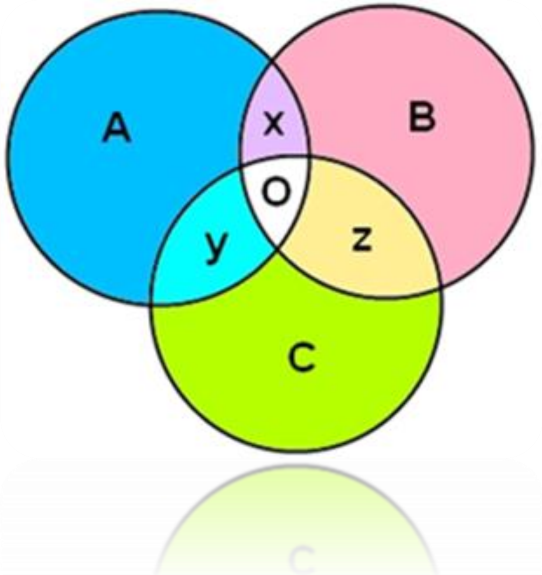
KS141203 MATEMATIKA DISKRIT
(*DISCRETE MATHEMATICS*)



Sets Operations

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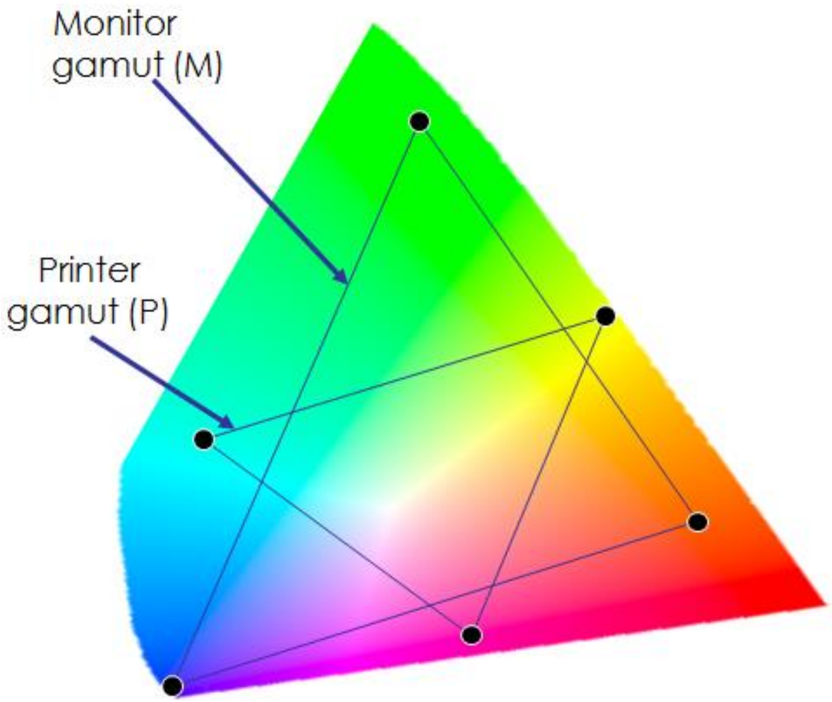
Sets of Colors

Pick any 3 “primary” colors

Triangle shows mixable color range (gamut) – the set of colors



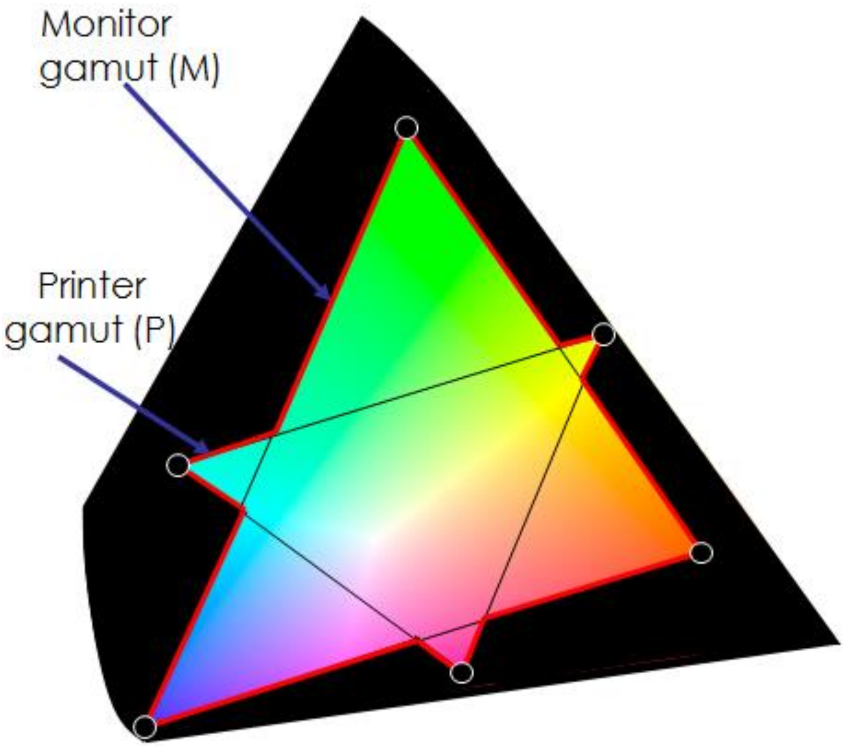
Set operations: Union (Gabungan)

