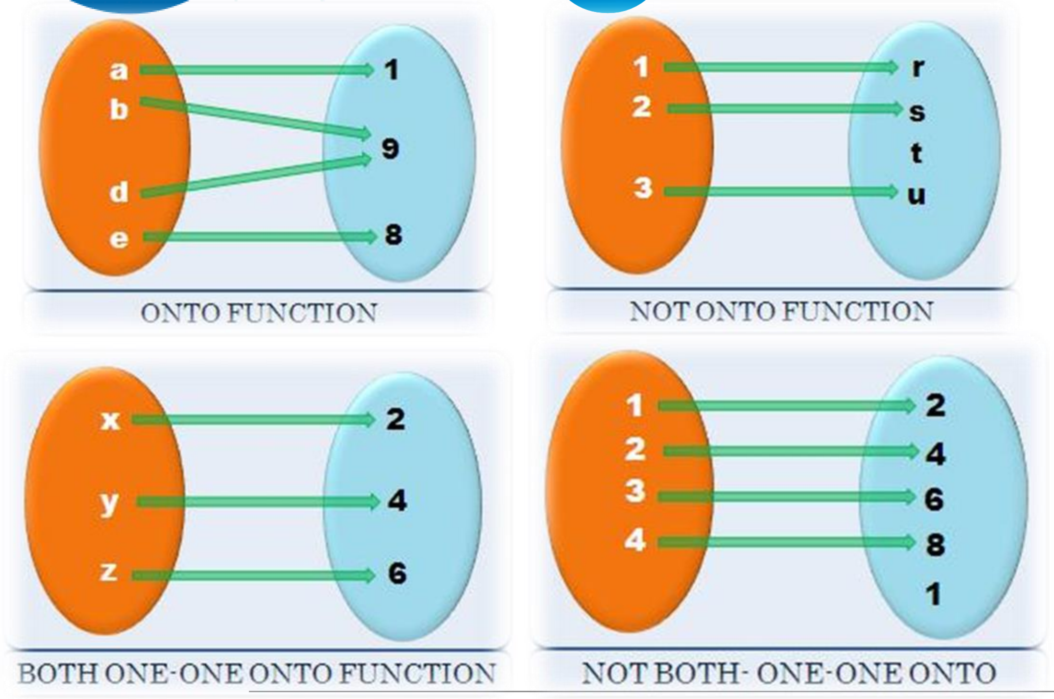
KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)



Functions

Ahmad Muklason, Ph.D.

