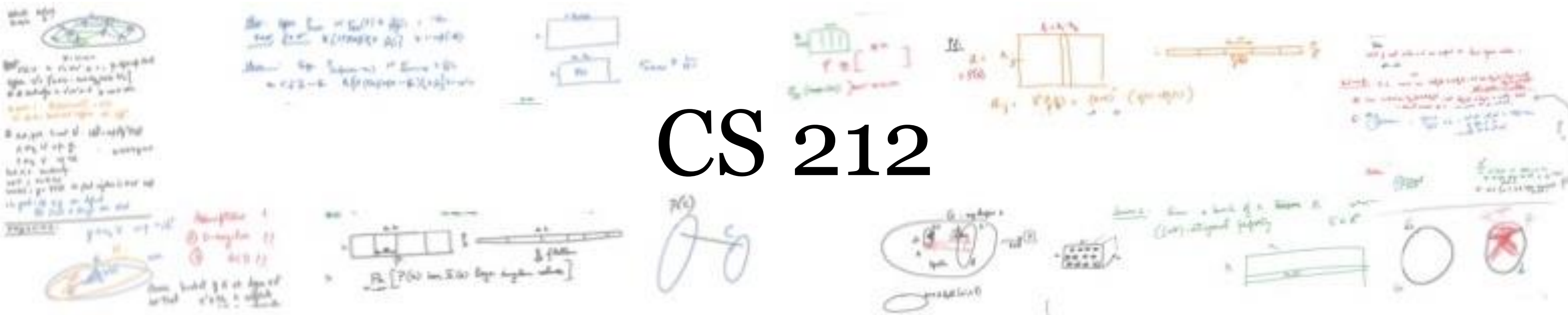


CS 212

Mathematical Foundations of Computer Science

Eigenvalues and Eigenvectors

Transpose

- **Transpose of a matrix:** Given $M \in \mathbb{R}^{m \times n}$, the transpose $M^T \in \mathbb{R}^{n \times m}$ has i, j entry equal to the j, i entry of M

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

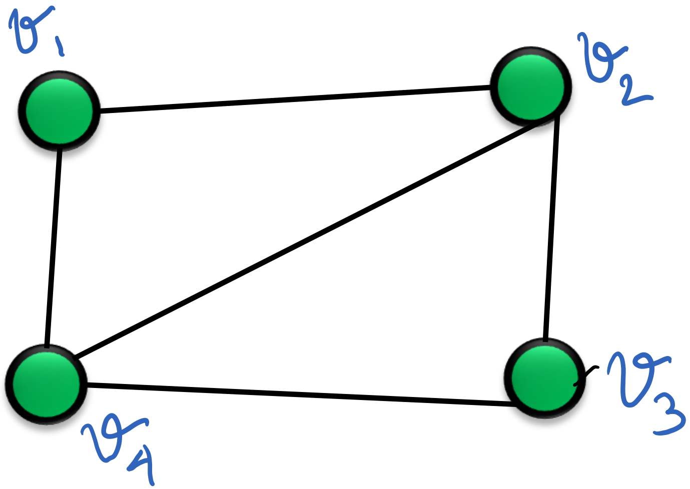
- M is **symmetric** if $M = M^T$

Adjacency Matrices

Graph G with n vertices.

The adjacency matrix is the n x n matrix A=[a_{ij}] with:

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge} \\ 0 & \text{if } (v_i, v_j) \text{ is not an edge} \end{cases}$$



Walks in a Graph

$A(i,j)$ tells us if there is an edge between vertex i and j.

Can we say something about connectivity? Longer paths?

Fact: The number of walks of length k from node i to node j is the entry in position (i,j) in the matrix A^k

Proof. By induction on length of walk. (Base case k=1).

