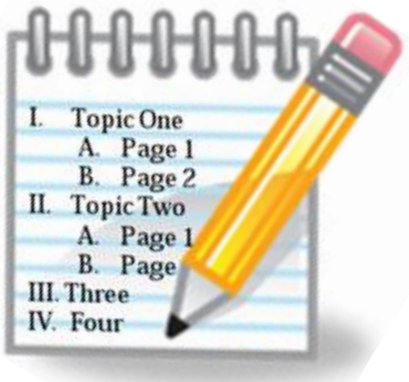

How Theorems are Stated

Many theorems assert that a property holds for **all elements in a domain**, such as the integers or real numbers.

Although the precise statement of a theorem needs to include **universal quantifier**, the standard convention in mathematics is to **omit** it.

For example:

- "If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$".

Really means:

- "**For all** positive real numbers x and y, if $x > y$, then $x^2 > y^2$".

Direct Proofs

Consider an implication: $p \rightarrow q$

- If ***p* is false**, then the implication is **always true**.

A direct proof shows that $p \rightarrow q$ is true by showing that if ***p*** is true, then ***q*** must also be true, so that the combination ***p*** is true and ***q*** is false never occurs.

To perform a direct proof, assume that ***p*** is true, and show that ***q*** must therefore be true.

Example of Direct Proofs

Definition: The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$.

Show that the square of an even number is an even number

- Rephrased: if n is even, then n^2 is even

Proof: Assume n is even

- Thus, $n = 2k$, for some integers k (definition of even numbers)
- $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
- As n^2 is 2 times an integer, n^2 is thus even

Indirect Proofs

Consider an implication: $p \to q$

- It's contraposition is $\neg q \to \neg p$
  - Is logically equivalent to the original implication!
  - Indirect proofs also known as proof by contraposition.
- If the antecedent ($\neg q$) is false, then the contrapositive is always true
- Thus, show that if $\neg q$ is true, then $\neg p$ is true

To perform an indirect proof, do a direct proof on the contraposition.

Example of Indirect Proofs

Prove that if n is an integer and $3n+2$ is odd, then n is odd.

- Prove the contraposition: If n is even, then $3n+2$ *is* is even.

Proof: $n=2k$ for some integers k (definition of even numbers)

- Assume that n is even. Then, $n=2k$ for some integer k.
- Substituting $2k$ for n, we find that $3n+2=3(2k)+2=6k+2=2(3k+1)$. This tells us that $3n+2$ is even (because it is a multiple of 2), and therefore not odd.
- This is the contraposition of the theorem.
- Because the contraposition of the conditional statement is true.
- Our proof by contraposition succeeded; we have proved the theorem "If 3n+2 is odd, then n is odd."

Which One to Use?

When do you use a direct proof versus an indirect proof?

- If it’s not clear from the problem, try direct first, then indirect second
- If indirect fails, try the other proofs

Example of Which to Use

Prove that if n is an integer and n^3+5 is odd, then n is even

Via direct proof

- $n^3+5 = 2k+1$ for some integer k (definition of odd numbers)
- $n^3 = 2k - 4$
- $n = \sqrt[3]{2k-4}$

So direct proof didn't work out.

Next up: indirect proof

Example of Which to Use

Prove that if n is an integer and n^3+5 is odd, then n is even

Via indirect proof

- Contrapositive: If n is odd, then n^3+5 is even
- Assume n is odd, and show that n^3+5 is even
- $n=2k+1$ for some integer k (definition of odd numbers)
- $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
- As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it is even

Proof by Contradiction

In a proof of $p \rightarrow q$ by contraposition (indirect proof), we assume that $\neg q$ is true and then show that $\neg p$ must also be true.

To proof by contradiction, we suppose that **both** p and $\neg q$ are true. Then we use the steps from the proof of $\neg q \rightarrow \neg p$ to show that $\neg p$ is true.

- Assume p and $\neg q$ are true
- Show $\neg p$ is true by using its contrapositive $\neg q \rightarrow \neg p$

This leads to the contradiction $\boldsymbol{p \land \neg p}$ completing the proof.

Example of Proof by Contradiction

Give a proof by contradiction of the theorem “If $3n + 2$ is odd, then n is odd.”

