Notes about the exam:

- *Answer format:* Please clearly indicate your answers by drawing a box around them. You do not need to show work to get full credit. However, if you do not have the right answer, you may get partial credit if you show your work.

- *Exam structure:* The exam is divided into two parts. Part 1 covers the same material as the midterm exam and is worth a maximum of 2 towards your final grade in the class. If you would like to apply your score from your midterm exam, follow the instructions at the beginning of Part 1. Part 2 is worth a maximum of 3 towards your final grade in the class. Everyone must do this part in order to pass the class.

Part 1

If you would like to apply your midterm exam grade to Part 1, write as your answer to problem 1: *"Use my midterm exam grade for part 1."* If you write that, we will not grade Part 1 on this exam. Your midterm exam grade will be your grade for Part 1. If you do not write that sentence, then we will grade what you have answered on the final exam and ignore the midterm exam score.

1. (2 pts) Solve the linear systems of equations: $\{3x_1 + 4x_2 = 6\}$ and $\{x_1 + 2x_2 = 4\}$ and $\{7x_1 + 5x_2 = 2\}$ for x_1 and x_2. Use whatever method you like. If there are free variables use s and p. If there is no answer, write "no solution". Final answers should be written so that x_1 and x_2 are written in terms of numbers and free variables if they exist. For example, an answer of "$x_1 = -x_2$ and $x_2 = s$" is not acceptable but "$x_1 = -s$ and $x_2 = s$"is acceptable.

2. (8 pts) Use the following matrices and column vectors for the tasks below.

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 5 & 4 & 3 \\ 2 & 2 & 1 \end{bmatrix}, \; b_1 = \begin{bmatrix} 9 \\ 16 \\ 5 \end{bmatrix}, \; b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

a. (4 pts) What is A^{-1}? If A is not invertible then write "does not exist."

b. (2 pts) Solve for x in $Ax = b_1$ if possible. If there is no solution then write "does not exist." If there is more than one solution then write in a general parametric form.

c. (2 pts) Solve for x in $Ax = b_2$ if possible. If there is no solution then write "does not exist." If there is more than one solution then write in a general parametric form.

3. (4 pts) Compute det(A) for A=

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 3 \\ 4 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 3 \\ 7 & 6 & 0 & 2 & 7 \\ 1 & 5 & 0 & 0 & 0 \end{bmatrix}$$

4. (8 pts) Use the following column vectors for the tasks below.

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \; v_2 = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \; v_3 = \begin{bmatrix} 1 \\ -7 \\ 1 \end{bmatrix}, \; v_4 = \begin{bmatrix} -1 \\ -2 \\ -1.5 \end{bmatrix},$$

a. (2 pts) Are these vectors linearly independent? Why or why not? (Full credit requires a reason)

b. (2 pts) List all pairs of vectors that are orthogonal.

c. (2 pts) Which pair of vectors have the largest angle between them?

d. (2 pts) Write down a set of 3 orthogonal vectors from $\{v_1, v_2, v_3, v_4\}$. If there is no such set then use as many of the vectors here as possible and generate whatever vector(s) are needed.

5. (2 pts) Write the parametric equation for a line in R^3 that passes through point $[3, 1, 4]$ and is parallel to $[-1, 0, 3]$. Use s, p and q for whatever parameters that are needed.

Part 2

1. (10 pts) Consider the equation $y = 1x_1^2 + 2x_2^2 - 1x_3^2 - 2\sqrt{18}x_2x_3$.

a. (2 pts) Write down the A matrix such that $y = x^T Ax$ where x is the column vector $x = [x_1, x_2, x_3]$.

b. (6 pts) Calculate the eigenvalues and their corresponding eigenvectors.

c. (2 pts) List all pairs of eigenvectors that are orthogonal.

2. (6 pts) Use A for the tasks below.

$$A = \begin{bmatrix} 3 & 1 & -3 \\ 0 & -2 & 0 \\ 6 & 2.5 & -3 \end{bmatrix}$$

a. (4 pts) A has an eigenvalue of 3i whose corresponding eigenvector is $[1, 0, 1-\mathrm{i}]$. Find the other eigenvalues and their eigenvectors.

b. (2 pts) If A can be diagonalized in the form $P^{-1}AP = D$ Write the matrix P and its diagonal matrix D. If A cannot be diagonalized, write "not diagonalizable".

3. (2 pts) What are the eigenvalues of A^4 if A=

$$\begin{bmatrix} 1 & 0 & 0 \\ 7 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

4. (6 pts) Use A for the tasks below.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 5 \\ 6 & 1 & -7 \\ 3 & -7 & -3 \end{bmatrix}$$

a. (4 pts) Write a basis for the vectors in the null space of A^T.

b. (2 pts) What is the dimension of the null space of A?

5. (2 pts) Prove the following: "If v is a vector in a real vector space R^n and w is the projection of v onto some other vector z in the same space then $||v|| \geq ||w||$."

6. (2 pts) Suppose the vector u in R^n can be expressed as a linear combination of linearly independent vectors $\{v_1, v_2, \ldots v_n\}$ such that $u = c_1v_1 + c_2v_2 + \ldots c_nv_n$. Prove the following: "The set of scalar coefficients $\{c_1, c_2, \ldots\}$ is unique such that there are no other ways to express u as a linear combination of vectors $\{v_1, v_2, \ldots v_n\}$." Hint: assume that u can be represented in two ways and prove by contradiction.

7. (4 pts) **BONUS** Find a P matrix that orthogonally diagonalizes the matrix A such that $D = P^T AP$ for

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$