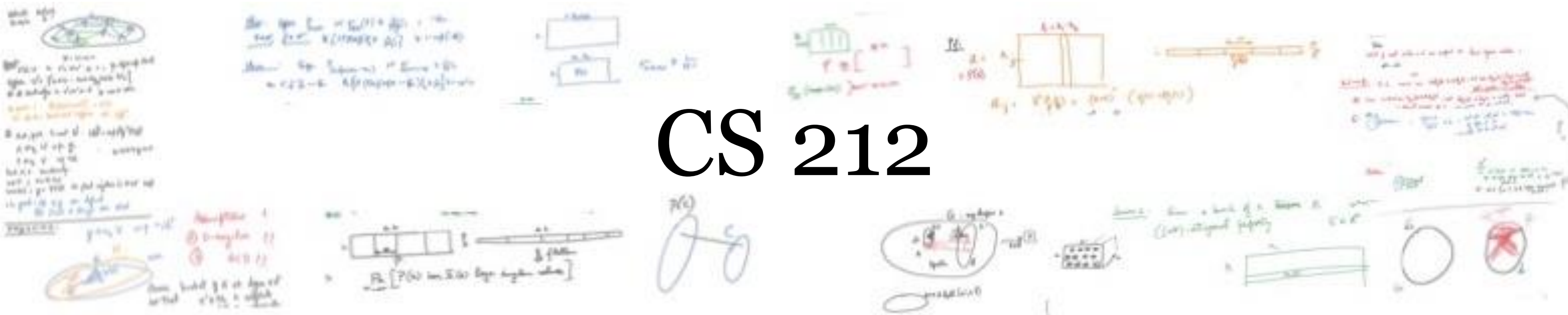
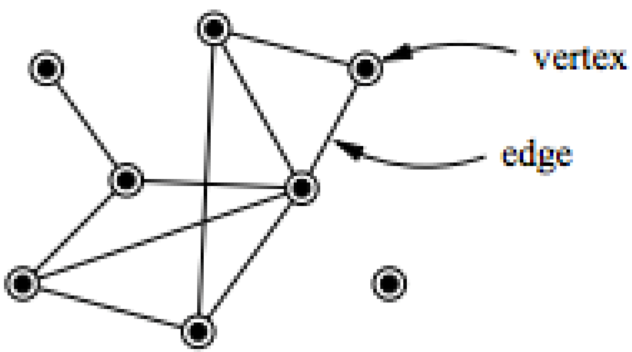


