CS 212 Homework 1

Each question carries 5 points. Recall that every question here requires a proof. You can discuss the questions in groups of 3 (please list other students you discussed the problems with). Please remember that the assignment solutions need to be submitted individually and not as a group (and written/ typset with no collaboration). Solutions are only accepted in PDF format. Also please be as clear and legible as possible; any step that is ambiguous because of unclear writing will be interpreted as a mistake. See the Canvas page for instructions on how to submit.

Problem 1 Let a and b be integers and assume there is a $n \geq 2$ that divides both a and b. That is, both a/n and b/n are integers. Are there integers x, y such that

$$ax + by = 1?$$

Hint: Try a proof by contradiction. Note that a, b, x and y can be negative.

Problem 2 A certain recursive algorithm FindBestAllocation has the following form, given an input arr of size n:

```
FindBestAllocation(arr[], n)
{
    if (n == 0)
     return;
    for (i = 0, ..., n − 1)
     FindBestAllocation(arr[], n − 1);
     something;
    something else;
}
```

Let $T(n)$ be the maximum number of steps taken by the algorithm on any input of size n (this is often called the running time of the algorithm). Suppose $T(0) = 0$ and that $T(n)$ is captured by the recurrence relationship

$$T(n) = nT(n-1) + 1$$

for all positive integers n. Prove that for all positive integers $n \geq 1$,

$$T(n) = \sum_{m=1}^{n} \frac{n!}{m!}.$$

Hint: Note that the code is meant to provide some motivation for analyzing algorithm run-times through induction. The details of the pseudocode are not important for this problem.

Problem 3 Prove that for any real number $x \geq 0$ and any integer $n > 0$ one has

$$(1 + 3x)^{2n} > \frac{2n}{2n + 1} + 6nx.$$

Hint: You may need to pick a stronger predicate/ induction statement. Try examples with small values of n to guess the correct induction statement.

Problem 4 Consider a local area network consisting of $2n$ computers where $n \geq 3$ is an odd integer. Each computer is directly wired to three other computers in the network. Due to an unfortunate bug in the network, anytime one computer is manually turned on or off, all three directly connected computers also switch their power state (i.e. on computers switch off and off computers switch on). Suppose the network starts with n computers being on and n computers being off. Is it possible to turn on all of the computers?

Hint: Try to identify an invariant that you can use for a proof by induction.