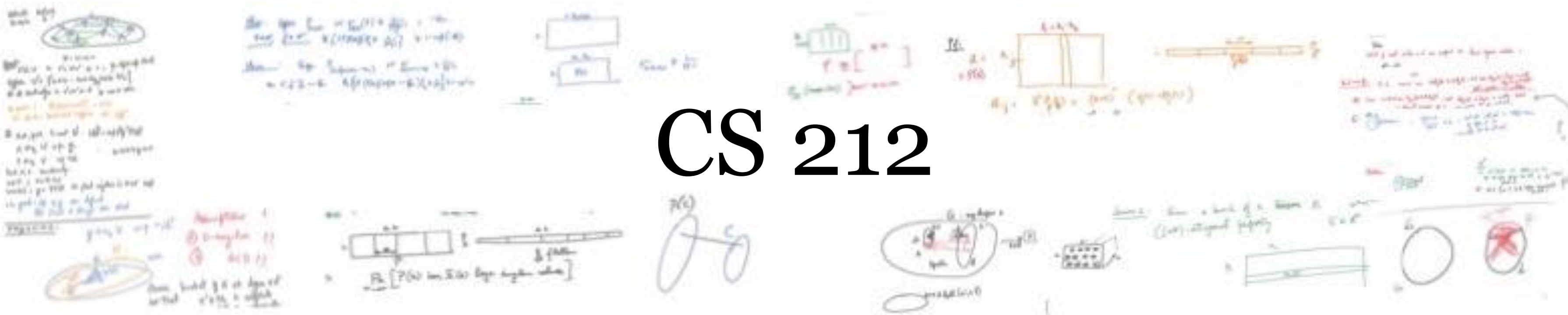


# CS 212

# Mathematical Foundations of Computer Science

## Lecture 10: More counting and the Binomial Theorem

# Recall: Counting With and Without Repetition

Number of ways of arranging n items in k positions:

1. With Repetition = $n^k$

$$\boxed{a_1}\boxed{a_2}\boxed{a_2}\boxed{a_3}\cdots\boxed{a_1}\boxed{a_3}$$

$$n \times n \times n \times n \cdots \times n \times n$$

2. Without Repetition = ${}^{n}P_{k} = n!/(n-k)!$

$$\boxed{a_2}\boxed{a_1}\boxed{a_3}\cdots\boxed{a_k}$$

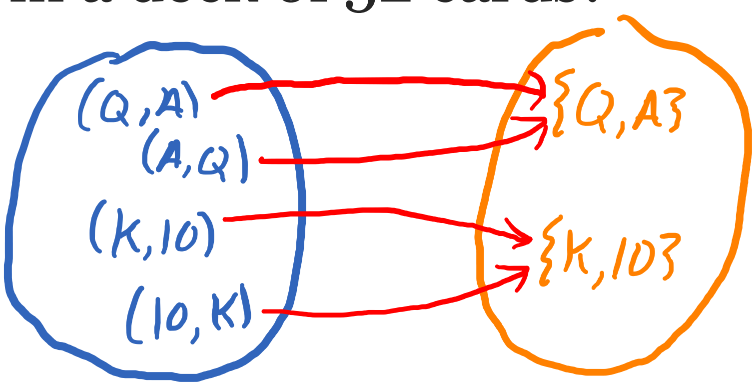
$$n \times n-1 \times n-2 \quad \times n-(k-1)$$

# Combinations: Choosing Items

How many ways of picking an ordered pair from a deck of 52 cards? $52 \times 51$

How many ways of *selecting a pair* i.e. picking an unordered pair from a deck of 52 cards?

Ordered $\xrightarrow{\text{2-to-1}}$ Unordered




$|\text{Ordered}| = 2\,|\text{Unordered}|$

$|\text{Unordered}| = \frac{51 \cdot 52}{2}$

Overcounting: If each configuration is counted *exactly* $m$ times, then the number of configurations $= \dfrac{\text{total count}}{m}$

Why? (remember m to 1 maps)!

