CS 212 Homework 3

Due 11:59PM on Tuesday, October 18, 2022.

Each question carries 5 points. Recall that every question here requires a proof. You can discuss the questions in groups of 3 (please list other students you discussed the problems with). Please remember that the assignment solutions need to be submitted individually and not as a group (and written/ typset with no collaboration). Solutions are only accepted in PDF format. Also please be as clear and legible as possible; any step that is ambiguous because of unclear writing will be interpreted as a mistake. See the Canvas page for instructions on how to submit.

Problem 1

There are n tennis players in Evanston. Two players are called tennis buddies if they have play at least one singles game with each other. Show that out of these n tennis players, there are at least two tennis players who have the same number of tennis buddies (in Evanston).

Problem 2

There are n families at a party, each family has exactly 2 adults and 1 child. They are to be seated in a circular table with $3n$ chairs.

(a) How many ways are there of seating all the guests at the table?

(b) How many ways can the guests be seated so that each family is seated together (adjacent), with the child in between the two parents?

Note: A circular table is different from a bench in the following sense: any cyclic rotation of a configuration gives rise to the same configuration.

Problem 3

The Boltzmann's equation $S(t) = k_B \log W(t)$ is a fundamental equation in Statistical Mechanics that relates the entropy $S(t)$ to $W(t)$, which is the number of microstates that constitute a state/configuration t. Here $k_B > 0$ is a universal constant called the Boltzmann constant. In this problem, you will use counting (and the Boltzmann equation) to identify the most stable configuration for a collection of n particles.

Suppose n is even, and we have n different particles, each of which can have one of two spins: *"up" spin* or *"down" spin*. The state of the system is the number of particles t that have spin "up". The number of microstates $W(t)$ of a state t corresponds to the number of ways in which exactly t out of the n particles have *spin "up"*.

(i) Give an expression for the number of microstates $W(t)$ corresponding to the state t. What is the number of microstates for the state $n-t$? (**1 point**)

(ii) Prove that the state t that has the maximum number of microstates is $t = n/2$ i.e. the value of t that maximizes " the number of ways in which exactly t out of n particles have state *up* " is $t = n/2$. (**2 points**)
Hint: *How does the number of microstates in state i compare to the number of microstates in state $(i-1)$?*

(iii) Prove that for the most stable state (the state with maximum number of microstates[1]), the number of microstates $W(t) \geq 2^n/(n+1)$. Then, use the Boltzmann equation to prove that the entropy of the most stable state is at least $k_B(n - \log(n+1))$. (**2 points**)
Hint: *What is the value of* $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n}$*?*

Problem 4

Prove that

$$\binom{2n}{4} = 2\binom{n}{4} + 2n \times \binom{n}{3} + \binom{n}{2} \times \binom{n}{2}.$$

using the following two approaches:

1. Prove this using a counting argument (counting the same quantity in two different ways).

2. Give an alternate proof using the Binomial Theorem.

[1] in the previous part you showed this was $t = n/2$