$A^k = A^{k-1} \times A \Longrightarrow A^k(i,j) = \sum_{\ell=1}^{n} A^{k-1}(i,\ell)A(\ell,j)$

$A^{k-1}(i,\ell) =$ #walks from vertex i to ℓ

#walks k length walks from i to j

$=$

$= \sum_{\ell=1}^{n} A^{k-1}(i,\ell)A(\ell,j) = A^k(i,j)$

Eigenvalues, Eigenvectors

Refresher: Eigenvalues, Eigenvectors

Given any matrix $M \in \mathbb{R}^{n \times n}$, $e \in \mathbb{R}^n$ is an eigenvector iff for some scalar $\lambda \in \mathbb{R}$, $\boldsymbol{M e = \lambda e}$.

$(\lambda, \boldsymbol{e})$ is called an eigenvalue, eigenvector pair.

- An eigenvector $\boldsymbol{e}$ is vector that $\boldsymbol{M}$ does not change the direction of. M just scales $\boldsymbol{e}$ by λ.
- An eigenvector $\boldsymbol{e}$ is a vector on which $\boldsymbol{M}$ acts as scalar multiplication.

Whats the big deal?

Roughly: Eigenvalues and eigenvectors are the building blocks that make up matrices

Spectral Thm. For any $n \times n$ symmetric matrix M (over reals)

1. All its eigenvalues are real.
2. Further, there are n real eigenvalues (and eigenvectors)

$$(\lambda_1, e_1), (\lambda_2, e_2) \dots (\lambda_n, e_n),$$

such that every pair of eigenvectors is orthogonal i.e.

$$\langle e_i, e_j \rangle = 0 \text{ for } i \neq j.$$

3. In addition,

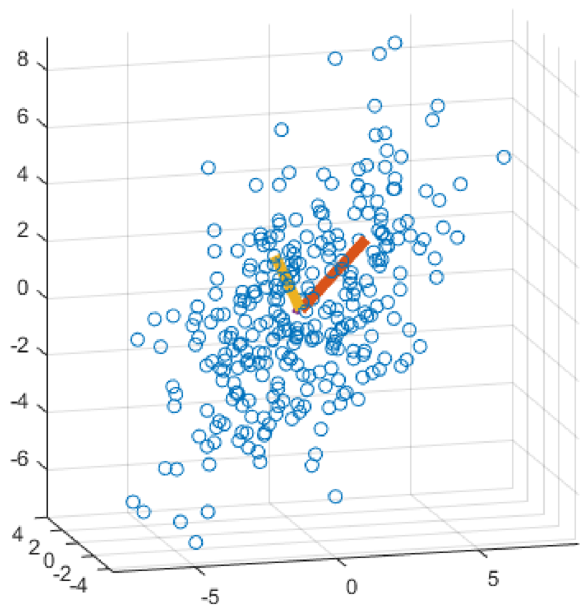
$$M = \sum_{i=1}^{n} \lambda_i e_i e_i^T$$