# Combinations: Choosing Items

How many ways of selecting $k$ items from $n$ items?

1. Put $k$ out of the $n$ items in $k$ positions
2. Account for overcounting (how much?)

Select 3 items: (1,2,3), (1,3,2), (2,1,3)
(2,3,1), (3,1,2), (3,2,1) $\xrightarrow[3!-1]{6-1}$ {1,2,3}

Select k items: Permutations (1,2, ..., k) $\xrightarrow{k!\text{ to-1}}$ {1,2, ..., k}

**Combinations:** Number of ways of choosing/selecting k out of $n$ items is ${}^nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{{}^nP_k}{k!}$

# Dealing with Identical Objects

How many ways of "dividing n identical objects into k parts"?

Hint: Think ….

1. Denote the n objects by 0s.
2. Denote partition into k regions with 1s between regions.

Donuts: $\underbrace{000}_{\text{type 1}}1\underbrace{00}_{\text{type 2}}1\underbrace{00}_{\text{type 3}}1\underbrace{000}_{\text{type 4}}1\underbrace{00}_{\text{type 5}}$

What is the # of bit sequences with exactly n zeros and k-1 ones.

We have n+k-1 positions. Choose where ones go

$$= \binom{n+k-1}{k-1} = \binom{n+k-1}{n} \qquad \binom{16}{4} = 1820$$

# Approximations for n choose k

How big is $\binom{n}{k}$ ? $= \dfrac{n!}{(n-k)!\,k!} = \dfrac{n(n-1)\cdots(n-k+2)(n-k+1)}{k(k-1)(k-2)\cdots(2)(1)}$

$\dfrac{\text{poly of deg } n^k}{\text{constant}}$

$= \underbrace{\dfrac{n}{k}\,\dfrac{n-1}{k-1}\,\dfrac{n-2}{k-2}\cdots\dfrac{n-k+2}{2}\,\dfrac{n-k+1}{1}}_{\text{poly of deg } n^k \,/\, \text{constant}} = \prod_{i=0}^{k-1}\dfrac{n-i}{k-i}$

$$\frac{n}{k} \le \frac{n-i}{k-i} \le n$$

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le n^k$$

$\binom{n}{k} = \Theta(n^k)$

when $k$ is a constant

# Counting Formulae

1. Arranging $n$ objects in $k$ positions (without repetition): ${}^{n}P_k = \frac{n!}{(n-k)!}$

2. Filling $k$ positions with $n$ objects (with repetition): $n^k$

3. Selecting $k$ out of $n$ objects (no ordering, no repetition): $\binom{n}{k}$

4. Selecting $n$ identical objects in $k$ different bins: $\binom{n+k-1}{k-1}$

Put one object in each bin then sort $n-k$ into $k$ bins

5. Selecting $n$ identical objects in $k$ different bins : with none being empty

$\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$

# Counting two ways

# Counting subsets two ways

Suppose we have a set X={1,2,..,n}.

How many subsets of X of size k? $\binom{n}{k}$

What is the number of all possible subsets of X? $|P(X)| = 2^n$

$\sum_{k=0}^{n}$ # subsets with $k$ elements

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

$|A \cup B| = |A| + |B|$
when $A \cap B = \emptyset$

# Proofs from Counting

**Thm.** For $n, k \in \mathbb{N},\ 1 \leq k \leq n$: $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

Proof By Induction: Exercise…

Cleverer Proof: What is $\binom{n+1}{k}$?

