KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)

Sequences and Summations

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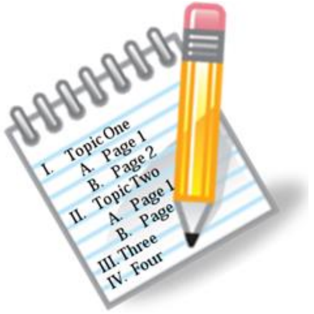
Outline

1. Definition of Sequence
2. Sequence Examples
3. Arithmetic Vs Geometric Sequences
4. Fibonacci Sequence
5. Determining Sequence Formula
6. Useful Sequences
7. Summations
8. Evaluating Sequences
9. Summation of A Geometric Series
10. Double Summation
11. Cardinality



Definitions of Sequence

Sequence: an ordered list of elements
- Like a set, but:
  - Elements can be duplicated
  - Elements are ordered

A sequence is a function from a subset of **Z** to a set S
- Usually from the positive or non-negative ints
- a_n is the image of n
  - a_n is a term in the sequence
  - $\{a_n\}$ means the entire sequence
- The same notation as sets!

Sequence Examples

$a_n = 3n$

- The terms in the sequence are $a_1, a_2, a_3, \ldots$
- The sequence $\{a_n\}$ is $\{ 3, 6, 9, 12, \ldots \}$

$b_n = 2^n$

- The terms in the sequence are $b_1, b_2, b_3, \ldots$
- The sequence $\{b_n\}$ is $\{ 2, 4, 8, 16, 32, \ldots \}$

Note that generally sequences are indexed from 1

Arithmetic vs. Geometric Sequences

Arithmetic sequences increase by a constant *amount*

- $a_n = 3n$
- The sequence $\{a_n\}$ is { 3, 6, 9, 12, … }

Arithmetic Progression

- $a, a+d, a+2d, \ldots, a+nd, \ldots$
- $a_n = a + (n-1)\, d$
- Discrete analogue of linear function $f(x) = dx + a$

Geometric sequences increase by a constant *factor*

- $b_n = 2^n$
- The sequence $\{b_n\}$ is { 2, 4, 8, 16, 32, … }

Geometric Progression

- $a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$
- $a_n = ar^{n-1}$
- Discrete analogue of exponential function $f(x) = ar^x$

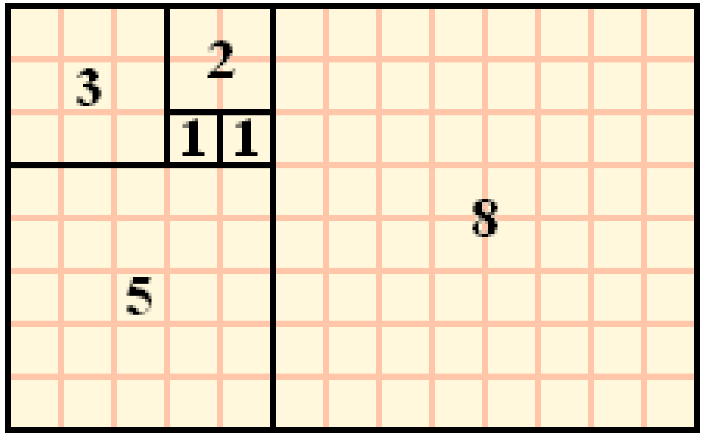
Fibonacci Sequence

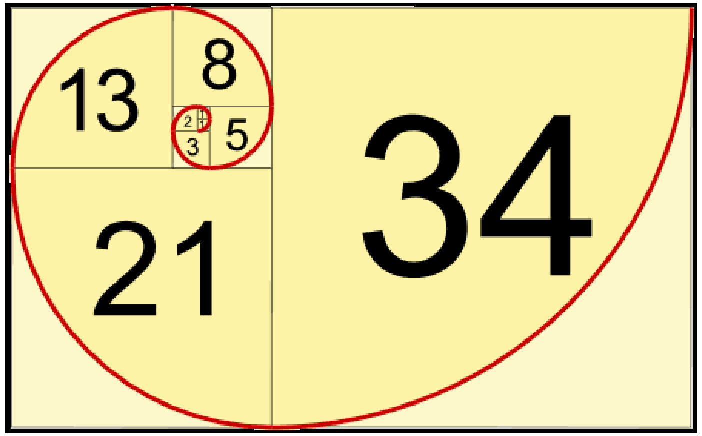
Sequences can be neither geometric nor arithmetic