A union of the sets contains all the elements in **EITHER** set

Union symbol is usually a $\cup$

Example:

- $C = M \cup P$



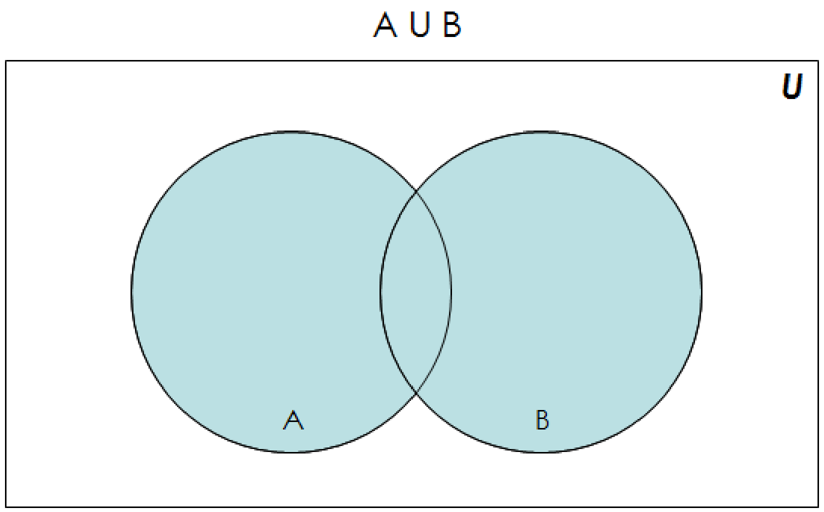
Set operations: Union (cont.)

Formal definition for the union of two sets:

$A \cup B = \{ x \mid x \in A \textbf{ or } x \in B \}$

Further examples

- $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$
- {New York, Washington} ∪ {3, 4} = {New York, Washington, 3, 4}
- $\{1, 2\} \cup \varnothing = \{1, 2\}$



Properties of the union operation

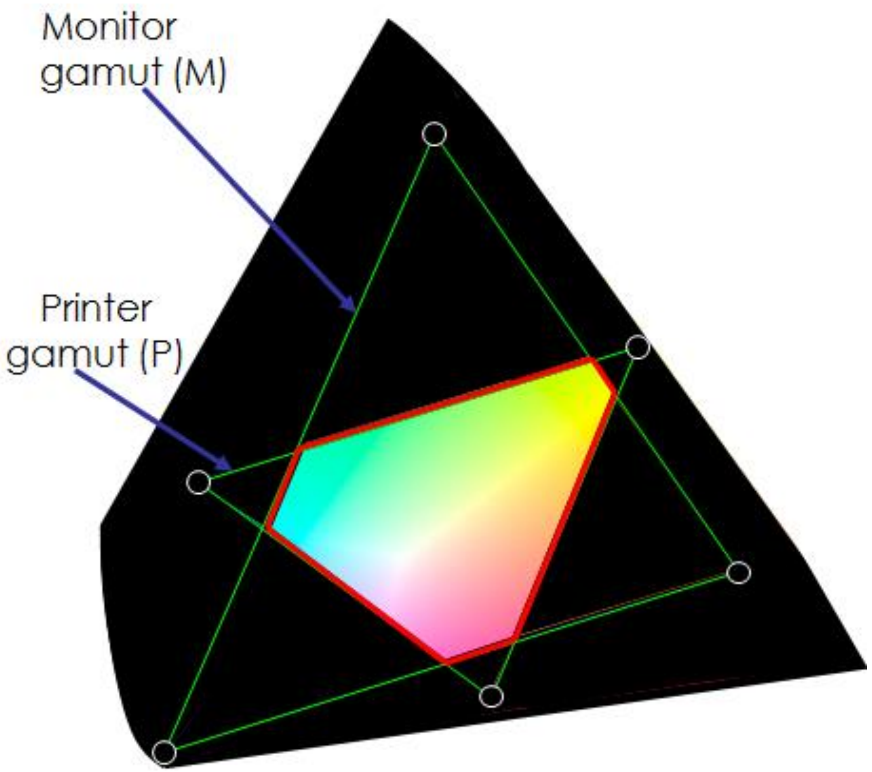
$A \cup \varnothing = A$	Identity law
$A \cup \boldsymbol{U} = \boldsymbol{U}$	Domination law
$A \cup A = A$	Idempotent law
$A \cup B = B \cup A$	Commutative law
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative law

Set operations: Intersection (Irisan)

