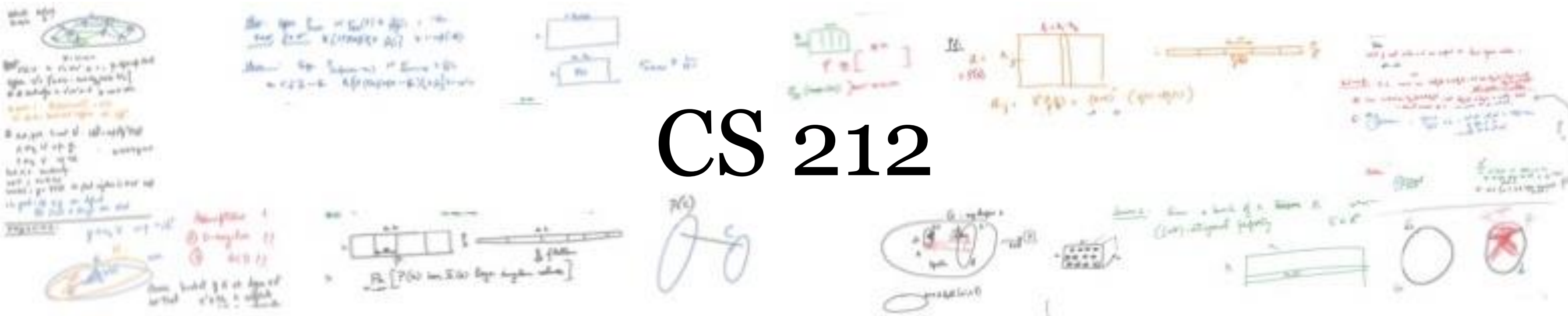


CS 212

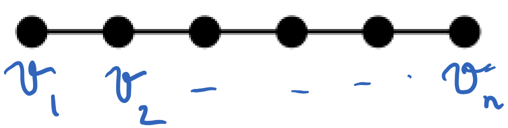
Mathematical Foundations of Computer Science

Connectivity, Trees

If $\exists$ walk from u to v
then $\exists$ path from u to v

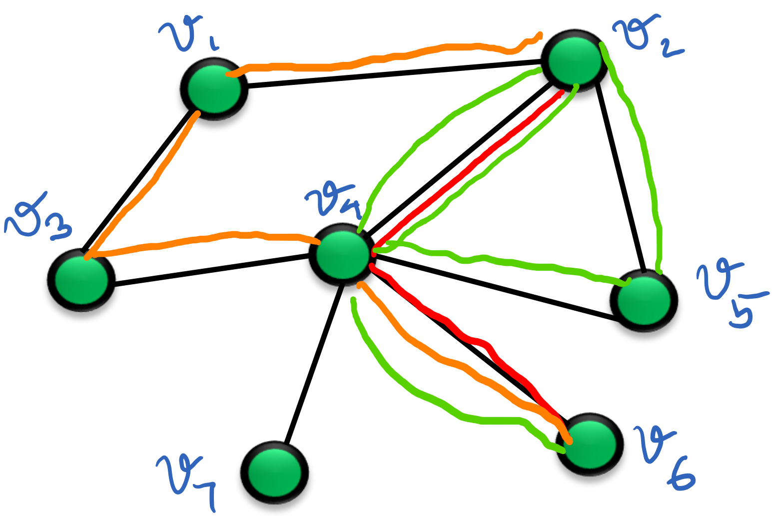
Paths

Path Graph or Linear Graph:



special graph in first vertex (v_1) and last vertex (v_n) have degree 1, and every other vertex has degree 2.

