212 Practice Mid-Term Exam

(from a previous year; note that the pattern can be different.)

20 points.

You have 50 minutes to complete the exam. Write clear proofs and arguments for each of the problems. Remember that all logarithms are to the base 2.

Problem 1

(6 points total)

Answer with explanations and proofs:

(a) If $f(n) = 4^{(\log n)^3}$ and $g(n) = n^4$, is $f(n) = o(g(n))$ or $g(n) = o(f(n))$? **(2 points)**

(b) All the DNA sequences of length k over the alphabet $\{A, C, G, T\}$ i.e., all strings of length k comprised of characters $'A', 'C', 'G', 'T'$ are distributed among $g(n) = n^4$ groups. How many DNA strings of length k exist? When $k = (\log n)^3$, can all these length k strings be distributed into a distinct group each i.e., can all the strings be distributed among $g(n) = n^4$ groups such that no group has more than one string ? **(2 points)**
Hint: *For the second part, focus on what happens as $n \to \infty$. Try to use the answer to the previous part, and use Pigeonhole principle to reason about this.*

(c) How many DNA sequences/strings of length k exist where at least two of the four characters from $\{A, C, G, T\}$ appear? **(2 points)**

Problem 2

(5 points) Use Induction to prove that for any $r > 0$ (and $r \neq 1$), $a \in \mathbb{R}$, and any natural number $n \geq 1$, we have

$$a + 2ar + 3ar^2 + \dots nar^{n-1} = \frac{a\left(nr^{n+1} - (n+1)r^n + 1\right)}{(r-1)^2}.$$

Problem 3

(4 points) Given two numbers $n \geq r$, we want to calculate the number of ways of distributing n identical objects into r distinct boxes such no box is empty.

1. Suppose we represent the n identical objects with n zeroes (n of them) and use $r - 1$ ones to represent the partitioning of the boxes. Explain in English (succinctly), what properties are satisfied by these $n+r-1$ long bit strings of zeros and ones. (**2 points**)

2. Using the previous part, prove that the number of ways of distributing n identical objects into r distinct boxes such no box is empty is $\binom{n-1}{r-1}$. Remember that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. **(2 points)**

Problem 4

(5 points total) For this problem, let n be an integer at least 3.

(a) A permutation of a set S is a sequence consisting of all the elements of S with no repetitions. Let S be the set of permutations of $[n]$, and consider the uniform distribution on S. For $i \in [n]$ let $E_i \subseteq S = \{s \mid s \in S, s_i = i\}$, that is, E_i is the set of permutations for which the ith element of each permutation is i. Give with proof the value of $\Pr[E_1]$ and $\Pr[E_2]$.

For example, if $n = 3$, then $E_2 = \{(1, 2, 3), (3, 2, 1)\}$.

(b) Give with proof the value of $\Pr[E_1 \cap E_2]$. Determine if E_1 and E_2 are independent, with a brief explanation why.