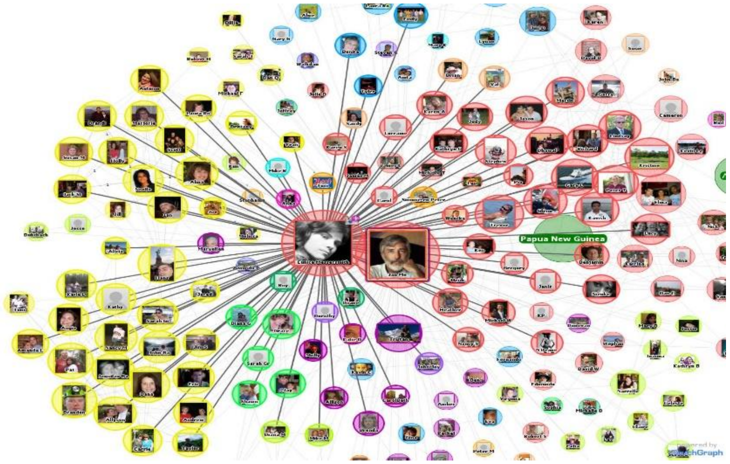
CS 212

Mathematical Foundations of Computer Science

Lecture 18: Introduction to Graphs

Abstraction: Graph





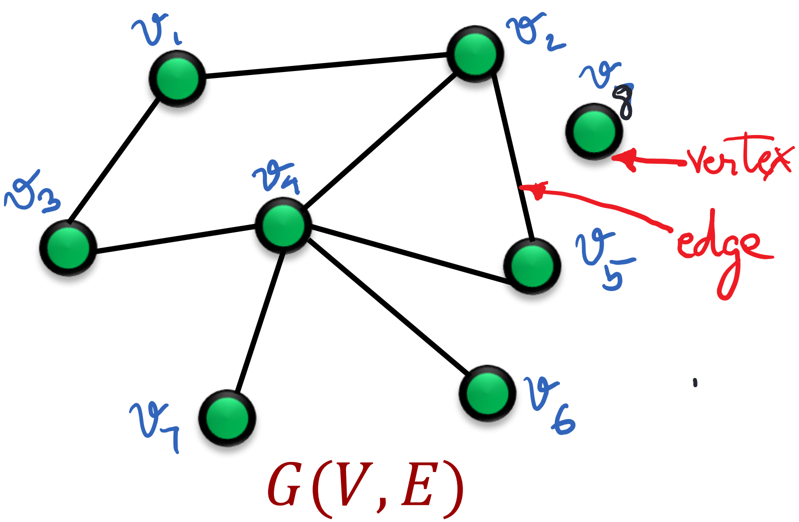
Representing objects and relations/ connections between objects

- Entities = Vertices
- Relations/ Connections = Edges

Social Networks :

- Vertices = People.
- Edges = Friendships.

Graphs



Graph $G = (V, E)$ is a pair of sets

$$V = \mathit{set\ of\ vertices},$$

$$\mathrm{E} \subseteq V \times V = \text{set of edges}$$

$$V \times V = \{(\mathrm{u}, \mathrm{v}) | \mathrm{u} \in V \text{ and } v \in V\}$$

Represented by $G = (V, E)$

Undirected graph: For any $u, v \in V$, if $(u, v) \in E$, then $(v, u) \in E$

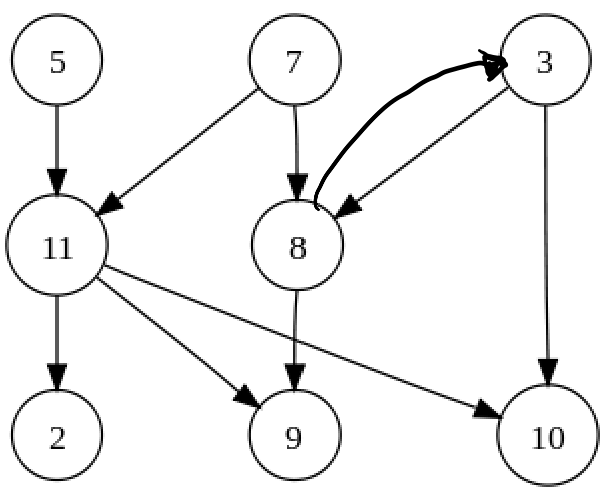
$|V| = n$ will denote the number of nodes in a graph

$|E| = m$ will denote the number of edges in a graph

Directed Graphs

Captures Asymmetric Relations between objects

Every edge has a direction associated with it.



$(u, v) \in E$ does not imply that $(v, u) \in E$

E.g. a graph representing NFL games
Edge from u to v if team u beats team v

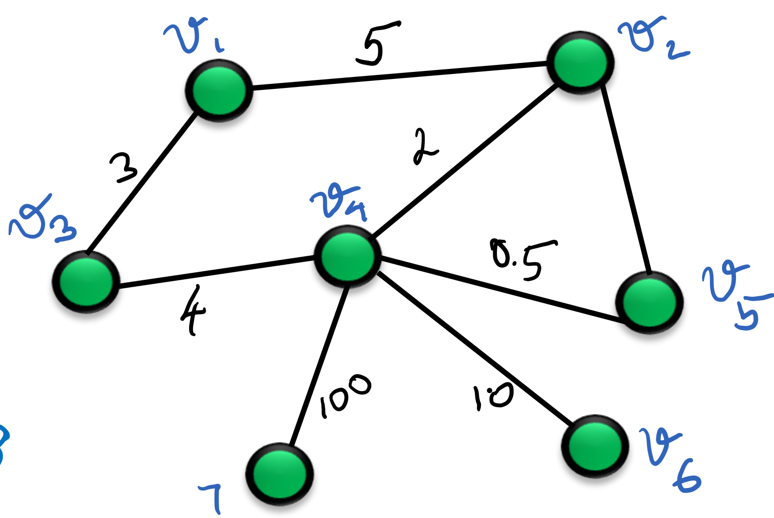
$V = \{2, 3, 5, 7, 9, 8, 11, 10\}$ $n = 8$

$E = \{(5,11), (11,2), (7,11), (7,8)\ (8,3), (3,8), (8,9), (3,10), (11,10)\}$ $m = 10$

