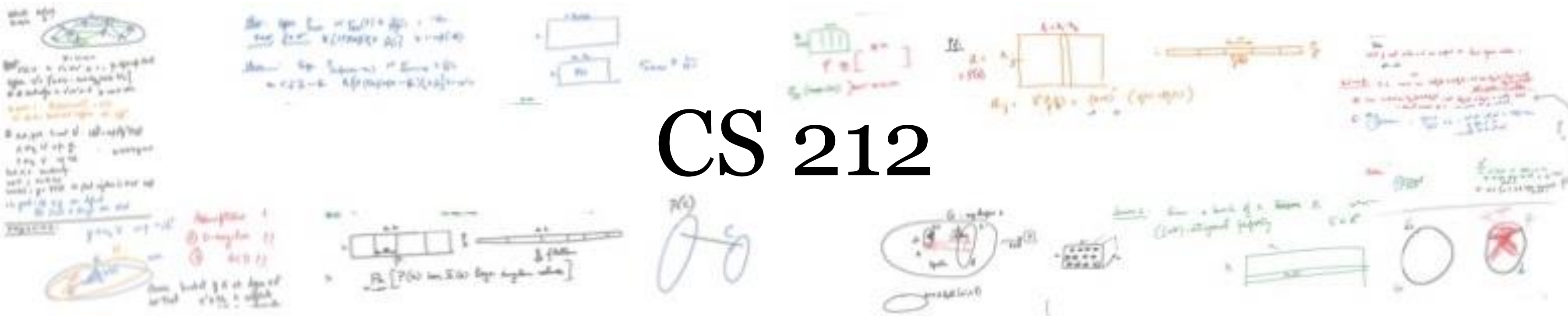


CS 212

Mathematical Foundations of Computer Science

Hall's theorem and Planar Graphs

Hall's Marriage Theorem

Neighbors of S: $N(S)$= set of nodes that has an edge with at least one vertex in S

Theorem. Bipartite graph $G(V = (L, R), E)$ has a perfect matching iff $|L| = |R| = n$ and for any subset $S \subseteq L, |N(S)| \geq |S|$.

- The condition of the theorem still holds if we swap roles of L, R

Theorem. Bipartite graph $G(V = (L, R), E)$ has a matching saturating L iff for any subset $S \subseteq L, |N(S)| \geq |S|$.

(similar statement for matching saturating R)

Proof of Hall's Theorem

Theorem. Bipartite graph $G(V = (L, R), E)$ has a ~~perfect~~ matching saturating L iff for every subset $S \subseteq L$, $|N(S)| \geq |S|$

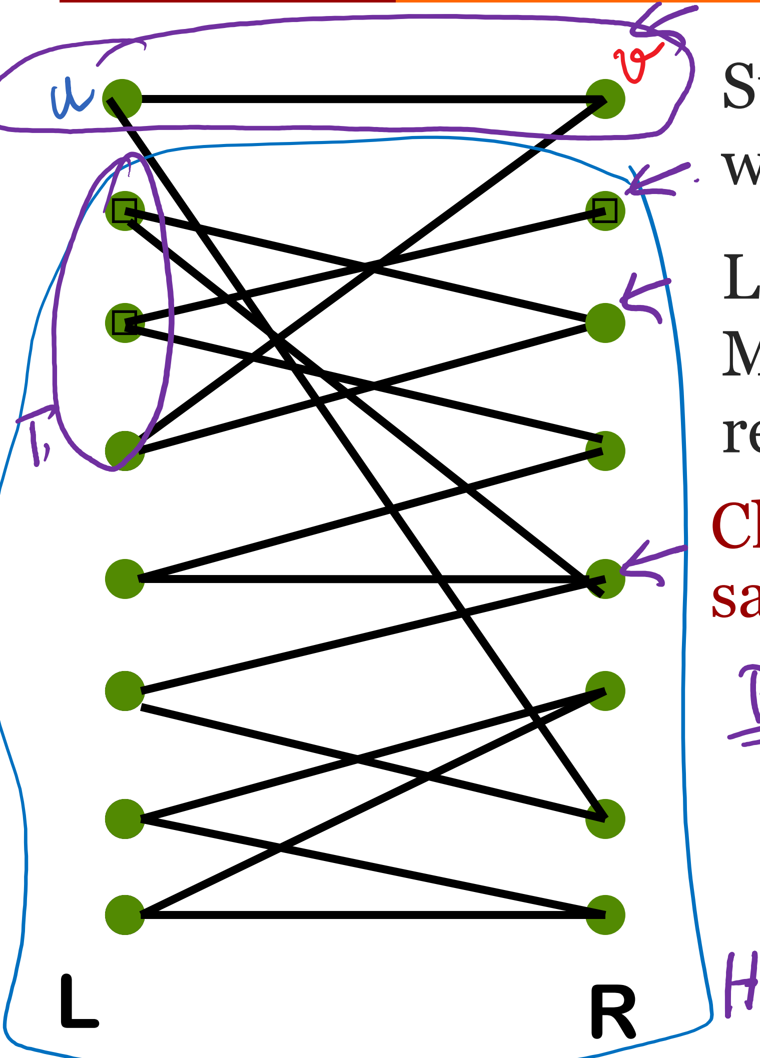
Proof: Strong induction on the size of L i.e., n. (Base case n=1 is easy)
I.H: Theorem is true for every bipartite graph G satisfying the conditions on < n vertices.
As with most inductive proofs, we'll try to reduce it to an instance of smaller size.

