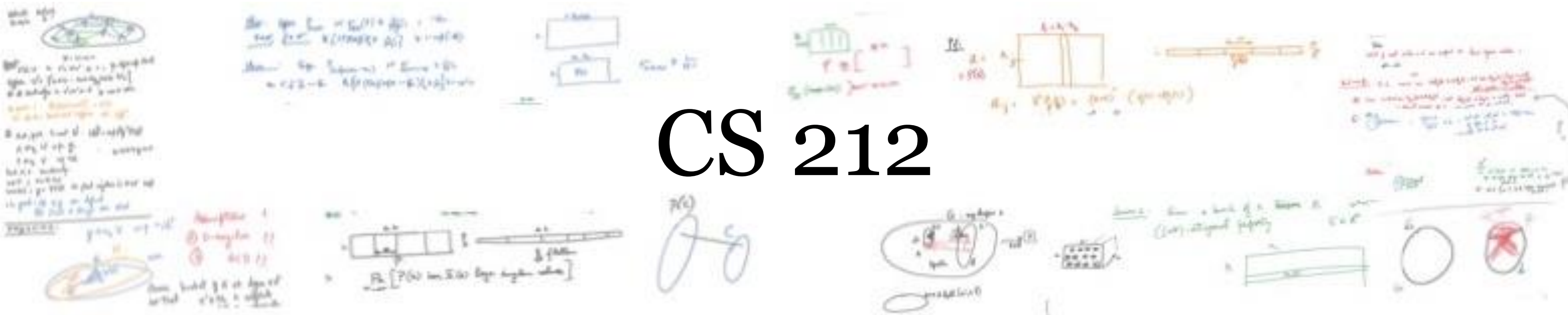


CS 212

Mathematical Foundations of Computer Science

Lecture 15: Deviation Bounds

Announcements

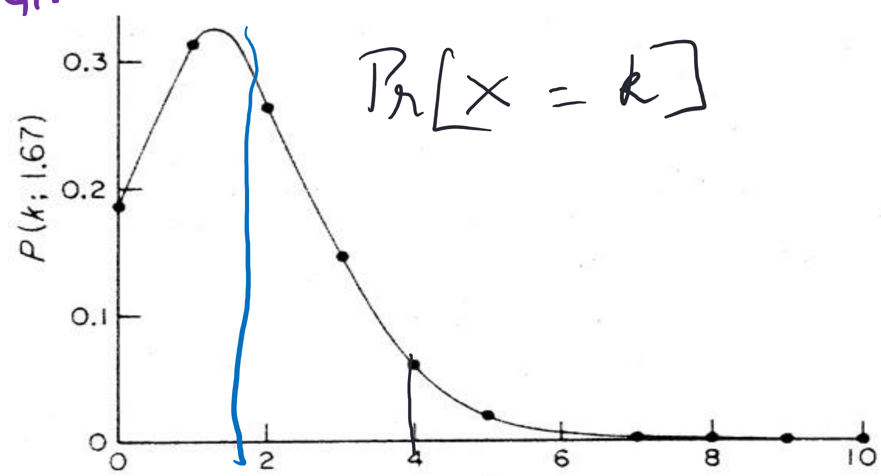
1. PS4 due on Tuesday. PS5 not due next week.
2. Midterm on Wednesday in class 3pm – 3:50pm.
3. Students with ANU considerations, please contact me/Eric/TAs about this. You will take the exam in CS dept. (and not ANU center)
4. Contact me if you have any issues.
5. Midterm review slides will be posted on Canvas.

Expectation

Average/mean value of the random variable X

The expectation, or expected value of a random variable X is

$$E[X] = \sum_{t \in S} Pr(t) \times X(t) = \sum_{k \in R} k \times \Pr[X = k]$$



Linearity of Expectation:

If X and Y are random variables ,

$$E[aX + bY] = aE[X] + bE[Y]$$

$$\mathbb{E}[XY] = \sum_{a \in \mathbb{R}} \sum_{b \in \mathbb{R}} \Pr[X=a \wedge Y=b]\, ab = \sum_a \sum_b \left(\Pr[X=a]\, a\right) \times \left(\Pr[Y=b] \cdot b\right)$$
$$= \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Independent R.V.s

If events A and event B are independent:

$$\Pr[\, A \wedge B \,] = \Pr[\, A \,]\, \Pr[\, B \,]$$

Two random variables X and Y are independent iff for every a,b, the events X=a and Y=b are independent

