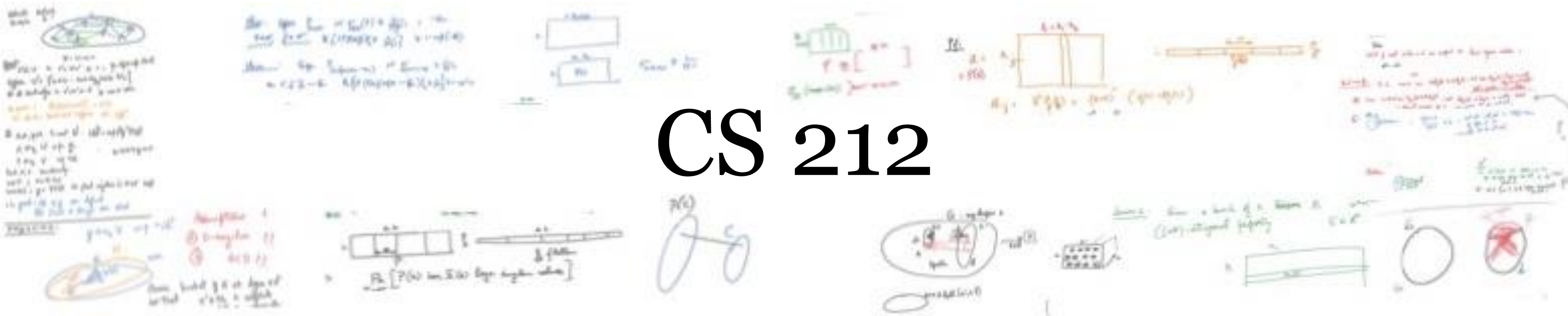


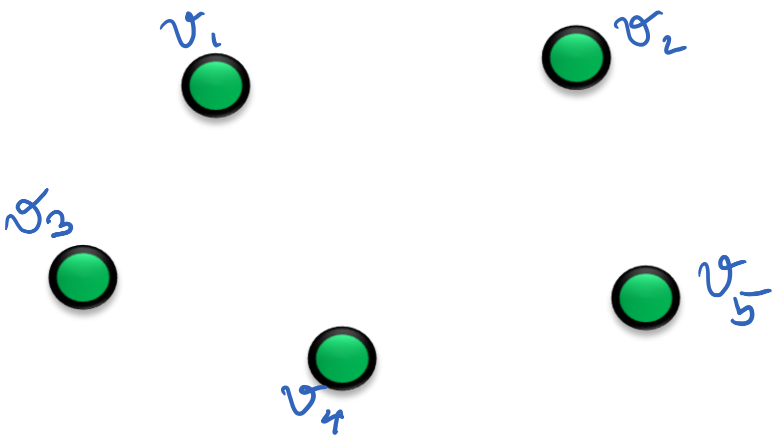
CS 212

Mathematical Foundations of Computer Science

Lecture 19: Graph Complements and Colorings

Independent Sets, Cliques, Graph Complements

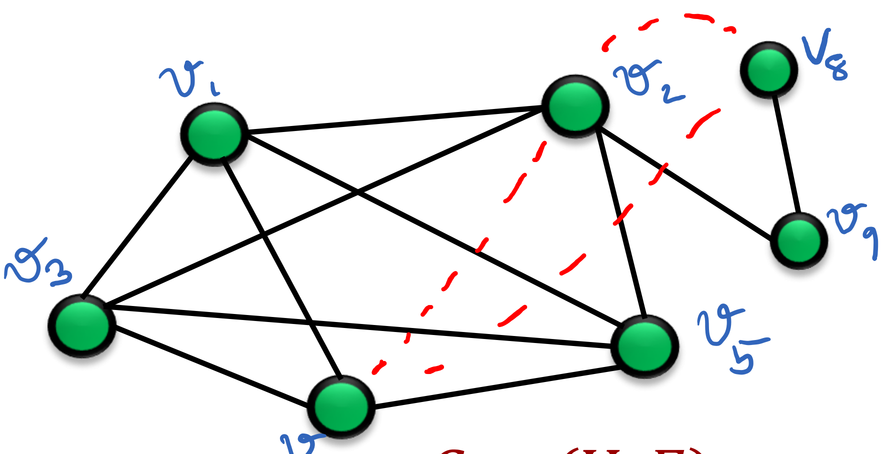
Independent Set



Empty graph: a graph with *no edges*

Independent Set:

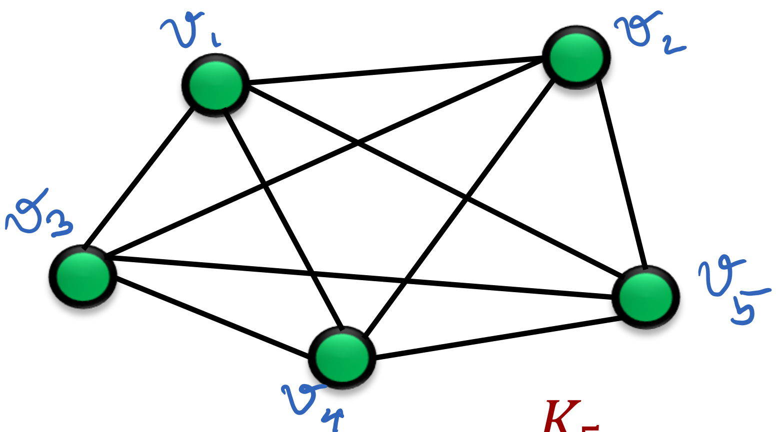
A subset of vertices with no edges between them in G (subset whose induced subgraph is empty). E.g., $S = \{v_2, v_4, v_8\}$



$G = (V, E)$

What is the size of the largest independent set in G? 3

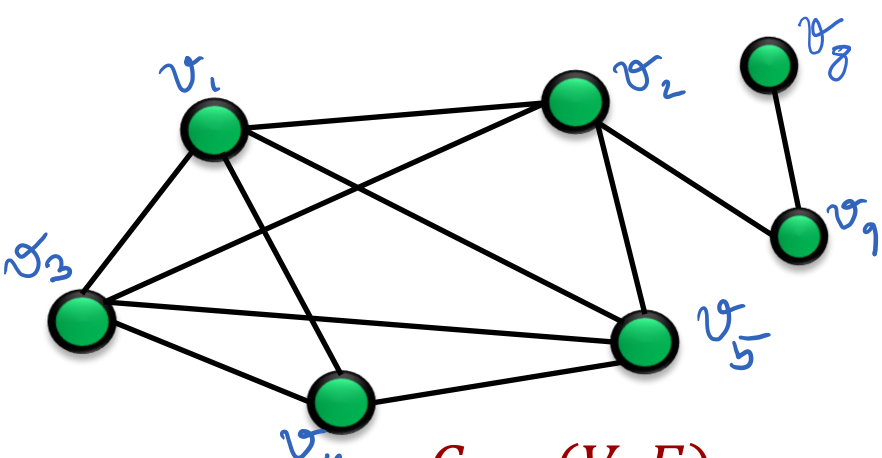
Recap: Complete Graphs, Cliques



K_5

K_n : complete graph on n-vertices.

Or also called an n-clique



$G = (V, E)$

Cliques in a graph:

A subgraph that is a clique.

