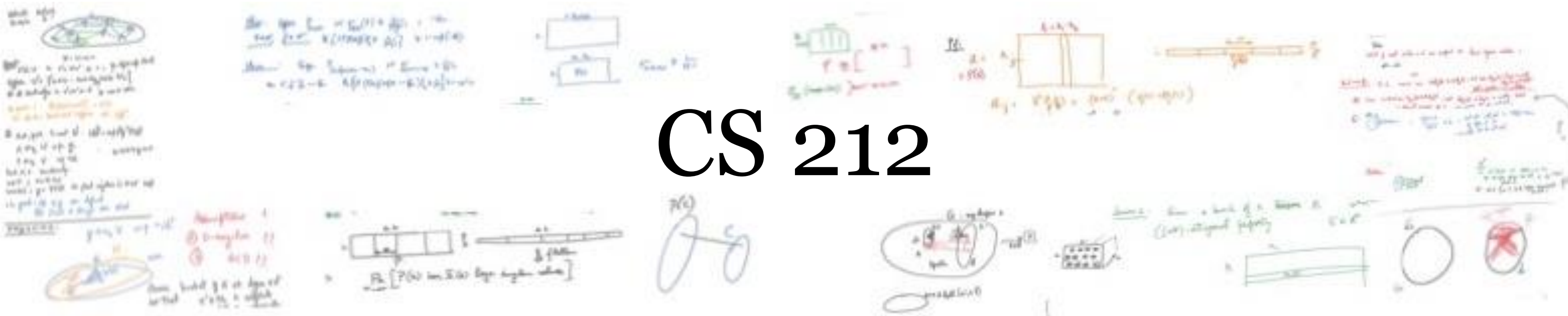


CS 212

Mathematical Foundations of Computer Science

Lecture 6: Counting, Permutations and Combinations

Counting

How do you count?

- *What is the number of 16 bit sequences with exactly 4 ones?*
- *Five kinds of donuts. How many ways to select a dozen?*

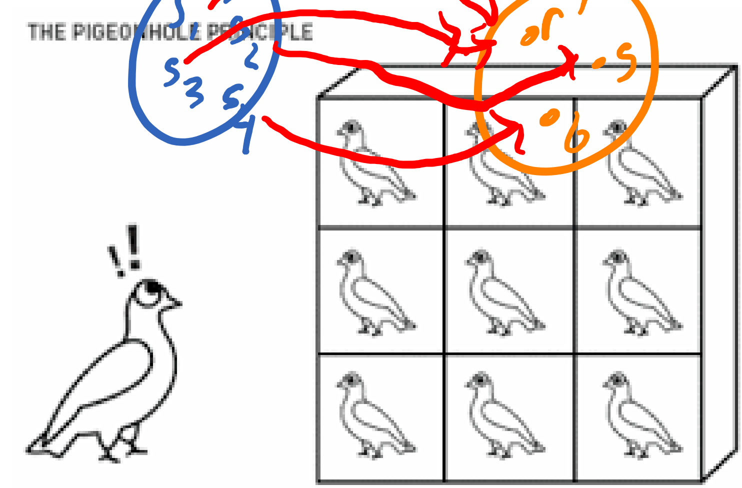
"Well, it's inventory time again — You do the rocks and I'll do the sticks."

Motivation:

- Combinatorial Proofs, Probability
- How many configurations does Algorithm search over?
- Time and Storage requirements of Algorithms

Counting and Concluding: Pigeonhole Principle

If socks can be either red, blue or green, how many socks do we need to find a matching pair? 4



Pigeonhole Principle (basic):

If there are n pigeons and $\leq n-1$ pigeon-holes, then there must be at least 2 pigeons in one of the holes.

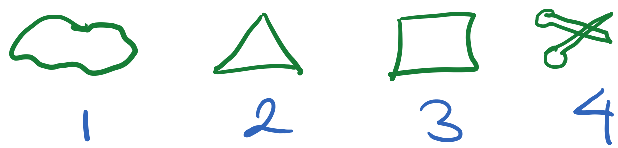
Pigeonhole Principle (general):

If $|Y| = n$ and $|X| \geq nk+1$, then for any function $f: X \to Y$, there exists a $y \in Y$ to which at least $k+1$ elements from X map to.

