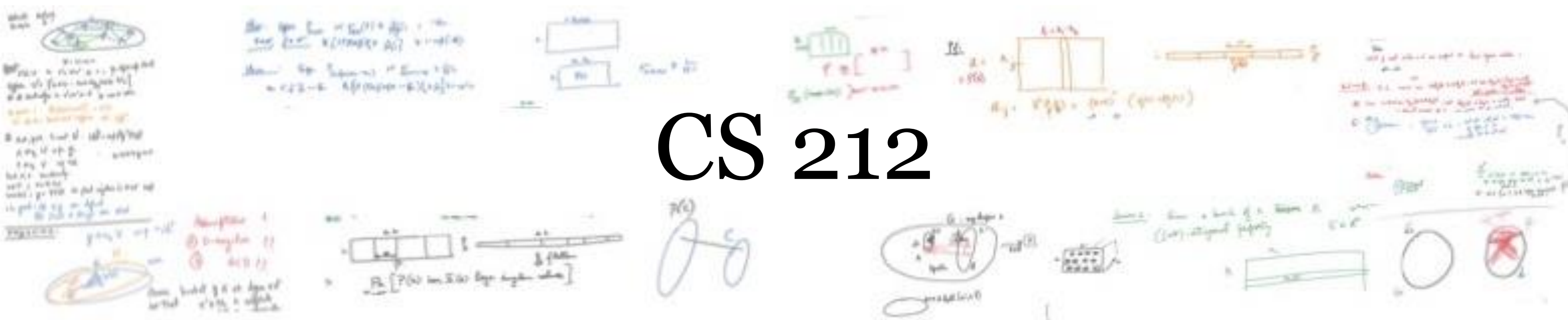


CS 212

Mathematical Foundations of Computer Science

Lecture 13: Bayes Rule, Birthday Paradox

Announcements

- Midterm on Wed, October 26th in class
- Not open book. One "cheat" sheet i.e., two sides where you can write down anything.
- PS4 is out. Due on Tuesday as usual.

$\Pr[A_2 \wedge A_7 \wedge A_{100} \wedge A_{101}] = \Pr[A_2] \cdot \Pr[A_7] \cdot \Pr[A_{100}] \Pr[A_{101}]$

Recap: Mutually Independent Events

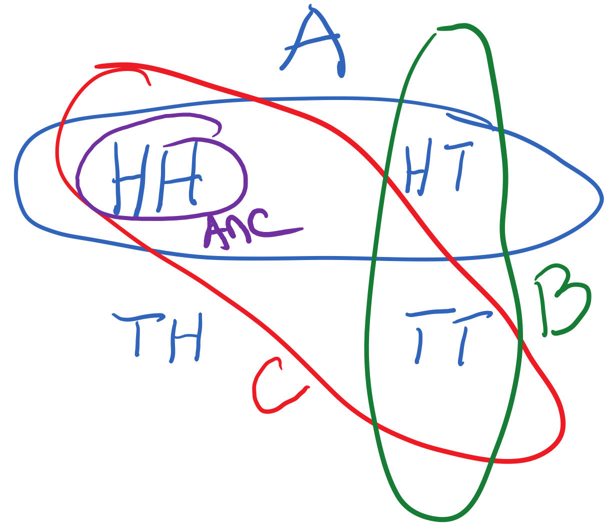
$A_1, A_2, \ldots, A_n$ are mutually independent events if knowing if some of them occurred does not change the probability of any of the others occurring

> $A_1, A_2, \ldots, A_n$ are mutually independent events iff for *any* *subset* of these n events
>
> every
>
> $$\Pr[A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}] = \Pr[A_{i_1}] \times \Pr[A_{i_2}] \times \ldots \times \Pr[A_{i_k}]$$

For n events, how many checks? $2^n - n - 1$

subsets

Are they Mutually Independent?



Two different coins both are fair coins.

Sample space: {HH,HT,TH,TT}

- A: 1st coin turns up heads.
- B: 2nd coin turns up tails.
- C: The two coins are equal.

$$\Pr[A] = \frac{2}{4} = \frac{1}{2} \qquad \Pr[B] = \frac{2}{4} = \frac{1}{2} \qquad \Pr[C] = \frac{2}{4} = \frac{1}{2}$$

$$\Pr[B \cap C] = \Pr[\{\mathrm{TT}\}] = \frac{1}{4} \qquad \Pr[A \cap B] = \Pr[\{\mathrm{HT}\}] = \frac{1}{4}$$

$= \Pr[B] \times \Pr[C]$

$\Pr[A \cap B \cap C] = 0$ $\qquad$ $\Pr[A]\Pr[B]\Pr[C] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Hence they are not mutually independent!

