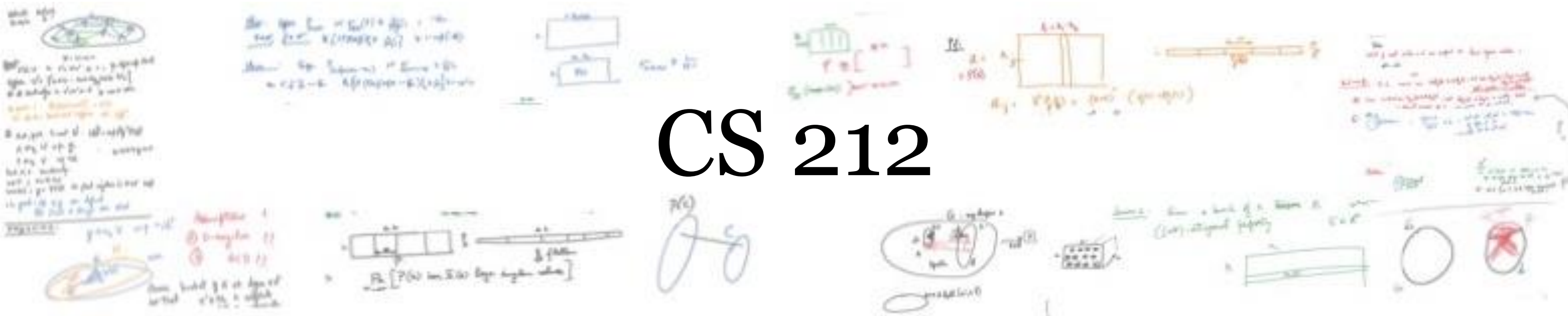


CS 212

Mathematical Foundations of Computer Science

Lecture 28: LP Duality and Applications

Announcements

1. No more problem sets!
2. No discussion sessions this week – thanksgiving week.
3. No office hours later this week. (i.e. on Wed - Sun)

Linear Programming

"Standard" LP Formulation

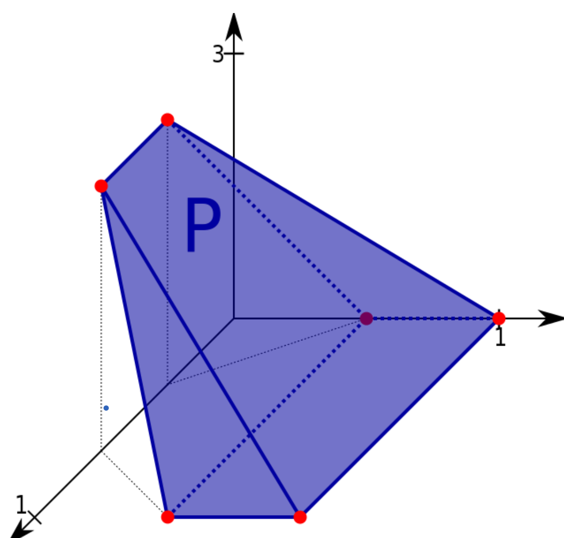
Variables: $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

Constraints: given by $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

$$\max \quad c^T x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\text{such that} \quad Ax \le b \quad \forall i \in [m]: \sum_{j=1}^{n} a_{ij} x_j \le b_j$$

$$x \ge 0 \quad \forall j \in [n]: x_j \geqslant 0$$



Claim: Standard LP formulation can capture general Linear programs (any linear objective subject to linear constraints).

$y_1 = 0, \; y_2 = 1, \; y_3 = \frac{1}{2}$

How do you know you are optimal?

I

Maximize $5x_1 - 3x_2$ $= 2$

$(x_1 + 3x_2 \leq 5) \times y_1$

$(3x_1 + x_2 \leq 4) \times y_2$

$(4x_1 - 8x_2 \leq -4) \times y_3$

$x_1, x_2 \geq 0$

$y_1, y_2, y_3 \geq 0$

Claim: (1,1) is optimal with max value =2.

How do you show optimality?

We need upper bound on objective

$5x_1 - 3x_2 \leq 2$????

+ (1,1) achieves value = 2

$5x_1 - 3x_2 \leq$

$x_1(y_1 + 3y_2 + 4y_3) + x_2(3y_1 + y_2 - 8y_3) \leq 5y_1 + 4y_2 - 4y_3$

II

minimize $5y_1 + 4y_2 - 4y_3$ $= 2$

$y_1 + 3y_2 + 4y_3 \geq 5$

$3y_1 + y_2 - 8y_3 \geq -3$

$y_1, y_2, y_3 \geq 0$

Ex: Best setting of y_1, y_2, y_3?