sistem informasi
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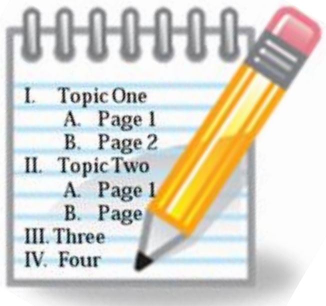
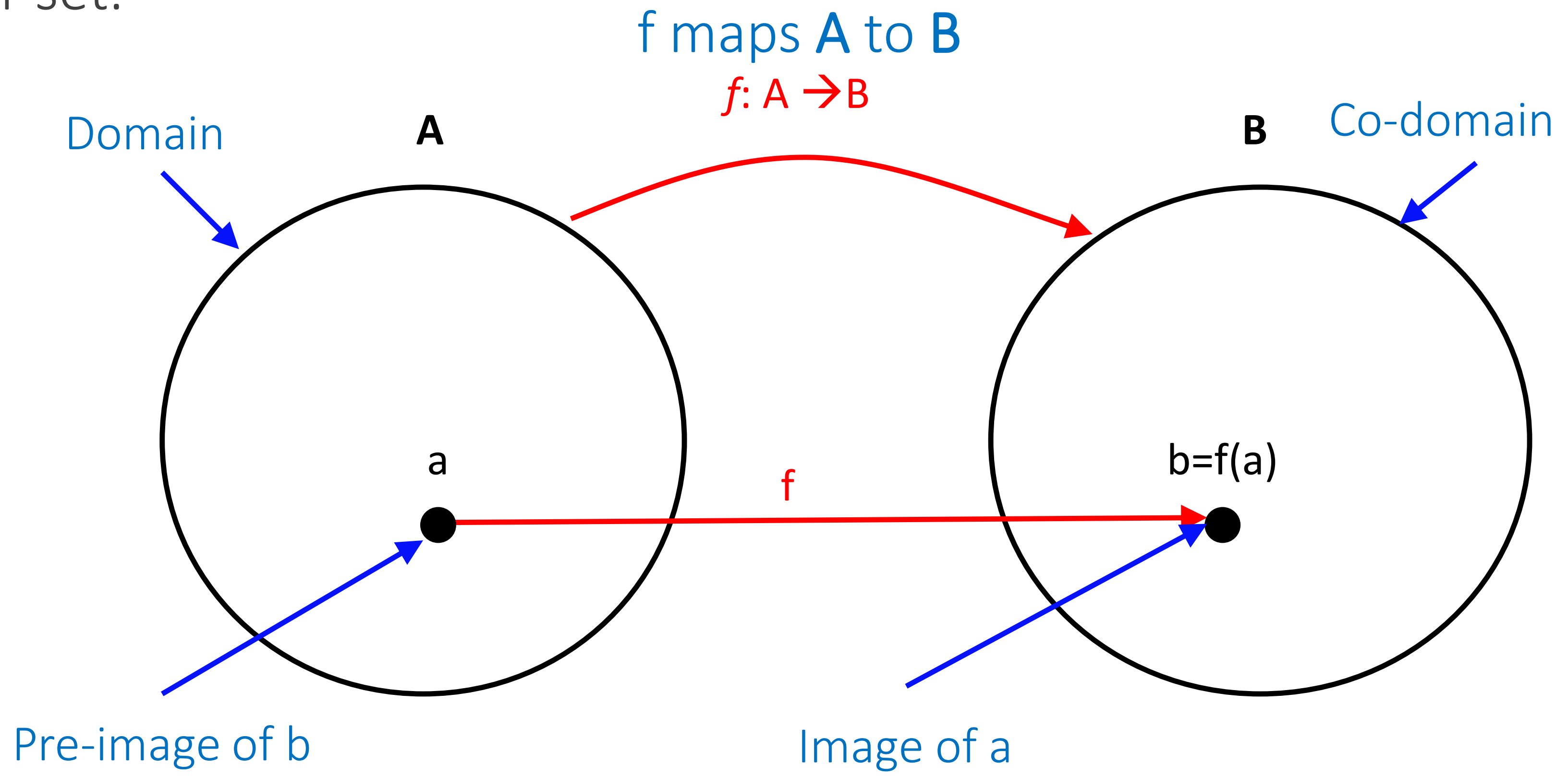
Outline

1. Definition of function
2. Function arithmetic
3. One-to-one functions
4. Onto functions
5. Bijections
6. Identity functions
7. Inverse functions
8. Composition of functions
9. Some useful functions
10. Proofing problems

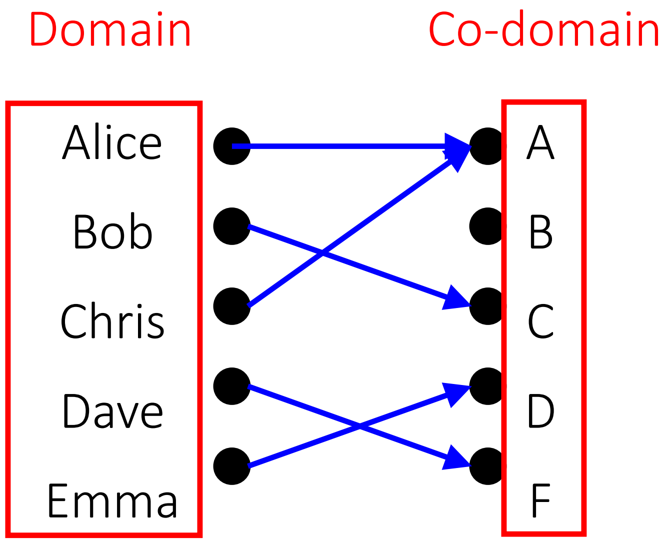


Definition of a function

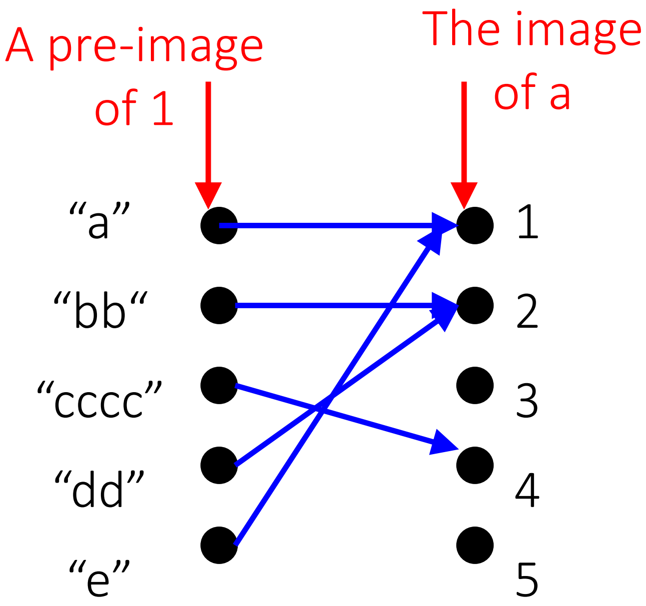
A function takes an element from a set and maps it to a **UNIQUE** element in another set.



