KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)

PROPOSITIONAL EQUIVALENCE

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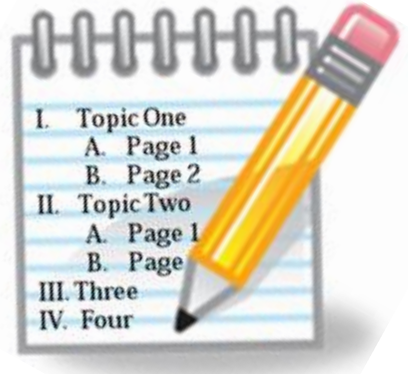
Outline

Tautology, Contradiction, Equivalence

Logical Equivalence

Using Logical Equivalence for Proofing



Tautology, Contradiction, Equivalence

Tautology: a statement (compound props.) that's always true no matter what the truth values of the propositions

- $p \vee \neg p$ will always be true

Contradiction: a statement (compound props.) that's always false

- $p \wedge \neg p$ will always be false

Contingency: a statement (compound props.) that's neither a tautology nor contradiction.

A logical equivalence means that the two sides always have the same truth values. Or *in other word* $p \rightarrow q$ is tautology

- Symbol is $\equiv$ or $\Leftrightarrow$ (we'll use $\equiv$)

Logical Equivalence

Identity law $p \wedge T \equiv p$

p	T	$p \wedge$T
T	T	T
F	T	F

Commutative law $p \wedge q \equiv q \wedge p$

p	q	p∧q	q∧p
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Logical Equivalence (cont.)

Associative law

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

p	q	r	p∧q	(p∧q)∧r	q∧r	p∧(q∧r)
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Summary of Logical Equivalence

$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws	$\neg (p \wedge q) \equiv \neg p \vee \neg q$ $\neg (p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$\neg(\neg p) \equiv p$	Double negation law	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws
$p \rightarrow q \equiv \neg p \vee q$	Definition of Implication	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Definition of Biconditional

Proof using Logical Equivalence

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$(p \to r) \lor (q \to r)$

$\equiv (\neg p \lor r) \lor (\neg q \lor r)$	Definition of implication
$\equiv \neg p \lor r \lor \neg q \lor r$	Associative
$\equiv \neg p \lor \neg q \lor r \lor r$	Commutative
$\equiv (\neg p \lor \neg q) \lor (r \lor r)$	Associative
$\equiv \neg (p \land q) \lor r$	De Morgan, Idempotent
$\equiv (p \land q) \to r$	Definition of implication

Proof using Logical Equivalence

- Show that $(p \land q) \rightarrow (p \lor q)$ is a Tautology.

(Proof)

$(p \land q) \rightarrow (p \lor q)$

$\equiv \neg (p \land q) \lor (p \lor q)$	Def. of Implication
$\equiv (\neg p \lor \neg q) \lor (p \lor q)$	De Morgan
$\equiv (\neg p \lor p) \lor (\neg q \lor q)$	Commutative, Associative
$\equiv T \lor T$	Negation
$\equiv T$	Identity

Proof using Logical Equivalence

- Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences.

(Proof)

$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$	by the second De Morgan law
$\equiv \neg p \land [\neg(\neg p) \lor \neg q]$	by the first De Morgan law
$\equiv \neg p \land (p \lor \neg q)$	by the double negation law
$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$	by the second distributive law
$\equiv F \lor (\neg p \land \neg q)$	because $\neg p \land p \equiv F$
$\equiv (\neg p \land \neg q) \lor F$	by the commutative law for disjunction
$\equiv \neg p \land \neg q$	by the identity law for F

Exercsise

Show that $p \leftrightarrow q$ is logically equivalent with $\sim p \leftrightarrow \sim q$ by using Table of Logical Equivalence Laws.

Answer

$p \leftrightarrow q \equiv {\sim}p \leftrightarrow {\sim}q$

$=(p \rightarrow q) \land (q \rightarrow p)$ Definition of Bi conditional

$= ({\backsim} p \lor q) \land ({\backsim} q \lor p)$ Definition of Implication

$= (p \lor {\sim}q) \land (q \lor {\sim}p)$ Commutative Law

$= ({\sim}p \rightarrow {\sim}q) \land ({\sim}q \rightarrow {\sim}p)$ Definition of Implication

