Classifying Equilibria with Unknown Parameters for a Dynamic Population Model

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Overview

We will investigate a model for population dynamics introduced by Bolker and Pacala.

- **Use a Mean Field Approximation to help figure out the domain for our unspecified parameters.**
  - **Generating a system of differential equations for the 2-Box model.**
  - **Find the Equilibrium Points.**
  - **EigenValue Method.**
  - **Find the conditions on the parameters by hand.**
- **We find that there is a stable equilibrium point for the 2-Box and 3-Box model.**
- **We find that there is an unstable equilibrium point always for the 2-Box and 3- Box model.**

*My mentor that I am working with is Professor Mariya Bessonov.

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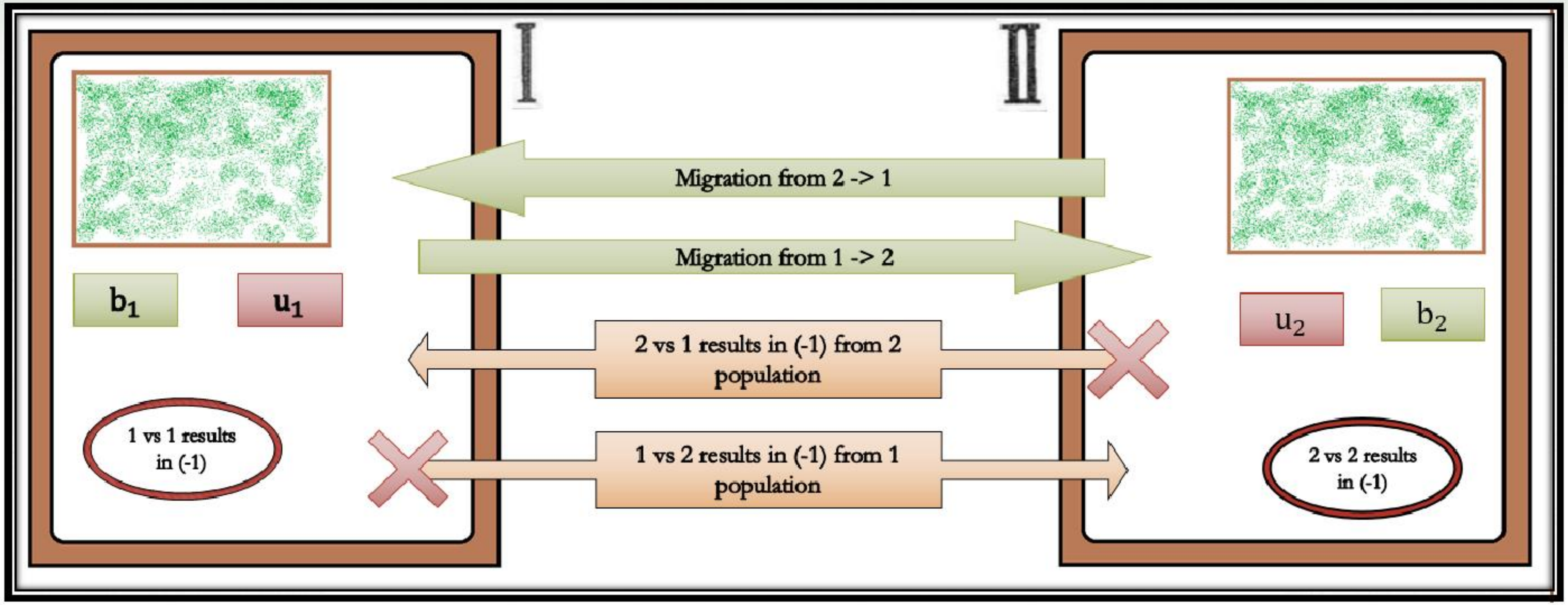
Introduction

2-Box Model



Analysis

2 – Box Model System

The function *f* represents the change in population density for the original 2 – Box model
z_i represents the population divided by L (location), which is the population density

$$f1 = \frac{dz_1}{dt} = b_1 z_1 - u_1 z_1 - z_1^2 a_{11}^{-} - z_1 a_{12}^{+} - z_1 z_2 a_{12}^{-} + z_2 a_{21}^{+}$$

$$f2 = \frac{dz_2}{dt} = b_2 z_2 - u_2 z_2 + z_1 a_{12}^{+} - z_2 a_{21}^{+} - z_1 z_2 a_{21}^{-} - z_2^2 a_{22}^{-}$$