We'll try to remove some vertices in L, R so that the graph on remaining vertices satisfies Hall's condition.

Strictly : $\forall S \subsetneq L \; |N(S)| \geq |S|$

Case 1: Strictly larger neighborhoods

$\forall S \; |N(S)| \geq |S|$



Suppose for every S s.t. $|S| \leq n-1$, we have $|N_G(S)| \geq |S| + 1$

Let $u \in L$ be any vertex on the left. Match it to one of its neighbors $v \in R$, remove both and continue.

Claim: Induced graph H on $(L-u, R-v)$ satisfies Hall's condition.

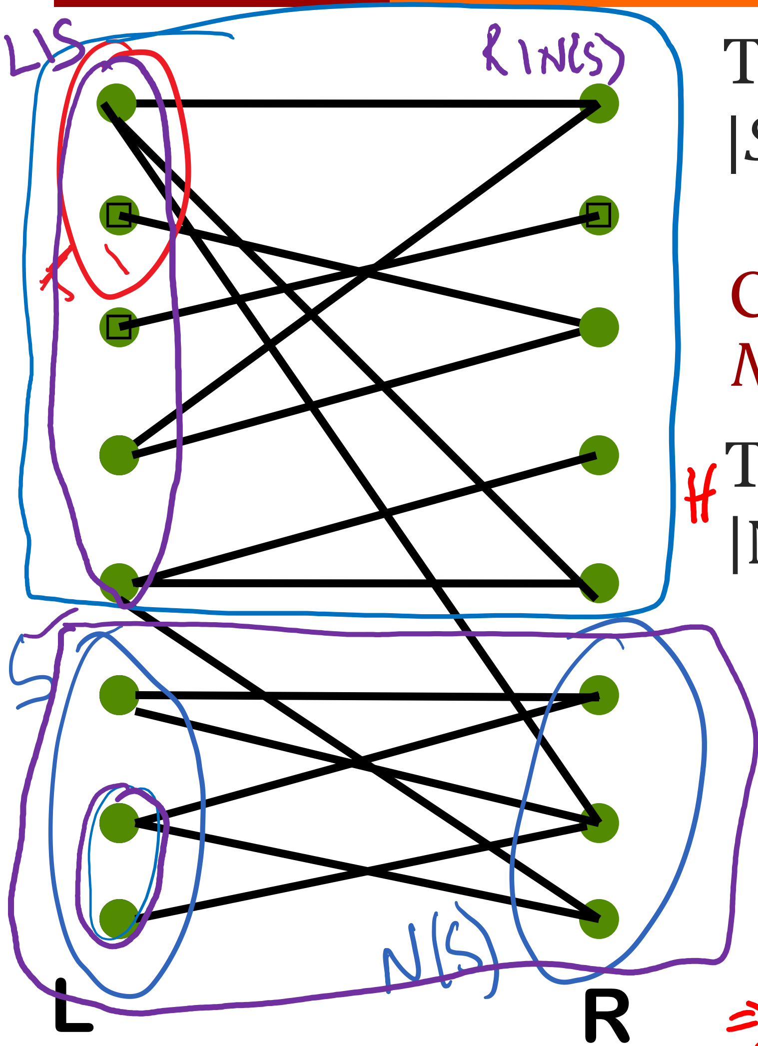
Pf: Consider $T \subseteq L \setminus \{u\}$

$|N_G(T)| \geq |T| + 1$

$|N_H(T)| = |N_G(T) \setminus \{v\}| \geq |N_G(T)| - 1$

$\geq (|T|+1) - 1 \geq |T|$.