Which parts are important

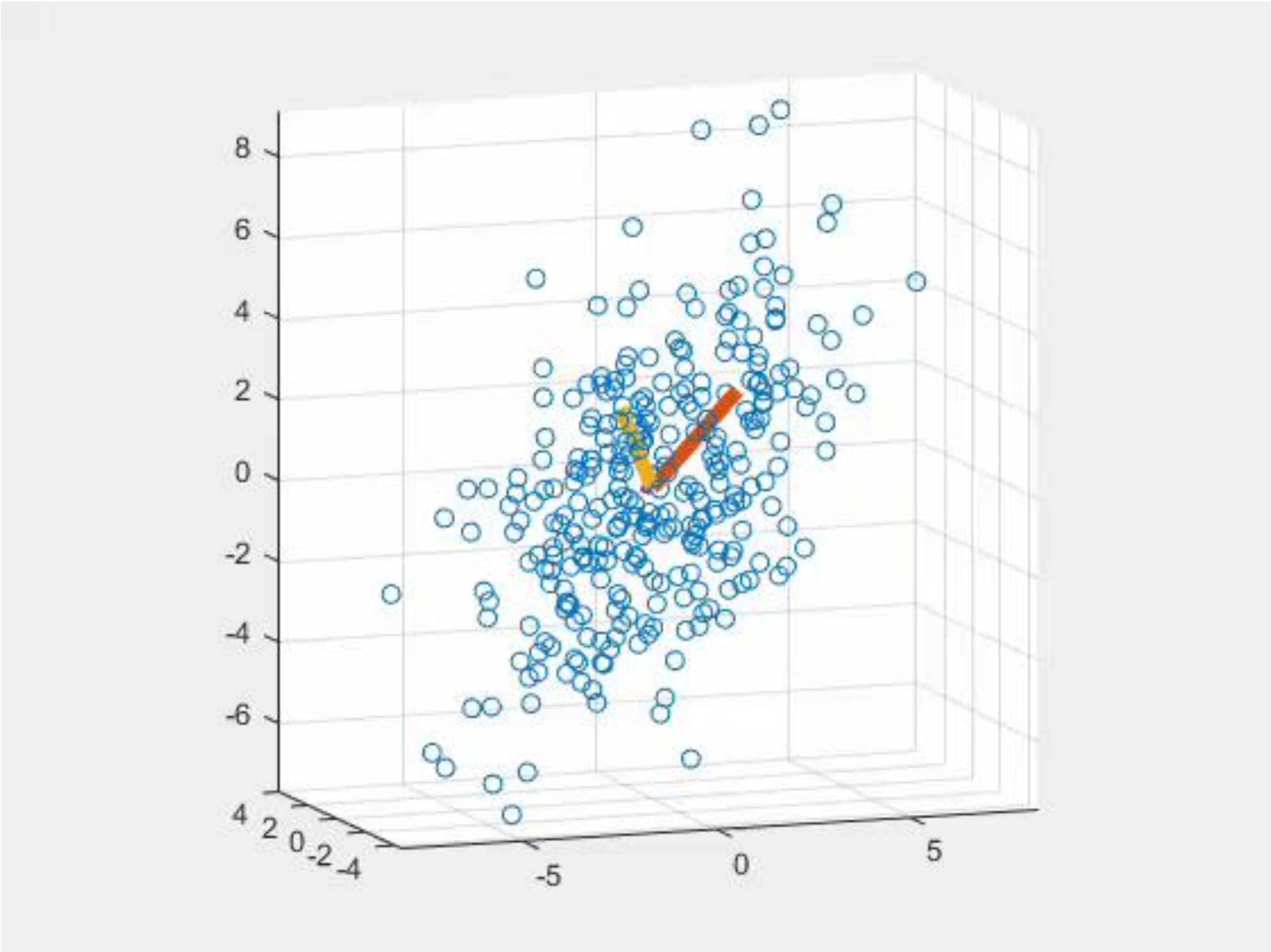
A concrete example: Suppose

$$M = 4e_1e_1^T + 2.5e_2e_2^T + \frac{1}{1000}e_3e_3^T \approx 4e_1e_1^T + 2.5e_2e_2^T$$

E.g., consider a data set of 300 points in three dimensions.