- $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
  - Alternative, $F(n) = F(n-1) + F(n-2)$
- Each term is the sum of the previous two terms
- Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... }
- This is the Fibonacci sequence
- Full formula:

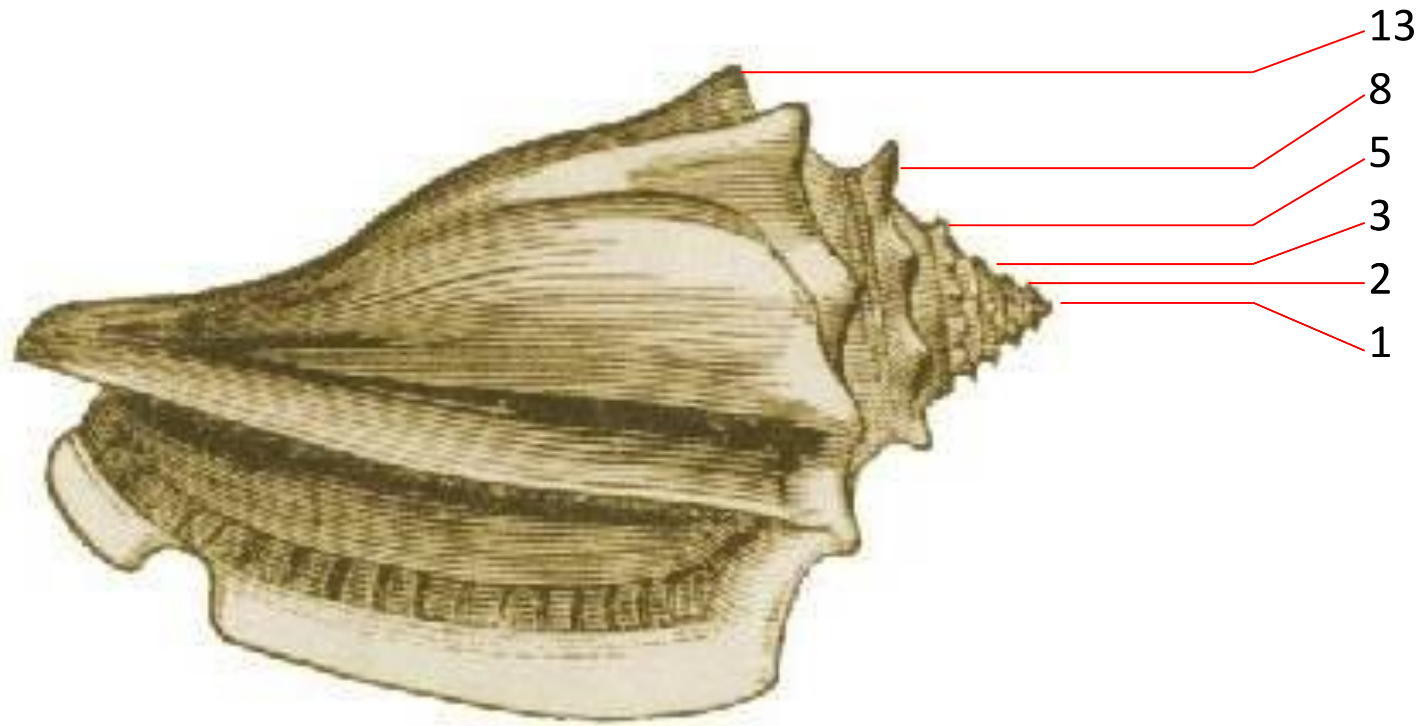
$$F(n) = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5}\cdot 2^n}$$

- Reference:
  - https://www.math.hmc.edu/funfacts/ffiles/10002.4-5.shtml
  - https://www.mathsisfun.com/numbers/fibonacci-sequence.html