Evaluating Conditional Probabilities

- In a rapid chess game between J. Polgar and V. Anand, they toss a fair coin to pick who plays white (or black)

(Anand plays white with prob ½)

- If Anand plays *black*, he has a 40% chance of winning and 60% chance of losing.

White/Black $\xrightarrow{?}$ Win/Losses

- If Anand plays *white*, he has 50% chance of winning & 50% of losing
- *Wh*=event Anand plays white, *S*: event Anand successfully wins.

Given Anand played white, what is probability that he wins? $\Pr[S|Wh]=0.5$

Given Anand won, what is probability that he played white? Pr[Wh|S]=?

Bayes Rule

$\bar{A}$: complement of A (or) not A (or) A does not occur.

- Need Pr[Wh|S] but easier to reason about Pr[S|Wh]

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[A]\Pr[B|A]}{\Pr[B]}$$

harder to reason

easy to think about.

$$= \frac{\Pr[A]\Pr[B|A]}{\Pr[A]\Pr[B|A] + \Pr[\bar{A}]\Pr[B|\bar{A}]}$$

$= \Pr[B] = \Pr[B \wedge A] + \Pr[B \wedge \bar{A}]$

- Wh=event Anand plays white, S: event Anand won.

$$\Pr[S|\text{Wh}] = \Pr[\mathit{Anand\ wins} \mid \mathit{Anand\ plays\ white}] = 0.5$$

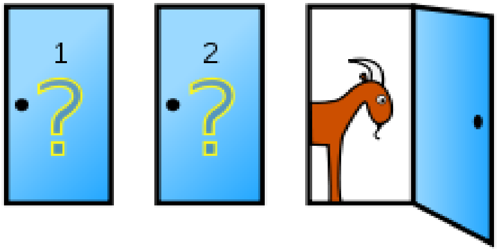
$$\Pr[S|\overline{\text{Wh}}] = \Pr[\mathit{Anand\ wins} \mid \mathit{Anand\ plays\ black}] = 0.4$$

A=Wh

B=S

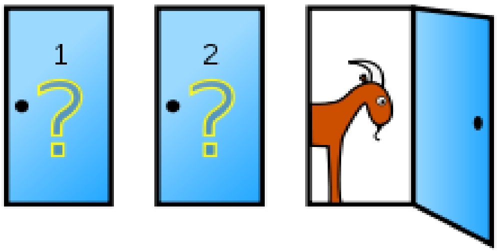
$$\Pr[\text{Wh}|S] = \frac{\Pr[S|\text{Wh}]\Pr[\text{Wh}]}{\Pr[S]} = \frac{0.5 \times \frac{1}{2}}{\Pr[\text{Wh}]\Pr[S|\text{Wh}] + \Pr[\overline{\text{Wh}}]\Pr[S|\overline{\text{Wh}}]} = \frac{0.5 \times \frac{1}{2}}{\frac{1}{2}\times 0.5 + \frac{1}{2}\times 0.4} = \frac{5}{9}$$

Bonus slide: Monty Hall Problem



In search of a new car, the player picks a door, say 1. The game host then opens one of the other doors, say 3, to reveal a goat and offers to let the player pick door 2 instead of door 1.

Bonus slide: Monty Hall Problem



In search of a new car, the player picks a door, say 1. The game host then opens one of the other doors, say 3, to reveal a goat and offers to let the player pick door 2 instead of door 1.

Sample space = { prize behind door 1, prize behind door 2, prize behind door 3 }

Each has probability 1/3

C : the event we chose correct door.