Solution:

- Assume that p and $\neg q$ are true: $3n + 2$ is odd, and n is not odd (it is even)
- Following the steps of proof by contraposition, we can show that if n is even, then $3n + 2$ is even
- $n = 2k \rightarrow 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$
- $3n + 2 = 2t$, where $t = (3k + 1)$, so $3n + 2$ is even ($\neg p$)
- Because both p and $\neg p$ are true, we have a contradiction here.
- This completes the proof by contradiction, proving that if $3n + 2$ is odd, then n is odd

Vacuous Proofs (Bukti Hampa)

Consider an implication: $p \to q$

If it can be shown that p is false, then the implication is always true

- By definition of an implication

Note that you are showing that the antecedent (p) is false

Remark: Conditional statement with false hypothesis is guaranteed to be true

Example of Vacuous Proofs

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution:

- Note that $P(0)$ is "If $0 > 1$, then $0^2 > 0$"
- We can show $P(0)$ using vacuous proof because the hypothesis $0 > 1$ is false.
- This tells us that $P(0)$ is automatically true.

Trivial Proofs (Bukti Mudah)

Consider an implication: $p \rightarrow q$

If it can be shown that q is true, then the implication is always true

- By definition of an implication

Note that you are showing that the conclusion (q) is true

Example of Trivial Proofs

Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all integers. Show that $P(0)$ is true.

Solution:

- The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$"
- Because $a^0 = b^0 = 1$, the conclusion of the conditional statement is true.
- Hence, the conditional statement, which is $P(0)$, is true.

Proof by Cases

Show a statement is true by showing all possible cases are true

Thus, you are showing a statement of the form: $(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$

is true by showing that:

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$$

Example of Proof by Cases

Prove that "$|xy| = |x||y|$" for all real numbers.

The possible cases:

	x	y	|xy|	|x| |y|
1	≥ 0	≥ 0	|xy|	|x||y|
2	≥ 0	< 0	|x(-y)|	|x||-y|
3	< 0	≥ 0	|(-x)y)|	|-x||y|
4	< 0	< 0	|(-x)(-y)|	|-x||-y|

We can conclude that $|xy| = |x||y|$ whenever x and y are real numbers.

The thing about proof by cases

Make sure you get ALL the cases

- The biggest mistake is to leave out some of the cases

Proofs of Equivalences

This is showing the definition of a bi-conditional

Given a statement of the form "*p* if and only if *q*"

- Show it is true by showing $(p \rightarrow q) \land (q \rightarrow p)$ is true

Proofs of Equivalences Example

Show that $m^2=n^2$ if and only if $m=n$ or $m=-n$

- Rephrased: $(m^2=n^2) \leftrightarrow [(m=n)\vee(m=-n)]$

Need to prove two parts:

- $(m^2=n^2) \rightarrow [(m=n)\vee(m=-n)]$
  - Subtract n^2 from both sides to get $m^2-n^2=0$
  - Factor to get $(m+n)(m-n) = 0$
  - Since that equals zero, one of the factors must be zero.
  - Thus, either $m+n=0$ (which means $m=-n$)
  - Or $m-n=0$ (which means $m=n$)
- $[(m=n)\vee(m=-n)] \rightarrow (m^2=n^2)$
  - Proof by cases!
  - Case 1: $(m=n) \rightarrow (m^2=n^2)$
    - $(m)^2 = m^2$, and $(n)^2 = n^2$, so this case is proven
  - Case 2: $(m=-n) \rightarrow (m^2=n^2)$
    - $(m)^2 = m^2$, and $(-n)^2 = n^2$, so this case is proven

Existence Proofs

Given a statement: $\exists x\, P(x)$

We only have to show that a $P(c)$ exists for some value of c.

Two types:

- **Constructive**: Find a specific value of c for which $P(c)$ exists.
- **Non-constructive**: Show that such a c exists, but don't actually find it.
  - Assume it does not exist, and show a contradiction.

Example Constructive Existence Proof

Show that a square exists that is the sum of two other squares

- Proof: $3^2 + 4^2 = 5^2$

Show that a cube exists that is the sum of three other cubes

- Proof: $3^3 + 4^3 + 5^3 = 6^3$

Example Non-constructive Existence Proof

Prove that either $2*10^{500}+15$ or $2*10^{500}+16$ is not a perfect square

- A perfect square is a square of an integer
- Rephrased: Show that a non-perfect square exists in the set $\{2*10^{500}+15, 2*10^{500}+16\}$

Proof: The only two perfect squares that differ by 1 are 0 and 1

- Thus, any other numbers that differ by 1 cannot both be perfect squares
- Thus, a non-perfect square must exist in any set that contains two numbers that differ by 1
- Note that we didn't specify which one it was!

