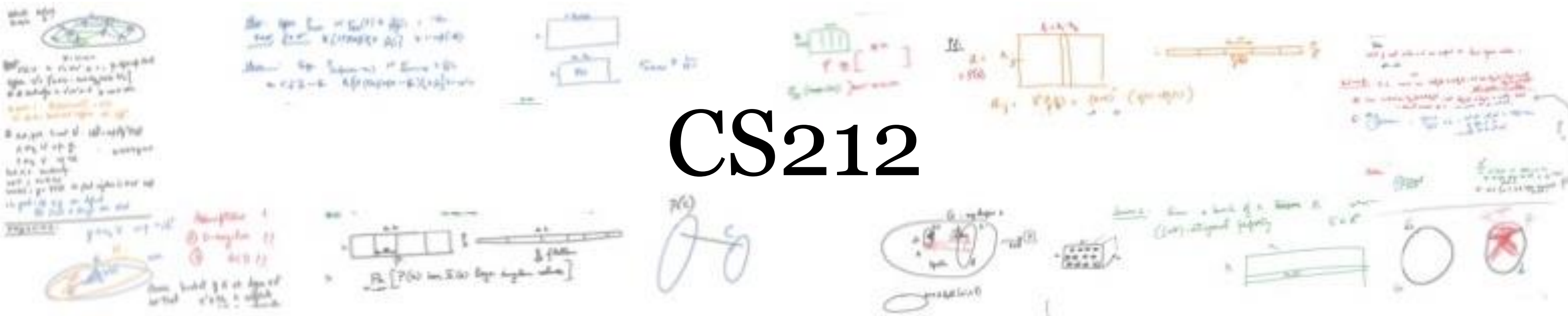


CS212

Mathematical Foundations of Computer Science

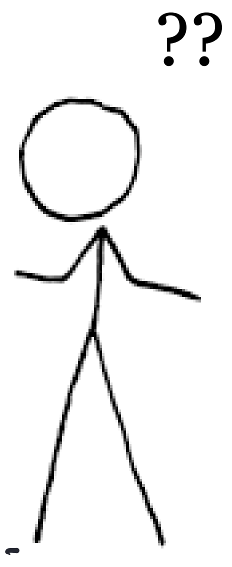
Lecture 2: Mathematical Proofs

Announcements

1. LaTex tutorial on Tuesday at 5PM over Zoom.
2. My office hours today is from 4:00PM – 5:00PM.
3. HW1 will be released next week
4. All discussions and office hours starting next week (mine starting today).

How to sort this stack?

5
2
3
4
1



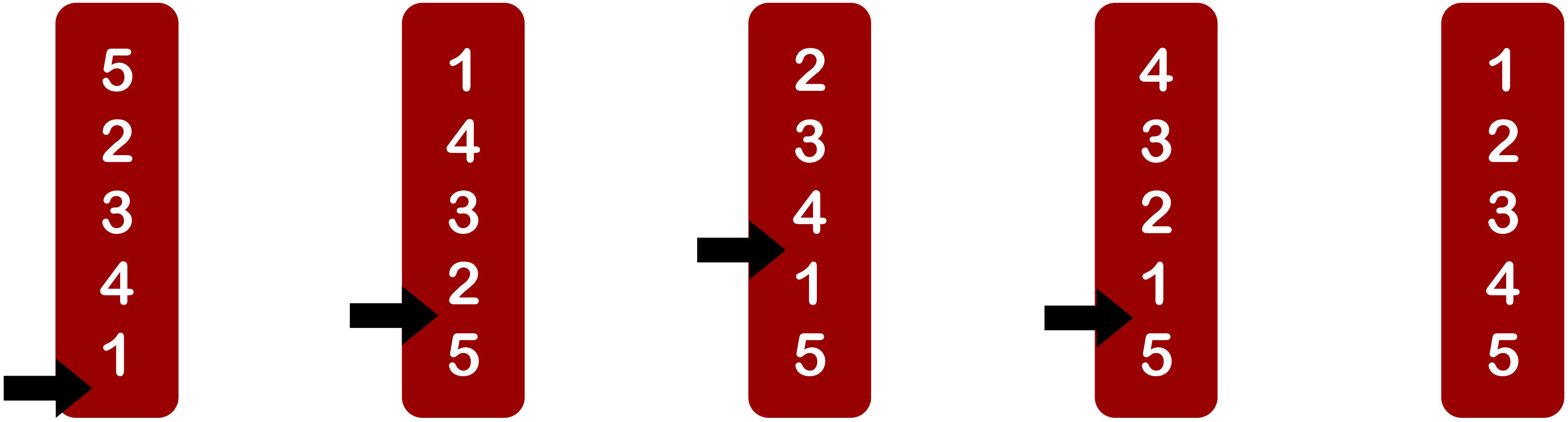
1
2
3
4
5

Four flips are sufficient

5	4	2	5	1
2	3	3	4	2
3	2	4	3	3
4	5	5	2	4
1	1	1	1	5

4 flips

4 Flips are Sufficient



Best Way to Sort

X = Smallest number of flips required to sort:

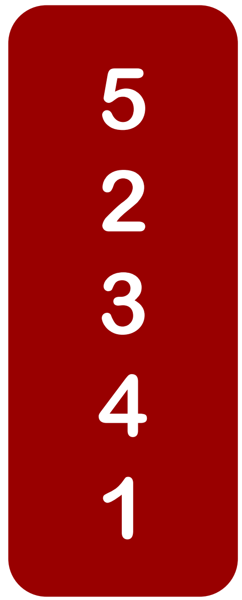
5
2
3
4
1

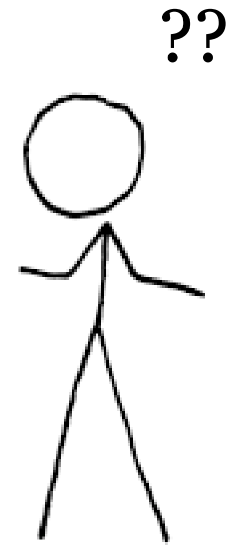
Lower Bound

$? \leq X \leq 4$

Upper Bound

Lower bound: can we do better?





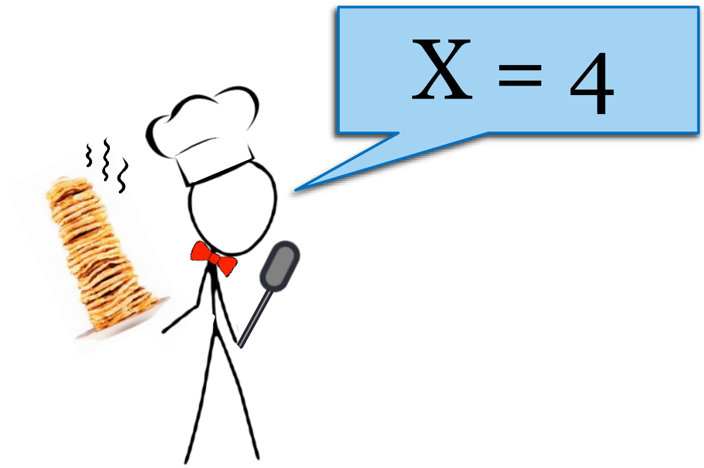
Lower bound: needs convincing argument that every way of sorting stack requires at least 4 flips

And the answer is…

$$4 \le X \le 4$$

Lower Bound

Upper Bound



where

X = Smallest number of flips required to sort:

5
2
3
4
1

5^{th} Pancake number

$P_5 =$ Number of flips required to sort the worst case stack of 5 pancakes

$$= \max_{X_s}[\text{min number of flips to sort } X_s]$$

1	5	4		5		2	3
2	4	2		2		3	2
3	3	3	…	3	…	4	5
4	2	5		4		5	4
5	1	1		1		1	1
X_1	X_2	X_3				X_{119}	X_{120}

Lower bound on P_5

$$P_5 = \max_{X_s}[\text{min number of flips to sort } X_s]$$

Lower Bound $\rightarrow$ $? \leq P_5 \leq ?$

What does proving a *Lower bound of* $P_5 \geq 4$ mean?

a. For every instance, prove a lower bound of 4

i.e. for every instance, every sorting algorithm takes ≥ 4 flips

b. Give one instance, where lower bound ≥ 4

i.e. for some instance, every sorting algorithm takes ≥ 4 flips

Upper bound on P_5

$$P_5 = \max_{X_s}[\text{min number of flips to sort } X_s]$$

$$? \leq P_5 \leq ?$$

Upper Bound

What does proving a *Upper bound of* $P_5 \leq 5$ mean?

a. For every instance, an upper bound of 5

i.e. one sorting algorithm which for every instance takes ≤ 5 flips

b. Give one instance where upper bound ≤ 5

i.e. one sorting algorithm, which for some instance takes ≤ 5 flips

5^{th} Pancake number

$$P_5 = \max_{X_s}[\text{min number of flips to sort } X_s]$$

Lower Bound

$$? \leq P_5 \leq ?$$

Upper Bound

Argument:
exhibit one instance where every way of sorting stack requires at least 4 flips

Algorithm:
Give sorting procedure that for *every* instance (stack) takes at most 4 flips

Values of P_n ?