$= {\sim}p \leftrightarrow {\sim}q \equiv {\sim}p \leftrightarrow {\sim}q$

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PREDICATE & QUANTIFIER

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Outline

Propositional function

Function with multiple variables

Quantifier

Universal quantifier

Existensial quantifier

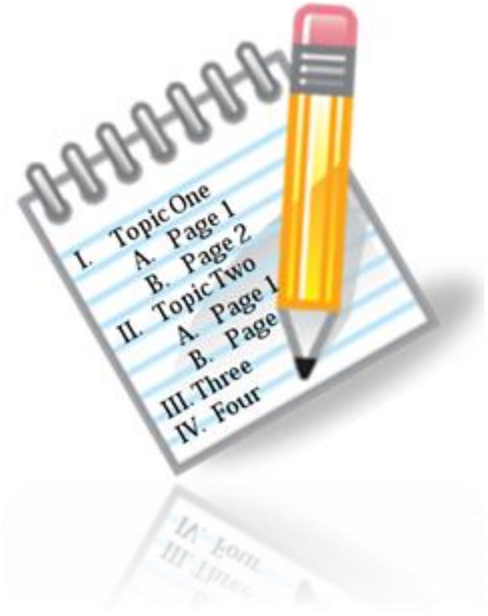
Binding variable

Negating quantifier

Multiple quantifiers

Order of quantifiers

Negating multiple quantifiers



Propositional Functions

Consider $P(x)$, as a symbolic notation of $x > 5$

- $P(x)$: propositional function P at x (fungsi proposisi P untuk x)
- x is subject
- > 5 is predicate
- $P(x)$ has no truth value when x is unknown
- $P(x)$ become a proposition when we assigned certain value to x
- The value given to x is taken from certain universe of discourse or domain (himpunan semesta)

Propositional Functions (cont.)

Example:

Consider $P(x) = x < 5$

- $P(x)$ has no truth values (x is not given a value)
- Let x be the integer; $P(1)$ is true: The proposition $1<5$ is true
- $P(10)$ is false: The proposition $10<5$ is false

Let $P(x) = x + 3 > x$

- For what values of x is $P(x)$ true?

Function with Multiple Variables

$P(x,y) = x + y = 0$

- $P(1,2)$ is false, $P(1,-1)$ is true

$P(x,y,z) = x + y = z$

- $P(3,4,5)$ is false, $P(1,2,3)$ is true

$P(x_1,x_2,x_3 \ldots x_n) = \ldots$

Quantifier

A quantifier is "an operator that limits the variables of a proposition"

- In some cases, it's a more accurate way to describe things than Boolean propositions

Process of bounding the variable *x* with a quantifier is called quantification

Two types of quantifier will be discussed:

- Universal quantifier
- Existential quantifier

Universal Quantifier

Represented by an upside-down *A*: $\forall$

- It means "for all"
- Let $P(x) = x+1 > x$

We can state the following:

- $\forall x\ P(x)$
- English translation: "for all values of x, $P(x)$ is true"
- English translation: "for all values of x, $x+1>x$ is true"

Universal Quantifier (cont.)

But is that always true?

- $\forall x\ P(x)$

Let x = the character ‘a’

- Is ‘a’+1 > ‘a’?

Let x = the state of East Java

- Is East Java+1 > East Java?

Don’t forget to specify your universe!

- What values x can represent
- Called the “domain” or “universe of discourse”

Universal Quantifier (cont.)

Let the universe be the real numbers.

Let $P(x) = x/2 < x$

- Not true for the negative numbers! (Called as counterexample)
- Thus, $\forall x\, P(x)$ is false
  - When the domain is all the real numbers

In order to prove that a universal quantification is true, it must be shown for ALL cases

In order to prove that a universal quantification is false, it must be shown to be false for only ONE case

Universal Quantifier (cont.)

Given some propositional function P(x)

and values in the universe $x_1 \ldots x_n$

The universal quantification $\forall x\, P(x)$ implies:

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

Existensial Quantifier

Represented by an backwards *E*: ∃

- It means “there exists”
- Let $P(x) = x^2 > 10$

We can state the following:

- $\exists x\ P(x)$
- English translation: “there exists (a value of) *x* such that *P*(*x*) is true”
- English translation: “for at least one value of *x*, $x^2 > 10$ is true”

Note that you still have to specify your universe

Existensial Quantifier (cont.)

Let $P(x) = x+1= x$

- There is no numerical value x for which $x+1= x$
- Thus, $\exists$x P(x) is false

Let $P(x) = x+1 = 0$

- There is a numerical value for which $x+1= 0$
- Thus, $\exists x\, P(x)$ is true

In order to show an existential quantification is **true**, you only have to find **ONE** value

In order to show an existential quantification is **false**, you have to show it's false for **ALL** values

Existensial Quantifier (cont.)