Path between $u, v \in V(G)$:



$G(V, E)$

A subgraph which is a path subgraph such that 1st vertex = u, last vertex=v.
e.g. path between v_2 and v_6 in G

$\{v_2, v_4, v_6\}$

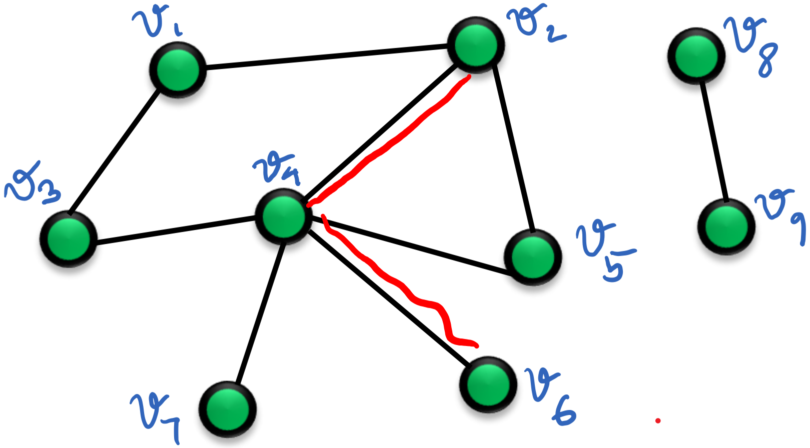
- No vertex is repeated in a path

Walk: Vertices, edges can be repeated

Connectivity, Connected Graphs

Connectivity between two vertices:

$u, v \in V(G)$ are connected iff there is a path from u to v in G



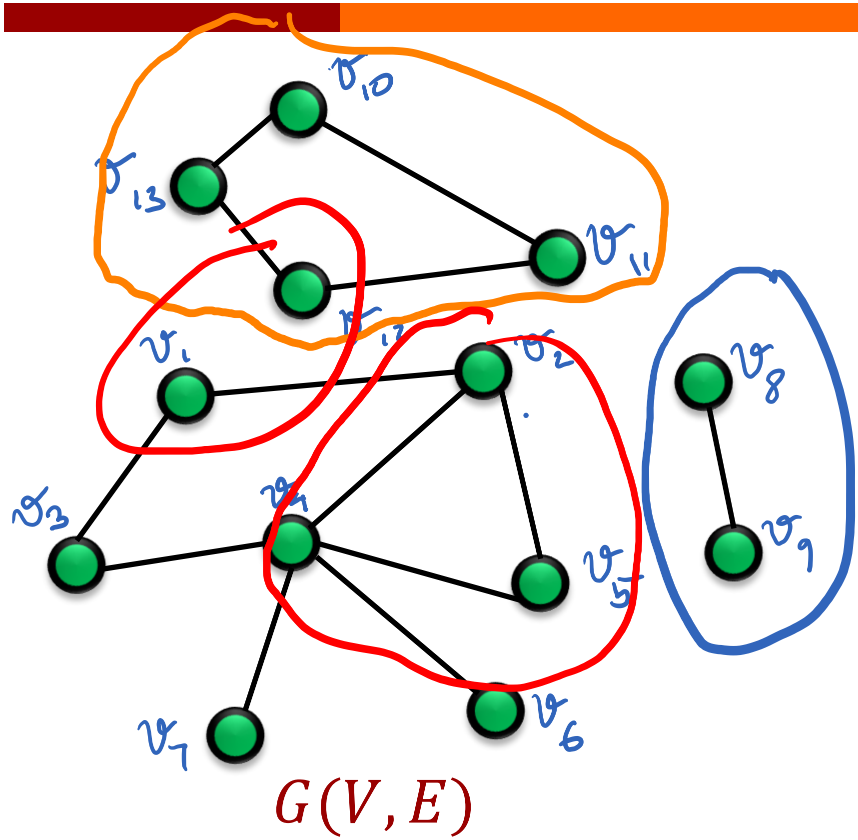
$G(V, E)$

e.g. v_2, v_6 are connected, v_8, v_9 are connected.

v_2, v_9 are *not* connected.

Connected Graph: Undirected graph $G(V, E)$ is connected iff every pair of nodes $u, v \in V(G)$ are connected.

