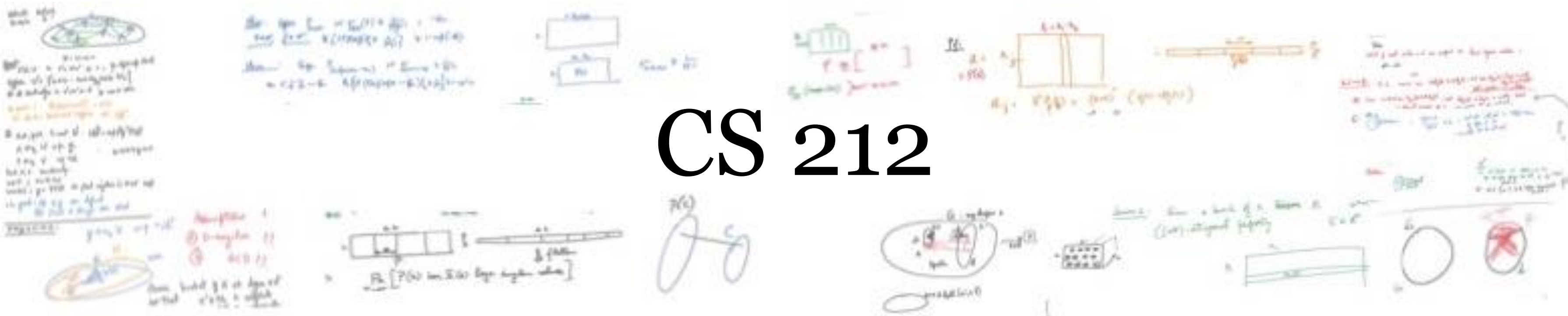


CS 212

Mathematical Foundations of Computer Science

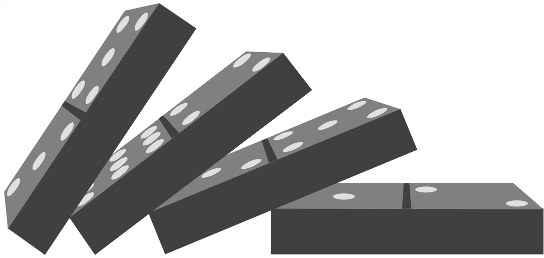
Lecture 5: Strong Induction and invariants

Announcements

- Slides will now be made available before class
- Proofs in homework should be formally written and logically correct.

Recall from last time

To prove: For all $k \in \mathbb{N}$, predicate $P(k)$ is true.



Two steps:

1. Base case: *Establish that* $P(0)$ *is true.*
2. For all $k \in \mathbb{N}$: $P(k) \implies P(k+1)$

Assume that P(k) is true. Establish that P(k+1) is true.

Making Induction Stronger

Factoring into Primes

Theorem. Every natural number $n \geq 2$ can be written as a product of primes (and powers of primes).

Proof. By induction on k. P(k) : 'k can be factored into primes'.

Base case: P(2) is true, since 2 is itself a prime.

Inductive Hypothesis (I.H): P(k) is true i.e.
k can be written as a product of primes.

Case 1: $k+1$ is prime. Then $P(k+1)$ is true.

$k + 1 = ?$

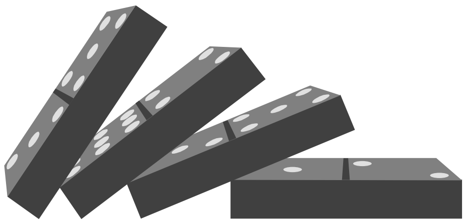
Case 2: $k+1 = ab$ where $2 \leq a, b < k$.

Hard to say anything about a, b from $P(k)$!

Strong Induction

To prove: For all $n \in \mathbb{N}$, predicate $P(n)$ is true.

Steps:



1. Base case: *Establish that* $P(0)$ *is true.*
2. Assume that $\mathrm{P}(0), P(1), \ldots, P(k)$ is true (Inductive Hypothesis)
3. Derive that $P(k+1)$ is true.

i.e. $\mathrm{P}(0), P(1), \ldots, P(k) \Rightarrow P(k+1)$

By Strong Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Fact: Can prove Principle of Strong Induction using Simple Induction (see Section 6.3 in book).

Use Strong Induction

Theorem. Every natural number $n \geq 2$ can be written as a product of primes (with repetition) i.e. powers of primes.

