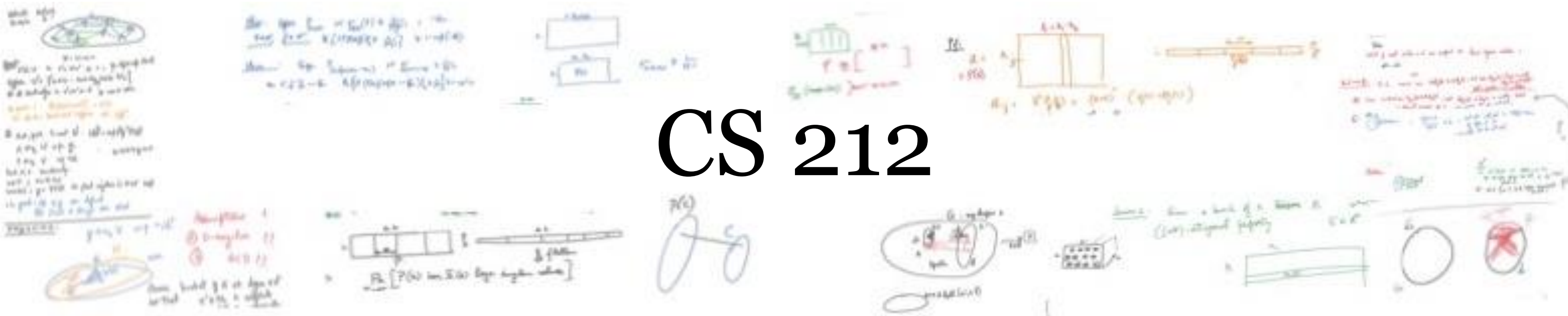


# CS 212

# Mathematical Foundations of Computer Science

## Lecture 27: LPs: general form and LP Duality

# Announcements

1. Next (last) assignment due tomorrow.
2. No discussion sessions this week – thanksgiving week.
3. No office hours later this week. (i.e. on Wed - Sun)

# Linear Programming

# Linear Optimization

Suppose you want to buy apples and oranges.

- Apple costs 2$/$lb$ and oranges cost 1$/$lb$.
- Apples have 1$mg/lb$ of vit-C, oranges have 2$mg/lb$ of vit-C.
- Apples and oranges both provide 1$Kcal/lb$.

How many pounds of apples & oranges can you buy with at most 3$, so that total Vitamin C intake $\leq 3\ mg$, in such a way to maximize calorie intake?

Let $x_1$ be the #lbs of apple bought, $x_2$ be #lbs of oranges bought

$$\max\ x_1 + x_2$$

\# money spent = $2x_1 + x_2 \leq 3$

\# vit-C consumed = $x_1 + 2x_2 \leq 3$

$x_1 \geq 0$, $x_2 \geq 0$

Linear Program.