$\Rightarrow 5x_1 - 3x_2 \leq 5y_1 + 4y_2 - 4y_3$.

Dual LP

- This Procedure possible for any LP!

Primal LP:

$$\max c_1x_1 + .. + c_jx_j + c_nx_n$$

s.t $\forall i \in [m]$

$y_i \times$ $$\left(a_{i1}x_1 + .. + a_{ij}x_j + a_{in}x_n \le b_i\right)$$

$$x \ge 0$$

$(a_{11}y_1 + a_{21}y_2 + \cdots + a_{n1}y_n)x_1 + \cdots + (a_{m1}y_1 + \cdots + a_{mn}y_m)x_n \le b_1y_1 + \cdots + b_my_m$

Dual LP: best upper bound

$$\boxed{\min \; \sum_{i=1}^{m} b_i y_i}$$

s.t $\forall j \in [n]$

constraint for coeff of variable x_j

$$\boxed{a_{1j}y_1 + .. + a_{ij}y_i + a_{nj}y_n \ge c_j}$$

$$y \ge 0$$

- One dual variable for each primal constraint.
- One dual constraint for each primal variable.

Primal LP: $\max \; c^T x$

such that $Ax \le b$

$x \ge 0$

Dual LP: $\min \; b^T y$

such that $A^T y \ge c$

$y \ge 0$

LP Duality

Primal LP: max $c^T x$

such that $Ax \leq b$

$x \geq 0$

Dual LP: min $b^T y$

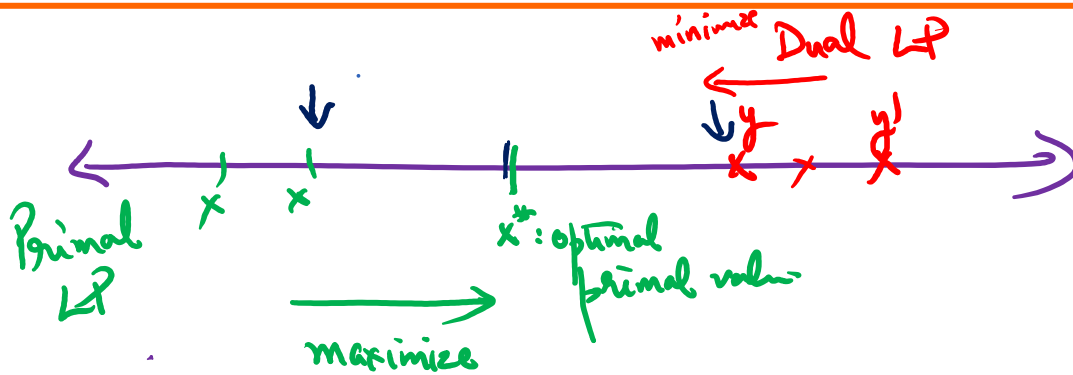
such that $A^T y \geq c$

$y \geq 0$

Weak Duality Theorem: For any feasible solution x to the primal LP, feasible solution y to the dual LP, we have

$$c^T x \leq b^T y$$

. In particular: $\max_{\substack{x: \\ Ax \leq b \\ x \geq 0}} c^T x \leq \min_{\substack{y: \\ A^T y \geq c \\ y \geq 0}} b^T y$



Strong LP Duality (no proof)

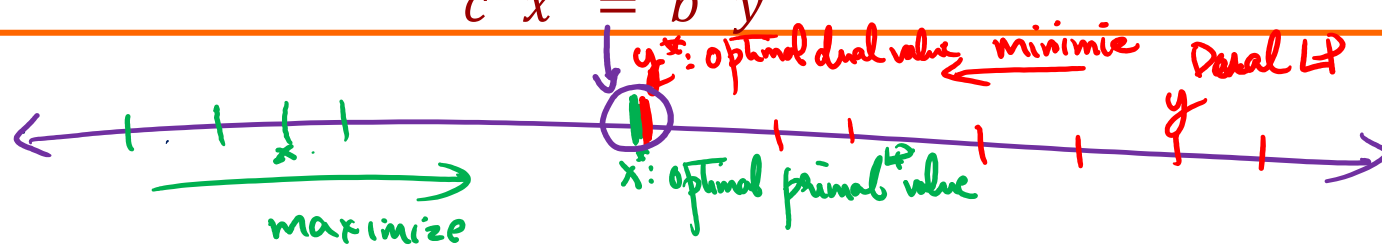
Primal LP: max $c^T x$
such that $Ax \leq b$
$x \geq 0$

