Sea $W=\{(x,x-y+z,y+2z,z)/\ x,y,z\in\mathbb{R}\}$.

a) Encuentre una base para W.

$$(x,x-y+z,y+2z,z)=x(1,1,0,0)+y(0,-1,1,0)+z(0,1,2,1)$$

$\Rightarrow B=\{(1,1,0,0),(0,-1,1,0),(0,1,2,1)\}$

b) Encuentre una base ortonormal para W.

Tenemos que aplicar el proceso de G-S, por ser una base ortonormal.

$$v_1=\frac{(1,1,0,0)}{\|(1,1,0,0)\|}=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right) \qquad S_1=c\ell\{v_1\}$$

$$v_2=\frac{u_2-proy_{S_1}^{u_2}}{\left\|u_2-proy_{S_1}^{u_2}\right\|}$$

$$proy_{S_1}^{u_2}=(u_2\bullet v_1)v_1=\left[(0,-1,1,0)\bullet\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)\right]\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$$

$$=\frac{-1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)=\left(\frac{-1}{2},\frac{-1}{2},0,0\right)$$

$$u_2-proy_{S_1}^{u_2}=(0,-1,1,0)-\left(\frac{-1}{2},\frac{-1}{2},0,0\right)=\left(\frac{1}{2},\frac{-1}{2},1,0\right)$$

$$v_2=\frac{\left(\frac{1}{2},\frac{-1}{2},1,0\right)}{\left\|\left(\frac{1}{2},\frac{-1}{2},1,0\right)\right\|}=\frac{\left(\frac{1}{2},\frac{-1}{2},1,0\right)}{\frac{\sqrt{6}}{2}}=\left(\frac{1}{\sqrt{6}},\frac{-1}{\sqrt{6}},\frac{2}{\sqrt{6}},0\right) \qquad S_2=c\ell\{v_1,v_2\}$$

$$v_3 = \frac{u_3 - proy_{S_2}^{u_3}}{\left\| u_3 - proy_{S_2}^{u_3} \right\|}$$

$$proy_{S_2}^{u_3} = (u_3 \bullet v_1)v_1 + (u_3 \bullet v_2)v_2$$

$$= \left[(0,1,2,1) \bullet \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)\right]\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) + \left[(0,1,2,1) \bullet \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right)\right]\left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) + \frac{3}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right) + \left(\frac{1}{2}, \frac{-1}{2}, 1, 0\right)$$

$$= (1,0,1,0)$$

$$u_3 - proy_{S_2}^{u_3} = (0,1,2,1) - (1,0,1,0) = (-1,1,1,1)$$

$$v_3 = \frac{(-1,1,1,1)}{\left\|(-1,1,1,1)\right\|} = \left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \qquad \Rightarrow B_{ortonormal} = \{v_1, v_2, v_3\}$$

c) Determine el complemento ortogonal de W $\left(i.e.\ W^{\perp}\right)$.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow[-f_2]{\substack{f_2+f_1 \\ f_2+f_3}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \xrightarrow{\frac{1}{3}f_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix} \xrightarrow[f_3+f_2]{-f_3+f_1} \begin{pmatrix} 1 & 0 & 0 & \frac{-1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix}$$

$$x - \frac{w}{3} = 0 \quad y + \frac{w}{3} = 0 \quad z + \frac{w}{3} = 0 \qquad \Rightarrow x = \frac{w}{3} \quad y = -\frac{w}{3} \quad z = -\frac{w}{3}$$

$$(x, y, z, w) = \left(\frac{w}{3}, -\frac{w}{3}, -\frac{w}{3}, w\right) = \frac{w}{3}(1,-1,-1,3)$$

$$\Rightarrow W^{\perp} = c\ell\{(1,-1,-1,3)\}$$