KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)

Mathematical Induction

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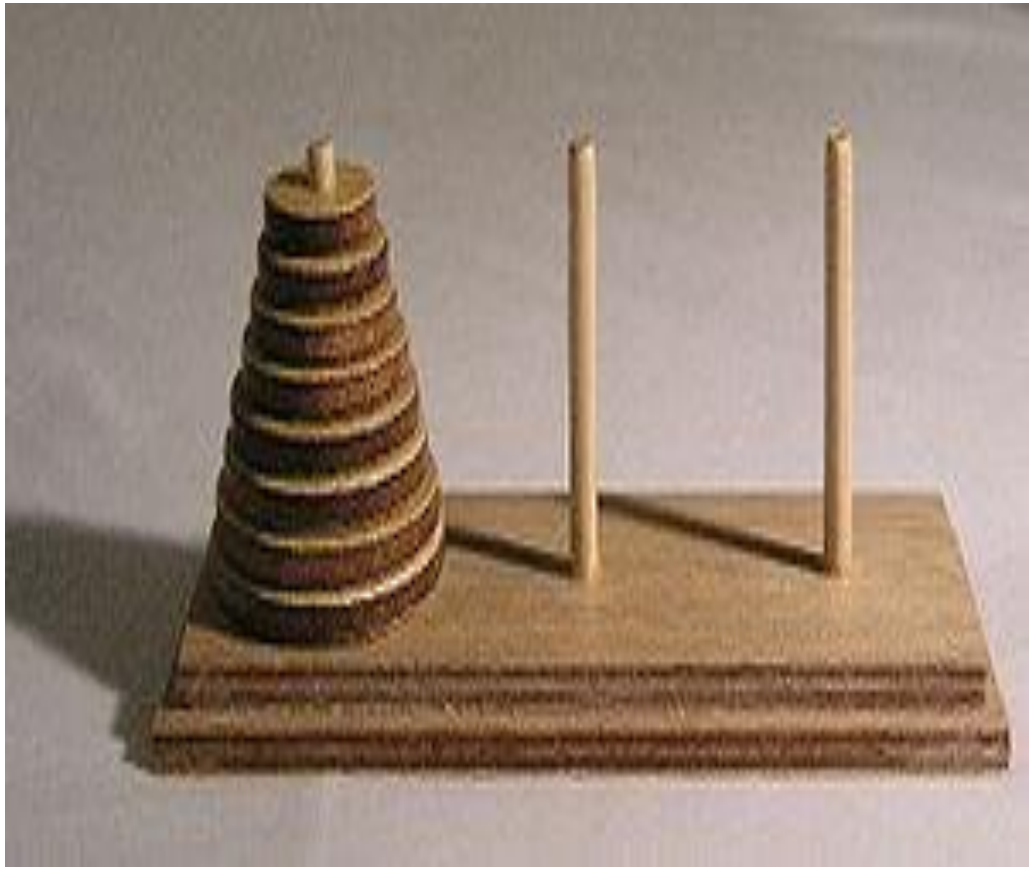
Outline

1. Hanoi Tower Simulation
2. Induction Definition
3. Induction Example
4. Weak Induction
5. Strong Induction

Simulation Game: Hanoi Tower



The objective of the puzzle is to move the entire stack to another rod, obeying the following rules:

1. Only one disk may be moved at a time.
2. Each move consists of taking the upper disk from one of the rods and sliding it onto another rod, on top of the other disks that may already be present on that rod.
3. No disk may be placed on top of a smaller disk.

Hanoi Tower Solution

For an even number of disks:

- make the legal move between pegs A and B
- make the legal move between pegs A and C
- make the legal move between pegs B and C
- repeat until complete

For an odd number of disks:

- make the legal move between pegs A and C
- make the legal move between pegs A and B
- make the legal move between pegs B and C
- repeat until complete

In each case, a total of 2^n-1 moves are made.

How do you climb infinite stairs?

Not a rhetorical question!

First, you get to the base platform of the staircase

Then repeat:

- From your current position, move one step up

Let's use that as a proof method

First, show P(x) is true for $x = 0$

- This is the base of the stairs

Then, show that if it's true for some value n, then it is true for n+1

- Show: $P(n) \rightarrow P(n+1)$
- This is climbing the stairs
- Let n=0. Since it's true for P(0) (base case), it's true for n=1
- Let n=1. Since it's true for P(1) (previous bullet), it's true for n=2
- Let n=2. Since it's true for P(2) (previous bullet), it's true for n=3
- Let n=3 …
- And onwards to infinity

Thus, we have shown it to be true for *all* non-negative numbers

What is induction?

A method of proof. It does not generate answers: it only can prove them

Three parts:

- **Base case(s)**: show it is true for one element
  - (get to the stair's base platform)
- **Inductive hypothesis**: assume it is true for any given element
  - (assume you are on a stair)
  - **Must be clearly labeled!!!**
- **Show that it true for the next highest element**
  - (show you can move to the next stair)

Induction example

Show that the sum of the first n odd integers is n^2

- Example: If $n = 5$, $1 + 3 + 5 + 7 + 9 = 25 = 5^2$
- Formally, show: $$\forall n \; P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i - 1 = n^2$$

Base case: Show that P(1) is true

$$P(1) = \sum_{i=1}^{1} 2(i) - 1 = 1^2$$

$$= 1 = 1$$

Induction example, continued

Inductive hypothesis: assume true for *k*

- Thus, we assume that P(*k*) is true, or that

$$\sum_{i=1}^{\boxed{k}} 2i - 1 = \boxed{k^2}$$

- Note: we don't yet know if this is true or not!

