KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)

GRAPHS

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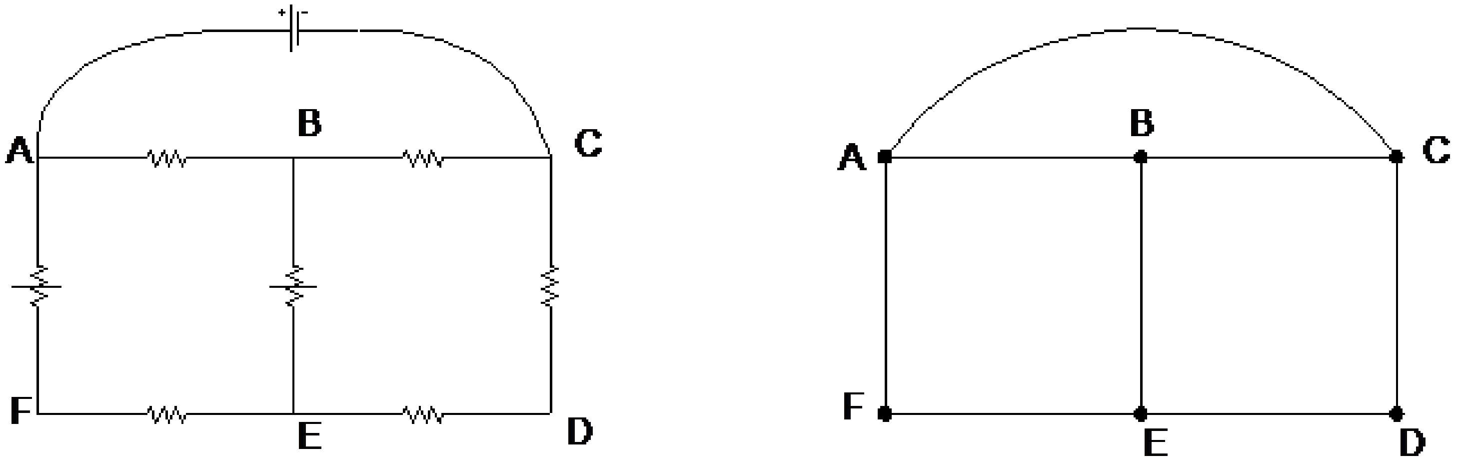
What are Graphs?

In mathematics and computer science, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects from a certain collection.

A "graph" in this context is a collection of "vertices" or "nodes" and a collection of edges that connect pairs of vertices.

Applications of Graphs

Electrical circuits

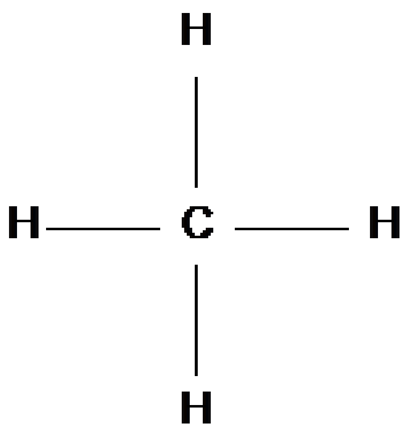


(a) (b)

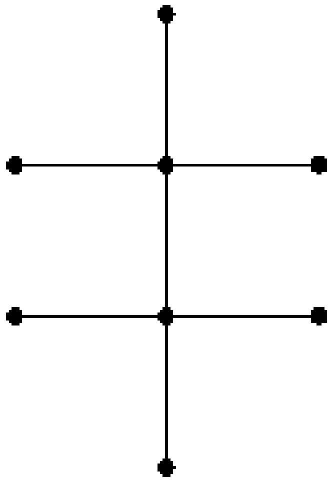
Applications of Graphs

Chemical compounds

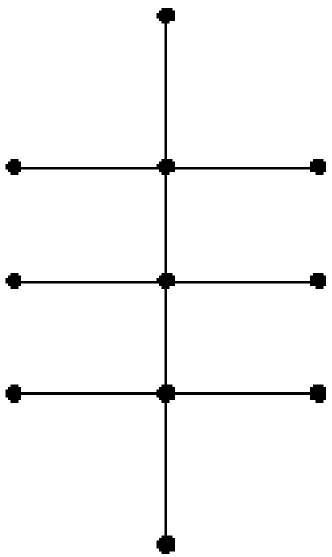
metana (CH_4)



etana (C_2H_6)