Other Generalizations Graphs

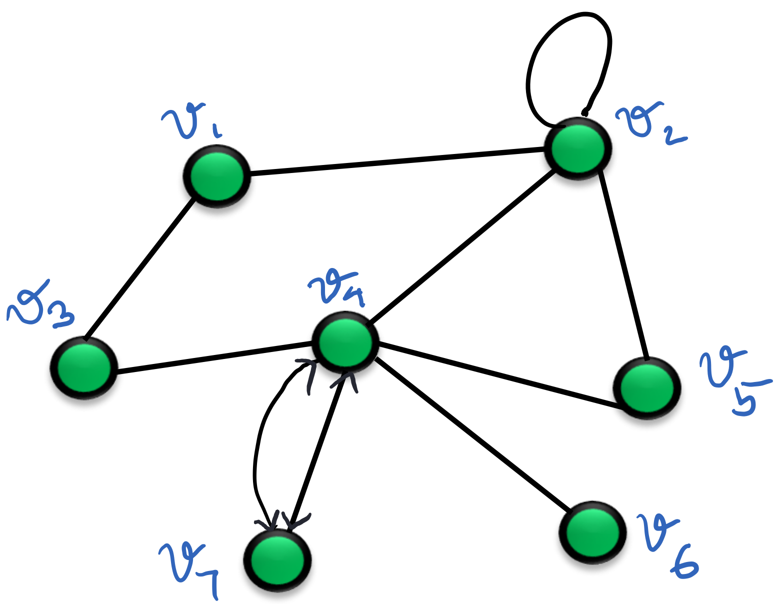
Weighted graphs: $G = (V, E, w)$

Edges have numbers associated with them, representing extent of relation e.g. maps with distances.



$G = (V, E)$

$|E| = 8$



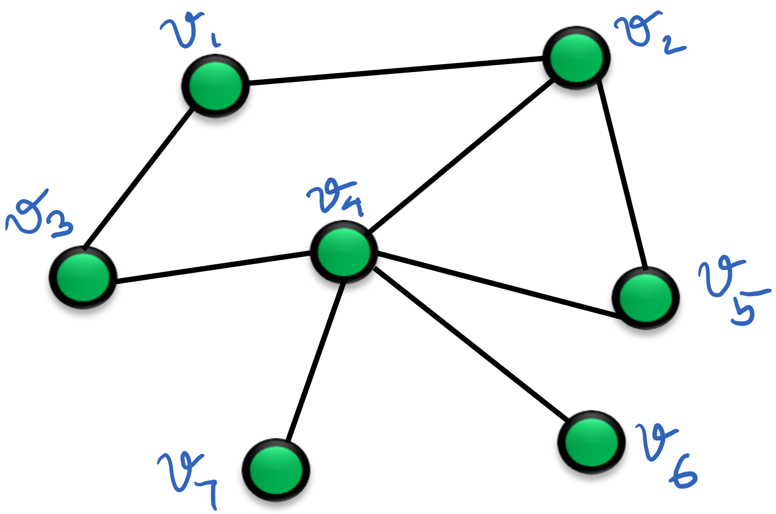
$G(V, E)$

Multigraph:

- Edges E is a multiset of $V \times V$
  i.e. can have parallel edges
- Can have self-loops too.

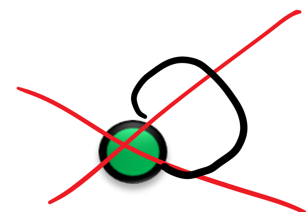
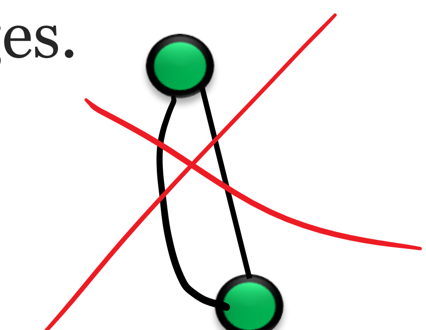
$E = \{(v_1, v_3)\ (v_3, v_4),\ (v_4, v_7),\ (v_4, v_5),\ (v_4, v_7),\ (v_2, v_2)\}$

