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Convergence rates of recombining trees for pricing options on stocks under GBM and regime-switching models with known cash dividends

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ABSTRACT

In the literature there appear various kinds of binomial trees for pricing options on stocks under geometric Brownian motions (GBMs) with known cash dividends. The aim of this paper is to compare the performance of the existing binomial trees in aspect of the convergence rates, which are usually used to measure precisely how fast the approximate values converge to the exact one, and to give a theoretical proof of the convergence rates for the interpolation binomial trees which are based on a model that excludes the arbitrage possibilities. Also the paper extends the studies to the regime-switching models with known cash dividend payment.

1. Introduction

If a continuously paid dividend yield is used or the future dividend is specified as a fixed percentage of the stock price at dividend dates, then the classical option pricing models of Merton (1973), Black and Scholes (1973) can be used with only some minor modifications; however in practice a fixed cash value of dividend (instead of percentage) is often paid discretely in time and this causes challenging in the option pricing. Q3

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In the literature, there are three basic models for the option pricing when the underlying stock pays fixed cash dividends in the future, namely, escrowed model, forward model and piecewise lognormal model (see for example the overview by Frishling (2002) or Vellekoop and Nieuwenhuis (2006)). Denote the stock price by S_t, the dividend payable at time t_i by D_{t_i}, the maturity date by T, the interest rate by r, and the volatility by σ.

1.1. Escrowed model

Assume that the stock price minus the present value of all dividends to be paid until the maturity of the option follows a geometric Brownian motion. More precisely, the model for the price and capital price will be:

$$dC_t = rC_t dt + \sigma C_t dW_t, \quad S_t = C_t + \sum_{t<t_i<T} D_{t_i} e^{-r(t_i - t)}, \quad S_T = C_T,$$

where W_t is the standard Brownian motion and C_t the capital process (see also Roll (1977), Geske (1979) and Whaley (1981)).

1.2. Forward model

Assume that the stock price plus the future value of all dividends (from past dividend dates to today) follows a geometric Brownian motion. That is

$$dA_t = rA_t dt + \sigma A_t dW_t, \quad S_t = A_t - \sum_{0<t_i<t} D_{t_i} e^{r(t_i - t)}, \quad S_0 = A_0.$$

(see also Heath and Jarrow (1988)).

1.3. Piecewise lognormal model

Assume that the stock price jumps downward at dividend dates with jump size equal to the cash dividend payments at those dates and follows a geometric Brownian motion in between those dividend dates. That is

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t_i < t < t_{i+1}, \quad S_{t_i}^{+} = S_{t_i}^{-} - D_{t_i}, \tag{1}$$

where $S_{t_i}^{+}$ and $S_{t_i}^{-}$ are stock prices immediately before and after the dividend respectively.

In the escrowed model and forward model, the price of European options can be calculated explicitly using the Black–Scholes' formula with an adjusted values of the current stock price or strike price (see Frishling (2002)). But there are serious problems in these two models. As the stock price does not follow the geometric Brownian motion anymore in these two models, there is mispricing risk for the options with known cash dividend (see for example Frishling (2002)). More seriously this escrowed model admits arbitrage opportunity as discussed by Haug, Haug, and Lewis (2003) (also the paper discusses the stock dependent dividend). As argued by Haug et al. (2003), the piecewise lognormal model can exclude the arbitrage opportunity.

In the literature there are modifications of the escrowed or forward models – mixed escrowed and forward model and volatility adjustment model.

1.4. Mixed Escrowed and forward model

Bos and Vandermark (2002) propose to split the dividend linearly and then reduce the stock price by one part of the dividend and raise the strike price by the other part of the dividend.

1.5. Volatility adjustment model

This adjustment model is to modify the value of volatility which incorporates the dividend payments and use the value in an escrowed model (see Beneder and Vorst (2002), Bos, Gairat, and Shepeleva (2003) and Dai and Lyuu (2009)).

These modifications of the escrowed and forward models usually give prices that are close to those generated by the piecewise lognormal model. But they do not specify the stock price process, but only adjust the parameters of the Black–Scholes formulas. This model may give good results in the case of a single dividend, but the performance deteriorates when more than one cash dividend payment is considered.

Based on the above model assumptions, the binomial trees are proposed in the literature. Based on the piecewise lognormal model, the non-recombining binomial trees are proposed by Dai (2009), Areal and Rodrigues (2013, 2014). Based on the escrowed model, a recombining binomial tree is proposed by Hull (2012). Based on the mixed escrowed and forward model, a new re-combining binomial tree is constructed by Guo and Liu (2015). Based on the piecewise lognormal model, a recombining binomial tree with interpolation (which is named by interpolation binomial tree) is proposed by Vellekoop and Nieuwenhuis (2006). Moreover Guo and Liu (2015) show that their re-combining binomial tree (which is named by Guo–Liu's recombining binomial tree) is more accurate than the recombining binomial tree of Hull (2012). It appears that the interpolation binomial tree of Vellekoop and Nieuwenhuis (2006) is the most accurate and reliable one, as its underlying model assumption is the piecewise lognormal model that is free of arbitrage. Although the convergence for the interpolation binomial tree is proved by Vellekoop and Nieuwenhuis (2006), the convergence rates are not studied. The convergence rates are very important aspects for measuring how accurate an approach is.

In this paper, we mainly study the convergence rates of the interpolation binomial trees proposed by Vellekoop and Nieuwenhuis (2006) and the Guo–Liu's recombining binomial tree by Guo and Liu (2015). The numerical results show that the interpolation binomial tree has stable convergence rates; however the Guo–Liu's recombining tree has the oscillating rates. This means that the interpolation binomial tree is much more reliable than the Guo–Liu's recombining binomial tree. We provide a rigorous proof of the convergence rates for the interpolation binomial tree which are observed by the numerical results.

As the regime-switching models are the very important calibration to GBM models, we also extend the studies to the regime-switching models with known cash dividends. The regime-switching model was first introduced by Hamilton (1989). Hardy (2001) provided empirical analysis of the model. The regime-switching model allows the parameters of the market model to depend on a Markov chain and the model can reflect the information of the market environment. The Markov chain can ensure that the parameters change according to the market environment and at the same time preserve the simplicity of the model. The regime-switching model with fixed cash dividend payment is described as follows. Let the underlying asset prices S_t follow a two-states regime switching model under risk-neutral measure:

$$dS_t = r(X(t))S_t dt + \sigma(X(t))S_t dW_t, \quad t_i < t < t_{i+1}, \quad S_{t_i}^{+} = S_{t_i}^{-} - D_{t_i}, \tag{2}$$

where W_t is a standard Brownian motion, $X(t)$ is a continuous-time Markov chain with two states (x_1, x_2). Assume that at each state $X(t)=x_\kappa$, $\kappa=1, 2$, the interest rates $r(x_\kappa)=r_\kappa \geq 0$ and volatility $\sigma(x_\kappa)=\sigma_\kappa$ for $\kappa=1, 2$ are constants. Let $A=(a_{\kappa\ell})_{\kappa,\ell=1,2}$ be the generator matrix of the Markov chain process whose elements are constants satisfying $a_{\kappa\ell} \geq 0$ for $\kappa \neq \ell$ and $a_{\kappa 1}+a_{\kappa 2}=0$ for $\kappa=1$, 2. The plain regime-switching models without cash dividends can be seen in Hamilton (1989), Hardy (2001), Yao, Zhang, and Zhou (2006), Yuen and Yang (2010) and Liu (2012). In this paper, we design the recombining trees for the option pricing with stock price following the regime-switching model with fixed cash dividend payment (2) and study the convergence rates both numerically and theoretically.

The rest of the paper is arranged as follows: in Section 2, we describe the interpolation binomial trees and Guo–Liu's recombining binomial trees, recombing trees for regime-switching models, and numerically study the convergence rates of these approaches; In Section 3, we prove the convergence rates of the interpolation binomial trees without/with regime-switching; conclusions are given in the final section; proofs, tables and figures are collected in the appendix.

2. Recombining trees and numerical comparisons

2.1. Comparisons of recombining trees without regime-switching

We first revisit the Guo–Liu's recombining binomial trees (Guo & Liu, 2015), Hull's recombining trees (Hull, 2012), the interpolation binomial trees (Vellekoop & Nieuwenhuis, 2006), and the exact non-recombining binomial trees (Areal & Rodrigues, 2013) for valuation of the European options with known cash dividend and compare their performance of convergence rates.

For simplicity of presentation, we assume there is only once cash dividend payment with amount D_τ at ex-dividend date τ. It will be easy to manipulate for the multiple dividend payments.

The idea of Guo–Liu's recombining binomial trees is described as follows. The known cash dividend D_τ is divided into two parts as in Bos and Vandermark (2002):

$$D_0 = D_\tau \frac{T-\tau}{T} e^{-r\tau}, \quad D_T = D_\tau \frac{\tau}{T} e^{r(T-\tau)},$$

where the subscripts 0, τ and T denote the valuation date, the ex-dividend date and the expiration date, respectively.