Dual LP: min $b^T y$
such that $A^T y \geq c$
$y \geq 0$

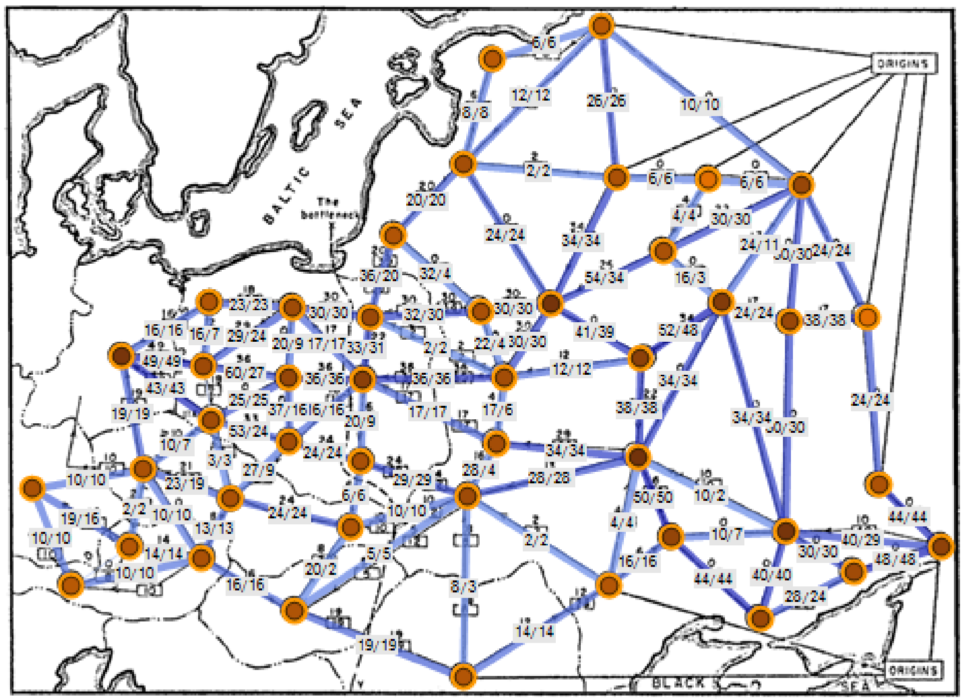
Strong Duality Theorem: One of these four holds:

1) Both are infeasible.
2) Primal is infeasible, Dual is unbounded.
3) Dual is infeasible, Primal is unbounded
4) Both feasible, and if x^* is the optimal solution to the primal LP, y^* is optimal solution to the dual LP,