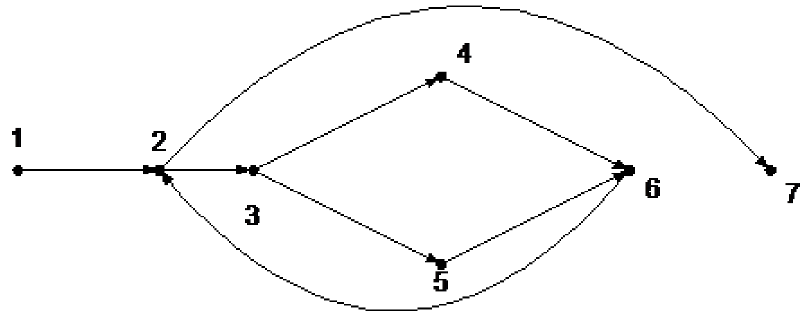
propana (C_3H_8)



Applications of Graphs

Program testing

```
read(x);
while x <> 9999 do
 begin
   if x < 0 then
      writeln('Masukan tidak boleh negatif')
   else
     x:=x+10;
   read(x);
 end;
writeln(x);
```



1 : read(x)
2 : x <> 9999
3 : x < 0
4 : writeln('Masukan tidak boleh negatif');
5 : x := x + 10
6 : read(x)
7 : writeln(x)

Applications of Graphs

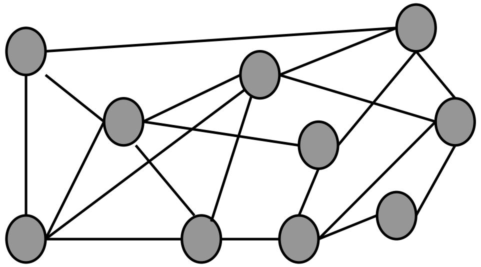
Potentially anything (graphs can represent relations; relations can describe the extension of any predicate).

More apps in networking, scheduling, flow optimization, circuit design, path planning.

Genealogy analysis, computer game-playing, program compilation, object-oriented design

Simple Graphs

Correspond to symmetric binary relations R.



A simple graph G = (V, E) consists of:

- a set *V* of *vertices* or *nodes* (*V* corresponds to the universe of the relation *R*),
- a set *E* of *edges* / *arcs* / *links*: unordered pairs of elements $u, v \in V$, such that uRv.

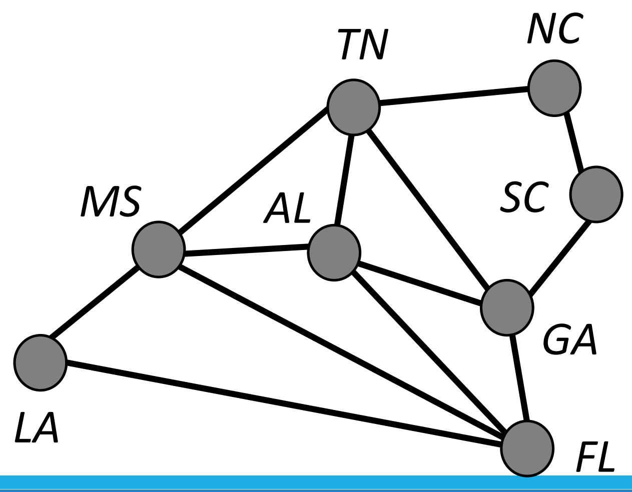
Example of a Simple Graph

Let V be the set of states in the far-southeastern U.S.:

- V = {FL, GA, AL, MS, LA, SC, TN, NC}

Let E = {{u, v} | u adjoins v}

= {{FL,GA}, {FL,AL}, {FL,MS}, {FL,LA}, {GA,AL}, {AL,MS}, {MS,LA}, {GA,SC}, {GA,TN}, {SC,NC}, {NC,TN}, {MS,TN}, {MS,AL}}

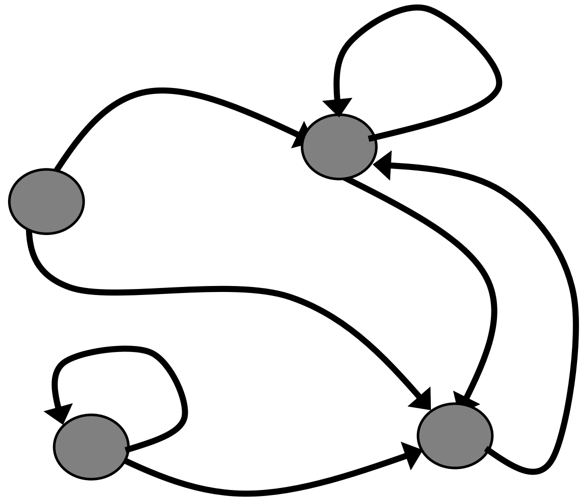


Directed Graphs

Correspond to arbitrary binary relations R, which need not be symmetric.

A directed graph (V, E) consists of a set of vertices V and a binary relation E on V.