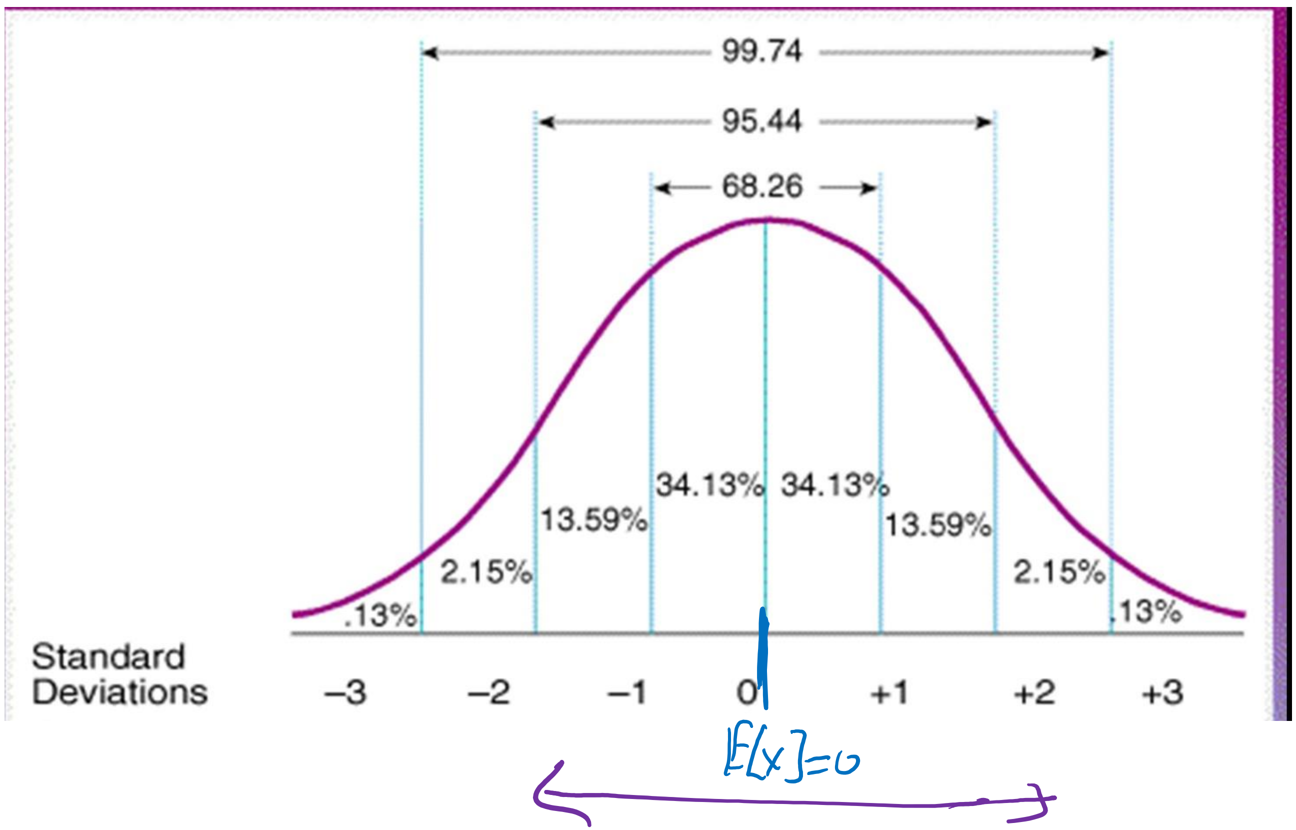
$$\Pr[(X=a) \wedge (Y=b)] = \Pr[X=a] \times \Pr[Y=b]$$

Thm: If random variables X and Y are independent:

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

(Converse is not necessary ~~true~~)

Aside: Cov(X, Y) $= \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$ is measure of dependence



Deviation Bounds

True or False

If the students in class on average spent 10 hours on PS3, can there be more than 50% of them who spent more than 20 hours?

True or False? False.

No! If there were, the average would be higher.

