

Classifying Equilibria with Unknown Parameters for a Dynamic Population Model

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Overview

We will investigate a model for population dynamics introduced by Bolker and Pacala.

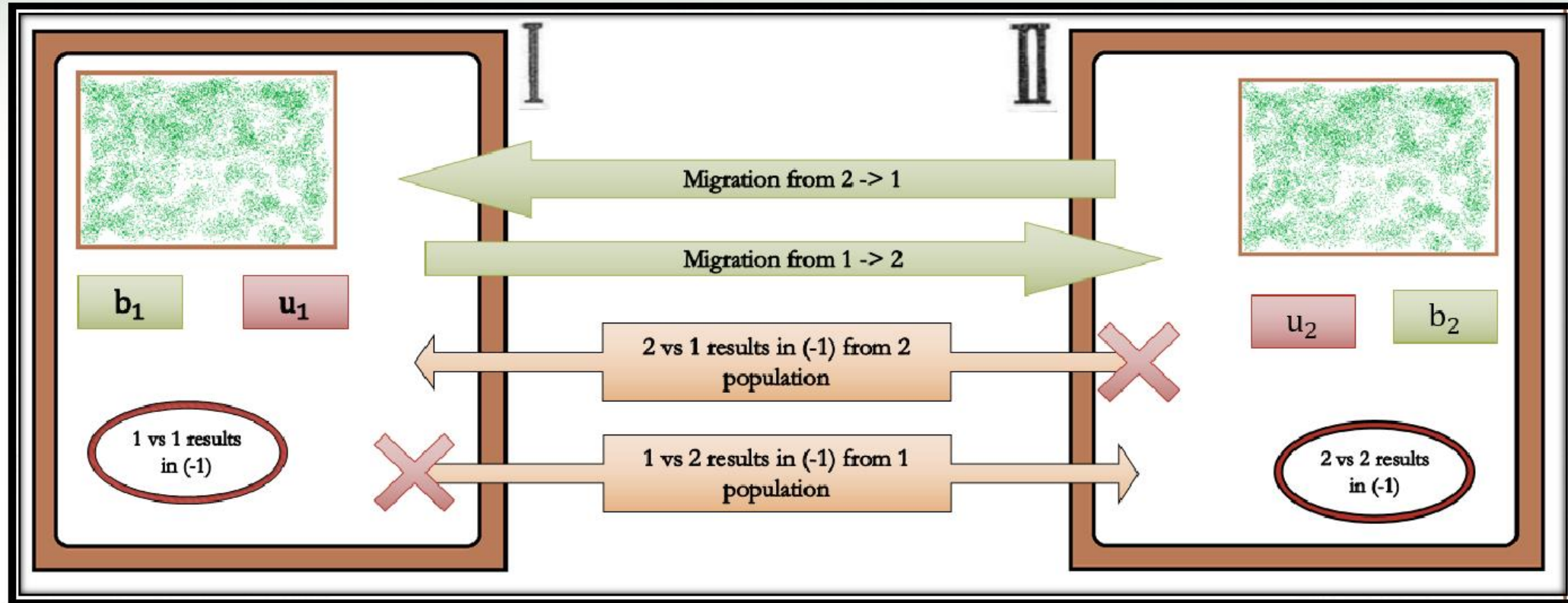
- ☞ Use a Mean Field Approximation to help figure out the domain for our unspecified parameters.
 - ☞ Generating a system of differential equations for the 2-Box model.
 - ☞ Find the Equilibrium Points.
 - ☞ EigenValue Method.
 - ☞ Find the conditions on the parameters by hand.
- ☞ We find that there is a stable equilibrium point for the 2-Box and 3-Box model.
- ☞ We find that there is an unstable equilibrium point always for the 2-Box and 3-Box model.

*My mentor that I am working with is Professor Mariya Bessonov.

