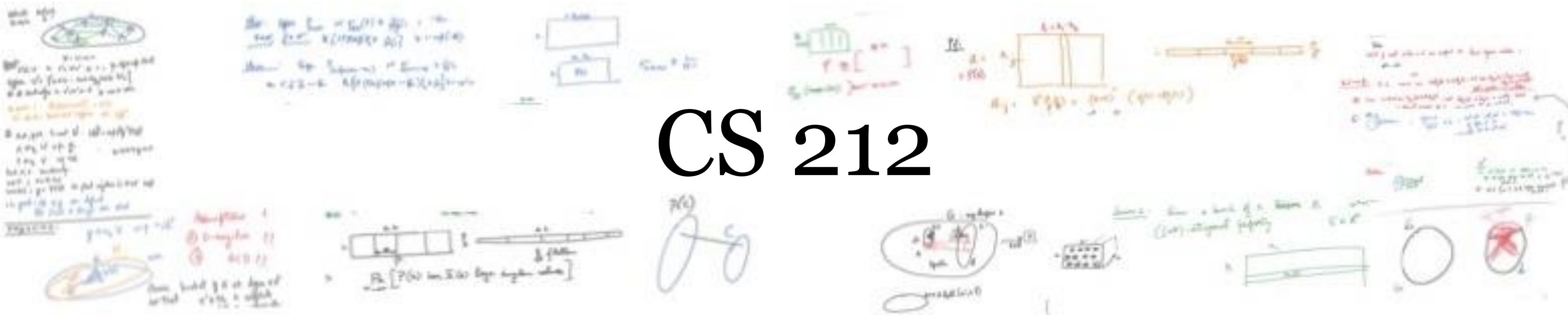


CS 212

Mathematical Foundations of Computer Science

Lecture 26: Introduction to Linear Programming

Announcements

1. Last problem set PS7 due on Tuesday before Thanksgiving.
2. There is class on Wednesday before Thanksgiving

Linear Programming

Linear Optimization

Suppose you want to buy apples and oranges.

- Apple costs $2\$/lb$ and oranges cost $1\$/lb$.
- Apples have $1mg/lb$ of vit-C, oranges have $2mg/lb$ of vit-C.
- Apples and oranges both provide $1Kcal/lb$.

How many pounds of apples & oranges can you buy with at most 3$, so that total Vitamin C intake $\leq 3\ mg$, in such a way to maximize calorie intake?

Let x_1 be the #lbs of apple bought, x_2 be #lbs of oranges bought

$$\boxed{\begin{array}{lll} \max_{x_1, x_2 \in \mathbb{R}} & x_1 + x_2 & \text{(calorie intake)} \\ & x_1 + 2x_2 \leq 3 & \text{(Vit-C constraint)} \\ & 2x_1 + x_2 \leq 3 & \text{(Budget)} \\ & x_1 \geq 0,\ x_2 \geq 0 & \text{(non-negative quantities)} \end{array}}$$