Case 2: Exists S where $|N(S)| = |S|$



Take any S with $|S| \leq n-1$ s.t. $|N(S)| = |S|$. Match vertices in S to $N(S)$. Why?

(Verify Hall's condition holds for subgraph induced by S, N(S)).

Claim: Induced graph H on $(L \setminus S, R \setminus N(S))$ satisfies Hall's condition.

H

Take any $T \subseteq L \setminus S$. We will show that $|N_H(T)| \geq |N_G(S \cup T) \setminus N_G(S)| \geq |T|$.

$|N_G(S \cup T)| \geq |S \cup T| = |S| + |T|$

But $|N_G(S)| = |S|$.

$|N_H(T)| \geq |N_G(S \cup T)| - |N(S)| \geq \cancel{|S|} + |T| \cancel{-|N(S)|}$

$\geq |T|$. Hence the claim.

$\Rightarrow$ Matching saturating L\S in H. $\Rightarrow$ Matching saturating L in G.

Hall's Marriage Theorem

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Theorem. Bipartite graph $\mathrm{G}(V = (L, R), E)$ has a matching saturating L iff for any subset $S \subseteq L, |N(S)| \geq |S|$.
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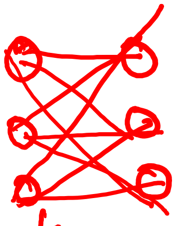
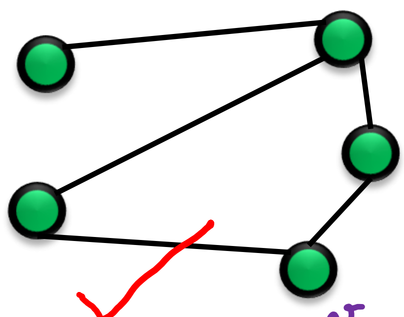


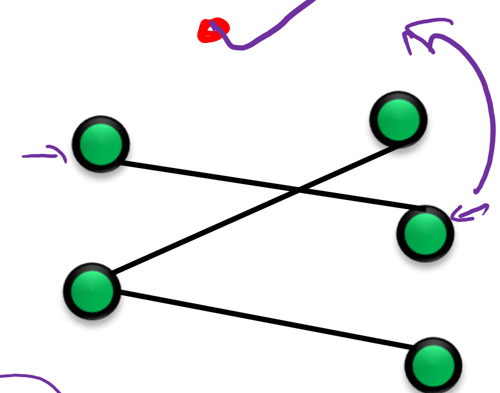
$K_{3,3}$

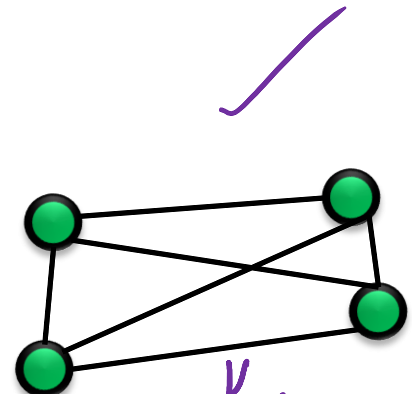
Planar Graphs

A graph is planar if it can be drawn (represented) on the plane without any crossing edges (no edges intersect).

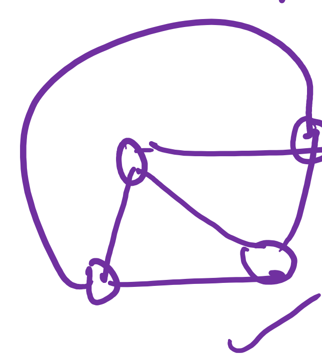
Which are planar?



