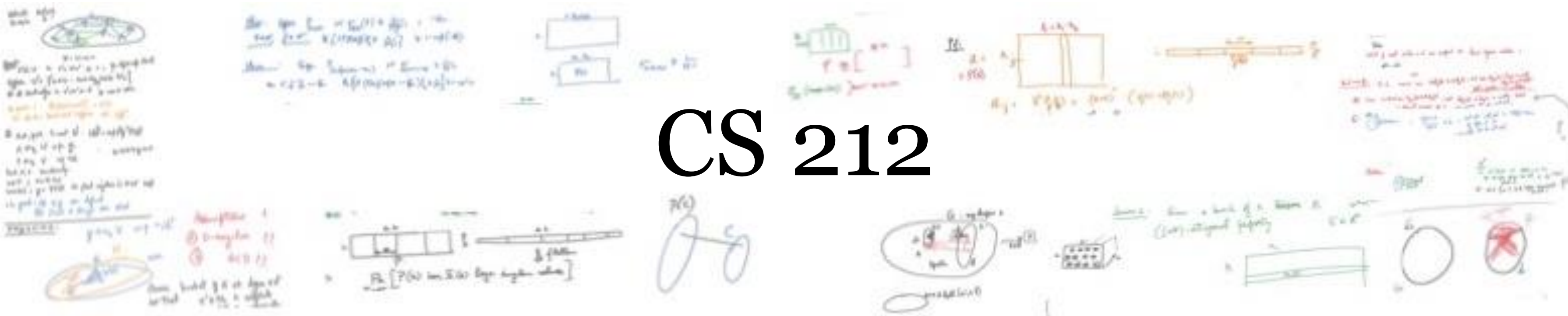


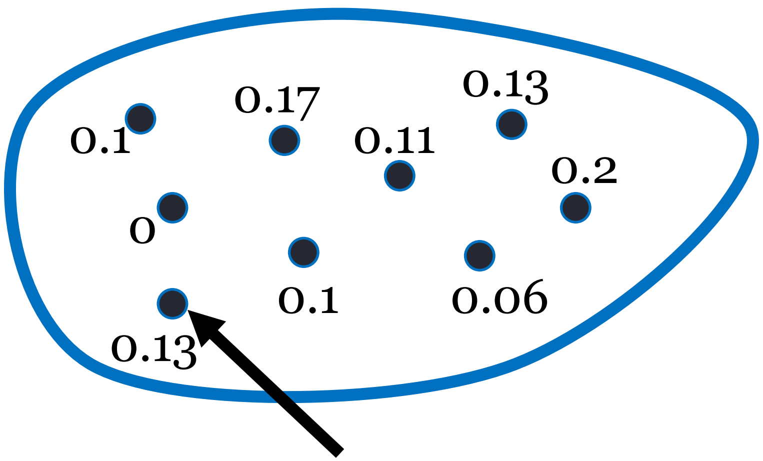
CS 212

Mathematical Foundations of Computer Science

Lecture 12: Probability – Conditional Probability, Independence

Recap: Probability Distribution

Sample space S



weight or probability
p(t) = 0.13

A (finite) probability distribution D is a finite set S of elements, where each element t in S has a non-negative real weight, proportion, or probability $p(t)$

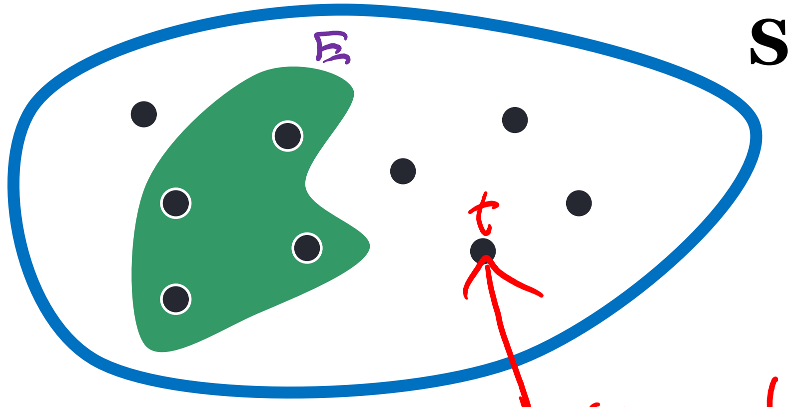
Weights must satisfy: $\sum_{t \in S} p(t) = 1$

S is sample space , elements t ∈ S are called samples/atoms.

Recap: Uniform Distribution

If each element (atomic event) has equal probability, the distribution is said to be uniform

$$\Pr_D[E] = \sum_{t \in E} p(t) = \frac{|E|}{|S|}$$



$p(t) = \frac{1}{|S|}$

This is where Counting comes in!

