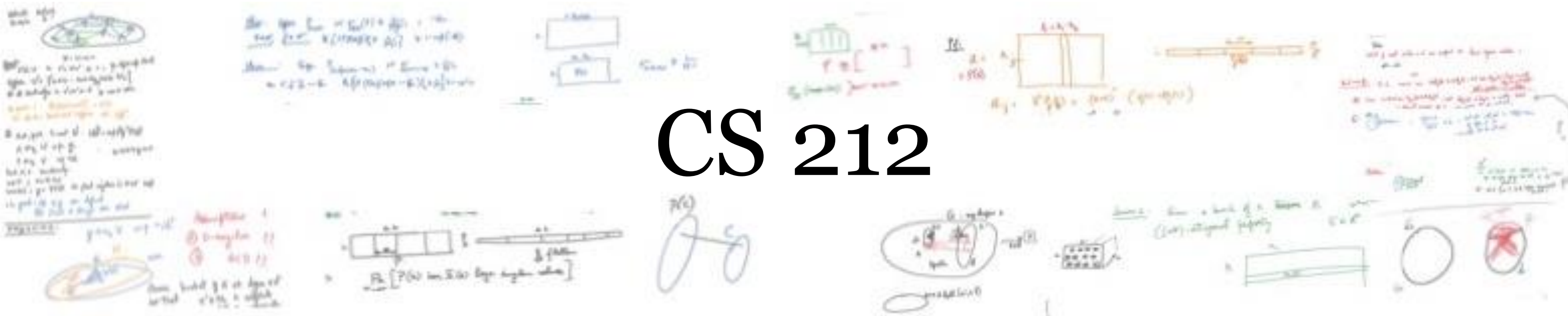


CS 212

Mathematical Foundations of Computer Science

Lecture 6: Set Theory, Relations, and functions

Announcements

- Homework 1 is due tomorrow (Oct. 4) at 11:59 P.M.

- Question 3: Prove that for any real number $x \geq 0$ and any integer $n > 0$ one has
$$(1+3x)^{2n} > \frac{2n}{2n+1} + 6nx.$$

- Homework 2: Will be posted tomorrow. Is due Tuesday Oct. 11 at 11:59 P.M.

Review of Contradiction

Proof by Contradiction

Proposition. If R, then Q.

Proof: Assume R is true but Q is false. Arrive at a contradiction.
I.e. show R true but Q false is not possible. Conclude R $\Longrightarrow$ Q.

Thm: If $P()$ is a predicate on $\mathbb{N}$ and (a) $P(0)$ is true and (b) $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{N}$, then for all k $\in \mathbb{N}$, predicate $P(k)$ is true.

- Here R is the statement: $P()$ is a predicate on $\mathbb{N}$ and (a) $P(0)$ is true and (b) $P(k) \Rightarrow P(k+1)$
- Q is the statement: for all k $\in \mathbb{N}$, predicate $P(k)$ is true.
- Not Q is: There exists an $m \in \mathbb{N}$ such that $P(m)$ is not true.

A Quick Review of Sets
(Read Chapter 2.6 in LLM)

Basic Set Theory [Cantor]

A **set** is an unordered collection of objects, where each *element* of the set is considered to included only once.

Note $\{1,2,2,3,3\}$ is not a set. (Multiset)

Roster Notation

$$\{o, s, c, a, r\} = \{r, a, s, c, o\}$$

$$\{\text{NY, ME, MA, NH, VT, PA}\}$$

Set Builder Notation

{things that are in the set | restrictions that elems satisfy}

s.t

square

$$\{x^2 \mid \underbrace{x \in \mathbb{Z}}_{x \text{ is an integer}}\} = \{\text{set of squares of integers}\}$$

$$\{S \mid \underbrace{S \subseteq \mathbb{Z}}_{S \text{ is a subset of natural numbers}} \wedge \underbrace{0 \notin S}_{0 \text{ is not in } S}\} = \{\{1\}, \{3,5\}, \{-400\}, \{\text{odd numbers}\}\}$$

and

Common Logical Symbols

$\forall$: for all	$\forall$: \forall
$\exists$: there exists	$\exists$: \exists
$\in$: element of / belongs to	$\in$: \in
$\notin$: not an element of	$\notin$: \notin
$\subseteq$: subset of	$\subseteq$: \subseteq
$\wedge$: and	$\wedge$: \wedge
$\vee$: or	$\vee$: \vee
$\vert$ or s.t. : such that	