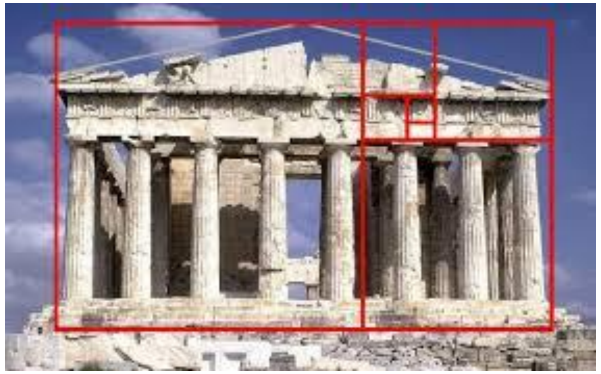


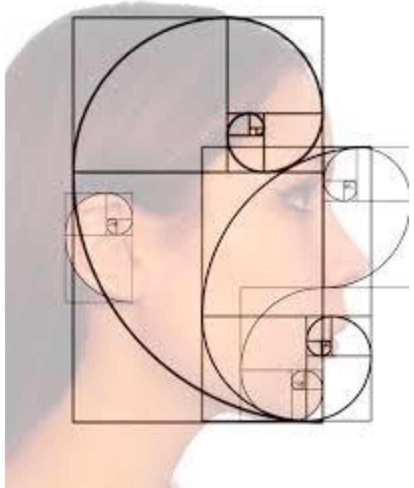


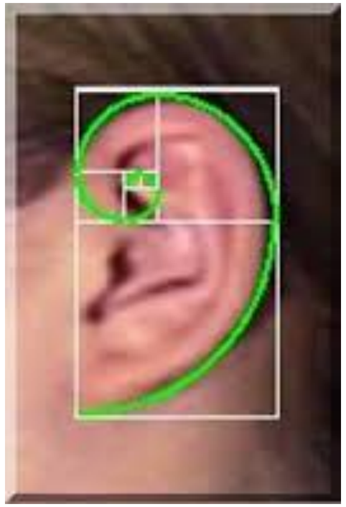
Fibonacci Sequence in Nature



Fibonacci Sequence in Nature







Determining the Sequence Formula

Given values in a sequence, how do you determine the formula?

Steps to consider:

- Is it an arithmetic progression (each term a constant amount from the last)?
- Is it a geometric progression (each term a factor of the previous term)?
- Does the sequence repeat (or cycle)?
- Does the sequence combine previous terms?
- Are there runs of the same value?

Determining the Sequence Formula

1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …

- The sequence alternates 1’s and 0’s, increasing the number of 1’s and 0’s each time

1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …

- This sequence increases by one, but repeats all even numbers once

1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …

- Each term is twice the cube of *n*. The non-0 numbers are a geometric sequence (2^n) interspersed with zeros

3, 6, 12, 24, 48, 96, 192, …

- Each term is twice the previous: geometric progression
- $a_n = 3*2^{n-1}$

Determining the sequence formula

15, 8, 1, -6, -13, -20, -27, …

- Each term is 7 less than the previous term
- $a_n = 22 - 7n$

3, 5, 8, 12, 17, 23, 30, 38, 47, …

- The difference between successive terms increases by one each time, $a_1 = 3$, $a_n = a_{n-1} + n$
- $a_n = n(n+1)/2 + 2$

2, 16, 54, 128, 250, 432, 686, …

- Each term is twice the cube of n
- $a_n = 2*n^3$

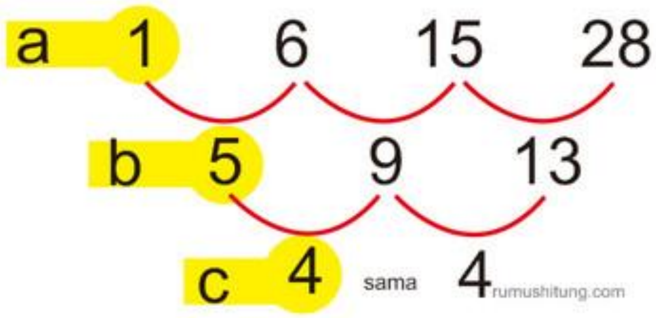
2, 3, 7, 25, 121, 721, 5041, 40321

- Each successive term is about n times the previous
- $a_n = n! + 1$
- Alternatively: $a_n = a_{n-1} * n - n + 1$