More functions

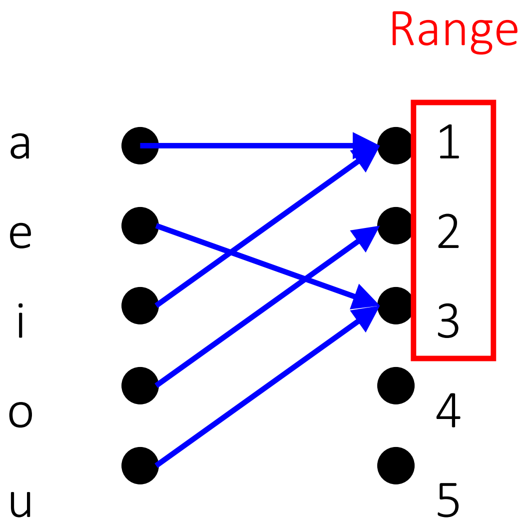


A class grade function

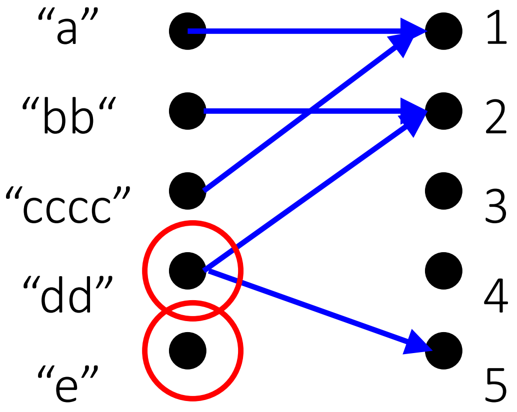


A string length function

Even More functions ☺



Some function…



Not a valid function!

Also not a valid function!

Function arithmetic

Let $f_1(x) = 2x$

Let $f_2(x) = x^2$

Let f_1 and f_2 are function from A to **R**.

Then $f_1 + f_2$ and $f_1 * f_2$ are also function from A to **R** defined by:

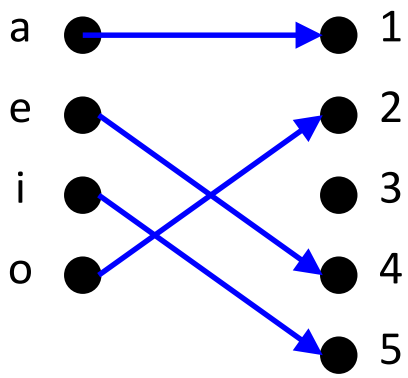
$$f_1 + f_2 = (f_1 + f_2)(x) = f_1(x) + f_2(x) = 2x + x^2$$

$$f_1 * f_2 = (f_1 * f_2)(x) = f_1(x) * f_2(x) = 2x * x^2 = 2x^3$$

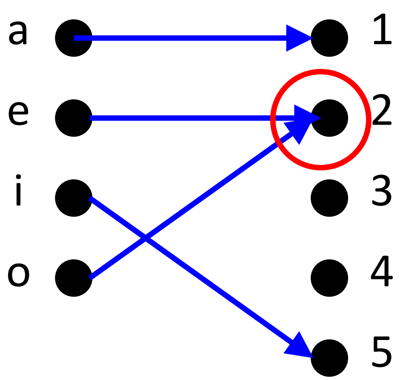
One-to-one functions

A function *f*: A → B is one-to-one if each element in the co-domain has a unique pre-image

- f is one to one ↔ $\forall a \forall b\ [\ f(a) = f(b) \rightarrow a = b]$
- Or equivalently $\forall a\ \forall b\ [a \neq b \rightarrow f(a) \neq f(b)]$



A one-to-one function



A function that is not one-to-one

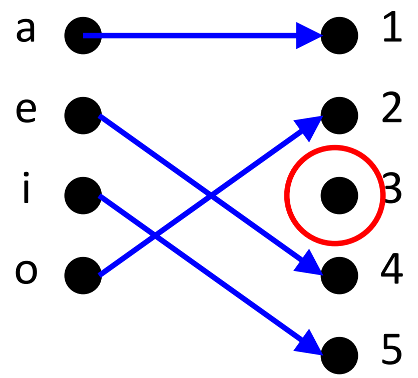
More on one-to-one

Injective is synonymous with one-to-one

A function is an injection if it is one-to-one

- Meaning no 2 values map to the same result

Note that there can be un-used elements in the co-domain



A one-to-one function

Example

Determine whether the function $f(x) = x^2$ from **Z** to **Z** is one to one.