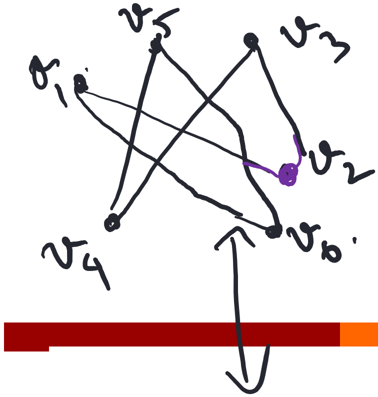
Simple Graphs: The Default Case



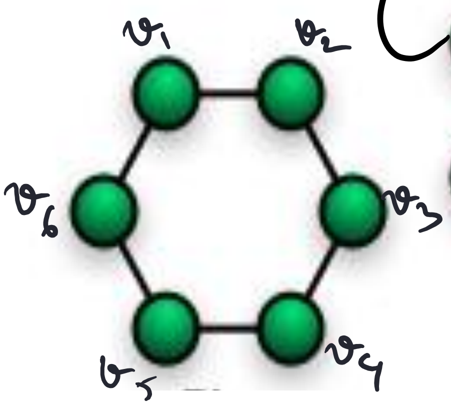
$G(V, E)$

G=(V,E) is a simple graph iff:

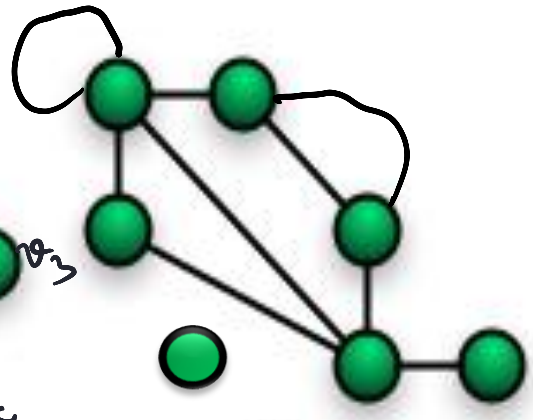
- Undirected, unweighted graph
  $$(u, v) \in E \Rightarrow (v, u) \in E$$
- No self-loops
  
  $$(u, u) \notin E$$
- No parallel edges.
  