An example

replace a with 2b for a cleaner presentation

$\{a \mid \exists\, b \in \mathbb{Z} \text{ s.t. } a = 2b\}$

{a s.t. theres exists an integer b such that a=2b}

$= \{2, 6, \ldots\} = \{\text{even integers}\}$

$2\cdot 1 \quad 3\cdot 2$

$= \{2b \mid b \in \mathbb{Z}\}$

$\{a \mid a < 0 \;\wedge\; \exists\, b \in \mathbb{R} \text{ s.t. } a = b^2\}$

= {a st. a is negative and $a=b^2$ for some $b \in \mathbb{R}$}

$= \{b^2 \mid b^2 < 0 \text{ and } b \in \mathbb{R}\} = \{\,\} = \emptyset$

$\neq \{-b^2 \mid b \in \mathbb{R}\}$

$S = \{1,2,3\}$ $P(S) = \{\{1\},\{2\},\{3\},\{1,2\},$
$\{2,3\},\{1,3\},\{1,2,3\},\phi\}$

Common Sets

Empty Set: $\emptyset = \{\}$

Power set: $\mathcal{P}(S) = \{A \mid A \subseteq S\}$

Cartesian product: $S_1 \times S_2 = \{(u,v) \mid u \in S_1, v \in S_2\}$ $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

Natural Numbers: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Rational numbers: $\mathbb{Q} = \{\ldots, 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, -\frac{447}{449}, \ldots\}$

Real Numbers: $\mathbb{R} = \{\ldots, 0, 1, \frac{1}{2}, -\frac{447}{449}, \sqrt{2}, \pi, e^{\frac{1+\sqrt{5}}{2}} - 20, \ldots\}$

Counting sets: $[n] = \{1, 2, 3, \ldots, n\}$

Open range: $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

Closed range: $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$

Common Set Operations

Term	Description	In symbols
Empty set	The set with nothing in it	$\emptyset$
Universe	The set with everything in it	U
Containment	x is an element of S	$x \in S$
Non-containment	x is not an element of S	$x \notin S$
Union	The set of objects in S_1 or S_2	$S_1 \cup S_2$
Intersection	The set of objects in S_1 and S_2	$S_1 \cap S_2$
Set difference	The set of objects in S_1 but not S_2	$S_1 \setminus S_2$
Complement	The set of objects not in S	S^c $U \setminus S$
Subset	All of S_1 is contained inside S_2	$S_1 \subseteq S_2$
Superset	S_1 contains all of the elements of S_2	$S_1 \supseteq S_2$

Some simple laws for union and intersection

Commutativity:

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Associativity:

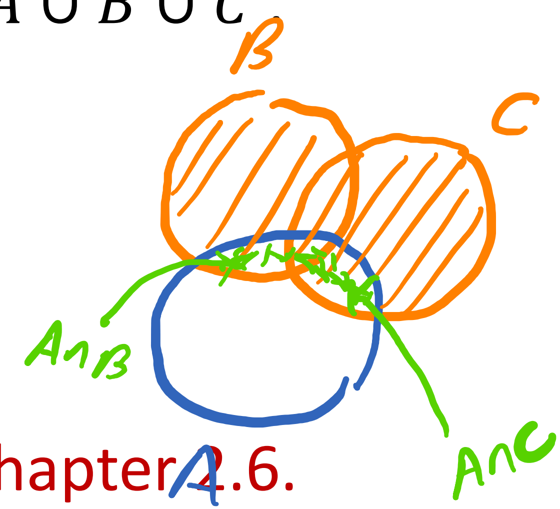
$A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$

Hence you can skip the parentheses in $A \cup B \cup C$.

Distributivity:

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



See also laws for set complement in Chapter 2.6.

$A \setminus B$ is those elements of A which are not in B

Cardinality/ Size of sets

Prove using $P(\mathbb{N})$

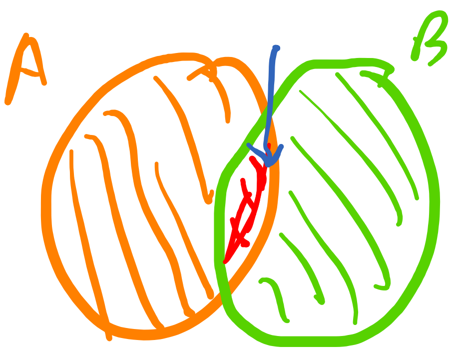
Fun fact $|\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}|$

Finite sets: number of elements in set (more subtle if infinite)

For set S, denoted by |S|

Size of the union

$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$A \cup B = A \setminus B \cup B \setminus A \cup A \cap B$$

$$|A \cup B| = |A \setminus B| + |B \setminus A| + |A \cap B|$$

$$= |A| - |A \cap B| + |B| - |B \cap A| + |A \cap B|$$

Proof Templates for Set Properties

How do I prove that a set is empty / universal?

$$S = \emptyset \text{ iff } \forall x \in U, x \notin S \qquad S = U \text{ iff } \forall x \in U, x \in S$$

How do I prove that one set is a subset of another?

$$S_1 \subseteq S_2 \text{ iff } \forall x \in S_1, x \in S_2$$

How do I prove that two sets are equal? Not equal?