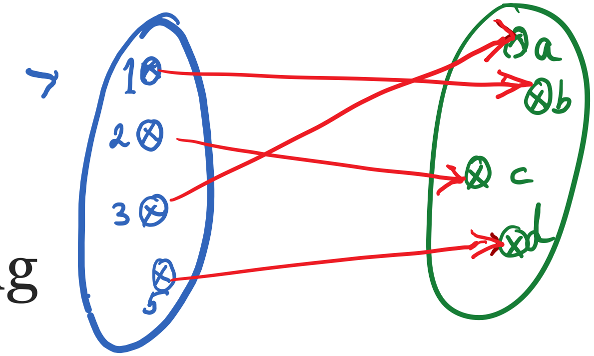
$$nk+1 \leq |X| = \left|\bigcup_{y \in Y} \mathrm{Preimage}(\{y\})\right| = \sum_{y \in Y} |\mathrm{Preimages}(\{y\})|$$

Rules and Tools for Counting

Counting One Thing by Another



Bijective function: one-to-one onto mapping



Thm. If $f : A \rightarrow B$ is a bijective function, then $|A|=|B|$

Five kinds of donuts. How many ways to select a dozen? no type 4

∃ a bijection between dozens and are 16 bit strings w/exactly 4 ones

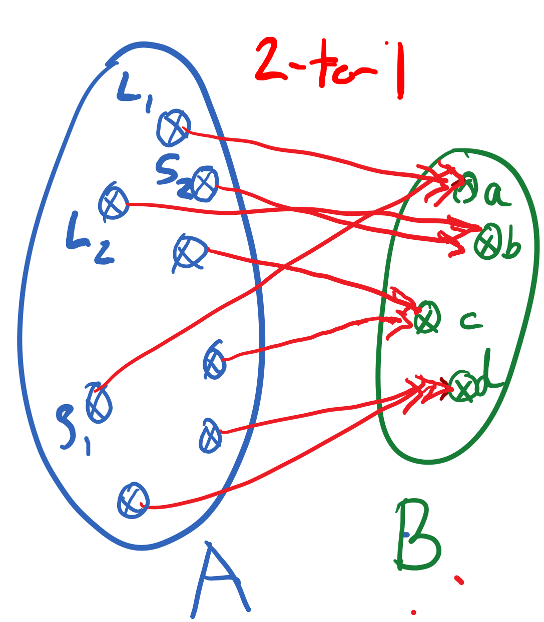
What is the number of 16 bit sequences with exactly 4 ones?

Counting using m-to-1 maps

Five kinds of donuts. You can either buy them all in small or all of them in large size. How many ways to select a dozen?

Twice as many as before

m-to-1 function: $f: A \rightarrow B$ is a m-to-1 function iff for every $z \in B$ there are exactly m elements $x \in A$ such that $f(x) = z$



Thm. If $f: A \rightarrow B$ is m-to-1 function, then $|A| = m \cdot |B|$

A and B disjoint. $A \cap B \neq \phi$

$|A \cup B| = |A| + |B| - |A \cap B|$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = \phi$

Simple Rules: Sum Rule

Sum Rule: If there are two disjoint or non-overlapping events that have n_1 and n_2 outcomes respectively, the total number of possible outcomes is $n_1 + n_2$.

If there are 6 republicans, 11 democrats in the primaries then the total number of non-independent candidates= $6+11=17$

- Candidate is either democrat or republican (disjoint)
- Sum up the counts

Sum Rule: If A and B are disjoint events, then $|A \cup B| = |A| + |B|$

Simple Rules : Product rule

Product Rule: If there are two independent events that have n_1 and n_2 outcomes respectively, the total number of possible pairs outcomes is $n_1 n_2$.