An N-step binomial tree is built up from $S = S_0 - D_0$ forward as usual, with up-step ratio $u = e^{\sigma\sqrt{\Delta t}}$ and down-step ratio $d = e^{-\sigma\sqrt{\Delta t}}$, where σ is the volatility of the stock price and $\Delta t = T/N$ the length of the time step. Suppose that throughout the paper, the ex-dividend date τ is between $m\Delta t$ and $(m+1)\Delta t$. Then the prices at nodes between the valuation date 0 and the ex-dividend date τ are modified by adding back the proper future value of D_0:

$$Su^k d^{n-k} + D_0 e^{rn\Delta t}, \quad k = 0, 1, \ldots, n; \qquad n \le m.$$

For all the nodes after the dividend is paid, the stock prices are reduced by the appropriate present value of D_T:

$$Su^k d^{n-k} - D_T e^{-r(T-n\Delta t)}, \quad k = 0, 1, \ldots, n; \qquad n > m.$$

Then the binomial price of an European option with payoff $f(S_T) = (S_T - K)^+$ (call option) or $f(S_T) = (K - S_T)^+$ (put option) is given by the following formula:

$$V^0(S_0) = e^{-rT} \sum_{k=0}^{N} C(N, k) p^k q^{N-k} f(Su^k d^{N-k} - D_T),$$

where $C(N, k) = N!/[k!(N-k)!]$ (the binomial coefficient), $p = [\exp(r\Delta t) - d]/(u - d)$ (the probability of the stock price being up), and $q = 1 - p$.

For comparison, the Hull's recombining binomial tree (Hull, 2012) builds a binomial tree from $S_0 - \overline{D}$ onward instead, where $\overline{D}$ is the present value of the dividend. Prices are later modified by adding back the proper future value of $\overline{D}$ only for the nodes between the valuation date and the ex-dividend date.

The idea of exact non-recombining tree is that it builds a standard binomial tree between the valuation date 0 and the dividend date τ starting from S_0 and evaluates an European option using the usual binomial formula with the values at the last step of the binomial tree given by the Black–Scholes formula. More precisely, for $n \le m$, it builds a standard binomial tree with nodes at time $n\Delta t$ corresponding to stock prices:

$$S_0 u^k d^{n-k}, \quad k = 0, 1, \ldots, n,$$

and the European option is evaluated by the formula

$$V^0(S_0) = e^{-rT}\sum_{k=0}^{m} C(m,k)p^k q^{m-k}[\mathcal{BS}(S_0u^kd^{m-k} - D_T)],$$

where $C(m, k) = m!/[k!(m-k)!]$ (the binomial coefficient), $p = [\exp(r\Delta t) - d]/(u - d)$ (the probability of the stock price being up), $q = 1 - p$, and function $\mathcal{BS}(S)$ is the standard Black–Scholes formula for a Vanilla European option without dividend paying for the stocks. The pricing results from the exact non-recombining tree will be taken as the benchmark of the exact option prices in the following comparisons.

The interpolation binomial tree builds an N-step binomial tree from S_0 forward as usual. The European option before the ex-dividend date is evaluated by the standard binomial formula with the last-step option values given by evaluating the interpolation function of the binomial option values right after the ex-dividend date at points $S_0u^kd^{m-k} - D_\tau$, $k = 0, 1, \ldots, m$. More precisely, the price of the European option computed by the interpolation binomial tree approach is given by

$$V^0(S_0) = e^{-rT}\sum_{k=0}^{m} C(m,k)p^k q^{m-k}[\widetilde{V}^m(S_0u^kd^{m-k} - D_\tau)], \tag{3}$$

where $\widetilde{V}^m(S)$ is the piecewise polynomial interpolation function based on pairs:

$$(S_0u^kd^{m+1-k}, \widehat{V}^{m+1}(S_0u^kd^{m+1-k})), \quad k = 0, 1, \ldots, m+1,$$

and

$$\widehat{V}^i(S) = e^{-r\Delta t}[p\widehat{V}^{i+1}(uS) + q\widehat{V}^{i+1}(dS)], \quad i = m+1, \ldots, N-1, \tag{4}$$

with

$$\widehat{V}^N(S) = f(S). \tag{5}$$

Numerical results are given by Tables C.1–C.3 and Fig. C.1.

The exact non-recombining binomial tree (Areal & Rodrigues, 2013) is abbreviated by exact Non-RBT, Hull's recombining binomial tree (Hull, 2012) by Hull RBT, Guo–Liu's recombining binomial tree (Guo & Liu, 2015) by Guo–Liu RBT, interpolation binomial tree (Vellekoop & Nieuwenhuis, 2006) by interpolation RBT. The error is the difference between the computed option value and benchmark value of the option, where the benchmark value is computed by the exact non-recombining binomial tree using number of time steps $N = 6000$. The convergence rates are calculated by the commonly used formula (Ma & Zhu, 2015a)

$$\text{Rate} = \log\left|\frac{\text{Error with number of time step } N_1}{\text{Error with number of time step } N_2}\right| / \log\left(\frac{N_2}{N_1}\right).$$

We study the convergence rates for at-the-money (ATM) call option with one fixed cash dividend payment in Table C.1 and with two fixed cash dividend payments in Table C.3. From numerical comparisons in Tables C.1 and C.3, we observe that the convergence rates of exact Non-RBT and interpolation RBT are stable at about 1, the convergence rates of Guo–Liu RBT and Hull RBT are oscillating (for some number of time steps N, the convergence rates are negative, which means that the absolute value of the error becomes larger for a bigger N). Exact Non-RBT performs the best among these binomial trees; however since the Black–Scholes formula is not always available, Exact Non-RBT has very narrow applications. Interpolation RBT performs the second best and does not relies on the Black–Scholes formula. Therefore interpolation RBT will be a good candidate to use in practice. In Table C.2, we study the convergence rates for out-of-the-money (OTM) call option. From Table C.2, we can see that the exact Non-RBT has the smallest error and its convergence rate is stable at about 1; The interpolation RBT has the second smallest error and the error decays overall with rate 1, although the convergence rate is oscillating.

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We also plot the errors of the binomial trees in log-scale. Since if the convergence rate is p, then

$$\log(|\text{Error}|) \approx C - p\log(N),$$

the negative value of the slope of the tangent line of the curve for the error in log-scale is the rate of convergence. From Fig. C.1 (the left and right subfigures), we again confirm that both exact Non-RBT and interpolation RBT have the best performance in the convergence rates (stable at about 1) for the ATM call option. From the middle one of Fig. C.1, we observe that both exact Non-RBT and interpolation RBT have the smallest error for the OTM call option.

2.2. Design of recombining tree with regime-switching

For the regime-switching option pricing, we first build a trinomial tree as in Yuen and Yang (2010). Let $\Delta t = T/N$ be the time step-size. Then for all the regimes, the jump ratios of the lattice are taken as

$$u = e^{\sigma\sqrt{\Delta t}}, \quad m = 1, \quad d = e^{-\sigma\sqrt{\Delta t}}, \tag{6}$$

where σ must satisfy

$$\sigma > \max_{\kappa=1,2}\{\sigma_\kappa\}$$

such that the risk-neutral probability measure exists. As suggested by Yuen and Yang (2010), one possible value is

$$\sigma = \max_{\kappa=1,2}\{\sigma_\kappa\} + (\sqrt{1.5} - 1)\overline{\sigma},$$

where $\overline{\sigma}$ is the arithmetic mean or the root mean square of σ_κ, $\kappa = 1, 2$. For regime κ, let π_u^κ, π_m^κ, π_d^κ be risk neutral probabilities corresponding to when the stock price increases, remains the same and decreases, respectively. Then the values of the probabilities are given by, for $\kappa = 1, 2$,

$$\pi_m^\kappa = 1 - \frac{1}{\lambda_\kappa^2}, \tag{7}$$

$$\pi_u^\kappa = \frac{e^{r_\kappa \Delta t} - e^{-\sigma\sqrt{\Delta t}} - (1 - 1/\lambda_\kappa^2)(1 - e^{-\sigma\sqrt{\Delta t}})}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}, \tag{8}$$

$$\pi_d^\kappa = \frac{e^{\sigma\sqrt{\Delta t}} - e^{r_\kappa \Delta t} - (1 - 1/\lambda_\kappa^2)(e^{\sigma\sqrt{\Delta t}} - 1)}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}, \tag{9}$$

where $\lambda_\kappa = \sigma/\sigma_\kappa$. Then the trinomial option value for regime $\kappa = 1, 2$ at asset price S and time $t_i = i\Delta t$ is given by