Uniqueness Proofs

A theorem may state that only one such value exists.

To prove this, you need to show:

- Existence: that such a value does indeed exist.
  - Either via a constructive or non-constructive existence proof.
- Uniqueness: that there is only one such value.

Uniqueness Proof Example

Show that if a and b are real number and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Existence

- Since $a(-b/a) + b = 0$, consequently the real number $r = -b/a$ is a solution of $ar + b = 0$
- A real number r exist for which $ar + b = 0$

Uniqueness

- Suppose that s is real number such that $as + b = 0$. Then $ar + b = as + b$, where $r = -b/a$
- Substracting b from both sides we find that $ar = as$.
- Dividing both sides by a, which is non zero, we see that $r = s$.
- This means that if $r \neq s$, then $as + b \neq 0$
  - Thus, the one solution is unique!

Counterexamples

Given a universally quantified statement, find a single example which it is not true.

Note that this is DISPROVING a UNIVERSAL statement by a counterexample.

$\forall x \neg R(x)$, where $R(x)$ means "x has red hair"

- Find one person (in the domain) who has red hair.

Every positive integer is the square of another integer

- The square root of 5 is 2.236, which is not an integer.

Mistakes in Proofs

If n^2 is an even integer, then n is an even integer.

(Proof) :

- Suppose n^2 is even.
- Then $n^2 = 2k$ for some integer k.
- Let $n = 2l$ for some integer l.
- Then n is an even integer.

What's wrong with this proof?

- Many incorrect arguments are based on a fallacy called begging the question/circular reasoning. This fallacy occurs when a statement is proved using itself or a statement that equivalent to it.
- The result is correct only the method of proof is wrong.

Exercise ☺

Show that "If n^2 is an odd integer, then n is an odd integer" by using indirect proofs.

Prove the contrapositive: If n is an even integer, then n^2 is an even integer.

Proof:

$n=2k$ for some integers k (definition of even numbers)

$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

Since n^2 is 2 times an integer, it is even

Exercise ☺

Show that these statements are equivalent:

- $p1$: n is an even integer
- $p2$: n-1 is an odd integer
- $p3$: n^2 is an even integer

Show that $p1 \rightarrow p2$, $p2 \rightarrow p3$, and $p3 \rightarrow p1$ are TRUE.

Proof of $p1 \rightarrow p2$: If n is an even integer, then n-1 is an odd integer (direct proof)

- Assume $n = 2k$ (even)
- $n-1 = 2k-1 = 2(k-1) + 1$ (odd)
- Proved!

Exercise ☺

Proof of $p2 \rightarrow p3$: if n-1 is an odd integer, then n^2 is an even integer (direct proof)

- Assume $n - 1 = 2k + 1$ (odd)
- $n - 1 = 2k + 1$
- $n = 2k + 2$
- $n^2 = (2k + 2)^2 = 4k^2 + 8k + 4 = 2(2k^2 + 4k + 2)$ (even)
- Proved!

Proof of $p3 \rightarrow p1$: If n^2 is an even integer, then n is an even integer (indirect proof)

- Contraposition: If n is an odd integer, then n^2 is an odd integer
- Assume $n = 2k + 1$ (odd)
- $n^2 = (2k+1)^2 = 2k^2 + 4k + 1 = 2(k^2 + 2k) + 1$ (odd)
- Proved!

Glossary

Teorema (*theorem*): pernyataan yang dapat dibuktikan kebenarannya

Argumen (*argument*): rangkaian pernyataan yang membentuk bukti

Aksioma (*axiom*): pernyataan yang digunakan dalam suatu bukti, yang kebenarannya bisa diasumsikan, diketahui, atau telah dibuktikan sebelumnya

Aturan penentuan kesimpulan (*rule of inference*): cara menarik kesimpulan dari pernyataan-pernyataan sebelumnya

Lemma: teorema sederhana yang digunakan dalam membuktikan teorema lain

Corollary: proposisi yang merupakan akibat langsung dari teorema yang dibuktikan

Conjecture: pernyataan yang nilai kebenarannya belum diketahui