CS 212 Homework 7

Due 11:59PM on Tuesday, November 22, 2022.

Each question carries 5 points. Recall that every question here requires a proof. You can discuss the questions in groups of 3 (please list other students you discussed the problems with). Please remember that the assignment solutions need to be submitted individually and not as a group (and written/ typset with no collaboration). Solutions are only accepted in PDF format. Also please be as clear and legible as possible; any step that is ambiguous because of unclear writing will be interpreted as a mistake. See the Canvas page for instructions on how to submit.

Problem 1

Prove that a connected planar graph that contains no triangle, with m edges and n vertices satisfies: $m \leq 2n - 4$, for $n \geq 4$.

Problem 2

Alice is visiting a website with n pages. The structure of the website and Alice's browsing process is captured by a matrix A as follows: every hour if Alice is browsing the page i, she will goes to the page j next hour with probability A_{ji} (assume that each entry of A is non-negative, and the entries in each column add up to 1).

(a) Show that $\lambda = 1$ is an eigenvalue of $B = A^\top$ (B is the matrix transpose of A).

(b) If Alice starts browsing from page i, show that after t hours, the probability that she ends up browsing page j is $(A^t)_{ji}$.

Problem 3

Let n be a positive integer. The trace of an n $\times$ n square matrix A is defined as

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn},$$

where a_{ii} denotes the entry on the ith row and ith column of A.

Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix with n distinct positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $tr(A^2) = \sum_{j=1}^{n} \lambda_j^2$.

You can use the following fact (without proof): if $T \in \mathbb{R}^{n\times n}$ is a symmetric matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then $tr(T) = \sum_{j=1}^{n} \lambda_j$.

Problem 4

Given a simple graph $G = (V, E)$ with $|V| = n$ vertices, let A be its adjacency matrix. Let D be the matrix with D_{ii} being the degree of vertices i in G and 0 for all other entries. Consider the matrix $L = D - A$.

(a) Show that $\mathbf{1} = (\underbrace{1, 1, \ldots, 1}_{n \text{ 1s}})$ is an eigenvector of L with eigenvalue 0.

(b) Show that for every, $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{x}^\intercal L \mathbf{x} = \sum_{(i,j) \in E} (x_i - x_j)^2$$

(c) Show that if the graph is connected, all eigenvectors of L with eigenvalue 0 must be of the form $c\mathbf{1}$ for some number c, i.e., a vector with value c in all of its components.

Hint: What can we know about y_1, y_2 if $y_1^2 + y_2^2 = 0$?