The function $f(x) = x^2$ is not one to one, because for instance $f(1) = f(-1) = 1$, but $1 \neq -1$.

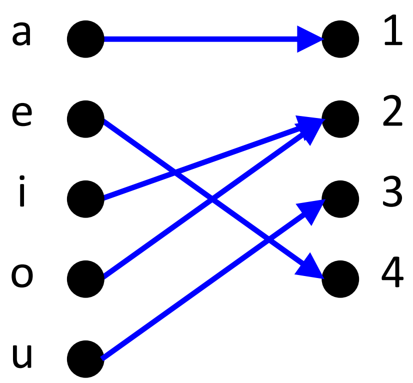
Is the function $f(x) = x + 1$ one to one?

The function is one to one. **(Note:** $x + 1 \neq y + 1$ if $x \neq y$**).**

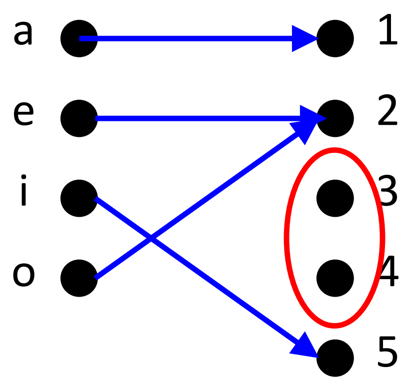
Onto functions

A function f: A → B is onto if each element in the co-domain is an image of some pre-image

- f is onto ↔ $\forall y \, \exists x \, [\, f(x) = y \,]$



An onto function



A function that is not onto

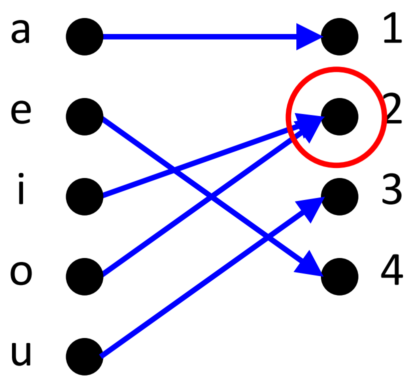
More on onto

Surjective is synonymous with onto

A function is an surjection if it is onto

Meaning all elements in the right are mapped to

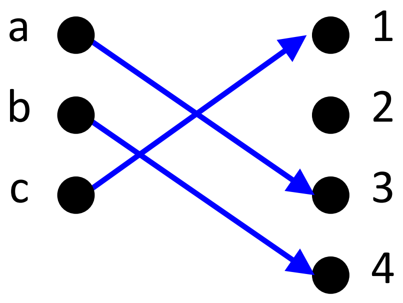
Note that there can be multiply used elements in the co-domain



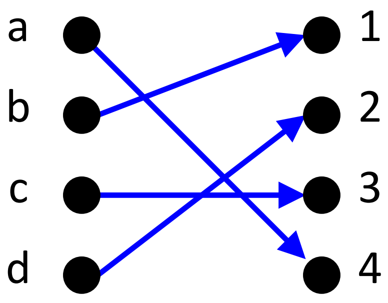
An onto function

Onto vs. one-to-one

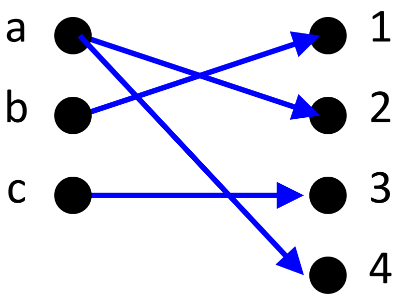
Are the following functions onto, one-to-one, both, or neither?