Given some propositional function $P(x)$

And values in the universe $x_1 \ldots x_n$

The existential quantification $\exists x\, P(x)$ implies:

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Conclusion

Statement	When True	When False
$\forall x\, P(x)$		
$\exists\, x\, P(x)$		

Conclusion

Statement	When True	When False
$\forall x\, P(x)$	P(x) is TRUE for every x	There is an x for which P(x) is FALSE
$\exists\, x\, P(x)$	There is an x for which P(x) is TRUE	P(x) is FALSE for every x

Notes

Recall that $P(x)$ is a propositional function

- Let $P(x)$ be "$x > 0$"

Recall that a proposition is a statement that is either true or false

- $P(x)$ is not a proposition

There are two ways to make a propositional function into a proposition:

- **Assign a certain value**
  - For example, $P(-1)$ is false, $P(1)$ is true
- **Provide a quantification**
  - For example, $\forall x\, P(x)$ is false and $\exists x\, P(x)$ is true
    - Let the universe of discourse be the real numbers

Binding Variable

Let $P(x, y)$ be $x > y$

Consider: $\forall x\ P(x, y)$

- This is not a proposition!
- What is y?
  - If it's 5, then $\forall x$ P(x, y) is false
  - If it's x-1, then $\forall x$ P(x, y) is true

Note that y is not "bound" by a quantifier

Binding Variable (cont.)

$(\exists x\, P(x)) \vee Q(x)$

- The x in $Q(x)$ is not bound; thus not a proposition

$(\exists x\, P(x)) \vee (\forall x\, Q(x))$

- Both x values are bound; thus it is a proposition

$(\exists x\, P(x) \wedge Q(x)) \vee (\forall y\, R(y))$

- All variables are bound; thus it is a proposition

$(\exists x\, P(x) \wedge Q(y)) \vee (\forall y\, R(y))$

- The y in $Q(y)$ is not bound; this not a proposition

Negating Quantifiers

Consider the statement:

- All students in this class have Acer Laptop

What is required to show the statement is false?

- There exists a student in this class that does NOT has Acer Laptop

To negate a universal quantification:

- You negate the propositional function
- AND you change to an existential quantification
- $\neg(\forall x\, P(x)) = \exists x\, \neg P(x)$

Negating Quantifiers (cont.)

Consider the statement:

- There is a student in this class with Acer Laptop.

What is required to show the statement is false?

- All students in this class do not have Acer Laptop.

Thus, to negate an existential quantification:

- negate the propositional function
- AND change to a universal quantification
- $\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$

Conclusion

Proposition	Negation	TRUE	FALSE
$\neg\exists x\, P(x)$	$\forall x\, \neg P(x)$	For all x, $P(x)$ is false	There is a value of x for which $P(x)$ is true
$\neg\forall x\, P(x)$	$\exists x\, \neg P(x)$	There is a value of x for which $P(x)$ is false	For all x, $P(x)$ is true

Translating from English

What about if the universe of discourse is all people?

- $S(x)$ be "x is a student in this class"
- $C(x)$ be "x has studied Calculus"

- Every student in this class has studied Calculus.
- $\forall x\,(S(x) \wedge C(x))$
  - This is wrong! Why?
  - It means that "All people are students in this class and have studied Calculus"

- $\forall x\,(S(x) \rightarrow C(x))$
  - It means that "For every person x, if x is student in this class, then x has studied Calculus"

Translating from English

Consider:

- “Every student in this class has visited Manado or Cianjur”

Let:

- S(x) be “x is a student in this class”
- M(x) be “x has visited Manado”
- C(x) be “x has visited Cianjur”

Translating from English

Consider: “Every student in this class has visited Cianjur or Manado”

$\forall x\ (M(x) \vee C(x))$

- When the universe of discourse is all students in this class

$\forall x\ (S(x) \rightarrow (M(x) \vee C(x)))$

- When the universe of discourse is all people

Translating from English

Consider: "Some students have visited Manado"

- Rephrasing: "There exists a student who has visited Manado"

$\exists x\, M(x)$

- True if the universe of discourse is all students

What about if the universe of discourse is all people?

- $\exists x\, (S(x) \rightarrow M(x))$
  - **This is wrong!** Why?
  - The statement is true although there is someone not in the class
- $\exists x\, (S(x) \wedge M(x))$
  - There is a person x who is a student in this class and who has visited Manado

Exercises

Let P(x), Q(x), R(x), and S(x) be the statements "*x is a baby*," "*x is logical*," "*x is able to manage a crocodile*," and "*x is despised*," respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and P(x), Q(x), R(x), and S(x).

a) Babies are illogical.
b) Illogical persons are despised.
c) Babies cannot manage crocodiles.