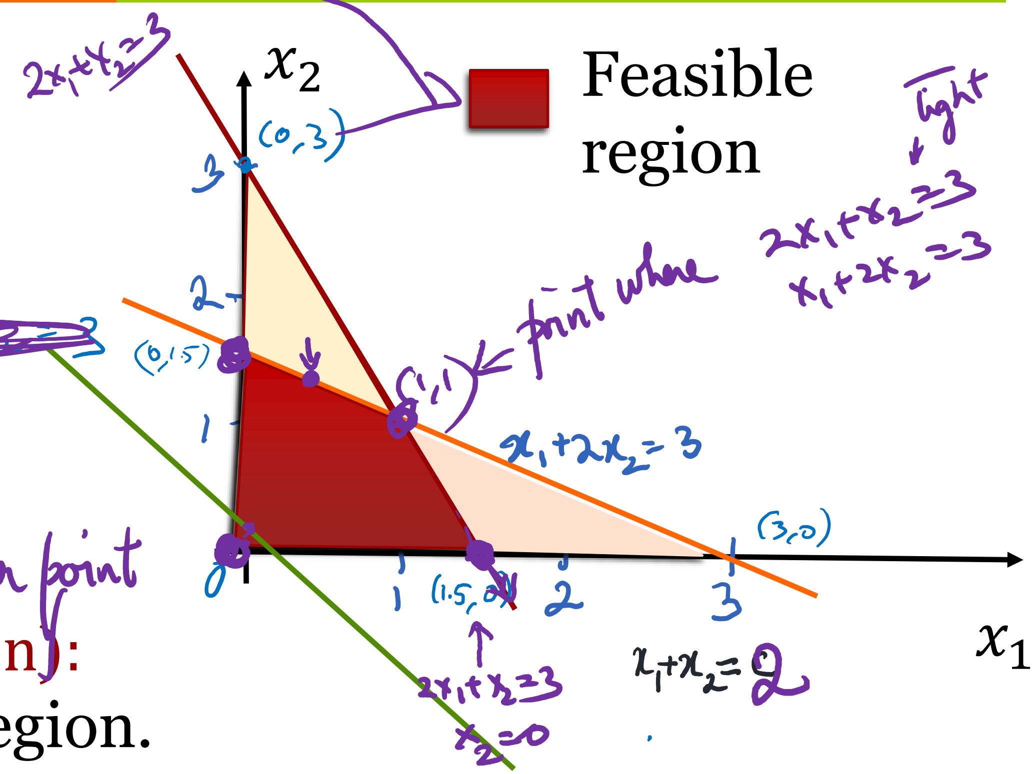
Linear Program

Optimize that…

Value of this LP = 2.

Maximize $x_1 + x_2$

$$\boxed{\begin{aligned} 2x_1 + x_2 &\le 3 \\ x_1 + 2x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}}$$



Corner point

Vertex (Basic feasible solution):
The corners of the feasible region.

What is the optimal solution? (1,1)

Linear Equalities

Linear equalities: $variables\ x_1, x_2, \ldots, x_n$

fixed vector fixed

$$a_{11}x_1 + a_{12}x_2 + \ \ldots + a_{1n}x_n = b_1 \quad i.e. \langle a_1, x\rangle = b_1$$

vars

……

$$a_{i1}x_1 + a_{i2}x_2 + \ \ldots + a_{in}x_n = b_i \qquad i.e. \langle a_i, x\rangle = b_i$$

$$a_{n1}x_1 + a_{n2}x_2 + \ \ldots + a_{nn}x_n = b_n$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$(A\,x)_{n\times 1} = (b)_{n\times 1}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

How do you solve Linear Equations?

How to Solve Linear Equations

How do you solve Linear Equations?

$$A\,x = b$$

$$\begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & a_{ij} & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\underbrace{A^{-1} A}\, x = A^{-1} b$$

$$x = A^{-1} b$$

If A has rank n, then solution $x = A^{-1}\, b$ (Cramer's rule).

Another simple procedure: Gaussian elimination.

LP Formulation

Variables: $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$

linear objective

$\forall j \in \{1,2,\dots,m\}: \langle a_j, x\rangle \le b_j$

$$\max \quad c^T x = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{such that} \quad \underbrace{(Ax)}_{m\times 1} \le (b)_{m\times 1}$$

$\forall i \in [m]: \langle a_i, x\rangle = \sum_{j=1}^{n} a_{ij} x_j \le b_i$

inequalities and not equalities

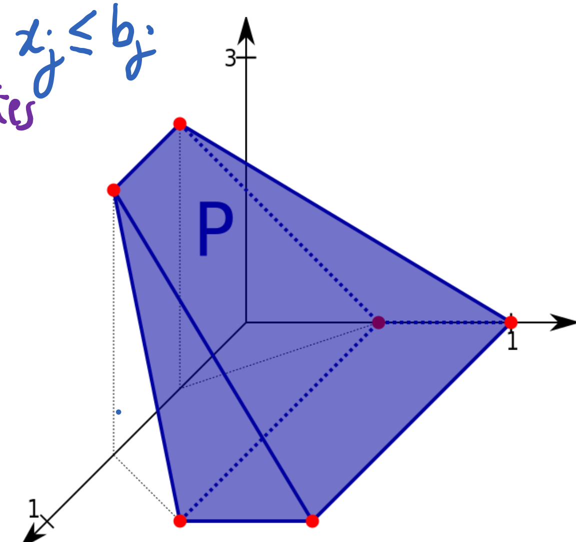
m linear constraints (inequalities)

$$x \ge 0$$

$\forall j \in [n]: x_j \ge 0$

each of the n variables is non-negative

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



LP Formulation

Eg: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Variables: $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

