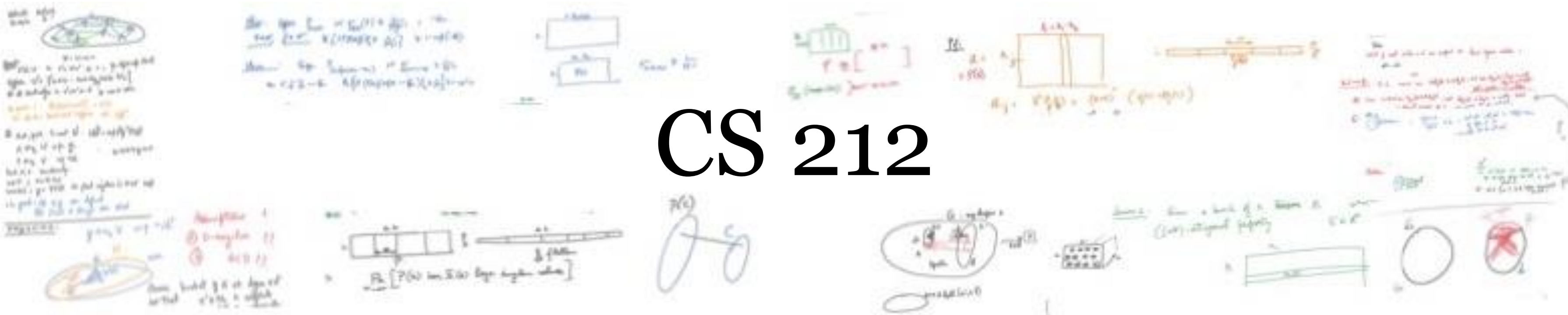


CS 212

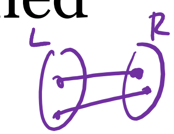
Mathematical Foundations of Computer Science

Lecture 22: Matchings

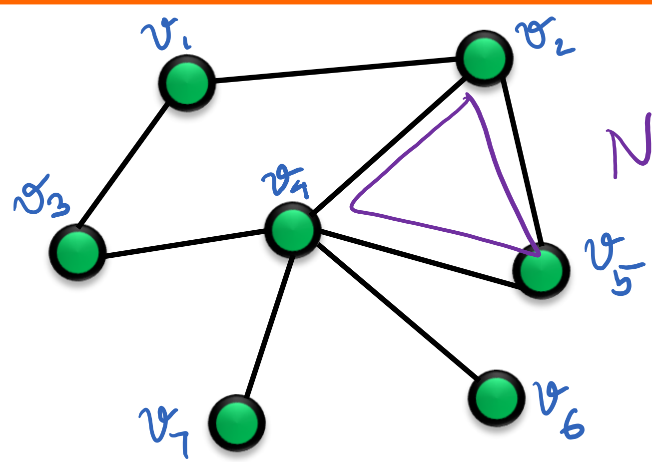
Recap: Characterization of Bipartite Graphs

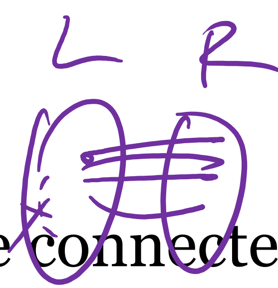
Vertex set

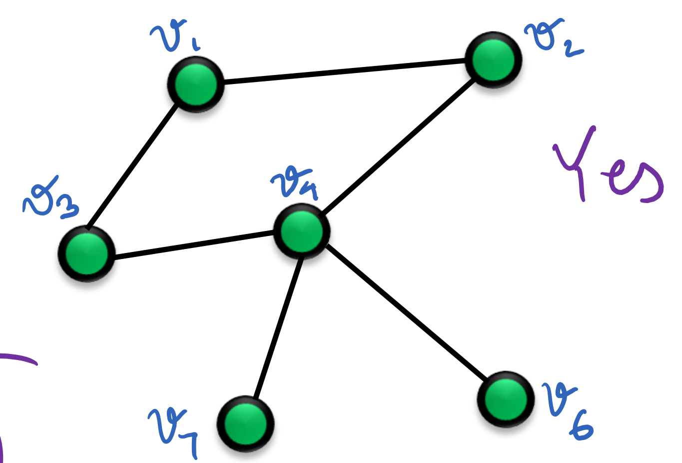
Bipartite graph: 2-colorable graphs i.e., V can be partitioned into L and R, where every edges goes between L and R.