$\Pr[C] = 1/3, \qquad \Pr[\bar{C}] = 2/3$

Staying: we win if we chose correct door

$\Pr[win] = \Pr[C]\Pr[win|C] + \Pr[\bar{C}]\Pr[Win|\bar{C}] = \frac{1}{3}$

Switching: win if we chose incorrect door

$\Pr[win] = \Pr[C]\Pr[win|C] + \Pr[\bar{C}]\Pr[Win|\bar{C}] = \frac{2}{3}$

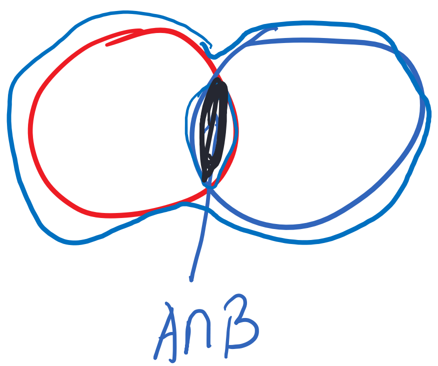
Why Tricky? "After one door is opened, others are equally likely…" But his action not independent of yours!

Sum Rule Revisited

Probability: If A, B are two disjoint or mutually exclusive events, then $\Pr[A \cup B] = \Pr[A] + \Pr[B]$ $- \Pr[A \cap B]$

$= 0$

What if the two events/ choices are not disjoint?



Union: If A and B are events, then

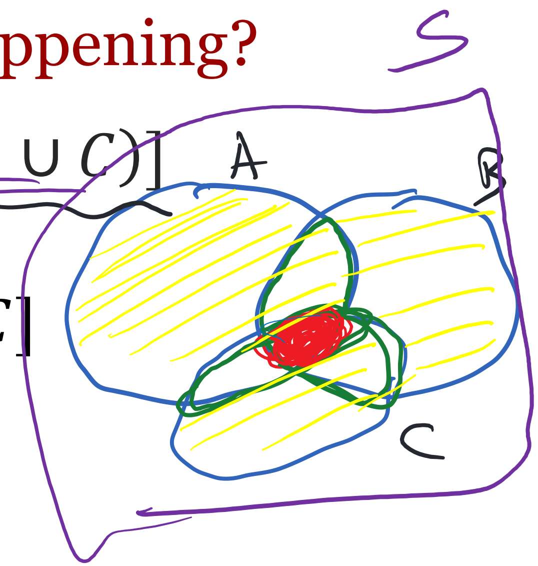
$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

Union of k events?

What is the probability of event A or B or C happening?

$$\Pr[A \cup \overbrace{(B \cup C)}^{D}] = \Pr[A] + \Pr[B \cup C] - \Pr[A \cap (B \cup C)]$$

$$\Pr[A \cup B \cup C] = \Pr[A] + \Pr[B] + \Pr[C]$$
$$-\Pr[A \cap B] - \Pr[B \cap C] - \Pr[A \cap C]$$
$$+\Pr[A \cap B \cap C]$$



Inclusion-Exclusion: For events $A_1, A_2, \ldots, A_n$

$$\Pr[\cup_{i=1}^{n} A_i] = \sum_{i=1}^{n} \Pr[A_i] - \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \Pr[A_i \cap A_j] + \sum_{i \neq j \neq k} \Pr[A_i \cap A_j \cap A_k] - \cdots$$

Union Bound

Inclusion-Exclusion: For events $A_1, A_2, \ldots, A_n$,

$$\Pr[\cup_{i=1}^{n} A_i] = \sum_{i=1}^{n} \Pr[A_i] - \sum_{i,j=1:i<j}^{n} \Pr[A_i \cap A_j] + \sum_{i,j,k=1:i<j<k}^{n} \Pr[A_i \cap A_j \cap A_k] - \cdots$$

+ − + −

How many terms are there? $2^n - 1$ (since we don't care about empty set)

What if there are a bunch of (non-independent, non-disjoint) events that have very complicated intersections etc. ?

Can we give an upper bound on the probability of the union?

Union Bound (Boole's inequality): For events $A_1, A_2, \ldots, A_k$

$$\Pr[A_1 \cup A_2 \cup \cdots \cup A_k] \leq \Pr[A_1] + \Pr[A_2] + \cdots + \Pr[A_k]$$

Useful for getting upper bounds on probabilities of bad events

Dealing with Murphy's laws

"Anything that can go wrong will go wrong" – Murphy's law

$$Prob[Failure] \leq \sum_{bad\ events\ E_i} \Pr[E_i]$$

Bad winter for Chicago :

E1: Avg. temperature < $20F$ $\Pr[E_1] = 0.1$

E2: Snow storm of > 1 foot $\Pr[E_2] = 0.05$

E3: Cloudy on half the days $\Pr[E_3] = 0.2$

Note: events E_1, E_2, E_3 not independent!