An intersection of the sets contains all the elements in **BOTH** sets

Intersection symbol is a ∩

Example:
C = M ∩ P

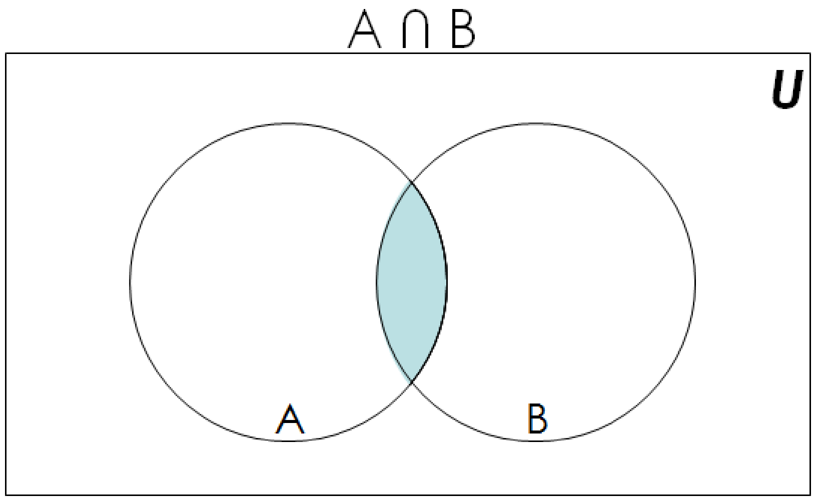


Set operations: Intersection

Formal definition for the intersection of two sets: $A \cap B = \{ x \mid x \in A$ **and** $x \in B \}$

Further examples

- $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$
- $\{\text{New York, Washington}\} \cap \{3, 4\} = \varnothing$
  - No elements in common
- $\{1, 2\} \cap \varnothing = \varnothing$
  - Any set intersection
    with the empty set
    yields the empty set



Properties of the intersection operation

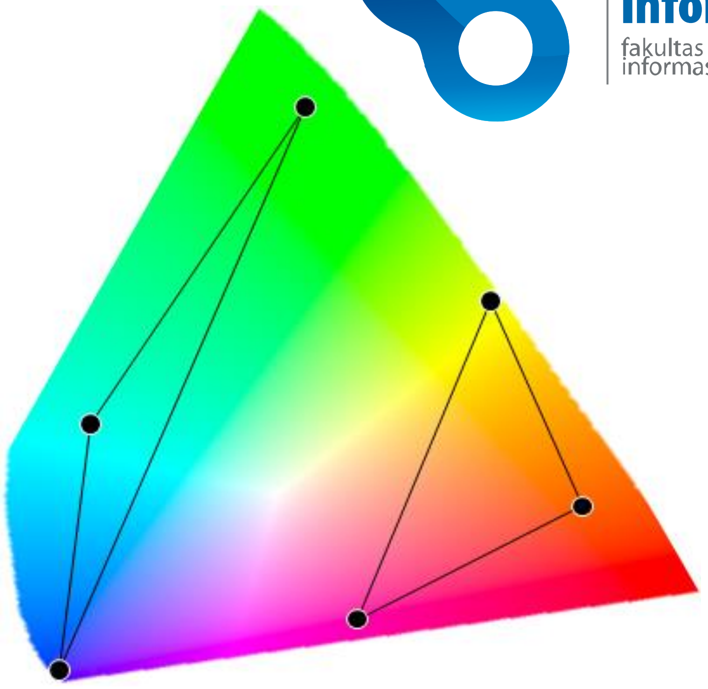
$A \cap \boldsymbol{U} = A$	Identity law
$A \cap \varnothing = \varnothing$	Domination law
$A \cap A = A$	Idempotent law
$A \cap B = B \cap A$	Commutative law
$A \cap (B \cap C) = (A \cap B) \cap C$	Associative law

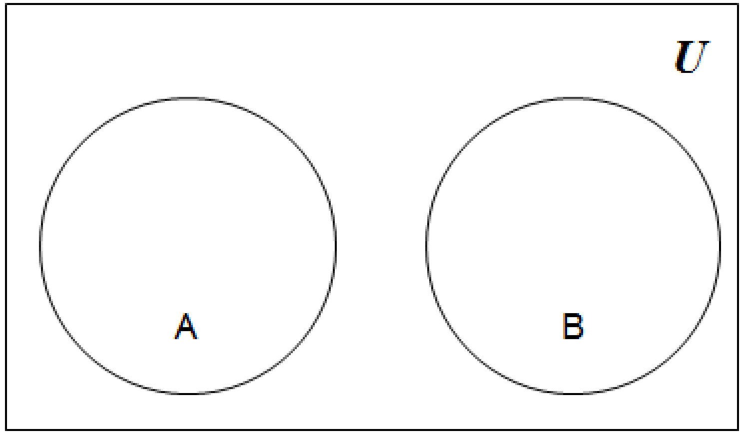
Disjoint sets

Two sets are disjoint if they have NO elements in common

Formally, two sets are disjoint if their intersection is the empty set

Another example: the set of the even numbers and the set of the odd numbers





Disjoint sets (cont.)