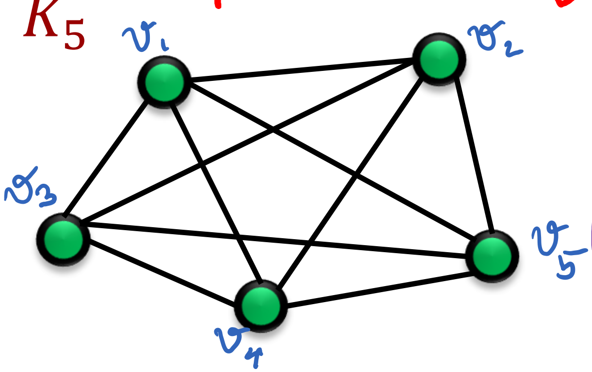


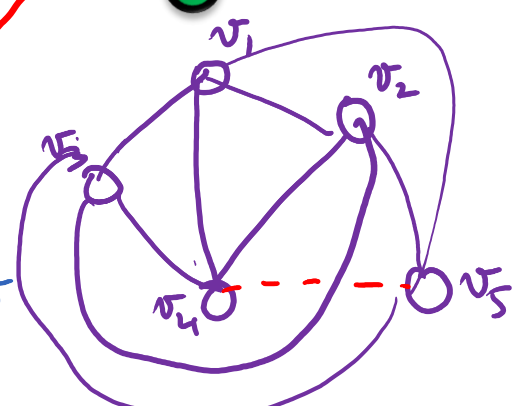


K_4



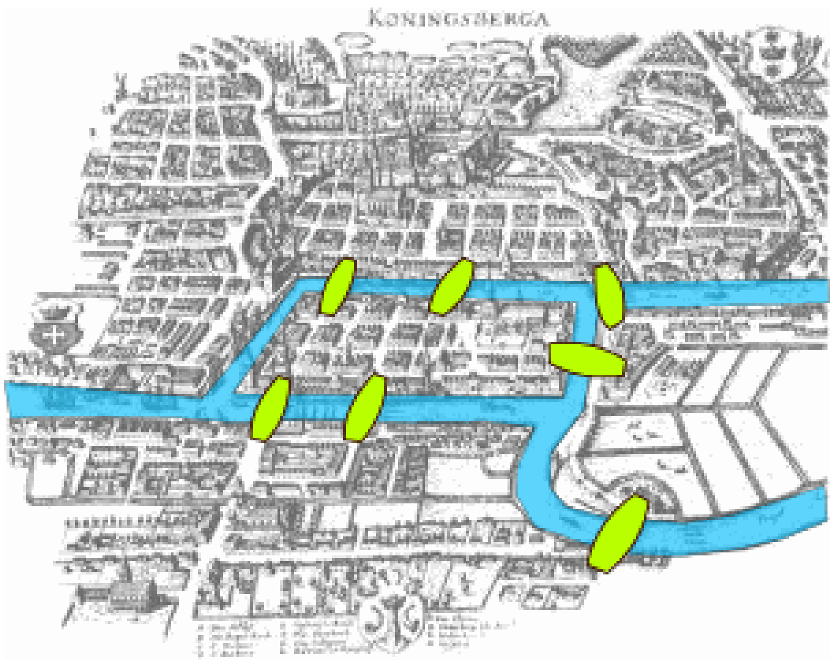
K_5





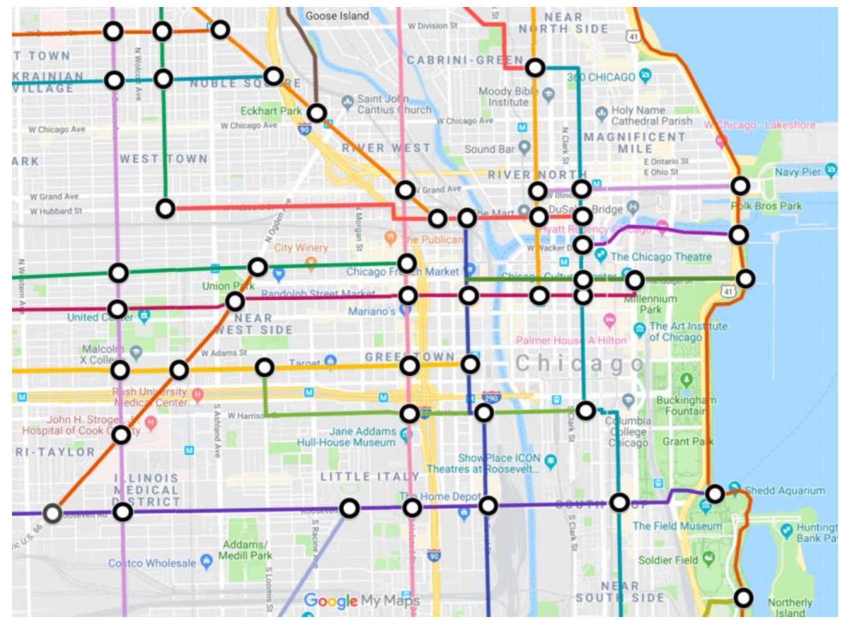
Examples of Planar Graphs

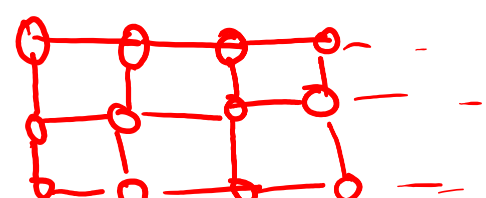
Inspired creation of graph theory by Euler

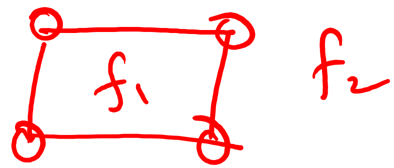


Konigsberg 7-bridge problem

Transportation networks





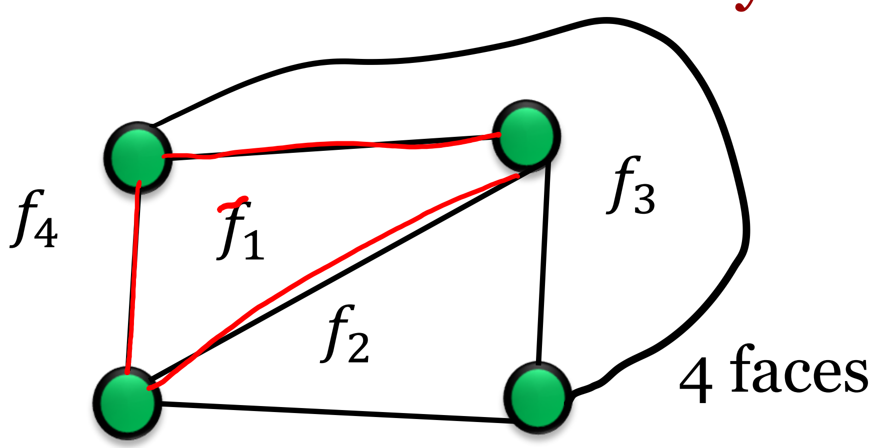


Faces

