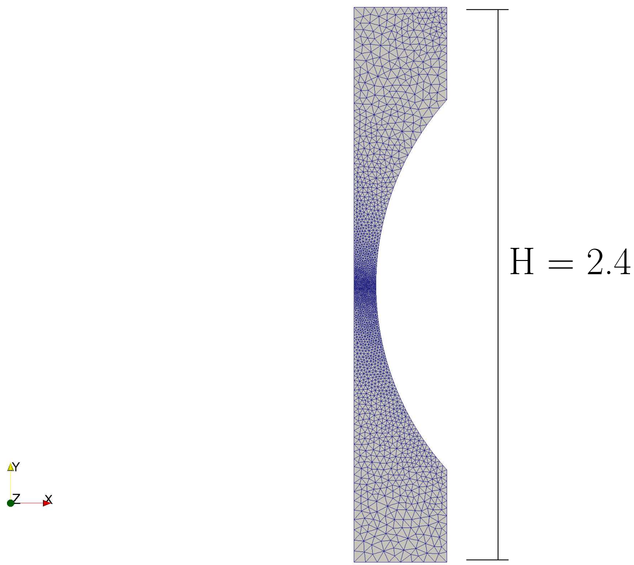
# Computational exercice 4: Linear Elasticity

Consider the computational domain $\Omega \subset \mathbb{R}^2$ shown in the figure below.




The elastostatic problem to be solved on this domain reads:

$$
\begin{cases}
\boldsymbol{\sigma} = \lambda \operatorname{div} \mathbf{u}\, \mathbf{I} + \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^\mathsf{T}\right) \\
\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \\
-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f} & \text{in } \Omega \\
\mathbf{u} = \mathbf{u}_{\text{bottom}} & \text{in } \Gamma_{\text{bottom}} = \{(x_1, x_2) \in \partial\Omega,\ x_2 = 0\} \\
\mathbf{u} = \mathbf{u}_{\text{top}} & \text{in } \Gamma_{\text{top}} = \{(x_1, x_2) \in \partial\Omega,\ x_2 = H\} \\
\boldsymbol{\sigma} \cdot \mathbf{n} = \boldsymbol{\mathcal{F}} & \text{in } \Gamma_N = \partial\Omega \setminus (\Gamma_{\text{bottom}} \cup \Gamma_{\text{top}})
\end{cases}
$$

where

- $E = 10$ (Young modulus), $\nu = 0.3$ (Poisson ratio)
- $\mathbf{f} = (0,0)^\mathsf{T}$
- $\boldsymbol{\mathcal{F}} = (0,0)^\mathsf{T}$
- $\mathbf{u}_{\text{bottom}} = (0,0)^\mathsf{T}$
- $\mathbf{u}_{\text{top}} = (0,0.1)^\mathsf{T}$

1. Write the continuous variational formulation. Define the linear and bilinear forms and the trial and test function spaces to be considered.

2. Write the discrete variational formulation. Considering continuous $\mathbb{P}_1$ elements, define the discrete spaces to be used.

3. Implement a Fenics script to solve the problem above. The mesh is available in the repository under the name `neck_2Dcorpo.xml` and can be loaded into the Fenics script using

```
.
mesh = Mesh("neck_2Dcorpo.xml")
.
```

4. Given the deviatoric stresses

$$\boldsymbol{s} = \boldsymbol{\sigma} - \frac{\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}}{3}$$

Compute the scalar quantity known as the Von Mises stress defined as the second invariant of the deviatoric stresses:

$$\sigma_V = \sqrt{\frac{3}{2}\boldsymbol{s} : \boldsymbol{s}}$$

The criterion is that yielding of a ductile material begins when $\sigma_V$ reaches a critical value.

Implement in Fenics. For visualization of results, project $\sigma_V$ onto a space of elementwise constant functions.

5. Repeat the previous points but now assuming the only component to be specified on $\Gamma_{\text{top}}$ is the vertical one, meaning that $u_1$ is free (no restrictions) and $u_2 = 0.1$. This can be done in Fenics by adding the suffix `sub(k)` to `W`. For our case:

```
.
bc = DirichletBC(W.sub(1), Constant(0.1), top)
.
```

6. Include the additional Dirichlet boundary condition: $u_1 = 0$ on $\Gamma_{\text{left}} = \{(x_1, x_2) \in \partial\Omega,\ x_1 = 0\}$ and solve.