Formal definition for disjoint sets: two sets are disjoint if their intersection is the empty set

Further examples

- {1, 2, 3} and {3, 4, 5} are not disjoint
- {New York, Washington} and {3, 4} are disjoint
- {1, 2} and $\varnothing$ are disjoint
  - Their intersection is the empty set
- $\varnothing$ and $\varnothing$ are disjoint!
  - Their intersection is the empty set

Set operations: Difference (Selisih)

A difference of two sets is the elements in one set that are **NOT** in the other

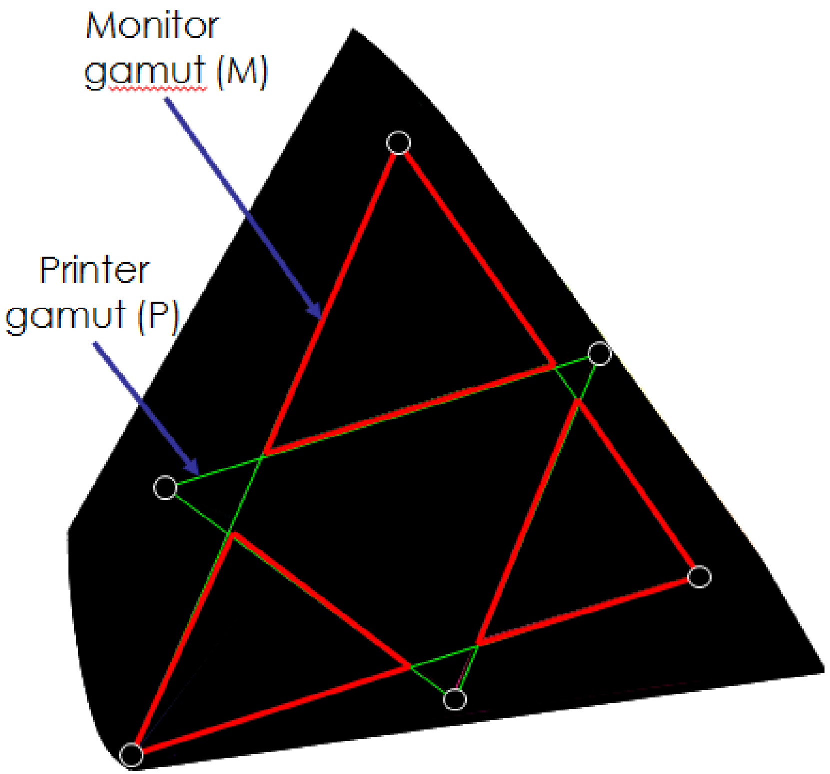
Difference symbol is a **minus sign**

Example:

- C = M – P

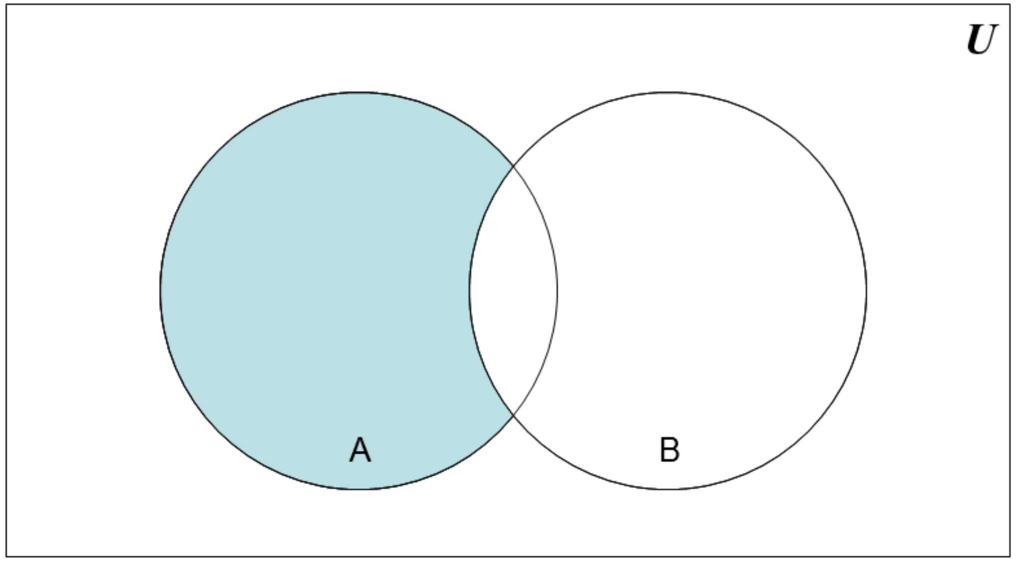
Also visa-versa:

- C = P – M

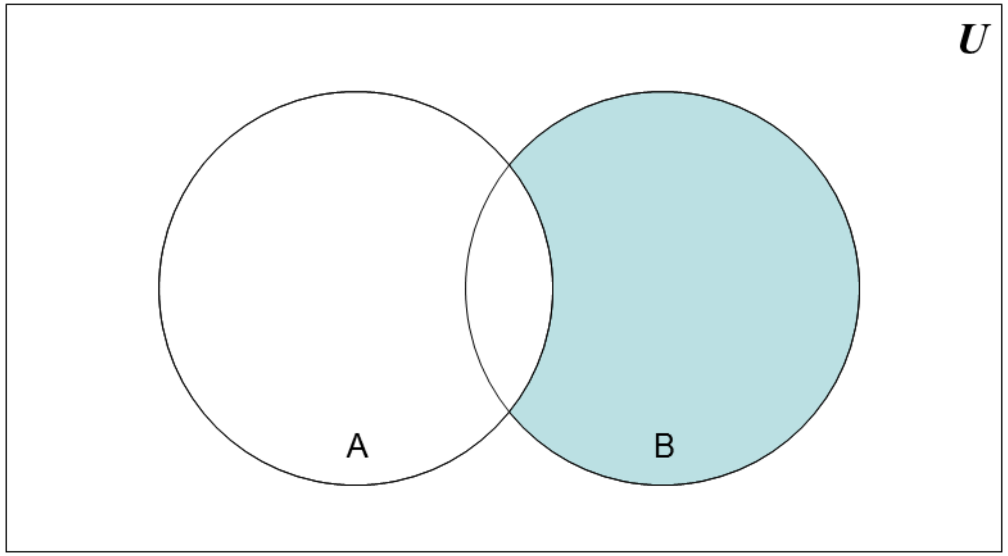


Set operations: Difference (cont.)