The function *g* is the basic case of 2 – Box model when each box has equal birth death, inner, outer and migration competition rates. These are our initial conditions on the parameters

$$g1 = \frac{dz_1}{dt} = b z_1 - u z_1 - z_1 a_{12}^{+} + z_2 a_{12}^{+} - z_1^2 a_i^{-} - z_1 z_2 a_o^{-}$$

$$g2 = \frac{dz_2}{dt} = b z_2 - u z_2 + z_1 a_{12}^{+} - z_2 a_{12}^{+} - z_2^2 a_i^{-} - z_1 z_2 a_o^{-}$$

2 - Box Model Equilibrium Points

To find our EQ points we must set the system equal to zero. We generate four EQ points.

$$\Big\{\{z1 \to 0,\ z2 \to 0\},$$

$$\Big\{z1 \to \frac{b-u}{a_i^- + a_o^-},\ z2 \to \frac{b-u}{a_i^- + a_o^-}\Big\},$$

$$\Big\{z1 \to \frac{1}{2\,a_i^-\,(a_i^- - a_o^-)}\Big(\big(b-u-2\,a_{12}^+\big)\,a_i^- - b\,a_o^- + u\,a_o^- + 2\,a_{12}^+\,a_o^- + \sqrt{\Big(\big(b-u-2\,a_{12}^+\big)\,(a_i^- - a_o^-)\,\Big(\big(b-u+2\,a_{12}^+\big)\,a_i^- + \big(-b+u+2\,a_{12}^+\big)\,a_o^-\Big)\Big)}\Big),$$

$$z2 \to -\frac{1}{2\,a_i^-\,(a_i^- - a_o^-)}\Big(\big(-b+u+2\,a_{12}^+\big)\,a_i^- + b\,a_o^- - u\,a_o^- - 2\,a_{12}^+\,a_o^- + \sqrt{\Big(\big(b-u-2\,a_{12}^+\big)\,(a_i^- - a_o^-)\,\Big(\big(b-u+2\,a_{12}^+\big)\,a_i^- + \big(-b+u+2\,a_{12}^+\big)\,a_o^-\Big)\Big)}\Big)\Big\},$$

$$\Big\{z1 \to \frac{1}{2\,a_i^-\,(a_i^- - a_o^-)}\Big(\big(b-u-2\,a_{12}^+\big)\,a_i^- - b\,a_o^- + u\,a_o^- + 2\,a_{12}^+\,a_o^- - \sqrt{\Big(\big(b-u-2\,a_{12}^+\big)\,(a_i^- - a_o^-)\,\Big(\big(b-u+2\,a_{12}^+\big)\,a_i^- + \big(-b+u+2\,a_{12}^+\big)\,a_o^-\Big)\Big)}\Big),$$

$$z2 \to \frac{1}{2\,a_i^-\,(a_i^- - a_o^-)}\Big(\big(b-u-2\,a_{12}^+\big)\,a_i^- - b\,a_o^- + u\,a_o^- + 2\,a_{12}^+\,a_o^- + \sqrt{\Big(\big(b-u-2\,a_{12}^+\big)\,(a_i^- - a_o^-)\,\Big(\big(b-u+2\,a_{12}^+\big)\,a_i^- + \big(-b+u+2\,a_{12}^+\big)\,a_o^-\Big)\Big)}\Big)\Big\}\Big\}$$

EigenValue Method

We need the Jacobian matrix of the system *f*. Then we find the eigenvalues of the Jacobian at each equilbirum point.

$$J_g\left(Z_1, Z_2\right) = \begin{pmatrix} \frac{\partial g1}{\partial z_1} & \frac{\partial g1}{\partial z_2} \\ \frac{\partial g2}{\partial z_1} & \frac{\partial g2}{\partial z_2} \end{pmatrix}$$

$$J_g = \begin{pmatrix} b - u - a^{\dagger}_{12} - 2\, z1\, a^{-}_{i} - z2\, a^{-}_{o} & a^{\dagger}_{12} - z1\, a^{-}_{o} \\ a^{\dagger}_{12} - z2\, a^{-}_{o} & b - u - a^{\dagger}_{12} - 2\, z2\, a^{-}_{i} - z1\, a^{-}_{o} \end{pmatrix}$$

EigenValue Method (continued)

We define a few general conditions because they have to make sense biologically.