Inductive step: show true for *k*+1

- We want to show that:

$$\sum_{i=1}^{\boxed{k+1}} 2i - 1 = \boxed{(k+1)^2}$$

Induction example, continued

Recall the inductive hypothesis: P(*k*) = $$\sum_{i=1}^{k} 2i-1 = k^2$$

Proof of inductive step: P(*k* + 1) = $$\sum_{i=1}^{k+1} 2i-1 = (k+1)^2$$

$$\sum_{i=1}^{k+1} 2i-1 = (k+1)^2$$

$$2(k+1)-1+\boxed{\sum_{i=1}^{k} 2i-1} = k^2+2k+1$$

$$2(k+1)-1+\boxed{k^2} = k^2+2k+1$$

$$k^2+2k+1 = k^2+2k+1$$

What did we show?

Base case: P(1)

If P(k) was true, then P(k+1) is true

- i.e., P(k) → P(k+1)

We know it's true for P(1)

- Because of P(k) → P(k+1), if it's true for P(1), then it's true for P(2)
- Because of P(k) → P(k+1), if it's true for P(2), then it's true for P(3)
- Because of P(k) → P(k+1), if it's true for P(3), then it's true for P(4)
- Because of P(k) → P(k+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

Thus, it is true for all possible values of n

In other words, we showed that: $\left[\mathrm{P}(1) \wedge \forall k\left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)\right] \rightarrow \forall n\, \mathrm{P}(n)$

The idea behind inductive proofs

Show the base case

Show the inductive hypothesis

Manipulate the inductive step so that you can substitute in part of the inductive hypothesis

Show the inductive step

Second induction example

Show the sum of the first n positive even integers is $n^2 + n$

- Rephrased:

$$\forall n \; P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i = n^2 + n$$

The three parts:

- Base case
- Inductive hypothesis
- Inductive step

Second induction example, continued

Base case: Show P(1): $$P(1) = \sum_{i=1}^{1} 2(i) = 1^2 + 1$$
$$= 2 = 2$$

Inductive hypothesis: Assume $$P(k) = \sum_{i=1}^{k} 2i = k^2 + k$$

Inductive step: Show $$P(k+1) = \sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$$

Second induction example, continued

Recall our inductive hypothesis:

$$P(k) = \sum_{i=1}^{k} 2i = k^2 + k$$

$$\sum_{i=1}^{k+1} 2i = (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} = (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} = (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 = k^2 + 3k + 2$$

Notes on proofs by induction

We manipulate the *k*+1 case to make part of it look like the *k* case

We then replace that part with the other side of the *k* case

$$\sum_{i=1}^{k+1} 2i = (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} = (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} = (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 = k^2 + 3k + 2$$

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i = k^2 + k$$

Strong induction

Weak mathematical induction assumes P(k) is true, and uses that (and only that!) to show P(k+1) is true

Strong mathematical induction assumes P(1), P(2), …, P(k) are all true, and uses that to show that P(k+1) is true.

$$\left[P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k)\right] \to P(k+1)$$

Strong induction example 1

Show that any number > 1 can be written as the product of one or more primes

Base case: P(2)

- 2 is the product of 2 (remember that 1 is not prime!)

Inductive hypothesis: assume P(2), P(3), …, P(k) are all true

Inductive step: Show that P(k+1) is true

Strong induction example 1

Inductive step: Show that P(k+1) is true

There are two cases:

- k+1 is prime
  - It can be written as the product of k+1
- k+1 is composite
  - It can be written as the product of two composites, a and b, where $2 \leq a \leq b < k+1$
  - By the inductive hypothesis, both P(a) and P(b) are true

Strong induction vs. non-strong induction

Show that every postage amount 12 cents or more can be formed using only 4 and 5 cent stamps

Answer via mathematical induction

Show base case: P(12):

- $12 = 4 + 4 + 4$

Inductive hypothesis: Assume P(k) is true

Inductive step: Show that P(k+1) is true

- If P(k) uses a 4 cent stamp, replace that stamp with a 5 cent stamp to obtain $P(k+1)$
- If P(k) does not use a 4 cent stamp, it must use only 5 cent stamps
  - Since $k > 10$, there must be at least three 5 cent stamps
  - Replace these with four 4 cent stamps to obtain $k+1$

Note that only $P(k)$ was assumed to be true

Answer via strong induction

Show base cases: P(12), P(13), P(14), and P(15)

- 12 = 4 + 4 + 4
- 13 = 4 + 4 + 5
- 14 = 4 + 5 + 5
- 15 = 5 + 5 + 5

Inductive hypothesis: Assume P(12), P(13), …, P(k) are all true

- For $k \geq 15$

Inductive step: Show that P(k+1) is true

- We will obtain P(k+1) by adding a 4 cent stamp to P(k+1-4)
- Since we know P(k+1-4) = P(k-3) is true, our proof is complete

Note that P(12), P(13), …, P(k) were all assumed to be true

Refreshment Test ☺

Using mathematical induction, show that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Don't forget:

- Base case: $n = 1$
- Inductive hypothesis
- Inductive step

Answer

Base case: n = 1

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$$

$$1^2 = \frac{6}{6}$$

$$1 = 1$$

Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive step: show P(k+1) is true

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$$

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$