E.g.: V = people, E = {(x, y) | x loves y}



Graph Terminology

1. Adjacent
2. Connects
3. Endpoints
4. Degree
5. Initial
6. Terminal
7. In-degree, out-degree
8. Sub graph, union.

Adjacency (ketetanggan)

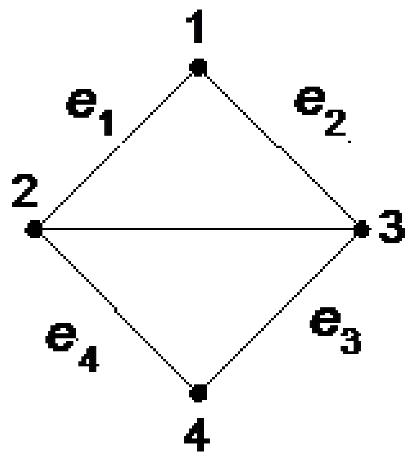
Let G be an undirected graph with edge set E. Let e $\in$ E be (or map to) the pair {u, v}. Then we say:

u, v are adjacent / neighbors / connected.

Edge e is incident with vertices u and v.

Edge e connects u and v.

Vertices u and v are endpoints of edge e.



v_1 is adjacent to v_2 and v_3, but not adjacent to v_4
e_1 is incident with v_1 and v_2

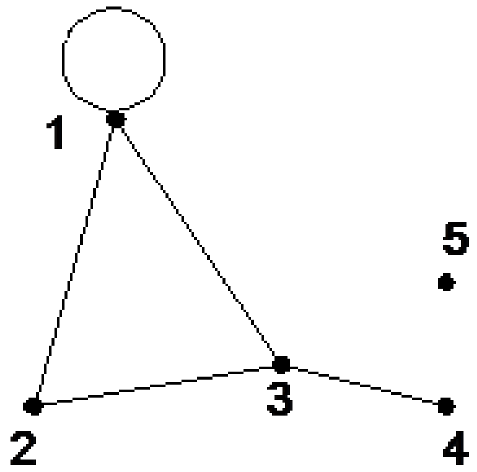
Degree of a Vertex

Let G be an undirected graph, $v \in V$ a vertex.

The degree of v, deg(v), is its number of incident edges. (Except that any self-loops *(simpul gelang)* are counted twice.)

A vertex with degree 0 is isolated *(simpul terpencil)*.

A vertex of degree 1 is pendant *(simpul anting-anting)*.



Handshaking Theorem

Let G be an undirected graph with vertex set V and edge set E. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

This theorem indicates that the sum of degree of all vertices in a graph is even (i.e. two times the number of its edges)

Corollary: Any undirected graph has an even number of vertices of odd degree.

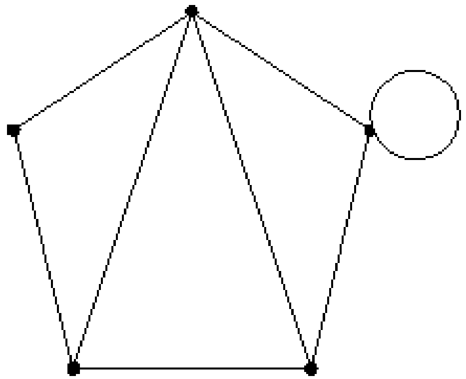
Handshaking Theorem Example

Can we draw a graph which has 5 vertices if each vertex has the following degree:

a) 2, 3, 1, 1, 2
b) 2, 3, 3, 4, 4

Solution:

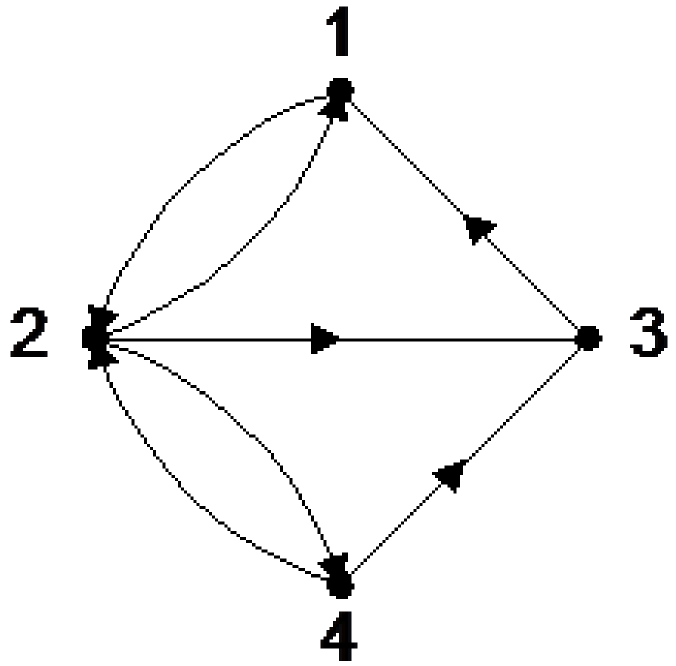
a) No, we can't, since the sum of the degree of vertices is odd. In addition, it has odd number of vertices with odd degree.
b) Yes, we can.