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Introduction

2-Box Model



Analysis

2 – Box Model System

The function f represents the change in population density for the original 2 – Box model

z_i represents the population divided by L (location), which is the population density

$$f_1 = \frac{dz_1}{dt} = b_1 z_1 - u_1 z_1 - z_1^2 a_{11}^- - z_1 a_{12}^+ - z_1 z_2 a_{12}^- + z_2 a_{21}^+$$

$$f_2 = \frac{dz_2}{dt} = b_2 z_2 - u_2 z_2 + z_1 a_{12}^+ - z_2 a_{21}^+ - z_1 z_2 a_{21}^- - z_2^2 a_{22}^-$$

The function g is the basic case of 2 – Box model when each box has equal birth death, inner, outer and migration competition rates. These are our initial conditions on the parameters

$$g_1 = \frac{dz_1}{dt} = b z_1 - u z_1 - z_1 a_{12}^+ + z_2 a_{12}^+ - z_1^2 a_i^- - z_1 z_2 a_o^-$$

$$g_2 = \frac{dz_2}{dt} = b z_2 - u z_2 + z_1 a_{12}^+ - z_2 a_{12}^+ - z_2^2 a_i^- - z_1 z_2 a_o^-$$

2 - Box Model Equilibrium Points

To find our EQ points we must set the system equal to zero. We generate four EQ points.

$$\{ \{ z1 \rightarrow 0, z2 \rightarrow 0 \},$$

|

$$\{ z1 \rightarrow \frac{b-u}{a_1^- + a_0^-}, z2 \rightarrow \frac{b-u}{a_1^- + a_0^-} \},$$

$$\left\{ z1 \rightarrow \frac{1}{2 a_1^- (a_1^- - a_0^-)} \left((b-u-2 a_{12}^+) a_1^- - b a_0^- + u a_0^- + 2 a_{12}^+ a_0^- + \sqrt{((b-u-2 a_{12}^+) (a_1^- - a_0^-) ((b-u+2 a_{12}^+) a_1^- + (-b+u+2 a_{12}^+) a_0^-))} \right), \right. \\ \left. z2 \rightarrow -\frac{1}{2 a_1^- (a_1^- - a_0^-)} \left((-b+u+2 a_{12}^+) a_1^- + b a_0^- - u a_0^- - 2 a_{12}^+ a_0^- + \sqrt{((b-u-2 a_{12}^+) (a_1^- - a_0^-) ((b-u+2 a_{12}^+) a_1^- + (-b+u+2 a_{12}^+) a_0^-))} \right) \right\},$$

$$\left\{ z1 \rightarrow \frac{1}{2 a_1^- (a_1^- - a_0^-)} \left((b-u-2 a_{12}^+) a_1^- - b a_0^- + u a_0^- + 2 a_{12}^+ a_0^- - \sqrt{((b-u-2 a_{12}^+) (a_1^- - a_0^-) ((b-u+2 a_{12}^+) a_1^- + (-b+u+2 a_{12}^+) a_0^-))} \right), \right. \\ \left. z2 \rightarrow \frac{1}{2 a_1^- (a_1^- - a_0^-)} \left((b-u-2 a_{12}^+) a_1^- - b a_0^- + u a_0^- + 2 a_{12}^+ a_0^- + \sqrt{((b-u-2 a_{12}^+) (a_1^- - a_0^-) ((b-u+2 a_{12}^+) a_1^- + (-b+u+2 a_{12}^+) a_0^-))} \right) \right\}$$

EigenValue Method

We need the Jacobian matrix of the system f . Then we find the eigenvalues of the Jacobian at each equilibrium point.

$$J_g(Z_1, Z_2) = \begin{pmatrix} \frac{\partial g_1}{\partial z_1} & \frac{\partial g_1}{\partial z_2} \\ \frac{\partial g_2}{\partial z_1} & \frac{\partial g_2}{\partial z_2} \end{pmatrix}$$

$$J_g = \begin{pmatrix} b - u - a_{12}^\dagger - 2 z_1 a_i^- - z_2 a_o^- & a_{12}^\dagger - z_1 a_o^- \\ a_{12}^\dagger - z_2 a_o^- & b - u - a_{12}^\dagger - 2 z_2 a_i^- - z_1 a_o^- \end{pmatrix}$$

EigenValue Method (continued)

We define a few general conditions because they have to make sense biologically.