A - B



B - A



Set operations: Difference (cont.)

Formal definition for the difference of two sets:

$A - B = \{ x \mid x \in A$ **and** $x \notin B \}$

$A - B = A \cap \overline{B}$ ← Important!

Further examples

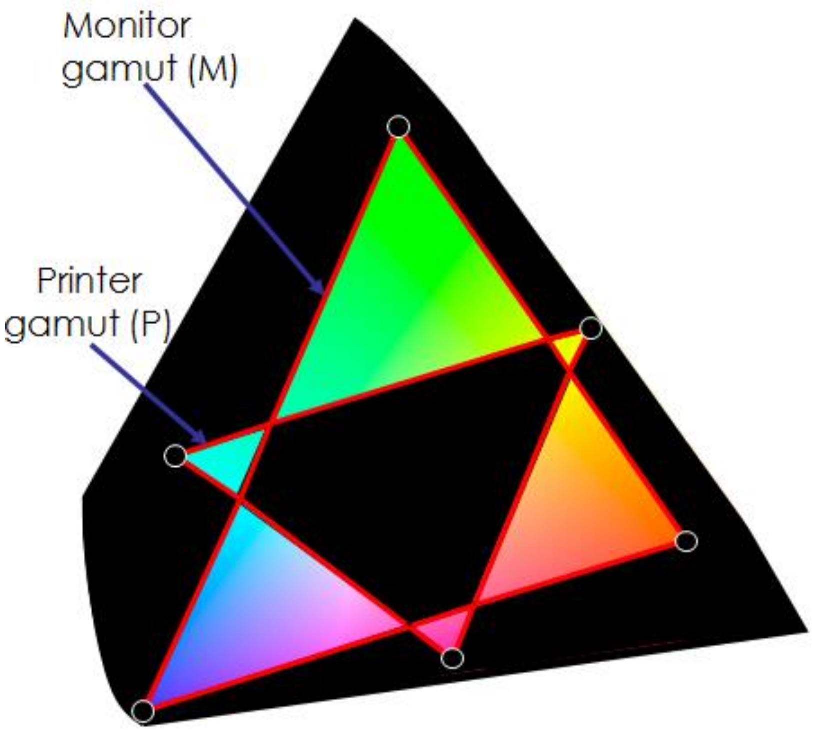
- {1, 2, 3} - {3, 4, 5} = {1, 2}
- {New York, Washington} - {3, 4} = {New York, Washington}
- {1, 2} - ∅ = {1, 2}
  - The difference of any set S with the empty set will be the set S

Set operations: Symmetric Difference

A symmetric difference of the sets contains all the elements in either set but **NOT both**

Symmetric diff. symbol is a ⊕

Example: C = M ⊕ P



Set operations: Symmetric Difference

Formal definition for the symmetric difference of two sets:

$A \oplus B = \{ x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$

$A \oplus B = (A \cup B) - (A \cap B)$ ← Important!

Further examples

- $\{1, 2, 3\} \oplus \{3, 4, 5\} = \{1, 2, 4, 5\}$
- $\{\text{New York, Washington}\} \oplus \{3, 4\} = \{\text{New York, Washington}, 3, 4\}$
- $\{1, 2\} \oplus \varnothing = \{1, 2\}$
  - The symmetric difference of any set S with the empty set will be the set S