What can be say about $\Pr[bad\ winter\ for\ Chicago]$?

$= \Pr[E_1 \vee E_2 \vee E_3] \leq \Pr[E_1] + \Pr[E_2] + \Pr[E_3] = 0.1 + 0.05 + 0.2$

$= 0.35$

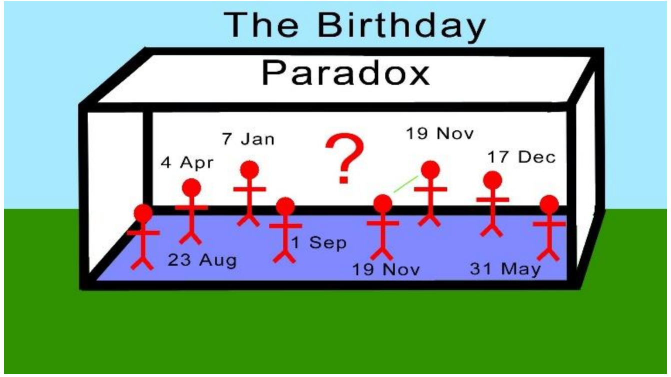
Birthday Paradox: Coincidences are more likely than you think!

Birthday Paradox

How many people do you need in the class so that two of the students have the same birthday?

By pigeonhole principle, it is enough to have 366 students.

What if we don't need certainty but good chance is enough?



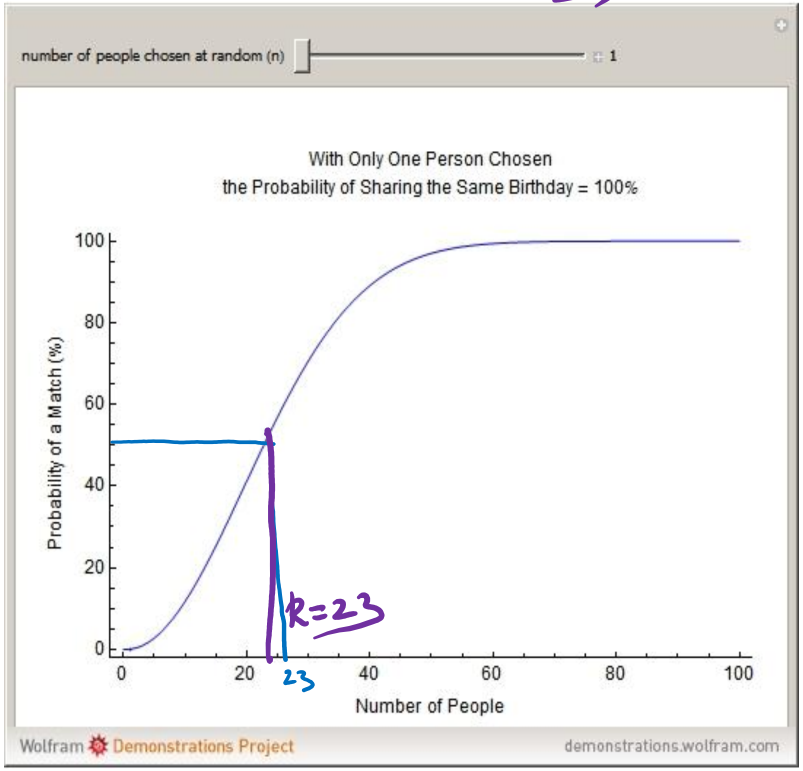
Paradox: If there are 23 students in class, then with probability > 50% there are at least two students who have same birthday.

If there are n days, with $O(\sqrt{n})$ students, we have a collision with prob $\geq \frac{1}{2}$.

The Explanation

Given k students what is the probability at *least two* of them share a birthday? $\Pr[\bar{E}] = 1 - \Pr[E]$

Complementary event: all k students have different birthdays E



Answer= 1 – Pr[all of them have different birthdays]=1-Pr[E]

k=2: $\Pr[E] = 1 \times \frac{364}{365}$

k=3: $\Pr[E] = 1 \times \underbrace{\frac{364}{365}} \times \underbrace{\frac{363}{365}}$

General k: $\Pr[E] = \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{2}{365}\right) \times \cdots \left(1 - \frac{k-1}{365}\right)$

For k=23, Pr[E]~ 0.493.. Hence answer ≥ 50%