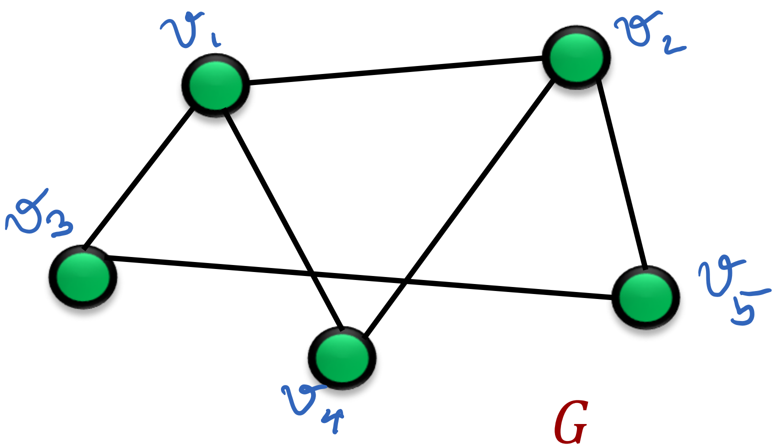
e.g. $\{v_1, v_3, v_4, v_5\}$

What is the size of the largest clique in G? 4

Set $H = \bar{G}$

$\bar{H} = (\bar{\bar{G}}) = G$

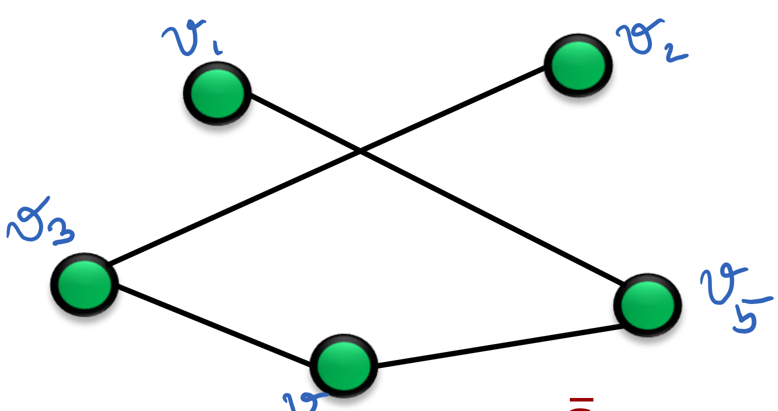
Graph Complements



G

$\bar{G} = (V, \bar{E})$=Complement of $G = (V, E)$

- Graph on the same of vertices
- $(u, v) \in \bar{E}$ *iff* $(u, v) \notin$ E



$\bar{G}$

Thm. S is an independent set in G iff S is a clique in $\bar{G}$

Pf. S is an independent set. So,
For every $u, v \in S$, $(u, v) \notin E$
i.e., $\forall u, v \in S$, $(u, v) \in \bar{E}$
Hence S is a clique in $\bar{G}$

Relations b/w Graph Properties

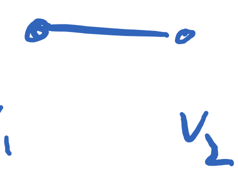
Given graph $G(V, E)$:

what is the size of the maximum independent set in G (independent set with largest number of vertices)?

a) = size of the maximum clique in $\bar{G}$? ✓

b) = size of the maximum clique in G ?

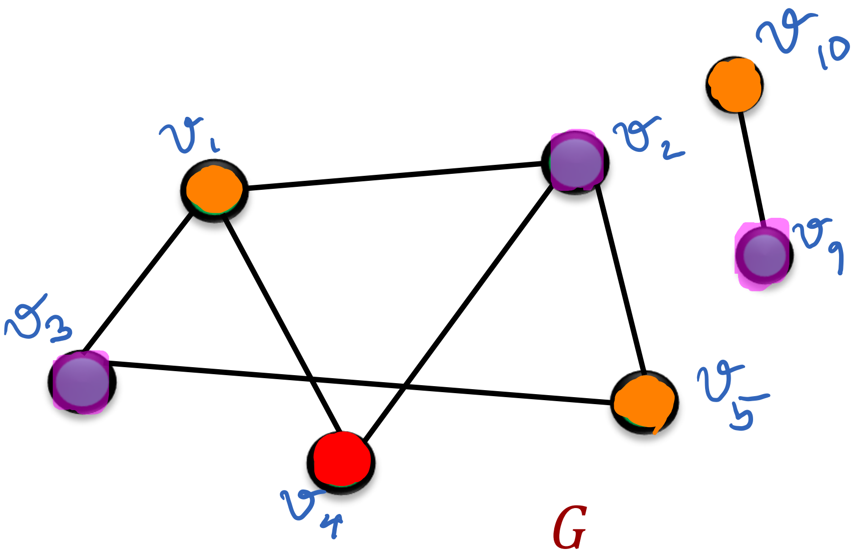
c) = size of the maximum independent set in $\bar{G}$?