Which parts are important



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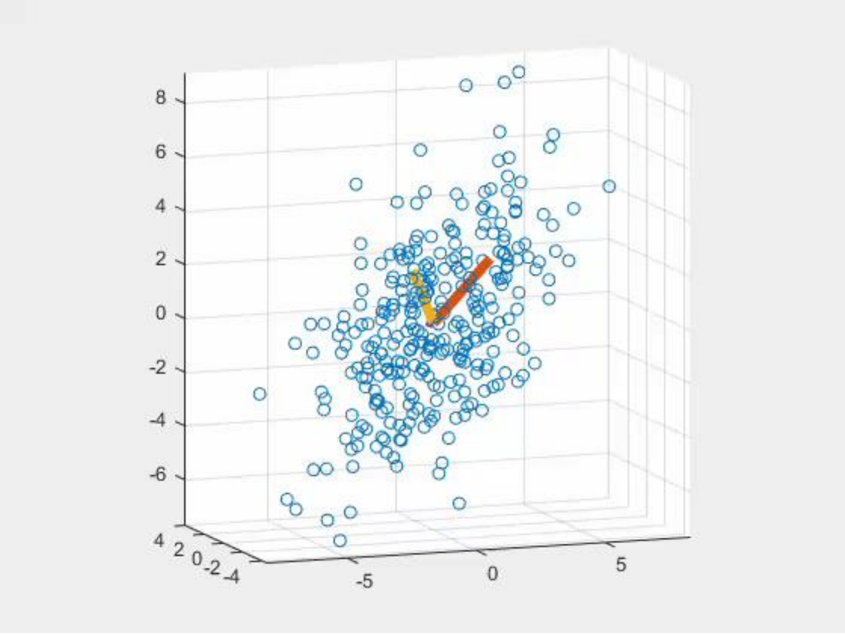
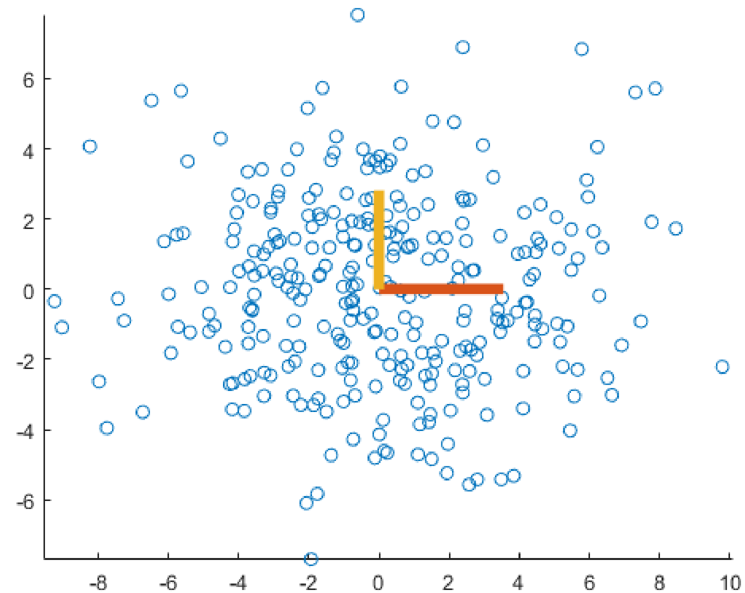