$b > u > 0$

$a^{\dagger} \geq 0$, $a_i \geq 0$, $a_O \geq 0$

$a_i \vee a_o > 0$

We now we find the eigenvalues of each of the equilibrium points we just found.

```
z1 = 0;
z2 = 0;
MatrixForm[A]
Simplify[Eigenvalues[A]]
```

$$\begin{pmatrix} b - u - a^{\dagger}_{12} & a^{\dagger}_{12} \\ a^{\dagger}_{12} & b - u - a^{\dagger}_{12} \end{pmatrix}$$

$$\{b - u, \ b - u - 2\, a^{\dagger}_{12}\}$$

$$z1 = \frac{b - u}{a^{-}_{i} + a^{-}_{o}}; \quad z2 = \frac{b - u}{a^{-}_{i} + a^{-}_{o}};$$

$$\left\{-b + u, \ \frac{1}{a^{-}_{i} + a^{-}_{o}} \left(\left(-b + u - 2\, a^{\dagger}_{12}\right) a^{-}_{i} + \left(b - u - 2\, a^{\dagger}_{12}\right) a^{-}_{o}\right)\right\}$$

$$z1 = \frac{1}{2\, a_i^- \left(a_i^- - a_o^-\right)} \left(\left(b - u - 2\, a_{12}^\dagger\right) a_i^- - b\, a_o^- + u\, a_o^- + 2\, a_{12}^\dagger\, a_o^- + \sqrt{\left(\left(b - u - 2\, a_{12}^\dagger\right) \left(a_i^- - a_o^-\right) \left(\left(b - u + 2\, a_{12}^\dagger\right) a_i^- + \left(-b + u + 2\, a_{12}^\dagger\right) a_o^-\right)\right)}\right);$$

$$z2 = -\frac{1}{2\, a_i^- \left(a_i^- - a_o^-\right)} \left(\left(-b + u + 2\, a_{12}^\dagger\right) a_i^- + b\, a_o^- - u\, a_o^- - 2\, a_{12}^\dagger\, a_o^- + \sqrt{\left(\left(b - u - 2\, a_{12}^\dagger\right) \left(a_i^- - a_o^-\right) \left(\left(b - u + 2\, a_{12}^\dagger\right) a_i^- + \left(-b + u + 2\, a_{12}^\dagger\right) a_o^-\right)\right)}\right);$$

$$\Bigg\{-\frac{1}{2\, a_i^- \left(a_i^- - a_o^-\right)} \Big(b\, a_i^-\, a_o^- - u\, a_i^-\, a_o^- - b \left(a_o^-\right)^2 + u \left(a_o^-\right)^2 - 2\, a_{12}^\dagger \left(\left(a_i^-\right)^2 - \left(a_o^-\right)^2\right) + \sqrt{\left(\left(a_i^- - a_o^-\right)^2 \left(4 \left(\left(b - u\right)^2 - 3 \left(a_{12}^\dagger\right)^2\right) \left(a_i^-\right)^2 - 4 \left(\left(b - u\right)^2 - 3 \left(b - u\right) a_{12}^\dagger + 2 \left(a_{12}^\dagger\right)^2\right) a_i^-\, a_o^- + \left(-b + u + 2\, a_{12}^\dagger\right)^2 \left(a_o^-\right)^2\right)\right)}\Big),$$
$$\frac{1}{2\, a_i^- \left(a_i^- - a_o^-\right)} \Big(-b\, a_i^-\, a_o^- + u\, a_i^-\, a_o^- + b \left(a_o^-\right)^2 - u \left(a_o^-\right)^2 + 2\, a_{12}^\dagger \left(\left(a_i^-\right)^2 - \left(a_o^-\right)^2\right) + \sqrt{\left(\left(a_i^- - a_o^-\right)^2 \left(4 \left(\left(b - u\right)^2 - 3 \left(a_{12}^\dagger\right)^2\right) \left(a_i^-\right)^2 - 4 \left(\left(b - u\right)^2 - 3 \left(b - u\right) a_{12}^\dagger + 2 \left(a_{12}^\dagger\right)^2\right) a_i^-\, a_o^- + \left(-b + u + 2\, a_{12}^\dagger\right)^2 \left(a_o^-\right)^2\right)\right)}\Big)\Bigg\}$$

$$z1 = \frac{1}{2\, a_i^- \left(a_i^- - a_o^-\right)} \left(\left(b - u - 2\, a_{12}^\dagger\right) a_i^- - b\, a_o^- + u\, a_o^- + 2\, a_{12}^\dagger\, a_o^- - \sqrt{\left(\left(b - u - 2\, a_{12}^\dagger\right) \left(a_i^- - a_o^-\right) \left(\left(b - u + 2\, a_{12}^\dagger\right) a_i^- + \left(-b + u + 2\, a_{12}^\dagger\right) a_o^-\right)\right)}\right);$$