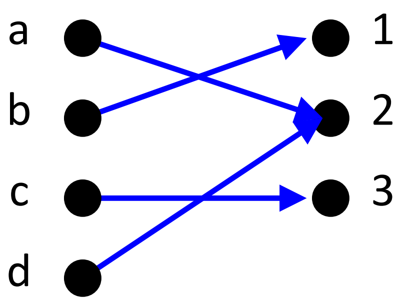
1-to-1, not onto



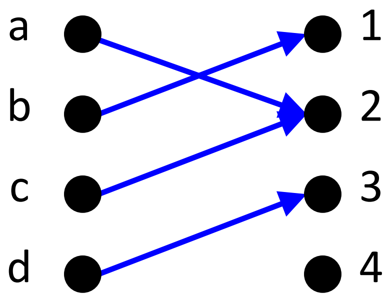
Both 1-to-1 and onto



Not a valid function



Onto, not 1-to-1



Neither 1-to-1 nor onto

Onto vs. one-to-one

Suppose that $f : A \rightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Example

Determine whether the function $f(x) = x^2$ from **Z** to **Z** is onto.

The function $f(x) = x^2$ is not onto because there is no integer x with $x^2 = -1$ for instance.

Is the function $f(x) = x + 1$ onto?

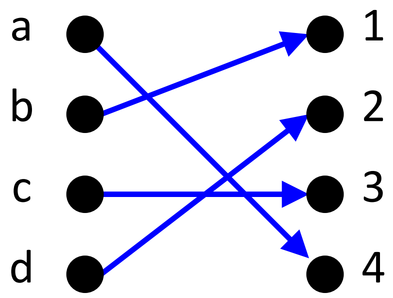
The function is onto because for every integer y there is an integer x such that $f(x) = y$.

(Note: $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$**)**

Bijections

Consider a function that is both one-to-one and onto:

Such a function is a one-to-one correspondence, or a bijection



Identity functions

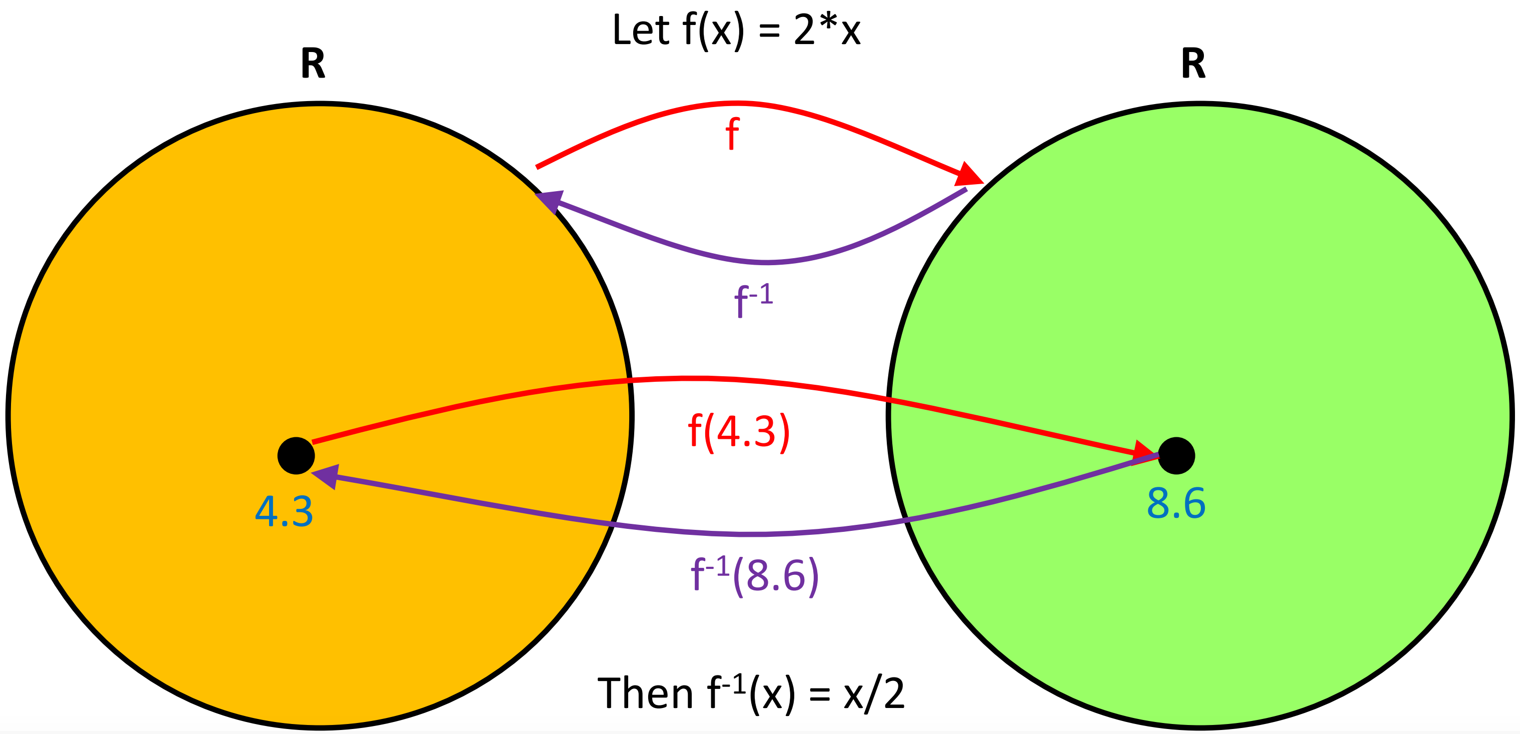
A function such that the image and the pre-image are ALWAYS equal