$$V^i(S,\kappa) = e^{-r_\kappa \Delta t}\sum_{\ell=1}^{2} p_{\kappa\ell}[\pi_u^\kappa V^{i+1}(uS,\ell) + \pi_m^\kappa V^{i+1}(S,\ell) + \pi_d^\kappa V^{i+1}(dS,\ell)],$$
$$\text{for} \quad i = 0, 1, \ldots, m-1, \tag{10}$$

with

$$V^m(S,\kappa) = \widetilde{V}^m(S - D_\tau, \kappa), \quad \kappa = 1, 2, \tag{11}$$

where $\widetilde{V}^m(S,\kappa)$ is the piecewise polynomial interpolation of $\widehat{V}^{m+1}(S,\kappa)$ based on the trinomial tree nodes at time t_{m+1}, and for $\kappa = 1, 2$,

$$\widehat{V}^i(S,\kappa) = e^{-r_\kappa \Delta t}\sum_{\ell=1}^{2} p_{\kappa\ell}[\pi_u^\kappa \widehat{V}^{i+1}(uS,\ell) + \pi_m^\kappa \widehat{V}^{i+1}(S,\ell) + \pi_d^\kappa \widehat{V}^{i+1}(dS,\ell)],$$
$$\text{for} \quad i = m+1, \ldots, N-1, \tag{12}$$

with $\widehat{V}^N(S,\kappa)=f(S)$. In the above formula $p_{\kappa\ell}$ is the transition probability from regime state κ to state ℓ for the time interval with length Δt which is given by

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = e^{A\Delta t} = \mathrm{I} + \sum_{l=1}^{\infty} (\Delta t)^l A^l / l!, \tag{13}$$

where I is the identity matrix and A is the generator matrix of the Markov chain process.

The numerical results computed by the trinomial trees for the regime-switching are given by Tables C.4–C.6 and Fig. C.2.

Tables C.4, C.6 and Fig. C.2 (the left and right subfigures) show that the trinomial tree methods have the first-order convergence rates for the ATM call option pricing with fixed dividend payment under regime-switching models. From Table C.5 and the middle one of Fig. C.2, we observe that the error of the trinomial tree methods for the OTM call option decays overall with rate 1, although the convergence rate is oscillating.

3. Theoretical results of the convergence rates

3.1. Theoretical results without regime-switching

In this section, we prove the convergence rates of the interpolation binomial trees without regime-switching. To this end, we need to reformulate the interpolation binomial tree approach (3)–(5) into a more convenient (mathematical) form. Denote the nodes of binomial trees by

$$S_{j+1} = S_j u, \quad j = 0, 1, \ldots, N-1, \tag{14}$$

$$S_{j-1} = S_j d, \quad j = 0, -1, \ldots, -(N-1). \tag{15}$$

Then the interpolation binomial tree approach (3)–(5) is equivalently rewritten as

$$V^i(S_j) = e^{-r\Delta t}[pV^{i+1}(S_{j+1}) + qV^{i+1}(S_{j-1})], \quad i = 0, 1, \ldots, m-1; \quad j = 0, \pm 1, \ldots, \pm i \tag{16}$$

with

$$V^m(S_j) = \widetilde{V}^m(S_j - D_\tau), \quad j = 0, \pm 1, \ldots, \pm m, \tag{17}$$

where $\widetilde{V}^m(S)$ is the piecewise quadratic polynomial interpolation function (see Atkinson and Han (2005))

$$\widetilde{V}^m(S) = \Pi \widehat{V}^{m+1}(S),$$

with

$$\Pi \widehat{V}^{m+1}(S)|_{S\in[S_{j-1},S_{j+1}]} = \widehat{V}^{m+1}(S_j)\frac{(S-S_{j-1})(S-S_{j+1})}{(S_j-S_{j-1})(S_j-S_{j+1})} + \widehat{V}^{m+1}(S_{j-1})\frac{(S-S_j)(S-S_{j+1})}{(S_{j-1}-S_j)(S_{j-1}-S_{j+1})} + \widehat{V}^{m+1}(S_{j+1})\frac{(S-S_{j-1})(S-S_j)}{(S_{j+1}-S_{j-1})(S_{j+1}-S_j)}, \quad j = 0, \pm 1, \ldots, \pm(m-1).$$

Here

$$\widehat{V}^i(S_j) = e^{-r\Delta t}[p\widehat{V}^{i+1}(S_{j+1}) + q\widehat{V}^{i+1}(S_{j-1})], \quad i = m+1, \ldots, N-1; \quad j = 0, \pm 1, \ldots, \pm i, \tag{18}$$

with

$$\widehat{V}^N(S_j) = f(S_j), \quad j = 0, \pm 1, \ldots, \pm N. \tag{19}$$

The exact value of the European options with payoff $f(S)$ and the known cash dividend payment under the piecewise lognormal model (1) can be formulated into the following piecewise PDEs (see for example Bos and Vandermark (2002))

$$\frac{\partial V(S,t)}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V(S,t)}{\partial S^2}+rS\frac{\partial V(S,t)}{\partial S}-rV(S,t)=0,\quad t\in[0,\tau], \tag{20}$$

$$V(S,\tau)=\widehat{V}(S-D_\tau,\tau), \tag{21}$$

$$\frac{\partial \widehat{V}(S,t)}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2\widehat{V}(S,t)}{\partial S^2}+rS\frac{\partial \widehat{V}(S,t)}{\partial S}-r\widehat{V}(S,t)=0,\quad t\in[\tau,T], \tag{22}$$

$$\widehat{V}(S,T)=f(S), \tag{23}$$

where T is the maturity date, r the interest rate, and σ the volatility.

We now present the convergence rates of the interpolation binomial tree approach as follows.

Theorem 3.1. *Denote the errors of the interpolation binomial methods by*

$$\epsilon_j^i=V(S_j,t_i)-V^i(S_j),\quad i=0,1,\ldots,m;\quad j=0,\pm1,\ldots,\pm i,$$

and

$$\widehat{\epsilon}_j^i=\widehat{V}(S_j,t_i)-\widehat{V}^i(S_j),\quad i=m+1,\ldots,N-1;\quad j=0,\pm1,\ldots,\pm i.$$

Then the convergence rates are estimated by

$$\max_{-i\le j\le i}|\widehat{\epsilon}_j^i|=O(\Delta t),\quad i=m+1,\ldots,N-1, \tag{24}$$

and

$$\max_{-i\le j\le i}|\epsilon_j^i|=O(\Delta t),\quad i=0,1,\ldots,m. \tag{25}$$

Proof. See Appendix A. □

3.2. *Theoretical results with regime-switching*

In this section, we prove the convergence rates of the interpolation trinomial trees. To this end, we need to reformulate the interpolation trinomial tree approach (10)–(12) into a more convenient (mathematical) form. Denote the nodes of trinomial trees by

$$S_{j+1}=\ S_ju,\quad j=0,1,\ldots,N-1, \tag{26}$$

$$S_{j-1}=\ S_jd,\quad j=0,-1,\ldots,-(N-1). \tag{27}$$

Then the interpolation trinomial tree approach (10)–(12) is equivalently rewritten as

$$V^i(S_j,\kappa)=e^{-r_\kappa\Delta t}\sum_{\ell=1}^{2}p_{\kappa\ell}[\pi_u^\kappa V^{i+1}(S_{j+1},\ell)+\pi_m^\kappa V^{i+1}(S_j,\ell)+\pi_d^\kappa V^{i+1}(S_{j-1},\ell)],$$
$$\text{for}\quad i=0,1,\ldots,m-1;\quad j=0,\pm1,\ldots,\pm i;\quad \kappa=1,2, \tag{28}$$

with

$$V^m(S_j,\kappa)=\widetilde{V}^m(S_j-D_\tau,\kappa),\quad j=0,\pm1,\ldots,\pm m;\quad \kappa=1,2, \tag{29}$$

where $\widetilde{V}^m(S,\kappa)$ is the piecewise quadratic polynomial interpolation function (see Atkinson and Han (2005))

$$\widetilde{V}^m(S,\kappa)=\Pi\widehat{V}^{m+1}(S,\kappa),\quad \kappa=1,2,$$

with

$$\Pi \widehat{V}^{m+1}(S,\kappa)|_{S\in[S_{j-1},S_{j+1}]} = \widehat{V}^{m+1}(S_j,\kappa)\frac{(S-S_{j-1})(S-S_{j+1})}{(S_j-S_{j-1})(S_j-S_{j+1})} + \widehat{V}^{m+1}(S_{j-1},\kappa)\frac{(S-S_j)(S-S_{j+1})}{(S_{j-1}-S_j)(S_{j-1}-S_{j+1})}$$
$$+\widehat{V}^{m+1}(S_{j+1},\kappa)\frac{(S-S_{j-1})(S-S_j)}{(S_{j+1}-S_{j-1})(S_{j+1}-S_j)}, \quad \text{for} \quad j=0,\pm1,\ldots,\pm(m-1);$$
$$\kappa = 1,2.$$

