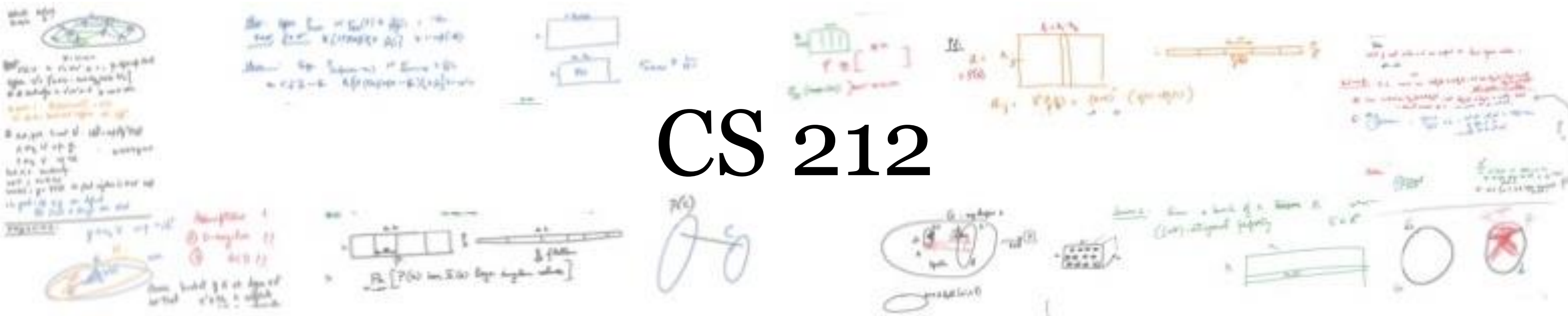


CS 212

Mathematical Foundations of Computer Science

Graphs and Linear Algebra

Announcements

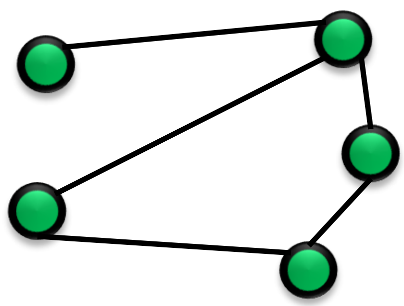
- In question 1 part (ii), assume $k \geq 2$.

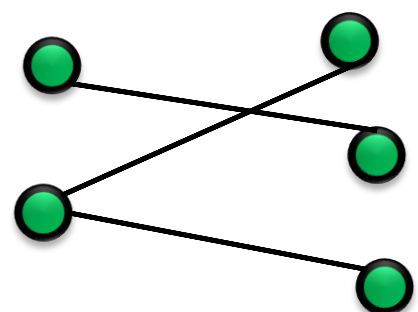
- In question 4, $\Delta =$ maximum degree of a vertex in the graph.

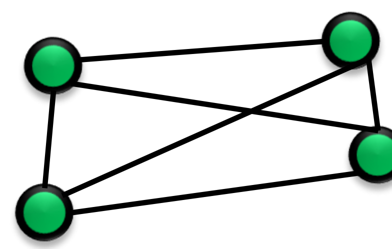
Planar Graphs

A graph is planar if it can be drawn (represented) on the plane without any crossing edges (no edges intersect).

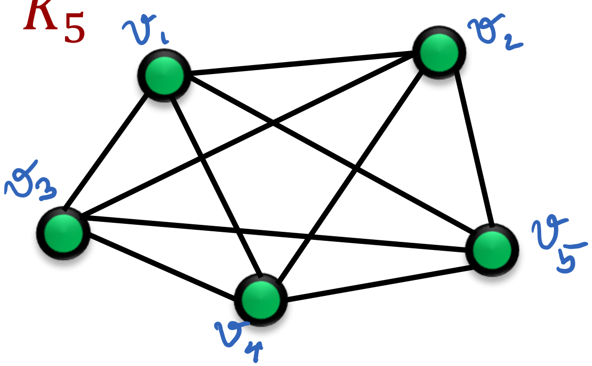
Which are planar?