Theorem: A graph $G(V, E)$ is bipartite iff there is G has no odd cycle.

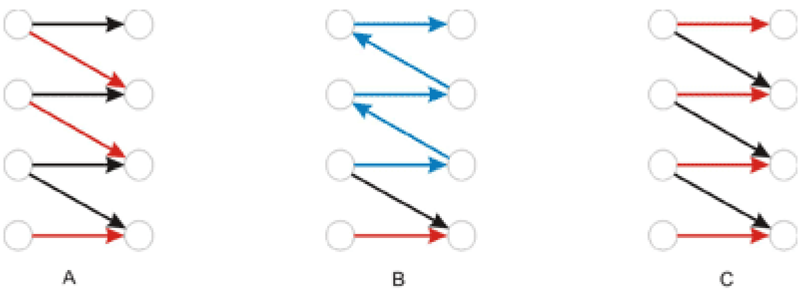






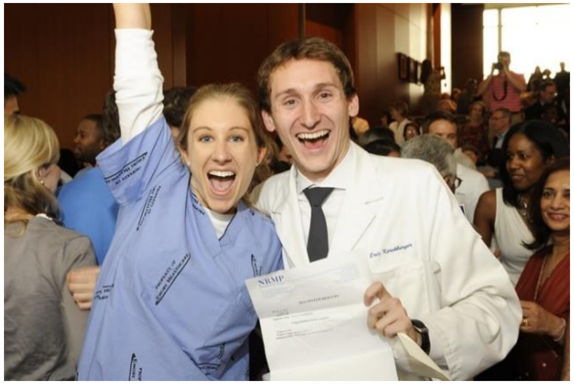
Note: In a graph G, if the connected components are bipartite, then G is bipartite.

