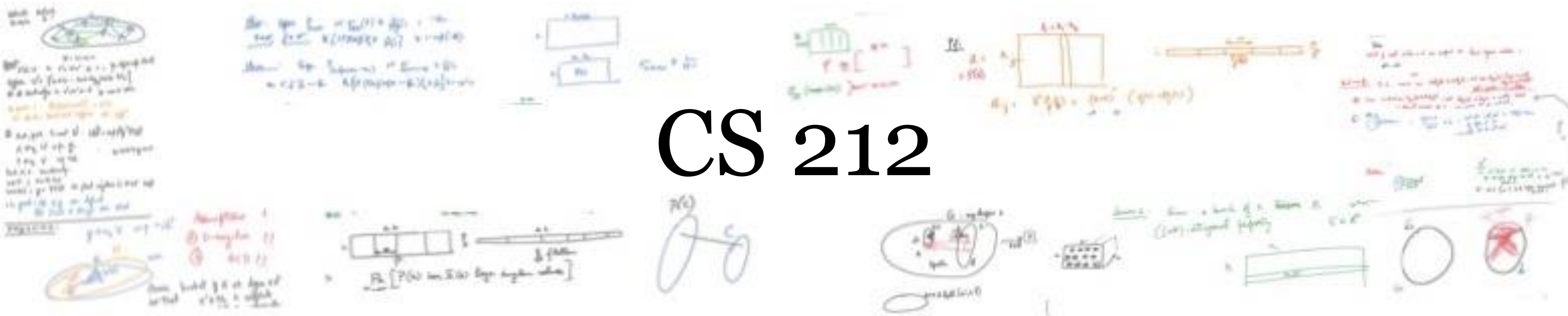


CS 212

Mathematical Foundations of Computer Science

Lecture 8: Asymptotics

Announcements

- Homework 2 is out. See Canvas Syllabus page for PS2.pdf
  To be submitted on Crowdmark by Oct. 11 (Tues) night

- After today's class on Big-Oh, Asymptotics you should be able to do all the questions.

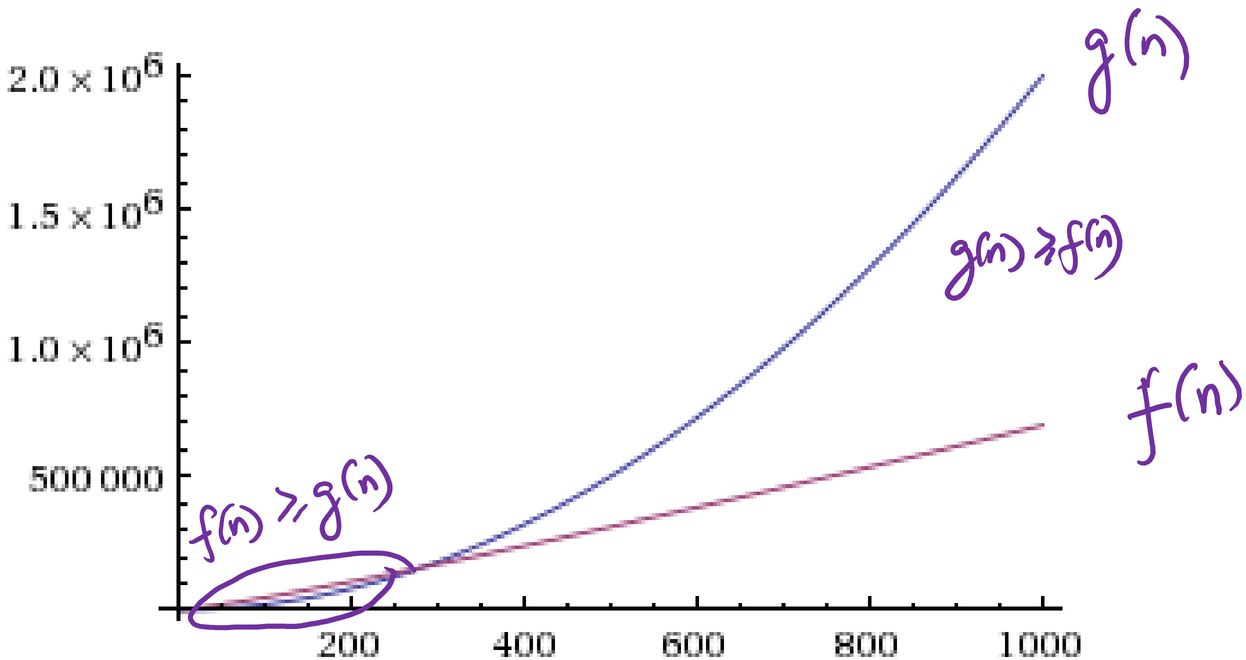
- See the Canvas Syllabus page for Discussion section notes.