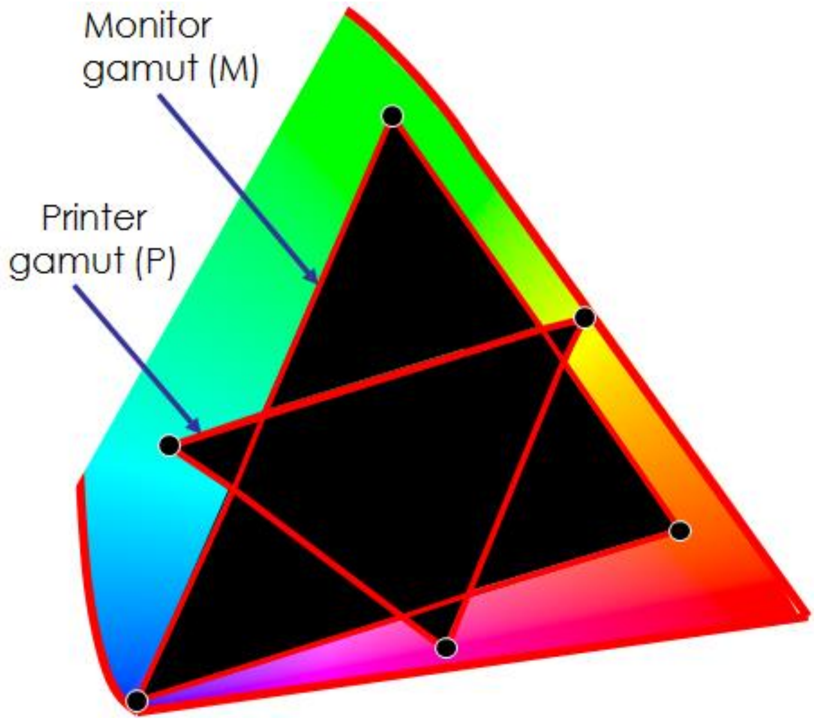
Complement sets

A complement of a set is all the elements that are NOT in the set

Complement symbol is a bar above the set name: $\overline{P}$ or $\overline{M}$

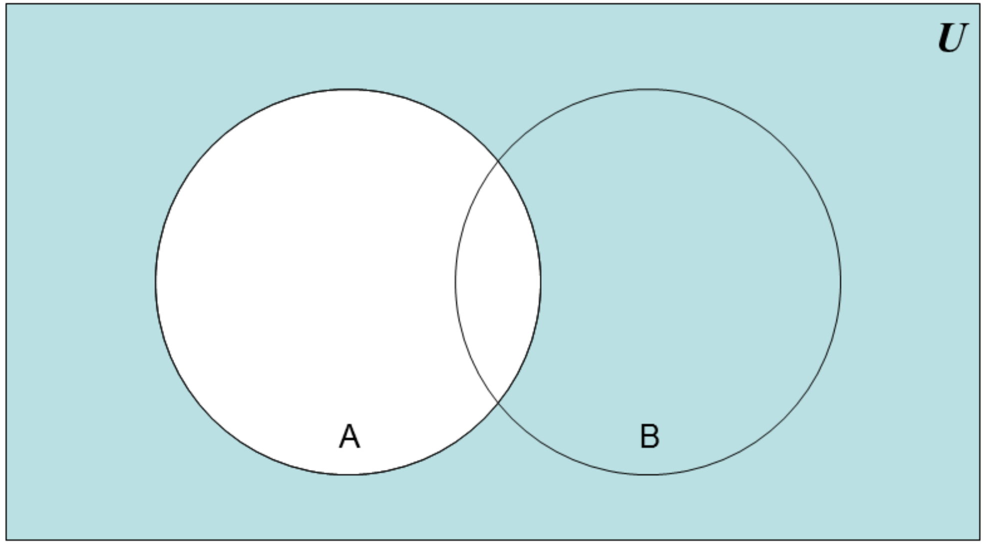
Alternative symbol:

- P^C or M^C

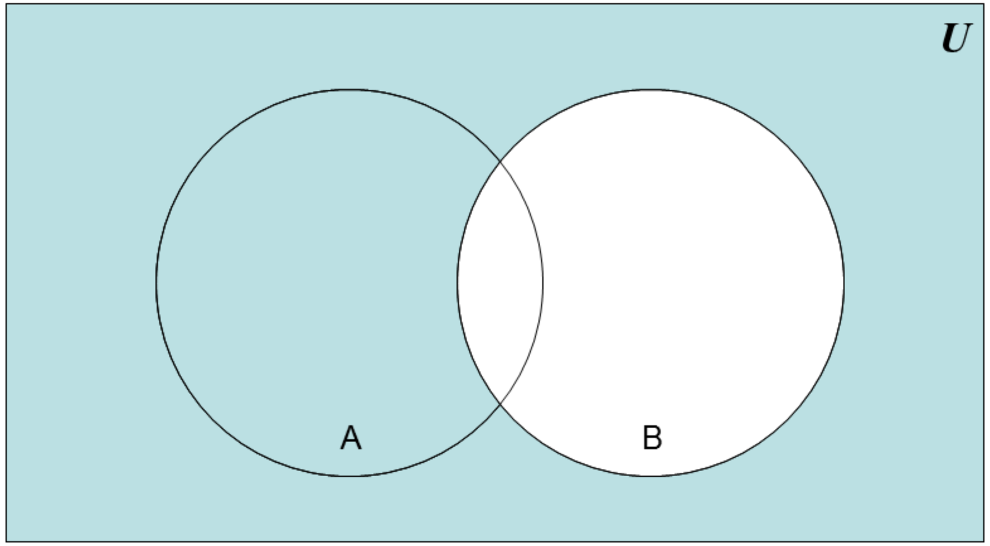


Complement sets (cont.)

$\overline{A}$



$\overline{B}$



Complement sets (cont.)

Formal definition for the complement of a set: $\overline{A} = \{\, x \mid x \notin A \,\} = A^c$

- Or $\boldsymbol{U} - A$, where $\boldsymbol{U}$ is the universal set

Further examples (assuming $\boldsymbol{U} = \mathbf{Z}$)

- $\overline{\{1, 2, 3\}} = \{\, \dots, -2, -1, 0, 4, 5, 6, \dots \,\}$

Properties of complement sets

- $\overline{\overline{A}} = A$ Complementation law
- $A \cup \overline{A} = \boldsymbol{U}$ Complement law
- $A \cap \overline{A} = \varnothing$ Complement law

Set identities

Set identities are basic laws on how set operations work

- Many have already been introduced on previous slides

Just like logical equivalences!