Directed Adjacency

Let G be a directed graph, and let e be an edge of G that is (or maps to) (u, v). Then we say:

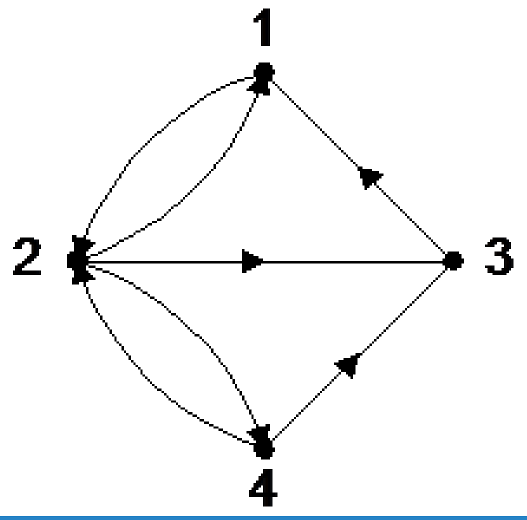
- u is *adjacent to* v, v is *adjacent from* u
- e *comes from* u, e *goes to* v.
- e *connects* u *to* v, e *goes from* u *to* v
- the *initial vertex* of e is u
- the *terminal vertex* of e is v



Directed Degree

Let G be a directed graph, v a vertex of G.

- The *in-degree* of v, $\deg^{-}(v)$, is the number of edges going to v.
- The *out-degree* of v, $\deg^{+}(v)$, is the number of edges coming from v.
- The *degree* of v, $\deg(v) \equiv \deg^{-}(v) + \deg^{+}(v)$, is the sum of v's in-degree and out-degree.



$$d_{in}(1) = 2;\ d_{out}(1) = 1$$
$$d_{in}(2) = 2;\ d_{out}(2) = 3$$
$$d_{in}(3) = 2;\ d_{out}(3) = 1$$
$$d_{in}(4) = 1;\ d_{out}(3) = 2$$

Directed Handshaking Theorem

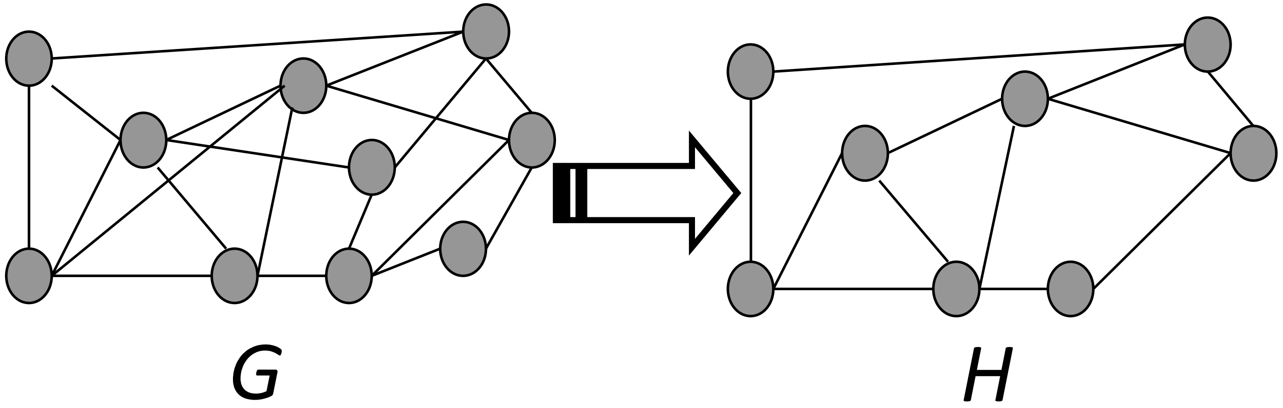
Let G be a directed graph with vertex set V and edge set E. Then:

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = \frac{1}{2} \sum_{v \in V} \deg(v) = |E|$$

Note that the degree of a node is unchanged by whether we consider its edges to be directed or undirected.]

Subgraphs

A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.



Graph Unions

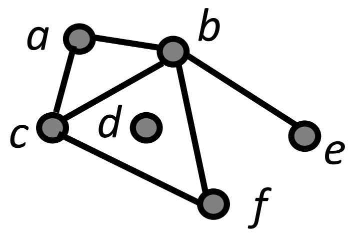
The union $G_1 \cup G_2$ of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Graph Representations

1. Adjacency lists (*senarai ketetanggaan*).
2. Adjacency matrices (*matriks ketetanggaan*).
3. Incidency matrices (*matriks bersisian*).