- Midterm on October 26^{th} (Wed) in class.

Which is Bigger?

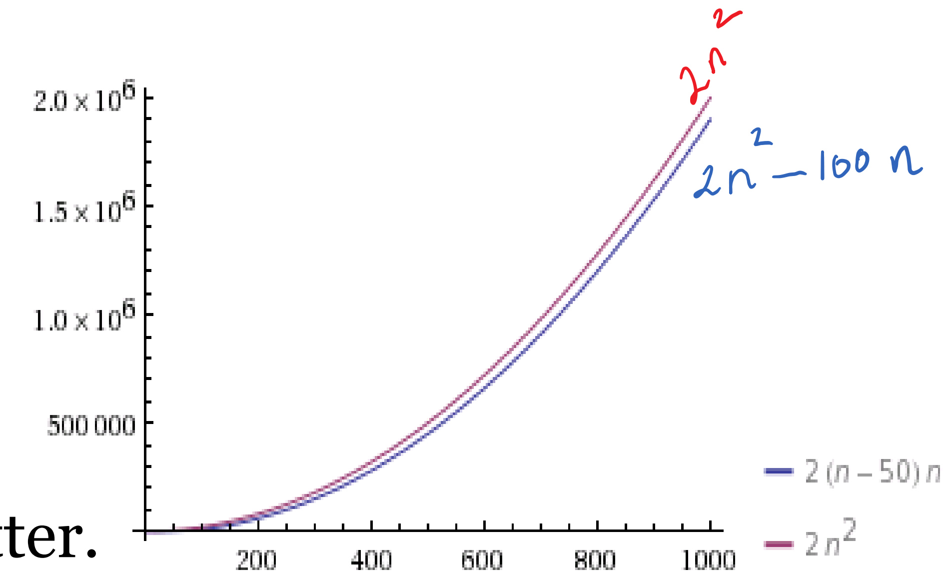
$$f(n) = 100n \log n \quad \text{or} \quad g(n) = 2n^2 - n?$$

$$n = 1,2,3, \ldots ?$$



How do we compare?

- Consider large values of n (reason about large inputs)
- Mathematically well-defined and elegant



Asymptotic Analysis:

- Gives good qualitative understanding of function
- Only higher order terms matter.

$g(n) = 2n^2 - 100.03n + \sqrt{1001} - \sqrt{1003}\, n \lg n$

Big-Oh Notation

- $g(n) = O(f(n))$: g is upper bounded on the order of f(n)

  Eg. g(n)=O(logn) : “g is Big-Oh of logn”.

$g(n) = O\big(f(n)\big)$ **iff there exists** $c \geq 0, n_0 \in \mathbb{N}$,
such that $g(n) \leq c \cdot f(n)$ for all sufficiently large $n \geq n_0$

Prop. Suppose $g(n) = 1000n + 2000000$, then $g(n) = O(n)$.

Proof. $c = 2000$, $n_0 = 2000$ $n \geq n_0 \geq 2000$ (since $n \geq n_0$)

$g(n) = 1000n + 2000 \times 1000 = 1000n + 1000 \times n_0 \leq 1000n + 1000n$

$\leq 2000n = cn$

Hence $g(n) = O(n)$ ∎

$O(\ ) \longleftrightarrow$ " $\leq$ "

$\Omega(\ ) \longleftrightarrow$ " $\geq$ "

$\Theta(\ ) \longleftrightarrow$ " $=$ "

$\exists$ constants c, n_0, $g(n) \geq c \cdot f(n)$ for all $n \geq n_0$

Big-Oh Notation continued

- $g(n) = \Omega(f(n))$: g is asymptotically lower bounded by f(n)

  $g(n) = \Omega\big(f(n)\big)$ iff $f(n) = O(g(n))$.