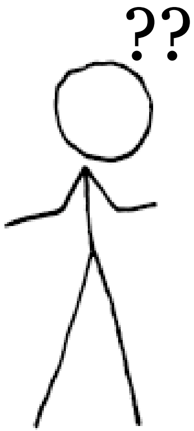
n	1	2	3	4	5	…			n
P_n	0	1	3						

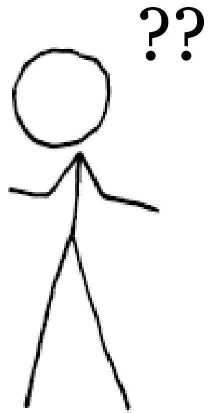
We don't know the value of P_n for large n !

Bracketing

What are the best lower bounds and upper bounds that I can prove?

$$[\quad\quad \le f(n) \le \quad\quad]$$





Bracketing

What are the best lower bounds and upper bounds that I can prove?

$$n \quad \leq \quad P_n \leq 2n - 3$$

These bounds can be proved using simple arguments

Improved bounds

$$(17/16)n \leq P_n \leq (5n + 5)/3$$

William Gates and Christos Papadimitriou. Bounds For Sorting By Prefix Reversal. *Discrete Mathematics*, vol 27, pp 47-57, 1979.

[CFMMSSV 08] Improvements in recent years.

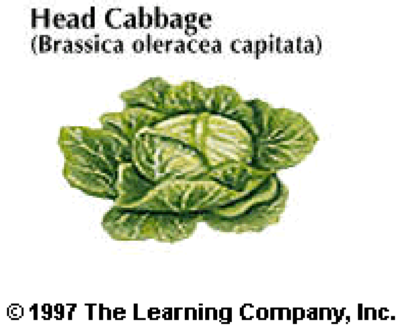
Can you do better than best known results?

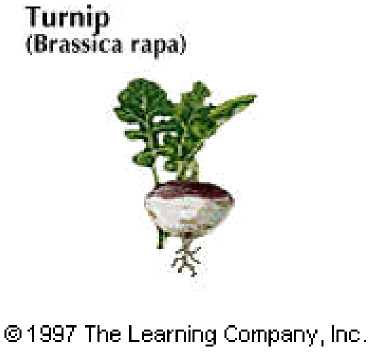
Pancakes… Relevance to CS?

Related to host of problems and applications at the frontiers of science

- Sorting by Prefix Reversal. *American Mathematics Monthly 82 (1) (1975), Jacob Goodman.*

- Mutation Distance:





Outline

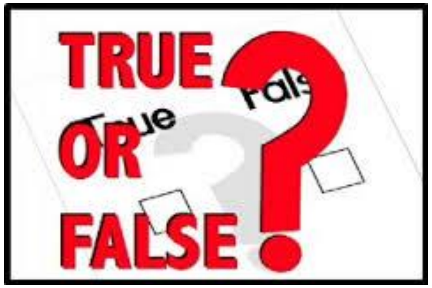
- Logical Reasoning and Proofs
- Different Kinds of proofs
- Proof by Contrapositivity
- Proof by Contradiction

Propositions & Predicates

- Proposition is a statement.
- Examples?

  e.g. 2022 is a prime number.

  Jack lives in Evanston



- Predicate: Proposition with parameters x

e.g. $P(x) =$ " 2x is an even number"

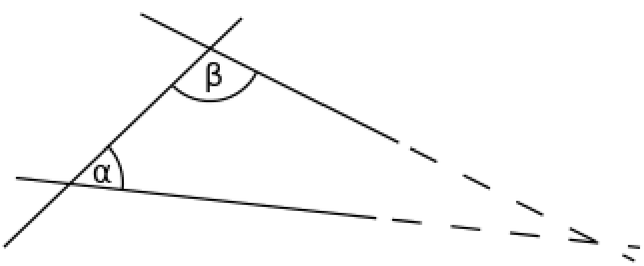
$P(x)=$"x is a prime"

$P(x)=$"x lives in Evanston"

Axioms

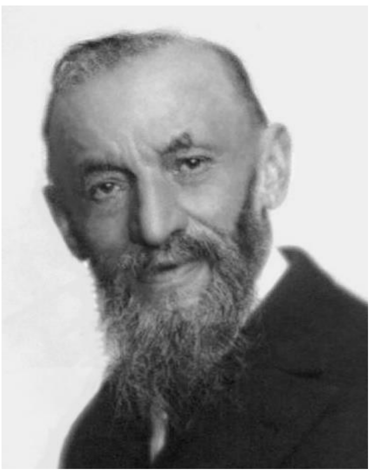
- Underlying mathematical assumptions
- Used to derive true propositions.
- Examples?