If $S_1 \subseteq S_2$ And $S_2 \subseteq S_1$ then $S_1 = S_2$

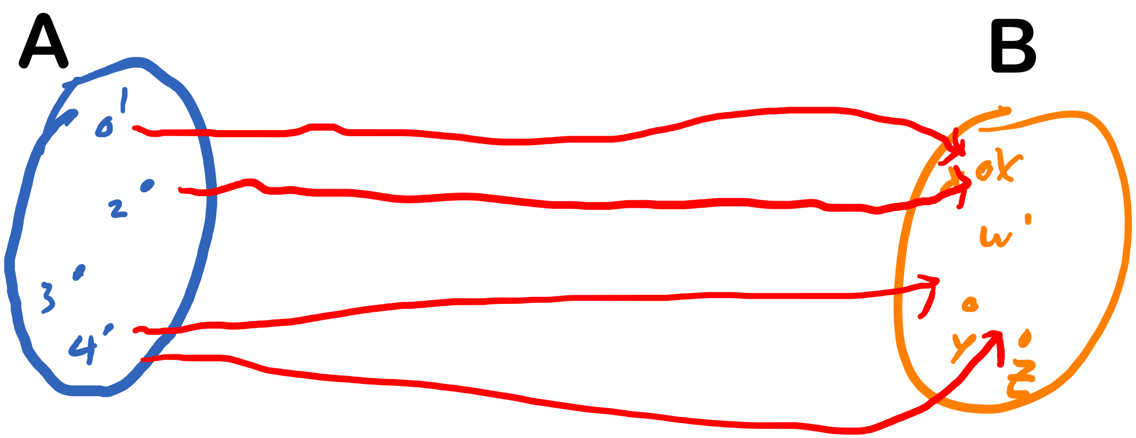
Not equal: Show either S_2 not a subset of S_1
or S_1 not a subset of S_2

Relations

Binary Relations

Binary relation $R\colon A \to B$ associates elements of A with B

xRy or $x \sim_R y$ or $(x, y) \in R$ means x related to y



$\text{Image}(\{1\}) = \{x\}$

$\text{Image}(\{4\}) = \{y, z\}$

$\text{Preimage}(\{a\}) = \{1, 2\}$

$\text{Preimage}(\{w\}) = \emptyset$

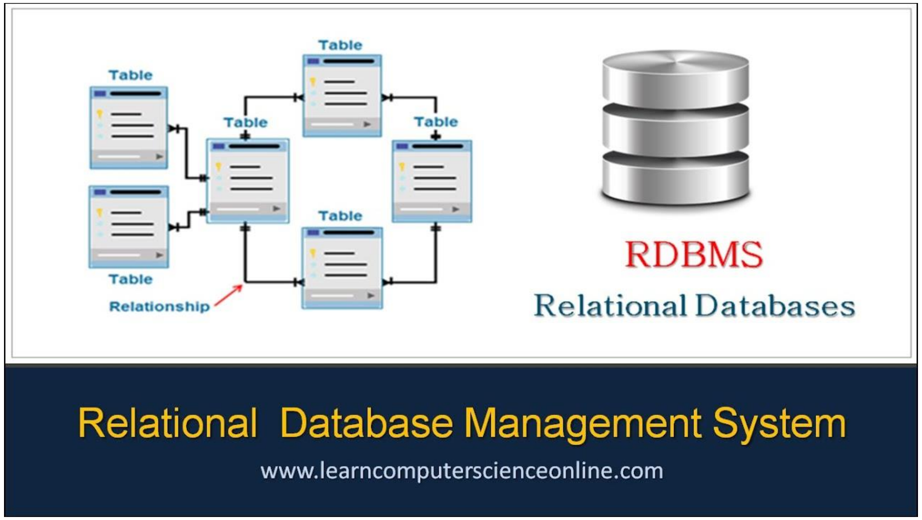
Domain=A

Co-domain=B

Range is the image of the relation $\{x, y, z\}$

i.e. the subset of co-domain that the domain is related to.

Relations



Captures connections and relations betweens entities.

Common Properties of Relations

Consider a relation $R: A \to A$ (relating elements within A)

Reflexivity:

not symmetric, not transit

aRa for every element a in the domain. E.g., R="has heard of"

Symmetric:

Reflexive, not transitive

$aRb \iff bRa$ E.g., R="shared the same class as"

Transitivity:

For any a, b, c, if aRb and bRc, then aRc

E.g., R="ancestor of"

not reflexive or symmetric

a=a

if a=b then b=c

Equivalence Relation

if a=b and b=c then a=c

- Any relation that is reflexive, symmetric and transitive. A very special kind of relation.

e.g., R="lives in the same city/town as" over set of people

R="leaves same remainder when divided by 2" over $\mathbb{N}$

Thm. Any equivalence relation R on domain A partitions the elements of A into disjoint classes $A_1, A_2, \ldots, A_k, \ldots$ called *equivalence classes*, in which for any two elements aRb iff they belong to the same class.