Big-Theta

- $g(n) = \Theta(f(n))$: f and g are asymptotically of the same order of magnitude (same upto constant factors)

  $g(n) = \Omega(f(n))$

  $g(n) = \Theta\big(f(n)\big)$ iff $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

E.g. $a, b, c \geq 0$, $g(n) = an^2 + bn + c$. Then g(n) vs n^2 ?

$g(n) = \Theta(n^2)$

$$\lim_{n\to\infty} \frac{an^2+bn+c}{n^2} = \lim_{n\to\infty}\left(a+\frac{b}{n}+\frac{c}{n^2}\right) = a$$

Relation to Limits (Calculus 101)

If $\boxed{\lim_{n\to\infty} \frac{f(n)}{g(n)}} \leq c$ for some constant $0 \leq c < \infty$, then $f(n) = O(g(n))$

E.g. $f(n) = 1000n + 2000000$. $g(n) = 2n^2$. Prove $f(n) = O(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{1000n + 2000000}{2n^2} = \lim_{n\to\infty} \frac{500}{n} + \frac{1000000}{n^2}$$

$$= \lim_{n\to\infty} \frac{500}{n} + \lim_{n\to\infty} \frac{1000000}{n^2}$$

$$= 0 + 0 = 0$$

$$f(n) = O(g(n)).$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty}\left(\frac{2n^2}{1000n+2000000}\right) = \lim_{n\to\infty}\left(\frac{n}{500 + \frac{1000000}{n}}\right) = \infty$$

$g(n) = \Omega(f(n))$

In this case $g(n) = \omega(f(n))$

little-oh

littleomega

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n) = o(g(n))$

Limits and Big-Oh

f, g

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

$f(n) = O(g(n))$

$f(n) = \Omega(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \in (0, \infty)$$

$c \in (0, \infty)$

$c = \infty$

$f(n) = o(g(n))$ "<"

$f(n) = \Theta(g(n))$ "="

$f(n) = \omega(g(n))$ ">"

Limits and Big-Oh

f, g $\quad f(n), g(n) \geq 0$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

(can include ∞)

$f(n) = O(g(n))$

Big Omega

$f(n) = \Omega(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \in (0, \infty)$$

$$c \in (0, \infty)$$

$$c = \infty$$

$f(n) = o(g(n))$

Big-Theta

$f(n) = \Theta(g(n))$

$f(n) = \omega(g(n))$

little-omega

Dealing with Logarithms

$$\mathrm{f(n)} = n \log n \quad \text{vs} \quad \mathrm{g(n)} = n^2$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\cancel{n}\log n}{n^{\cancel{2}}} = \lim_{n\to\infty} \frac{\log n}{n}$$

Hôpital rule: If $n \to \infty, f(n), g(n) \to \infty$: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$

$\frac{d(\log n)}{dn} = \frac{1}{n}$

$\frac{d}{dn}(n) = 1$

$$\lim_{n\to\infty} \frac{(\log n)}{(n)} = \lim_{n\to\infty} \frac{(1/n)}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$f(n) = o(g(n))$

- Sometimes easier to compare logarithms of functions i.e. compare $\log f(n)$, $\log g(n)$ e.g. exponentials (but be careful).

Simple Rules for Big-Oh

1. Transitivity:

$$g(n) = O(f(n)),\ h(n) = O\big(g(n)\big) \Rightarrow \ h(n) = O(f(n))$$

2. Constant factors don't matter: $c \cdot f(n) = \Theta(f(n))$

3. Smaller terms don't matter e.g. $an^2 + bn + c = \Theta(n^2)$.

4. Among polynomials, exponent is most important.

i.e. $n^a = o(n^b)$ if $a < b$.

$\lim_{n\to\infty} \frac{n^a}{n^b} = \lim_{n\to\infty} \frac{1}{n^{b-a}}$ e.g. $n^{2.5} = o(n^{2.500001})$

5. Logarithms are dominated by polynomials

i.e. $\log n = o(n^a)$ if $a > 0$

$\log n = o(n^{0.000000001})$

Limits and Big-Oh

f, g

$f(n), g(n) \geq 0$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

(can include ∞)

$f(n) = O(g(n))$

Big Omega

$f(n) = \Omega(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \in (0, \infty)$$

$c \in (0, \infty)$

$c = \infty$

Big-Theta

$f(n) = o(g(n))$

$f(n) = \Theta(g(n))$

$f(n) = \omega(g(n))$

little-omega

Taking Logarithms

How do you compare $e^{\sqrt{n}}$ and n^{100} ?