$$z2 = \frac{1}{2\, a_i^- \left(a_i^- - a_o^-\right)} \left(\left(b - u - 2\, a_{12}^\dagger\right) a_i^- - b\, a_o^- + u\, a_o^- + 2\, a_{12}^\dagger\, a_o^- + \sqrt{\left(\left(b - u - 2\, a_{12}^\dagger\right) \left(a_i^- - a_o^-\right) \left(\left(b - u + 2\, a_{12}^\dagger\right) a_i^- + \left(-b + u + 2\, a_{12}^\dagger\right) a_o^-\right)\right)}\right);$$

$$\Bigg\{-\frac{1}{2\, a_i^- \left(a_i^- - a_o^-\right)} \Big(b\, a_i^-\, a_o^- - u\, a_i^-\, a_o^- - b \left(a_o^-\right)^2 + u \left(a_o^-\right)^2 - 2\, a_{12}^\dagger \left(\left(a_i^-\right)^2 - \left(a_o^-\right)^2\right) + \sqrt{\left(\left(a_i^- - a_o^-\right)^2 \left(4 \left(\left(b - u\right)^2 - 3 \left(a_{12}^\dagger\right)^2\right) \left(a_i^-\right)^2 - 4 \left(\left(b - u\right)^2 - 3 \left(b - u\right) a_{12}^\dagger + 2 \left(a_{12}^\dagger\right)^2\right) a_i^-\, a_o^- + \left(-b + u + 2\, a_{12}^\dagger\right)^2 \left(a_o^-\right)^2\right)\right)}\Big),$$
$$\frac{1}{2\, a_i^- \left(a_i^- - a_o^-\right)} \Big(-b\, a_i^-\, a_o^- + u\, a_i^-\, a_o^- + b \left(a_o^-\right)^2 - u \left(a_o^-\right)^2 + 2\, a_{12}^\dagger \left(\left(a_i^-\right)^2 - \left(a_o^-\right)^2\right) + \sqrt{\left(\left(a_i^- - a_o^-\right)^2 \left(4 \left(\left(b - u\right)^2 - 3 \left(a_{12}^\dagger\right)^2\right) \left(a_i^-\right)^2 - 4 \left(\left(b - u\right)^2 - 3 \left(b - u\right) a_{12}^\dagger + 2 \left(a_{12}^\dagger\right)^2\right) a_i^-\, a_o^- + \left(-b + u + 2\, a_{12}^\dagger\right)^2 \left(a_o^-\right)^2\right)\right)}\Big)\Bigg\}$$

2-Box Model : Special Conditions

Now that we have a list of eigenvalues for each equilibrium point, we make several special cases to further constrict the solution set for stability.

(a) b, u , a_o, $a^\dagger$, $a_i = 0$

(b) b, u , a_i, $a^\dagger$, $a_o = 0$

(c) b, u , a_o, a_i , $a^\dagger = 0$

$$z1 = \frac{b - u}{a_i^- + a_o^-}; \quad z2 = \frac{b - u}{a_i^- + a_o^-};$$

$$\left\{-b + u, \frac{1}{a_i^- + a_o^-} \left(\left(-b + u - 2 a_{12}^\dagger\right) a_i^- + \left(b - u - 2 a_{12}^\dagger\right) a_o^-\right)\right\}$$

Results

EQ Point (0,0) is unstable in every special case.

EQ Point $\left(\frac{b-u}{a_i+a_o}, \frac{b-u}{a_i+a_o}\right)$ is stable in these scenarios:

i) When $a_i = 0$, if $(b-u-2a^\dagger) < 0$

ii) **When $a_o = 0$, if $(-b+u-2a^\dagger) < 0$** **{always Stable}**

iii) When a_i and $a_o > 0$, if $a_i > a_o$.

EQ Point 3 and 4 have several scenarios also:

i) Is unstable when $a_i = 0$.

ii) Is stable **when $a_o = 0$, if $\left(a^\dagger + -\frac{1}{2} a_i \sqrt{-4\left(b-u+3a_i^2-\alpha^\dagger\right)\alpha^\dagger + (b-u)^2\left[1+4a_i^2\right]} < 0\right)$** **and if $\left(-a^\dagger - \frac{1}{2} a_i \sqrt{-4\left(b-u+3a_i^2-\alpha^\dagger\right)\alpha^\dagger + (b-u)^2\left[1+4a_i^2\right]} < 0\right)$**

iii) When $a^\dagger = 0$, if $\frac{(-b+u)\left(a_o+\sqrt{1-4a_oa_i+4a_i^2}\right)}{2a_i} < 0$ and if $\frac{-(b+u)a_ia_o-(b-u)a_o^2}{2a_i(a_i-a_o)} - \frac{\left((b-u)\sqrt{1-4a_oa_i+4a_i^2}\right)}{2a_i} < 0$

We have stability when the following inequalities hold.