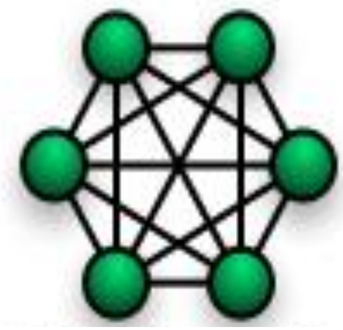
Common Graphs



Cycle graph

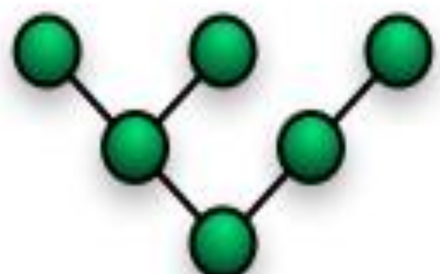


Star



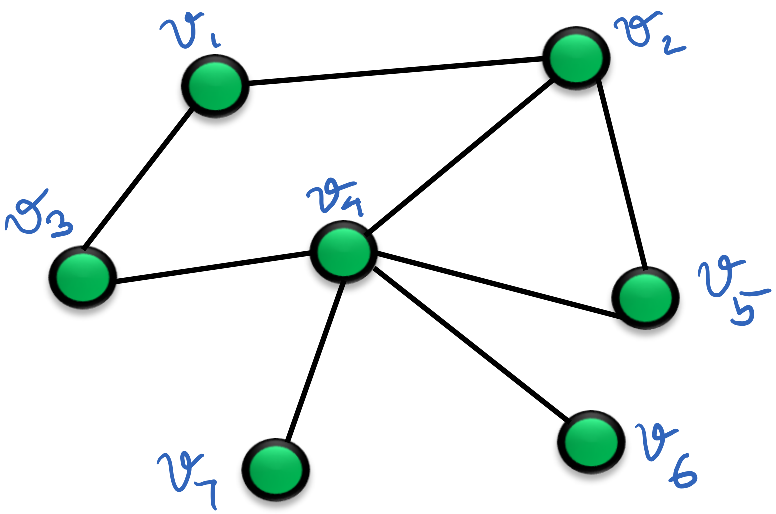
Complete graph / Clique

Path

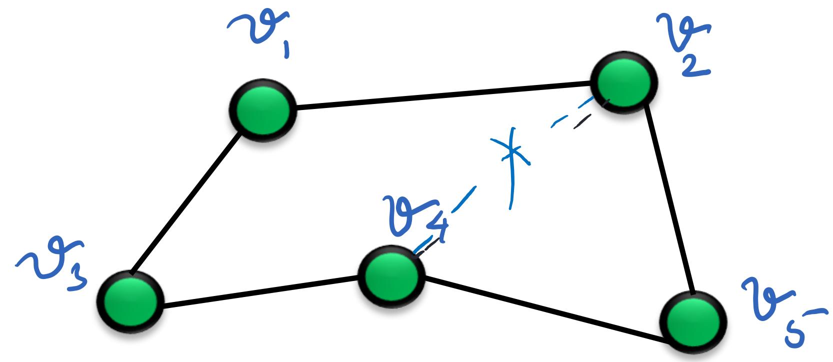


Tree

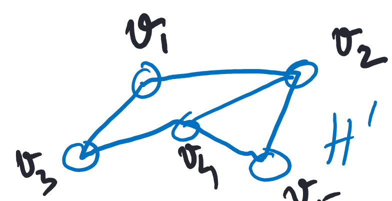
Subgraph of a Graph



$G = (V, E)$



$H = (U, F)$



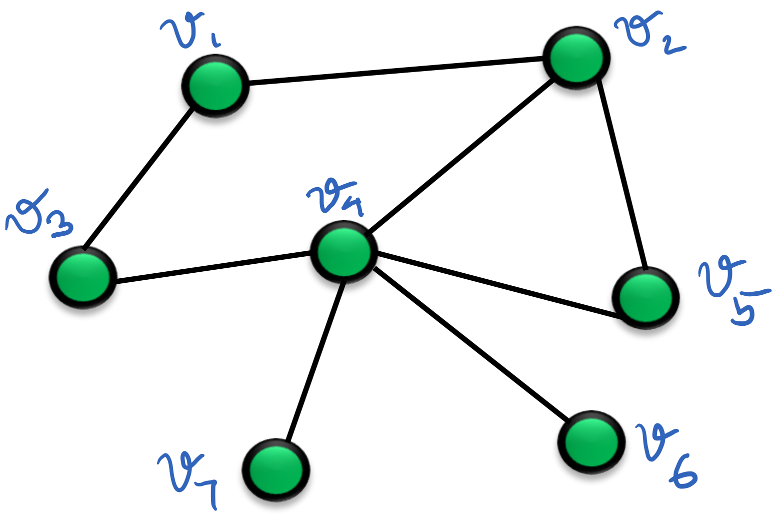
H' is an induced subgraph.

A *subgraph* $H = (U, F)$ of a graph $G = (V, E)$ is a graph where both $U \subseteq V$ and $F \subseteq E$.

- Subgraph doesn't need to contain all the edges incident on U

Induced subgraph: when $U \subseteq V$ and $F = \{(u, v) \in E : u, v \in U\}$.

Degree of a Vertex



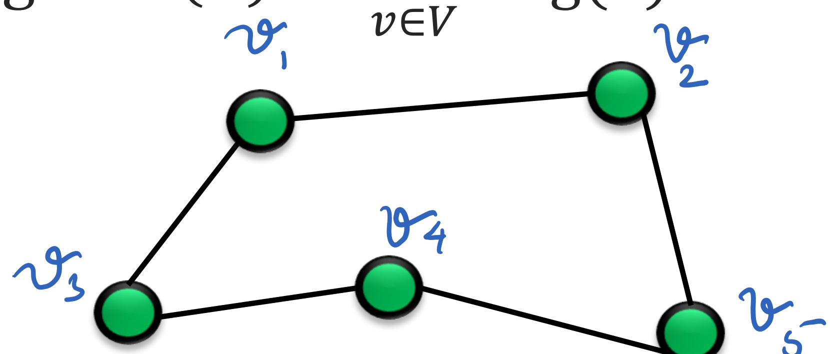
$G(V, E)$

Degree (or valency) of a vertex v (represented by $\deg(v)$) in graph $G(V, E)$ is the number of edges in E incident on v.