$f(x) = 1 * x$

$f(x) = x + 0$

The domain and the co-domain must be the same set

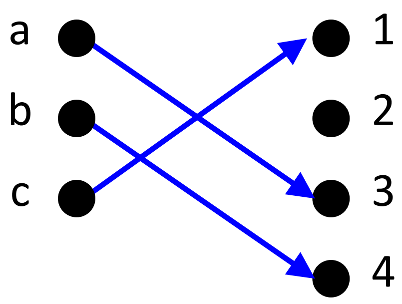
Inverse functions



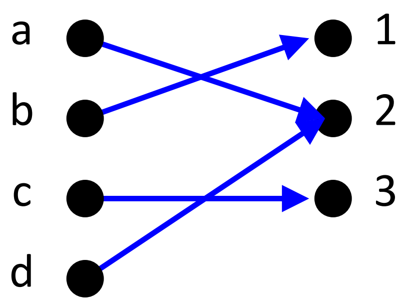
Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

More on inverse functions

Can we define the inverse of the following functions?



What is $f^{-1}(2)$?
Not onto!



What is $f^{-1}(2)$?
Not 1-to-1!

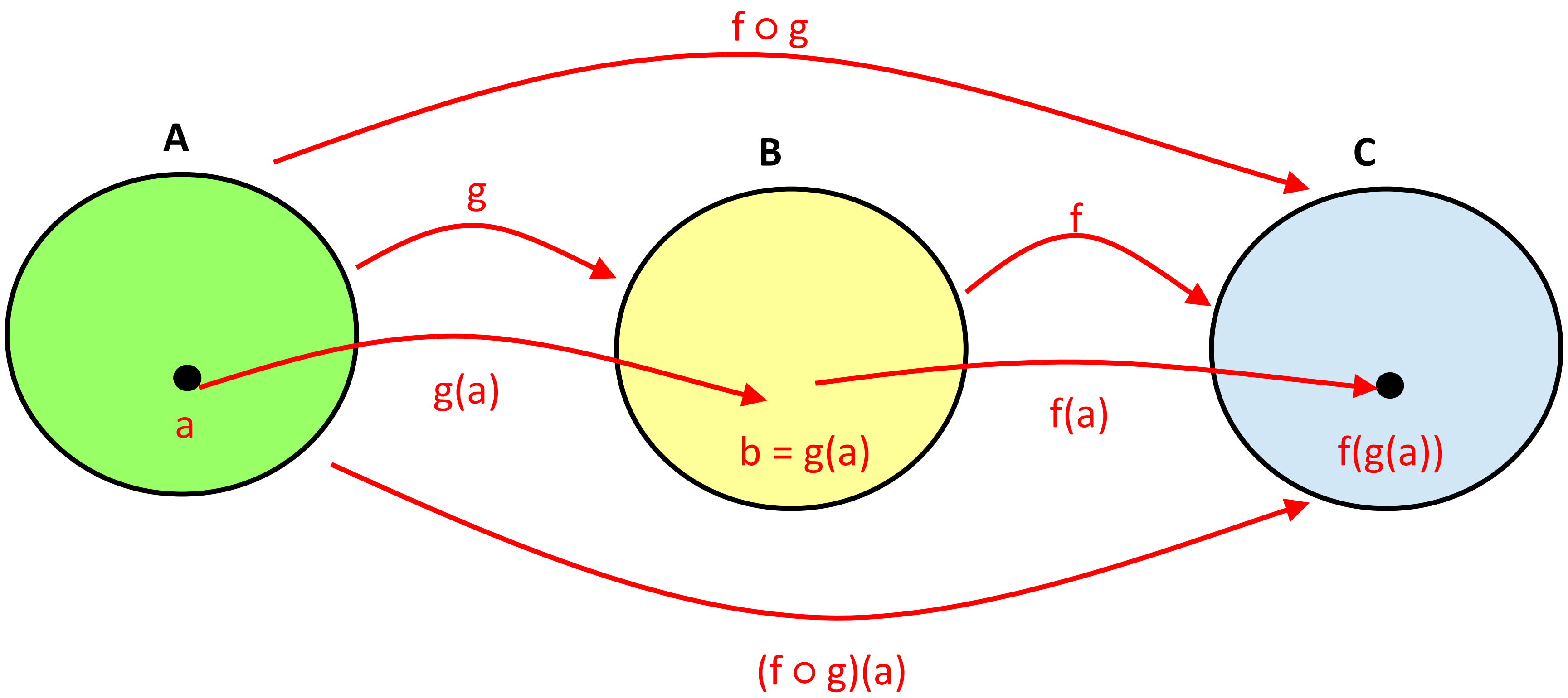
An inverse function can ONLY be defined on a bijection