Analysis

3 – Box Model System

The function *f* represents the change in population density for the original 3 – Box model
z_i represents the population divided by L (location), which is the population density.

$$f1 = \frac{dz_1}{dt} = b_1 z_1 - u_1 z_1 - z_1^2 a_{11}^- - z_1 a_{12}^\dagger - z_1 z_2 a_{12}^- - z_1 a_{13}^\dagger - z_1 z_3 a_{13}^- + z_2 a_{21}^\dagger + z_3 a_{31}^\dagger$$

$$f2 = \frac{dz_2}{dt} = b_2 z_2 - u_2 z_2 + z_1 a_{12}^\dagger - z_2 a_{21}^\dagger - z_1 z_2 a_{21}^- - z_2^2 a_{22}^- - z_2 a_{23}^\dagger - z_2 z_3 a_{23}^- + z_3 a_{32}^\dagger$$

$$f3 = \frac{dz_3}{dt} = b_3 z_3 - u_3 z_3 + z_1 a_{13}^\dagger + z_2 a_{23}^\dagger - z_3 a_{31}^\dagger - z_1 z_3 a_{31}^- - z_3 a_{32}^\dagger - z_2 z_3 a_{32}^- - z_3^2 a_{33}^-$$

The function *g* is the basic case o f 2 – Box model when each box has equal birth death, inner, outer and migration competition rates. These are our initial conditions on the parameters.

$$g1 = \frac{dz_1}{dt} = b\, z1 - u\, z1 - z1^2\, a_i^- - 2\, z1\, a_m^\dagger + z2\, a_m^\dagger + z3\, a_m^\dagger - z1\, z2\, a_o^- - z1\, z3\, a_o^-$$

$$g2 = \frac{dz_2}{dt} = b\, z2 - u\, z2 - z2^2\, a_i^- + z1\, a_m^\dagger - 2\, z2\, a_m^\dagger + z3\, a_m^\dagger - z1\, z2\, a_o^- - z2\, z3\, a_o^-$$

$$g3 = \frac{dz_3}{dt} = b\, z3 - u\, z3 - z3^2\, a_i^- + z1\, a_m^\dagger + z2\, a_m^\dagger - 2\, z3\, a_m^\dagger - z1\, z3\, a_o^- - z2\, z3\, a_o^-$$

3 - Box Model Equilibrium Points

To find our EQ points we must set the system equal to zero.

$$\left\{\{z1 \to 0,\ z2 \to 0,\ z3 \to 0\},\ \left\{z1 \to \frac{b-u}{a_i^- + 2\,a_o^-},\ z2 \to \frac{b-u}{a_i^- + 2\,a_o^-},\ z3 \to \frac{b-u}{a_i^- + 2\,a_o^-}\right\},\right.$$

$$\left\{z1 \to \frac{a_i^-\left(b-u-4\,a_m^\top\right) - b\,a_o^- + u\,a_o^- + 4\,a_m^\top\,a_o^- + \sqrt{\left(-a_i^- + a_o^-\right)\left(-a_i^-\left((b-u)^2 - 8\left(a_m^\top\right)^2\right) + \left(-b+u+4\,a_m^\top\right)^2 a_o^-\right)}}{2\,a_i^-\left(a_i^- - a_o^-\right)},\right.$$

$$z2 \to -\frac{a_i^-\left(-b+u+2\,a_m^\top\right) + b\,a_o^- - u\,a_o^- - 2\,a_m^\top\,a_o^- + \sqrt{\left(-a_i^- + a_o^-\right)\left(-a_i^-\left((b-u)^2 - 8\left(a_m^\top\right)^2\right) + \left(-b+u+4\,a_m^\top\right)^2 a_o^-\right)}}{2\left(\left(a_i^-\right)^2 - \left(a_o^-\right)^2\right)},$$

$$\left.z3 \to -\frac{a_i^-\left(-b+u+2\,a_m^\top\right) + b\,a_o^- - u\,a_o^- - 2\,a_m^\top\,a_o^- + \sqrt{\left(-a_i^- + a_o^-\right)\left(-a_i^-\left((b-u)^2 - 8\left(a_m^\top\right)^2\right) + \left(-b+u+4\,a_m^\top\right)^2 a_o^-\right)}}{2\left(\left(a_i^-\right)^2 - \left(a_o^-\right)^2\right)}\right\},$$