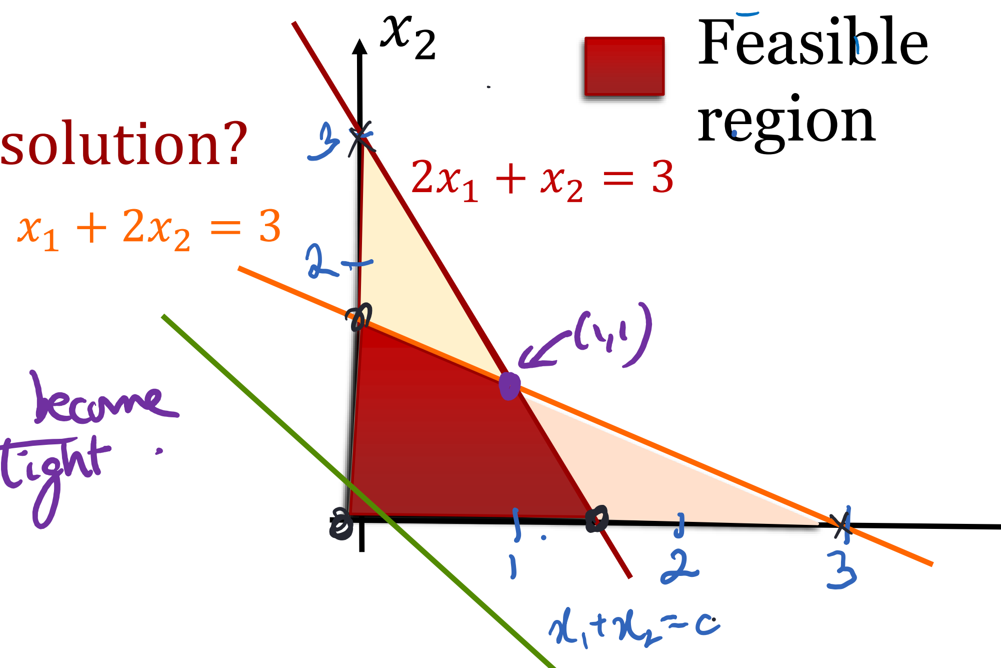
$$\boxed{\begin{array}{ll} & \max \quad c^T x \\ \text{s.t.} & Ax \leq b,\ x \geq 0 \end{array}}$$

What can we say about optimal solution?

One of the corner points.

What is a corner point?

pt where n of the inequalities become tight.



Optimal Solution always attained at one of the corner points

Variables: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

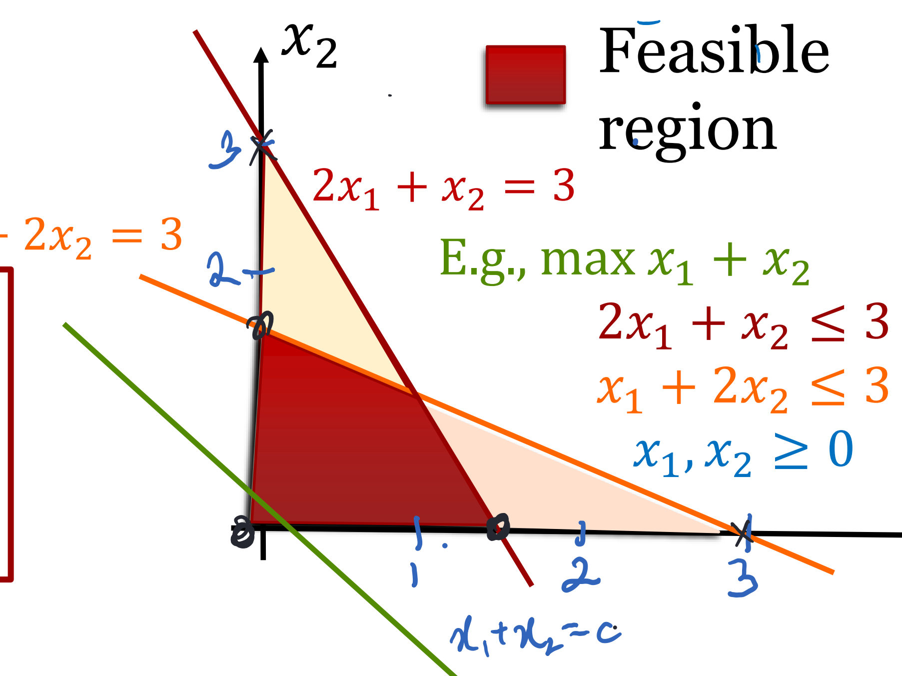
$$\max \quad c^T x$$

$$\text{s.t.} \qquad Ax \le b, x \ge 0$$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Theorem. For a given Linear Program, if the maximum is not infinite, then there exists a corner point where optimum is attained.