For n coin tosses,

$\Pr[\text{exactly } \frac{n}{2} \text{ heads } \& \frac{n}{2} \text{ tails}] = \frac{\binom{n}{n/2}}{2^n} \sim \frac{1}{\sqrt{n}}$

Probability and Counting

uniform

A ~~fair~~ coin is tossed 100 times in a row.

#heads = 50

#tails = 50

Qn: What is the probability that we get exactly half heads?

$|S| = 2^{100}$

S = set of all outcomes $\{H,T\}^{100}$

$\{H,T\} \times \{H,T\} \times \{H,T\} \times \cdots \times \{H,T\}$

Each sequence in S is equally likely, and hence has probability $1/|S| = 1/2^{100}$

E: event ~~that there~~ are 50 H and 50 T in the 100 coin tosses

Probability of event E = proportion of E in S= $\Pr[E] = \frac{|E|}{|S|}$

$= \frac{\binom{100}{50}}{2^{100}} \sim 0.1$

Conditional Probability & Independence

Conditional Probability

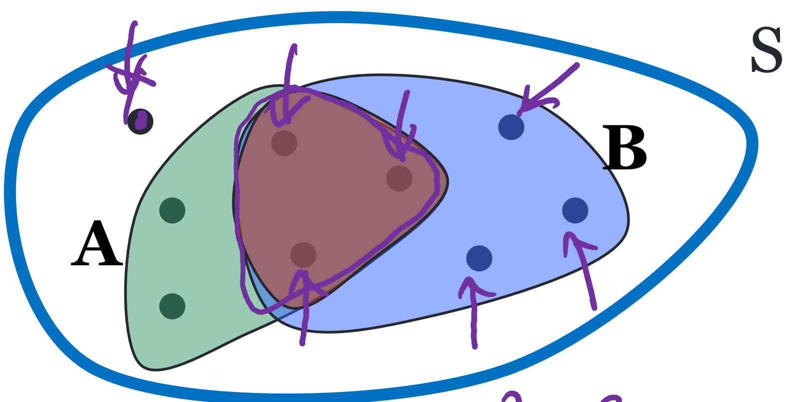
conditioned on event B having occured already

The probability of event A given event B is written as
$\Pr[A \mid B]$

conditioned on

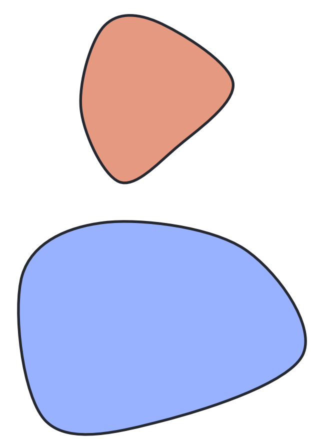
Pr[event A and event B occur]

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$



proportion of $A \cap B$

to B



If uniform, $\Pr[A \mid B] = \frac{3}{6} = \frac{1}{2}$

An Example

Qn: Suppose we roll two dice (say red and blue). What is the probability that first die is 1 given that the total is 7?

A: Event that 1st die is 1. B: Event that total is 7.

Want $\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$

A

$\Pr[A] = \frac{6}{36}$, $\Pr[B] = \frac{6}{36}$

B

$$
S = \left\{ \begin{array}{llllll}
\boxed{(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6),} \\
(2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\
(3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\
(4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\
(5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\
(6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6)
\end{array} \right\}
$$

$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$

$= \frac{1/36}{6/36} = \frac{1}{6}$

$\wedge$: logical AND $\equiv$ set intersection
$\vee$: logical OR $\equiv$ set union.

Other Simple Rules

$\bar{A}$ or A^c
$= S \setminus A$
↑ sample space

Complementary events:

complement of A

$$\Pr[\bar{A}] = 1\text{-}\ \Pr[A]\ \text{(or)}$$
$$\Pr[\bar{A}] + \Pr[A] = 1$$

$$\Pr[B|A] = \frac{\Pr[A \cap B]}{\Pr[A]}$$

$$\Pr[A \cap B] = \Pr[A]\Pr[B|A]$$

Conditioning: Given any events A & B,

$$\Pr[B] = \Pr[B \wedge \bar{A}] + \Pr[B \wedge A]$$
$$= \Pr[B \mid \bar{A}]\ \underline{\Pr[\bar{A}]} + \Pr[B|A]\ \underline{\Pr[A]}$$

If A & B are disjoint, $A \cap B = \emptyset$
events A & B are disjoint $\Pr[A \wedge B] = \Pr[A \cap B] = 0$

What are Independent Events?