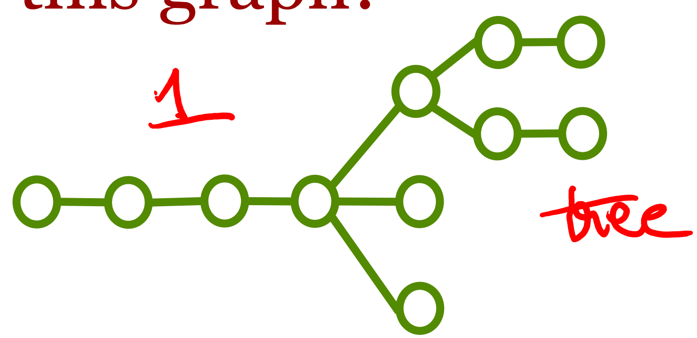
An embedding of planar graph splits the plane into disjoint faces

Face: A region bounded by a set of edges, vertices in embedding

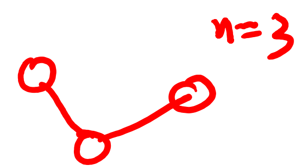
How many faces in this graph?



4 faces



Tree has only 1 face



- When $n \geq 4$, each face borders at least 3 edges.
- One "outside" face (do not forget the outside face).

Notation: in a planar graph, F denotes the set of faces.

