Problem 1 (5 points)

Answer with explanations and proof. Suppose $f(n) = 2^{(\log n)^2}$, $g(n) = n^3$, $h(n) = \binom{n}{2}$. You need to first arrange $f(n), g(n), h(n)$ in increasing order of asymptotics as $f_1(n), f_2(n), f_3(n)$, and explain whether the consecutive terms $f_i(n), f_{i+1}(n)$ (for $i = 1, 2$) satisfy a $f_i(n) = o(f_{i+1}(n))$ or $f_i(n) = \Theta(f_{i+1}(n))$?

First observe that $h(n) = \binom{n}{2} = \frac{n!}{2(n-2)!} = \frac{n(n-1)}{2}$.

It follows that

$$\lim_{n\to\infty} \frac{h(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2-n}{2n^3} = \lim_{n\to\infty} \frac{1}{2n} - \frac{1}{2n^2} = 0.$$

Hence $h(n) = o(g(n))$.

We next compare $g(n)$ and $f(n)$. Observe that

$$\lim_{n\to\infty} \frac{\log g(n)}{\log f(n)} = \lim_{n\to\infty} \frac{\log n^3}{\log 2^{((\log n)^2)}} = \log_{n\to\infty} \frac{3\log n}{(\log n)^2}$$

$$= \lim_{n\to\infty} \frac{3}{\log n} = 0$$

Thus $\log(g(n)) = o(\log(f(n)))$. Using a theorem from class we obtain $g(n) = o(f(n))$

Problem 2 (5 points)

The running time of an algorithm is given by the function $T(n)$ which satisfies the following recurrence relation $T(n) = 5T(n-1) - 6T(n-2)$. Also $T(1) = 8, T(2) = 22$ time units. Prove that $T(n) = 2 \times 3^n + 2^n$ (time units) is a solution for the above recurrence for integer $n \geq 1$.

Since the recurrence uses $T(n-1)$ and $T(n-2)$ we must first show that the formula holds at $n=1$ and $n=2$. We have

$$2 \times 3^1 + 2^1 = 8 = T(1)$$
$$2 \times 3^2 + 2^2 = 22 = T(2)$$

Now, fix $k \geq 3$ and assume that

$$T(k-1) = 2 \times 3^{k-1} + 2^{k-1} \quad \text{and that}$$
$$T(k-2) = 2 \times 3^{k-2} + 2^{k-2}$$

From our recurrence relation and using our induction hypothesis we have

$$\begin{aligned} T(k) &= 5T(n-1) - 6T(n-2) \\ &= 5 \cdot 2 \cdot 3^{k-1} + 5 \cdot 2^{k-1} - 6 \cdot 2 \cdot 3^{k-2} - 6 \cdot 2^{k-2} \\ &= 10 \cdot 3^{k-1} + 5 \cdot 2^{k-1} - (2 \cdot 2 \cdot 3) \cdot 3^{k-2} - (2 \cdot 3) \cdot 2^{k-2} \\ &= 10 \cdot 3^{k-1} + 5 \cdot 2^{k-1} - 4 \cdot 3^{k-1} - 3 \cdot 2^{k-1} \end{aligned}$$

$= 6 \cdot 3^{k-1} + 2 \cdot 2^{k-1}$

$= 2 \cdot 3^k + 2^k$

We conclude using induction that $T(n) = 2 \cdot 3^n + 2^n$ for all $n \geq 1$

Problem 3 (3 points total)

There are m red balls and n blue balls in a bag, and balls are picked at random from the bag. Prove that the probability that we pick at least one ball of each color conditioned on picking a total of k balls is $1 - \frac{\binom{m}{k}+\binom{n}{k}}{\binom{m+n}{k}}$. You can assume that $k \leq \min\{m, n\}$.

First note that the probability in question is equal to one minus the probability that either all balls selected are blue or all balls selected are red.

There are $\binom{m}{k}$ ways to pick only blue balls and $\binom{n}{k}$ ways to pick only red balls. These two events are disjoint, so it follows that there are $\binom{m}{k} + \binom{n}{k}$ ways to only pick red balls or blue balls

The total number of ways to pick k balls out of our $n+m$ balls is $\binom{n+m}{k}$

from which we obtain that the probability of only pick red or only picking blue balls is $$\frac{\binom{n}{k}+\binom{m}{k}}{\binom{n+m}{k}}.$$

Therefore, the probability that at least one ball of each color is chosen is

$$1-\frac{\binom{n}{k}+\binom{m}{k}}{\binom{n+m}{k}}$$ as claimed.

Problem 4 (3 points total)

Give a counting based proof that that $(1000!) \times (1023!)$ divides $2023!$. **(2 points)**

Note that $\frac{2023!}{(1000!)(1023!)} = \binom{2023}{1000}$ which is the number of ways to select 1000 objects of of 2023. For any positive integers $n \geq k$, number of ways to select k objects out of n is always a positive integer.

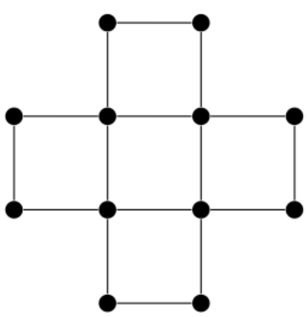
We conclude that $\binom{2023}{1000}$ is a positive integer.

That is $(1000!)(1023!)$ divides $2023!$

Problem 5 (2+2 points)

Identify if the following proofs are correct. If not, explain what is wrong in them.

1. We start with the following 12 points connected by links.



Each of these links is deleted independently with probability $\frac{1}{3}$.

Claim. *The expected number of squares remaining after the links are deleted is $\frac{80}{81}$.*

Proof. Before we start, there are 5 possible squares that could remain intact. Let S_1 be an indicator random variable that is 1 if the first square remains intact and 0 otherwise. Similarly define S_2, S_3, S_4, S_5 for the other squares.

We have that for $i = 1, 2, 3, 4, 5$,

$$\Pr\{S_i = 1\} = \left(1 - \frac{1}{3}\right)^4 = \frac{16}{81},$$

Let S be the number of squares that remain. We get

$$\mathbb{E}[S] = \sum_{i=1}^{5} \mathbb{E}[S_i] = 5 \cdot \frac{16}{81} = \frac{80}{81}.$$

□

The proof is correct.

2. Suppose there are 1000 students in the 2023 Northwestern Freshman class. Say two students are related if they share the same birthday.

Claim. *Every student is related to some other student.*

Proof. We give a proof of the claim using the pigeonhole principle. There are 1000 students, and the birthdays of the students have to be one among the 365 days in the year. In the setting of the pigeonhole principle, there are 1000 pigeons, and at most 365 pigeonholes. By the pigeonhole principle the students can not have distinct birthdays. Furthermore since $1000 > 2 \times 365$, for every student there is at least one other student who has the same birthday. Hence the claim follows. $\square$

The proof is incorrect. The pigeonhole principal only allows us to conclude that there is at least one day on which multiple students have a birthday.

In particular, the line "Furthermore, since $1000 > 2 \cdot (365)$, for every student there is at least one other student who has the same birthday" is False.