- Replace $\cup$ with $\vee$
- Replace $\cap$ with $\wedge$
- Replace $\varnothing$ with F
- Replace $\boldsymbol{U}$ with T

Recap of set identities

$A \cup \varnothing = A$ $A \cap U = A$	Identity Law	$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination law
$A \cup A = A$ $A \cap A = A$	Idempotent Law	$(A^c)^c = A$	Complement Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Law	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	De Morgan’s Law
$A \cup (B \cup C)$ $= (A \cup B) \cup C$ $A \cap (B \cap C)$ $= (A \cap B) \cap C$	Associative Law	$A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) =$ $(A \cup B) \cap (A \cup C)$	Distributive Law
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Law	$A \cup A^c = U$ $A \cap A^c = \varnothing$	Complement Law

How to prove a set identity?

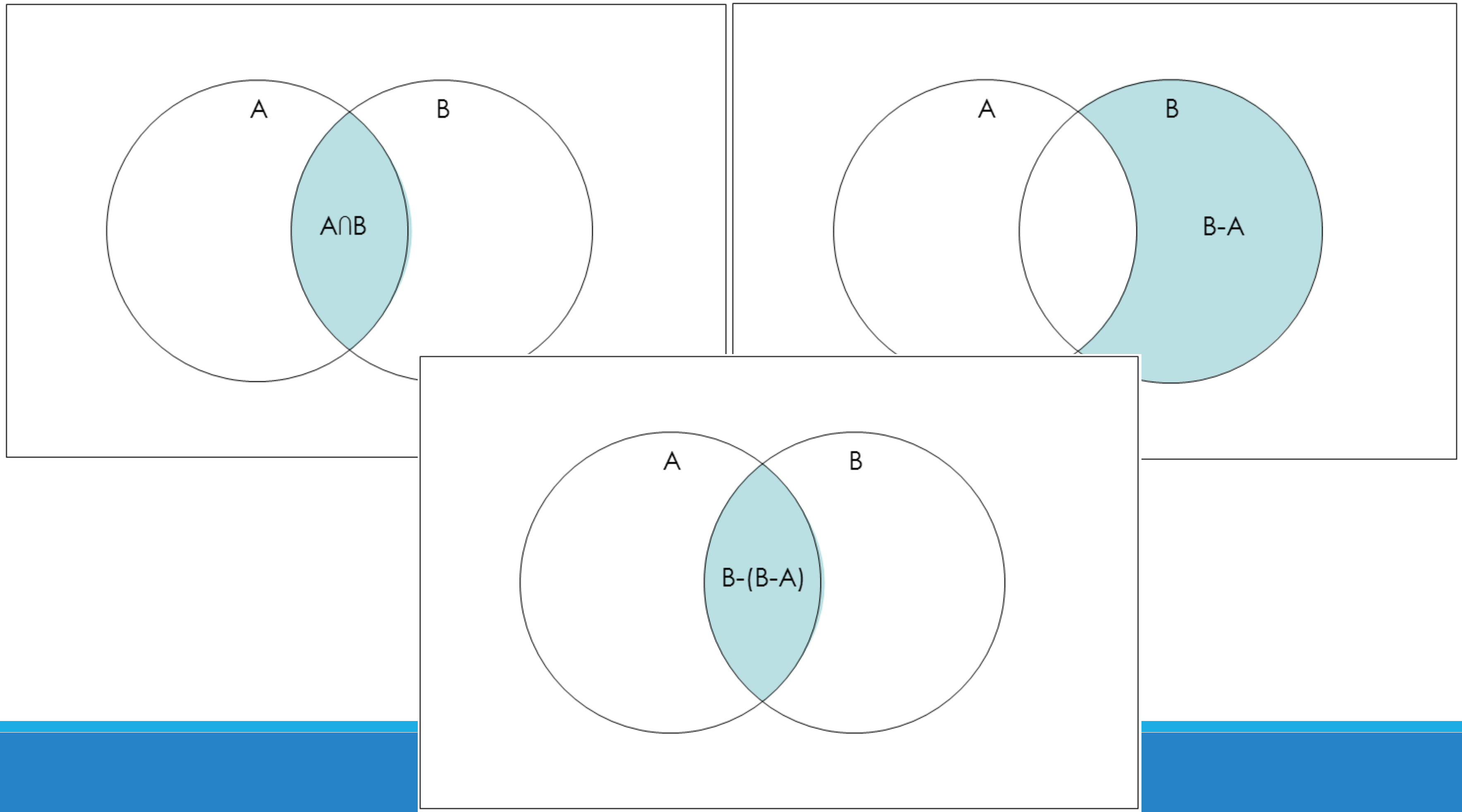
For example: $A \cap B = B - (B - A)$

There are four methods to prove:

- Use the basic set identities
- Use membership tables
- Prove each set is a subset of each other
  - This is like proving that two numbers are equal by showing that each is less than or equal to the other
- Use set builder notation and logical equivalences

What we are going to prove?

$$A \cap B = B - (B - A)$$



Proof by Set Identities

Prove that A ∩ B = B - (B - A)

$A \cap B = B\text{-}(B \cap \overline{A})$	Definition of difference
$= B \cap \overline{(B \cap \overline{A})}$	Definition of difference
$= B \cap (\overline{B} \cup \overline{\overline{A}})$	DeMorgan’s law
$= B \cap (\overline{B} \cup A)$	Complementation law
$= (B \cap \overline{B}) \cup (B \cap A)$	Distributive law
$= \varnothing \cup (B \cap A)$	Complement law
$= (B \cap A)$	Identity law
$= A \cap B$	Commutative law

What is a membership table?