# Optimize this…

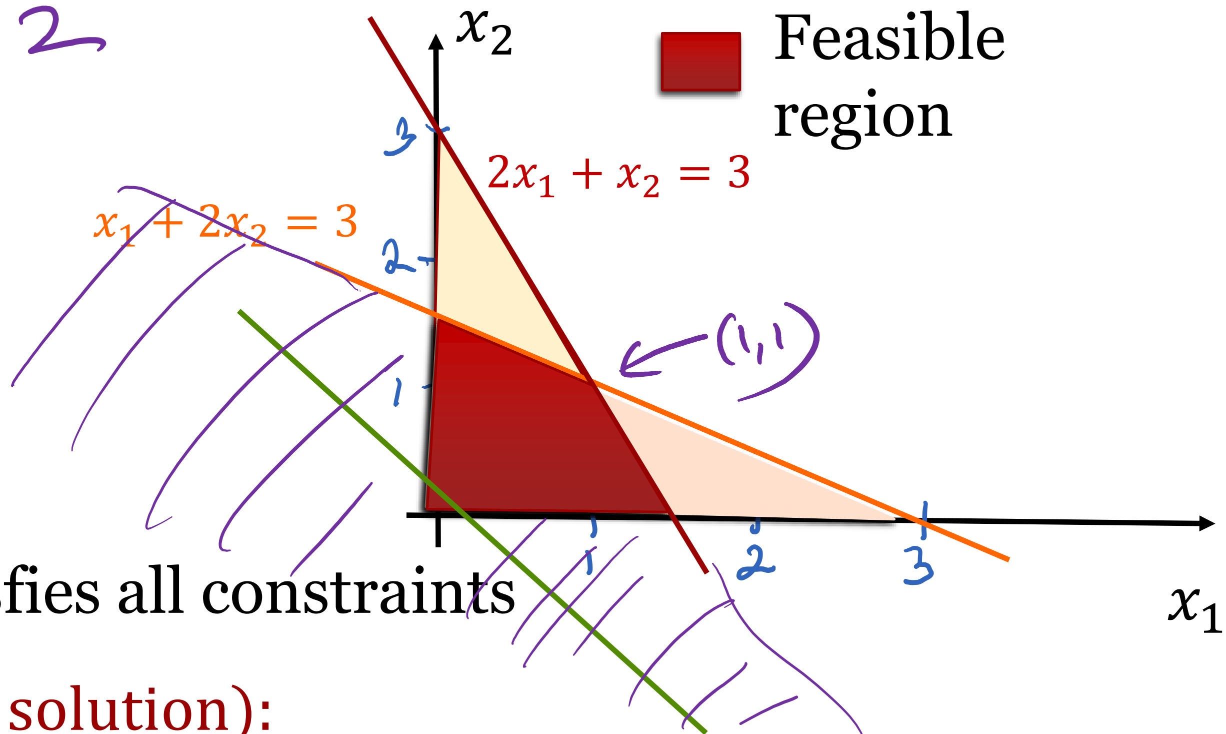
$=2$

Maximize $x_1 + x_2$

$2x_1 + x_2 \leq 3$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$




**Feasible Solution.**
Any vector $x$ that satisfies all constraints

**Vertex (Basic feasible solution):**
The corners of the feasible region.

**What is the optimal solution?**
One of the corner points.

# LP Formulation

Variables: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\max \quad \underline{c^T x} = \sum_{i=1}^{n} c_i x_i$$

$$\text{such that} \quad (Ax) \leq b$$

m-dim vector

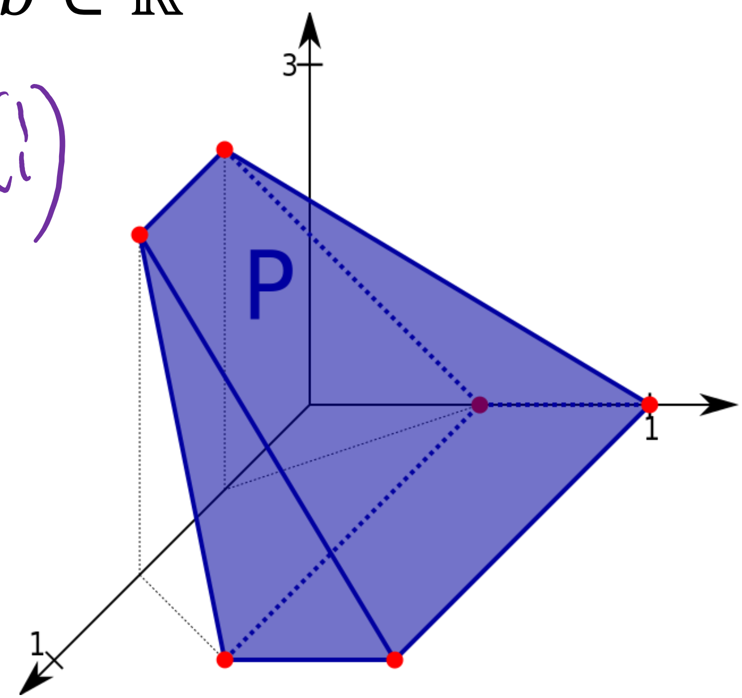
entrywise inequality

m-dim vector

$$x \geq 0$$

$\forall i \; x_i \geq 0$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \text{ith row} & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$