Here

$$\widehat{V}^i(S_j,\kappa) = e^{-r_\kappa \Delta t}\sum_{\ell=1}^{2} p_{\kappa\ell}[\pi_u^\kappa \widehat{V}^{i+1}(S_{j+1},\ell) + \pi_m^\kappa \widehat{V}^{i+1}(S_j,\ell) + \pi_d^\kappa \widehat{V}^{i+1}(S_{j-1},\ell)],$$
$$\text{for} \quad i=m+1,\ldots,N-1; \quad j=0,\pm1,\ldots,\pm i; \quad \kappa=1,2, \tag{30}$$

with

$$\widehat{V}^N(S_j,\kappa) = f(S_j), \quad j=0,\pm1,\ldots,\pm N; \quad \kappa=1,2. \tag{31}$$

The exact value of the European options with payoff $f(S)$ and the known cash dividend payment under the regime-switching model (2) can be formulated into the following piecewise PDEs using Yao et al. (2006) and Bos and Vandermark (2002),

$$\frac{\partial V(S,t,\kappa)}{\partial t} + \frac{1}{2}\sigma_\kappa^2 S^2 \frac{\partial^2 V(S,t,\kappa)}{\partial S^2} + r_\kappa S\frac{\partial V(S,t,\kappa)}{\partial S} - r_\kappa V(S,t,\kappa)$$
$$+ a_{\kappa 1}V(S,t,1) + a_{\kappa 2}V(S,t,2) = 0, \quad t\in[0,\tau]; \quad \kappa=1,2, \tag{32}$$

$$V(S,\tau,\kappa) = \widehat{V}(S-D_\tau,\tau,\kappa), \quad \kappa=1,2, \tag{33}$$

$$\frac{\partial \widehat{V}(S,t,\kappa)}{\partial t} + \frac{1}{2}\sigma_\kappa^2 S^2 \frac{\partial^2 \widehat{V}(S,t,\kappa)}{\partial S^2} + r_\kappa S\frac{\partial \widehat{V}(S,t,\kappa)}{\partial S} - r_\kappa \widehat{V}(S,t,\kappa)$$
$$+ a_{\kappa 1}\widehat{V}(S,t,1) + a_{\kappa 2}\widehat{V}(S,t,2) = 0, \quad t\in[\tau,T]; \quad \kappa=1,2, \tag{34}$$

$$\widehat{V}(S,T,\kappa) = f(S), \quad \kappa=1,2. \tag{35}$$

We now present the convergence rates of the interpolation trinomial tree approach for the option pricing with the stocks governed by the regime-switching model and paying fixed cash dividends.

Theorem 3.2. *Denote the errors of the interpolation trinomial methods by*

$$\epsilon_j^i(\kappa) = V(S_j,t_i,\kappa) - V^i(S_j,\kappa), \quad i=0,1,\ldots,m; \quad j=0,\pm1,\ldots,\pm i; \quad \kappa=1,2,$$

and

$$\widehat{\epsilon}_j^i(\kappa) = \widehat{V}(S_j,t_i,\kappa) - \widehat{V}^i(S_j,\kappa), \quad i=m+1,\ldots,N-1; \quad j=0,\pm1,\ldots,\pm i; \quad \kappa=1,2.$$

Then the convergence rates are estimated by

$$\max_{-i\le j\le i}|\widehat{\epsilon}_j^i(\kappa)| = O(\Delta t), \quad i=m+1,\ldots,N-1; \quad \kappa=1,2, \tag{36}$$

and

$$\max_{-i\le j\le i}|\epsilon_j^i(\kappa)| = O(\Delta t), \quad i=0,1,\ldots,m; \quad \kappa=1,2. \tag{37}$$

Proof. See Appendix B. □

4. Conclusions

In this paper we studied the convergence rates of recombining binomial trees for option pricing with fixed cash dividend payment and gave the proof of convergence rates for the interpolation binomial trees. The interpolation binomial tree has stable convergence rate 1 while Guo–Liu RBT of Guo and Liu (2015) has oscillating convergence rates. We also designed the recombining trinomial method for the option pricing with fixed cash dividend under regime-switching models and prove the convergence rates. The numerical results show that the convergence rates are 1 which is further confirmed by the theory. In the future we shall extend the studies to the derivative pricing with path-dependent options and investigate the optimal dividend.

Uncited reference

Veiga and Wystup (2009).

Acknowledgement

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Appendix A. Proof of Theorem 3.1

Define reminders

$$R_j^i = V(S_j, t_i) - e^{-r\Delta t}[pV(S_{j+1}, t_{i+1}) + qV(S_{j-1}, t_{i+1})],$$
$$\text{for} \quad i = 0, 1, \ldots, m-1; \quad j = 0, \pm 1, \ldots, \pm i, \tag{A.1}$$

$$\widehat{R}_j^i = \widehat{V}(S_j, t_i) - e^{-r\Delta t}[p\widehat{V}(S_{j+1}, t_{i+1}) + q\widehat{V}(S_{j-1}, t_{i+1})],$$
$$\text{for} \quad i = m+1, \ldots, N-1; \quad j = 0, \pm 1, \ldots, \pm i. \tag{A.2}$$

Then using the techniques of Taylor's expansion and the PDEs (20) and (22), it can be estimated that (see Kwok (1998) for the detailed derivations)

$$|R_j^i| = O((\Delta t)^2), \quad |\widehat{R}_j^i| = O((\Delta t)^2). \tag{A.3}$$

Adding (18) to (A.2) gives that

$$\widehat{\epsilon}_j^i = e^{-\Delta t}[p\widehat{\epsilon}_{j+1}^{i+1} + q\widehat{\epsilon}_{j-1}^{i+1}] + \widehat{R}_j^i. \tag{A.4}$$

Since $p+q=1$, (A.4) gives that

$$|\widehat{\epsilon}_j^i| \le e^{-r\Delta t} \max_{-(i+1)\le j\le i+1} |\widehat{\epsilon}_j^{i+1}| + |\widehat{R}_j^i|.$$

Therefore,

$$\max_{-i\le j\le i} |\widehat{\epsilon}_j^i| \le e^{-r\Delta t} \max_{-(i+1)\le j\le i+1} |\widehat{\epsilon}_j^{i+1}| + |\widehat{R}_j^i|. \tag{A.5}$$

Then iterating of (A.5) leads to

$$\max_{-i\le j\le i} |\widehat{\epsilon}_j^i| \le e^{-r(N-i)\Delta t} \max_{-N\le j\le N} |\widehat{\epsilon}_j^N| + \sum_{k=0}^{N-i-1} |\widehat{R}_j^{i+k}| e^{-kr\Delta t}. \tag{A.6}$$

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From (19) and (23) with $S = S_j$, $t_N = T$, we have that

$$\widehat{\epsilon}_j^N \equiv \widehat{V}(S_j, t_N) - \widehat{V}^N(S_j) = 0. \tag{A.7}$$

From (A.3) we estimate that

$$\begin{aligned}\sum_{k=0}^{N-i-1} |\widehat{R}_j^{i+k}| e^{-kr\Delta t} &\leq O((\Delta t)^2) \cdot \frac{1-e^{-(N-k)r\Delta t}}{1-e^{-r\Delta t}} \leq O((\Delta t)^2) \cdot \frac{1-e^{-Nr\Delta t}}{1-e^{-r\Delta t}} \\ &= O((\Delta t)^2) \cdot \frac{1-e^{-rT}}{1-e^{-r\Delta t}} \leq O((\Delta t)^2) \cdot \frac{1-e^{-rT}}{r\Delta t - r^2(\Delta t)^2/2} \leq O((\Delta t)^2) \cdot \left(\frac{1-e^{-rT}}{r-r^2/2}\right)\frac{1}{\Delta t} = O(\Delta t).\end{aligned} \tag{A.8}$$

Combining (A.7) and (A.8) into (A.6) gives the estimation (24).