7. Implement the 3D version of the previous problem. The mesh is shown in the figure below and the file is in the repository under the name `neck_3Dcorpo.xml`. Consider:

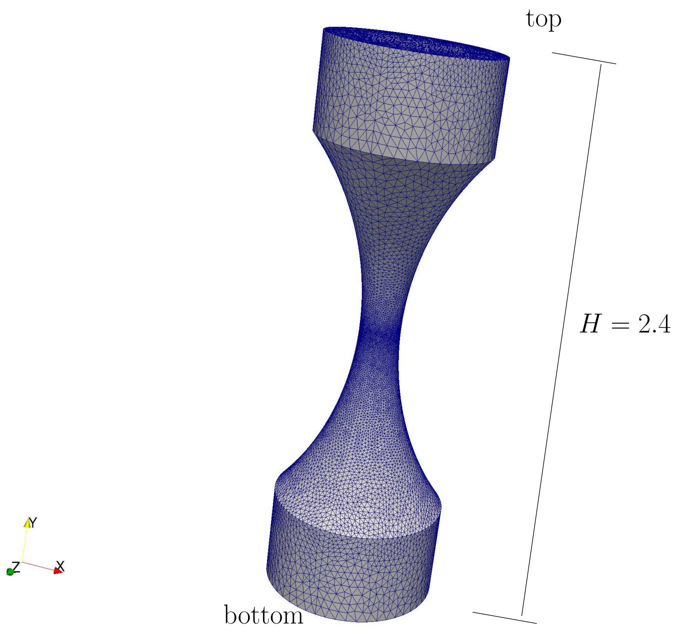
   - $E = 10$ (Young modulus), $\nu = 0.3$ (Poisson ratio)
   - $\mathbf{f} = (0, 0, 0)^\intercal$
   - $\boldsymbol{\mathcal{F}} = (0, 0, 0)^\intercal$
   - $\mathbf{u}_{\text{bottom}} = (0, 0, 0)^\intercal$
   - $\mathbf{u}_{\text{top}} = (0, 0.1, 0)^\intercal$

8. In problems with symmetry of revolution as in the previous example, it is convenient to solve the equations in the axisymmetric form. A three dimensional vector field $\mathbf{v}$ in such case has the form $[v_r,\ v_y,\ 0]^T$, where $r$ stands for the radial component (the distance to the symmetry axis) and $y$ the vertical component. It is assumed that both $v_r$ and $v_y$ are functions only of $(r, y)$. Hence,

$$\nabla \mathbf{v} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\nabla \mathbf{v}_{\text{2D}}} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline 0 & 0 & \dfrac{v_r}{r} \end{array}\right], \qquad \nabla \mathbf{v}_{\text{2D}} = \begin{bmatrix} v_{r,r} & v_{r,y} \\ & \\ v_{y,r} & v_{y,y} \end{bmatrix}.$$

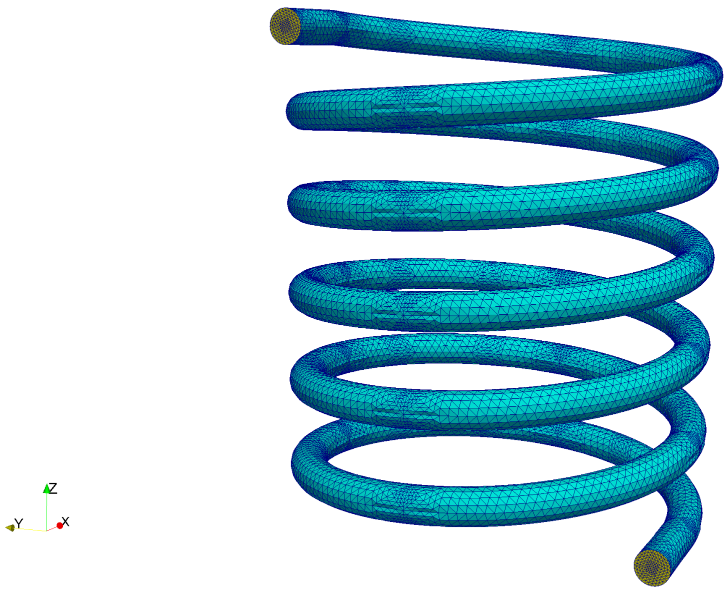
Therefore giving

$$\begin{aligned} \int_{\Omega_{3D}} 2\mu\, \boldsymbol{\epsilon}(\mathbf{u}) : \nabla \mathbf{v}\, d\Omega &= 2\,\pi \int_{\Omega_{2D}} 2\mu\, \left( \boldsymbol{\epsilon}_{\text{2D}}(\mathbf{u}) : \nabla \mathbf{v}_{\text{2D}} + \frac{u_r\, v_r}{r^2} \right) r\, dx_{\text{2D}}, \\ \int_{\Omega_{3D}} \lambda\, (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})\, d\Omega &= 2\,\pi \int_{\Omega_{2D}} \lambda \left( \nabla \cdot \mathbf{u}_{\text{2D}} + \frac{u_r}{r} \right) \left( \nabla \cdot \mathbf{v}_{\text{2D}} + \frac{v_r}{r} \right) r\, dx_{\text{2D}} \end{aligned}$$




Implement in Fenics and compare results to the 3D full case.

9. Consider the mesh of an elastic coil shown in the figure below. Solve the elasticity problem by applying a Dirichlet restriction $\mathbf{u} = 0$ on the lower yellow region (`surface_tag=3004`) and a surface force distribution in the upper yellow region (`surface_tag=3005`). The necessary files are in the repository under the name `ocoil.xml` and `ocoil_facet_region.xml`.


Z
Y
X