# “Standard” LP Formulation

**Variables:** $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

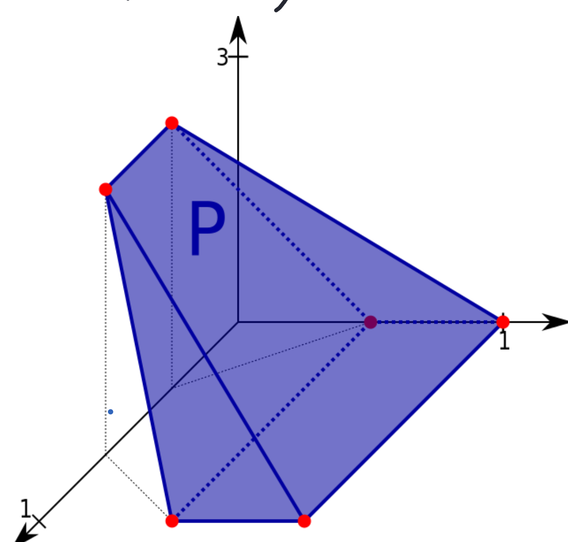
**Constraints:** given by $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ & given $c \in \mathbb{R}^n$

$$\max \quad c^T x = c_1x_1 + c_2x_2 + \cdots + c_n x_n$$

$$\text{such that} \quad Ax \le b \qquad \forall i \in [m]: \sum_{j=1}^{n} a_{ij} x_j \le b_j$$

$$x \ge 0 \qquad \forall j \in [n]: x_j \ge 0$$

$\le$




> Claim: Standard LP formulation can capture general Linear programs.

max $c^T x$
$Ax \le b$
$x \ge 0$

# Capturing a minimization problem

Minimize $2y_1 + 4y_2 + 2y_3$ $\quad = \text{maximize } -(2y_1 + 4y_2 + 2y_3)$ $= -2y_1 - 4y_2 - 2y_3$

$$y_1 + y_2 \le 5 \qquad\qquad y_1 + y_2 \le 5$$

$$y_2 + y_3 \le 4 \qquad\qquad y_2 + y_3 \le 4$$

$$y_1, y_2, y_3 \ge 0 \qquad\qquad y_1, y_2, y_3 \ge 0$$

Variables: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

$$\min \quad c^T x \qquad = \max \quad (-c)^T x$$

$$\text{s.t} \quad Ax \le b \qquad \text{s.t} \quad Ax \le b$$

$$x \ge 0 \qquad\qquad x \ge 0$$

← Standard form

$y_2 + y_3 = 4$

$\Leftrightarrow \left\{ \begin{array}{l} y_2 + y_3 \le 4 \\ -y_2 - y_3 \le -4 \end{array} \right\}$

# Capturing other inequality, equality constraints

$$\max\ 2y_1 + 4y_2 + 2y_3$$
$$y_1 + y_2 \le 5$$
$$(-1) \times (y_2 + y_3 \ge 4)$$
$$y_1, y_2, y_3 \ge 0$$

$$= \max\ 2y_1 + 4y_2 + 2y_3$$
$$y_1 + y_2 \le 5$$
$$-y_2 - y_3 \le -4$$
$$y_1, y_2, y_3 \ge 0$$

Standard form

Variables: $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

Constraint $\sum_{j=1}^{n} a_{ij} x_j \ge b_j$ is the same as $\sum_{j=1}^{n} (-a_{ij}) x_j \le -b_j$

How do you capture constraint $\sum_{j=1}^{n} a_{ij} x_j = b_j$ ?

It is captured by following two constraints:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_j \text{ and } \sum_{j=1}^{n} (-a_{ij}) x_j \le -b_j$$

# Unconstrained Variables

$\max 2y_1 + 4y_2 + 2y_3$

$y_1 + y_2 \leq 5$

$y_2 + y_3 \leq 4$

$y_2, y_3 \geq 0$

$y_1 \in \mathbb{R}$

Variable $y_1$ is unconstrained. How to bring it to standard form?

$$\left.\begin{aligned} &\text{Max } 2(z_1 - z_2) + 4y_2 + 2y_3 \\ &z_1 - z_2 + y_2 \leq 5 \\ &y_2 + y_3 \leq 4 \\ &z_1, z_2, y_2, y_3 \geq 0 \end{aligned}\right\} \text{Standard form}$$

Solution: Think of two non-negative variables $z_1, z_2$ such that $y_1 = z_1 - z_2$