Connected Components



Connected component: a subset S of vertices that are connected in G, and connected to no other vertices of G (i.e. a maximal connected subset)

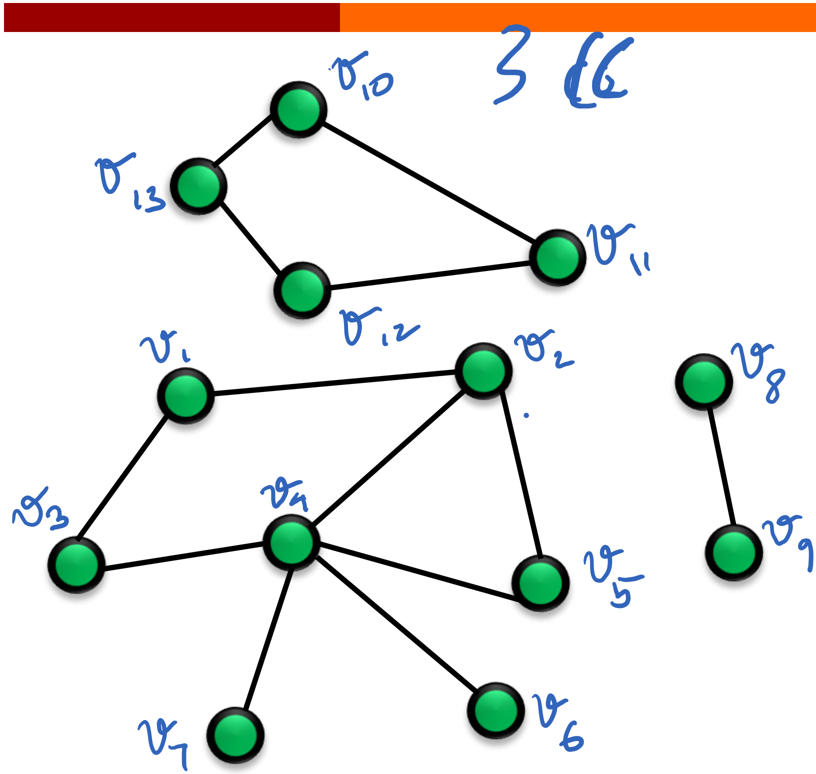
$\{v_8, v_9\}$ is a CC

$\{v_{10}, v_{11}, v_{12}, v_{13}\}$ yes

$\{v_4, v_5, v_2\}$ no

$\{v_1, v_{12}\}$ no

Connected Components of a graph G



$G(V, E)$

- Connectivity is an equivalence relation!

  Reflexive $u \sim u$

  symmetry $u \sim v \Rightarrow v \sim u$

  transitive $u \sim v, v \sim w \Rightarrow u \sim w$

- Connected components gives a partition of the vertices. Why?

How many connected components does a connected graph have?

How do we find them? DFS BFS

$u \sim v$ to say
u connected to v

Connected Components

Lemma. Every vertex belongs to exactly one connected component.

Proof 1: Follows from connectivity is equivalence relation.

Proof 2: Given u Define $S = \{v \in V \mid u \sim v\}$. Claim S is a CC.

If $v, w \in S$. Then $u \sim v$, $w \sim u$ hence $v \sim w$.

To show S is maximal assume $z \notin S$. Then $z \not\sim u$

If $S \cup \{z\}$ is connected. Then $\exists\, y \in S$ s.t. $y \sim z$.

$u \sim y$, $y \sim z \Rightarrow u \sim z \Rightarrow$ contradiction.

If S_1 is connected and $S_1 \ni u$ then $S_1 \subseteq S$

A simple bound on Components

Thm. Number of connected components is at least $|V| - |E|$

$|E| \geq |V| - \#CC$

Proof. By induction on the number of edges m.

Base case is true.

IH: Any graph with (m-1) edges has at least |V|-m+1 components.

$t \geq |V| - m + 1$

Add edge (u,v) to G

Case 1: (u,v) is same CC. Then # of CC remains the same

$t \geq |V| - m + 1 \geq |V| - m$

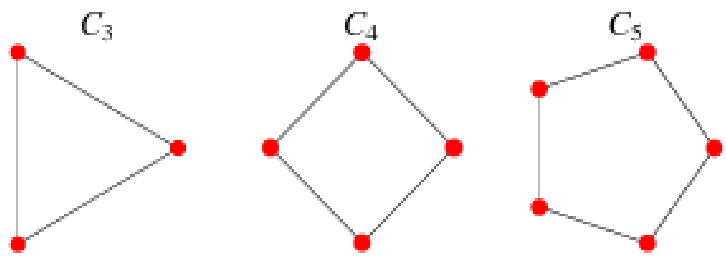
Case 2: (u,v) in different CC. Then # CC drops by 1.

