Computational exercice 3: 2nd order problems

Consider the computational domain $\Omega = (0, L) \times (0, B)$ and the Diffusion-Convection equation with homogeneous right hand side

$$\begin{aligned}
-\nabla \cdot (\kappa \nabla u) + \boldsymbol{\beta} \cdot \nabla u &= 0 \quad \text{in} \quad \Omega \\
\frac{\partial u}{\partial y}(x, 0) &= 0, \quad x \in [0, L] \\
\frac{\partial u}{\partial y}(x, B) &= 0, \quad x \in [0, L] \\
u(0, y) &= 0, \quad y \in [0, B] \\
u(L, y) &= 1, \quad y \in [0, B]
\end{aligned}$$

in which κ is the scalar diffusion coefficient and $\boldsymbol{\beta} = [\beta_1, \beta_2]^\intercal$ the convection velocity field.

1. Write the continuous variational formulation. Define the linear and bilinear forms and the trial and test function spaces to be considered.

2. Write the discrete variational formulation. Considering Lagrange polynomial elements of degree k, define the discrete spaces to be used.

3. Implement a Fenics script to solve the problem. Consider the following points:

   (a) **Boundary layer at exit**

   Although the problem we are solving above is two dimensional, with those boundary conditions and $\boldsymbol{\beta} = [\beta, 0]^\intercal$, in fact, the exact solution only depends on x. To find this exact solution we

can solve the one dimensional problem

$$-\kappa\frac{d^2u}{dx^2}+\beta\frac{du}{dx} \;=\; 0 \;\text{ in }\; (0,1), \quad u(0)=0, \quad u(L)=1$$

whose solution is

$$u(x)=\frac{e^{Pe\,\frac{x}{L}}-1}{e^{Pe}-1}$$

which depends on the Peclet number that is defined as $Pe=\|\boldsymbol{\beta}\|L/\kappa$. A boundary layer is observed in this problem near the exit boundary ($x=L$). The characteristic length scale being

$$\delta_c\sim\frac{\kappa}{\|\boldsymbol{\beta}\|}$$

- This 1D solution is just for comparison. Consider a 2D implementation. Consider $L=2$, $B=1$, the velocity field $\boldsymbol{\beta}=[1,0]^\intercal$ and a mesh refinement with $h\sim 0.1$. Compute the solution with values of κ equal to 10, 1, 0.1, 0.01 and 0.001. For each case plot the solution and analyse the behavior of the finite element solution compare to the exact one.
- Taking $\kappa=0.01$, increasingly refine the mesh **uniformly** and observe how the solution improves. Repeat, but now refining the mesh only **locally** near the exit, so as to better capture the boundary layer. Provide a criterion to chose the mesh refinement.

(b) **Stabilized formulation: Streamline Upwind Petrov-Galerkin (SUPG)**

Besides intensive mesh refinement, a remedy to the situation observed above is adding stabilization terms. Consider the following stabilized (consistent) formulation: “Find $u_h\in V_h$

$$a(u_h,v_h)+r(u_h,v_h)=\ell(v_h)\quad\forall v_h\in V_{h0}\text{”}$$

where $a(\cdot,\cdot)$ and $\ell(\cdot)$ are the bilinear and linear forms respectively of the original problem and the perturbation term r is a bilinear form given by

$$r(u_h,v_h)=\sum_{K\in\mathcal{T}_h}\int_K \mathscr{P}(v_h)\,\tau_K\,\mathscr{R}(u_h)\,dx$$

where for this case $\mathscr{R}(u) = -\nabla \cdot (\kappa \nabla u) + \boldsymbol{\beta} \cdot \nabla u$, is the residual of the original differential equation, the elementwise stabilization parameter τ_K being defined as

$$\tau_K = \left[\frac{4\,\kappa}{h_K^2} + \frac{2\,\|\boldsymbol{\beta}\|}{h_K} \right]^{-1}$$

and the "perturbed" test function $\mathscr{P}(v)$ is

$$\mathscr{P}(v) = \boldsymbol{\beta} \cdot \nabla v$$

Other choices for $\mathscr{P}(v)$ lead to different methods, such as the Galerkin Least Square (GLS) or the Algebraic Subgrid Scale (ASGS) methods (see Codina, 1997, CMAME).

(a) Modify the original script by adding the additional term to the variational formulation

(b) Recompute the previous cases and compare results to the unstabilized case.

4. **Bonus: A tensorial diffusion coefficient**

Introduce the necessary lines to consider the case of a tensorial diffusion coefficient.

Consider the domain is $\Omega = (0,1) \times (0,1)$, $\boldsymbol{\beta} = [0,0]$. In order to build an analytical solution we use the following tensor

$$\kappa = \begin{bmatrix} (x+1)^2 + y^2 & \sin(x\,y) \\ \sin(x\,y) & (x+1)^2 \end{bmatrix}$$

and build a source term $f(x,y)$ such that the solution is

$$u(x,y) = x^3\,y^4 + x^2 + \sin(x\,y)\,\cos(y)$$

To solve this problem, impose the exact solution as Dirichlet condition on the whole boundary.