$$c^T x^* = b^T y^*$$



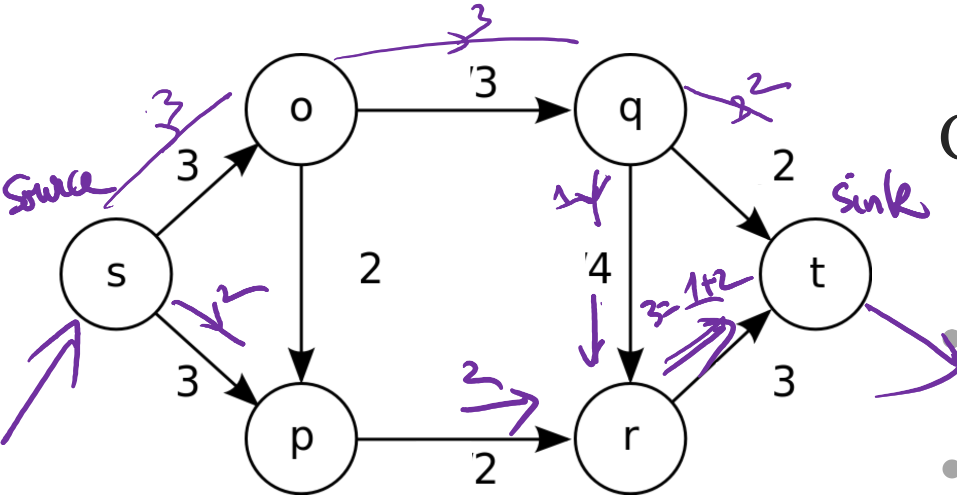
Transportation Networks

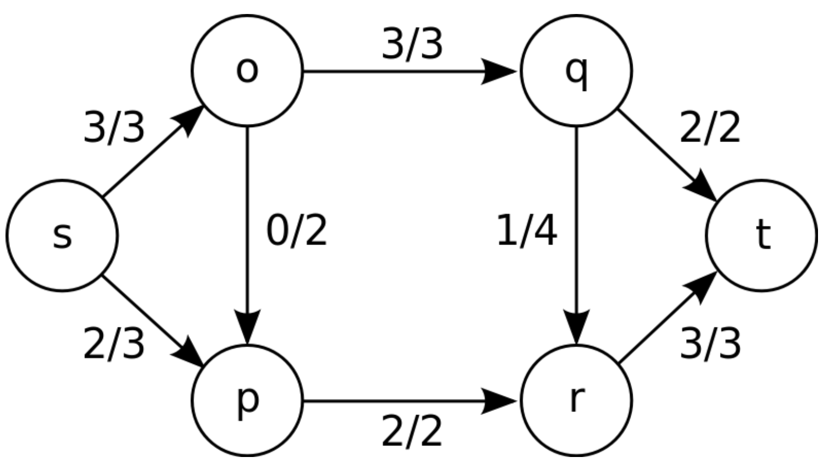


- In a road network, roads have capacities (number of cars/ minutes).
- How best to direct flow of heavy traffic during World Cup?

st

Maximum Flow Problem





Graph $G(V, E)$: directed or undirected.

- Capacity $b(u, v)$ for each edge.
- Two special nodes source s, sink t.
- **Flow:** for any vertex $v \in V$, flow into v = flow out of v
- What is the maximum amount of flow from s to t ? 5

Linear Programming Formulation

Variable for each edge e=(u,v).

f_{uv} : flow on any edge $e = (u, v)$

$$\max \boxed{\sum_{v:(s,v)\in E} f_{sv}}$$

flow going out of s.

$$\forall (u,v)\in E: \boxed{f_{uv} \le b(u,v)}$$

capacity constraint

$$\forall v \in V \setminus \{s,t\}:$$

$$\sum_{u:(u,v)\in E} f_{uv} = \sum_{w:(v,w)\in E} f_{vw}$$

flow conservation

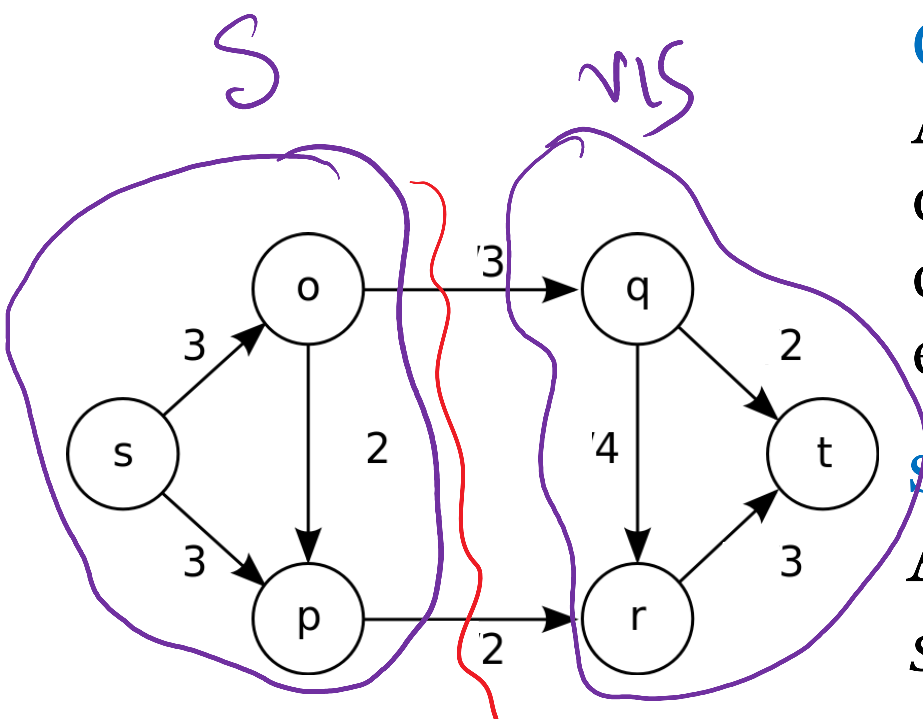
Graph $G(V, E)$: directed or undirected.