Adjacency Lists

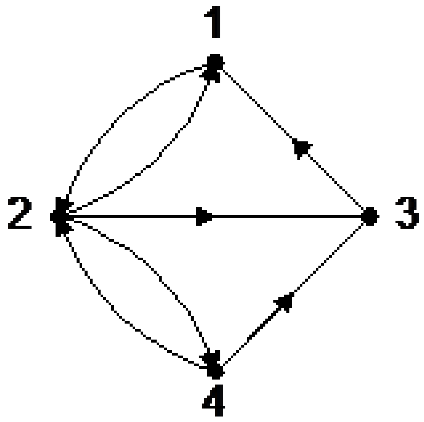
A table with 1 row per vertex, listing its adjacent vertices.



Vertex	Adjacent Vertices
a	b, c
b	a, c, e, f
c	a, b, f
d	
e	b
f	c, b

Directed Adjacency Lists

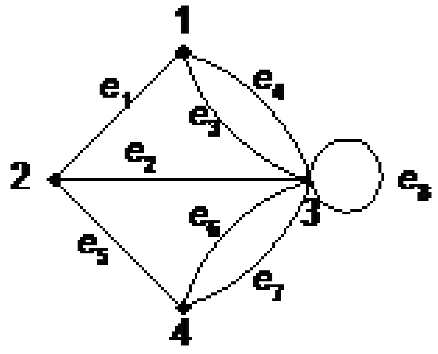
1 row per node, listing the terminal nodes of each edge incident from that node.



Node	Terminal Node
1	2
2	1, 3, 4
3	1
4	2, 3

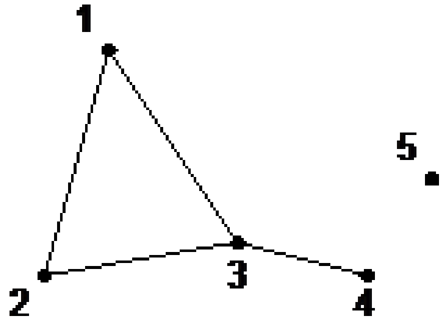
Adjacency Matrices

Matrix $A=[a_{ij}]$, where a_{ij} is 1 if $\{v_i, v_j\}$ is an edge of G, 0 otherwise.



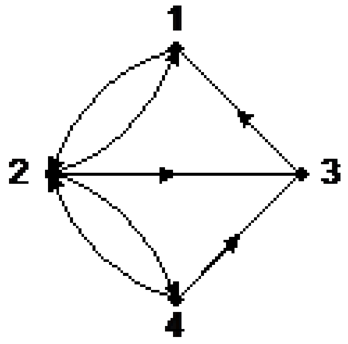
$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

(a)



$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(b)

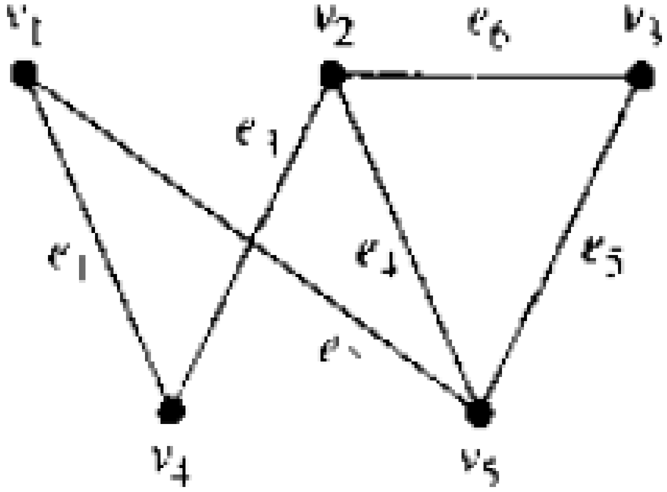


$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(c)

Incidency Matrices

Matrix $A=[a_{ij}]$, where a_{ij} is 1 if vertex v_i is incident with edge e_j, 0 otherwise.



$$
\begin{array}{c c}
 & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} &
\left[\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{array}\right].
\end{array}
$$

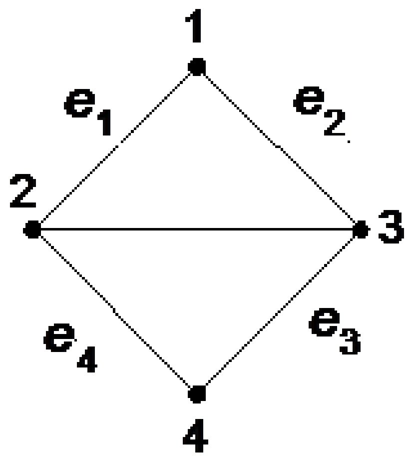
Connectivity

In an undirected graph, a path of length *n* from *u* to *v* is a sequence of adjacent edges going from vertex *u* to vertex *v*.

A path is a circuit if *u* = *v*.

A path traverses the vertices along it.

A path is simple if it contains no edge more than once.