Graph Matching

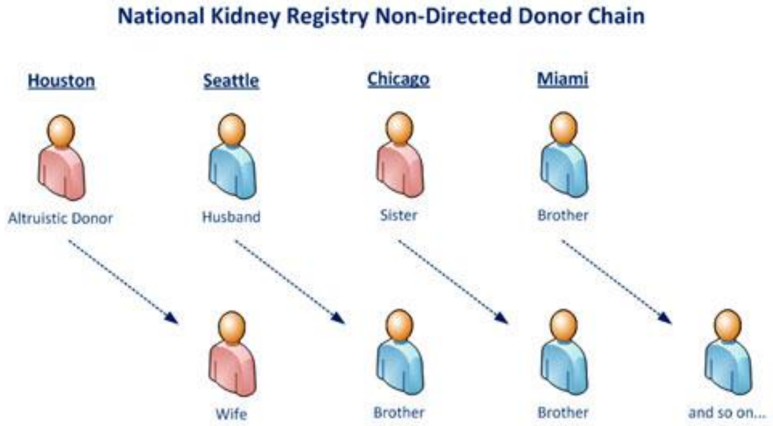


Algorithmic subroutine

Matching couples



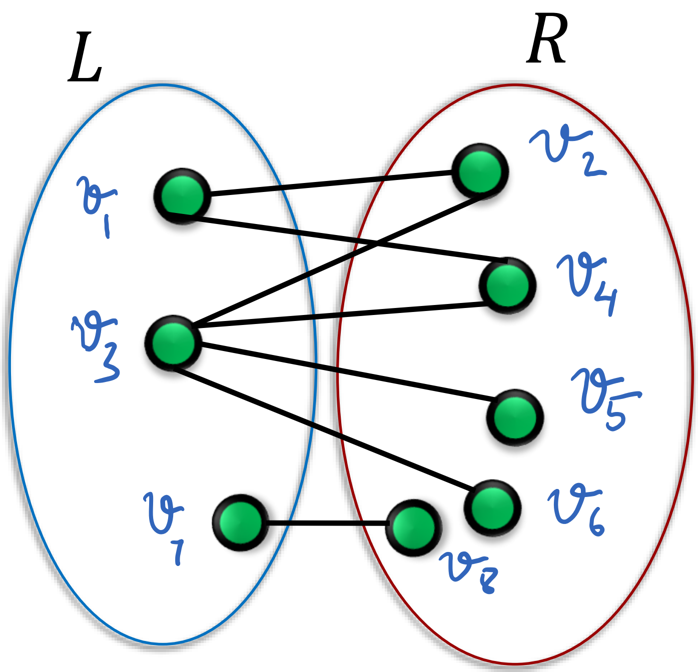
Residency matching



Kidney exchanges

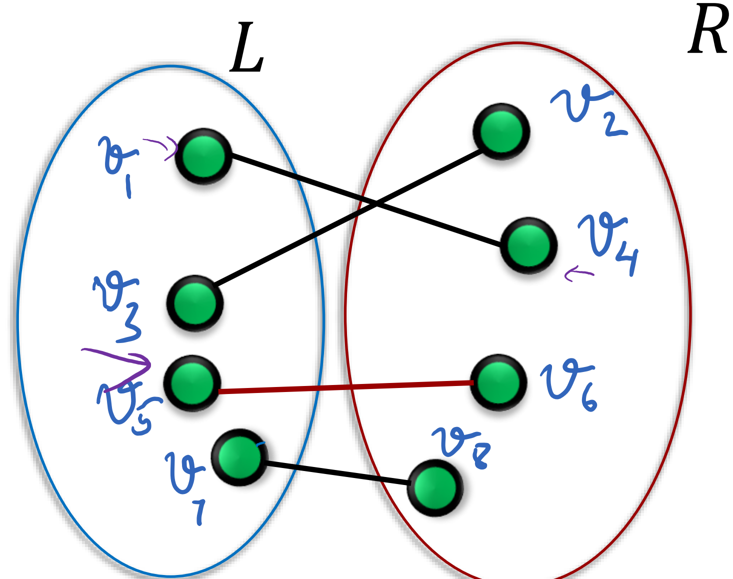
Eg: High School Dance Matching

A group of 100 boys and girls attend a formal dance. Every boy knows 5 girls, and every girl knows 5 boys. Can they be matched into dance partners so that each pair knows each other?



Matchings, Perfect Matchings

Matching: A set of edges, no two of which share a vertex.



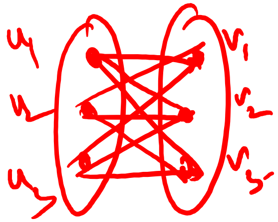
Matching saturating L (or R):

A matching that touches every vertex of L (or R respectively).

Perfect Matching: A matching is perfect if it includes every vertex i.e., saturates both L and R (it is necessary that |L|=|R| for a perfect matching to exist).

$S = \{u_1\}$

$N(S) = \{u_1, u_3, u_4\}$



$S = \{u_2, u_3\}$

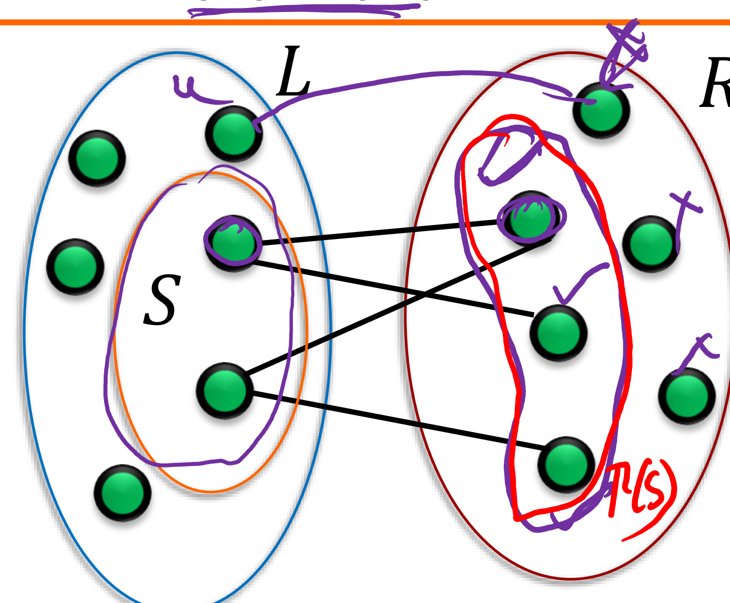
$N(S) = \{v_1, v_2, v_3\}$

Hall's Theorem

Neighbors of S: $N(S)$= set of nodes that has an edge with at least one vertex in S $N(S)$: neighborhood of S in the entire graph G edges: E

Theorem. Bipartite graph $G = (V = (L, R), E)$ has a perfect matching iff (i) $\forall S \subseteq L, |N(S)| \geq |S|$, and (ii) $|L| = |R|$

Hall's Criterion: Every set of k vertices in L have at least k neighbors in R (for all k)