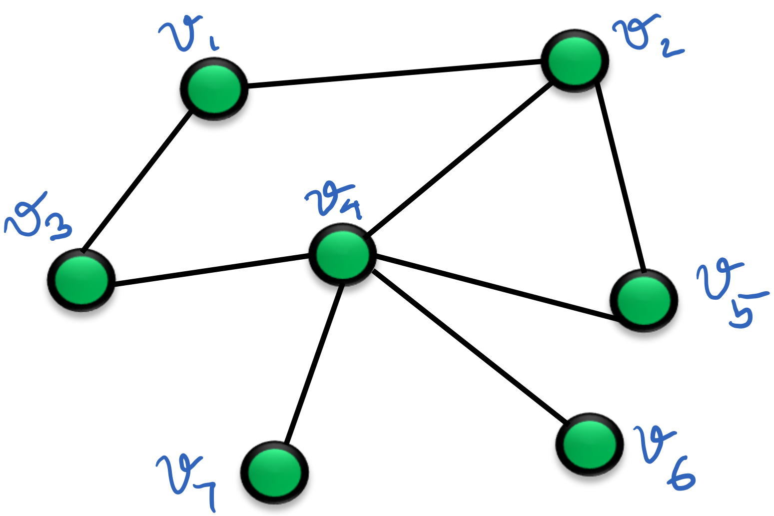
$t - 1 \geq |V| - m + 1 - 1 = |V| - m$

Corollary. Every connected graph has at least $n - 1$ edges

Cycles

Simple cycle: A connected graph where every vertex has degree exactly 2.





$G(V, E)$

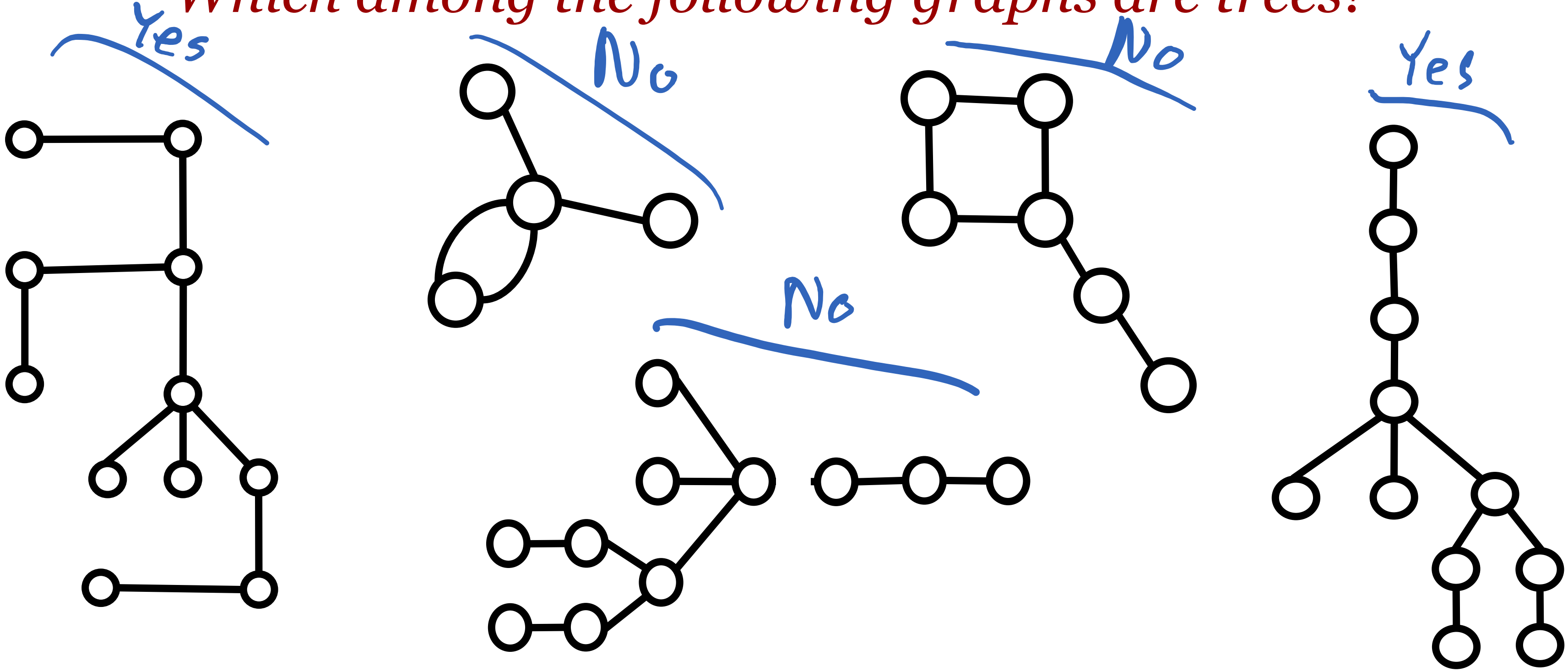
Cycles in a Graph G:

