KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)

Relations

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Outline

Definition of Relation

Representing Relation

Relation Vs Function

Relation Properties

What is a relation

Relation generalizes the notion of functions.

Recall: A function takes **EACH** element from a set and maps it to a **UNIQUE** element in another set

$f: X \to Y$

$\forall x \in X, \exists y$ such that $f(x) = y$

Let A and B be sets. A binary relation (sets of ordered pair) R is a subset of $A \times B$

Recall: $A \times B = \{(a, b) \mid a \in A, b \in B\}$

$a \, R \, b$: $(a, b) \in R$

Application: Relational database model is based on the concept of relation.

Relation examples

Let A be the students in a CS major

- A = {Alice, Bob, Claire, Dan}

Let B be the courses the department offers

- B = {CS101, CS201, CS202}

We specify relation $R \subset A \times B$ as the set that lists all students $a \in A$ enrolled in class $b \in B$

R = { (Alice, CS101), (Bob, CS201), (Bob, CS202),
(Dan, CS201), (Dan, CS202) }

Relation examples (cont'd)

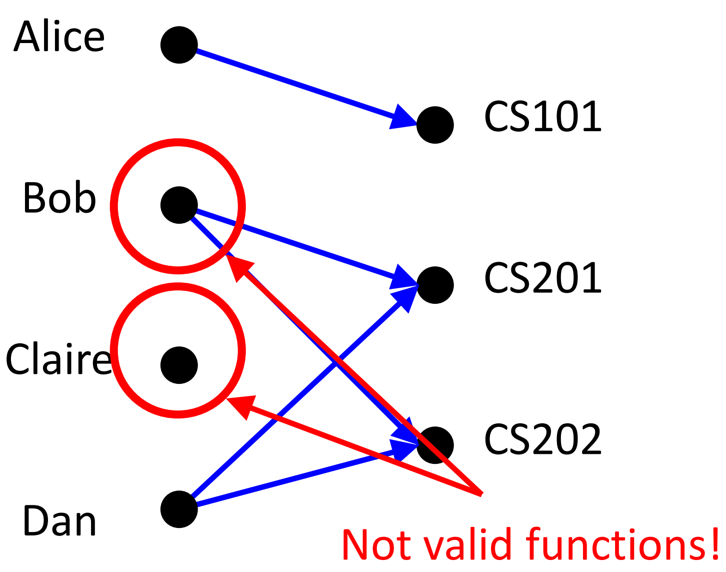
Another relation example:

- Let A be the cities in the US
- Let B be the states in the US
- We define R to mean a is a city in state b
- Thus, the following are in our relation:
  - (C'ville, VA)
  - (Philadelphia, PA)
  - (Portland, MA)
  - (Portland, OR)
  - etc…

Most relations we will see deal with ordered pairs of integers

Representing relations

We can represent relations graphically:



We can represent relations in a table:

	CS101	CS201	CS202
Alice	X		
Bob		X	X
Claire			
Dan		X	X

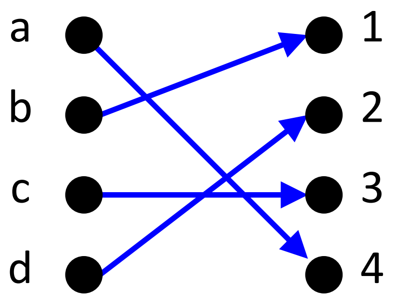
Relations vs. functions

If $R \subset X \times Y$ is a relation, then is R a function?

If $f: X \to Y$ is a function, then is f a relation?

Not all relations are functions

But consider the following function:



All functions are relations!

When to use which?

A function is used when you need to obtain a SINGLE result for any element in the domain

- Example: sin, cos, tan

A relation is used when there are multiple mappings between the domain and the co-domain

- Example: students enrolled in multiple courses

Relations on a set

A relation on the set A is a relation from A to A

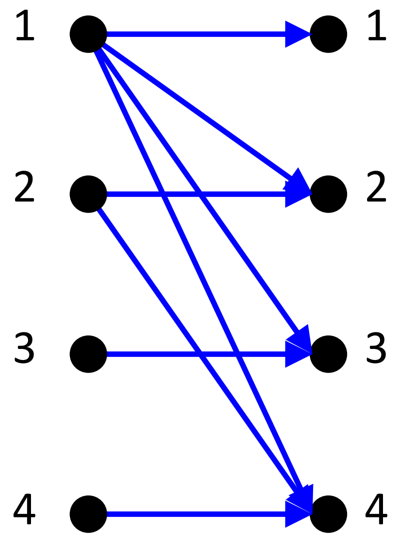
- In other words, the domain and co-domain are the same set
- We will generally be studying relations of this type

Relations on a set

Let A be the set { 1, 2, 3, 4 }

Which ordered pairs are in the relation $R = \{ (a, b) \mid a \text{ divides } b \}$

R = { (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) }



R	1	2	3	4
1	X	X	X	X
2		X		X
3			X	
4				X

More examples

Consider some relations on the set **Z.** Are the following ordered pairs in the relation?