Corner points

Variables: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

$$\max \quad c^T x \quad \text{s.t.} \qquad Ax \le b, x \ge 0$$

E.g., maximize $x_1 + x_2$

$$2x_1 + x_2 \le 3$$
$$x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

Corner point: intersection of n tight constraints

① $x_1 = 0, x_2 = 0$: $(0,0)$ is intersection. Feasible.
Objective value $= 0$

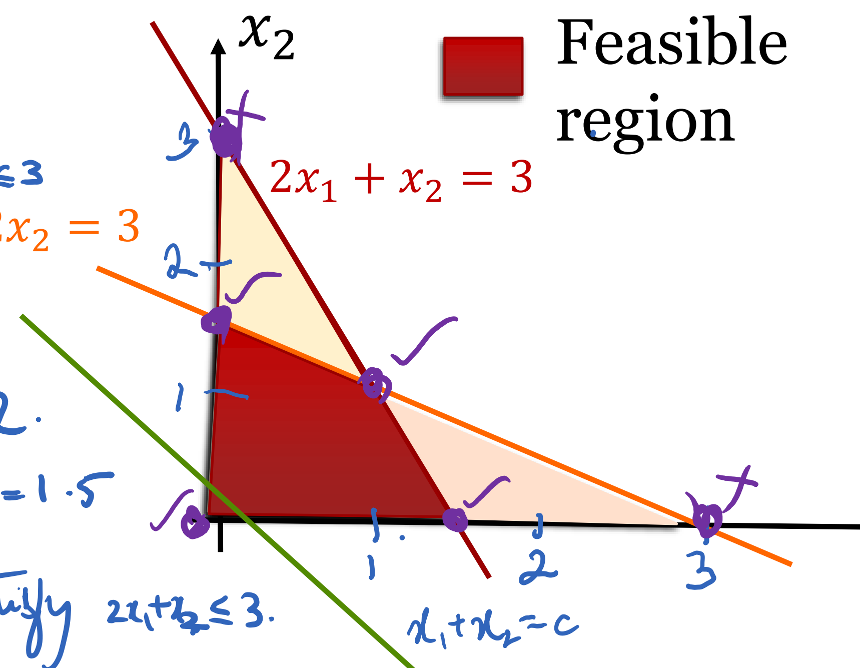
② $2x_1 + x_2 = 3, x_1 = 0$: $(0,3)$. Does not satisfy $x_1 + 2x_2 \le 3$

③ $2x_1 + x_2 = 3, x_2 = 0$: $(1.5, 0)$. Feasible.
Obj. value $= 1.5$

④ $2x_1 + x_2 = 3, x_1 + 2x_2 = 3$: $(1,1)$. Feasible. Obj value $= 2$.

⑤ $x_1 + 2x_2 = 3, x_1 = 0$: $(0, 1.5)$. Feasible. Obj value $= 1.5$

⑥ $x_1 + 2x_2 = 3, x_2 = 0$: $(3,0)$. Not feasible: not satisfy $2x_1 + x_2 \le 3$.



LP Formulation

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Variables: $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

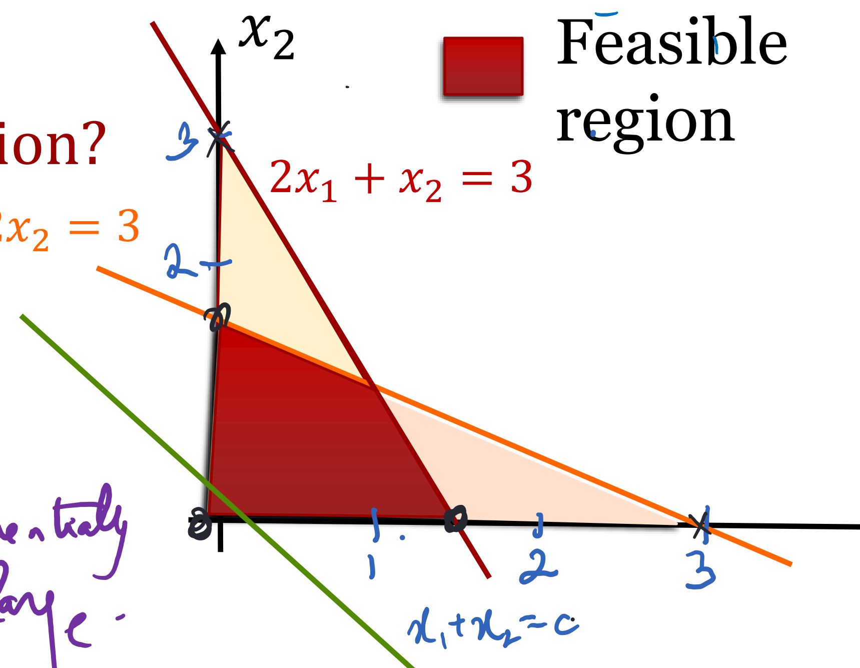
$$\max \quad c^T x$$

$$\text{s.t.} \qquad \underbrace{Ax \le b}_{m \text{ constraints}}, \underbrace{x \ge 0}_{n \text{ constraints}}$$

What can we say about optimal solution?