Graph coloring

Graph Coloring

A graph $G(V, E)$ is k- colorable (vertex) if each vertex can be colored with one of k colors such that each edge is not monochromatic i.e. if $(u, v) \in E$ then u, v have different colors.

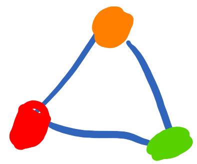


Is this graph 2-colorable?

No

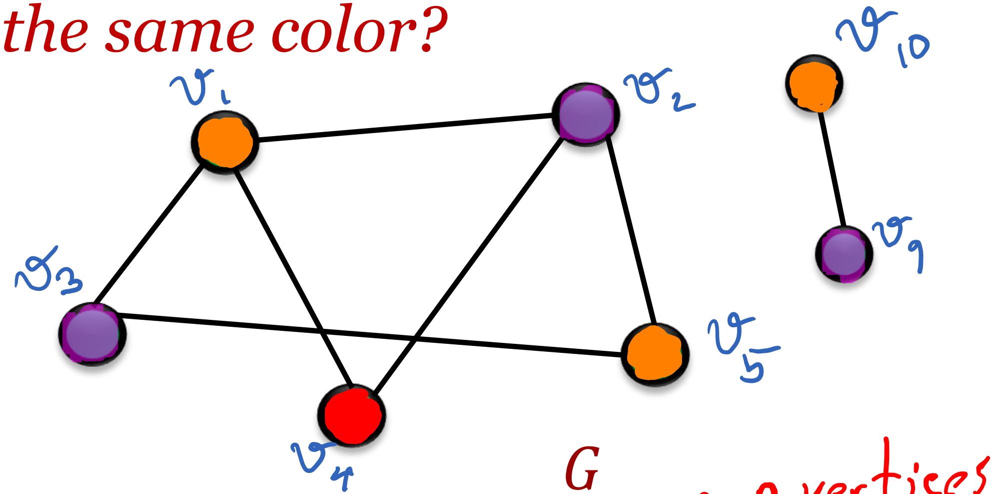
Is this graph 3-colorable?

Yes



Color Classes

What can you say about each color class i.e. all the vertices of the same color?



Each color class is an independent set

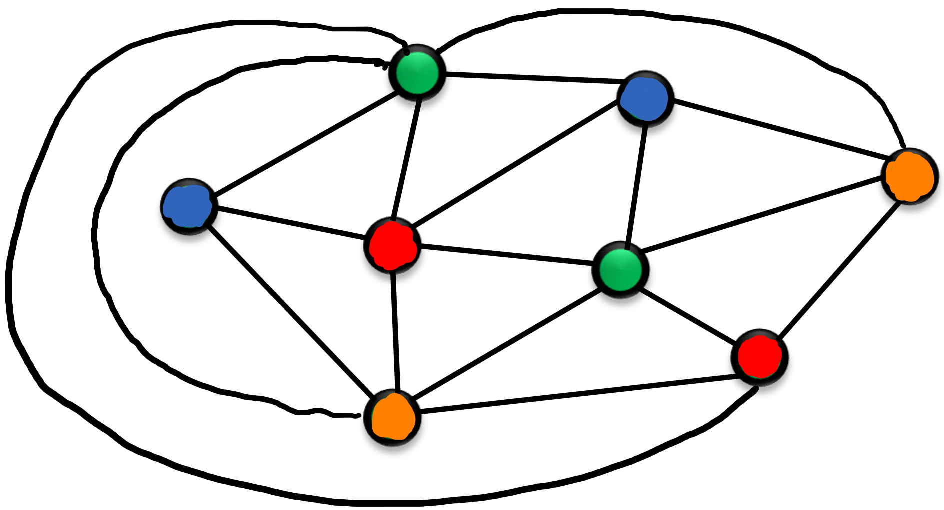
on n vertices

Theorem. If a graph is k-colorable, then the size of the maximum independent set $\geq \frac{n}{k}$?

