CS 212 Homework 4

Due 11:59PM on Tuesday, October 25, 2022.

Each question carries 5 points. Recall that every question here requires a proof. You can discuss the questions in groups of 3 (please list other students you discussed the problems with). Please remember that the assignment solutions need to be submitted individually and not as a group (and written/ typset with no collaboration). Solutions are only accepted in PDF format. Also please be as clear and legible as possible; any step that is ambiguous because of unclear writing will be interpreted as a mistake. See the Canvas page for instructions on how to submit.

Problem 1

There is a theft in the town of Evanston which has a population of $N = 90000$, and anyone could be the culprit with equal probability ($\frac{1}{N}$). The forensics team has a finger-print test that for a given person says there is a match with probability 0.90 if the person is guilty, and with probability 10^{-4} if the person is not guilty. The finger-print test outputs a match with a certain Harry in Evanston. Compute

$$\mathbb{P}[\text{Harry is innocent} \mid \text{Harry is tested positive }].$$

Problem 2

In a network there are n servers. k client machines are trying to connected to the network. Each machine connects to one of the n servers independently at random.

Assume that for any $0 < x < 1$, $1 - x \le e^{-x}$. Prove that when $k \ge 2\sqrt{n \ln n + 1} + 1$, the probability that each client machine gets dedicate service from one server is at most $1/n^2$.

Problem 3

Alice and Bob are playing a game with 10 rounds. Each round Alice wins with probability 1/4 and Bob wins with probability 3/4. The winner of each round is independent. A run is a maximal contiguous sequence of either Alice wins or Bob wins. For example, in the winning sequence ABBAAABBBB, there are 4 runs A, BB, AAA, BBBB. Let E_i be the event that a new run starts at the position i. In the example we just given, the events E_1,E_2, E_4, E_7 occurred while any other E_i did not occur.

Please determine if the following events are mutually independent and explain why.

(a) E_2 and E_3

(b) E_2 and E_4

Hint: To check if two events are independent, evaluate the joint probability and compare with the product of the probability.

Problem 4

A miner is mining in his mine. Everyday he tosses a fair coin to decide whether to use a pickaxe or dynamite to mine. If he uses a pickaxe, he can get gold worth 100 dollars for the day. If he decides to use dynamite, however, he will first need to buy dynamite for $1,000$ dollars. Then, with a $1/3$ chance he will get gold worth $12,000$ for that day and with a $2/3$ chance he will get nothing and end up blowing up his mine which makes him unable to mine in the future. Assume the miner always has enough money to buy dynamite.

(a) Let T denote the number of days he mines before the mine collapses. What is $\mathbb{E}[T]$?

(b) Let R denote the profit he makes from the mine until it collapses, i.e. the total worth of gold yield from the mine less the money he spent on dynamite. What is $\mathbb{E}[R]$?