One of the corner points.

How many corner points? $\binom{m+n}{n}$ ← exponentially large.



“Standard” LP Formulation

Variables: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

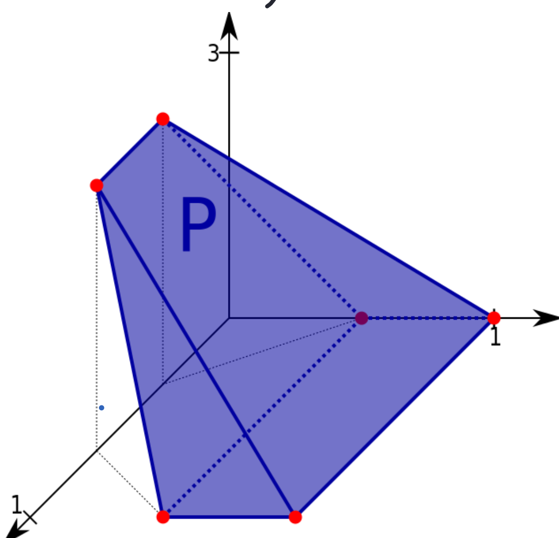
Constraints: given by $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ & given $c \in \mathbb{R}^n$

$$\max \quad c^T x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\text{such that} \quad Ax \le b \quad \forall i \in [m]: \sum_{j=1}^{n} a_{ij} x_j \le b_j$$

$$x \ge 0 \quad \forall j \in [n]: x_j \geq 0$$

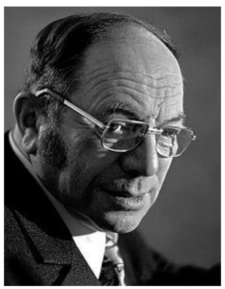
$\leq$



Claim: Standard LP formulation can capture general Linear programs.

How do you optimize/solve a Linear Program?

This Class: No algorithms for LP.

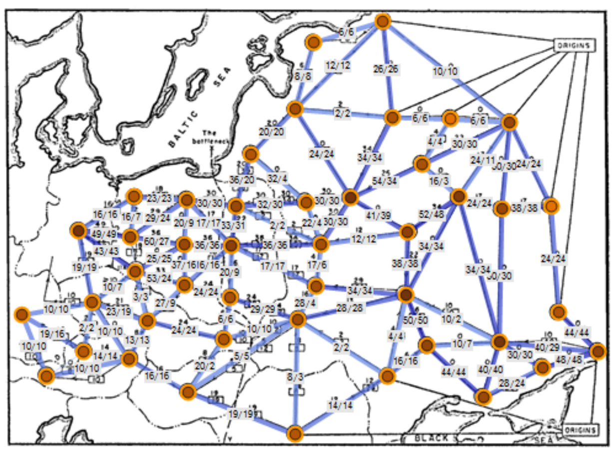


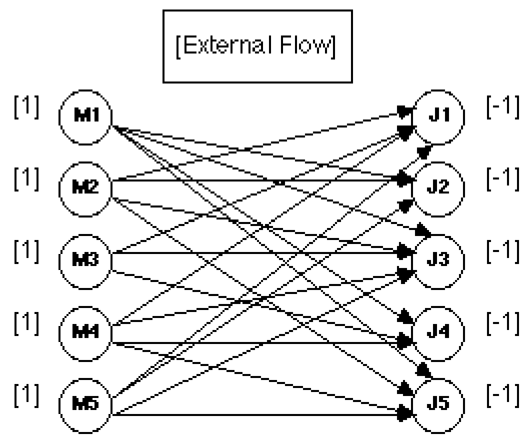
LPs introduced by Kantorovich, Koopmans, Dantzig.
(Won Nobel Prize in 1971.)

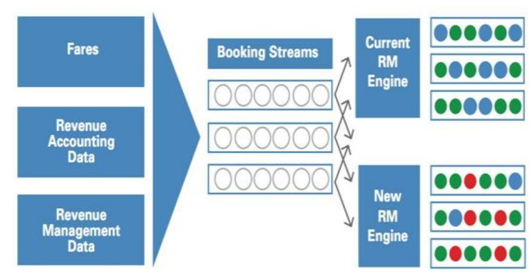
Finding efficient (polynomial time) algorithms for Linear Programming was an open problem from 1940s till it was solved in 1979 by Khachiyan.

Importance of LPs

- Important in Business, Industry and Economics.







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Thank you!