- Capacity $b(u, v)$ for each edge.
- Two special nodes source s, sink t.
- **Flow:** for any $v \in V - \{s, t\}$, flow into v = flow out of v
- What is the maximum amount of flow from s to t ?

Bottlenecks

How do we say that our Maximum flow is optimal?

An intuitive graph theoretic quantity that captures this?



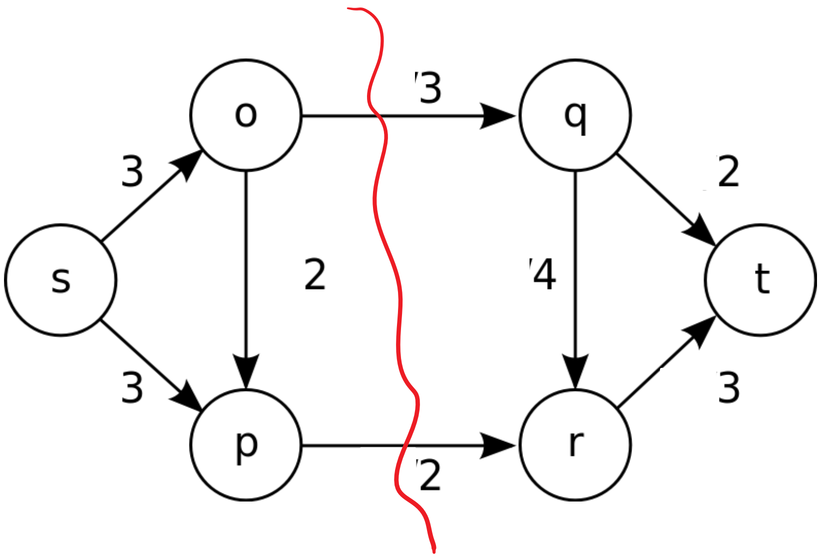
Cut in a graph: s-t Cut

Any subset of vertices $S \subset V$ defines a cut $(S, V \setminus S)$. Cost of cut = total weight/ capacity of edges (u, v) with $u \in S, v \in V \setminus S$

s-t cut:

Any subset $S \subset V$ such that $s \in S,\ t \in V \setminus S$.

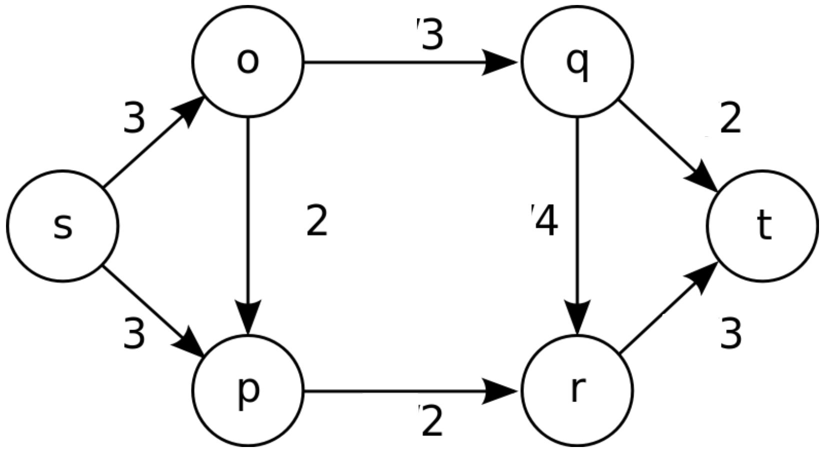
Minimum $s - t$ Cut



Given an undirected graph $G(V, E)$ with capacities on edges, what is the size of the minimum cut that separates s, t

Every $s - t$ cut gives an obstruction. Minimum $s - t$ cut is the worst bottleneck.

Aside: Max-Flow Min-Cut Theorem



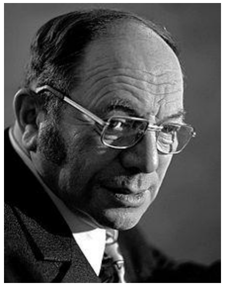
Max Flow- Min Cut Theorem: The maximum flow in a network between s, t = capacity of the Minimum $s - t$ cut in G.

*Similar to Strong LP duality (Algorithms course).

Weak LP Duality can be used to show Max-Flow ≤ Min-Cut

Linear Programming

This Class: No algorithms for LP.



LPs introduced and studied by Kantorovich, Koopmans, Dantzig.
LP Duality was discovered by Von Neumann.
(Kantorovich and Koopmans won Nobel Prize in 1971 for the "theory of optimal allocation of resources" i.e., linear programming)

Thank you!