	(1,1)	(1,2)	(2,1)	(1,-1)	(2,2)
$R_1 = \{ (a,b) \mid a \leq b \}$	X	X			X
$R_2 = \{ (a,b) \mid a > b \}$			X	X	
$R_3 = \{ (a,b) \mid a = \lvert b \rvert \}$	X			X	X
$R_4 = \{ (a,b) \mid a = b \}$	X				X
$R_5 = \{ (a,b) \mid a = b+1 \}$			X		
$R_6 = \{ (a,b) \mid a+b \leq 3 \}$	X	X	X	X	

Relation properties

Six properties of relations we will study:

- Reflexive
- Irreflexive
- Symmetric
- Asymmetric
- Antisymmetric
- Transitive

Reflexivity

A relation is reflexive if every element is related to itself

- Or, $(a, a) \in R$

Examples of reflexive relations:

- $=, \leq, \geq$

Examples of relations that are not reflexive:

- $<, >$

Irreflexivity

A relation is irreflexive if every element is *not* related to itself

- Or, $(a, a) \notin R$
- Irreflexivity is the opposite of reflexivity

Examples of irreflexive relations:

- $<$, $>$

Examples of relations that are not irreflexive:

- $=$, $\leq$, $\geq$

Reflexivity vs. Irreflexivity

A relation can be neither reflexive nor irreflexive

- Some elements are related to themselves, others are not

Example

A = {1, 2}, R = {(1, 1)}

It is not reflexive, since $(2, 2) \notin R$,

It is not irreflexive, since $(1, 1) \in R$.

Symmetric, Asymmetric, Antisymmetric

A relation is symmetric if

for all a, b $\in$ A, $(a, b) \in R \Rightarrow (b,a) \in R$

A relation is asymmetric if

for all a, b $\in$ A, $(a, b) \in R \Rightarrow (b,a) \notin R$

A relation is antisymmetric if

for all a, b $\in$ A, $((a, b) \in R \wedge (b,a) \in R) \Rightarrow a=b$

(Second definition) for all a, b $\in$ A, $((a,b) \in R \wedge a \neq b) \Rightarrow (b,a) \notin R)$

Example: Consider relations on {1, 2, 3, 4}

- R1 = {(1,1), (1,2), (2,1)} is symmetric
- R2 = {(1,2), (1,3), (1,4)} is asymmetric
- R3 = {(1,1), (2,1}, (2,2), (3,1), (3,2), (4,1), (4,2), (4,3)} is antisymmetric

Notes on *symmetric relations

A relation can be neither symmetric or asymmetric

- R = { (a,b) | a=|b| }
- This is not symmetric
  - -4 is not related to itself
- This is not asymmetric
  - 4 is related to itself
- Note that it is antisymmetric

Transitivity

A relation is transitive if, for every $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

If $a < b$ and $b < c$, then $a < c$

- Thus, $<$ is transitive

If $a = b$ and $b = c$, then $a = c$

- Thus, $=$ is transitive

Transitivity examples

Consider isAncestorOf()

- Let Alice be Bob’s parent, and Bob be Claire’s parent
- Thus, Alice is an ancestor of Bob, and Bob is an ancestor of Claire
- Thus, Alice is an ancestor of Claire
- Thus, isAncestorOf() is a transitive relation

Consider isParentOf()

- Let Alice be Bob’s parent, and Bob be Claire’s parent
- Thus, Alice is a parent of Bob, and Bob is a parent of Claire
- However, Alice is *not* a parent of Claire
- Thus, isParentOf() is *not* a transitive relation