- In simple graph with n vertices: $\leq \deg(v) \leq$

Max. degree $\Delta(G) = \max_{v \in V} \deg(v)$

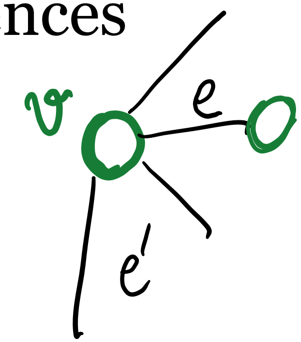
Regular graph: graph where every vertex has same degree.



Sum of Degrees/ Handshake Lemma

Thm. In any undirected graph $G = (V, E)$, the sum of the degrees is equal to twice the number of edges: $\sum_{v \in V(G)} \deg(v) = 2\,|E(G)|$

Proof. By "Double Counting" # of (vertex, edge) incidences
i.e. pairs (v, e) where $v \in V$, e $\in E$ and e is incident on v

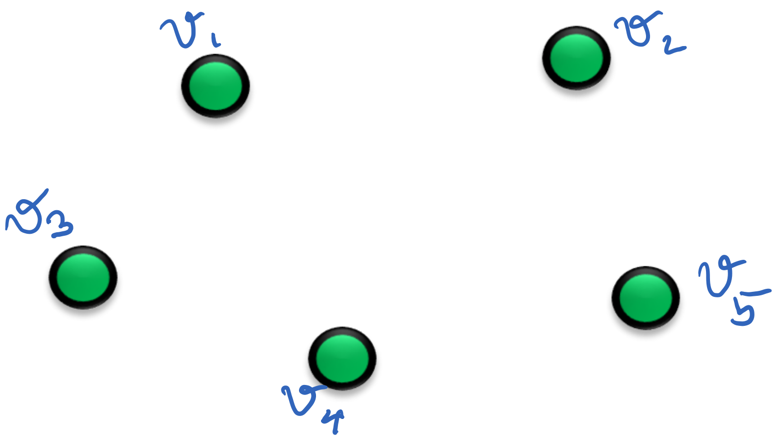


How much does each vertex contribute?

How much does each edge contribute?

Independent Sets, Cliques, Graph Complements

Independent Set

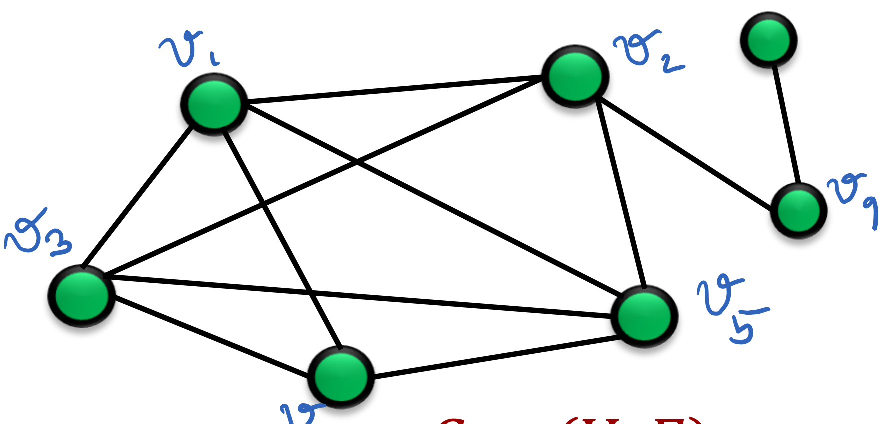


Empty graph: a graph with *no edges*

Independent Set:

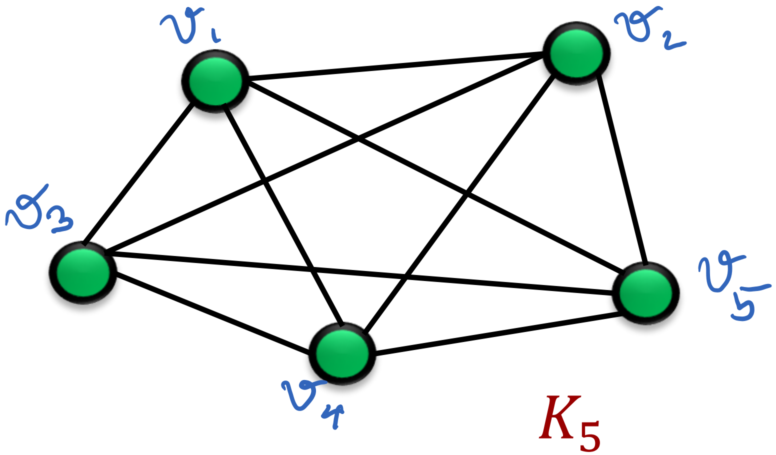
A subset of vertices with no edges between them in G (subset whose induced subgraph is empty). E.g., $S = \{v_2, v_4, v_8\}$

What is the size of the largest independent set in G?