Any subgraph of G which is a cycle

e.g. $(v_1, v_2, v_5, v_4, v_3, v_1)$ forms a cycle in G

Trees

Which among the following graphs are trees?



Trees: A connected graph with no cycles

Equivalent Definitions of Trees

Theorem: Let G be a graph with n vertices and m edges

The following are equivalent:

1. G is a connected and acyclic (i.e. G is a tree)

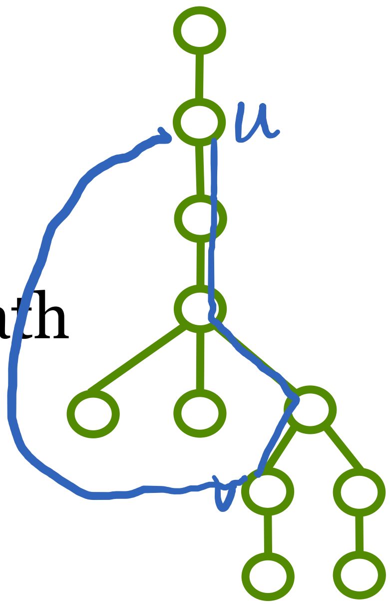
2. Every two vertices of G are joined by a unique path

3. G is connected and $m = n - 1$

   i.e. A tree is connected with minimal # of edges

4. G is acyclic and $m = n - 1$

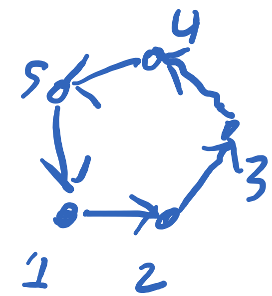
5. G is acyclic and if any two non-adjacent nodes are joined by an edge, the resulting graph has exactly one cycle



Proof of the Equivalence

How many implications do we need to show?

$$5 \times 4 = 20?$$



To prove this, it suffices to show

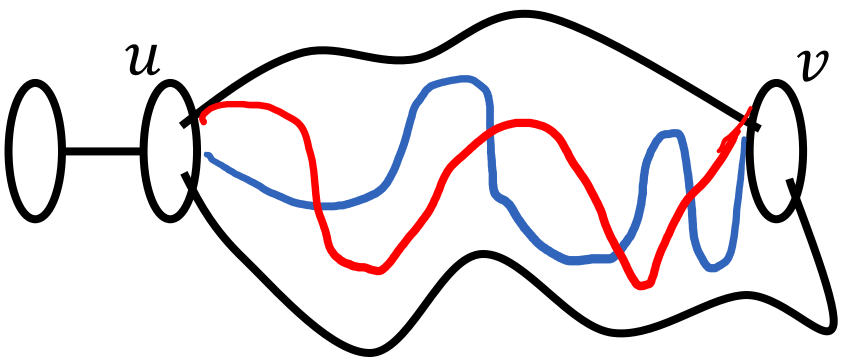
$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$$

Proof of 1 $\Rightarrow$ 2

Claim: If G is a tree (connected, acyclic), then every two nodes are joined by unique path.

Proof: (by contradiction). Suppose not.

Assume G is a tree that has two nodes u, v connected by two different paths:



Then there exists a cycle (formally: a closed walk. Then use PS5 #2)