# "Standard" LP Formulation

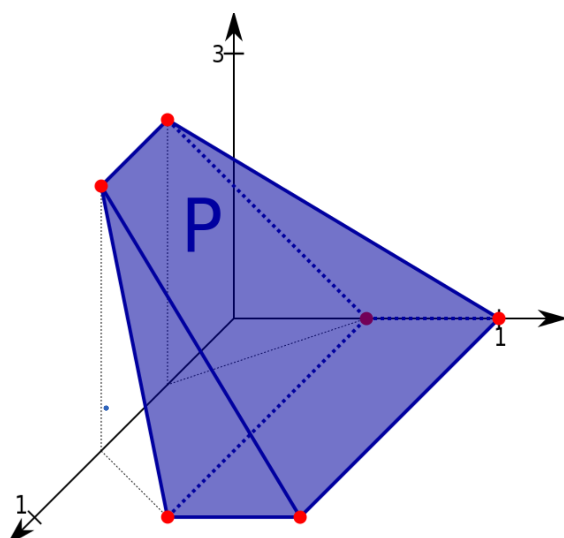
Variables: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

$$\max \quad c^T x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\text{such that} \quad Ax \le b \quad \forall i \in [m]: \ \sum_{j=1}^{n} a_{ij} x_j \le b_i$$

$$x \ge 0 \quad \forall j \in [n]: \ x_j \geq 0$$




Claim: Standard LP formulation can capture general Linear programs (any linear objective subject to linear constraints).

# How do you know you are optimal?

Maximize $x_1 + x_2$

$1 \times (2x_1 + x_2 \leq 3)$

$1 \times (x_1 + 2x_2 \leq 3)$

$x_1, x_2 \geq 0$

feasible

How do we know that (1,1) is optimal? i.e. value of LP $= 2$

To prove: for every feasible $(x_1, x_2)$ we have $x_1 + x_2 \leq 2$

$2x_1 + x_2 \leq 3$

$x_1 + 2x_2 \leq 3$

$3x_1 + 3x_2 \leq 6$

$\Rightarrow \quad x_1 + x_2 \leq 2.$

$\Rightarrow \quad \text{Max} \leq 2$

# Another Example

Maximize $2y_1 + 4y_2 + 2y_3$

$\left( y_1 + y_2 \le 5 \right) \times 2$

$\left( y_2 + y_3 \le 4 \right) \times 2$

$y_1, y_2, y_3 \ge 0$

Guess for optimal solution:

$$y_1 = 1, y_2 = 4, y_3 = 0.$$

objective=18.

*Is this optimal?*

$2y_1 + 2y_2 + 2y_2 + 2y_3 \le 18$

$2y_1 + 4y_2 + 2y_3 \le 18$

# Another Example

$y_1 \geq 0 \quad y_2 \geq 0$

Maximize $2x_1 + 4x_2 + x_3$

$(x_1 + x_2 \leq 5) \times y_1$

$(x_2 + x_3 \leq 4) \times y_2$

$x_1, x_2, x_3 \geq 0$

$y_1 x_1 + y_1 x_2 + y_2 x_2 + y_2 x_3 \leq 5y_1 + 4y_2.$

$y_1 x_1 + (y_1 + y_2) x_2 + y_2 x_3 \leq 5y_1 + 4y_2$

- Say you didn't know to multiply by 2, 2.
- How do you show optimality?

Ex: What is the best value of $y_1, y_2, y_3$ ?

# How do you know you are optimal?

Claim: (1,1) is optimal with max value =2.

Maximize $5x_1 - 3x_2$

$$(x_1 + 3x_2 \le 5) \times y_1$$
$$(3x_1 + x_2 \le 4) \times y_2$$
$$(4x_1 - 8x_2 \le -4) \times y_3$$
$$x_1, x_2 \ge 0 \qquad y_1, y_2, y_3 \ge 0$$

*How do you show optimality?*

We need upper bound on objective

$$5x_1 - 3x_2 \le \ ????$$

$5x_1 - 3x_2 \le$

$$x_1 (y_1 + 3y_2 + 4y_3) + x_2 (3y_1 + y_2 - 8y_3) \le 5y_1 + 4y_2 - 4y_3$$

minimize $5y_1 + 4y_2 - 4y_3$

$$y_1 + 3y_2 + 4y_3 \ge 5 \rightarrow \text{coeff of } x_1 \ge 5$$
$$3y_1 + y_2 - 8y_3 \ge -3 \rightarrow \text{coeff of } x_2 \ge -3$$
$$y_1, y_2, y_3 \ge 0$$

$5x_1 - 3x_2 \le 5y_1 + 4y_2 - 4y_3$

also an LP!

Thank you!