Summary of relation's properties

reflexive	$\forall a\ (a, a) \in R$
irreflexive	$\forall a\ (a, a) \notin R$
symmetric	$\forall\ a, b \in A, (a, b) \in R \Rightarrow (b, a) \in R$
asymmetric	$\forall\ a, b \in A, (a, b) \in R \Rightarrow (b, a) \notin R$
antisymmetric	$\forall\ a, b \in A, ((a, b) \in R \wedge (b, a) \in R) \Rightarrow a=b$ (*) $\forall\ a, b \in A, ((a, b) \in R \wedge a \neq b) \Rightarrow (b, a) \notin R)$
transitive	$\forall\ a, b, c \in A, ((a, b) \in R \wedge (b, c) \in R) \Rightarrow (a, c) \in R$

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Equivalence Relations

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Outline

1. Combining Relation
2. Equivalence Relation
3. Equivalence relation examples
4. Related items
   - Equivalence class
   - Partitions

Combining relations

There are two ways to combine relations R_1 and R_2

- Via Boolean operators
- Via relation "composition"

Combining relations via Boolean operators

Consider two relations $R_{\geq}$ and $R_{\leq}$

We can combine them as follows:

- $R_{\geq} \cup R_{\leq}$ = all numbers $\geq$ OR $\leq$
  - That's all the numbers
- $R_{\geq} \cap R_{\leq}$ = all numbers $\geq$ AND $\leq$
  - That's all numbers equal to
- $R_{\geq} \oplus R_{\leq}$ = all numbers $\geq$ or $\leq$, but not both
  - That's all numbers not equal to
- $R_{\geq} - R_{\leq}$ = all numbers $\geq$ that are not also $\leq$
  - That's all numbers strictly greater than
- $R_{\leq} - R_{\geq}$ = all numbers $\leq$ that are not also $\geq$
  - That's all numbers strictly less than

Note that it's possible the result is the empty set

Combining relations via relational composition

Let R be a relation from A to B, and S be a relation from B to C

- Let $a \in A$, $b \in B$, and $c \in C$
- Let $(a, b) \in R$, and $(b, c) \in S$
- Then the composite of R and S consists of the ordered pairs (a, c)
  - We denote the relation by $S \circ R$
  - Note that S comes first when writing the composition!

Combining relations via relational composition

Let M be the relation "is mother of"

Let F be the relation "is father of"

What is $M \circ F$?

- If $(a, b) \in F$, then a is the father of b
- If $(b, c) \in M$, then b is the mother of c
- Thus, $M \circ F$ denotes the relation "maternal grandfather"

What is $F \circ M$?

- If $(a, b) \in M$, then a is the mother of b
- If $(b, c) \in F$, then b is the father of c
- Thus, $F \circ M$ denotes the relation "paternal grandmother"

- What is $M \circ M$?
  - If $(a, b) \in M$, then a is the mother of b
  - If $(b, c) \in M$, then b is the mother of c
  - Thus, $M \circ M$ denotes the relation "maternal grandmother"
- Note that M and F are **not transitive** relations!!!

Combining relations via relational composition

Given relation R

- $R \circ R$ can be denoted by R^2
- $R^2 \circ R = (R \circ R) \circ R = R^3$
- Example: M^3 is your mother's mother's mother

Equivalence relations

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Consider relation $R = \{ (a, b) \mid len(a) = len(b) \}$

- Where $len(a)$ means the length of string a
- It is reflexive: $len(a) = len(a)$
- It is symmetric: if $len(a) = len(\mathrm{b})$, then $len(b) = len(a)$
- It is transitive: if $len(a) = len(\mathrm{b})$ and $len(\mathrm{b}) = len(\mathrm{c})$, then $len(\mathrm{a}) = len(\mathrm{c})$
- Thus, R is an equivalence relation

Equivalence relation example

Consider the relation $R = \{ (a, b) \mid m \mid a - b \}$
- Called "congruence modulo m"

Is it reflexive: (a, a) $\in R$ means that $m \mid a - a$
- $a - a = 0$, which is divisible by m

Is it symmetric: if $(a, b) \in R$ then $(b, a) \in R$
- (a, b) means that $m \mid a - b$
- Or that $km = a - b$. Negating that, we get $b - a = - km$
- Thus, $m \mid b - a$, so (b, a) $\in R$

- Is it transitive: if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$
  - (a, b) means that $m \mid a - b$, or that $km = a - b$
  - (b, c) means that $m \mid b - c$, or that $lm = b - c$
  - (a, c) means that $m \mid a - c$, or that $nm = a - c$
  - Adding these two, we get $km + lm = (a - b) + (b - c)$
  - Or $(k + l)m = a - c$
  - Thus, m divides $a - c$, where $n = k + l$
- Thus, congruence modulo m is an equivalence relation

Sample questions

Which of these relations on {0, 1, 2, 3} are equivalence relations?