Compositions of functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

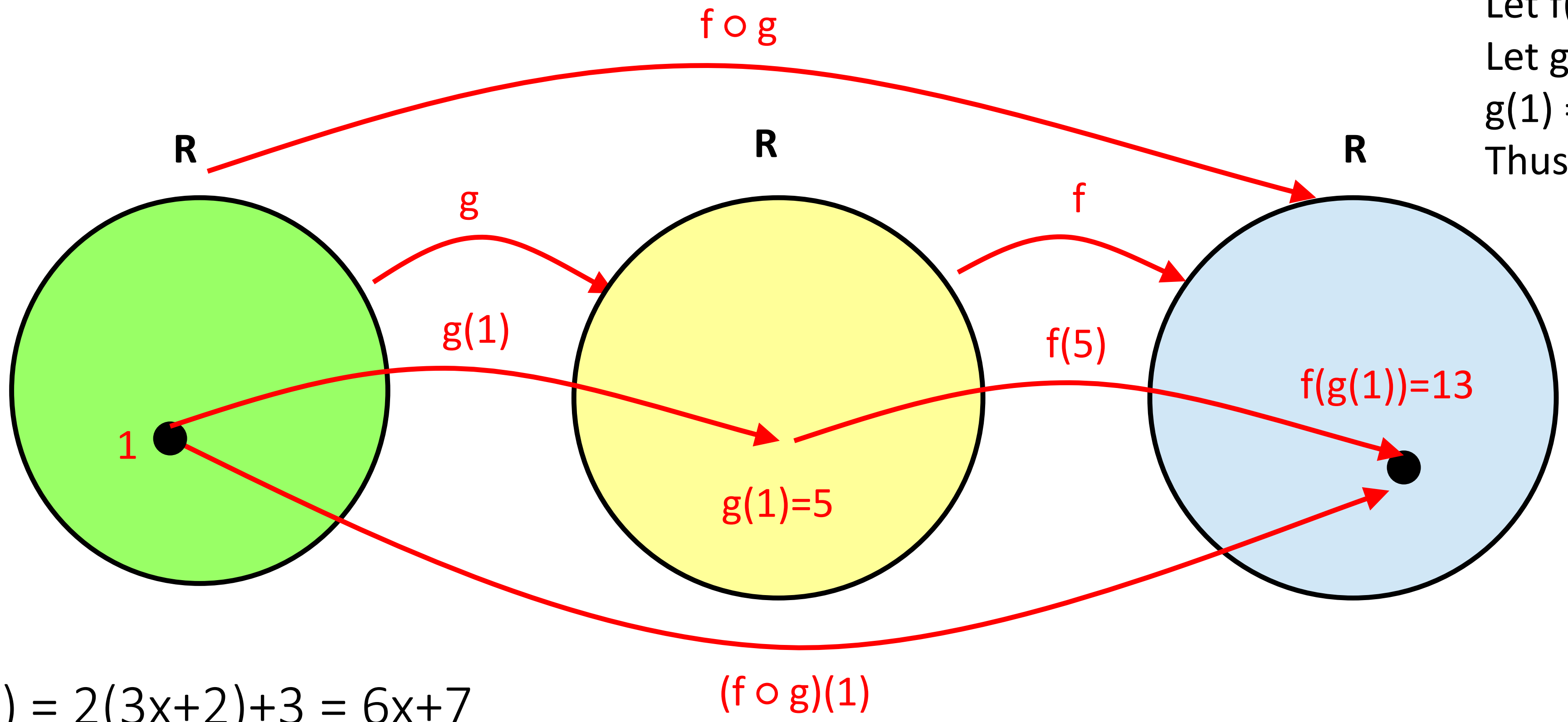
Compositions of functions



$$(f \circ g)(x) = f(g(x))$$

Compositions of functions

Let f(x) = 2x+3 Let g(x) = 3x+2



Let (f ○ g)(x) = f(g(x))
Let f(x) = 2x + 3
Let g(x) = 3x + 2
g(1) = 5, f(5) = 13
Thus, (f ○ g)(1) = f(g(1)) = 13

f(g(x)) = 2(3x+2)+3 = 6x+7

Compositions of functions

Does $f(g(x)) = g(f(x))$?