$$\left\{z1 \to \frac{a_i^-\left(b-u-4\,a_m^\top\right) - b\,a_o^- + u\,a_o^- + 4\,a_m^\top\,a_o^- - \sqrt{\left(-a_i^- + a_o^-\right)\left(-a_i^-\left((b-u)^2 - 8\left(a_m^\top\right)^2\right) + \left(-b+u+4\,a_m^\top\right)^2 a_o^-\right)}}{2\,a_i^-\left(a_i^- - a_o^-\right)},\right.$$

$$z2 \to \frac{a_i^-\left(b-u-2\,a_m^\top\right) - b\,a_o^- + u\,a_o^- + 2\,a_m^\top\,a_o^- + \sqrt{\left(-a_i^- + a_o^-\right)\left(-a_i^-\left((b-u)^2 - 8\left(a_m^\top\right)^2\right) + \left(-b+u+4\,a_m^\top\right)^2 a_o^-\right)}}{2\left(\left(a_i^-\right)^2 - \left(a_o^-\right)^2\right)},$$

$$\left.z3 \to \frac{a_i^-\left(b-u-2\,a_m^\top\right) - b\,a_o^- + u\,a_o^- + 2\,a_m^\top\,a_o^- + \sqrt{\left(-a_i^- + a_o^-\right)\left(-a_i^-\left((b-u)^2 - 8\left(a_m^\top\right)^2\right) + \left(-b+u+4\,a_m^\top\right)^2 a_o^-\right)}}{2\left(\left(a_i^-\right)^2 - \left(a_o^-\right)^2\right)}\right\},$$

$$\{z1\to\frac{a_1^-(b-u-2a_n^1)-ba_0^-+ua_0^-+2a_n^1a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}}{2((a_1^-)^2-(a_0^-)^2)},$$

$$z2\to\frac{1}{4((a_1^-)^3-a_1^-(a_0^-)^2)}\Bigg(2b(a_1^-)^2-2u(a_1^-)^2-6(a_1^-)^2a_n^1-ba_1^-a_0^-+ua_1^-a_0^-+2a_1^-a_n^1a_0^--b(a_0^-)^2+u(a_0^-)^2+4a_n^1(a_0^-)^2-a_0^-\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}+\sqrt{2}a_1^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)-(b-u-4a_n^1)(a_0^-)^2\Big((b-u-4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)+2(a_1^-)^2a_n^1\Big(7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(b-u-5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}+\sqrt{2}a_0^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)-(b-u-4a_n^1)(a_0^-)^2\Big((b-u-4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)+2(a_1^-)^2a_n^1\Big(7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(b-u-5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}\Bigg),$$

$$z3\to-\frac{1}{4((a_1^-)^3-a_1^-(a_0^-)^2)}\Bigg(-2b(a_1^-)^2+2u(a_1^-)^2+6(a_1^-)^2a_n^1+ba_1^-a_0^--ua_1^-a_0^--2a_1^-a_n^1a_0^-+b(a_0^-)^2-u(a_0^-)^2-4a_n^1(a_0^-)^2+a_0^-\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}+\sqrt{2}a_1^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)-(b-u-4a_n^1)(a_0^-)^2\Big((b-u-4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)+2(a_1^-)^2a_n^1\Big(7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(b-u-5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}+\sqrt{2}a_0^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)-(b-u-4a_n^1)(a_0^-)^2\Big((b-u-4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)+2(a_1^-)^2a_n^1\Big(7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(b-u-5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}\Bigg)\},$$

$$\{z1\to\frac{a_1^-(b-u-2a_n^1)-ba_0^-+ua_0^-+2a_n^1a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}}{2((a_1^-)^2-(a_0^-)^2)},$$

$$z2\to-\frac{1}{4((a_1^-)^3-a_1^-(a_0^-)^2)}\Bigg(-2b(a_1^-)^2+2u(a_1^-)^2+6(a_1^-)^2a_n^1+ba_1^-a_0^--ua_1^-a_0^--2a_1^-a_n^1a_0^-+b(a_0^-)^2-u(a_0^-)^2-4a_n^1(a_0^-)^2+a_0^-\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}+\sqrt{2}a_1^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)-(b-u-4a_n^1)(a_0^-)^2\Big((b-u-4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)+2(a_1^-)^2a_n^1\Big(7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(b-u-5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}+\sqrt{2}a_0^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)-(b-u-4a_n^1)(a_0^-)^2\Big((b-u-4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)+2(a_1^-)^2a_n^1\Big(7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(b-u-5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}\Bigg),$$