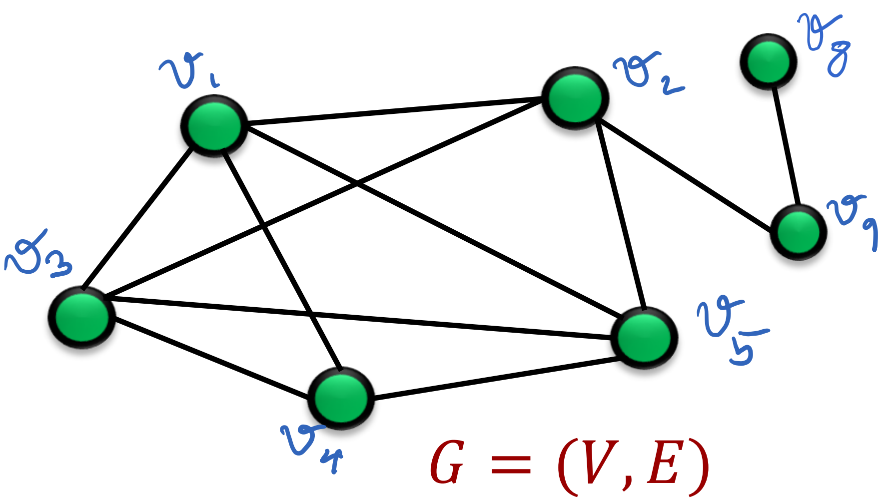
$G = (V, E)$

Complete Graphs, Cliques



K_n : complete graph on n-vertices.

Or also called an n-clique

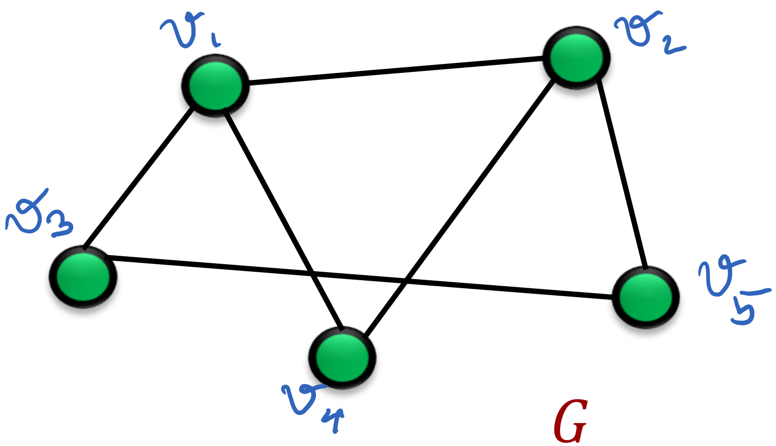


Cliques in a graph:

A subgraph that is a clique.

What is the size of the largest clique in G?

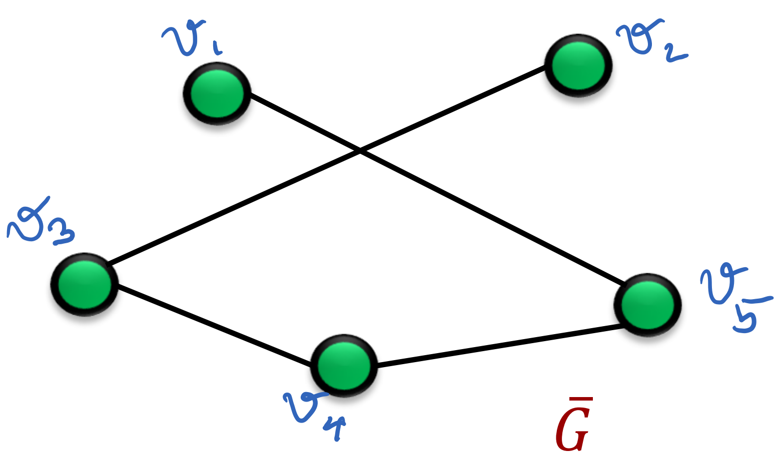
Graph Complements



$\bar{G} = (V, \bar{E})$=Complement of $G = (V, E)$

- Graph on the same of vertices
- $(u, v) \notin$ E *iff* $(u, v) \in \bar{E}$

Thm. S is an independent set in G iff S is a clique in $\bar{G}$



Pf. S is an independent set. So,
For every $u, v \in S$, $(u, v) \notin E$
i.e., $\forall u, v \in S$, $(u, v) \in \bar{E}$
Hence S is a clique in $\bar{G}$