$$b > u > 0$$

$$a^{\dagger} \geq 0, a_i \geq 0, a_o \geq 0$$

$$a_i \vee a_o > 0$$

We now we find the eigenvalues of each of the equilibrium points we just found.

$$z1 = 0;$$

$$z2 = 0;$$

$$\text{MatrixForm}[\mathbf{A}]$$

$$\text{Simplify}[\text{Eigenvalues}[\mathbf{A}]]$$

$$\begin{pmatrix} b - u - a_{12}^{\dagger} & a_{12}^{\dagger} \\ a_{12}^{\dagger} & b - u - a_{12}^{\dagger} \end{pmatrix}$$

$$\{b - u, b - u - 2 a_{12}^{\dagger}\}$$

$$z1 = \frac{b - u}{a_i^{-} + a_o^{-}}; \quad z2 = \frac{b - u}{a_i^{-} + a_o^{-}};$$

$$\left\{ -b + u, \frac{1}{a_i^{-} + a_o^{-}} \left((-b + u - 2 a_{12}^{\dagger}) a_i^{-} + (b - u - 2 a_{12}^{\dagger}) a_o^{-} \right) \right\}$$

$$z1 = \frac{1}{2 a_i^- (a_i^- - a_o^-)} \left((b - u - 2 a_{12}^\dagger) a_i^- - b a_o^- + u a_o^- + 2 a_{12}^\dagger a_o^- + \sqrt{((b - u - 2 a_{12}^\dagger) (a_i^- - a_o^-) ((b - u + 2 a_{12}^\dagger) a_i^- + (-b + u + 2 a_{12}^\dagger) a_o^-))} \right);$$

$$z2 = -\frac{1}{2 a_i^- (a_i^- - a_o^-)} \left((-b + u + 2 a_{12}^\dagger) a_i^- + b a_o^- - u a_o^- - 2 a_{12}^\dagger a_o^- + \sqrt{((b - u - 2 a_{12}^\dagger) (a_i^- - a_o^-) ((b - u + 2 a_{12}^\dagger) a_i^- + (-b + u + 2 a_{12}^\dagger) a_o^-))} \right);$$

$$\left\{ -\frac{1}{2 a_i^- (a_i^- - a_o^-)} \left(b a_i^- a_o^- - u a_i^- a_o^- - b (a_o^-)^2 + u (a_o^-)^2 - 2 a_{12}^\dagger ((a_i^-)^2 - (a_o^-)^2) + \sqrt{((a_i^- - a_o^-)^2 (4 ((b - u)^2 - 3 (a_{12}^\dagger)^2) (a_i^-)^2 - 4 ((b - u)^2 - 3 (b - u) a_{12}^\dagger + 2 (a_{12}^\dagger)^2) a_i^- a_o^- + (-b + u + 2 a_{12}^\dagger)^2 (a_o^-)^2)} \right), \right. \\ \left. \frac{1}{2 a_i^- (a_i^- - a_o^-)} \left(-b a_i^- a_o^- + u a_i^- a_o^- + b (a_o^-)^2 - u (a_o^-)^2 + 2 a_{12}^\dagger ((a_i^-)^2 - (a_o^-)^2) + \sqrt{((a_i^- - a_o^-)^2 (4 ((b - u)^2 - 3 (a_{12}^\dagger)^2) (a_i^-)^2 - 4 ((b - u)^2 - 3 (b - u) a_{12}^\dagger + 2 (a_{12}^\dagger)^2) a_i^- a_o^- + (-b + u + 2 a_{12}^\dagger)^2 (a_o^-)^2)} \right) \right\}$$

$$z1 = \frac{1}{2 a_i^- (a_i^- - a_o^-)} \left((b - u - 2 a_{12}^\dagger) a_i^- - b a_o^- + u a_o^- + 2 a_{12}^\dagger a_o^- - \sqrt{((b - u - 2 a_{12}^\dagger) (a_i^- - a_o^-) ((b - u + 2 a_{12}^\dagger) a_i^- + (-b + u + 2 a_{12}^\dagger) a_o^-))} \right);$$