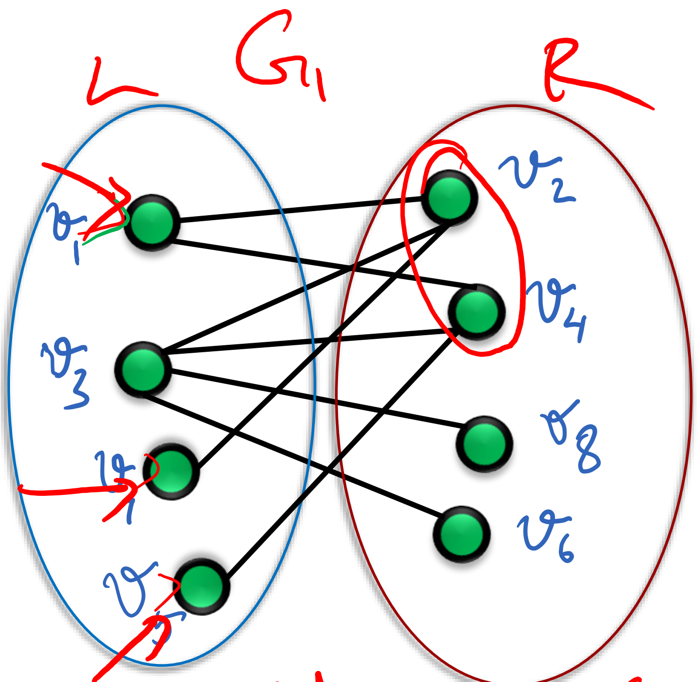


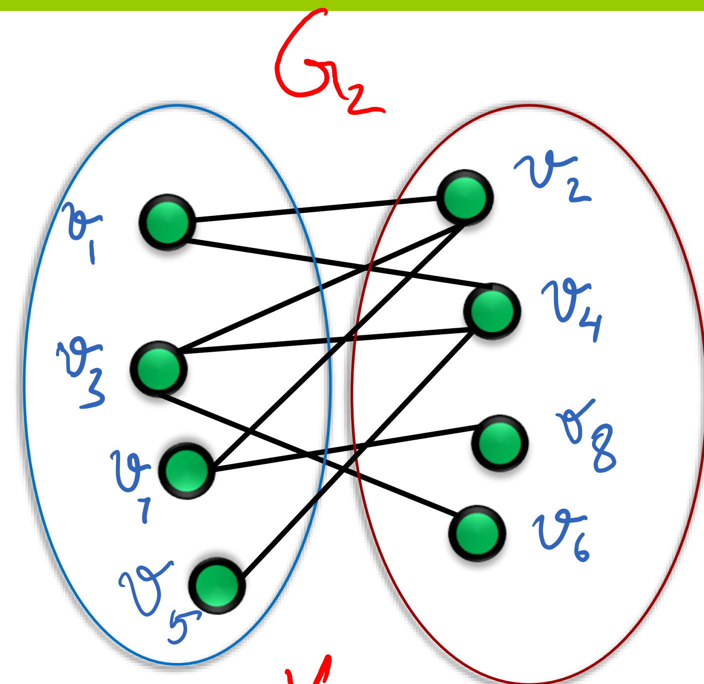
When no perfect matching?

- Either if for some $\underset{}{S}^{\subseteq L}$, $|N(S)| < |S|$ "bottleneck"
- Or if $|L| \neq |R|$

If Hall's criterion holds (but $|L| \neq |R|$) there is a matching saturating L.

Is there a perfect matching?





No.