Relations b/w Graph Properties

Given graph $G(V,E)$:

what is the size of the maximum independent set in G (independent set with largest number of vertices)?

a) = size of the maximum clique in $\bar{G}$?

b) = size of the maximum clique in G ?

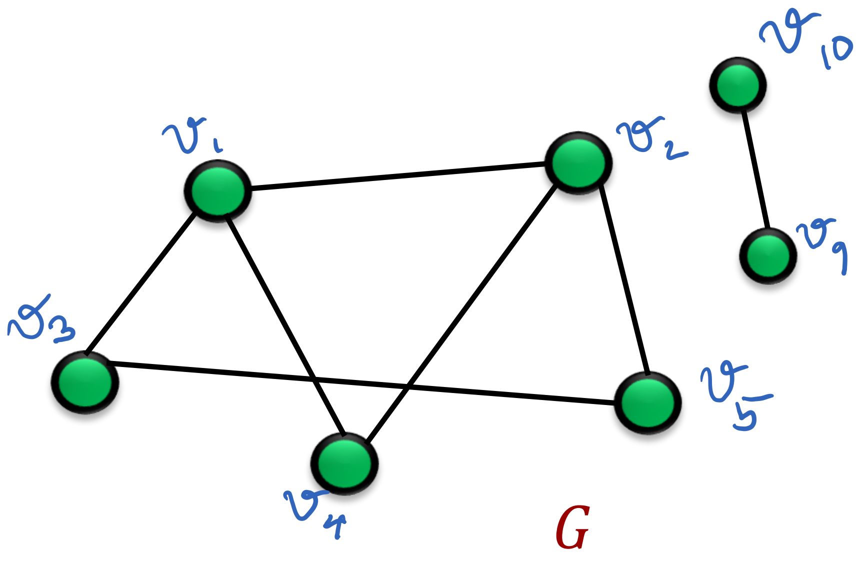
c) = size of the maximum independent set in $\bar{G}$?

d) None of the above

e) All of the above

Graph Coloring

A graph $G(V, E)$ is k- colorable (vertex) if each vertex can be colored with one of k colors such that each edge is not monochromatic i.e. if $(u, v) \in E$ then u, v have different colors.

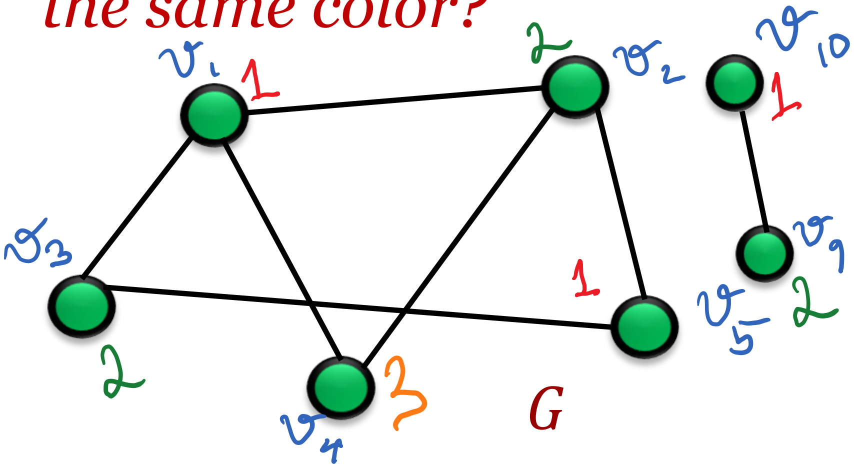


Is this graph 3-colorable?

Is this graph 2-colorable?

Color Classes

What can you say about each color class i.e. all the vertices of the same color?



Each color class is an independent set

Theorem. If a graph on n vertices is k-colorable, then the size of the maximum independent set ≥ ?

Thank you!