$$z3\to\frac{1}{4((a_1^-)^3-a_1^-(a_0^-)^2)}\Bigg(2b(a_1^-)^2-2u(a_1^-)^2-6(a_1^-)^2a_n^1-ba_1^-a_0^-+ua_1^-a_0^-+2a_1^-a_n^1a_0^--b(a_0^-)^2+u(a_0^-)^2+4a_n^1(a_0^-)^2-a_0^-\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}+\sqrt{2}a_1^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)-(b-u-4a_n^1)(a_0^-)^2\Big((b-u-4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)+2(a_1^-)^2a_n^1\Big(7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(b-u-5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}+\sqrt{2}a_0^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)-(b-u-4a_n^1)(a_0^-)^2\Big((b-u-4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)+2(a_1^-)^2a_n^1\Big(7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(b-u-5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}\Bigg)\},$$

$$\{z1\to\frac{a_1^-(b-u-2a_n^1)-ba_0^-+ua_0^-+2a_n^1a_0^--\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}}{2((a_1^-)^2-(a_0^-)^2)},$$

$$z2\to\frac{1}{4((a_1^-)^3-a_1^-(a_0^-)^2)}\Bigg(2b(a_1^-)^2-2u(a_1^-)^2-6(a_1^-)^2a_n^1-ba_1^-a_0^-+ua_1^-a_0^-+2a_1^-a_n^1a_0^--b(a_0^-)^2+u(a_0^-)^2+4a_n^1(a_0^-)^2+a_0^-\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}+\sqrt{2}a_1^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)+(b-u-4a_n^1)(a_0^-)^2\Big((-b+u+4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-2(a_1^-)^2a_n^1\Big(-7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(-b+u+5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}+\sqrt{2}a_0^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)+(b-u-4a_n^1)(a_0^-)^2\Big((-b+u+4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-2(a_1^-)^2a_n^1\Big(-7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(-b+u+5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}\Bigg),$$

$$z3\to\frac{1}{4((a_1^-)^3-a_1^-(a_0^-)^2)}\Bigg(2b(a_1^-)^2-2u(a_1^-)^2-6(a_1^-)^2a_n^1-ba_1^-a_0^-+ua_1^-a_0^-+2a_1^-a_n^1a_0^--b(a_0^-)^2+u(a_0^-)^2+4a_n^1(a_0^-)^2+a_0^-\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}-\sqrt{2}a_1^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)+(b-u-4a_n^1)(a_0^-)^2\Big((-b+u+4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-2(a_1^-)^2a_n^1\Big(-7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(-b+u+5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}-\sqrt{2}a_0^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)+(b-u-4a_n^1)(a_0^-)^2\Big((-b+u+4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-2(a_1^-)^2a_n^1\Big(-7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(-b+u+5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}\Bigg)\},$$

$$\{z1\to\frac{a_1^-(b-u-2a_n^1)-ba_0^-+ua_0^-+2a_n^1a_0^--\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}}{2((a_1^-)^2-(a_0^-)^2)},$$

$$z2\to\frac{1}{4((a_1^-)^3-a_1^-(a_0^-)^2)}\Bigg(2b(a_1^-)^2-2u(a_1^-)^2-6(a_1^-)^2a_n^1-ba_1^-a_0^-+ua_1^-a_0^-+2a_1^-a_n^1a_0^--b(a_0^-)^2+u(a_0^-)^2+4a_n^1(a_0^-)^2+a_0^-\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}-\sqrt{2}a_1^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)+(b-u-4a_n^1)(a_0^-)^2\Big((-b+u+4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-2(a_1^-)^2a_n^1\Big(-7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(-b+u+5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}-\sqrt{2}a_0^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)+(b-u-4a_n^1)(a_0^-)^2\Big((-b+u+4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-2(a_1^-)^2a_n^1\Big(-7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(-b+u+5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}\Bigg),$$

$$z3\to\frac{1}{4((a_1^-)^3-a_1^-(a_0^-)^2)}\Bigg(2b(a_1^-)^2-2u(a_1^-)^2-6(a_1^-)^2a_n^1-ba_1^-a_0^-+ua_1^-a_0^-+2a_1^-a_n^1a_0^--b(a_0^-)^2+u(a_0^-)^2+4a_n^1(a_0^-)^2+a_0^-\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}+\sqrt{2}a_1^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)+(b-u-4a_n^1)(a_0^-)^2\Big((-b+u+4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-2(a_1^-)^2a_n^1\Big(-7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(-b+u+5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}+\sqrt{2}a_0^-\sqrt{\Big(\frac{1}{(a_1^-+a_0^-)^2}(a_1^--a_0^-)\Big(2(a_1^-)^3((b-u)^2-7(a_n^1)^2)+(b-u-4a_n^1)(a_0^-)^2\Big((-b+u+4a_n^1)a_0^-+\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-2(a_1^-)^2a_n^1\Big(-7(b-u-3a_n^1)a_0^-+2\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)-a_1^-a_0^-\Big(((b-u)^2-14(b-u)a_n^1+36(a_n^1)^2)a_0^-+2(-b+u+5a_n^1)\sqrt{(-a_1^-+a_0^-)(-a_1^-((b-u)^2-8(a_n^1)^2)+(-b+u+4a_n^1)^2a_0^-)}\Big)\Big)\Big)}\Bigg)\}\}$$