$X \geq 0$

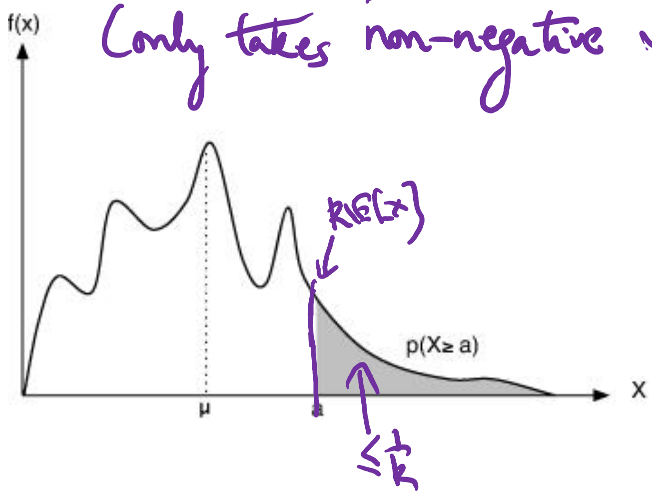
$\Pr[X \geq a] \leq \frac{E[X]}{a}$

$a = tE[X] \Leftrightarrow t = \frac{a}{E[X]}$

Markov's inequality

If X is a *non-negative r.v.* with mean E[X], then

(only takes non-negative values)



$$\Pr[\, X \geq 2E[X] \,] \leq \tfrac{1}{2}$$

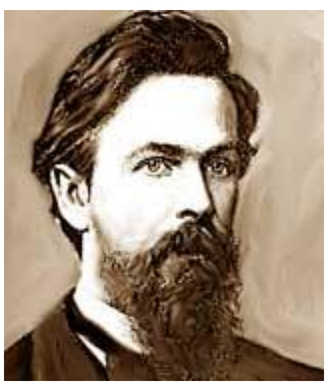
$$\forall k \geq 1, \qquad \Pr[\, X \geq k \cdot E[X] \,] \leq 1/k$$

Proof. Let $\mu = E[X]$. Suppose (*) $\Pr[X \geq k\mu] > \frac{1}{k}$ (for contradiction)

$E[X] = \Pr[X \geq k\mu] \times \underbrace{E[X \mid X \geq k\mu]} + \Pr[X < k\mu]\, E[X \mid X < k\mu]$

$\geq \underline{\Pr[X \geq k\mu]} \cdot (k\mu) + \Pr[X < k\mu] \cdot 0$ (since $X \geq 0$)

$E[X] \overset{(*)}{>} \frac{1}{k} \times k\mu = \mu = E[X]$. Hence a contradiction



Variance of random variable X

How much is the deviation from the mean $\mu = E[X]$?

Deviation from average/expectation -

Let $Y = X - E[X]$. What about $E[Y]$? 0

≥0

$$Var[X] = E[(X - EX)^2] = E[Y^2]$$

"average squared deviation from mean"

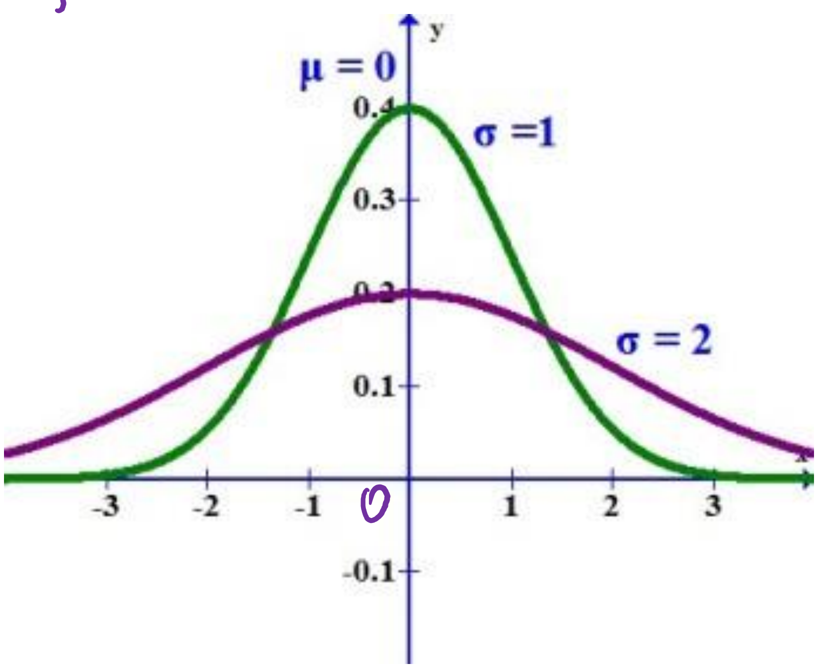
Fact: $\mathrm{Var}[X] = E[X^2] - (E[X])^2$

Suppose $\mu = \mathbb{E}[X]$

$$Var[X] = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu \cdot \mathbb{E}[X] + \mu^2$$

$$= \mathbb{E}[X^2] - 2\mu \cdot \mu + \mu^2 = \mathbb{E}[X^2] - \mu^2$$



Standard deviation $\sigma(X) = \sqrt{\mathrm{Var}(X)}$