Answer

a) $\forall x(P(x) \rightarrow \neg Q(x))$
b) $\forall x(\neg Q(x) \rightarrow S(x))$
c) $\forall x(P(x) \rightarrow \neg R(x))$

Multiple Quantifiers

You can have multiple quantifiers on a statement

$\forall x \exists y\, P(x, y)$

- "For all *x*, there exists a y such that $P(x,y)$"
- Example: $\forall x \exists y\, (x+y = 0)$

$\exists x \forall y\, \mathrm{P}(x,y)$

- There exists an *x* such that for all *y* $P(x,y)$ is true"
- Example: $\exists x \forall y\, (x*y = 0)$

Order of quantifiers

$\exists x \forall y$ and $\forall x \exists y$ are not equivalent!

$\exists x \forall y\ P(x,y)$

- $P(x, y) = (x+y = 0)$ is false

$\forall x \exists y\ P(x,y)$

- $P(x,y) = (x+y = 0)$ is true

Quantifications of Two Variables

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

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Negating multiple quantifiers

Recall negation rules for single quantifiers:

- $\neg \forall x \, P(x) \equiv \exists x \, \neg P(x)$
- $\neg \exists x \, P(x) \equiv \forall x \, \neg P(x)$
- Essentially, you change the quantifier(s), and negate what it's quantifying

Examples:

- $\neg(\forall x \exists y \, P(x,y)) \equiv \exists x \forall y \, \neg P(x,y)$
- $\neg(\forall x \exists y \forall z \, P(x,y,z)) \equiv \exists x \forall y \exists z \, \neg P(x,y,z)$

Negating multiple quantifiers (cont.)

Consider $\neg(\forall x \exists y\ P(x,y)) = \exists x \forall y\ \neg P(x,y)$

- The left side is saying "for all *x*, there exists a *y* such that *P* is true"
- To disprove it (negate it), you need to show that "there exists an *x* such that for all *y*, *P* is false"

Consider $\neg(\exists x \forall y\ P(x,y)) = \forall x \exists y\ \neg P(x,y)$

- The left side is saying "there exists an *x* such that for all *y*, *P* is true"
- To disprove it (negate it), you need to show that "for all *x*, there exists a *y* such that *P* is false"

Translating Quantifiers

Let N(x) be the statement "x has visited North Dakota", where he domain consist of the students in your school. Express each of these quantifications in English.

a) $\exists x\ N(x)$

Some students in the school have visited North Dakota.

There exists a student in the school who has visited N.D.

b) $\forall x\ N(x)$

Every student in the school has visited North Dakota.

All students in the school have visited North Dakota.

c) $\neg \exists x\ N(x)$: negation of part a)

No student in the school has visited North Dakota.

There does not exist a student in the school who has visited N.D.

Translating Quantifiers

Let N(x) be the statement "x has visited North Dakota", where he domain consist of the students in your school. Express each of these quantifications in English.

d) $\exists x \neg N(x)$

Some students in the school have not visited North Dakota.

There exists a student in the school who has not visited N.D.

e) $\neg \forall x\, N(x)$: negation of part b)

It is not true that every student in the school has visited N.D.

Not all students in the school have visited N.D.

f) $\forall x \neg N(x)$

All students in the school have not visited North Dakota.

(common English sentence takes this sentence, incorrectly, the answer of part e)

Note: c) and f) are equivalent; d) and e) are also equivalent. But both pairs are not equivalent to each other.

Translating Quantifiers

Note: The domain is all integers

The product of two negative integers is positive

- $\forall x \forall y\, ((x<0) \land (y<0) \rightarrow (xy > 0))$
- Why conditional instead of and?

The average of two positive integers is positive

- $\forall x \forall y\, ((x>0) \land (y>0) \rightarrow ((x+y)/2 > 0))$

The difference of two negative integers is not necessarily negative

- $\exists x \exists y\, ((x<0) \land (y<0) \land (x-y \geq 0))$
- Why and instead of conditional?

The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers

- $\forall x \forall y\, (|x+y| \leq |x| + |y|)$

Translating Quantifiers

Note:The domain is all real numbers

$\exists x \forall y\ (x+y = y)$

- There exists an additive identity for all real numbers

$\forall x \forall y\ (((x \geq 0) \wedge (y<0)) \rightarrow (x-y > 0))$

- A non-negative number minus a negative number is greater than zero

$\exists x \exists y\ (((x \leq 0) \wedge (y \leq 0)) \wedge (x-y > 0))$

- The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)

$\forall x \forall y\ (((x \neq 0) \wedge (y \neq 0)) \leftrightarrow (xy \neq 0))$

- The product of two non-zero numbers is non-zero if and only if both factors are non-zero