$$z2 = \frac{1}{2 a_i^- (a_i^- - a_o^-)} \left((b - u - 2 a_{12}^\dagger) a_i^- - b a_o^- + u a_o^- + 2 a_{12}^\dagger a_o^- + \sqrt{((b - u - 2 a_{12}^\dagger) (a_i^- - a_o^-) ((b - u + 2 a_{12}^\dagger) a_i^- + (-b + u + 2 a_{12}^\dagger) a_o^-))} \right);$$

$$\left\{ -\frac{1}{2 a_i^- (a_i^- - a_o^-)} \left(b a_i^- a_o^- - u a_i^- a_o^- - b (a_o^-)^2 + u (a_o^-)^2 - 2 a_{12}^\dagger ((a_i^-)^2 - (a_o^-)^2) + \sqrt{((a_i^- - a_o^-)^2 (4 ((b - u)^2 - 3 (a_{12}^\dagger)^2) (a_i^-)^2 - 4 ((b - u)^2 - 3 (b - u) a_{12}^\dagger + 2 (a_{12}^\dagger)^2) a_i^- a_o^- + (-b + u + 2 a_{12}^\dagger)^2 (a_o^-)^2)} \right), \right. \\ \left. \frac{1}{2 a_i^- (a_i^- - a_o^-)} \left(-b a_i^- a_o^- + u a_i^- a_o^- + b (a_o^-)^2 - u (a_o^-)^2 + 2 a_{12}^\dagger ((a_i^-)^2 - (a_o^-)^2) + \sqrt{((a_i^- - a_o^-)^2 (4 ((b - u)^2 - 3 (a_{12}^\dagger)^2) (a_i^-)^2 - 4 ((b - u)^2 - 3 (b - u) a_{12}^\dagger + 2 (a_{12}^\dagger)^2) a_i^- a_o^- + (-b + u + 2 a_{12}^\dagger)^2 (a_o^-)^2)} \right) \right\}$$

2-Box Model : Special Conditions

Now that we have a list of eigenvalues for each equilibrium point, we make several special cases to further constrict the solution set for stability.

(a) b, u, a_o, a^+ , $a_i = 0$

(b) b, u, a_i, a^+ , $a_o = 0$

(c) b, u, a_o, a_i , $a^+ = 0$

$$z1 = \frac{b-u}{a_i^- + a_o^-}; \quad z2 = \frac{b-u}{a_i^- + a_o^-};$$

$$\left\{ -b + u, \frac{1}{a_i^- + a_o^-} \left((-b + u - 2 a_{12}^+) a_i^- + (b - u - 2 a_{12}^+) a_o^- \right) \right\}$$

Results

EQ Point (0,0) is unstable in every special case.

EQ Point $(\frac{b-u}{a_i+a_o}, \frac{b-u}{a_i+a_o})$ is stable in these scenarios:

- i) When $a_i = 0$, if $(b-u-2a^\dagger) < 0$
- ii) When $a_o = 0$, if $(-b+u-2a^\dagger) < 0$ {always Stable}
- iii) When a_i and $a_o > 0$, if $a_i > a_o$.

EQ Point 3 and 4 have several scenarios also:

- i) Is unstable when $a_i = 0$.
- ii) Is stable when $a_o = 0$, if $(a^\dagger + -\frac{1}{2} a_i \sqrt{-4 (b-u+3 a_i^2 - \alpha^\dagger) \alpha^\dagger + (b-u)^2 [1+4 a_i^2]} < 0)$
and if $(-a^\dagger - \frac{1}{2} a_i \sqrt{-4 (b-u+3 a_i^2 - \alpha^\dagger) \alpha^\dagger + (b-u)^2 [1+4 a_i^2]} < 0)$
- iii) When $a^\dagger = 0$, if $\frac{(-b+u) (a_o + \sqrt{1-4 a_o a_i + 4 a_i^2})}{2 a_i} < 0$ and if $\frac{-(b+u) a_i a_o - (b-u) a_o^2}{2 a_i (a_i - a_o)} - \frac{((b-u) \sqrt{1-4 a_o a_i + 4 a_i^2})}{2 a_i} < 0$

We have stability when the following inequalities hold.