Pf. By strong induction on n. P(k) : 'k can be factored into primes'.
Base case: P(2) is true, since 2 is itself a prime.

I.H: Assume P(2),…,P(k) is true i.e. $2,3,\ldots k$ can all be written as a product of primes (with repetition).

Case 1: $k+1$ is prime. Then $P(k+1)$ is true

Case 2: $k+1=ab$ where $2 \leq a,b<k$. In this case $P(a)$ and $P(b)$ are true, so a and b are products of primes. It follows that $k+1$ is a product of products of primes. That is $k+1$ is a product of primes and $P(k+1)$ is true. We conclude that $P(k)$ is true for all $k \in \mathbb{N}$ by induction.

Mathematical Induction vs. Strong Induction

Base case needs to be established for both

Mathematical Induction (simple)

- Induction Hypothesis: “P(k) is true”

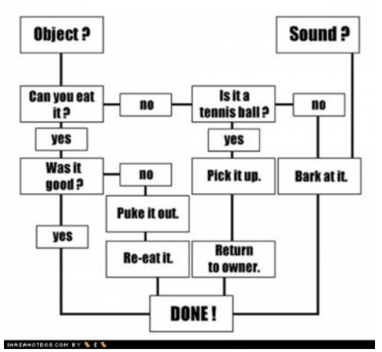
Strong Induction

- Induction Hypothesis: “P(0), P(1), P(2),…, P(k) are all true”

Invariants

Proving Invariants

- Not varying; constant. Unaffected by any operation.
- Very useful in Program Analysis esp. for Loops



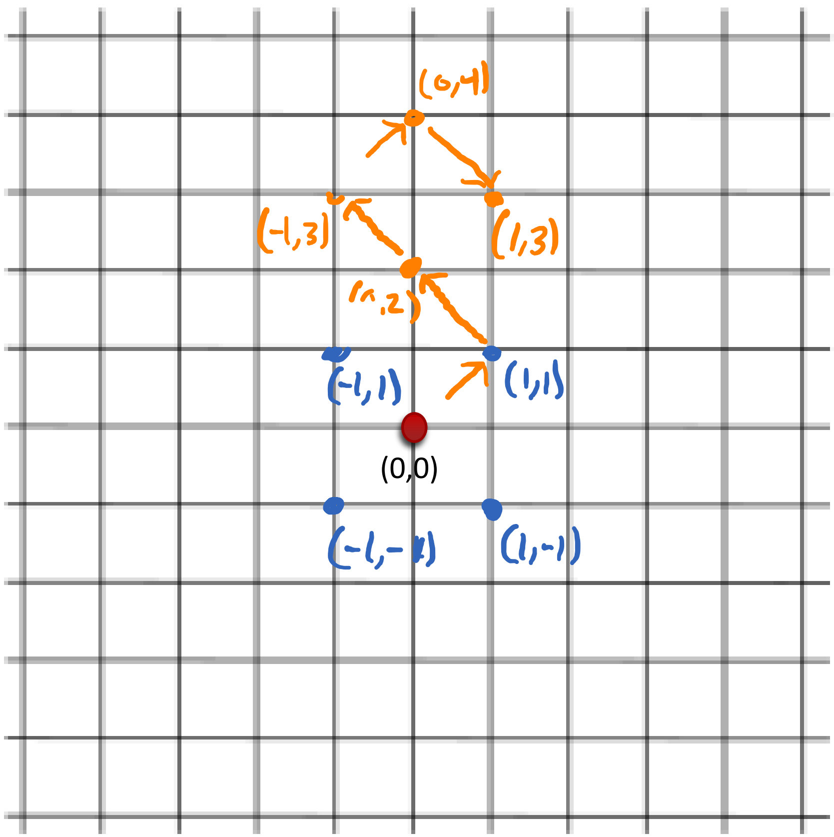
How to Prove an Invariant:

- P(k): “Invariant holds in kth step of the algorithm”
- Show using Mathematical Induction

Example: Insertion Sort Algorithm for sorting n numbers.
After first k steps, the first k numbers are sorted.

An example

- A robot on an infinite grid. At each time takes 1 step up/down, and 1 step left/right.
- E.g., after 1 step, possible locations are (1,1), (-1,1), (1,-1), (-1,-1).



Suppose start at (0,0). **Can you reach (124,-625)?**

Thm: If (x, y) can be reached, then $x + y$ is even.

Proof by induction on the number of steps k.