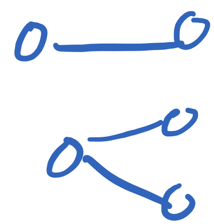


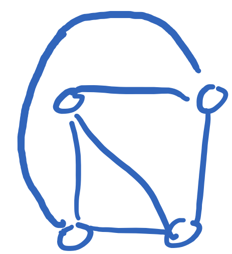




K_5





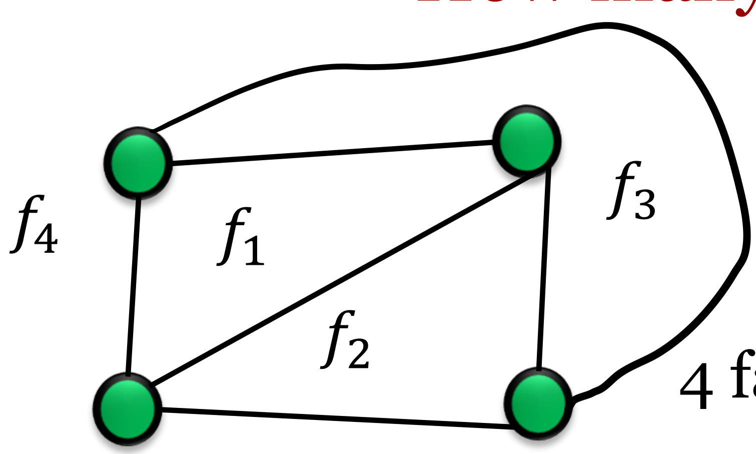


Faces

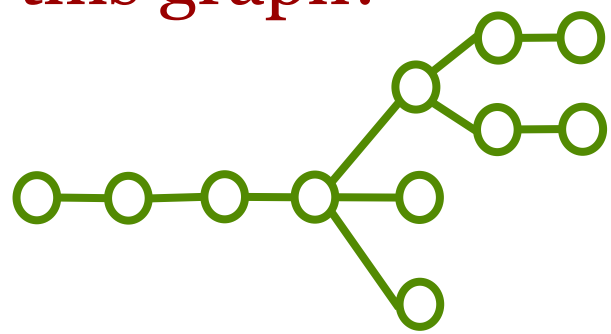
An embedding of planar graph splits the plane into disjoint faces

Face: A region bounded by a set of edges, vertices in embedding

How many faces in this graph?



4 faces



Tree has only 1 face

- When $n \geq 4$, each face borders at least 3 edges.
- One "outside" face (do not forget the outside face).

Euler's Formula

$n - 2 = m - f$

Thm. If G is a connected planar graph G with vertex set V (size n), edges E (m of them) and faces F (f of them), then

$$|V| - |E| + |F| = n - m + f = 2$$

Recall total $\binom{n}{2}$ possible edges $3n-6 = o\left(\binom{n}{2}\right)$

Average degree of a planar graph ≤ 6

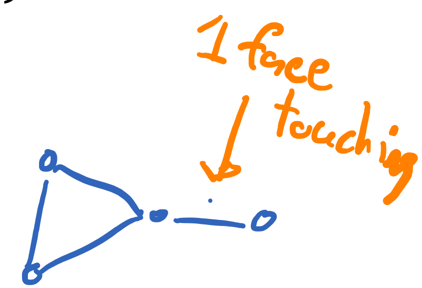
Thm. In a connected planar graph $G(V,E)$ on $n \geq 4$ vertices, the number of edges $m \leq 3n - 6$.

Proof. By Euler's formula, $n - 2 = m - f$.

Want to bound f in terms of m. Count #(edge,face) incidences

Every face has how many edges? ≥ 3

How many faces can an edge belong to? ≤ 2



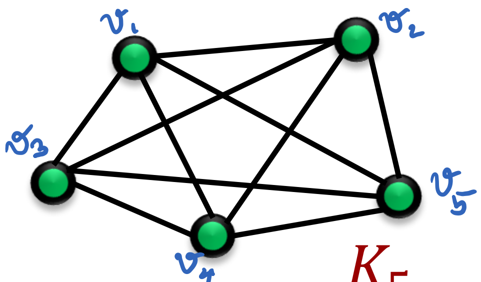
$3f \leq$ #edge-face incidences $\leq 2m \Longrightarrow$ $3f \leq 2m \Rightarrow f \leq \frac{2m}{3}$

