CS 212 Homework 6

Due 11:59PM on Tuesday, November 15, 2022.

Each question carries 5 points. Recall that every question here requires a proof. You can discuss the questions in groups of 3 (please list other students you discussed the problems with). Please remember that the assignment solutions need to be submitted individually and not as a group (and written/ typset with no collaboration). Solutions are only accepted in PDF format. Also please be as clear and legible as possible; any step that is ambiguous because of unclear writing will be interpreted as a mistake. See the Canvas page for instructions on how to submit.

Problem 1 (5 points)

Let $G(V, E)$ be a simple undirected graph. Define the distance $d(T_1, T_2)$ as the number of edges in T_1 that do not belong to T_2 i.e., $d(T_1, T_2) = |E(T_1)| - |E(T_1) \cap E(T_2)|$. Let T, T' be two **spanning** trees such that $d(T, T') = k \geq 1$.

(i) Prove that there exists two edges $e, f \in E$ such that the following hold simultaneously: (a) $T - e + f$ is a **spanning** tree, (b) $d(T, T - e + f) = 1$, (c) $d(T - e + f, T') = d(T, T') - 1$.

(ii) Suppose $d(T, T') = k$. Use the previous part to prove that there exists a sequence of **spanning** trees $T'_1, T'_2, \ldots, T'_{k-1}$ such that $d(T'_i, T) = i$ and $d(T'_i, T') = k - i$ for all $1 \leq i \leq k - 1$.

Definition (Spanning tree): A spanning tree T of an undirected graph G is a subgraph that is a tree which includes all of the vertices of G.

Problem 2 (5 points)

Show that any bipartite graph that is d-regular (i.e. every vertex has exactly d edges incident on it) has a perfect matching.

Problem 3 (5 points)

Let G be a simple undirected graph with $n \geq 11$ vertices, show that either G or its graph complement $\overline{G}$ is not planar.

Hint: You can use the following theorem without proof in this problem:

Any simple planar graph with m edges and n vertices where $n \geq 3$ satisfies $m \leq 3n - 6$.

Problem 4 (5 points)

Suppose we are given a simple graph $G = (V, E)$ on n vertices, and $\chi(G)$ be the minimum number of colors needed to color the graph. For a vertex $u \in V$, the neighborhood $N(u) = \{v : (u, v) \in E\}$. (Note that $u \notin N(u)$ since the graph is simple.)

(a) Show that $\chi(G) \leq \Delta + 1$.
Hint: *To show this consider the colors* $\{1, 2, \ldots, k\}$ *where* $k = \Delta + 1$. *Consider the vertices one-by-one in (an arbitrary) order, and assign to vertex* u, *the smallest available color that hasn't been assigned yet to one of its neighbors.* **(2 points)**

(b) Consider the following polynomial time algorithm. Given a graph 3-colorable graph H:

- if every vertex in $H(V, E)$ has degree $< \sqrt{n}$, then we use the algorithm in part (a).
- Otherwise there is some vertex, say u, of degree $deg(u) \geq \sqrt{n}$. Since G is 3-colorable, the neighborhood of u denoted by $N(u)$ (remember $N(u) \subseteq V$) is 2-colorable. Hence we use a new color for u and two different colors for the neighbors of u. Then we remove these vertices $V(H') = V \setminus N(u) \cup \{u\}$ and recurse on H'.

Show that this algorithm colors a 3-colorable graph using at most $O(\sqrt{n})$ colors.
Note: *The graph* H *is* 3-*colorable, hence there exists a* 3-*coloring, but it may be hard to find such a* 3 *coloring. The goal of the exercise is to show that the above simple, efficient algorithm finds a coloring that uses* $O(\sqrt{n})$ *colors.* **(2 points)**