EigenValue Method

We need the Jacobian matrix of the system *f*. Then we find the eigenvalues of the Jacobian at each equilbirum point.

$$J_g\left(Z_1, Z_2, Z_3\right) = \begin{pmatrix} \frac{\partial g1}{\partial z_1} & \frac{\partial g1}{\partial z_2} & \frac{\partial g1}{\partial z_3} \\ \frac{\partial g2}{\partial z_1} & \frac{\partial g2}{\partial z_2} & \frac{\partial g2}{\partial z_3} \\ \frac{\partial g3}{\partial z_1} & \frac{\partial g3}{\partial z_2} & \frac{\partial g3}{\partial z_3} \end{pmatrix}$$

$$J_g = \begin{pmatrix} b - u - 2\,z1\,a_i^- - 2\,a_m^\dagger - z2\,a_o^- - z3\,a_o^- & a_m^\dagger - z1\,a_o^- & a_m^\dagger - z1\,a_o^- \\ a_m^\dagger - z2\,a_o^- & b - u - 2\,z2\,a_i^- - 2\,a_m^\dagger - z1\,a_o^- - z3\,a_o^- & a_m^\dagger - z2\,a_o^- \\ a_m^\dagger - z3\,a_o^- & a_m^\dagger - z3\,a_o^- & b - u - 2\,z3\,a_i^- - 2\,a_m^\dagger - z1\,a_o^- - z2\,a_o^- \end{pmatrix}$$

EigenValue Method (continued)

We define a few general conditions because they have to make sense biologically.

$b > u > 0$

$a^{\dagger} \geq 0$, $a_i \geq 0$, $a_O \geq 0$

$a_i \vee a_o > 0$

We now we find the eigenvalues of each of the equilibrium points we just found.

```
z1 = 0; z2 = 0; z3 = 0;
MatrixForm[A]
Simplify[Eigenvalues[A]]
```

$\{b - u,\ b - u - 3\, a_m^{\dagger},\ b - u - 3\, a_m^{\dagger}\}$

(b – u) will never be $<$ 0 so this point is unstable.

$$z1 = \frac{b - u}{a_i^- + 2\, a_o^-};\ z2 = \frac{b - u}{a_i^- + 2\, a_o^-};\ z3 = \frac{b - u}{a_i^- + 2\, a_o^-};$$

$$\left\{-b + u,\ \frac{a_i^- \left(-b + u - 3\, a_m^{\dagger}\right) + \left(b - u - 6\, a_m^{\dagger}\right) a_o^-}{a_i^- + 2\, a_o^-},\ \frac{a_i^- \left(-b + u - 3\, a_m^{\dagger}\right) + \left(b - u - 6\, a_m^{\dagger}\right) a_o^-}{a_i^- + 2\, a_o^-}\right\}$$

[illegible]

Results

3-Box Model : Special Conditions

Now that we have a list of eigenvalues for each equilibrium point, we make several special cases to further constrict the solution set for stability.

(a) b, u , a_o, $a^\dagger > 0$, $a_i = 0$
(b) b, u , a_i, $a^\dagger > 0$, $a_o = 0$
(c) b, u , a_o, $a_i > 0$, $a^\dagger = 0$

EQ Point (0,0) is unstable in every special case.

EQ Point $\left(\frac{b-u}{a_i^- + 2\, a_o^-}, \frac{b-u}{a_i^- + 2\, a_o^-}, \frac{b-u}{a_i^- + 2\, a_o^-}\right)$ **is stable in these scenarios:**

i) When $a_i = 0$, if $\left(b - u - 6\, a_m^\dagger\right) < 0$

ii) When $a_o = 0$, **if** $\left(-b + u - 3\, a_m^\dagger\right) < 0$ **{always Stable}**

iii) When a_i and $a_o > 0$, if $a_i > a_o$ and $b > a_m^\dagger$.

Classifying EQ Point 3-8 and higher are still in progress.

We have stability when the following inequalities hold.

References

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Thank You!

Discussion and Questions