Adding (16) to (A.1) gives that

$$\epsilon_j^i = e^{-\Delta t}\left[p\epsilon_{j+1}^{i+1} + q\epsilon_{j-1}^{i+1}\right] + R_j^i. \tag{A.9}$$

Analogously to the derivation of (A.6), we can derive that

$$\max_{-i \leq j \leq i} |\epsilon_j^i| \leq e^{-r(m-i)\Delta t} \max_{-m \leq j \leq m} |\epsilon_j^m| + \sum_{k=0}^{m-i-1} |R_j^{i+k}| e^{-kr\Delta t}, \tag{A.10}$$

with

$$\begin{aligned}|\epsilon_j^m| &= |V(S_j, t_m) - V^m(S_j)| = |\widehat{V}(S_j - D_\tau, t_m) - \widetilde{V}^m(S_j - D_\tau)| = |(\widehat{V}(S_j - D_\tau, t_m) - \Pi\widehat{V}(S)|_{S=S_j-D_\tau}) \\ &\quad + (\Pi\widehat{V}(S)|_{S=S_j-D_\tau} - \widetilde{V}^m(S_j - D_\tau))| \leq |\widehat{V}(S_j - D_\tau, t_m) \\ &\quad - \Pi\widehat{V}(S)|_{S=S_j-D_\tau}| + |\Pi\widehat{V}(S)|_{S=S_j-D_\tau} - \widetilde{V}^m(S_j - D_\tau)|.\end{aligned} \tag{A.11}$$

From the convergence theory of the piecewise interpolation (see Atkinson and Han (2005)), we know that

$$|\widehat{V}(S_j - D_\tau, t_m) - \Pi\widehat{V}(S)|_{S=S_j-D_\tau}| = O((S_{j+1} - S_{j-1})^3) = O((\Delta t)^{3/2}), \tag{A.12}$$

where we used in the last step that

$$S_{j+1} - S_{j-1} = (u-d)S_j = (e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}})S_j = O(\sqrt{\Delta t}).$$

Assuming $S_j^* \equiv S_j - D_\tau$ falls in interval $[S_{\widehat{j}-1}, S_{\widehat{j}+1}]$ for some $\widehat{j} \in \{0, \pm 1, \ldots, \pm(m-1)\}$, we estimate the second term in (A.11) as

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$$
\begin{aligned}
&|\Pi\widehat{V}(S)|_{S=S_j-D_\tau}-\widetilde{V}^m(S_j-D_\tau)|=|\Pi\widehat{V}(S)|_{S=S_j^*}-\widetilde{V}^m(S_j^*)|=|\Pi\widehat{V}(S)|_{S=S_j^*}-\Pi\widehat{V}^{m+1}(S)|_{S=S_j^*}|\le|\widehat{V}(S_{\widehat{j}})\\
&-\widehat{V}^{m+1}(S_{\widehat{j}})|\left|\frac{(S_j^*-S_{\widehat{j-1}})(S_j^*-S_{\widehat{j+1}})}{(S_{\widehat{j}}-S_{\widehat{j-1}})(S_{\widehat{j}}-S_{\widehat{j+1}})}\right|+|\widehat{V}(S_{\widehat{j-1}})-\widehat{V}^{m+1}(S_{\widehat{j-1}})|\left|\frac{(S_j^*-S_{\widehat{j}})(S_j^*-S_{\widehat{j+1}})}{(S_{\widehat{j-1}}-S_{\widehat{j}})(S_{\widehat{j-1}}-S_{\widehat{j+1}})}\right|\\
&+|\widehat{V}(S_{\widehat{j+1}})-\widehat{V}^{m+1}(S_{\widehat{j+1}})|\left|\frac{(S_j^*-S_{\widehat{j-1}})(S_j^*-S_{\widehat{j}})}{(S_{\widehat{j+1}}-S_{\widehat{j-1}})(S_{\widehat{j+1}}-S_{\widehat{j}})}\right|\le\frac{1}{4}(|\widehat{V}(S_{\widehat{j}})-\widehat{V}^{m+1}(S_{\widehat{j}})|+|\widehat{V}(S_{\widehat{j-1}})\\
&-\widehat{V}^{m+1}(S_{\widehat{j-1}})|+|\widehat{V}(S_{\widehat{j+1}})-\widehat{V}^{m+1}(S_{\widehat{j+1}})|)=O(\Delta t), \qquad (A.13)
\end{aligned}
$$

where in the last step we used (24). Combining (A.12) and (A.13) with (A.11), we obtain that

$$|\epsilon_j^m|=O(\Delta t),\quad j=0,\pm1,\ldots,\pm m. \qquad (A.14)$$

Finally using (A.3) and (A.14) in (A.10), we arrive at (25). Thus Theorem 3.1 is proved.

Appendix B. Proof of Theorem 3.2

We note that the estimation (36) is exactly the result of Ma and Zhu (2015a). Moreover following Ma and Zhu (2015a, 2015b), we have that

$$\begin{pmatrix}\max\limits_{-i\le j\le i}|\epsilon_j^i(1)|\\ \max\limits_{-i\le j\le i}|\epsilon_j^i(2)|\end{pmatrix}\le D^{m-i}\begin{pmatrix}\max\limits_{-m\le j\le m}|\epsilon_j^m(1)|\\ \max\limits_{-m\le j\le m}|\epsilon_j^m(2)|\end{pmatrix}+\left(\mathrm{I}+\sum_{l=1}^{m-i-1}D^l\right)\begin{pmatrix}|O\left((\Delta t)^2\right)|\\ |O\left((\Delta t)^2\right)|\end{pmatrix}, \qquad (B.1)$$

where for positive integer l,

$$D^l=\begin{pmatrix}1+l(a_{11}-r_1)\Delta t+O((\Delta t)^2) & la_{12}\Delta t+O((\Delta t)^2)\\ la_{21}\Delta t+O((\Delta t)^2) & 1+l(a_{22}-r_2)\Delta t+O((\Delta t)^2)\end{pmatrix},$$

and

$$\mathrm{I}+\sum_{l=1}^{m-i-1}D^l=\begin{pmatrix}(m-i)+\frac{(m-i)(m-i-1)}{2}(a_{11}-r_1)\Delta t+O\left((\Delta t)^2\right) & \frac{(m-i)(m-i-1)}{2}a_{12}\Delta t+O\left((\Delta t)^2\right)\\ \frac{(m-i)(m-i-1)}{2}a_{21}\Delta t+O\left((\Delta t)^2\right) & (m-i)+\frac{(m-i)(m-i-1)}{2}(a_{22}-r_2)\Delta t+O\left((\Delta t)^2\right)\end{pmatrix},$$

with

$$|\epsilon_j^m(\kappa)|=|V(S_j,t_m,\kappa)-V^m(S_j,\kappa)|=|\widehat{V}(S_j-D_\tau,t_m,\kappa)-\widetilde{V}^m(S_j-D_\tau,\kappa)|,\quad \kappa=1,2.$$

Similarly to the derivation of (A.14), using (29) and (36) we derive that

$$|\epsilon_j^m(\kappa)|=O(\Delta t),\quad j=0,\pm1,\ldots,\pm m;\qquad \kappa=1,2. \qquad (B.2)$$

Therefore combining (B.2) with (B.1) we obtain (37). Thus the proof of Theorem 3.2 is complete.

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Appendix C. Tables and figures

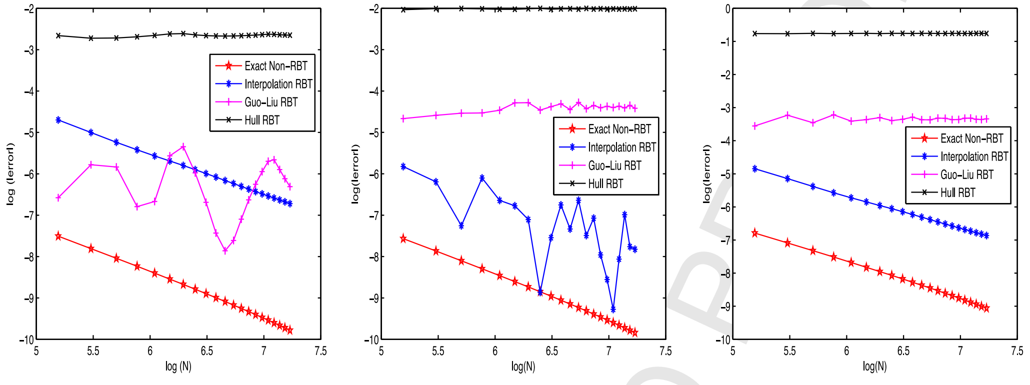

Fig. C.1. The log-scale plots of the errors for binomial trees for European call option pricing with fixed cash dividends without regime-switching. Left figure is for ATM call option with the parameters being taken as the same as that in Table C.1; middle figure is for OTM call option with the parameters being taken as the same as that in Table C.2; right figure is for call option with two fixed cash dividends and the parameters being taken as the same as that in Table C.3.