$V = \bigcup_{i=1}^{k}$ ith color class $\Rightarrow |V| = \sum_{i=1}^{k} |$ith color class$| \leq k|$max color class$|$

Four color theorem

Theorem. If $G = (V, E)$ is a simple "planar" graph (i.e., no edges cross) then G is 4- colorable.



Proof: Hard!

Bipartite Graphs

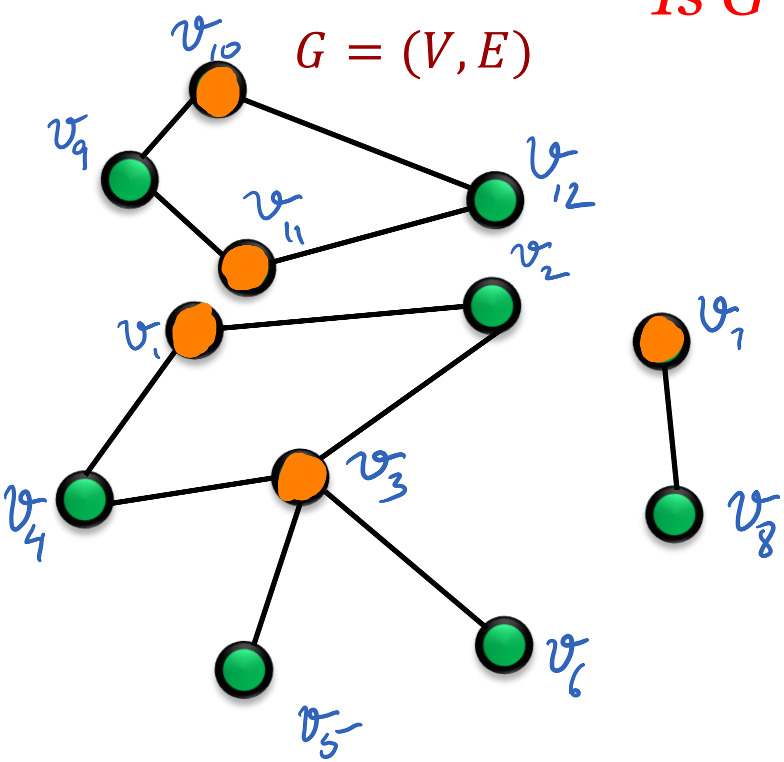
$L = \{v_{10}, v_{11}, v_1, v_3, v_7\}$
$R = V \setminus L$