Euclid’s axioms of Geometry



1. "To draw a straight line from any point to any point."
2. "To produce [extend] a finite straight line continuously in a straight line."
3. "To describe a circle with any centre and distance [radius]."
4. "That all right angles are equal to one another."
5. *“The parallel postulate*: if two lines lying on another line…”

Peano's axioms: Natural numbers $\mathbb{N}$



Giuseppe Peano

- 0 is a natural number.
- The next four axioms describe the equality relation. The remaining axioms define the arithmetical properties of the natural numbers. The naturals are assumed to be closed under a single-valued successor function S.
- For every natural number n, $S(n)$ is a natural number (this is represented by $n + 1$).
- For all natural numbers m and n, $m = n$ if and only if $S(m) = S(n)$. That is, S is an injection.
- There is no natural number whose successor is 0.

Theorems, Lemmas… and Proofs

- Our goal is to derive whether Propositions are true or not..

Axioms → True Propositions

Proofs

- Logical reasoning to infer Propositions
- Also called Theorems, Lemmas…

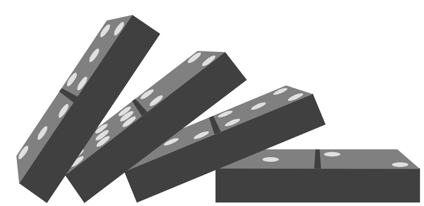
Types of Proofs

- Direct Proof

Axioms $\Rightarrow S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow$ Proposition

- Proof by Cases
- Proof by Contrapositivity

- Proof by Contradiction



- Proof by Mathematical Induction

Types of Proofs

Proof by Obviousness	The proof is so clear that it need not be mentioned.
Proof by General Agreement	All in favor??...
Proof by Convenience	It would be nice if this is true, so..."
Proof by Necessity	It had better be true or the whole structure of mathematics would crumble to the ground.
Proof by Plausibility	It sounds good, so it must be true.
Proof by Intimidation	Don't be stupid. Of course it's true.
Proof by Lack Of Time	Because of the time constraint, I'll leave the proof to you.

~~Proof by example~~

Your Favorite Proof

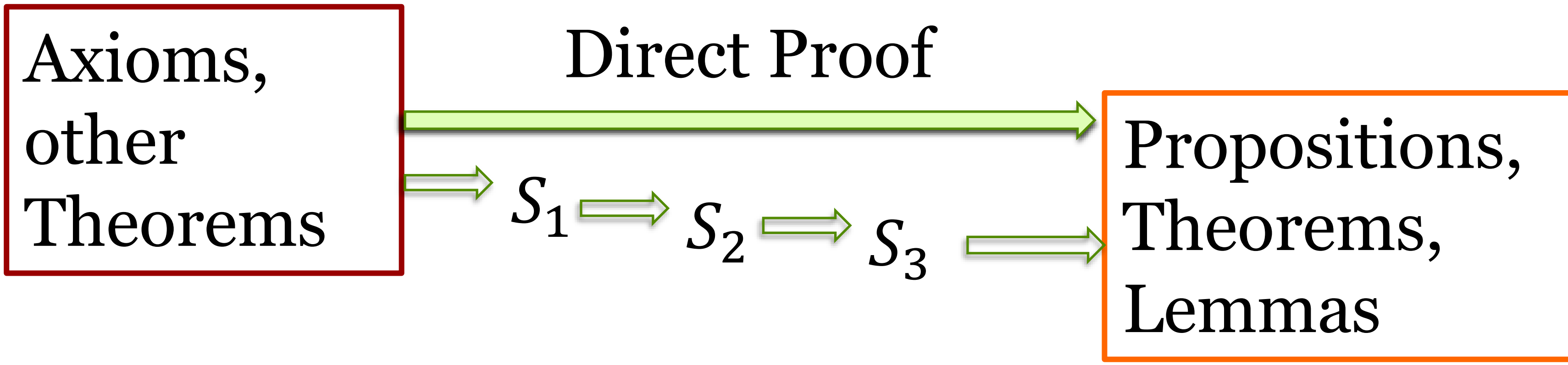
What's your favorite math proof from high school?

Proof of infiniteness of primes.

Pythagoras theorem.

Godel's theorem.

Direct Proofs



Eg. Axioms $A_1, A_2, .. A_k$ implies propositions S_1, S_2, S_3 which in turn implies required Theorem/ Proposition.

Most high school proofs are direct proofs
(e.g. Pythagoras theorem, many Geometry proofs).

Summary and Takeaways

- Logical Reasoning and proofs
- Lots of proofs! Embrace proofs! Don’t accept any statement without proof!

- Sort pancakes and become the world’s richest person!