$\lim_{n\to\infty} \frac{e^{\sqrt{n}}}{n^{100}}$

Theorem. If f(n), g(n) ≥ 1 $\forall n \in \mathbb{N}$, *and* $\log f(n) = o(\log g(n))$, then $f(n) = o\big(g(n)\big)$.

*not clear for $O()$: needs to be done more carefully

$f(n) = e^{\sqrt{n}}$

$g(n) = n^{100}$

Pf:

$\log f(n) = \log(e^{\sqrt{n}}) = \sqrt{n} \log e$

$\log g(n) = \log(n^{100}) = 100 \log n$.

$\log n = o(\sqrt{n})$ by the rules in prev. slides $\Rightarrow$ $\log g(n) = o(\log f(n))$

$\Rightarrow g(n) = o(f(n))$.

Simple rules: 1. $n = 2^{\log n}$ 2. $(x^{a})^{b} = x^{ab}$

3. $n^{a} = (2^{\log n})^{a} = 2^{a \log n}$ 4. $a^{x} = (2^{\log a})^{x} = 2^{x \log a}$

Taking Logarithms (Proof)

Theorem. If f(n), g(n) ≥ 1 $\forall n \in \mathbb{N}$, *and* $\log f(n) = o(\log g(n))$, then $f(n) = o\big(g(n)\big)$.

Proof is for interested folks. You don't need to know this.

Pf. Let $p(n) = \log f(n)$, $q(n) = \log g(n)$.

Given: $p(n) = o(q(n)) \Longrightarrow \lim_{n \to \infty} \frac{p(n)}{q(n)} = 0 \Rightarrow \exists n_0 \in \mathbb{N}$ s.t for every constant $c > 0$, $p(n) \leq c \cdot q(n) \; \forall n \geq n_0$.

Pick $c = \frac{1}{2}$ $\Rightarrow p(n) \leq \frac{1}{2} q(n) \; \forall n \geq n_0$. ①

$\lim_{n \to \infty} \frac{p(n)}{q(n)} = 0$; $p(n) \geq 1 \Rightarrow \lim_{n \to \infty} q(n) = \infty$ ②

$\frac{f(n)}{g(n)} = \frac{2^{p(n)}}{2^{q(n)}} \leq 2^{-q(n)/2}$ (from ①); $\lim_{n \to \infty} \frac{f(n)}{g(n)} \leq \lim_{n \to \infty} \frac{1}{2^{q(n)/2}} = 0$ (from ②) ∎

Alternatively: Comparing exponents

How do you compare $e^{\sqrt{n}}$ and n^{100} ?

Theorem. If $\mathrm{f(n)}, \mathrm{g(n)} \geq 1 \ \forall n \in \mathbb{N}, and\ f(n) = o(g(n))$, then for any constant $a > 1$, $\mathrm{a}^{f(n)} = o\left(a^{g(n)}\right)$.

Big-Oh, Theta, Small-oh

How do we compare two non-negative functions $f(n), g(n)$? that are increasing

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$$

$f(n) = o(g(n))$

$g(n) = \omega(f(n))$

Also,

$f(n) = O(g(n))$

$g(n) = \Omega(f(n))$

$$0 < \lim_{n\to\infty}\frac{f(n)}{g(n)} < \infty$$

$f(n) = \Theta(g(n))$

$g(n) = \Theta(f(n))$

$f(n) = O(g(n))$

$f(n) = \Omega(g(n))$

$g(n) = \Omega(f(n))$

$g(n) = O(f(n))$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \infty$$

$f(n) = \omega(g(n))$

$g(n) = o(f(n))$

$g(n) = O(f(n))$

$f(n) = \Omega(g(n))$

?

$\lim_{n\to\infty}\frac{f(n)}{g(n)}$ doesn't exist!

Eg: $g(n) = n$

$f(n) = n\,|\sin(n)|$

$\lim_{n\to\infty}\frac{f(n)}{g(n)} = |\sin(n)|$