In this graph 1, 2, 4, 3 is a path with lenght 3 (simple path).
1, 2, 3, 1 is a circuit.

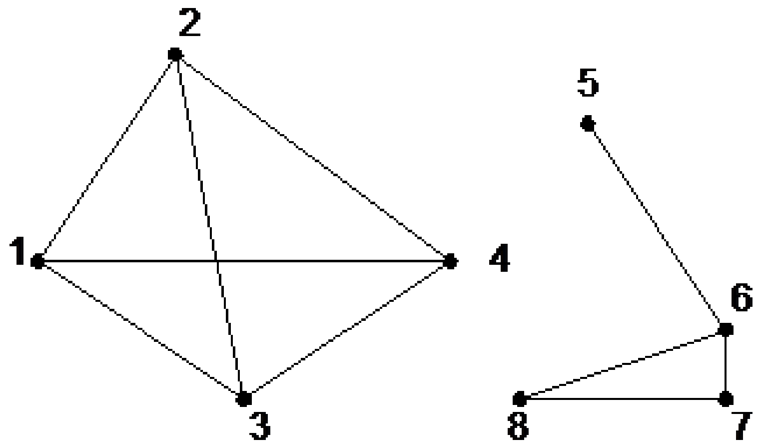
Paths in Directed Graphs

Same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness

An undirected graph is connected iff there is a path between every pair of distinct vertices in the graph.

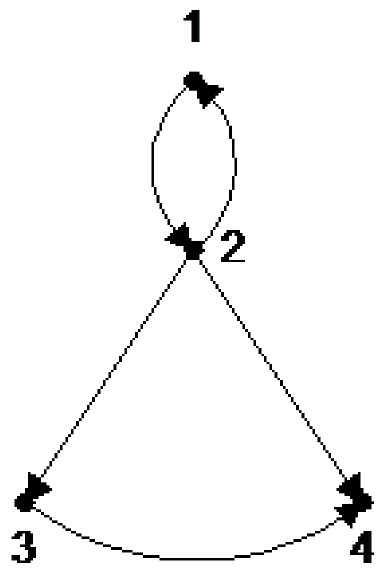
Theorem: There is a simple path between any pair of vertices in a connected undirected graph.

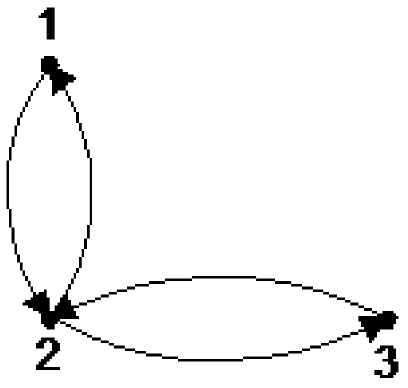


(disconnected graph)

More on Connectedness

In the following figures, the first graph is weakly connected, but the second one is strongly connected

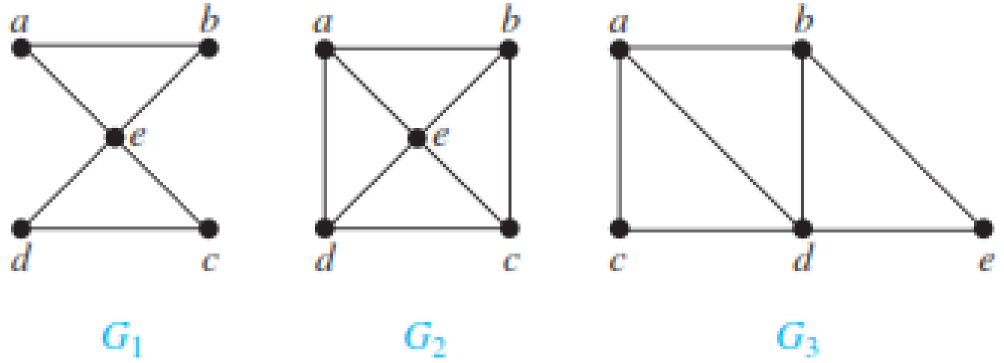




Euler Paths and Circuits

An *Euler circuit in a graph G is a simple circuit containing every edge of G.*

An Euler path in G is a simple path containing every edge of G.

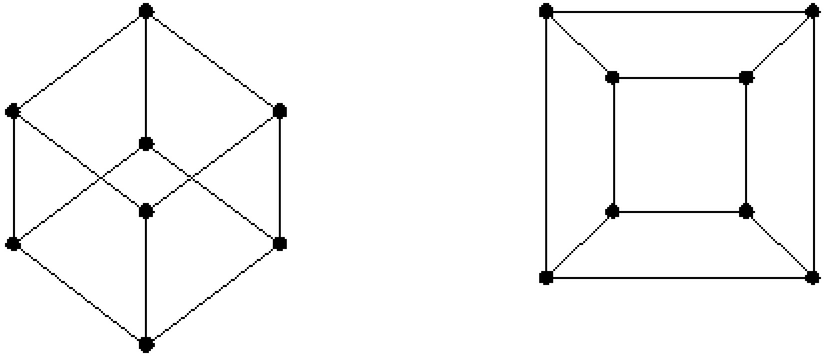


Which of the undirected graphs in the figure above have an Euler circuit? Of those that do not, which have an Euler path?