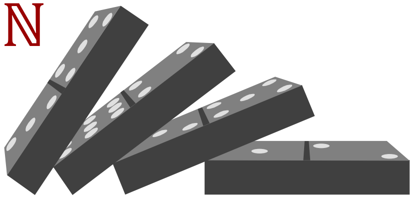
$P(k)$: After k steps, the location has co-ordinates adding up to an even number $P(0)$ is true.

Assume $P(k)$ is true. That is assume that if the robot is at (x,y) at the kth time step, then $x+y$ is even. At time $k+1$, the robot is at $(x+1,y+1)$, $(x+1,y-1)$, $(x-1,y+1)$, $(x-1,y-1)$ which have of digits $x+y+2$, $x+y$, or $x+y-2$. By assumption, $x+y$ is even, so these are all even. We conclude $P(k+1)$ is true. There Thm is true by induction

Why is Induction True?

Thinking about proof of Induction

Theorem: For any predicate $P()$ on natural numbers, suppose (a) $P(0)$ is true and (b) $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{N}$ then all $\mathrm{k} \in \mathbb{N}$, predicate $P(k)$ is true.



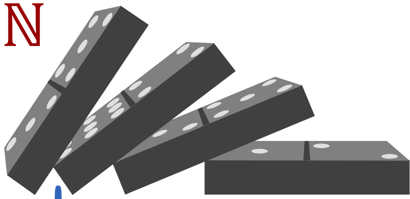
Assume towards a contradiction that $\exists$ an n s.t. $P(n)$ is not true.

To get ideas for the proof, let's assume $P(2), P(3), P(4)$ are not true but $P(k)$ true for all other k. So $P(0), P(1), P(5), P(6), \ldots$ are true. But then $P(1)$ true $\Rightarrow P(2)$ true which is a contradiction.

Note that out of 2,3,4, the smallest is 2.
So let's see if the smallest integer s.t. $P(\ )$ not true gives us a proof by contradiction

Proof of Mathematical Induction

Theorem: For any predicate $P()$ on natural numbers, suppose (a) $P(0)$ is true and (b) $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{N}$ then all $k \in \mathbb{N}$, predicate $P(k)$ is true.



Assume toward a contradiction that $\exists$ some m s.t. $P(m)$ is not true. Let $s \in \mathbb{N}$ be the smallest natural number s.t. $P(s)$ is not true. I.e. If $k \in \mathbb{N}$ and $k < s$, then $P(k)$ is true.

Case 1: $s \geq 1$: In this case, $P(s-1)$ is true so (b) implies that $P(s)$ is true which contradicts our assumption that $P(s)$ is false.

Case 2: $s = 0$: In this case $P(0)$ is true by (a) so this is a contradiction.

In either case, we arrive at a contradiction, so our assumption that $\exists$ an m s.t. $P(m)$ is false cannot be correct. Therefore $P(k)$ is true $\forall k \in \mathbb{N}$

Summary

- Principle of Mathematical Induction – proof by contradiction
- Strong Induction
- Invariants

Using induction to define mathematical objects

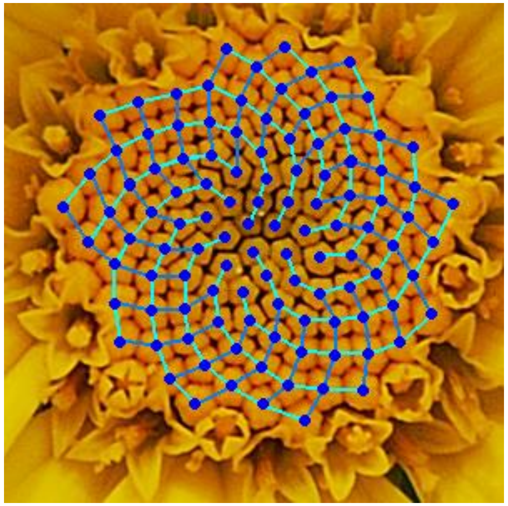
Defining Objects Inductively

- Defining and building step by step.
- Easy to prove properties using Induction!

Powers of 2: $F(0) = 1.$ $F(n) = 2 * F(n-1)$

Fibonacci numbers: $F(1) = 1; F(2) = 1; F(n) = F(n-1) + F(n-2)$

- Fibonacci numbers come up a lot in math, nature.
- Very interesting properties (proved using induction)



Yellow Chamomile

Koch Snowflake