Let $f(x) = 2x+3$

Let $g(x) = 3x+2$

$f(g(x)) = 2(3x+2)+3 = 6x+7$

$g(f(x)) = 3(2x+3)+2 = 6x+11$

Not equal!

Function composition is not commutative!

Note: fungsi yang paling kanan dioperasikan paling awal, selanjutnya fungsi di samping kirinya, and so forth.

Useful functions

Floor: $\lfloor x \rfloor$ means take the greatest integer less than or equal to the number

Ceiling: $\lceil x \rceil$ means take the lowest integer greater than or equal to the number

Round(x) = floor(x+0.5) = $\lfloor x+0.5 \rfloor$

Sample floor/ceiling questions

Find these values

$\lfloor 1.1 \rfloor$	1
$\lceil 1.1 \rceil$	2
$\lfloor -0.1 \rfloor$	-1
$\lceil -0.1 \rceil$	0
$\lceil 2.99 \rceil$	3
$\lceil -2.99 \rceil$	-2
$\lfloor ½+\lceil ½ \rceil \rfloor$	$\lfloor ½+1 \rfloor = \lfloor 3/2 \rfloor = 1$
$\lceil \lfloor ½ \rfloor + \lceil ½ \rceil + ½ \rceil$	$\lceil 0 + 1 + ½ \rceil = \lceil 3/2 \rceil = 2$

Ceiling and floor properties

Let n be an integer

(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$

(1b) $\lceil x \rceil = n$ if and only if $n-1 < x \leq n$

(1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \leq x$

(1d) $\lceil x \rceil = n$ if and only if $x \leq n < x+1$

(2) $x-1 < \lfloor x \rfloor \leq x \leq = \lceil x \rceil < x+1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x+n \rceil = \lceil x \rceil + n$

Ceiling property proof

Prove rule 4a: $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

- Where n is an integer
- Will use rule 1a: $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$

Direct proof!

- Let $m = \lfloor x \rfloor$
- Thus, $m \le x < m+1$ (by rule 1a)
- Add n to both sides: $m+n \le x+n < m+n+1$
- By rule 4a, $m+n = \lfloor x+n \rfloor$
- Since $m = \lfloor x \rfloor$, $m+n$ also equals $\lfloor x \rfloor + n$
- Thus, $\lfloor x \rfloor + n = m+n = \lfloor x+n \rfloor$

Proving function problems

Let f be an invertible function from Y to Z

Let g be an invertible function from X to Y

Show that the inverse of $f \circ g$ is:

- $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Proving function problems

Thus, we want to show, for all $z \in Z$ and $x \in X$

$$\left((f \circ g) \circ \left(g^{-1} \circ f^{-1}\right)\right)(z) = z$$

$$\left(\left(g^{-1} \circ f^{-1}\right) \circ (f \circ g)\right)(x) = x$$

$$\begin{aligned}
\left((f \circ g) \circ \left(g^{-1} \circ f^{-1}\right)\right)(z) &= (f \circ g)\left(\left(g^{-1} \circ f^{-1}\right)(z)\right) \\
&= (f \circ g)\left(g^{-1}\left(f^{-1}(z)\right)\right) \\
&= f\left(g\left(g^{-1}\left(f^{-1}(z)\right)\right)\right) \\
&= f\left(f^{-1}(z)\right) \\
&= z
\end{aligned}$$

The second equality is similar

Compositions of functions

When the composition of a function and its inverse is formed, in either order, an identity function is obtained.

Suppose that f is a one-to-one correspondence from the set A to the set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A.

The inverse function reverses the correspondence of the original function, so $f^{-1}(b)=a$ when $f(a)=b$, and $f(a)=b$ when $f^{-1}(b)=a$. Hence,

$(f^{-1} \circ f)(a)=f^{-1}(f(a))=f^{-1}(b)=a$,

and

$(f \circ f^{-1})(b)=f(f^{-1}(b))=f(a)=b$.

Exercise

Determine whether function from Z to Z is onto if $f(m, n) = m + n$.

Show that the function $f(x) = |x|$ from the set of real numbers to the set of non-negative real numbers is not invertible, but if the domain is restricted to the set of non-negative real numbers the function is invertible.