Analysis

3 – Box Model System

The function f represents the change in population density for the original 3 – Box model
 z_i represents the population divided by L (location), which is the population density.

$$f_1 = \frac{dz_1}{dt} = b_1 z_1 - u_1 z_1 - z_1^2 a_{11}^- - z_1 a_{12}^+ - z_1 z_2 a_{12}^- - z_1 a_{13}^+ - z_1 z_3 a_{13}^- + z_2 a_{21}^+ + z_3 a_{31}^+$$

$$f_2 = \frac{dz_2}{dt} = b_2 z_2 - u_2 z_2 + z_1 a_{12}^+ - z_2 a_{21}^+ - z_1 z_2 a_{21}^- - z_2^2 a_{22}^- - z_2 a_{23}^+ - z_2 z_3 a_{23}^- + z_3 a_{32}^+$$

$$f_3 = \frac{dz_3}{dt} = b_3 z_3 - u_3 z_3 + z_1 a_{13}^+ + z_2 a_{23}^+ - z_3 a_{31}^+ - z_1 z_3 a_{31}^- - z_3 a_{32}^+ - z_2 z_3 a_{32}^- - z_3^2 a_{33}^-$$

The function g is the basic case of 2 – Box model when each box has equal birth death, inner, outer and migration competition rates. These are our initial conditions on the parameters.

$$g_1 = \frac{dz_1}{dt} = b z_1 - u z_1 - z_1^2 a_i^- - 2 z_1 a_m^+ + z_2 a_m^+ + z_3 a_m^+ - z_1 z_2 a_o^- - z_1 z_3 a_o^-$$

$$g_2 = \frac{dz_2}{dt} = b z_2 - u z_2 - z_2^2 a_i^- + z_1 a_m^+ - 2 z_2 a_m^+ + z_3 a_m^+ - z_1 z_2 a_o^- - z_2 z_3 a_o^-$$

$$g_3 = \frac{dz_3}{dt} = b z_3 - u z_3 - z_3^2 a_i^- + z_1 a_m^+ + z_2 a_m^+ - 2 z_3 a_m^+ - z_1 z_3 a_o^- - z_2 z_3 a_o^-$$

3 - Box Model Equilibrium Points

To find our EQ points we must set the system equal to zero.

$$\left\{ \{z_1 \rightarrow 0, z_2 \rightarrow 0, z_3 \rightarrow 0\}, \left\{ z_1 \rightarrow \frac{b-u}{a_i^- + 2a_o^-}, z_2 \rightarrow \frac{b-u}{a_i^- + 2a_o^-}, z_3 \rightarrow \frac{b-u}{a_i^- + 2a_o^-} \right\}, \right.$$

$$\left\{ z_1 \rightarrow \frac{a_i^- (b-u-4a_m^+) - b a_o^- + u a_o^- + 4 a_m^+ a_o^- + \sqrt{(-a_i^- + a_o^-) (-a_i^- ((b-u)^2 - 8(a_m^+)^2) + (-b+u+4a_m^+)^2 a_o^-}}{2 a_i^- (a_i^- - a_o^-)}, \right.$$

$$z_2 \rightarrow -\frac{a_i^- (-b+u+2a_m^+) + b a_o^- - u a_o^- - 2 a_m^+ a_o^- + \sqrt{(-a_i^- + a_o^-) (-a_i^- ((b-u)^2 - 8(a_m^+)^2) + (-b+u+4a_m^+)^2 a_o^-}}{2 ((a_i^-)^2 - (a_o^-)^2)},$$

$$z_3 \rightarrow -\frac{a_i^- (-b+u+2a_m^+) + b a_o^- - u a_o^- - 2 a_m^+ a_o^- + \sqrt{(-a_i^- + a_o^-) (-a_i^- ((b-u)^2 - 8(a_m^+)^2) + (-b+u+4a_m^+)^2 a_o^-}}{2 ((a_i^-)^2 - (a_o^-)^2)} \left. \right\},$$