Multilevel Arithmetic Progression

1. Find the general pattern by using this formula:

$$U_n = \frac{a}{0!} + \frac{(n-1)b}{1!} + \frac{(n-1)\,(n-2)c}{2!} + \frac{(n-1)\,(n-2)(n-3)d}{3!} + \;dst$$



2. Another way, using n degree polynomial: http://www.algebra.com/algebra/homework/Sequences-and-series/Sequences-and-series.faq.question.825537.html

Useful Sequences

$n^2 = 1, 4, 9, 16, 25, 36, \ldots$

$n^3 = 1, 8, 27, 64, 125, 216, \ldots$

$n^4 = 1, 16, 81, 256, 625, 1296, \ldots$

$2^n = 2, 4, 8, 16, 32, 64, \ldots$

$3^n = 3, 9, 27, 81, 243, 729, \ldots$

$n! = 1, 2, 6, 24, 120, 720, \ldots$

Summations

A summation:

upper limit

$$\sum_{j=m}^{n} a_j \quad \text{or} \quad \sum\nolimits_{j=m}^{n} a_j$$

lower limit

index of summation

is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
        sum += a(j);
```

Evaluating Sequences

$\Sigma^{5}_{k=1} (k+1)$

$= 2 + 3 + 4 + 5 + 6 = 20$

$\Sigma^{4}_{k=0} (-2)^{k}$

$= (-2)^{0} + (-2)^{1} + (-2)^{2} + (-2)^{3} + (-2)^{4} = 11$

$\Sigma^{10}_{k=1} 3$

$= 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30$

$\Sigma^{10}_{k=1} (2^{k} - 2^{k-1})$

$= (2^{1}-2^{0}) + (2^{2}-2^{1}) + (2^{3}-2^{2}) + \ldots (2^{10}-2^{9}) = 1023$

- Note that each term (except the first and last) is cancelled by another term

Evaluating Sequences

Let S = { 1, 3, 5, 7 }

What is $\Sigma_{j \in S} j$

- $1 + 3 + 5 + 7 = 16$

What is $\Sigma_{j \in S} j^2$

- $1^2 + 3^2 + 5^2 + 7^2 = 84$

- What is $\Sigma_{j \in S} (1/j)$
  - $1/1 + 1/3 + 1/5 + 1/7 = 176/105$
- What is $\Sigma_{j \in S} 1$
  - $1 + 1 + 1 + 1 = 4$

Summation of A Geometric Series

Sum of a geometric series:

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Example:

$$\sum_{j=0}^{10} 2^{n} = \frac{2^{10+1} - 1}{2-1} = \frac{2048-1}{1} = 2047$$

Proof

$$S = \sum_{j=0}^{n} ar^{j}$$

If r = 1, then the sum is:

$$\boxed{S = \sum_{j=0}^{n} a = (n+1)a}$$

$$rS = r\sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} ar^{j+1}$$

$$= \sum_{k=1}^{n+1} ar^{k}$$

Shifting the index wih $k = j+1$

$$= \sum_{k=0}^{n} ar^{k} + \left(ar^{n+1} - a\right)$$

$$rS = S + \left(ar^{n+1} - a\right)$$

$$rS - S = \left(ar^{n+1} - a\right)$$

$$S(r-1) = \left(ar^{n+1} - a\right)$$

$$S = \frac{\left(ar^{n+1} - a\right)}{r-1}$$

Double Summations

Like a nested for loop

$$\boxed{\sum_{i=1}^{4}\sum_{j=1}^{3} ij}$$

Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
    for ( int j = 1; j <= 3; j++ )
        sum += i*j;
```

Cardinality

For finite sets (only), cardinality is the number of elements in the set

For finite and infinite sets, two sets A and B have the same cardinality if there is a one-to-one correspondence from A to B

Cardinality

Example on finite sets:

- Let $S = \{ 1, 2, 3, 4, 5 \}$
- Let $T = \{ a, b, c, d, e \}$
- There is a one-to-one correspondence between the sets

Example on infinite sets:

- Let S = **Z**+
- Let T = $\{ x \mid x = 2k \text{ and } k \in \mathbf{Z}^+ \}$
- One-to-one correspondence:

  $1 \leftrightarrow 2$ $2 \leftrightarrow 4$ $3 \leftrightarrow 6$ $4 \leftrightarrow 8$

  $5 \leftrightarrow 10$ $6 \leftrightarrow 12$ $7 \leftrightarrow 14$ $8 \leftrightarrow 16$

  Etc.