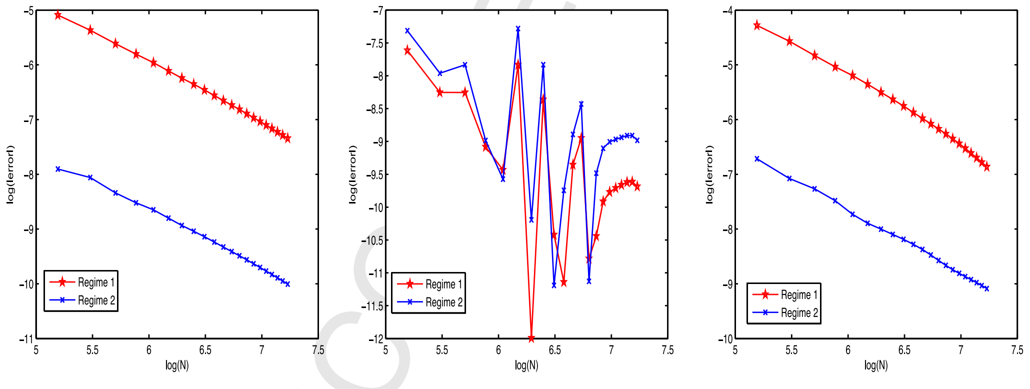

Fig. C.2. The log-scale plots of the errors for binomial trees for European call option pricing with fixed cash dividends without regime-switching. Left figure is for ATM call option with the parameters being taken as the same as that in Table C.4; middle figure is for OTM call option with the parameters being taken as the same as that in Table C.5; right figure is for call option with two fixed cash dividends and the parameters being taken as the same as that in Table C.6.

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Table C.1
Computational results of binomial trees for ATM European call option pricing with fixed cash dividends without regime-switching.

N	Exact Non-RBT			Hull RBT			Guo–Liu RBT			Interpolation RBT		
	Error	Rate	Time (s)	Error	Rate	Time (s)	Error	Rate	Time (s)	Error	Rate	Time (s)
180	5.49e−4	–	0.0441	−6.99e−02	–	0.2203	−1.39e−03	–	0.1837	−9.11e−03	–	0.2052
240	4.07e−4	1.0397	0.0640	−6.57e−02	0.2180	0.4231	3.09e−03	−2.7742	0.4135	−6.73e−03	1.0551	0.5057
300	3.22e−4	1.0499	0.0769	−6.60e−02	−0.0195	0.7331	2.92e−03	0.2450	0.7496	−5.32e−03	1.0543	0.7878
360	2.65e−4	1.0603	0.0948	−6.79e−02	−0.1539	1.1891	1.12e−03	5.2622	1.1948	−4.42e−03	1.0065	1.3497
420	2.25e−4	1.0713	0.1045	−7.03e−02	−0.2290	1.8920	−1.270e−3	−0.8161	1.8941	−3.82e−03	0.9516	2.0390
480	1.95e−4	1.0829	0.1223	−7.29e−02	−0.2708	2.5115	−3.83e−03	−8.2593	2.8744	−3.38e−03	0.9192	2.9835
540	1.71e−4	1.0941	0.1416	−7.33e−02	−0.0532	3.7040	−4.77e−03	−1.8702	4.0629	−3.03e−03	0.9238	4.1595
600	1.52e−4	1.1062	0.1595	−7.12e−02	0.2817	4.8354	−2.54e−03	5.9957	5.7521	−2.74e−03	0.9508	5.6362
660	1.37e−4	1.1190	0.1819	−7.00e−02	0.1817	7.5200	−1.24e−03	7.4793	7.5580	−2.50e−03	0.9816	7.3244
720	1.24e−4	1.1308	0.2301	−6.94e−02	0.0981	9.6737	−5.92e−04	8.5307	9.8240	−2.29e−03	1.0043	9.4832
780	1.13e−4	1.1431	0.2587	−6.92e−02	0.0287	12.2319	−3.86e−04	5.3352	12.4423	−2.11e−03	1.0144	13.2497
840	1.04e−4	1.1577	0.2845	−6.94e−02	−0.0286	13.8827	−4.95e−04	−3.3423	15.4299	−1.96e−03	1.0129	15.4691
900	9.59e−5	1.1700	0.3079	−6.97e−02	−0.0759	18.6228	−8.27e−04	−7.4456	18.9803	−1.83e−03	1.0036	19.0660
960	8.88e−5	1.1835	0.3125	−7.03e−02	−0.1150	20.8545	−1.32e−03	−7.2347	23.0740	−1.71e−03	0.9898	23.1923
1020	8.26e−5	1.1996	0.3725	−7.09e−02	−0.1472	25.5713	−1.93e−03	−6.2388	28.5929	−1.61e−03	0.9777	28.5922
1080	7.71e−5	1.2125	0.3838	−7.16e−02	−0.1737	29.7414	−2.61e−03	−5.3418	33.6317	−1.53e−03	0.9684	32.8888
1140	7.21e−5	1.2279	0.4747	−7.24e−02	−0.1957	51.1366	−3.36e−03	−4.6359	38.2238	−1.45e−03	0.9630	37.5741
1200	6.77e−5	1.2418	0.5304	−7.21e−02	0.0570	58.7807	−3.48e−03	−0.6742	45.0546	−1.38e−03	0.9610	44.4051
1260	6.36e−5	1.2587	0.5568	−7.14e−02	0.1980	70.2676	−2.75e−03	4.8374	51.3114	−1.32e−03	0.9658	51.1466
1320	6.00e−5	1.2734	0.5813	−7.09e−02	0.1535	68.8494	−2.20e−03	4.7219	59.6128	−1.26e−03	0.9690	59.3064
1380	5.66e−5	1.2931	0.6065	−7.06e−02	0.1128	77.5897	−1.82e−03	4.3134	67.9021	−1.21e−03	0.9720	69.7042

Model parameters are set as follows: strike $K = 50$, maturity date $T = 1$, ex-dividend date $\tau = 1/6$, dividend value $D_\tau = 3$, interest rate $r = 0.03$, volatility $\sigma = 0.35$, initial stock price $S_0 = 50$. Exact Non-RBT: exact non-recombining binomial tree (Areal & Rodrigues, 2013), Hull RBT: Hull's recombining binomial tree (Hull, 2012), Guo–Liu RBT: Guo–Liu's recombining binomial tree (Guo & Liu, 2015), Interpolation RBT: interpolation binomial tree (Vellekoop & Nieuwenhuis, 2006). The error is the difference between the computed option value and benchmark value of the option (5.980264), where the benchmark value is computed by the exact non-recombining binomial tree using number of time steps $N = 6000$.

Table C.2
Computational results of binomial trees for OTM European call option pricing with fixed cash dividends without regime-switching.

N	Exact Non-RBT			Hull RBT			Guo–Liu RBT			Interpolation RBT		
	Error	Rate	Time (s)	Error	Rate	Time (s)	Error	Rate	Time (s)	Error	Rate	Time (s)
180	−5.17e−04	–	0.0478	−1.31e−01	–	0.1891	−9.41e−03	–	0.1867	2.96e−03	–	0.2178
240	−3.83e−04	1.0406	0.0608	−1.34e−01	−0.0871	0.3774	−1.02e−02	−0.2720	0.3975	2.04e−03	1.2918	0.4677
300	−3.03e−04	1.0505	0.0881	−1.35e−01	−0.0191	0.6848	−1.07e−02	−0.2304	0.7395	7.06e−04	4.7526	0.8257
360	−2.50e−04	1.0609	0.0987	−1.33e−01	0.0471	1.1156	−1.08e−02	−0.0394	1.1856	−2.24e−03	−6.3283	1.3533
420	−2.12e−04	1.0717	0.1022	−1.32e−01	0.0582	1.7666	−1.15e−02	−0.4277	1.7993	−1.30e−03	3.5084	2.0676
480	−1.83e−04	1.0833	0.1282	−1.32e−01	0.0050	2.6707	−1.38e−02	−1.3322	2.7573	1.14e−03	0.9726	3.0336
540	−1.61e−04	1.0943	0.1487	−1.34e−01	−0.1057	3.8481	−1.39e−02	−0.0435	4.0660	−8.20e−04	2.8289	4.3966
600	−1.43e−04	1.1067	0.1590	−1.35e−01	−0.0648	5.3184	−1.15e−02	1.7271	5.4471	−1.41e−04	16.6930	5.9805
660	−1.29e−04	1.1188	0.1730	−1.32e−01	0.1783	7.1288	−1.26e−02	−0.8920	6.5881	5.29e−04	−13.8533	8.0818
720	−1.17e−04	1.1313	0.1914	−1.33e−01	−0.0782	9.1519	−1.34e−02	−0.7652	8.9175	−1.17e−03	−9.1072	10.5208
780	−1.07e−04	1.1442	0.2078	−1.34e−01	−0.0531	11.7709	−1.17e−02	1.7186	11.3776	6.54e−04	7.2580	13.2226
840	−9.78e−05	1.1575	0.2160	−1.33e−01	0.1365	14.9489	−1.39e−02	−2.3244	14.1608	−1.31e−03	−9.3685	16.4196
900	−9.02e−05	1.1693	0.2221	−1.35e−01	−0.2396	18.2237	−1.20e−02	2.1859	17.4933	5.57e−04	12.3761	20.3421
960	−8.36e−05	1.1845	0.2521	−1.33e−01	0.2423	22.1889	−1.29e−02	−1.1904	21.3170	−8.48e−04	−6.4998	25.3478
1020	−7.77e−05	1.1990	0.2873	−1.34e−01	−0.1459	27.6453	−1.22e−02	0.8822	26.0174	3.49e−04	14.6537	30.5196
1080	−7.25e−05	1.2140	0.2901	−1.33e−01	0.1299	31.4401	−1.26e−02	−0.5612	30.0636	−1.92e−04	10.4005	37.5328
1140	−6.79e−05	1.2258	0.3761	−1.34e−01	−0.0994	37.2826	−1.23e−02	0.4922	35.2618	−9.38e−05	13.2931	44.3888
1200	−6.37e−05	1.2456	0.4295	−1.33e−01	0.0969	43.1605	−1.27e−02	−0.5955	41.8574	3.15e−04	−23.6443	49.9781
1260	−5.99e−05	1.2581	0.4989	−1.34e−01	−0.1080	51.3200	−1.22e−02	0.7905	51.8539	−9.23e−04	−22.0060	59.3806
1320	−5.64e−05	1.2720	0.5830	−1.33e−01	0.1219	61.4684	−1.30e−02	−1.2920	56.2080	4.26e−04	16.6300	64.7251
1380	−5.33e−05	1.2922	0.6086	−1.34e−01	−0.1722	70.8450	−1.21e−02	1.5858	64.0132	−4.00e−04	1.3879	74.8824