Which parts are important

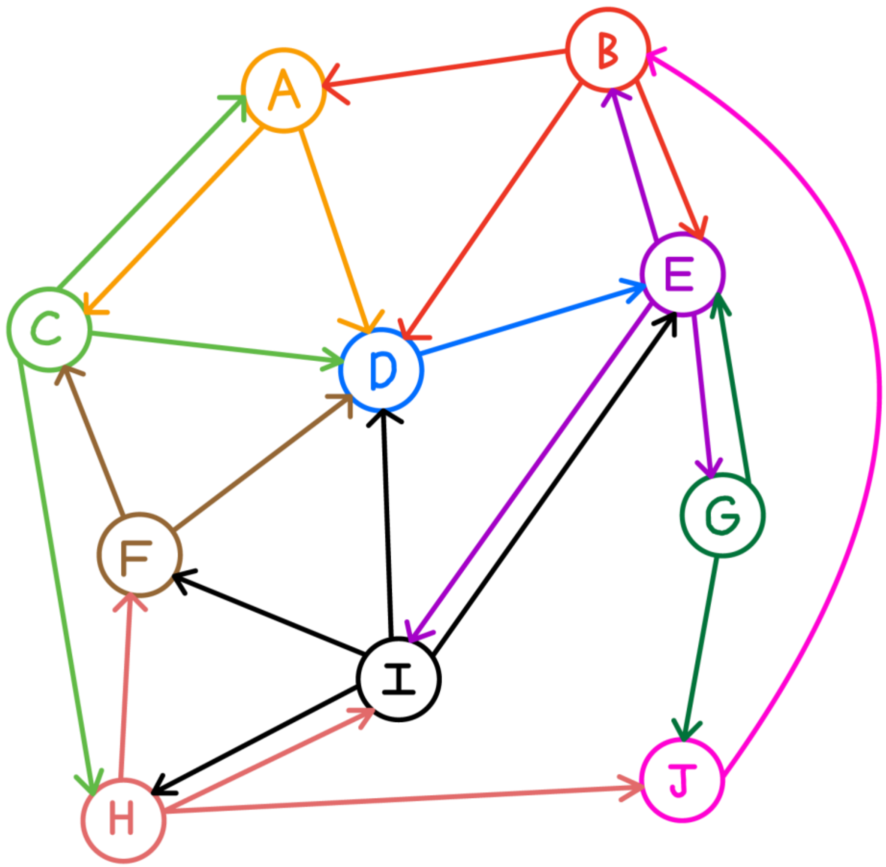
A concrete example: Suppose

$$M = 4e_1e_1^T + 2.5e_2e_2^T + \frac{1}{1000}e_3e_3^T \approx 4e_1e_1^T + 2.5e_2e_2^T$$

Can reduce to two dimensions!



Which Vertex is most important?



Applications of eigenvectors include

1. Graph theory
2. Page rank
3. Principal component analysis
4. Machine Learning
5. Convex optimization
6. Quantum mechanics
7. Control Theory
8. Infectious disease
9. Differential equations

Back to the start

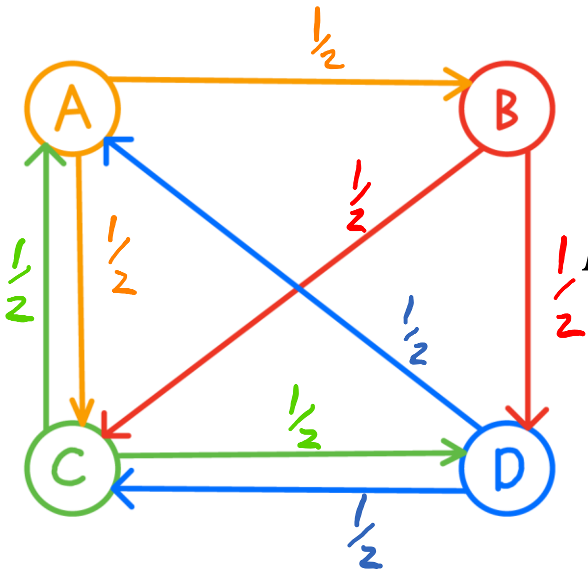
Given any matrix $M \in \mathbb{R}^{n \times n}$, $e \in \mathbb{R}^n$ is an eigenvector iff for some scalar $\lambda \in \mathbb{R}$, $\boldsymbol{Me} = \lambda\, \boldsymbol{e}$.

If e is an eigenvector of M and $c \in \mathbb{R}$, then ce is an eigenvector!

To check if a vector e is an eigenvector of M, just check if there is a $\lambda \in \mathbb{R}$ so that $\boldsymbol{Me} = \lambda\, \boldsymbol{e}$.

Is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ an eigenvector of the matrix $\begin{pmatrix} -2 & 6 \\ -2 & 5 \end{pmatrix}$?