$$\left\{ z_1 \rightarrow \frac{a_i^- (b-u-4a_m^+) - b a_o^- + u a_o^- + 4 a_m^+ a_o^- - \sqrt{(-a_i^- + a_o^-) (-a_i^- ((b-u)^2 - 8(a_m^+)^2) + (-b+u+4a_m^+)^2 a_o^-}}{2 a_i^- (a_i^- - a_o^-)}, \right.$$

$$z_2 \rightarrow \frac{a_i^- (b-u-2a_m^+) - b a_o^- + u a_o^- + 2 a_m^+ a_o^- + \sqrt{(-a_i^- + a_o^-) (-a_i^- ((b-u)^2 - 8(a_m^+)^2) + (-b+u+4a_m^+)^2 a_o^-}}{2 ((a_i^-)^2 - (a_o^-)^2)},$$

$$z_3 \rightarrow \frac{a_i^- (b-u-2a_m^+) - b a_o^- + u a_o^- + 2 a_m^+ a_o^- + \sqrt{(-a_i^- + a_o^-) (-a_i^- ((b-u)^2 - 8(a_m^+)^2) + (-b+u+4a_m^+)^2 a_o^-}}{2 ((a_i^-)^2 - (a_o^-)^2)} \left. \right\},$$

[illegible]

EigenValue Method

We need the Jacobian matrix of the system f . Then we find the eigenvalues of the Jacobian at each equilibrium point.

$$J_g(z_1, z_2, z_3) = \begin{pmatrix} \frac{\partial g_1}{\partial z_1} & \frac{\partial g_1}{\partial z_2} & \frac{\partial g_1}{\partial z_3} \\ \frac{\partial g_2}{\partial z_1} & \frac{\partial g_2}{\partial z_2} & \frac{\partial g_2}{\partial z_3} \\ \frac{\partial g_3}{\partial z_1} & \frac{\partial g_3}{\partial z_2} & \frac{\partial g_3}{\partial z_3} \end{pmatrix}$$

$$J_g = \begin{pmatrix} b - u - 2 z_1 a_i^- - 2 a_m^+ - z_2 a_o^- - z_3 a_o^- & a_m^+ - z_1 a_o^- & a_m^+ - z_1 a_o^- \\ a_m^+ - z_2 a_o^- & b - u - 2 z_2 a_i^- - 2 a_m^+ - z_1 a_o^- - z_3 a_o^- & a_m^+ - z_2 a_o^- \\ a_m^+ - z_3 a_o^- & a_m^+ - z_3 a_o^- & b - u - 2 z_3 a_i^- - 2 a_m^+ - z_1 a_o^- - z_2 a_o^- \end{pmatrix}$$

EigenValue Method (continued)

We define a few general conditions because they have to make sense biologically.

$$b > u > 0$$

$$a^{\dagger} \geq 0, a_i \geq 0, a_0 \geq 0$$

$$a_i \vee a_0 > 0$$

We now we find the eigenvalues of each of the equilibrium points we just found.

$$z1 = 0; z2 = 0; z3 = 0;$$

$$\text{MatrixForm}[A]$$

$$\text{Simplify}[\text{Eigenvalues}[A]]$$

$$\{b - u, b - u - 3 a_m^{\dagger}, b - u - 3 a_m^{\dagger}\}$$

$(b - u)$ will never be < 0 so this point is unstable.

$$z1 = \frac{b - u}{a_i^{-} + 2 a_0^{-}}; z2 = \frac{b - u}{a_i^{-} + 2 a_0^{-}}; z3 = \frac{b - u}{a_i^{-} + 2 a_0^{-}};$$

$$\left\{ -b + u, \frac{a_i^{-} (-b + u - 3 a_m^{\dagger}) + (b - u - 6 a_m^{\dagger}) a_0^{-}}{a_i^{-} + 2 a_0^{-}}, \frac{a_i^{-} (-b + u - 3 a_m^{\dagger}) + (b - u - 6 a_m^{\dagger}) a_0^{-}}{a_i^{-} + 2 a_0^{-}} \right\}$$