Model parameters are set as follows: strike $K=70$, maturity date $T=1$, ex-dividend date $\tau=1/2$, dividend value $D_\tau=3$, interest rate $r=0.03$, volatility $\sigma=0.35$, initial stock price $S_0=50$. Exact Non-RBT: exact non-recombining binomial tree (Areal & Rodrigues, 2013), Hull RBT: Hull's recombining binomial tree (Hull, 2012), Guo–Liu RBT: Guo–Liu's recombining binomial tree (Guo & Liu, 2015), Interpolation RBT: interpolation binomial tree (Vellekoop & Nieuwenhuis, 2006). The error is the difference between the computed option value and benchmark value of the option (1.61666927), where the benchmark value is computed by the exact non-recombining binomial tree using number of time steps $N=6000$.

Table C.3
Computational results of binomial trees for European call option pricing with two fixed cash dividends without regime-switching.

N	Exact Non-RBT			Hull RBT			Guo–Liu RBT			Interpolation RBT		
	Error	Rate	Time (s)	Error	Rate	Time (s)	Error	Rate	Time (s)	Error	Rate	Time (s)
180	1.13e−03	–	2.8294	−4.66e−01	–	0.1866	−2.86e−02	–	0.2028	−7.86e−03	–	0.2759
240	8.36e−04	1.0385	5.1539	−4.65e−01	0.0068	0.4110	−3.96e−02	−1.1320	0.3945	−5.83e−03	1.0419	0.5308
300	6.61e−04	1.0488	8.5537	−4.69e−01	−0.0387	0.7270	−3.15e−02	1.0202	0.7304	−4.60e−03	1.0606	0.8923
360	5.45e−04	1.0596	13.0596	−4.66e−01	0.0302	1.1972	−4.01e−02	−1.3154	1.1805	−3.81e−03	1.0370	1.4694
420	4.62e−04	1.0707	19.2726	−4.67e−01	−0.0102	1.9822	−3.30e−02	1.2487	1.9384	−3.27e−03	0.9939	2.1924
480	4.00e−04	1.0821	26.8265	−4.69e−01	−0.0330	2.8741	−3.47e−02	−0.3627	2.8831	−2.88e−03	0.9421	3.2530
540	3.52e−04	1.0939	36.6930	−4.66e−01	0.0520	4.1411	−3.68e−02	−0.4982	4.0603	−2.59e−03	0.9119	4.7057
600	3.13e−04	1.1056	49.2557	−4.69e−01	−0.0509	5.5264	−3.37e−02	0.8408	5.6598	−2.35e−03	0.9142	6.3432
660	2.81e−04	1.1180	58.2307	−4.68e−01	0.0111	7.4476	−3.49e−02	−0.3842	7.3938	−2.15e−03	0.9435	8.4094
720	2.55e−04	1.1309	75.7677	−4.67e−01	0.0362	9.6938	−3.73e−02	−0.7530	9.7539	−1.97e−03	0.9728	10.8076
780	2.33e−04	1.1433	95.0546	−4.68e−01	−0.0406	12.3097	−3.45e−02	0.9770	12.4043	−1.82e−03	0.9981	13.8766
840	2.14e−04	1.1569	118.5368	−4.69e−01	−0.0200	15.5757	−3.44e−02	0.0360	15.3190	−1.69e−03	1.0077	17.2060
900	1.97e−04	1.1697	148.9498	−4.67e−01	0.0543	19.1693	−3.63e−02	−0.7733	19.0434	−1.58e−03	1.0120	21.7590
960	1.83e−04	1.1844	181.3254	−4.67e−01	−0.0090	23.0085	−3.60e−02	0.1229	22.9283	−1.48e−03	0.9975	26.1522
1020	1.70e−04	1.1973	223.5310	−4.69e−01	−0.0636	28.4117	−3.46e−02	0.6457	28.3622	−1.39e−03	0.9838	31.0042
1080	1.58e−04	1.2129	266.7740	−4.68e−01	0.0396	32.5734	−3.47e−02	−0.0706	32.8196	−1.32e−03	0.9652	36.7836
1140	1.48e−04	1.2275	320.3133	−4.67e−01	0.0338	38.4646	−3.61e−02	−0.6948	38.4704	−1.25e−03	0.9608	43.4175
1200	1.39e−04	1.2424	403.8761	−4.68e−01	−0.0152	44.7453	−3.62e−02	−0.0811	44.6931	−1.19e−03	0.9469	49.9781
1260	1.31e−04	1.2582	505.9598	−4.69e−01	−0.0589	51.3200	−3.50e−02	0.7104	51.3695	−1.14e−03	0.9487	57.8292
1320	1.23e−04	1.2748	575.0049	−4.68e−01	0.0310	59.5962	−3.48e−02	0.1317	59.7371	−1.09e−03	0.9428	67.6553
1380	1.16e−04	1.2903	672.0754	−4.68e−01	0.0382	68.0367	−3.54e−02	−0.3999	68.5086	−1.04e−03	0.9497	76.7858

Model parameters are set as follows: strike $K = 50$, maturity date $T = 1$, two ex-dividend dates $\tau_1 = 1/6$ and $\tau_2 = 1/2$ with dividend values $D_{\tau_1} = 3$ and $D_{\tau_2} = 6$, respectively, interest rate $r = 0.03$, volatility $\sigma = 0.35$, initial stock price $S_0 = 50$. Exact Non-RBT: exact non-recombining binomial tree (Areal & Rodrigues, 2013), Hull RBT: Hull's recombining binomial tree (Hull, 2012), Guo–Liu RBT: Guo–Liu's recombining binomial tree (Guo & Liu, 2015), Interpolation RBT: interpolation binomial tree (Vellekoop & Nieuwenhuis, 2006). The error is the difference between the computed option value and benchmark value of the option (3.66256430), where the benchmark value is computed by the exact non-recombining binomial tree using number of time steps $N = 6000$.

Table C.4
Convergence rates of interpolation binomial trees for ATM European call option pricing with fixed cash dividends with regime-switching.

N	Regime 1				Regime 2			
	Value	Error	Rate	Time (s)	Value	Error	Rate	Time (s)
180	2.96599699	6.15e−03	–	4.9865	4.18018989	3.70e−04	–	4.9865
240	2.96748897	4.66e−03	0.9652	7.4728	4.18024334	3.16e−04	0.5430	7.4728
300	2.96850563	3.64e−03	1.1025	13.5304	4.18032062	2.39e−04	1.2562	13.5304
360	2.96909813	3.01e−03	1.0421	22.5888	4.18033772	1.99e−04	0.9909	22.5888
420	2.96958330	2.58e−03	1.0202	35.8347	4.18037425	1.75e−04	0.8417	35.8347
480	2.96991602	2.22e−03	1.1251	54.6755	4.18039823	1.50e−04	1.1361	54.6755
540	2.97018102	1.95e−03	1.1078	77.6072	4.18041679	1.32e−04	1.1281	77.6072
600	2.97040436	1.74e−03	1.0582	104.5270	4.18043268	1.19e−04	0.9981	104.5270
660	2.97058270	1.57e−03	1.1045	144.4776	4.18044568	1.07e−04	1.0399	144.4777
720	2.97073516	1.42e−03	1.1624	229.7129	4.18046213	9.73e−05	1.1303	229.7129
780	2.97086024	1.29e−03	1.1559	286.6993	4.18047065	8.88e−05	1.1444	286.6993
840	2.97096396	1.19e−03	1.1308	330.8824	4.18047771	8.18e−05	1.1176	330.8824
900	2.97105422	1.10e−03	1.1467	545.6731	4.18048378	7.57e−05	1.1181	545.6731
960	2.97113524	1.02e−03	1.1896	602.5420	4.18048919	7.03e−05	1.1491	602.5420
1020	2.97120745	9.43e−04	1.2169	714.1153	4.18049406	6.54e−05	1.1845	714.1153
1080	2.97127081	8.80e−04	1.2167	942.2011	4.18049838	6.11e−05	1.1954	942.2011
1140	2.97132667	8.24e−04	1.2133	1079.5784	4.18050223	5.72e−05	1.2040	1079.5784
1200	2.97137694	7.74e−04	1.2274	1169.5027	4.18050569	5.38e−05	1.2156	1169.5027
1260	2.97142287	7.28e−04	1.2545	1429.8414	4.18050882	5.07e−05	1.2290	1429.8414
1320	2.97146500	6.86e−04	1.2820	1623.5992	4.18051169	4.78e−05	1.2539	1623.5992
1380	2.97150343	6.47e−04	1.2978	1767.7442	4.18051431	4.52e−05	1.2687	1767.7442