Euler’s Formula

Thm. If G is a connected planar graph G with vertex set V (size n), edges E (m of them) and faces F (f of them), then

$$|V| - |E| + |F| = n - m + f = 2 \qquad f = m - n + 2$$

Proof of Euler’s formula

Proof approach 1: Induction

Proof approach 2: Non-inductive proof using “Dual graphs”.

(see https://www.ics.uci.edu/~eppstein/junkyard/euler/interdig.html for the proof.)

[FYI:] Proof of Euler's formula

Proof by Induction on the number of edges m.

Connected + : Tree

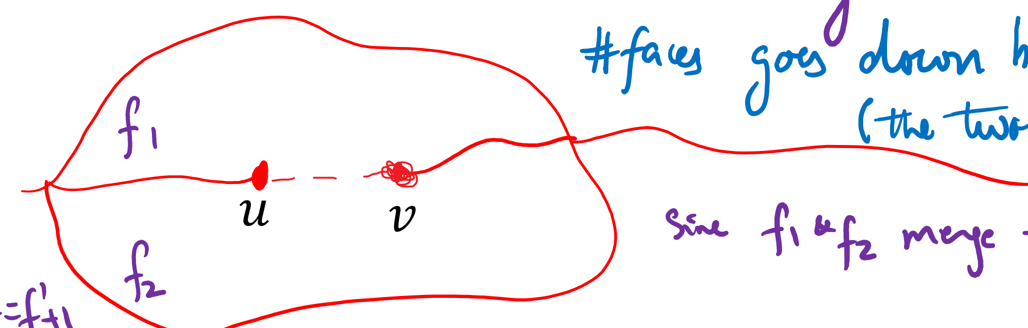
$n-(n-1)+1=2$ as reqd.

Base case: $m = n - 1$. Tree has only 1 face ($f = 1$)! $n - m + f = 2$

IH: True for any connected graph with < m edges.

As $m \geq n$, there is a cycle. Let e $= (u, v)$ be any edge on the cycle

Suppose we delete $e = (u, v)$



#faces goes down by exactly 1.

(the two faces incident on e become one face)

Since f_1 & f_2 merge to form one face.

$n'-m'+f'=2$ by I.H

$m'=m-1$

$n'=n$

$f'=f-1$

$m=m'+1$

$n=n'$, $f=f'+1$

Hence, $n - m + f = n' - (m' + 1) + (f' + 1)$

$= n' - m' + f' = 2$ by inductive hypothesis

$m \le \frac{n(n-1)}{2}$

Average degree of a planar graph ≤ 6

Thm. In a connected planar graph $G(V,E)$ on $n \geq 4$ vertices, the number of edges $m \leq 3n - 6$.

Average degree $= \frac{2|E|}{|V|} \le \frac{2(3n-6)}{n} \le 6$

Proof. By Euler's formula, $n - 2 = m - f$.
If $f = 1$, then $m \leq n - 1 \leq 3n - 6$ since $n \geq 2.5$

Want to bound f in terms of m. Count #(edge,face) incidences

Every face has how many edges? ≥ 3 #(edge, faces) incidences $\geq 3f$

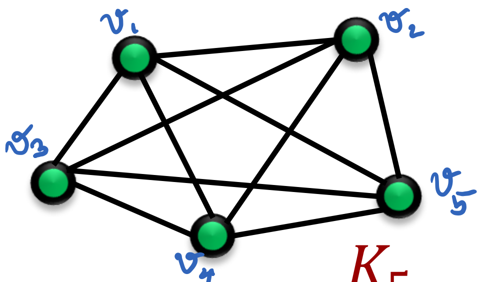
How many faces can an edge belong to? ≤ 2 #(edge, face) incidences $\leq 2m$

$3f \leq$ #edge-face incidences $\leq 2m \Longrightarrow$

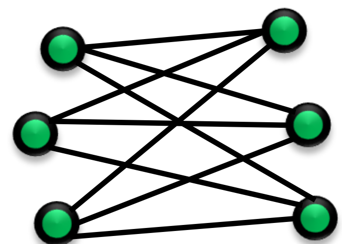
[Aside] Non planar graphs

How do you say when a graph is non-planar?

It clearly should not contain K_5 and $K_{3,3}$



K_5



$K_{3,3}$

Thm [Wagner]. Any graph that does not "contain" K_5 and $K_{3,3}$ is a planar graph.

"Contain": graph minor.

Thank you!