A and B are independent events iff

$$\Pr[\, A \mid B \,] = \Pr[\, A \,]$$

$\Pr[A] = \Pr[A \mid B] = \dfrac{\Pr[A \cap B]}{\Pr[B]}$ independent

$\Pr[A \mid B] = \dfrac{\Pr[A \cap B]}{\Pr[B]}$

$$\Leftrightarrow$$

$$\boxed{\Pr[\, A \cap B \,] = \Pr[\, A \,]\,\Pr[\, B \,]}$$

$$\Leftrightarrow$$

$$\Pr[\, B \mid A \,] = \Pr[\, A \,]$$

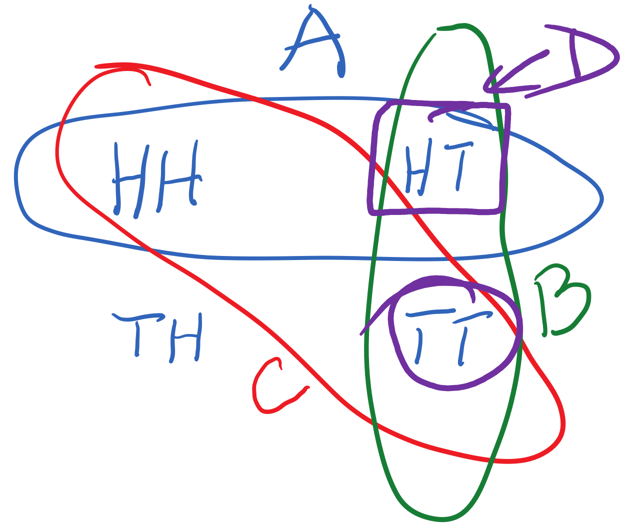
Example: Probability that for 2 different coins,
E1: 1st coin turns up heads, E2: 2nd coin turns up tails

A&B are independent

An Example

Two different coins both are fair coins.

Sample space: {HH,HT,TH,TT}



- A: 1st coin turns up heads.
- B: 2nd coin turns up tails.
- C: The two coins are equal.

$$\Pr[A] = \frac{2}{4} = \frac{1}{2} \qquad \Pr[B] = \frac{2}{4} = \frac{1}{2} \qquad \Pr[C] = \frac{2}{4} = \frac{1}{2}$$

$$\Pr[A \cap B] = \Pr[\{\mathrm{HT}\}] = \frac{1}{4} \qquad \Pr[B \cap C] = \Pr[\{\mathrm{TT}\}] = \frac{1}{4} = \Pr[B] \cdot \Pr[C]$$

A and B are independent (so too with B and C; also A and C)

What about the events $D = A \cap B$ and C? Not independent

Mutually Independent Events

$A_1, A_2, \ldots, A_n$ are mutually independent events if knowing if some of them occurred does not change the probability of any of the others occurring

every

$A_1, A_2, \ldots, A_n$ are mutually independent events iff for *any* *subset* of these n events

$$\Pr\left[A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right] = \Pr\left[A_{i_1}\right] \times \Pr\left[A_{i_2}\right] \times \ldots \times \Pr\left[A_{i_k}\right]$$

For n events, how many checks? 2^n

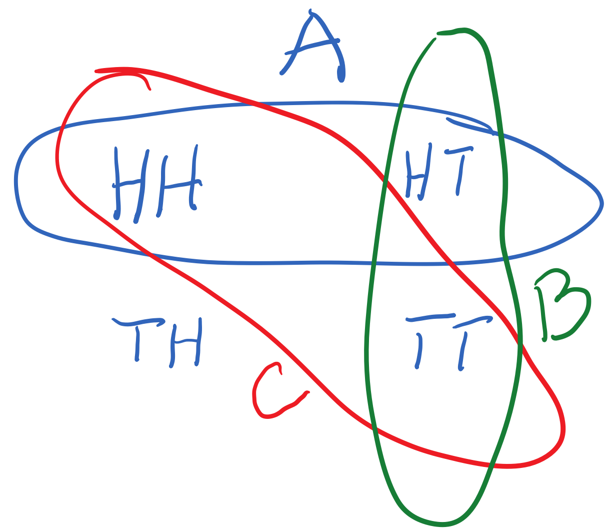
subsets

Are they Mutually Independent?

Two different coins both are fair coins.

Sample space: {HH,HT,TH,TT}

- A: 1st coin turns up heads.
- B: 2nd coin turns up tails.
- C: The two coins are equal.



$$\Pr[A] = \frac{2}{4} = \frac{1}{2} \qquad \Pr[B] = \frac{2}{4} = \frac{1}{2} \qquad \Pr[C] = \frac{2}{4} = \frac{1}{2}$$

$$\Pr[B \cap C] = \Pr[\{\mathrm{TT}\}] = \frac{1}{4} \qquad \Pr[A \cap B] = \Pr[\{\mathrm{HT}\}] = \frac{1}{4}$$

Hence they are not mutually independent!