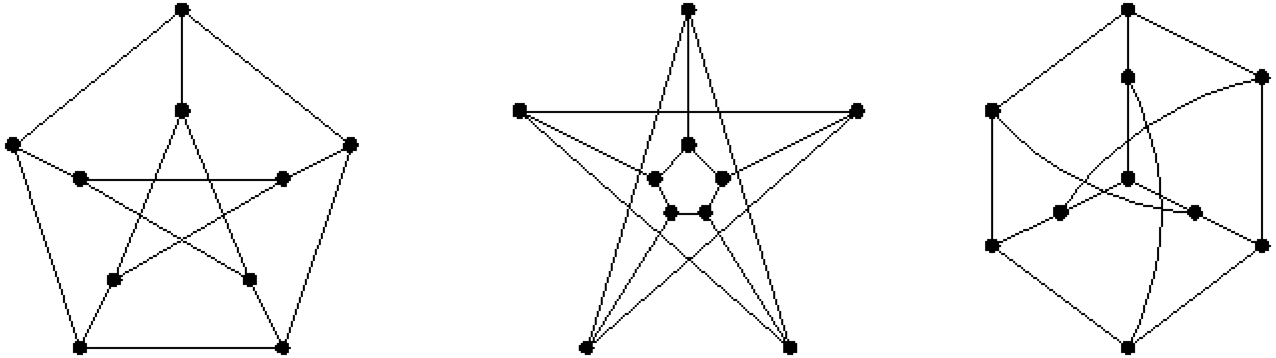
- The graphs G1 has an Euler circuit, for example, a, e, c, d, e, b, a.
- Neither of the graph G2 or G3 has an Euler circuit.
- However, G3 has an Euler path, namely, a, c, d, e, b, d, a, b.

Isomorphic Graphs

The same graphs can have different geometric representation, which called isomorphic graphs.



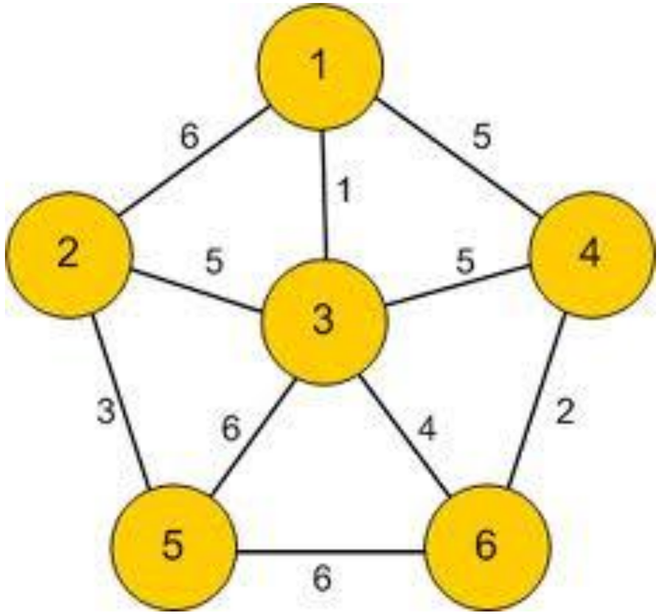
(a)



Weighted Graphs

A weighted graph associates a label (weight) with every edge in the graph.

Such weights might represent, for example, costs, lengths or capacities, etc. depending on the problem at hand



Graph Application in CS

Shortest path (Lintasan terpendek) → Djikstra algorithm

- The shortest path problem is the problem of finding a path between two vertices (or nodes) in a graph such that the sum of the weights of its constituent edges is minimized.
- This is analogous to the problem of finding the shortest path between two intersections on a road map: the graph's vertices correspond to intersections and the edges correspond to road segments, each weighted by the length of its road segment.

Graph Application in CS

Traveling salesperson problem (Persoalan pedagang keliling)

- Given a list of cities and their pairwise distances, the task is to find the shortest possible route that visits each city exactly once and returns to the origin city.

Graph Application in CS

Chinese postman problem (Persoalan tukang pos Cina)

- Chinese postman problem (CPP), postman tour or route inspection problem, is to find a shortest closed path or circuit that visits every edge of a (connected) undirected graph.

