Notes about the exam:

- For each question, write only your answer in the box on the exam sheet and turn in the exam sheet. Answers elsewhere will not be graded. Correct answers get full credit. You might get partial credit for incorrect answers if you show your work clearly on attached pages. No calculators are allowed.
- The exam is divided into two parts. Part 1 is worth 20 points and Part 2 is worth 40 points. The exam grade is the sum of both parts. To pass the exam, you need at least 30 points. If you passed the midterm (score of 14 and above) then write "I passed the midterm" in one of the answer boxes in Part 1. You can then skip Part 1 and receive 20 points.

Part 1 (20 pts)

For the questions on Part 1 of the exam use the following matrices and vectors.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 7 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 9 \end{bmatrix}, \; u = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \; v = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \; b = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}, \; c = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

1. (4 pts) Compute $u \cdot (v \times u) + v \cdot (v \times u) + (v \times u) \cdot (v \times u)$

2. (4 pts) Find the general solution to $Ax = b$, if it exists.

3. (4 pts) Compute $\det(AB)+\det(A + B)$

4. (4 pts) Compute B^{-1}

5. (4 pts) Use u, v, b, and c to answer 5a and 5b:

a. (2 pts) List all pairs of vectors that are linearly independent.

b. (2 pts) Write a set of vectors that form a basis for $\mathbb{R}^3$. If none exists, say why.

Part 2 (40 pts)

For the questions on Part 2 of the exam use the following matrices and vectors.

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad C = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

1. (12 pts) Use the A matrix defined above to answer questions 1a-1f.

a. (2 pts) Write a basis for the vectors in the null space of A

b. (2 pts) What is rank(A)?

c. (2 pts) What are the eigenvalues of A?

d. (2 pts) Write a basis for the vectors in the null space of A^2 (Hint: Compute A^2 first.)

e. (2 pts) What is the rank of A^2?

f. (2 pts) Compute A^3

2. (6 pts) Use the B matrix defined above to answer questions 2a-2c.

a. (2 pts) What are the eigenvalues of B if $\theta = \pi/2$?

b. (2 pts) What are the eigenvalues of B^2 if $\theta = \pi/2$?

c. (2 pts) What is B^4 if $\theta = \pi/2$?

3. (8 pts) Use C matrix defined above to answer questions 3a-3b.

a. (4 pts) What are the eigenvalues and eigenvectors of C?

b. (4 pts) If C can be diagonalized, write the matrices D and P such that $C = PDP^{-1}$. If C cannot be diagonalized, say why not.

4. (4 pts) Consider the equation $y = 2x_1^2 + 4x_2^2 - 2x_3^2 - 7x_1x_2 + 4x_1x_3$.

a. (2 pts) Write down a symmetric matrix Q such that $y = x^TQx$ where x is the column vector $x = [x_1, x_2, x_3]^T$, if possible. If it is not possible give a short reason why.

b. (2 pts) Write down a triangular matrix Q such that $y = x^TQx$ where x is the column vector $x = [x_1, x_2, x_3]^T$, if possible. If it is not possible give a short reason why.

5. (8 pts) Use the Euclidean inner product $\langle x, y\rangle = x \cdot y$ and the vectors $\{u_1, u_2, u_3\}$ defined above to answer questions 5a-5b.

a. (6 pts) Use the Gram-Schmidt process to create an orthonormal basis $\{v_1, v_2, v_3\}$ from vectors $\{u_1, u_2, u_3\}$. Note: they need to be normalized.

b. (2 pts) Compute $||v_1 + v_2|| - \langle v_1, v_2\rangle + ||v_2 + v_3|| - \langle v_2, v_3\rangle + ||v_1 + v_3|| - \langle v_1, v_3\rangle$. Hint: the answer is the same for any orthonormal set of vectors $\{v_1, v_2, v_3\}$.

6 (2 pts) What are two equivalent statements to the $\det(A) \neq 0$ for some square matrix A with n rows.