  - Note that here the '$\leftrightarrow$' symbol means that there is a correspondence between them, not the biconditional

More Definitions

A set that is either finite or has the same cardinality as the set of Z^+ is called countable.

A set that is not countable is called uncountable.

Countably infinite: elements can be listed

- Anything that has the same cardinality as the positive integer
- Example: rational numbers, odd integers, all integers

Uncountably infinite: elements cannot be listed

- Example: real numbers

When an infinite set *S* is countable, we denote the cardinality of *S* by $\aleph_0$ (aleph null) -- $|S| = \aleph_0$

Showing a Set is Countably Infinite

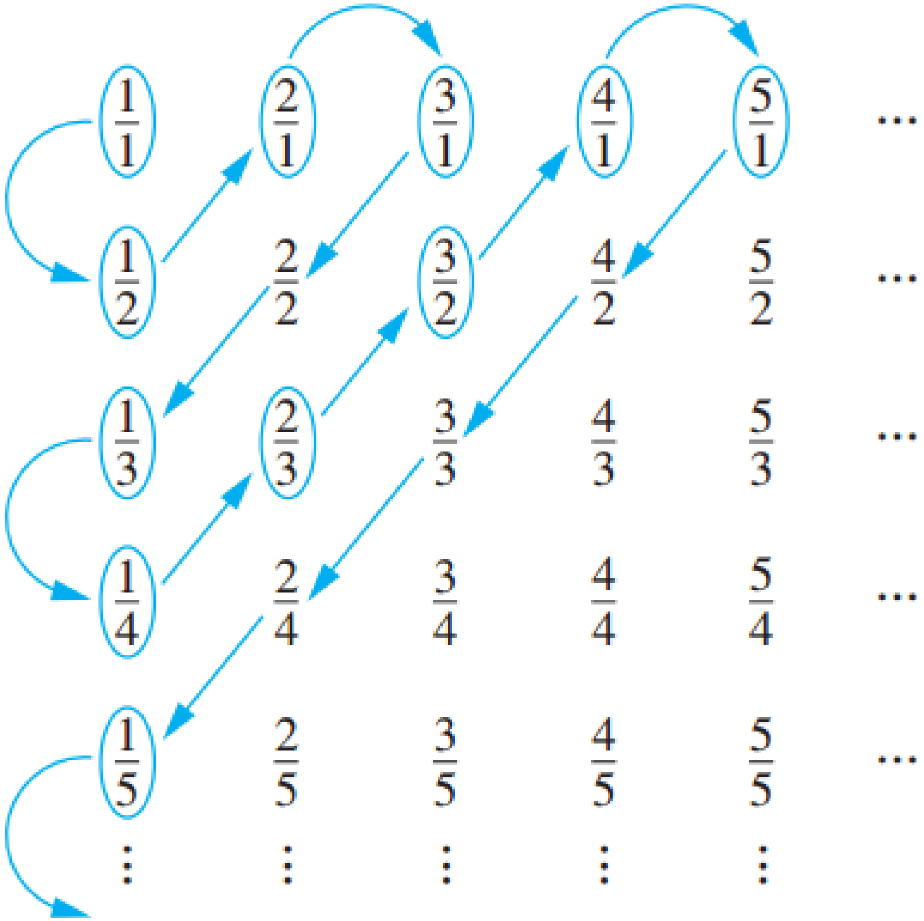
Done by showing there is a one-to-one correspondence between the set and the positive integers

Examples

- Even numbers
  - Shown two slides ago
- Rational numbers
  - Shown next slide
- Ordered pairs of integers
  - Shown next two slides