Page rank: Where do people spend the most time

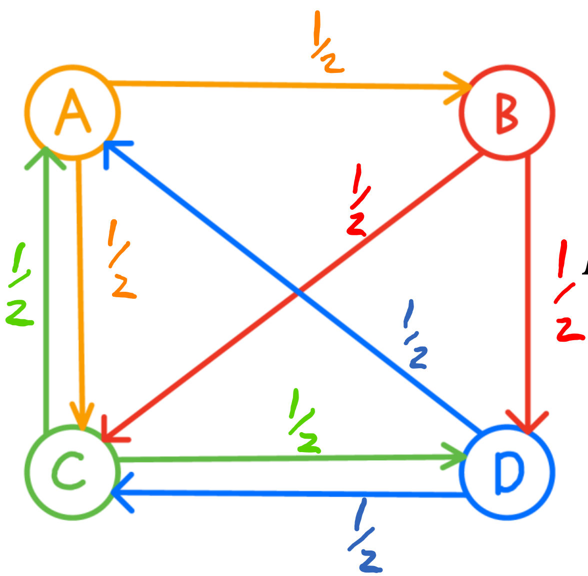


$$M = \begin{pmatrix} 0 & 0 & .5 & .5 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & .5 \\ 0 & .5 & .5 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$M(M^n v) \approx M^n v$ for large n. Solve M$e = e$!

	$Hr\ 0$	$Hr\ 1$	$Hr\ 2$	$Hr\ 3$	$Hr\ 4$	$Hr\ 5$	$Hr\ 6$	$Hr\ 7$
A	1	0	.25	.375	.25	.281	.297	.281
B	0	.5	0	.125	.188	.125	.141	.148
C	0	.5	.25	.375	.312	.344	.328	.336
D	0	0	.5	.125	.25	.25	.234	.242
	e	Me	M^2e	M^3e	M^4e	M^5e	M^6e	M^7e

Page rank: Where do people spend the most time



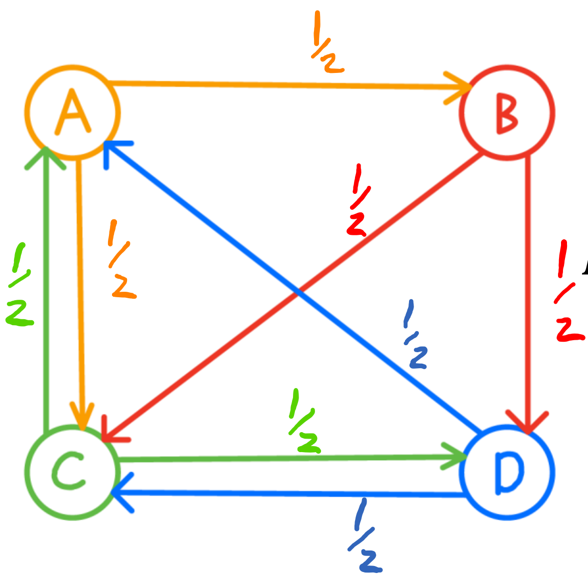
Solution to M$e = e$ is

$$M = \begin{pmatrix} 0 & 0 & .5 & .5 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & .5 \\ 0 & .5 & .5 & 0 \end{pmatrix} \qquad e = \begin{pmatrix} 0.286 \\ 0.143 \\ 0.333 \\ 0.238 \end{pmatrix}$$

Page C is the most popular!

	$Hr\ 0$	$Hr\ 1$	$Hr\ 2$	$Hr\ 3$	$Hr\ 4$	$Hr\ 5$	$Hr\ 6$	$Hr\ 7$
A	1	0	.25	.375	.25	.281	.297	.281
B	0	.5	0	.125	.188	.125	.141	.148
C	0	.5	.25	.375	.312	.344	.328	.336
D	0	0	.5	.125	.25	.25	.234	.242
	e	Me	M^2e	M^3e	M^4e	M^5e	M^6e	M^7e