$$d_{\mathbf{A}}^{\mathbf{B}} = \frac{\sqrt{|\langle \mathbf{a}^{\mathbf{B}} | \mathbf{a}^{\mathbf{A}} \rangle|^2 + |\langle \mathbf{a}^{\mathbf{B}} | \mathbf{a}^{\mathbf{A}'} \rangle|^2}}{3\sqrt{|\langle \mathbf{a}^{\mathbf{B}} | \mathbf{a}^{\mathbf{A}} \rangle|^2 + |\langle \mathbf{a}^{\mathbf{B}} | \mathbf{a}^{\mathbf{A}'} \rangle|^2}}, \quad d_{\mathbf{A}'}^{\mathbf{B}} = \frac{\sqrt{|\langle \mathbf{a}^{\mathbf{B}} | \mathbf{a}^{\mathbf{A}} \rangle|^2 + |\langle \mathbf{a}^{\mathbf{B}} | \mathbf{a}^{\mathbf{A}'} \rangle|^2}}{3\sqrt{|\langle \mathbf{a}^{\mathbf{B}} | \mathbf{a}^{\mathbf{A}} \rangle|^2 + |\langle \mathbf{a}^{\mathbf{B}} | \mathbf{a}^{\mathbf{A}'} \rangle|^2}}.$$

[illegible]

$$\frac{2}{2|k_1|^{2s} + 2\epsilon_1|k_1|^{2s}}$$

[illegible][illegible][illegible][illegible][illegible][illegible][illegible]
$$k^2 \left[(|b_1|^2 d + 2|b_2|^2 \tilde{a} + 2|b_3|^2 \tilde{a} + 4|b_4|^2 d + \tilde{a} + \tilde{a} d) |b_1|^2 + 2|b_2|^2 + 2|b_3|^2 + 4d |b_4|^2 + 3|b_1|^2 \sqrt{|b_2^2 \tilde{a} + |b_3|^2 (|b_1|^2 d + 2|b_2|^2 \tilde{a} + 2|b_3|^2 \tilde{a} + 4d) |b_4|^2} + 3|b_1|^2 \sqrt{|b_2^2 \tilde{a} + |b_3|^2 (|b_1|^2 d + 2|b_2|^2 \tilde{a} + 2|b_3|^2 \tilde{a} + 4d) |b_4|^2} \right]$$
[illegible][illegible]

$$\frac{2}{2|k_1|^2 + 2k_1|k_2|}$$

[illegible][illegible][illegible][illegible]

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260	261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	291	292	293	294	295	296	297	298	299	300	301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320	321	322	323	324	325	326	327	328	329	330	331	332	333	334	335	336	337	338	339	340	341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360	361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379	380	381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399	400	401	402	403	404	405	406	407	408	409	410	411	412	413	414	415	416	417	418	419	420	421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439	440	441	442	443	444	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459	460	461	462	463	464	465	466	467	468	469	470	471	472	473	474	475	476	477	478	479	480	481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499	500	501	502	503	504	505	506	507	508	509	510	511	512	513	514	515	516	517	518	519	520	521	522	523	52
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$$\frac{1}{\sqrt{\pi}} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) dx \right)^2 = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) dx$$

$$\frac{1}{2|a_1| + 2|a_2|}$$

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Results

3-Box Model : Special Conditions

Now that we have a list of eigenvalues for each equilibrium point, we make several special cases to further constrict the solution set for stability.

(a) $b, u, a_o, a^+ > 0, a_i = 0$

(b) $b, u, a_i, a^+ > 0, a_o = 0$

(c) $b, u, a_o, a_i > 0, a^+ = 0$

EQ Point (0,0) is unstable in every special case.

EQ Point $(\frac{b-u}{a_i+2a_o}, \frac{b-u}{a_i+2a_o}, \frac{b-u}{a_i+2a_o})$ is stable in these scenarios:

i) When $a_i = 0$, if $(b - u - 6a_m^+) < 0$

ii) When $a_o = 0$, if $(-b + u - 3a_m^+) < 0$ {always Stable}

iii) When a_i and $a_o > 0$, if $a_i > a_o$ and $b > a_m^+$.

Classifying EQ Point 3-8 and higher are still in progress.

We have stability when the following inequalities hold.

References

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Thank You!

Discussion and
Questions