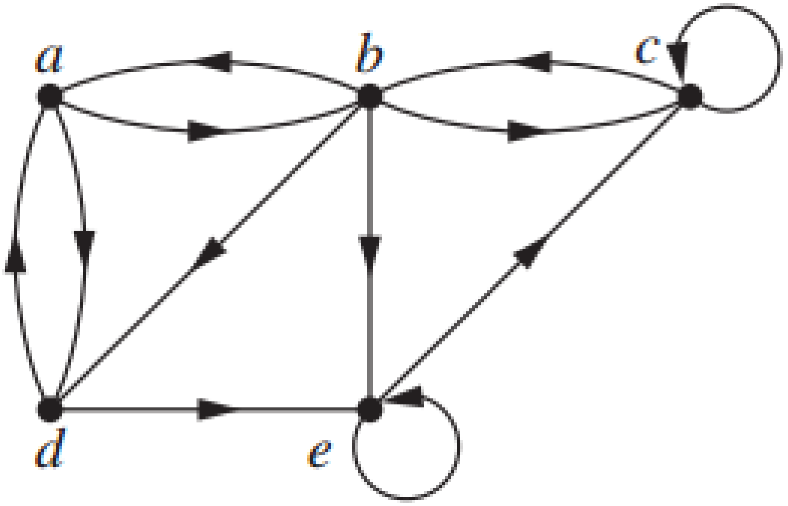
Graph Application in CS

Graph coloring (Pewarnaan graf)

- Graph coloring is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints.
- In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a vertex coloring.
- Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges share the same color, and a face coloring of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.

Example

Represent the following graph with an adjacency matrix.



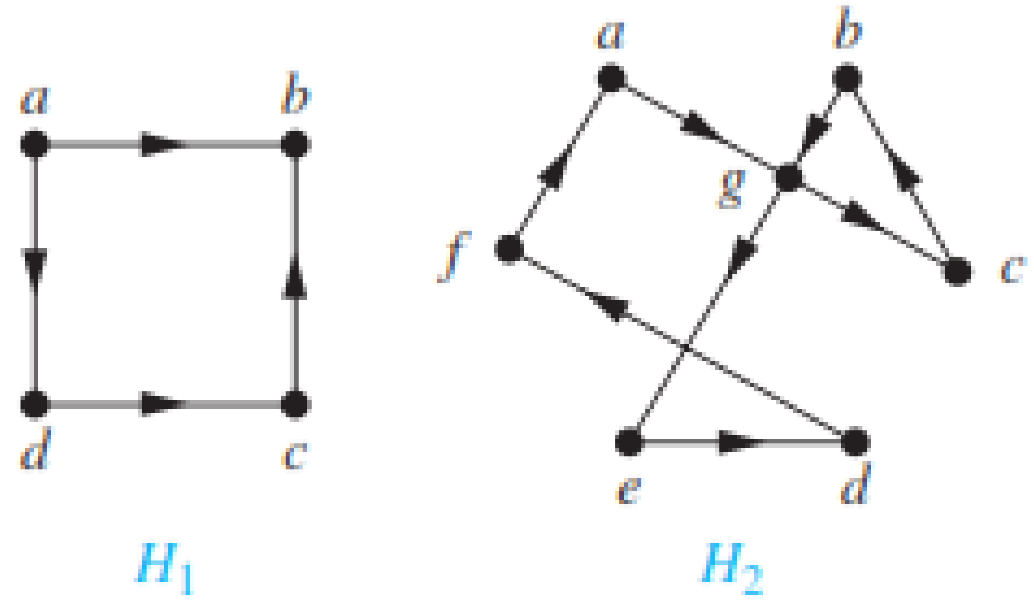
Example

Draw an undirected graph represented by the given adjacency matrix.

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

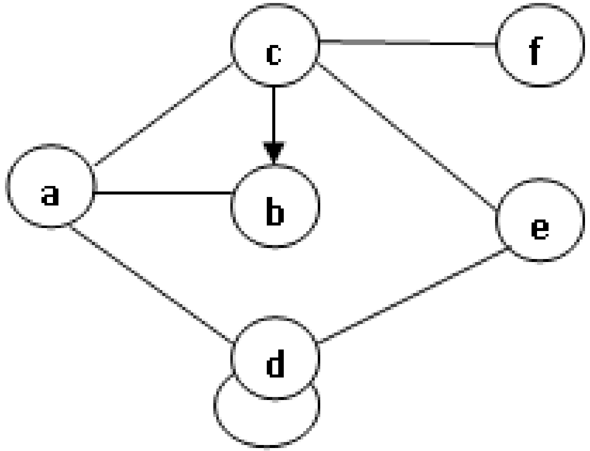
Example

Which of the directed graphs in the following figure have an Euler circuit? Of those that do not, which have an Euler path?



Exercise

1. Given the following graph *G*.



   a. Define the adjacency matrix for graph *G*.
   b. Do graph *G* has an Euler path? If it has, specify the path.