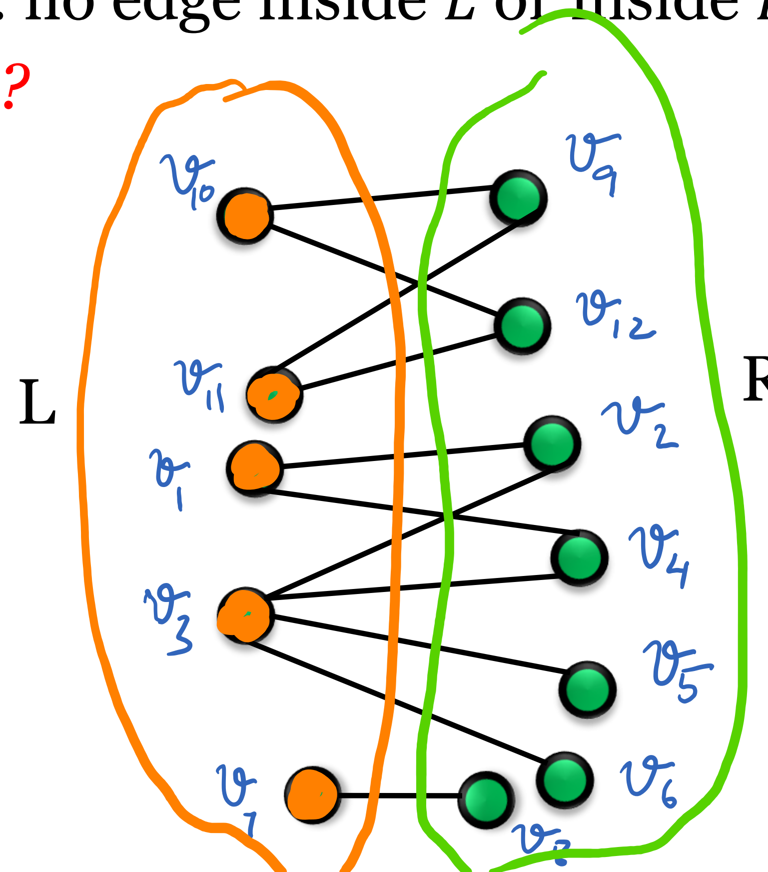
Bipartite Graphs

A graph is bipartite iff it is 2-colorable.

I.e. the nodes can be partitioned into sets L and R such that *all edges* go only between L and R. I.e. no edge inside L or inside R

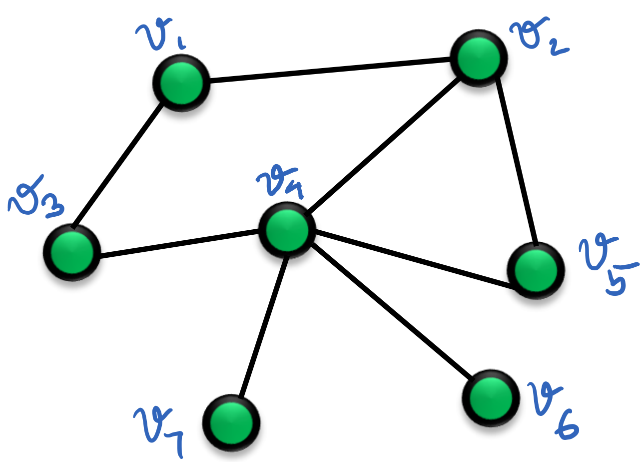
Is G bipartite?





Cycles in Graphs

Given $G = (V, E)$ a cycle is a sequence of $k \geq 3$ vertices $(v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_1})$ such that is an edge between any two consecutive vertices and such that v_{i_1} is the only repeated vertex.



$\{v_1, v_2, v_4, v_3, v_1\}$ Length 4

$\{v_2, v_4, v_5, v_2\}$ Length 3

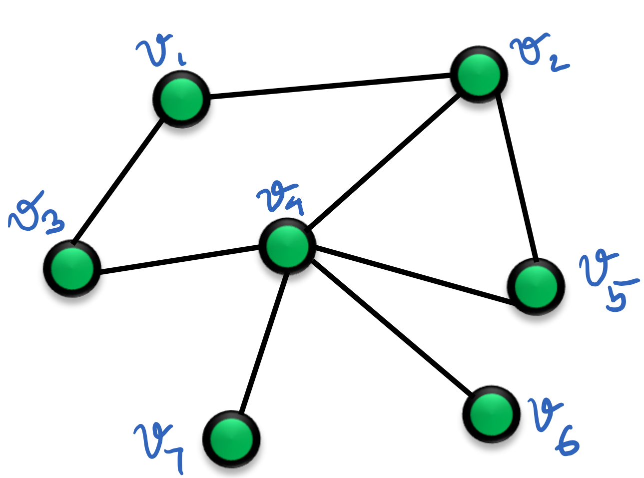
$\{v_1, v_2, v_5, v_4, v_3, v_1\}$ Length 5

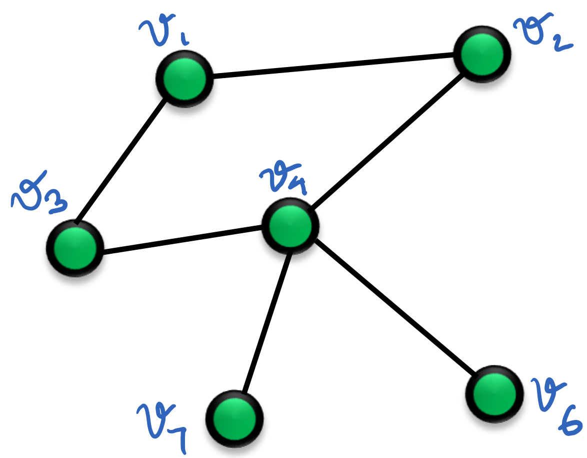
The length of a cycle $(v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_1})$ is k.