$n-2 = m - f \geq m - \frac{2m}{3} = \frac{1}{3}m \Rightarrow 3(n-2) \geq m$

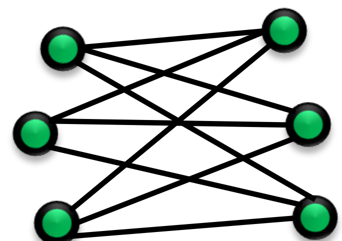
[Aside] Non planar graphs

How do you say when a graph is non-planar? If $m \overset{?}{>} 3n-6$

It clearly should not contain K_5 and $K_{3,3}$



K_5



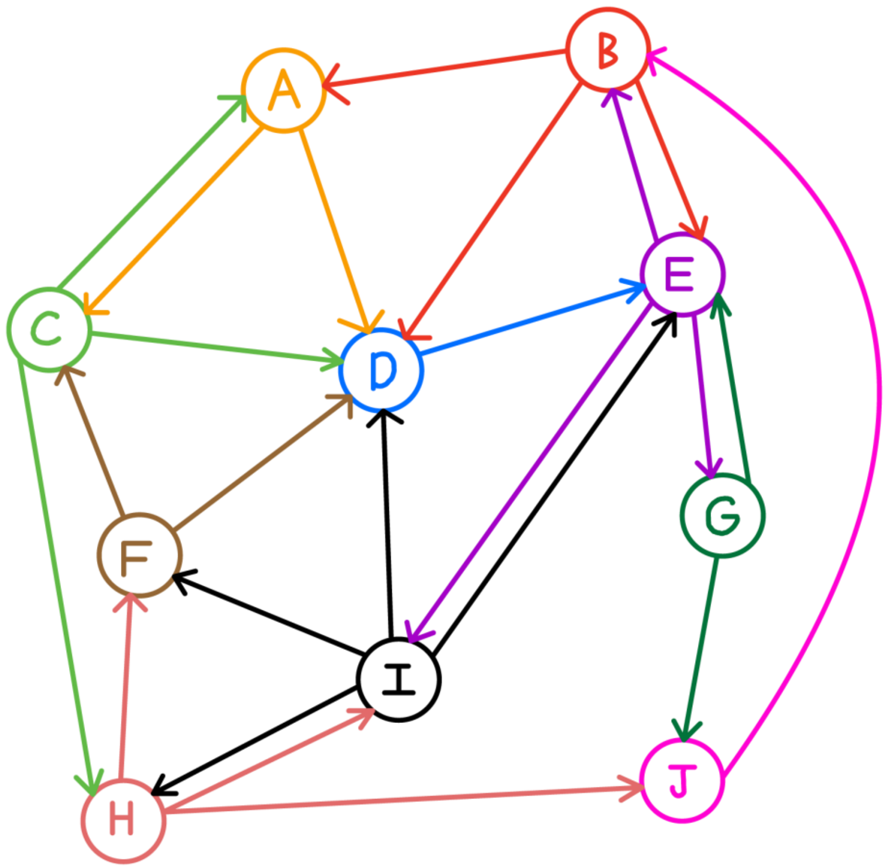
$K_{3,3}$

Thm [Wagner]. Any graph that does not "contain" K_5 and $K_{3,3}$ is a planar graph.

"Contain": graph minor.

Linear algebra and graphs

Which Vertex is most important?



Intro to Linear algebra

Vectors and Inner Products

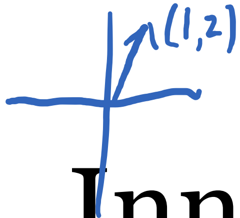
Vector $x \in \mathbb{R}^n$ usually refers to a n-dimensional column vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

each x_i is a real number

Inner product: $x, y \in \mathbb{R}^n \Rightarrow \langle x, y \rangle = \sum_{\ell=1}^{n} x_\ell y_\ell$

$$\langle x, y \rangle = x^T y = (x_1, x_2, \dots, x_n)_{1 \times n} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_{n \times 1} = \sum_{\ell=1}^{n} x_\ell y_\ell$$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$



Inner products give geometry!

Vector $x \in \mathbb{R}^n$ usually refers to a n-dimensional column vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

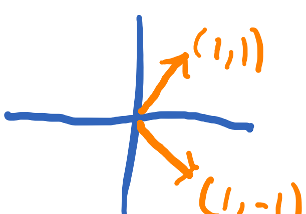
Inner product: $x, y \in \mathbb{R}^n \Rightarrow \langle x, y \rangle = \sum_{\ell=1}^{n} x_\ell y_\ell$

i.e. perpendicular

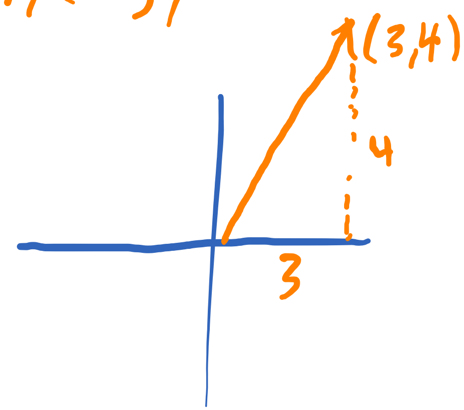
$x, y \in \mathbb{R}^n$ *are orthogonal iff* $\langle x, y \rangle = 0$

Length of $x \in \mathbb{R}^n$ *is* $\sqrt{\langle x, x \rangle}$

$\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = 1 \cdot 1 + 1 \cdot (-1)$
$= 0$



$\sqrt{\left\langle \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\rangle} = \sqrt{3 \cdot 3 + 4 \cdot 4} = \sqrt{25}$
$= 5$



Inner Products vs Rank-1 matrices

Inner product: $x, y \in \mathbb{R}^n \Rightarrow \langle x, y \rangle = \sum_{\ell=1}^{n} x_\ell y_\ell$

$$\langle x, y \rangle = x^T y = (x_1, x_2, \dots, x_n)_{1 \times n} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_{n \times 1} = \sum_{\ell=1}^{n} x_\ell y_\ell$$