Model parameters are set as follows: strike $K = 50$, maturity date $T = 1$, ex-dividend date $\tau = 1/6$, dividend value $D_\tau = 3$, interest rate $r_1 = 0.05$ for regime 1 and $r_2 = 0.05$ for regime 2, volatility $\sigma_1 = 0.15$ for regime 1 and $\sigma_2 = 0.25$ for regime 2, initial stock price $S_0 = 50$. The generator of the regime-switching process is $\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$. The error is the difference between the computed option value and benchmark value of the option (2.9721506 for regime 1 and 4.1805595 for regime 2), where the benchmark values are computed with number of time steps $N = 6000$.

Table C.5
Convergence rates of interpolation trinomial trees for OTM European call option pricing with fixed cash dividends with regime-switching.

N	Regime 1				Regime 2			
	Value	Error	Rate	Time (s)	Value	Error	Rate	Time (s)
180	0.119359187	−4.93e−04	–	3.1965	0.472326983	−6.66e−04	–	3.1965
240	0.119591923	−2.60e−04	2.2199	7.4679	0.472644443	−3.49e−04	2.2503	7.4679
300	0.119592215	−2.60e−04	0.0050	14.1880	0.472596728	−3.96e−04	−0.5748	14.1880
360	0.119738652	−1.14e−04	4.5413	31.1162	0.472867724	−1.25e−04	6.3132	31.1162
420	0.119772335	−8.00e−05	2.2807	47.1783	0.472923521	−6.96e−05	3.8200	47.1783
480	0.119455972	−3.96e−04	−11.9881	69.7014	0.472304886	−6.88e−04	−17.1617	69.7014
540	0.119846084	−6.20e−06	35.2966	99.7150	0.473030511	3.74e−05	24.7248	99.7150
600	0.119618894	−2.33e−04	−34.4327	116.3023	0.472596032	−3.97e−04	−22.4199	116.3023
660	0.119822500	−2.98e−05	21.5998	154.2175	0.472979316	−1.38e−05	35.2589	154.2175
720	0.119866783	1.45e−05	8.2759	200.2827	0.473051762	5.87e−05	−16.6437	200.2827
780	0.119765697	−8.66e−05	−22.3285	287.2813	0.472855602	−1.37e−04	−10.6423	287.2813
840	0.119722491	−1.30e−04	−5.4621	333.8819	0.472774791	−2.18e−04	−6.2382	333.8819
900	0.119831560	−2.07e−05	26.5907	393.7498	0.472978434	−1.47e−05	39.1391	393.7498
960	0.119881465	2.92e−05	−5.3001	483.3380	0.473069289	7.62e−05	−25.5290	483.3380
1020	0.119901755	4.95e−05	−8.7076	575.7088	0.473104306	1.11e−04	−6.2379	575.7088
1080	0.119909375	5.71e−05	−2.5065	681.9540	0.473115844	1.23e−04	−1.7271	681.9540
1140	0.119912943	6.07e−05	−1.1213	823.5822	0.473120362	1.27e−04	−0.6686	823.5822
1200	0.119915960	6.37e−05	−0.9463	977.0164	0.473124378	1.31e−04	−0.6057	977.0164
1260	0.119918579	6.63e−05	−0.8262	1051.7555	0.473128099	1.35e−04	−0.5729	1051.7555
1320	0.119919045	6.68e−05	−0.1506	1210.2861	0.473128052	1.35e−04	0.0075	1210.2861
1380	0.119914468	6.22e−05	1.5978	1424.6903	0.473118660	1.26e−04	1.6228	1424.6903

Model parameters are set as follows: strike $K = 70$, maturity date $T = 1$, ex-dividend date $\tau = 1/2$, dividend value $D_\tau = 3$, interest rate $r_1 = 0.05$ for regime 1 and $r_2 = 0.05$ for regime 2, volatility $\sigma_1 = 0.15$ for regime 1 and $\sigma_2 = 0.25$ for regime 2, initial stock price $S_0 = 50$. The generator of the regime-switching process is $\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$. The error is the difference between the computed option value and benchmark value of the option (0.119852286 for regime 1 and 0.472993101 for regime 2), where the benchmark values are computed with number of time steps $N = 6000$.

Table C.6
Convergence rates of interpolation trinomial trees for European call option pricing with two fixed cash dividends with regime-switching.

N	Regime 1				Regime 2			
	Value	Error	Rate	Time (s)	Value	Error	Rate	Time (s)
180	4.73012665	−1.38e−02	–	3.4847	7.16288086	1.21e−03	–	3.4847
240	4.73360732	−1.04e−02	1.0062	8.7590	7.16251415	8.44e−04	1.2536	8.7590
300	4.73599725	−7.98e−03	1.1742	16.4704	7.16236933	7.00e−04	0.8431	16.4704
360	4.73748176	−6.49e−03	1.1292	28.5054	7.16223303	5.63e−04	1.1885	28.5054
420	4.73844002	−5.54e−03	1.0358	46.7168	7.16210888	4.39e−04	1.6150	46.7168
480	4.73924889	−4.73e−03	1.1830	66.8935	7.16204232	3.73e−04	1.2308	66.8935
540	4.73989566	−4.08e−03	1.2494	91.6993	7.16200446	3.35e−04	0.9097	91.6993
600	4.74039406	−3.58e−03	1.2367	117.6754	7.16197377	3.04e−04	0.9126	117.6754
660	4.74080537	−3.17e−03	1.2800	155.0388	7.16194731	2.78e−04	0.9552	155.0388
720	4.74115509	−2.82e−03	1.3434	201.6984	7.16192341	2.54e−04	1.0346	201.6984
780	4.74144542	−2.53e−03	1.3572	254.8332	7.16190045	2.31e−04	1.1850	254.8332
840	4.74168359	−2.29e−03	1.3341	320.4227	7.16187841	2.09e−04	1.3546	320.4227
900	4.74188590	−2.09e−03	1.3395	385.3513	7.16185869	1.89e−04	1.4385	385.3513
960	4.74206598	−1.91e−03	1.3965	466.3823	7.16184236	1.73e−04	1.4002	466.3823
1020	4.74222957	−1.75e−03	1.4774	565.1137	7.16182935	1.60e−04	1.2921	565.1137
1080	4.74237786	−1.60e−03	1.5530	680.8153	7.16181897	1.49e−04	1.1761	680.8153
1140	4.74251169	−1.46e−03	1.6182	792.0087	7.16181033	1.41e−04	1.1027	792.0087
1200	4.74263254	−1.34e−03	1.6800	927.1257	7.16180278	1.33e−04	1.0757	927.1257
1260	4.74274173	−1.23e−03	1.7382	1069.1929	7.16179578	1.26e−04	1.1074	1069.1929
1320	4.74284050	−1.13e−03	1.7938	1286.1665	7.16178910	1.19e−04	1.1701	1286.1665
1380	4.74292959	−1.05e−03	1.8391	1423.8627	7.16178261	1.13e−04	1.2572	1423.8627

Model parameters are set as follows: strike $K = 50$, maturity date $T = 1$, ex-dividend dates $\tau_1 = 1/6$ and $\tau_2 = 1/2$ with dividend values $D_{\tau_1} = 3$ and $D_{\tau_2} = 6$, respectively, interest rate $r_1 = 0.05$ for regime 1 and $r_2 = 0.05$ for regime 2, volatility $\sigma_1 = 0.15$ for regime 1 and $\sigma_2 = 0.25$ for regime 2, initial stock price $S_0 = 50$. The generator of the regime-switching process is $\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$. The error is the difference between the computed option value and benchmark value of the option (4.74397541 for regime 1 and 7.16166969 for regime 2), where the benchmark values are computed with number of time steps $N = 6000$.

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