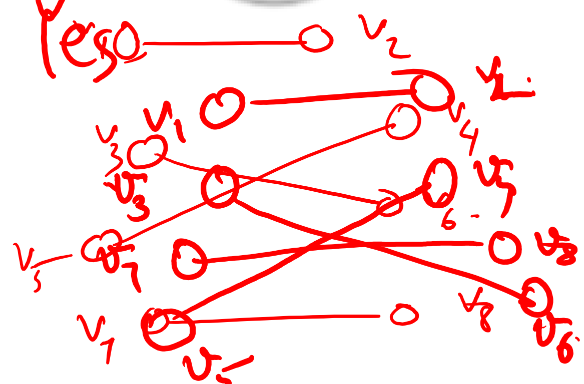
$S = \{v_6, v_8\}$

$N(S) = \{v_3\}$

$S = \{v_1, v_7, v_5\}$

$N(S) = \{v_2, v_4\}$

Yes

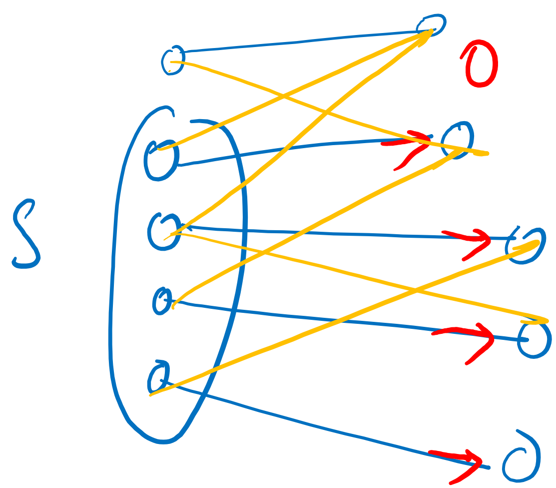


Proof of Hall's Theorem

Theorem. Bipartite graph $G(V = (L, R), E)$ has a matching saturating L iff for any subset $S \subseteq L$, $|N(S)| \geq |S|$, ~~and |L| = |R|~~.

Easy direction. Matching saturating L implies:
for any subset $S \subseteq L$, there are at least $|S|$ nodes of R connected to at least one node in S.

— : edges of matching



$|N(S)| \geq |S|$

Proof of Hall's Theorem

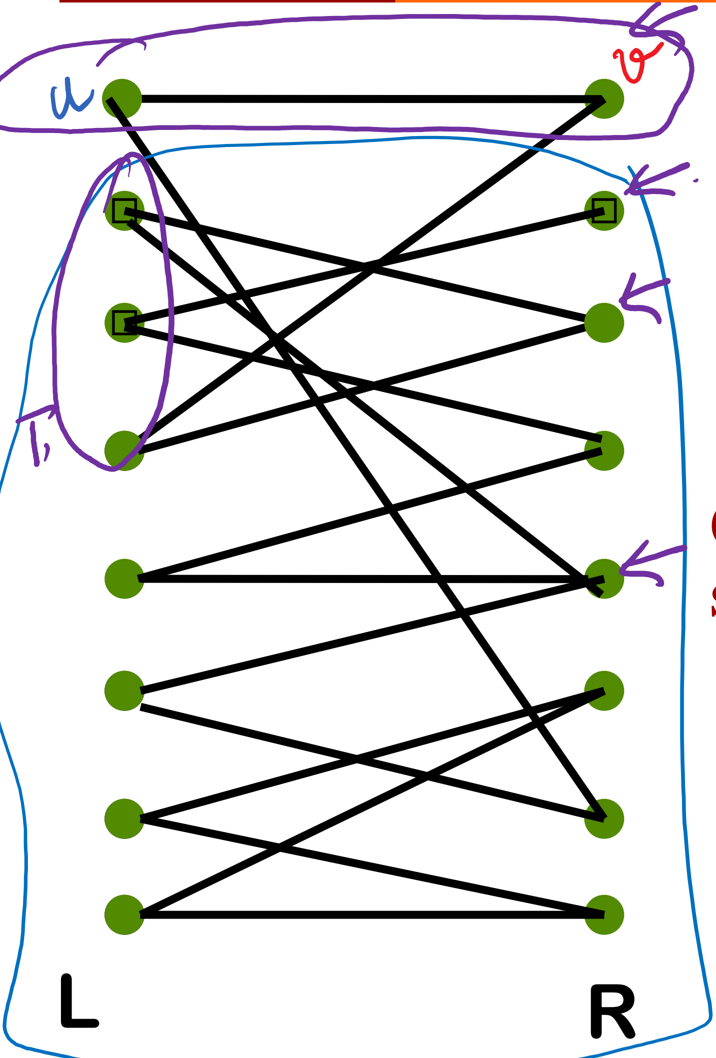
Theorem. Bipartite graph $G(V = (L, R), E)$ has a ~~perfect~~ matching saturating L iff for every subset $S \subseteq L$, $|N(S)| \geq |S|$ ~~and |L| = |R|~~

Proof: Strong induction on the size of L i.e., n. (Base case n=1 is easy)
I.H: Theorem is true for every bipartite graph G satisfying the conditions on < n vertices.
As with most inductive proofs, we'll try to reduce it to an instance of smaller size.