Determine the properties of an equivalence relation that the others lack

{ (0,0), (1,1), (2,2), (3,3) }

Has all the properties, thus, is an equivalence relation

{ (0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3) }

Not reflexive: (1,1) is missing

Not transitive: (0,2) and (2,3) are in the relation, but not (0,3)

{ (0,0), (1,1), (1,2), (2,1), (2,2), (3,3) }

Has all the properties, thus, is an equivalence relation

{ (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2) (3,3) }

Not transitive: (1,3) and (3,2) are in the relation, but not (1,2)

{ (0,0), (0,1) (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }

Not symmetric: (1,2) is present, but not (2,1)

Not transitive: (2,0) and (0,1) are in the relation, but not (2,1)

Sample questions

Suppose that A is a non-empty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x,y) where $f(x) = f(y)$

- Meaning that x and y are related if and only if $f(x) = f(y)$

Show that R is an equivalence relation on A

Reflexivity: $f(x) = f(x)$

- True, as given the same input, a function always produces the same output

Symmetry: if $f(x) = f(y)$ then $f(y) = f(x)$

- True, by the definition of equality

Transitivity: if $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$

- True, by the definition of equality

Equivalence classes

Let R be an equivalence relation on a set A.

The set of all elements that are related to an element a of A is called the ***equivalence class*** of a.

The equivalence class of a with respect to R is denoted by $[a]_R$

When only one relation is under consideration, the subscript is often deleted, and $[a]$ is used to denote the equivalence class

Equivalence classes of two elements of A are either identical or disjoint.

More on equivalence classes

Consider the relation R = { (a, b) | a mod 2 = b mod 2 }

- Thus, all the even numbers are related to each other
- As are the odd numbers

The even numbers form an equivalence class

- As do the odd numbers

The equivalence class for the even numbers is denoted by [2] (or [4], or [784], etc.)

- [2] = { …, -4, -2, 0, 2, 4, … }
- 2 is a *representative* of it's equivalence class

There are only 2 equivalence classes formed by this equivalence relation

More on equivalence classes

Consider the relation $R = \{ (a, b) \mid a = b \text{ or } a = -b \}$

- Thus, every number is related to additive inverse

The equivalence class for an integer a:

- $[7] = \{ 7, -7 \}$
- $[0] = \{ 0 \}$
- $[a] = \{ a, -a \}$

There are an infinite number of equivalence classes formed by this equivalence relation

Partitions

Consider the relation R = { (a, b) | a mod 2 = b mod 2 }

This splits the integers into two equivalence classes: even numbers and odd numbers

Those two sets together form a partition of the integers

Formally, a partition of a set S is a collection of non-empty disjoint subsets of S whose union is S

In this example, the partition is { [0], [1] }

- Or { {…, -3, -1, 1, 3, …}, {…, -4, -2, 0, 2, 4, …} }

Sample questions

Which of the following are partitions of the set of integers?

a) The set of even integers and the set of odd integers

    Yes, it's a valid partition

b) The set of positive integers and the set of negative integers

    No: 0 is in neither set

c) The set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3

    Yes, it's a valid partition

d) The set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100

    Yes, it's a valid partition

e) The set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6

    The first two sets are not disjoint (2 is in both), so it's not a valid partition

Exercises

Which of these relation on {0, 1, 2, 3} are equivalence relation? Determine the properties of an equivalence relation that the others lack.

a) {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}
b) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0),(2, 2), (3, 3)}

Answer

a) Equivalence relation

b) Not symmetric, not transitive

reflexive	$\forall a\ (a, a) \in R$
irreflexive	$\forall a\ (a, a) \notin R$
symmetric	$\forall\ a, b \in A, (a, b) \in R \Rightarrow (b, a) \in R$
asymmetric	$\forall\ a, b \in A, (a, b) \in R \Rightarrow (b, a) \notin R$
antisymmetric	$\forall\ a, b \in A, ((a, b) \in R \wedge (b, a) \in R) \Rightarrow a=b$ (*)for all $a, b \in A, ((a, b) \in R \wedge a \neq b) \Rightarrow (b, a) \notin R)$
transitive	$\forall\ a, b, c \in A, ((a, b) \in R \wedge (b, c) \in R) \Rightarrow (a, c) \in R$