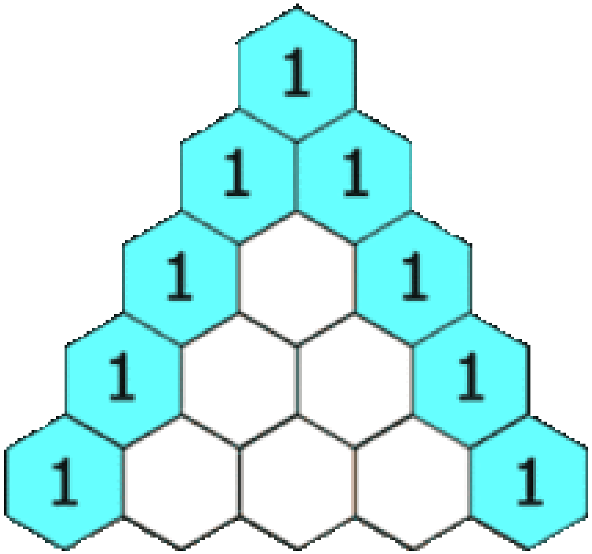
$= \binom{n+1}{k}$

From $n+1$ items $a_1, a_2, \ldots, a_{n+1}$, number of ways we can select $k$ of them

Case 1: $a_1$ is chosen. Need to chose $k-1$ out of remaining $n$ $= \binom{n}{k-1}$

Case 2: $a_1$ is not chosen. Need to chose $k$ out of my remaining $n$ $= \binom{n}{k}$

$\binom{n+1}{k}$ = Total $= \binom{n}{k-1} + \binom{n}{k}$




# Binomial Theorem

# How do you expand $(a+b)^n$ ?

$(a+b)^0 = 1$

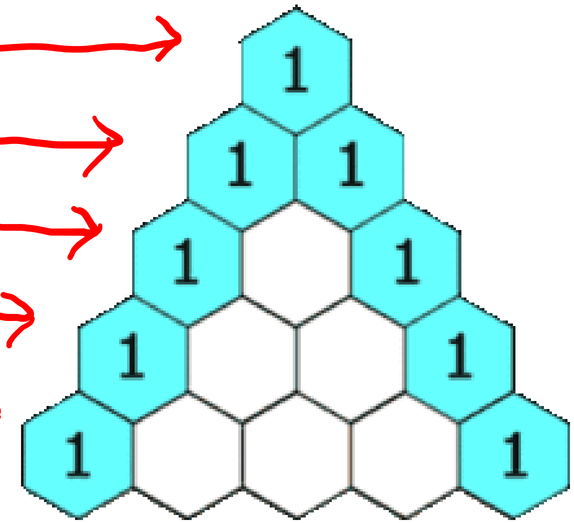
$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^n =$




## Simple observations:

1. Number of terms = n+1.

2. The total degree of each term=degree(a)+degree(b)= n (homogenous)

3. The co-efficient increases towards middle and decreases again

# Binomial Theorem

$x = \frac{b}{n}$ multiply by $a^n$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

$$= a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{i} a^{n-i} b^i + \cdots + \binom{n}{1} a b^{n-1} + b^n$$

Recall: $\binom{n}{i} = \binom{n}{n-i}$

Different forms:

$a = 1 \quad b = x$

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$(1-x)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i x^i$$

# Binomial Theorem

$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= a^3 + a^2b + aba + ab^2 + baa + bab + bba + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3 = \binom{3}{3}a^3 + \binom{3}{2}a^2b + \ldots$$

aba

$b^2a$

Proof: $(a+b)^n = (a+b)(a+b)(a+b)\ldots(a+b)$

$$= \binom{n}{n}a^n + \binom{n}{n-1}a^{n-1}b + \cdots + \binom{n}{n-i}a^{n-i}b^i + \cdots + \binom{n}{1}ab^{n-1} + \binom{n}{0}b^n$$

$(x+1)^{n+1} = (x+1)^n(x+1)$

Use $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ with induction

Alternate Proof:

# Identities Using Binomial Theorem

$\binom{n}{k}$= co-efficient of $x^{\mathrm{k}}$ in $(1+x)^n$ $= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots$
$+ \binom{n}{n-1}x^{n-1}$
$+ \binom{n}{n}x^n$

Fact: $2^n = 1 + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n-1} + \binom{n}{n}$

Proof: Plug in $x=1$

$$\binom{n}{0} + \binom{n}{2} + \cdots \textit{even terms} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots \textit{odd terms} + \cdots$$

Substitute $x = 1$ in $(1-x)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i x^i$

$$0 = (1-x)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i$$