Characterization of Bipartite Graphs

How do you tell if a given graph is bipartite or not?

Theorem: A graph $G = (V, E)$ is bipartite iff G has no odd length cycles.





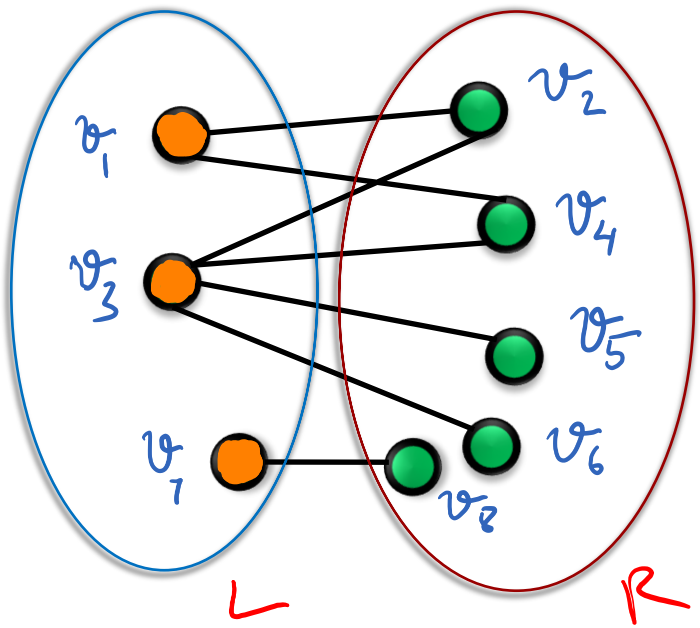
Proof – part 1

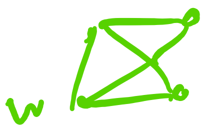
Thm: A graph $G = (V, E)$ is bipartite iff there is G has no odd cycle.

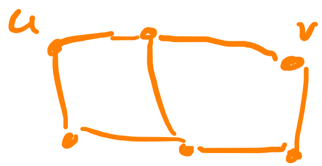
Proof of the easy direction: if bipartite , then no odd cycles.

Starting at a vertex $u \in L$, after odd number of steps, end up in R, (how do you prove this?)
1-step: any neighbor of a vertex in L is in R, and vice-versa

Hence, can not form an odd length path that starts in u and ends in u





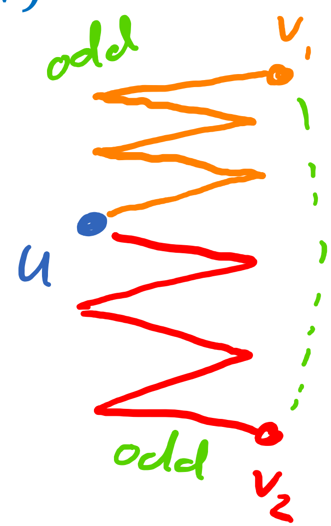


$d(u,v)=2$

Proof – part 2

Proof of the other direction: if no odd cycles, then bipartite.

Distance(u,v): length of the shortest path from u to v.
(min. number of edges on any path from u to v)



odd + odd + 1 = odd

Starting at a vertex $u \in V$.

$$R = \{v \in V: distance(u,v) = odd\}$$
$$L = \{v \in V: distance(u,v) = even\}$$

Claim: *No edge between any* $v_1 \in R, v_2 \in R$

Pf: Suppose not. If $(v_1, v_2) \in E$, then there is a closed walk of odd length
➔ a cycle of odd length

Similarly, *no edge between any* $v_1 \in L, v_2 \in L$

(do this for every piece or "connected component" separately)