Page rank: Where do people spend the most time



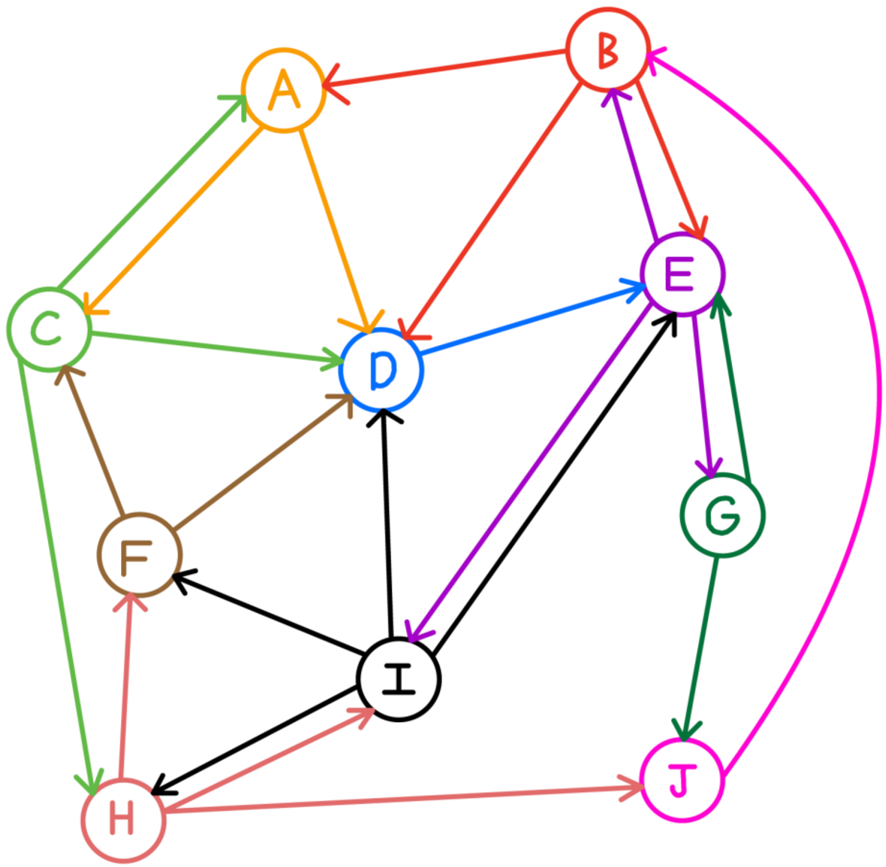
Solution to $Me = e$ is

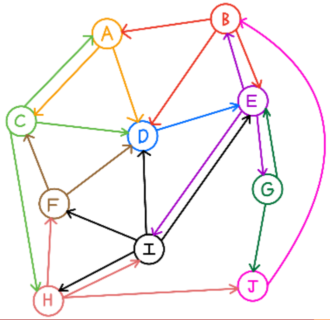
$$M = \begin{pmatrix} 0 & 0 & .5 & .5 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & .5 \\ 0 & .5 & .5 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0.286 \\ 0.143 \\ 0.333 \\ 0.238 \end{pmatrix}$$

Page C is the most popular!

	$Hr\ 8$	$Hr\ 9$	$Hr\ 10$	$Hr\ 11$	$Hr\ 12$	…	$Hr\ \infty$
A	.285	.287	.285	.286	.286	…	.286
B	.141	.143	.144	.143	.143	…	.143
C	.332	.334	.333	.333	.333	…	.333
D	.242	.236	.238	.238	.238	…	.238
	$M^8 e$	$M^9 v$	$M^{10} v$	$M^{11} v$	$M^{12} v$	…	$Me = e$

Page rank on our original graph





Page rank

For this graph, we again find e so that M$e = e$. This time

$$M = \begin{pmatrix} 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/3 & 1/3 & 0 & 0 & 1/2 & 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 1 & 0 & 0 & 1/2 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/3 & 0 & 0 \end{pmatrix} \qquad e = \begin{pmatrix} .067 \\ .145 \\ .053 \\ .144 \\ .261 \\ .040 \\ .087 \\ .043 \\ .102 \\ .058 \end{pmatrix}$$

Page E is the most popular!

Interpretation of the results