$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

$\boxed{\text{Product Rule: For sets A, B, one has } |A \times B| = |A| \times |B|}$

How many binary strings of size n? $= 2^n$

- 2 choices for the first bit (from {0,1})
- 2 choices for second bit (it doesn't depend on the first bit)
- .... n such choices

$\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\} = \{0,1\}^n$

$|\{0,1\}^n| = |\{0,1\}^{n-1} \times \{0,1\}| = |\{0,1\}^{n-1}|\,|\{0,1\}|$

Generalized Product Rule

8

8

A Chess Example:

In how many ways can you place two rooks on a chess board so they don't threaten each other?

$64 = n_1$ Event 1: Place black rook

$(64-15) = n_2$ Event 2: Place white rook

$64 \cdot (64-15) = 64 \cdot 49$

Generalized Product Rule: For a sequence of k events, if there are

n_1 possible outcomes for the first event,

n_2 possible outcomes for 2nd event for each outcome of 1st event,

....

n_k possibilities for k^{th} event for each sequence of outcomes for first (k-1) events,

then the total number of possible outcomes is $n_1 \times n_2 \times \cdots \times n_k$.

Ordering Items: Permutations

Given n different objects $a_1, a_2, a_3, \ldots, a_n$, how many different orderings are there? Answer= $\boxed{a_1}\boxed{a_2}\boxed{} \cdots \boxed{}\boxed{} = n!$

$n \times n-1 \times n-2 \times \cdots \times 2 \times 1$

Permutations and Factorial: Number of ways of arranging n items is $n! = 1 \times 2 \times 3 \times \cdots \times n$.

k-1 th box

How many ways to arrange n students in k chairs (one per chair)? $\boxed{a_1}\boxed{a_2}\boxed{a_3} \cdots \boxed{a_{k-1}}\boxed{} \leftarrow$ kth box

$n \times n-1 \times n-2 \quad \times\ n-(k-1)$

$n-k+1$

Permutations: Number of ways of arranging n items is k positions is ${}^{n}P_{k} = \mathrm{n(n-1)(n-2)} \ldots \mathrm{(n-k+1)} = \frac{n!}{(n-k)!}$

With or Without Repetition?

- Have n colors. Want to paint k tiles. How many ways?

  $n \times n \times \cdots \times n$ $\qquad n^k$

- Have n colors. Want to paint k tiles with different colors. How many ways?

  $n \times n-1 \times n-2 \times \cdots \times n-k+1$

Number of ways of arranging n items in k positions:

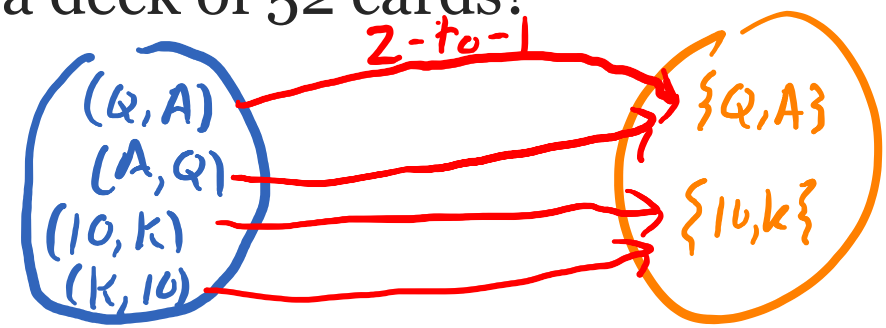
1. With Repetition = n^k
2. Without Repetition = ${}^{n}P_{k} = \frac{n!}{(n-k)!}$

Combinations: Choosing Items

How many ways of picking an ordered pair from a deck of 52 cards?

Event 1: Pick first card   Event 2: Pick second card 52×51

How many ways of *selecting a pair* i.e. picking an unordered pair from a deck of 52 cards?



$|\text{ordered pairs}| = 2\,|\text{unordered pairs}|$

$\frac{52 \cdot 51}{2} = \#\text{ unordered pairs}$

Overcounting: If each configuration is counted *exactly* m times, then the number of configurations $= \frac{\text{total count}}{m}$

Why? (remember m to 1 maps)!