CS 212 Homework 5

Due 11:59PM on Tuesday, November 8, 2022.

Each question carries 5 points. Recall that every question here requires a proof. You can discuss the questions in groups of 3 (please list other students you discussed the problems with). Please remember that the assignment solutions need to be submitted individually and not as a group (and written/ typset with no collaboration). Solutions are only accepted in PDF format. Also please be as clear and legible as possible; any step that is ambiguous because of unclear writing will be interpreted as a mistake. See the Canvas page for instructions on how to submit.

Problem 1

A random undirected graph on n vertices is built as follows: for each pair of vertices (i, j) with $i < j$, toss a fair coin (turns up heads with probability 1/2); if the outcome is heads, add an edge between i and j; otherwise, there is no edge between i and j. Let X be the number of edges in the graph. Note that X is a random variable: a function of the coin tosses used to construct the graph.

Prove that with probability at least 96%, we have

$$\tfrac{1}{2}\binom{n}{2} - \tfrac{5}{2}\sqrt{\binom{n}{2}} \leq X \leq \tfrac{1}{2}\binom{n}{2} + \tfrac{5}{2}\sqrt{\binom{n}{2}}.$$

Hint: First compute the mean and variance of X.

Problem 2

Let $G(V, E)$ be a simple undirected graph. A closed walk in an graph $G(V, E)$ from u to u is any sequence of vertices $(u = v_0, v_1, v_2, \ldots, v_{k-1}, u = v_k)$ (for some $k \in \mathbb{N}$) where $v_0, v_1, v_2, \ldots, v_{k-1}, v_k \in V$ (some could be repeated) and there is an edge between every consecutive pair of vertices (i.e. $\forall i \in \{1, 2, \ldots, k\}$, $(v_{i-1}, v_i) \in E$). Further, we require that every pair of consecutive edges in the closed walk are not the same (in other words, $\forall i \in \{2, \ldots, k\}$, we have $v_i \neq v_{i-2}$). Note however that some of the vertices or edges in the walk can be repeated. If $k \geq 3$ and $v_0, v_1, v_2, \ldots, v_{k-1}$ are distinct vertices , it is a simple cycle. Show that if the graph G has a closed walk, then G also has a simple cycle.

Problem 3

There are n students in a class, and the number of pairs of friends (edges) in the class is m.

a. What is the average degree of the graph G representing these friendships? **(1 point)**

b. Prove that there is a non-empty subset of students in which each of them know have at least $\frac{m}{n}$ friends among themselves.

Hint: When you model this problem using a graph, you are looking for a subgraph H with minimum degree at least $\frac{m}{n}$, where the degree is measured w.r.t H. Think about the vertices of degree $< \frac{m}{n}$, and see what happens when you remove them one at a time.

Problem 4

Given an undirected graph $G = (V, E)$, consider $\overline{G} = (V', E')$ defined as follows: $V' = V$, $(u, v) \in E'$ iff $(u, v) \notin E$. That is, $\overline{G}$ has the same vertices as G, but two vertices in $\overline{G}$ has an edge between them if and only if there is no edge between them in G.

Show that for any undirected graph G, either G or $\overline{G}$ is connected.