- Membership tables show all the combinations of sets an element can belong to
  - 1 means the element belongs, 0 means it does not
- Consider the following membership table:

A	B	A U B	A ∩ B	A - B
1	1	1	1	0
1	0	1	0	1
0	1	1	0	0
0	0	0	0	0

- The bottom row is all elements that belong to neither set A or set B
  - Thus, these elements are neither the union, the intersection, nor difference

Proof by membership tables

The following membership table shows that $A \cap B = B - (B - A)$

A	B	$A \cap B$	B-A	B-(B-A)
1	1	1	0	1
1	0	0	0	0
0	1	0	1	0
0	0	0	0	0

Because the two indicated columns have the same values, the two expressions are identical

This is similar to Propositional logic!

Proof by showing each set is a subset of the other

Assume that an element is a member of one of the identities
- Then show it is a member of the other

Repeat for the other identity

We are trying to show:
- $(x \in A \cap B \rightarrow x \in B-(B-A)) \wedge (x \in B-(B-A) \rightarrow x \in A \cap B)$
- This is the biconditional:
- $x \in A \cap B \leftrightarrow x \in B-(B-A)$

Not good for long proofs

Proof by showing each set is a subset of the other

Assume that $x \in B-(B-A)$

- By definition of difference, we know that $x \in B$ and $x \notin B-A$

Consider $x \notin B-A$

- If $x \in B-A$, then (by definition of difference) $x \in B$ and $x \notin A$
- Since $x \notin B-A$, then only one of the inverses has to be true (DeMorgan's law): $x \notin B$ or $x \in A$

So we have that $x \in B$ and ($x \notin B$ or $x \in A$)

- It cannot be the case where $x \in B$ and $x \notin B$
- Thus, $x \in B$ and $x \in A$
- This is the definition of intersection

Thus, if $x \in B-(B-A)$ then $x \in A \cap B$

Proof by showing each set is a subset of the other

Assume that $x \in A \cap B$

- By definition of intersection, $x \in A$ and $x \in B$

Thus, we know that $x \notin B-A$

- B-A includes all the elements in B that are also not in A not include any of the elements of A (by definition of difference)

Consider B-(B-A)

- We know that $x \notin B-A$
- We also know that if $x \in A \cap B$ then $x \in B$ (by definition of intersection)
- Thus, if $x \in B$ and $x \notin B-A$, we can restate that (using the definition of difference) as $x \in B-(B-A)$

Thus, if $x \in A \cap B$ then $x \in B-(B-A)$

Proof by set builder notation and logical equivalences

First, translate both sides of the set identity into set builder notation

Then modify one side to make it identical to the other

- Do this using logical equivalences

Proof by set builder notation and logical equivalences

$B-(B-A)$	Original statement
$=\{x \mid x\in B \wedge x \notin (B-A)\}$	Definition of difference
$=\{x \mid x\in B \wedge \neg(x \in (B-A))\}$	Negating “element of”
$=\{x \mid x\in B \wedge \neg(x \in B \wedge x \notin A)\}$	Definition of difference
$=\{x \mid x\in B \wedge (x \notin B \vee x \in A)\}$	DeMorgan’s Law
$=\{x \mid (x\in B \wedge x \notin B) \vee (x \in B \wedge x \in A)\}$	Distributive Law
$=\{x \mid (x\in B \wedge \neg(x \in B)) \vee (x \in B \wedge x \in A)\}$	Negating “element of”
$=\{x \mid F \vee (x \in B \wedge x \in A)\}$	Negation Law
$=\{x \mid x\in B \wedge x \in A\}$	Identity Law
$=A \cap B$	Definition of intersection

Example

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution:

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Exercise

1. For each of the following sets, determine whether 2 is an element of that set.
   a) {x ∈ R | x is an integer greater than 1}
   b) {x ∈ R | x is the square of an integer}
   c) {2,{2}}
   d) {{2},{2,{2}}}
2. If a set has *n* elements, what is the cardinality of its power set?
3. What can you say about the sets A and B if A ⊕ B = A?
4. Let A, B, and C be sets. Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$

   (Specify the law you used in every steps).

Answers

1. a) Yes b) No c) Yes d) No
2. 2^n elements
3. B =∅
4. $$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

Exercise

1. What is the cardinality of :
   a) {{a,a}}
   b) {a, {a}}
   c) {a, {a}, {a, {a}} }
2. If a set has *n* elements, what is the cardinality of its power set?
3. What can you say about the sets A and B if A $\oplus$ B = A?
4. Let A, B, and C be sets. Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$

   (Specify the law you used in every steps).

Answers

1. The cardinality:
   a) 1
   b) 2
   c) 3
2. 2^n elements
3. B =$\emptyset$
4. $$\begin{aligned}\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{by the first De Morgan law}\\ &= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law}\\ &= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections}\\ &= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}\end{aligned}$$