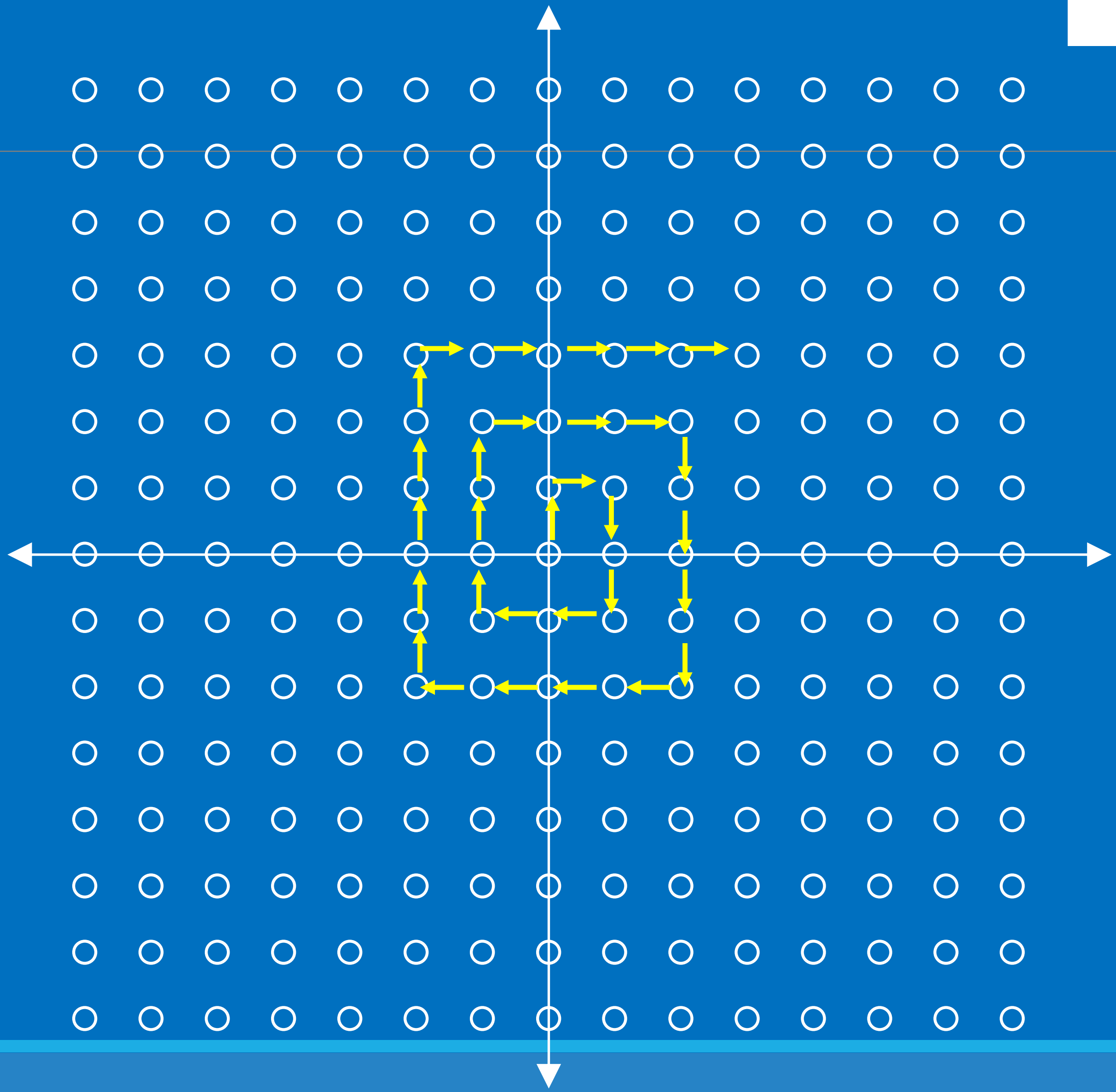
Show that the rational numbers are countably infinite

- First, let's show the positive rationals are countable
- See diagram:
- Can easily add 0 (add one column to the left)
- Can add negative rationals as well
- Whenever we encounter a number *p/q* that is already listed, we do not list it again. For example, when we come to 2/2 = 1 we do not list it because we have already listed 1/1 = 1. The uncircled numbers in the list are those we leave out because they are already listed.



Ordered Pairs of Integers: Countably Infinite

A one-to-one correspondence



Exercise

Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

- all bit strings not containing the bit 0
- all positive rational numbers that cannot be written with denominators less than 4

Answer

Countable: match n with the string of n 1s.

Countable. To find a correspondence, follow the path in Example 4, but omit fractions in the top three rows (as well as continuing to omit fractions not in lowest terms).