We'll try to remove some vertices in L, R so that the graph on remaining vertices satisfies Hall's condition.

Case 1: Strictly larger neighborhoods

$\forall S \; N(S) \geq |S|$



Suppose for every S s.t. $|S| \leq n-1$, we have $|N_G(S)| \geq |S| + 1$

Let $u \in L$ be any vertex on the left. Match it to one of its neighbors $v \in R$, remove both and continue.

Claim: Induced graph H on $(L-u, R-v)$ satisfies Hall's condition.

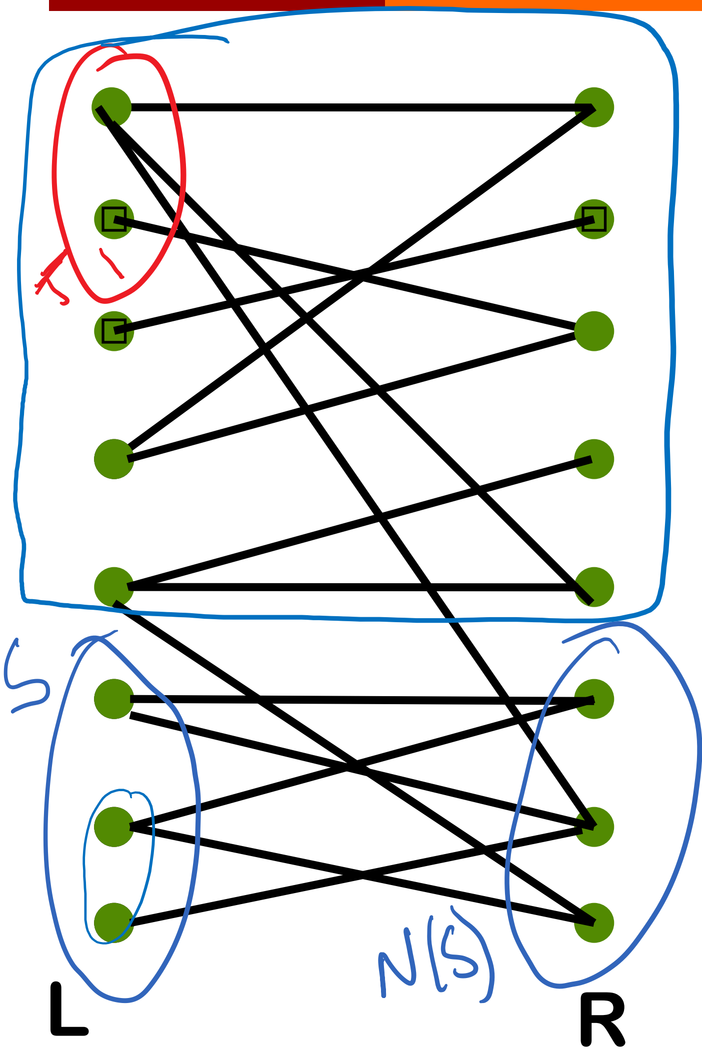
Pf: Consider $T \subseteq L \setminus \{u\}$

$|N_G(T)| \geq |T| + 1$

$|N_H(T)| = |N_G(T) \setminus \{v\}| \geq |N_G(T)| - 1$

$\geq (|T|+1) - 1 \geq |T|$.

Case 2: Exists S where $|N(S)| = |S|$



Take any S with $|S| \leq n - 1$ s.t. $|N(S)| = |S|$. Match vertices in S to $N(S)$. Why?

Claim: Induced graph H on $(L \setminus S, R \setminus N(S))$ satisfies Hall's condition.

Take any $T \subseteq L \setminus \text{S}$. We will show that $|\text{N}_\text{H}(\text{T})| \geq |N_G(S \cup T) \setminus \text{N}_G(S)| \geq |T|$.

Hall's Marriage Theorem

Neighbors of S: $N(S)$= set of nodes that has an edge with at least one vertex in S

Theorem. Bipartite graph $\mathrm{G}(V = (L, R), E)$ has a perfect matching iff $|L| = |R| = n$ and for any subset $S \subseteq L$, $|N(S)| \geq |S|$.

- The condition of the theorem still holds if we swap roles of L, R

Theorem. Bipartite graph $\mathrm{G}(V = (L, R), E)$ has a matching saturating L iff for any subset $S \subseteq L$, $|N(S)| \geq |S|$.

(similar statement for matching saturating R)

Thank you!