$x, y \in \mathbb{R}^n$ *are orthogonal iff* $\langle x, y \rangle = 0$ $\left(\, x^T y = \langle x, y \rangle \,\right)$

i,jth entry of xy^T is given by $x_i y_j$

Outer product or Rank-1 matrix product:

$$xy^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1} (y_1, y_2, \dots, y_n)_{1 \times n} = \begin{array}{cc} & j \\ i & \begin{pmatrix} & & \\ & x_i y_j & \\ & & \end{pmatrix}_{n \times n} \end{array}$$

$2n$ entries

n^2 entries

M_{ij} is i,j the entry of M

Matrix vector multiplication

- Matrix vector multiplication: $M \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n} \Rightarrow Mx \in \mathbb{R}^{m}$

The ith entry of Mx is $\sum_{j=1}^{n} M_{ij} x_j$

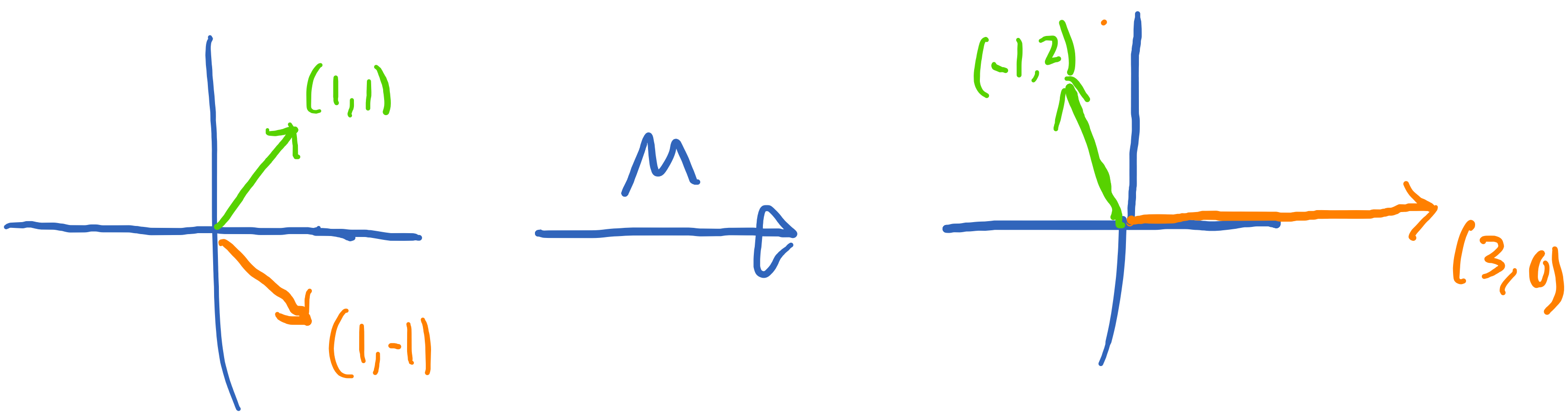
$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} m_{11}x_1 + m_{12}x_2 + m_{13}x_3 \\ m_{21}x_1 + m_{22}x_2 + m_{23}x_3 \\ m_{31}x_1 + m_{32}x_2 + m_{33}x_3 \end{pmatrix}$$

Matrix vector as linear maps

- Matrix vector multiplication: $M \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n} \Rightarrow Mx \in \mathbb{R}^{m}$

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + (-2)(1) \\ 1 \cdot 1 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + (-2)(-1) \\ 1 \cdot 1 + 1 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$



Matrices multiplication

- Matrix vector multiplication: $M \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \Rightarrow Mx \in \mathbb{R}^m$
- Matrix-matrix multiplication: $C_{m \times p} = A_{m \times n} \times B_{n \times p}$

Matrices multiplication

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$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$C_{ij} = \sum_{\ell=1}^{n} A(i,\ell) \times B(\ell,j) = \langle A_i^T, B_j \rangle$$

$$C_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$$

ith row of A
B_j is jth column of B

Matrices: A Refresher

- Matrix vector multiplication: $M \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \Rightarrow Mx \in \mathbb{R}^m$
- Matrix-matrix multiplication: $C_{n \times n} = A_{n \times n} \times B_{n \times n}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$A[B_1 \; B_2 \; B_3] = [AB_1 \; AB_2 \; AB_3]$$

B_j is jth column of B

Transpose

- **Transpose of a matrix:** Given $M \in \mathbb{R}^{m\times n}$, the transpose $M^T \in \mathbb{R}^{n\times m}$ has i, j entry equal to the j, i entry of M

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

- M is symmetric if $M = M^T$